Astérisque

LUC ILLUSIE

An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology

Astérisque, tome 279 (2002), p. 271-322

<http://www.numdam.org/item?id=AST_2002__279__271_0>

Astérisque
279, 2002, p. 271–322

AN OVERVIEW OF THE WORK OF K. FUJIWARA, K. KATO, AND C. NAKAYAMA ON LOGARITHMIC ÉTALE COHOMOLOGY

by

Luc Illusie

Abstract. — This paper is a report on the work of K. Fujiwara, K. Kato and C. Nakayama on log étale cohomology of log schemes. After recalling basic terminology and facts on log schemes we define and study a class of log étale morphisms of log schemes, called Kummer étale morphisms, which generalize the tamely ramified morphisms of classical algebraic geometry. We discuss the associated topology and cohomology. The main results are comparison theorems with classical étale cohomology and log Betti cohomology, a theorem of invariance of Kummer étale cohomology under log blow-ups (for which we provide a complete proof) and a local acyclicity theorem for log smooth log schemes over the spectrum of a henselian discrete valuation ring, which implies tameness for the corresponding classical nearby cycles. In the last section we state results of K. Kato on log étale cohomology, where localization by Kummer étale morphisms is replaced by localization by all log étale morphisms.

These notes are a slightly expanded version of lectures given at the Centre Émile Borel of the Institut Henri Poincaré in June, 1997. Their purpose is to present a survey of the theory of log (= logarithmic) étale cohomology developed by Fujiwara, Kato, and Nakayama in the past few years. Though the results obtained in this field are not of the same magnitude as those pertaining to log crystalline cohomology and the p-adic comparison theorems, reported on at other places of these proceedings ([**Tsu**], [**Br-M**]), they shed a new light on classical questions of étale cohomology, such as the tameness of nearby cycles. The log techniques provide more natural proofs to known theorems as well as interesting generalizations and refinements. In order to give a flavor of these, let us fix some notations. Let $S = \operatorname{Spec} A$ be a henselian trait, with generic point $\eta = \operatorname{Spec} K$ and closed point $s = \operatorname{Spec} k$. Let p be the

2000 ***Mathematics Subject Classification.*** — 14A99, 14D05, 14D06, 14D10, 14E05, 14E20, 14E22, 14F20, 14F35, 14G20.
Key words and phrases. — Logarithmic geometry, monoid, log structure, log scheme, Kummer, log étale, log smooth, divisor with normal crossings, semistable reduction, covering, fundamental group, Betti cohomology, étale cohomology, ℓ-adic cohomology, tame, log blow-up, toric variety, acyclicity, nearby cycles, vanishing cycles, monodromy, weights, regular, purity, base change.

characteristic exponent of s and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ where n is an integer invertible on S. Fix a geometric point $\overline{\eta} = \operatorname{Spec} \overline{K}$ above η and let I (resp. P) denote the inertia (resp. wild inertia) subgroup of $G = \operatorname{Gal}(\overline{K}/K)$. Let X be a scheme over S. By a theorem of Rapoport-Zink [**R-Z**], it is known that if X is regular, the generic fiber X_η is smooth and the special fiber X_s is a divisor with simple normal crossings and multiplicities m_i's prime to p, then P acts trivially on the sheaves of nearby cycles $R^q\Psi\Lambda$. As a consequence of the theory of logarithmic étale cohomology, Nakayama [**Na 2**] shows that this conclusion still holds in cases where some of the m_i's are divisible by p but X underlies a log smooth and vertical log scheme over S (1.5, 8.3). Another striking corollary is that under the same assumption the complex of nearby cycles $R\Psi\Lambda$, as an object of the derived category of Λ-modules on the geometric special fiber with continuous action of G, depends only on the special fiber X_s endowed with its natural log structure, and in particular, when X has semistable reduction, depends only on the first infinitesimal neighborhood of X_s in X. In this latter case, this implies the degeneration at E_2 of the weight spectral sequences of Raporport-Zink [**R-Z**] and Steenbrink ([**Ste 1**], [**Ste 2**]).

The paper is organized as follows. For the convenience of the reader we have collected in section 1 some basic terminology on log schemes, whose language we will use freely. The basic reference for this is [**Ka 1**] (see also [**I1**]). The definition and main properties of the Kummer étale topology, which replaces, on fs log schemes, the classical étale topology on schemes, are discussed in section 2. The theory of the corresponding log fundamental group is sketched in sections 3 and 4. In section 5 we begin the study of Kummer log étale cohomology. We compare it both to classical étale cohomology and, in the case of log schemes over $\mathbb{C}$, to the "log Betti" cohomology developed by Kato-Nakayama [**K-N**]. Section 6, the longest of this paper, is devoted to a fundamental result of Fujiwara-Kato [**F-K**], namely the invariance of Kummer étale cohomology under log blow-up. Because of the key role this result plays in the applications to nearby cycles — and also because [**F-K**] as it stands is still unpublished — we give the proof in detail, with a simplification due to Ekedahl, who suggested it to us during the lectures. Nakayama's results on nearby cycles mentioned above are discussed in section 8, after some premiminaries on (log) cohomological purity in section 7. One drawback of Kummer étale cohomology is that unlike classical étale cohomology it lacks good finiteness and base change theorems, as Nakayama pointed out in [**Na 1**]. In section 9 we present an attempt of Kato to remedy this by working with a finer topology, in which localization by any log étale map is permitted, and discuss some open problems in this direction.

1. Log schemes

1.1. All monoids are assumed to be commutative with units and maps of monoids to carry the unit to the unit. The group envelope of a monoid P is denoted P^{gp}. A monoid P is called *integral* if the canonical map $P \to P^{\mathrm{gp}}$ is injective, and *saturated* if it is integral and for any $a \in P^{\mathrm{gp}}$, a is in P if and only if there exists $n \geqslant 1$ such that $a^n \in P$. If P is a monoid, we denote by P^* the subgroup of its invertible elements and we set $\overline{P} = P/P^*$; thus $\overline{P}^{\mathrm{gp}} = \mathrm{Coker}(P^* \to P^{\mathrm{gp}})$. We say P is *sharp* if $P^* = \{1\}$.

1.2. A *pre-log structure* on a scheme X is a pair (M, α) where M is a sheaf of monoids on the étale site of X and α is a homomorphism from M to the multiplicative monoid $\mathcal{O}_X$. A pre-log structure (M, α) is called a *log structure* if α induces an isomorphism from $\alpha^{-1}(\mathcal{O}_X^*)$ to $\mathcal{O}_X^*$. The log structure defined by the inclusion $\mathcal{O}_X^* \subset \mathcal{O}_X$ is called the *trivial log structure.* A *log scheme* is a triple (X, M, α), usually simply denoted X, consisting of a scheme X and a log structure (M, α) on X. The sheaf of monoids of a log scheme X is generally denoted by M_X, and the sheaf $\mathcal{O}_X^*$ is considered as a subsheaf of M_X by means of α. To avoid confusion, it is sometimes convenient to denote the underlying scheme by $\overset{\circ}{X}$. For any pre-log structure (M, α) on a scheme X there is defined a log structure (M^a, α^a) and a map $M \to M^a$ (compatible with α and α^a) which is universal in the obvious sense; this log structure is called the *associated log structure.* A map of log schemes $f : (X, M, \alpha) \to (Y, N, \beta)$ is a map of schemes $f : X \to Y$ together with a map of sheaves of monoids $f^{-1}N \to M$ compatible in the natural way with α and β. If $Y = (Y, N, \beta)$ is a log scheme and $f : X \to \overset{\circ}{Y}$ is just a map of schemes, then the log structure on X associated to $(f^{-1}N, f^{-1}\beta)$ is called the *inverse image log structure* and denoted f^*N. A map of log schemes $f : X = (X, M, \alpha) \to Y = (Y, N, \beta)$ is called *strict* if the natural map from f^*N to M is an isomorphism.

1.3. If P is a monoid, the inclusion $P \subset \mathbb{Z}[P]$ defines a pre-log structure on $\operatorname{Spec} \mathbb{Z}[P]$, whose associated log structure is called the *canonical* log structure. A (global) *chart, modeled on P*, of a log scheme X is a strict map of log schemes $X \to \operatorname{Spec} \mathbb{Z}[P]$ for some monoid P, where $\operatorname{Spec} \mathbb{Z}[P]$ is endowed with its canonical log structure. Giving such a chart is the same as giving a monoid P and a homomorphism from the constant sheaf of monoids P_X on X to M_X inducing an isomorphism on the associated log structures. A log scheme X is called *integral* if the stalk of M_X at each geometric point of X is integral, *fine* (resp. *fine and saturated*, or fs for short) if in addition, locally for the étale topology it admits a chart modeled on a finitely generated and integral (resp. finitely generated and saturated) monoid. Any fs log scheme admits (étale locally) a chart modeled on a torsionfree, fs (*i.e.* finitely generated and saturated) monoid (such monoids are the basic stones of the theory of toric varieties). A *log point* is an fs log scheme whose underlying scheme is the spectrum of a field k. It is called *trivial* if its log structure is trivial, *standard* if its log structure is associated to $(\mathbb{N} \to k, 1 \mapsto 0)$. If $f : X \to Y$ is a map of fine log schemes, a *chart* of f is a triple (a, b, u) where $a : X \to \operatorname{Spec} \mathbb{Z}[P]$ and $b : Y \to \operatorname{Spec} \mathbb{Z}[Q]$ are charts of X and Y and $u : Q \to P$ is a map of monoids such that the corresponding square of log schemes commutes; a chart of f exists étale locally (and P and Q can be chosen to be fs if X and Y are fs). It turns out that fs log schemes are the most useful objects of log geometry. The category of fs log schemes admits finite inverse and direct limits; in particular one can perform base change in this category (the result being slightly different from that in the category of fine or arbitrary log schemes).

1.4. If X is an fs log scheme, there is a largest open Zariski subset of X (possibly empty) on which the log structure is trivial, *i.e.* $\alpha : M \to \mathcal{O}^*$ is an isomorphism. It is called the *open subset of triviality* of the log structure of X and is sometimes denoted X_{triv}. This is the first basic geometric invariant of X. For example, if X is the toric scheme $\operatorname{Spec}\mathbb{Z}[P]$ (with P a torsionfree, fs monoid), endowed with its canonical log structure, X_{triv} is the torus $T = \operatorname{Spec}\mathbb{Z}[P^{\mathrm{gp}}]$ canonically embedded in X. Finer invariants are obtained by considering $\overline{M}^{\mathrm{gp}} = M^{\mathrm{gp}}/\mathcal{O}^*$, which is a constructible sheaf of torsionfree abelian groups, and the stratification

$$X = X_0 \supset \cdots \supset X_i \supset \cdots$$

where X_i is the *closed* (Zariski) subset of X where $\operatorname{rk}\overline{M}^{\mathrm{gp}} \geqslant i$; in particular, $X_{\mathrm{triv}} = X_0 - X_1$: in the toric case above, this is just the stratification by the closures of orbits of the action of T on X.

1.5. $f : X \to Y$ be a map of fine log schemes. One says that f is *log smooth* (resp. *log étale*) if étale locally (on X and Y) f admits a chart $c = (a, b, u : Q \to P)$ such that the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of u^{gp} are finite groups of order invertible on X and the map from X to $X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[Q]$ deduced from c is a smooth (resp. étale) map on the underlying schemes. (For an intrinsic definition, in terms of local infinitesimal liftings such as in the classical case, see [**Ka 1**].) Log smooth (resp. log étale) maps are stable under composition and arbitrary base change (either in the category of fine or fs log schemes). If X and Y are log étale log schemes over a log scheme S, any S-morphism from X to Y is log étale. if $f : X \to Y$ is a map of schemes, viewed as a map of log schemes with trivial log structures, then f is log smooth (resp. log étale) iff f is classically smooth (resp. étale).

1.6. A map $h : Q \to P$ of fs monoids is said to be *Kummer* if h is injective and for all $a \in P$ there exists $n \in \mathbb{N}$, $n \geqslant 1$, such that $na \in h(Q)$ (the monoid laws written additively). A map $f : X \to Y$ of fs log schemes is said to be *Kummer* if for all geometric point $\overline{x}$ of X with image $\overline{y}$ in Y, the natural map $\overline{M}_{\overline{y}} \to \overline{M}_{\overline{x}}$ is Kummer. A map $f : X \to Y$ of fs log schemes is said to be *Kummer étale* if it is both log étale and Kummer. If f is log étale, then f is Kummer if and only if f is *exact*, which means that $f^*M_Y \to M_X$ is exact at each stalk (a map $h : Q \to P$ of integral monoids is called *exact* if $Q = (h^{\mathrm{gp}})^{-1}(P)$ in Q^{gp}) ([**Na 1**], 2.1.2). One can also show that f is Kummer étale if and only if f is étale locally deduced by strict base change and étale localization from a map $\operatorname{Spec}\mathbb{Z}[h] : \operatorname{Spec}\mathbb{Z}[P] \to \operatorname{Spec}\mathbb{Z}[Q]$, where $h : Q \to P$ is a Kummer map such that $nP \subset h(Q)$ for some integer n invertible on X ([**Vi 1**], 1.2).

Log blow-ups provide other examples of log étale maps, see 6.1.

1.7. Let X be a locally noetherian regular scheme and let $D \subset X$ be a divisor with normal crossings. Let $j : U = X - D \hookrightarrow X$ be the corresponding open immersion. Then the inclusion $M_X = \mathcal{O}_X \cap j_*\mathcal{O}_U^* \hookrightarrow \mathcal{O}_X$ is an fs log structure on X, which is said to be *defined by* $X - D$ (or, sometimes, by D); in the case the pair (X, D) is that of a trait and its closed point, this log structure is called the *canonical log structure.* Étale locally X has a chart modeled on $\mathbb{N}^r$ (if $\prod_{1\leqslant i\leqslant r} t_i^{a_i}$ is a local equation of D where $(t_i)_{1\leqslant i\leqslant r}$ is part of a system of local parameters on X, $\mathbb{N}^r \to \mathcal{O}_X$, $(n_i) \mapsto \prod t_i^{n_i}$ is a local chart).

Let $S = \operatorname{Spec} A$ be a trait endowed with the canonical log structure and let X be an fs S-log scheme locally of finite type. Then X is log smooth over S if and only if étale locally X is strict (1.2) and smooth over $T = \operatorname{Spec} A[P]/(x - \pi)$, where π is a uniformizing parameter of A, P is an fs monoid, x is an element of P such that the order of the torsion part of $P^{\mathrm{gp}}/\langle x\rangle$ is invertible on X and the log structure of T is associated to the canonical map $P \to A[P]/(x - \pi)$. It follows from the theory of resolution of singularities for toric varieties (*cf.* [**Ka 4**] and [**Na 2**]) that after some log blow up (see 6.1) we may obtain a local model as above with $P = \mathbb{N}^r$ (which model is then regular, generically smooth, with reduced special fiber a divisor with normal crossings). The case of *semistable reduction* corresponds to $P = \mathbb{N}^r$ and $x = (1, \ldots, 1)$.

2. Kummer étale topology

2.1. Let X be an fs log scheme. The *Kummer étale site* of X, denoted

$$X_{\mathrm{ket}}$$

is defined as follows. The objects of X_{ket} are the $X - fs$ log schemes which are Kummer étale (1.6). If T, T' are objects of X_{ket}, a morphism from T to T' is an X-map $T \to T'$; any such map is again Kummer étale ([**Vi 1**], 1.5). The category X_{ket} admits finite inverse limits. The *Kummer étale topology* is the topology on X_{ket} generated by the covering families $(u_i : T_i \to T)_{i\in I}$ of maps of X_{ket} such that T is set theoretically the union of the images of the u_i; the Kummer étale site of X is the category X_{ket} equipped with the Kummer étale topology; the *Kummer étale topos* of X, denoted $\operatorname{Top}(X_{\mathrm{ket}})$ (or simply X_{ket} again when no confusion can arise) is the category of sheaves on X_{ket} (we shall not use the more standard notation $\widetilde{X}_{\mathrm{ket}}$ for we will later need the tilda to denote some other object); here and in the sequel we neglect questions of universes, which should be treated as in the classical case of étale topology (*cf.* [**SGA 4**] VII 1), see [**Na 1**].

The datum for each object T of X_{ket} of the set of covering families of T as above defines a *pretopology* on X_{ket} in the sense of ([**SGA 4**] II 1.3), and as a result (loc. cit.) the Kummer étale topology is simple to describe (the sieves generated by the covering families are cofinal in the set of covering sieves of T). In the verification of the axioms of a pretopology the only nontrivial point is to check the stability of covering families under base change, in other words, since "Kummer étale" is stable under fs change, to check the *universal surjectivity* of covering families. This follows from Nakayama's "fourth point lemma" :

Lemma 2.2 **([Na 1], 2.2.2).** — *Let*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{2.2.1}$$

be a cartesian square of fs log schemes, and let $y' \in Y'$, $s \in X$ such that $g(y') = f(x)$. Assume that f or g is exact (1.6). *Then there exists $x' \in X'$ such that $h(x') = x$ and $f'(x') = y'$.*

See (loc. cit.) for the proof.

2.3. Let $f : X \to Y$ be a morphism of fs log schemes. Base-changing by f in the category of fs log schemes defines an inverse image functor

$$f^{-1}Y_{\mathrm{ket}} \longrightarrow X_{\mathrm{ket}},$$

which commutes with finite inverse limits, and by 2.2 is continuous (*i.e.* transforms covering families into covering families). Therefore by ([**SGA 4**] IV 4.9.2) f^{-1} defines a morphism of topoi

$$f_{\mathrm{ket}} : \mathrm{Top}(X_{\mathrm{ket}}) \longrightarrow \mathrm{Top}(X_{\mathrm{ket}})$$

(also denoted simply f) such that $f_*(E)(T) = E(f^{-1}T)$ for any sheaf E on X_{ket} and any object T of Y_{ket}. If $g : Y \to Z$ is a second morphism of fs log schemes, we have as usual canonical isomorphisms $(g \circ f)^{-1} \cong f^{-1} \circ g^{-1}$, $(g \circ f)_{\mathrm{ket}} \cong g_{\mathrm{ket}} \circ f_{\mathrm{ket}}$.

2.4. Let X be a scheme. Let us endow X with the trivial log structure. If $u : T \to X$ is an object of X_{ket}, then u is strict, the log structure of T is trivial and u is étale in the classical sense. It follows that the Kummer étale site X_{ket} can be identified with the classical étale site X_{et} of X, an identification which we will do in the sequel.

Let now X be an fs log scheme, and let $\overset{\circ}{X}$ denote the underlying scheme equipped with the trivial log structure. We have a natural map of log schemes

$$\varepsilon : X \longrightarrow \overset{\circ}{X},$$

which, by 2.3, defines a map of sites (resp. topoi)

$$\varepsilon : X_{\mathrm{ket}} \longrightarrow \overset{\circ}{X}_{\mathrm{et}} \ .$$

We shall sometimes call $\overset{\circ}{X}_{\mathrm{et}}$ the *classical étale site* (resp. *topos*) of X and denote it by X_{cl}. If F is a sheaf on the classical étale site of X, $\varepsilon^{-1}F$ induces on the classical étale site of any Kummer étale $u : Y \to X$ the classical inverse image $u^{-1}F$. As we shall see later 5.2, ε cohomologically behaves as a constructible fibration in tori.

2.5. Let X be an fs log scheme. Denote by fs/X the category of fs log schemes over X. We define the *Kummer étale topology* on fs/X as the topology generated by the covering families which are surjective families of Kummer étale maps $T_i \to T$ (as before such families define a pretopology on fs/X). The corresponding site (resp. topos), denoted

$$(fs/X)_{\mathrm{ket}}$$

is called the *big Kummer étale site* (resp. *topos*) of X, in contrast with X_{ket} sometimes called the *small* Kummer étale site (resp. topos). Similar to 2.4 we have a natural "forgetful" map of topoi

$$\varepsilon : (fs/X)_{\mathrm{ket}} \longrightarrow (fs/X)_{\mathrm{et}} \ .$$

The relations between the big and small sites (resp. topoi) are as good as in the case of the classical étale topology ([**SGA 4**] VII 2, 4). This is due to the following theorem of Kato:

Theorem 2.6 **([Ka 3], 3).** — *Let X be an fs log scheme. The Kummer étale topology on fs/X is coarser than the canonical topology.*

This means that representable functors on fs/X are sheaves for the Kummer étale topology, namely, if Y is an fs log scheme over X, the functor $T \mapsto \mathrm{Hom}_X(T, Y)$ on fs/X is a sheaf for the Kummer étale topology. Kato in fact shows that this functor is a sheaf for a finer topology on fs/X, the *log flat topology*, which we will not consider in these notes.

2.7. . Let X be an fs log scheme. By 2.6 any fs log scheme Y over X defines a sheaf $\mathrm{Hom}_X(-, Y)$ on $(fs/X)_{\mathrm{ket}}$, hence on X_{et} by restriction. Here are some important examples.

(a) *The sheaf $\mathcal{O}$.* Let Y be the affine line $\mathbb{A}^1_X$ endowed with the inverse image log structure by the canonical projection onto X. The sheaf on $(fs/X)_{\mathrm{ket}}$ (resp. X_{ket}) corresponding to Y is just the structural sheaf $\mathcal{O}$. Indeed, for any fs log scheme T over X, we have

$$\mathrm{Hom}_X(T, Y) = \mathrm{Hom}_{\overset{\circ}{X}}(\overset{\circ}{T}, \overset{\circ}{Y}) = \Gamma(T, \mathcal{O}_T).$$

More generally, let $\mathcal{E}$ be a quasi-coherent sheaf on $\overset{\circ}{X}$, and let $Y = \mathbb{V}(\mathcal{E})$ ($:= \operatorname{Spec}\operatorname{Sym}(\mathcal{E})$), endowed with the inverse image log structure of X. Then the sheaf corresponding to Y is $(u : T \to X) \mapsto \Gamma(T, u^*\mathcal{E})$.

(b) *The sheaf* M. Let Y be the affine line $\mathbb{A}^1_X$ endowed with the log structure obtained by fs pull-back by $X \to \operatorname{Spec}\mathbb{Z}$ from that of $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[\mathbb{N}]$ endowed with the canonical log structure; in other words, $\mathbb{A}^1_X$ is the product $X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[\mathbb{N}]$ in the category of fs log schemes over $\operatorname{Spec}\mathbb{Z}$ with trivial log structure. Then the sheaf corresponding to Y is the structural sheaf of monoids M. Indeed, for any fs log scheme T over X, we have

$$\operatorname{Hom}_X(T, Y) = \operatorname{Hom}(T, \operatorname{Spec}\mathbb{Z}[\mathbb{N}]) = \Gamma(T, M_T).$$

(Actually (a) and (b) are the key cases to which Kato reduces the proof of 2.6.)

(c) *The sheaf* M^{gp}. Consider the functor $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$ on fs/X. Though one can show that this functor is *not* representable (as soon as X is nonempty), it is easy to deduce from (b) that it is still a sheaf for the Kummer étale topology (*cf.* [**Ka 3**], 3.6, [**Ka 2**], 2.1.3).

(d) *The Kummer exact sequence.* Let n be an integer inversible on X. Let $\mathbb{Z}/n(1) = \mu_n$ denote the sheaf on $(fs/X)_{\mathrm{ket}}$ induced by the sheaf of n-th roots of unity on the classical big étale site of X, *i.e.* $T \mapsto \{z \in \Gamma(T, \mathcal{O}_T); z^n = 1\}$. Then the following sequence of sheaves on $(fs/X)_{\mathrm{ket}}$ (resp. X_{ket}) is exact (*Kummer exact sequence*)

$$0 \longrightarrow \mathbb{Z}/n(1) \longrightarrow M^{\mathrm{gp}} \xrightarrow{n} M^{\mathrm{gp}} \longrightarrow 0$$

([**K-N**], 2.3): one is reduced to showing that a section $a \in \Gamma(X, M)$ is Kummer étale locally an n-th power, but such a section corresponds to a map $a : X \to \mathbb{A}^1_{\mathbb{Z}}$ (*cf.* (b)) and the map $X' \to X$ deduced from the n-th power endomorphism of $\mathbb{A}^1_{\mathbb{Z}}$ by base change by a is a surjective Kummer étale map which makes a an n-th power.

Let $f : X \to Y$ be a morphism of schemes. If f is a *universal homeomorphism*, which means that f is a homeomorphism on the underlying spaces and remains so after any base change $Y' \to Y$, or equivalently ([**EGA IV**] 18.12.11) is radicial, integral and surjective, then the inverse image functor $f^{-1} : Y_{\mathrm{et}} \to X_{\mathrm{et}}$ is an equivalence ([**SGA 4**] VIII 1.1). This result (*topological invariance of the étale site*) plays a key role in the foundations of étale cohomology. The following analogue and generalization in the Kummer étale context has been established by I. Vidal :

Theorem 2.8 **([Vi 1], 4.2, [Vi 4]).** — *Let $f : X \to Y$ be a morphism of fs log schemes. Assume that f is Kummer* (1.6)*, is a homeomorphism on the underlying spaces and remains so after any fs base change $Y' \to Y$. Then the inverse image functor $f^{-1} : Y_{\mathrm{ket}} \to X_{\mathrm{ket}}$ is an equivalence.*

2.9. We shall say that a morphism $f : X \to Y$ of fs log schemes is a *universal Kummer homeomorphism* if it satisfies the hypotheses of 2.8. Here are some examples.

(a) Assume that f is strict and induces a universal homeomorphism on the underlying schemes. Then since fs base change by strict maps commutes with taking the underlying schemes, f is a universal Kummer homeomorphism. An important particular case is that of a *thickening* [**Ka 1**]. In this case, the conclusion of 2.8 is a consequence of the existence and uniqueness of infinitesimal liftings of log étale maps (loc. cit.).

(b) Assume that Y is an $\mathbb{F}_p$-log scheme, and that f is purely inseparable in the sense of Kato ([**Ka 1**], 4.9) and induces a universal homeomorphism on the underlying schemes. Then f is a universal Kummer homeomorphism ([**Vi 1**], 2.10). Basic examples are *absolute Frobenius* and *exact relative Frobenius* maps ([**Ka 1**], 4.12).

3. Finite Kummer étale covers

Definition 3.1. — Let X be an fs log scheme. A (finite) Kummer étale cover of X is an fs log scheme Y over X such that the sheaf it defines on X_{ket} (2.6) is finite locally constant, *i.e.* there exists a Kummer étale covering family $(X_i \to X)_{i \in I}$ of X such that for each $i \in I$ the log scheme Y_i over X_i deduced by base change is a finite sum of copies of X_i. If G is a finite étale group scheme over X (in the classical sense), a *Kummer étale Galois cover of X of group G* is a Kummer étale cover Y of X endowed with an action of G by X-automorphisms such that Y is a G-torsor on X.

When X has the trivial log structure, a Kummer étale cover of X is an étale cover of X in the classical sense, with the trivial log structure. We shall sometimes say "cover" instead of "Kummer étale cover", when no confusion can arise. If a finite étale group scheme G acts on a cover Y of X, for Y to be Galois of group G means that Y is, locally for the Kummer étale topology on X, isomorphic to G (with the log structure inverse image of that of X), G acting on itself by left translations; this is equivalent to saying that the map $G \times_X Y \to Y \times_X Y$, $(g, y) \mapsto (y, gy)$ is an isomorphism (where fiber products are taken in the fs sense). In this case, X is a sheaf-theoretic quotient of Y, *i.e.* the sequence $Y \times_X Y \rightrightarrows Y \to X$ is exact as a sequence of sheaves on Y_{ket}.

Here is a basic example ([**Ka 3**], 2.5).

Proposition 3.2. — *Let X be an fs log scheme, endowed with a global chart $X \to \operatorname{Spec} \mathbb{Z}[P]$, where P is an fs monoid. Let $u : P \to Q$ be a Kummer map of fs monoids* (1.6), *such that $Q^{\text{gp}}/u(P^{\text{gp}})$ is annihilated by an integer n invertible on X. Let $Y = X \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[Q]$. Then the natural projection $f : Y \to X$ is a Kummer étale Galois cover of group the (classical) étale diagonalizable group $G = D(Q^{\text{gp}}/u(P^{\text{gp}}))_X = \operatorname{Spec} \mathcal{O}_X[Q^{\text{gp}}/u(P^{\text{gp}})]$. Moreover, f is open, finite and surjective on the underlying schemes, and remains so after any fs base change $X' \to X$.*

The key point is the following elementary lemma (*cf.* [**Vi 1**], 1.8]):

Lemma 3.3. — *Let $u : P \to Q$ be a map of fs monoids, such that $\Gamma := Q^{\mathrm{gp}}/u(P^{\mathrm{gp}})$ is annihilated by an integer $n \geqslant 1$. Then the natural map*

$$(Q \oplus_P Q)^{\mathrm{fs}} \longrightarrow \Gamma \oplus Q,\ (a,b) \longmapsto (\bar{b}, a+b) \tag{3.3.1}$$

is an isomorphism, where the left hand side denotes the amalgamated sum in the category of fs monoids and $(^-)$ the class in Γ of an element of Q^{gp}.

For the convenience of the reader we insert a proof of 3.3. We have natural maps

$$Q \oplus_P Q \longrightarrow (Q \oplus_P Q)^{\mathrm{int}} \longrightarrow (Q \oplus_P Q)^{\mathrm{fs}}$$

where $M := (Q \oplus_P Q)^{\mathrm{int}}$ is the amalgamated sum in the category of fine monoids and $(Q \oplus_P Q)^{\mathrm{fs}}$ is the saturation M^{sat} of M. We know that M is the image of $Q \oplus Q$ in $(Q \oplus_P Q)^{\mathrm{gp}} = Q^{\mathrm{gp}} \oplus_{P^{\mathrm{gp}}} Q^{\mathrm{gp}}$ and that $M^{\mathrm{gp}} = Q^{\mathrm{gp}} \oplus_{P^{\mathrm{gp}}} Q^{\mathrm{gp}}$. Now the homomorphism

$$Q^{\mathrm{gp}} \oplus Q^{\mathrm{gp}} \longrightarrow \Gamma \oplus Q^{\mathrm{gp}},\ (a,b) \longmapsto (\bar{b}, a+b)$$

induces an isomorphism

$$Q^{\mathrm{gp}} \oplus_{P^{\mathrm{gp}}} Q^{\mathrm{gp}} \xrightarrow{\sim} \Gamma \oplus Q^{\mathrm{gp}}, \tag{$*$}$$

as is shown by the commutative diagram with exact rows

$$\begin{tikzcd}
P^{\mathrm{gp}} \arrow[r, "{(-u,u)}"] \arrow[d, "\mathrm{Id}"'] & Q^{\mathrm{gp}} \oplus Q^{\mathrm{gp}} \arrow[r] \arrow[d, "{(a,b) \mapsto (b, a+b)}"] & Q^{\mathrm{gp}} \oplus_{P^{\mathrm{gp}}} Q^{\mathrm{gp}} \arrow[r] \arrow[d] & 0 \\
P^{\mathrm{gp}} \arrow[r, "{(u,0)}"] & Q^{\mathrm{gp}} \oplus Q^{\mathrm{gp}} \arrow[r] & \Gamma \oplus Q^{\mathrm{gp}} \arrow[r] & 0
\end{tikzcd}$$

We shall identify the two sides by $(*)$. Thus M is the submonoid of $\Gamma \oplus Q^{\mathrm{gp}}$ consisting of pairs (x,y) of the form $(\bar{b}, a+b)$ for a, $b \in Q$. Hence we have $M^{\mathrm{sat}} \subset \Gamma \oplus Q$. Conversely, let $(x,y) \in \Gamma \oplus Q$. Let $n \geqslant 1$ such that $nQ^{\mathrm{gp}} \subset u(P^{\mathrm{gp}})$. Then $n(x,y) = (0, ny) \in M$, so $(x,y) \in M^{\mathrm{sat}}$ and $M^{\mathrm{sat}} = \Gamma \oplus Q$.

Remark 3.4

(a) Following Kato ([**Ka 2**], 3.4.1), let us call *small* a morphism $u : P \to Q$ of integral monoids such that $\operatorname{Coker} u^{\mathrm{gp}}$ is torsion. When Q is fine (a fortiori, fs), this is equivalent to saying that $\operatorname{Coker} u^{\mathrm{gp}}$ is annihilated by a positive integer. Here are examples of small morphisms : (i) a *Kummer morphism* of fs monoids; (ii) a "*partial blow-up*" (*cf.* 6.1) : let P be an fs monoid, $I \subset P$ a nonempty ideal, $a \in I$, Q the saturation of the submonoid of P^{gp} generated by P and the elements $b - a$ for $b \in I$; then the inclusion $P \hookrightarrow Q$ is small: $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is an isomorphism.

(b) Lemma 3.3 shows the interest of working in the category of fs monoids, for the analogous statement with the push-out taken in the category of fine monoids would not hold. For example, for $P = Q = \mathbb{N}$, u the multiplication by $n > 1$, the amalgamated sum $Q \oplus_P Q$ is an integral monoid strictly contained in $\mathbb{N} \oplus \mathbb{Z}/n\mathbb{Z}$.

(c) The map

$$Q \oplus Q \longrightarrow Q \oplus Q\ ,\ (a,b) \longmapsto (b, a+b)$$

corresponds to the map

$$D(Q) \times D(Q) \longrightarrow D(Q) \times D(Q)\ ,\ (g, x) \longmapsto (x, gx)$$

on the associated diagonalizable monoid schemes. The map (3.3.1) deduced by passing to the quotients corresponds to the induced map

$$D(\Gamma) \times D(Q) \longrightarrow D(Q) \times_{D(P)} D(Q)\ ,\ (g, x) \longmapsto (x, gx),$$

which is an isomorphism (the fibered product on the right hand side being taken in the category of fs log schemes).

Let us prove 3.2. By construction, $f : Y \to X$ is Kummer, and since $\operatorname{Coker} u^{\mathrm{gp}}$ is killed by n invertible on X, f is log étale. By 3.4 (c), the action of $D(\Gamma)$ on $D(Q)$ induces, by base change by $X \to D(P)$, an action of G on Y such that $G \times_X Y \to Y \times_X Y$, $(g, y) \mapsto (x, gy)$ is an isomorphism. So Y is a Kummer étale Galois cover of X of group G. As for the last assertions, it is enough to prove that f is finite, open and surjective. Since base change by a strict map commutes with taking the underlying schemes, we may replace X by $D(P)$, which we shall denote by X again. We follow the argument of Kato ([**Ka 3**], 2.5). Since $\operatorname{Coker} u^{\mathrm{gp}}$ is killed by n, $Y := D(Q)$ is finite over X, and since u is injective, the projection $f : Y \to X$ is surjective. Thus X has the quotient topology of Y. Let U be an open subset of Y. To show that $f(U)$ is open, we have to show that $f^{-1}(f(U))$ is open. But by Nakayama's fourth point lemma (2.2), $f^{-1}(f(U)) = pr_2(pr_1^{-1}(U))$, where $pr_1,\ pr_2 : Y \times_X Y \to Y$ are the two canonical projections from the fs fibered product. Composition with the isomorphism $G \times_X Y \to Y \times_X Y$, $(g, y) \mapsto (y, gy)$ (where $G := D(\operatorname{Coker} u^{\mathrm{gp}})_X$ transforms pr_2 into the action of G on Y. Since $G \times_X Y = D(\Gamma) \times Y$ (the product on the right hand side being taken over $\operatorname{Spec} \mathbb{Z}$), $f^{-1}f(U)$ is therefore the image of U by the action of $D(\Gamma)$ on Y, $\rho : D(\Gamma) \times Y \to Y$, $(g, y) \mapsto gy$. But the map

$$D(\Gamma) \times Y \to D(\Gamma) \times Y\ ,\ (g, y) \longmapsto (g, gy)$$

is an isomorphism, which transforms $pr_2 : D(\Gamma) \times Y \to Y$ into ρ. Now, since $D(\Gamma)$ is fppf over $\operatorname{Spec} \mathbb{Z}$, pr_2 is fppf, too, and in particular, open, so $f^{-1}f(U)$ is open, which concludes the proof.

Definition 3.5. — Let X be an fs log scheme. By a standard Kummer Galois cover of X, we shall mean a Kummer cover $Y \to X$ of the type defined in 3.3.

The existence, locally for the classical étale topology, of charts of Kummer étale maps $X \to Y$ subordinate to Kummer maps $u : P \to Q$ with $\operatorname{Coker} u^{\mathrm{gp}}$ annihilated by an integer invertible on X, combined with 3.3, shows:

Corollary 3.6. — *Let X be an fs log scheme. The Kummer étale topology on X is generated by the surjective classical étale families and the standard Kummer Galois covers.*

Corollary 3.7. — *Let $f : X' \to X$ be a Kummer étale map of fs log schemes. Then the induced map $\overset{\circ}{f}$ on the underlying schemes is open.*

The property of being open being local for the classical étale topology, we may assume that f is a standard Kummer Galois cover. Then the conclusion follows from 3.2.

3.8. Let S be a locally noetherian fs log scheme. Denote by

$$fs'/S$$

the full subcategory of fs/S consisting of fs log schemes over S which are locally of finite type over S as schemes. Consider a property $\mathcal{P}$ of morphisms $f : X \to Y$ in fs'/S. We shall say that $\mathcal{P}$ is *local for the Kummer étale topology* if the following conditions are satisfied:

(i) if $f : X \to Y$ satisfies $\mathcal{P}$, then so does the morphism $f' : X' \to Y'$ deduced from f by any Kummer étale base change $S' \to S$;

(ii) if $(S_i \to S)_{i \in I}$ is a covering family for X_{ket} (2.1), and if $f_i : X_i \to Y_i$ deduced from f by the base change $S_i \to S$ satisfies $\mathcal{P}$ for every i, then so does f.

When (i) is fulfilled, to check (ii) it is enough, in view of 3.6, to check the following: (a) $\mathcal{P}$ is local for the classical étale topology (b) if f_V is deduced from f by a classical étale base change $V \to S$, and if after base change by a standard Kummer Galois cover $V' \to V$, $f_{V'} = f \times_V V'$ satisfies $\mathcal{P}$, then f_V satisfies $\mathcal{P}$.

Here is an example:

Proposition 3.9. — *Let S be a locally noetherian fs log scheme. The property for a map f in fs'/S of inducing a separated (resp. proper, resp. finite) map on the underlying schemes is local for the Kummer étale topology.*

Since the property of being separated (resp. proper, resp. finite) on the underlying schemes is local for the classical étale topology ([**SGA 1**] IX 2.4), it suffices to show the following. Let $f : X \to Y$ be a map of fs log schemes, with Y locally noetherian and f locally of finite type, and let $g : Y' \to Y$ be a *standard* Kummer étale Galois cover of Y. Then f is closed (resp. quasi-finite) on the underlying schemes if and only if f' deduced by fs base change by g is so. But this follows from Nakayama's fourth point lemma 2.2 and the fact that g makes Y' a quotient topological space of Y.

In particular:

Corollary 3.10. — *Let Y be a locally noetherian fs log scheme, and let $f : X \to Y$ be a Kummer étale cover* 3.1. *Then f is finite and surjective.*

Remark 3.11. — One can show ([**Ka 3**] 10.2, [**Vi 2**] 1.2, [**Vi 2**]) that conversely, if $f : X \to Y$ is a Kummer étale map of locally noetherian fs log schemes which induces a finite map on the underlying schemes, then f is a Kummer étale cover (in the sense of 3.1).

Here are some other examples of properties which are local for the Kummer étale topology:

Proposition 3.12. — *Let S be a locally noetherian fs log scheme. The property for a morphism in fs'/S of being log smooth (resp. log étale, resp. Kummer étale) is local for the Kummer étale topology* (2.8).

See [**Vi 2**] for a proof.

Moreover, the following descent result, stated in ([**Ka 3**] 10.2), holds (see [**Vi 2**] for a proof):

Proposition 3.13. — *Let S be a locally noetherian fs log scheme. Let* $\mathrm{Kcov}(S)$ *(resp.* $\mathrm{Lcf}(S_{\mathrm{ket}})$*) denote the category of Kummer étale covers of S (resp. the category of sheaves on S_{ket} which are locally constant with finite fibers). Then the natural functor* $\mathrm{Kcov}(S) \to \mathrm{Lcf}(S)$ *is an equivalence of categories.*

4. Log geometric points and fundamental groups

Definition 4.1

(a) A *log geometric point* is a log scheme s which is the spectrum of a separably closed field k such that M_s is saturated and for every integer $n \geqslant 1$ prime to the characteristic of k, the multiplication by n on $\overline{M}_s$ $(= M_s/k^*)$ is bijective.

(b) Let X be an fs log scheme. A *log geometric point* of X is a map of log schemes $x : s \to X$, where s is a log geometric point. If $x : s \to X$ is a log geometric point, a *Kummer étale neighborhood* of x is a map of X-log schemes $s \to U$ where $U \to X$ is Kummer étale.

A log geometric point $x : s \to X$ defines a (classical) geometric point $\mathring{x} : \mathring{s} \to \mathring{X}$ of the underlying scheme, and a point $\mathring{x}(\mathring{s})$ of $\mathring{X}$. We say that x is *over* or *localized at* $\mathring{x}(\mathring{s})$. As in the classical case, we will often make the abuse of notation consisting in denoting by the same letter the log geometric point x and its source. Also, when y is a point of X, we will often denote by $\widetilde{y}$ a log geometric point over y and $\overline{y}$ the corresponding (classical) geometric point.

4.2. Let X be an fs log scheme, given with a global chart

$$X \longrightarrow \operatorname{Spec} \mathbb{Z}[P]$$

where P is an fs monoid. Let $\mathcal{N} = \mathcal{N}_X$ be the set of integers $\geqslant 1$ which are invertible on X, ordered by divisibility. For $n \in \mathcal{N}$ denote by P_n a copy of P, and for $m \leqslant n$ let u_{mn} be the multiplication by n/m. Write P for P_1 and u_n for u_{1n} = multiplication by n: $P \to P_n$. Let

$$X_n := X \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[P_n]$$

(with $\operatorname{Spec}\mathbb{Z}[P_n] \to \operatorname{Spec}\mathbb{Z}[P]$ given by u_n). The (P_n, u_{mn}) form an inductive system indexed by $\mathcal{N}$ and correspondingly the $(X_n, v_{mn} = X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[u_{mn}])$ a projective system indexed by $\mathcal{N}$. Each transition map

$$v_{mn} : X_n \longrightarrow X_m$$

is a standard Kummer cover of group $D(P^{\mathrm{gp}} \otimes \mathbb{Z}/d\mathbb{Z})_X = \operatorname{Hom}(P^{\mathrm{gp}}, (\mu_d)_X)$ (3.5) where $d = n/m$. In particular, X_n is a standard Kummer cover of X of group $\operatorname{Hom}(P^{\mathrm{gp}}, (\mu_n)_X)$.

These projective systems can be used to construct log geometric points. Let $s = \operatorname{Spec} k$ be an fs log point, where k is separably closed of characteristic p, and let $s \to \operatorname{Spec}\mathbb{Z}[P]$ be a chart modeled on the sharp fs monoid $P := \overline{M}_s$, associated to a chosen splitting $M_s = k^* \oplus P$, so that the log structure of s is associated to $P \to k$, $a \mapsto 0$ if $a \neq 0$. Let $\widetilde{P}$ be the limit of the inductive system P_n as above, indexed by $\mathcal{N} = \mathcal{N}_s$, and let $\widetilde{s} : (s \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[\widetilde{P}])_{\mathrm{red}}$, with its natural log structure. Then $M_{\widetilde{s}} = k^* \oplus \widetilde{P}$ is saturated, the log structure of $\widetilde{s}$ is associated to $\widetilde{P} \to k$, $a \mapsto 0$ if $a \neq 0$, and multiplication by n in $\mathcal{N}$ on $\widetilde{P} = \overline{M}_{\widetilde{s}}$ is invertible, so that the natural map $\widetilde{s} \to s$ is a log geometric point over s (which is the identity on the underlying schemes).

Definition 4.3. — Let $\widetilde{x}$ be a log geometric point of the fs log scheme X, and let $\mathcal{F}$ be a sheaf on X_{ket}. The set

$$\mathcal{F}_{\widetilde{x}} := \operatorname{ind.lim} \mathcal{F}(U),$$

where U runs through the category of Kummer étale neighborhoods (4.1) of $\widetilde{x}$ (where maps are pointed maps), is called the *stalk* of $\mathcal{F}$ at $\widetilde{x}$.

Proposition 4.4

(a) *Let $\widetilde{x}$ be a log geometric point of the fs log scheme X. The functor $\mathcal{F} \mapsto \mathcal{F}_{\widetilde{x}}$ is a fibre functor (or point)* ([**SGA 4**] IV 6.1, 6.2) *of the topos X_{ket}.*

(b) *Every fibre function on X_{ket} is isomorphic to one of the form described in* (a).

(c) *The fibre functors of the form described in* (a) *make a conservative system.*

Let $\mathcal{V}(\widetilde{x})$ be the category of Kummer étale neighborhoods of $\widetilde{x}$. Using 2.2 one checks that $\mathcal{V}(\widetilde{x})^o$ is filtering, whence (a). We have seen in 4.2 that for any classical geometric point $\overline{x}$ of X there is a log geometric point $\widetilde{x}$ over $\overline{x}$. This implies (c). By the known description of the points of the classical étale topos ([**SGA 4**] VIII 7.9), to check (b) we are reduced to the case where the underlying scheme of X is the spectrum of a separably closed field. The proof is then similar to that of (loc. cit.).

Remark 4.4.1. — If $\widetilde{s} \to s$ is the log geometric point constructed in 4.2, then the $s_m \to s$ for $m \in \mathcal{N}$ form a cofinal system of Kummer étale neighborhoods of $\widetilde{s}$. Therefore for a sheaf $\mathcal{F}$ on s_{ket} we have

$$\mathcal{F}_{\widetilde{s}} = \operatorname{ind.lim} \Gamma((s_m)_{\mathrm{ket}}, \mathcal{F}).$$

4.5. As in the classical case, one can consider log strictly local log schemes, log strict localizations, specializations of fibre functors associated to log geometric points, and eventually get a stronger statement than 4.4 (b), in the form of an equivalence similar to that of ([**SGA 4**] VIII 7.9). Here is a brief sketch. Details are left to the reader.

A log scheme S is called *log strictly local* if S is saturated, the underlying scheme $\overset{\circ}{S}$ is strictly local (in the classical sense), and for every integer $n \geqslant 1$ invertible on S, multiplication by n on $\overline{M}_s$ (where s is the closed point) is bijective. If $\widetilde{s}$ is a log geometric point of the fs log scheme X, the *log strict localization* of X at $\widetilde{x}$ is defined as the inverse limit, in the category of saturated log schemes, of the Kummer étale neighborhoods of $\widetilde{x}$. It is usually denoted by $X(\widetilde{x})$ or $X_{(\widetilde{x})}$. It is a log strictly local log scheme over X. Let x, y be points of $\overset{\circ}{X}$ such that x is a specialization of y, *i.e.* $x \in \{y\}^-$, and let $\widetilde{x}$ (resp. $\widetilde{y}$) be a log geometric point over x (resp. y). Then a *specialization map* $s : \widetilde{y} \to \widetilde{x}$ is defined as an X-map $X(\widetilde{y}) \to X(\widetilde{x})$, or equivalently an X-map $\widetilde{y} \to X(\widetilde{x})$. Given $\widetilde{x}$ and y there exists at least one specialization map $s : \widetilde{y} \to \widetilde{x}$ with $\widetilde{y}$ a log geometric point over y. To such a specialization map s is associated a map of the corresponding fibre functors

$$s^* : (-)_{\widetilde{x}} \longrightarrow (-)_{\widetilde{y}} \ ,$$

given by "inverse image by s": $\mathcal{F}(U) \to \mathcal{F}_{\widetilde{y}}$ for U a log étale neighborhood of $\widetilde{x}$. We obtain in this way a category $\mathrm{Pt}(X)$ of log geometric points of X and a functor from $\mathrm{Pt}(X)$ to the category $\mathrm{Pt}(X_{\mathrm{ket}})$ of points of the topos X_{ket}, which turns out to be an equivalence, as in ([**SGA 4**] VIII 7.9).

4.6. Let S be a locally noetherian, connected, fs log scheme and let $\widetilde{s}$ be a log geometric point of X. Using 3.13 it is easy to see ([**Vi 2**]) that the pair formed by $\mathrm{Kcov}(S)$ (3.13) and the fiber functor

$$F : \mathrm{Kcov}(S) \longrightarrow fsets, \quad X \longmapsto F(X) := X_{\widetilde{s}}$$

(where $fsets$ denotes the category of finite sets) satisfies the axioms (G1) to (G6) of ([**SGA 1**] V 4). Therefore, by (loc. cit.) the functor F is pro-representable by a pro-object $\mathcal{S}$ of $\mathrm{Kcov}(S)$ and if

$$\pi_1^{\log}(S, \widetilde{s})$$

denotes the profinite group $Aut(F)$ (opposite to $\mathrm{Aut}(\mathcal{S})$), then F induces an equivalence

$$\mathrm{Kcov}(S) \xrightarrow{\sim} \pi_1^{\log}(S, \widetilde{s}) - fsets \ ,$$

where the right hand side denotes the category of finite sets on which $\pi_1^{\log}(S, \widetilde{s})$ acts continuously. The group $\pi_1^{\log}(S, \widetilde{s})$ is called the *log fundamental group* of S at $\widetilde{s}$, and $\mathcal{S}$ a *log universal cover* of S. As in the classical case (loc. cit.), for X in $\mathrm{Kcov}(S)$, *i.e.* a Kummer étale cover of S, its connected components are again Kummer étale covers of S; X is connected if and only if $\pi_1^{\log}(S, \widetilde{s})$ acts transitively on the stalk $X_{\widetilde{s}}$. If G is a finite group, Kummer Galois covers of S of group G are pushed-out from $\mathcal{S}$ by (continuous) homomorphisms $\pi_1^{\log}(S,\widetilde{s}) \to G$; surjective homomorphisms correspond to connected covers. The log fundamental group is functorial with respect to pointed maps (and as usual, the definition – and the functoriality – can be extended to non necessarily connected, pointed log schemes). If $\widetilde{s} \to \widetilde{t}$ is a specialization map between log geometric points of S (4.5), the corresponding map $\pi_1^{\log}(S,\widetilde{s}) \to \pi_1^{\log}(S,\widetilde{t})$ is an isomorphism. If $\overline{s}$ is the (classical) geometric point defined by $\overline{s}$, the map $\varepsilon : S \to \mathring{S}$ induces a *surjective* homomorphism

$$\pi_1^{\log}(S,\widetilde{s}) \longrightarrow \pi_1(S,\overline{s}),$$

called the *forgetful homomorphism.*

Examples 4.7

(a) Let $s = \operatorname{Spec} k$ be an fs log point (1.3), and let $\widetilde{s} = \operatorname{Spec} \overline{k}$ be a log geometric point over s. Then the forgetful homomorphism defines an exact sequence

$$1 \longrightarrow I^{\log}(s,\widetilde{s}) \longrightarrow \pi_1^{\log}(s,\widetilde{s}) \longrightarrow \pi_1(s,\widetilde{s}) \longrightarrow 1\ , \tag{4.7.1}$$

where $I = I^{\log}(s,\widetilde{s})$ is called the *log inertia group* of s. Moreover, there is a natural isomorphism

$$I^{\log}(s,\widetilde{s}) \cong \operatorname{Hom}(\overline{M}_{\overline{s}}^{\mathrm{gp}}, \widehat{\mathbb{Z}}'(1)(\overline{k})) \tag{$*$}$$

and hence a noncanonical isomorphism

$$I^{\log}(s,\widetilde{s}) \cong \widehat{\mathbb{Z}}'(1)(\widetilde{k})^r,$$

where $r = \operatorname{rk} \overline{M}_{\overline{s}}^{\mathrm{gp}}$ and $\widehat{\mathbb{Z}}'(1)$ denotes the product of $\mathbb{Z}_\ell(1)$ for ℓ different from the characteristic exponent p of k; the isomorphism $(*)$ is given by the "tame character" pairing

$$I^{\log}(s,\widetilde{s}) \times M_{\overline{s}} \longrightarrow \widehat{\mathbb{Z}}'(1)(\overline{k})$$

associating to $\sigma \in I^{\log}(s,\widetilde{s})$ and $a \in M_{\overline{s}}$ the projective system $(\sigma(a^{1/n})/a^{1/n})_n \in \widehat{\mathbb{Z}}'(1)(\overline{k})$ where $(a^{1/n})_n$ is a compatible system of n-th roots of a in $M_{\overline{s}}$ (written multiplicatively), n running through the integers $\geqslant 1$ and prime to p. This follows from 4.2 and 4.4 (b).

The fiber functor $\mathcal{F} \mapsto \mathcal{F}_{\widetilde{x}}$ defines an equivalence of categories

$$\operatorname{Top}(s_{\mathrm{ket}}) \xrightarrow{\sim} \pi_1^{\log}(s,\widetilde{s}) - sets$$

where the right hand side denotes the category of sets endowed with a continuous action of $\pi_1^{\log}(s,\widetilde{s})$. For ℓ prime, we get an equivalence

$$\{\overline{\mathbb{Q}}_\ell - \text{sheaves on } s_{\mathrm{ket}}\} \xrightarrow{\sim} \operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi),$$

where the right hand side denotes the category of continuous finite dimensional $\overline{\mathbb{Q}}_\ell$-representations of $\pi = \pi_1^{\log}(s,\widetilde{s})$, and $\overline{\mathbb{Q}}_\ell$-sheaves on the Kummer étale site of s are defined as in the classical case (see below). For $\ell \neq p$, k "not too big" (*i.e.* no finite extension of k contains all the roots of unity of order a power of ℓ), and $\overline{M}_s^{\mathrm{gp}}$ of rank 1, the argument of Grothendieck ([**S-T**] Appendix) applied to the sequence

(4.7.1) shows that any V in $\mathrm{Rep}_{\overline{Q}_\ell}(\pi)$ is quasi-unipotent, *i.e.* an open subgroup of $I^{\log}(s, \widetilde{s})$ acts on V by quasi-unipotent automorphisms, and in this way V gives rise to a representation of a Weil-Deligne group associated to π (*cf.* [**Vi 3**] for details and further developments).

(b) Let $S = \operatorname{Spec} A$ be an henselian trait, with closed point $s = \operatorname{Spec} k$ and generic point $\eta = \operatorname{Spec} K$. Fix an algebraic closure $\overline{K}$ of K and denote by $\overline{A}$ the normalization of A in $\overline{k}$, a valuation ring whose residue field $\overline{k}$ is an algebraic closure of k. We put $\overline{S} = \operatorname{Spec} \overline{A}$, $\overline{s} = \operatorname{Spec} \overline{k}$, $\overline{\eta} = \operatorname{Spec} \overline{K}$. Endow S (resp. $\overline{S}$) with the natural log structure associated to the inclusion $A - \{0\} \to A$ (resp. $\overline{A} - \{0\} \to \overline{A}$), and denote by S (resp. $\widetilde{S}$) the resulting log scheme. While S is an fs log scheme, $\widetilde{S}$ is not, but is an integral and saturated log scheme, log strictly local 4.5, which is the inverse limit of the fs log schemes S_i associated to the normalizations of A in the finite extensions K_i of K. With the induced log structure s (resp. $\overline{s}$) is a discrete valuative log point (resp. a log geometric point above s), which we will denote by s (resp. $\widetilde{s}$). The maps of log schemes

$$\eta \longrightarrow S \longrightarrow S^{\mathrm{cl}}$$

pointed by the log geometric point $\overline{\eta}$ induce surjections

$$G = \mathrm{Gal}(\overline{K}/K) = \pi_1(\eta) \longrightarrow G^t := \pi_1^{\log}(S) \longrightarrow \mathrm{Gal}(\overline{k}/k) = \pi_1(S^{\mathrm{cl}}) = \pi_1(s^{\mathrm{cl}})$$

(we omit the base point $\overline{\eta}$ for brevity), with kernels

$$P = \mathrm{Ker}(G \to G^t), \quad I = \mathrm{Ker}(G \to \mathrm{Gal}(\overline{k}/k)), \quad I^t = \mathrm{Ker}(G^t \to \mathrm{Gal}(\overline{k}/k)) \cong \widehat{\mathbb{Z}}'(1),$$

the wild inertia, the inertia and the tame inertia respectively. The identification of G^t with the tame quotient G/P (P the pro-p-Sylow) comes from the calculation in (a) together with the following two facts: (i) the specialization map $\overline{\eta} \to \widetilde{s}$ induces an isomorphism

$$G^t = \pi_1^{\log}(S, \overline{\eta}) \xrightarrow{\sim} \pi_1^{\log}(S, \widetilde{s}),$$

(ii) the pointed map $(s, \widetilde{s}) \to (S, \widetilde{s})$ induces an isomorphism

$$\pi_1^{\log}(s, \widetilde{s}) \xrightarrow{\sim} \pi_1^{\log}(S, \widetilde{s}).$$

In other words, restriction from S to η defines an equivalence between Kummer étale covers of S and tame extensions of K, while restriction from S to s defines an equivalence between Kummer étale covers of S and Kummer étale covers of s.

(c) More generally, let X be a locally noetherian, regular scheme and let $D \subset X$ be a divisor with normal crossings, and $U := X - D$. Endow X with the log structure defined by D. Let $\widetilde{x}$ be a log geometric point of X above $x \in D$, with image $\overline{x}$ as a geometric point of D. Then one can show that there is a natural isomorphism

$$\pi_1^{\log}(X, \widetilde{x}) \cong \pi_1^t(U, \overline{x}),$$

where the right hand side is the tame fundamental group of Grothendieck-Murre [**G-M**], classifying the finite Galois covers of U tamely ramified along D ([**Ka 3**], 10.3) (see 7.6 for a generalization).

(d) The isomorphism in (b) (ii) holds without assuming S to be a trait. More precisely, let S be the spectrum of a local henselian ring with closed point s and let $\widetilde{s}$ be a log geometric point over s. Then the pointed map $(s,\widetilde{s}) \to (S,\widetilde{s})$ induces an isomorphism

$$\pi_1^{\log}(s,\widetilde{s}) \xrightarrow{\sim} \pi_1^{\log}(S,\widetilde{s}),$$

compatible with the classical isomorphism $\pi_1(s,\overline{s}) \to \pi_1(S,\overline{s})$ (and therefore an isomorphism on the corresponding log inertia groups). It is indeed easily seen, by reducing to the strictly local case and taking a chart $S \to \operatorname{Spec}\mathbb{Z}[P]$ with $P = \overline{M}_s$, that the Kummer covers $S_n \to S$ constructed in 4.2 form a cofinal system ([**Vi 1**], A 3.2). One checks moreover that when S is *classically* strictly local, for any sheaf $\mathcal{F}$ on S_{ket} the stalk map $\Gamma(S_{\mathrm{ket}},\mathcal{F}) \to \mathcal{F}_{\widetilde{s}}$ induces an isomorphism $\Gamma(S_{\mathrm{ket}},\mathcal{F}) \xrightarrow{\sim} (\mathcal{F}_{\widetilde{s}})^I$, where $I = \pi_1^{\log}(S,\widetilde{s})$ is the log inertia group.

(e) Grothendieck's specialization theorem and calculation of the prime to p fundamental group of a proper and smooth curve over an algebraically closed field ([**SGA 1**], X 3.10) can be revisited at the light of log fundamental groups: see Fujiwara [**F**], where this calculation is reduced to that of the prime to p fundamental group of the projective line minus three points, thanks to a log Van Kampen theorem and the fact that the prime to p log fundamental groups of the (log) geometric fibers of a semistable family of curves form, in a suitable sense, a locally constant family.

4.8. Let Λ be a noetherian ring. Let X be a locally noetherian fs log scheme and let $\mathcal{F}$ be a sheaf of Λ-modules on X_{ket}. One says that $\mathcal{F}$ is *constructible* if locally for the Zariski topology X is set theoretically a finite disjoint union of strict log subschemes Y_i over which $\mathcal{F}$ is locally constant and of finite type for the Kummer étale topology. (Note that in view of Vidal's result 2.8, the suscheme structure of the $\mathring{Y}_i$'s does not matter.) Constructible sheaves of Λ-modules form a full subcategory of the abelian category of all Λ-modules, which is stable under kernel, cokernel, extension, tensor product and inverse images (see ([**Na 1**], 3.1). They don't enjoy, however, the nice stability properties of the classical constructible sheaves with respect to the usual operations of homological algebra, e.g. they are not in general stable under $R^q f_*$ for f *proper*, even for $q = 0$, *cf.* ([**Na 1**], B 3 (i)). This defect can be partially corrected by either making further hypotheses on f ([**Na 1**], 5.5.2) or working with the *full* log étale site (see 9 below).

A formalism of $\overline{\mathbb{Q}}_\ell$-sheaves and L-functions on fs log schemes can be developed similar to the classical one, see [**Vi 3**].

5. Comparison theorems

(A) Kummer étale *vs* classical étale cohomology

Fix an integer $n \geqslant 1$ and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

5.1. Let X be an fs log scheme. Consider the canonical map

$$\varepsilon : X_{\mathrm{ket}} \longrightarrow X_{\mathrm{cl}}$$

from the Kummer étale site to the classical étale site of X (2.4). Assume that n is *invertible* on X. Recall that we then have the Kummer exact sequence of abelian sheaves on X_{ket} (2.7 (d))

$$0 \longrightarrow \Lambda(1) \longrightarrow M^{\mathrm{gp}} \xrightarrow{\ n\ } M^{\mathrm{gp}} \longrightarrow 0 \ , \tag{5.1.1}$$

where M^{gp} (resp. $\Lambda(1) = \mu_n$) is the sheaf on X_{ket} defined by $Y \mapsto \Gamma(Y, M^{\mathrm{gp}})$ (resp. $\Gamma(Y, \mu_n)$). Consider the composite map of abelian sheaves on X_{cl}:

$$c : M^{\mathrm{gp}} \longrightarrow \varepsilon_* M^{\mathrm{gp}} \longrightarrow R^1\varepsilon_*\Lambda(1) \ ,$$

where the first map is the adjunction map and the second one is the boundary map coming from (5.1.1). It is the sheafification of the map $M^{\mathrm{gp}}(U) \to H^1(U_{\mathrm{ket}}, \Lambda(1))$ associating to a section s of M^{gp} over an object U of X_{cl} the $\Lambda(n)$-Kummer étale torsor $s^{1/n}$ of its n^{th}-roots (M^{gp} written multiplicatively). When s is a section of $\mathcal{O}^*$, $s^{1/n}$ comes from a classical étale torsor, and hence c vanishes on $\mathcal{O}^*$, thus inducing a map of sheaves of Λ-modules, still denoted

$$c : \overline{M}^{\mathrm{gp}} \otimes \Lambda(-1) \longrightarrow R^1\varepsilon_*\Lambda \ .$$

By cup-product, we get maps

$$c : (\otimes^q \overline{M}^{\mathrm{gp}}) \otimes \Lambda(-q) \longrightarrow R^q\varepsilon_*\Lambda \tag{5.1.2}$$

for all integers $q \geqslant 0$. The following basic result, due to Kato-Nakayama ([**K-N**] 2.4), is easy:

Theorem 5.2. — *The maps c (5.1.2) induce isomorphisms*

$$c : (\Lambda^q \overline{M}^{\mathrm{gp}}) \otimes \Lambda(-q) \xrightarrow{\ \sim\ } R^q\varepsilon_*\Lambda \ .$$

Lemma 5.3. — *Let S be an fs log scheme whose underlying scheme is strictly local. Let $\widetilde{s}$ be a log geometric point over the closed point s, and $I = \pi_1^{\log}(S, \widetilde{s})$ the corresponding log inertia group. Then the stalk map induces an isomorphism*

$$R\Gamma(S_{\mathrm{ket}}, E) \xrightarrow{\ \sim\ } R\Gamma(I, E_{\widetilde{s}})$$

for E in $D^+(S_{\mathrm{ket}}, \Lambda)$ ($E \mapsto E_{\widetilde{s}}$ being viewed as a functor from $D^+(S, \Lambda)$ to $D^+(\Lambda[I])$ (derived category of Λ-modules endowed with a continuous action of I)).

This follows from 4.7 (d).

Let us prove 5.2. The assertions that c (5.1.2) factors through Λ^q and that the factored map is an isomorphism can both be checked on the stalks. So we may assume $X = S$ is classically strictly local. With the notations of 5.3 we then have

$$(R^q\varepsilon_*\Lambda)_s = H^q(S_{\mathrm{ket}}, \Lambda) = H^q(I, \Lambda).$$

By 4.7 (d) we have $I = \pi_1^{\log}(s, \widetilde{s})$, and by 4.7 (a) we have

$$I = \mathrm{Hom}(\overline{M}_s^{\mathrm{gp}}, \widehat{\mathbb{Z}}'(1)) \cong \widehat{\mathbb{Z}}'(1)^r, \ r = \mathrm{rk}\, \overline{M}_s^{\mathrm{gp}}.$$

Therefore the cup-product $H^1(I, \Lambda)^{\otimes q} \to H^q(I, \Lambda)$ factors through an isomorphism $\Lambda^q H^1(I, \Lambda) \xrightarrow{\sim} H^q(I, \Lambda)$ and

$$H^1(I, \Lambda) = \mathrm{Hom}(\mathrm{Hom}(\overline{M}_s^{\mathrm{gp}}, \widehat{\mathbb{Z}}'(1)), \Lambda) = \overline{M}_s^{\mathrm{gp}} \otimes \Lambda(-1).$$

It only remains to check that this last canonical isomorphism corresponds to c, which follows from its description in 4.7 (a) in terms of the tame character pairing.

5.4. For $E \in D^+(X_{\mathrm{cl}}, \Lambda)$ we have, by the projection formula,

$$E \overset{L}{\otimes} R\varepsilon_*\Lambda \cong R\varepsilon_*\varepsilon^* E$$

(note that by 5.2 $R\varepsilon_*\Lambda$ is stalkwise perfect), hence for E concentrated in degre 0, we have a natural isomorphism

$$(\Lambda^q\ \overline{M}^{\mathrm{gp}}) \otimes E(-q) \longrightarrow R^q\varepsilon_*\varepsilon^* E\ .$$

Intuitively, X_{ket} over X_{cl} behaves like a (constructible) fibration in tori with fiber $(\mathbb{G}_m)^{r(x)}$ over $\overline{x}$ with $r(x) = \mathrm{rk}(\overline{M}^{\mathrm{gp}})_{\overline{x}}$.

(B) Kummer étale *vs* log Betti cohomology

5.5. Let X be an fs log scheme locally of finite type over $\mathbb{C}$. Kato and Nakayama [**K-N**] associate to X a *topological space* $X^{\log}$ together with a continuous map

$$\tau : X^{\log} \longrightarrow X_{\mathrm{an}}$$

where $X_{\mathrm{an}} = X(\mathbb{C})$ is the analytic space associated to X (in the case of normal crossing varieties a similar construction had been done independently by Kawamata and Namikawa [**Kaw-Nam**]). As a set of points, $X^{\log}$ is the set of maps of log spaces $(\mathrm{Spec}\,\mathbb{C}, \pi) \to X_{\mathrm{an}}$, where π is the *polar* log structure $\mathbb{R}_{\geqslant 0} \times S^1 \to \mathbb{C}$, $(r, z) \mapsto rz$); the map τ is given by forgetting the log structures. A point x of $X^{\log}$ can be viewed as a pair $x = (y, h)$ where $y \in X_{\mathrm{an}}$ and h is a homomorphism $h : M_y^{\mathrm{gp}} \to S^1$ extending the "angle" homomorphism $f \mapsto f(y)/|f(y)|$ on $\mathcal{O}_y^*$.

When P is an fs monoid, and $X = \mathrm{Spec}\,\mathbb{C}[P]$, with the canonical log structure, $X^{\log} = \mathrm{Hom}_{\mathrm{monoids}}(P, \mathbb{R}_{\geqslant 0} \times S^1)$, and for $x \in X^{\log}$, $\tau(x) \in X$ is the point $\pi \circ x : P \to \mathbb{C}$ of X with value in $\mathbb{C}$; thus $X^{\log}$ has a natural topology of locally compact space (it's even a C^∞ manifold with corners), and τ is proper, with fiber $\tau^{-1}(y)$ at $y \in X$ a product of $r(y)$ copies of S^1, where $r(y)$ is the rank of $(P/P_y)^{\mathrm{gp}} = \overline{M}_y^{\mathrm{gp}}$,

$P_y = y^{-1}(\mathbb{C}^*) \subset P$ being the face associated to y: $P \to \mathbb{C}$. In general, using charts, one defines a topology on $X^{\log}$ which is functorial in X, and is characterized by the following properties: (i) for $X = \operatorname{Spec} \mathbb{C}[P]$ it is given by the above; (ii) for a strict map $X \to Y$, the square (with vertical maps τ)

$$\begin{array}{ccc} X^{\log} & \longrightarrow & Y^{\log} \\ \downarrow & & \downarrow \\ X_{\mathrm{an}} & \longrightarrow & Y_{\mathrm{an}} \end{array}$$

is a cartesian square of topological spaces. In particular, $\tau : X^{\log} \to X_{\mathrm{an}}$ is proper, and for $x \in X_{\mathrm{an}}$, $\tau^{-1}(x)$ is a product of $r(x)$ copies of S^1, where $r(x) = \operatorname{rk} \overline{M}_x^{\mathrm{gp}}$.

5.6. One can view $\tau : X^{\log} \to X_{\mathrm{an}}$ as an analogue of the map $\varepsilon : X_{\mathrm{ket}} \to X_{\mathrm{cl}}$ discussed in (A). Kato-Nakayama prove for τ a result similar to 5.2.

Consider the exponential sequence on $X^{\log}$

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \tau^{-1}(\mathcal{O}_{X_{\mathrm{an}}}) \xrightarrow{\exp} \tau^{-1}(\mathcal{O}^*_{X_{\mathrm{an}}}) \longrightarrow 0 ,$$

with the usual notation $\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z}$. Kato-Nakayama embed it in a larger one

$$(5.6.1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \tau^{-1}(\mathcal{O}_{X_{\mathrm{an}}}) & \xrightarrow{\exp} & \tau^{-1}(\mathcal{O}^*_{X_{\mathrm{an}}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \mathcal{L} & \xrightarrow{\exp} & \tau^{-1}((M^{\mathrm{gp}})_{X_{\mathrm{an}}}) & \longrightarrow & 0 \end{array}$$

Here X_{an} is endowed with the log structure naturally induced by that of X, and $\mathcal{L}$ is a certain sheaf of "logarithms" of local sections of $\tau^{-1}(M^{\mathrm{gp}})$ consisting of pairs $(i\theta, s)$ where θ is a local continuous $\mathbb{R}$-valued function on $X^{\log}$ and s a local section of $\tau^{-1}(M^{\mathrm{gp}})$ such that $\exp(i\theta) = h(s)$, in the notation of 5.5. The middle vertical map in the diagram above is given by $f \mapsto (iIm(f), \exp(f))$, and $\exp : \mathcal{L} \to \tau^{-1}(M^{\mathrm{gp}})$ is given by the second coordinate.

Using that for any sheaf $\mathcal{F}$ on X_{an} the adjunction map $\mathcal{F} \to \tau_* \tau^* \mathcal{F}$ is an isomorphism, one deduces from the bottom exact sequence of (5.6.1) a map $c : \overline{M}^{\mathrm{gp}} \otimes \mathbb{Z}(-1) \to R^1 \tau_* \mathbb{Z}$ of abelian sheaves on X_{an}, hence by cup-product a map

$$(5.6.2) \qquad c : (\otimes^q \overline{M}^{\mathrm{gp}})(-q) \longrightarrow R^q \tau_* \mathbb{Z}$$

for all $q \geqslant 0$. Since τ is proper with fibers products of S^1, one obtains the following analogue of 5.2:

Theorem 5.7 **([K-N], 1.5).** — *The maps* c (5.6.2) *induce isomorphisms*

$$c : (\Lambda^q \overline{M}^{\mathrm{gp}})(-q) \xrightarrow{\sim} R^q \tau_* \mathbb{Z} .$$

Corollaries similar to 5.4 hold.

5.8. Let X be a scheme locally of finite type over $\mathbb{C}$. Because an étale map $U \to X$ induces a local analytic isomorphism $U_{\mathrm{an}} \to X_{\mathrm{an}}$, we get a map of ringed sites (or topoi)

$$\eta : X_{\mathrm{an}} \longrightarrow X_{\mathrm{et}},$$

such that $(\eta_*\mathcal{F})(U) = \mathcal{F}(U_{\mathrm{an}})$ for any sheaf $\mathcal{F}$ on X_{an} and U étale over X. Let n be an integer $\geqslant 1$ and $\Lambda = \mathbb{Z}/n\mathbb{Z}$. The basic comparison theorem of Artin-Grothendieck ([**SGA 4**] XVI 4.1) asserts that for $\mathcal{G} \in D_c^+(X_{\mathrm{et}}, \Lambda)$ (the full subcategory of $D^+(X_{\mathrm{et}}, \Lambda)$ consisting of complexes with bounded below, constructible cohomology), the adjunction map

$$\alpha : \mathcal{G} \longrightarrow R\eta_*\eta^*\mathcal{G}$$

is an isomorphism, and consequently the natural map

$$H^q(X_{\mathrm{et}}, \mathcal{G}) \longrightarrow H^q(X_{\mathrm{an}}, \eta^*\mathcal{G})$$

from étale to Betti cohomology is an isomorphism.

Kato and Nakayama establish a similar comparison theorem between *Kummer étale cohomology* and *log Betti cohomology* (by the latter we mean cohomology of the spaces $X^{\log}$). As in 5.5, let now X be an fs log scheme locally of finite type over $\mathbb{C}$. The fact that the m^{th} power map on S^1 is a local homeomorphism implies, by taking charts, that any Kummer étale map $U \to X$ induces a local homeomorphism $U^{\log} \to X^{\log}$. Therefore we get a map of ringed sites (or topoi)

$$\eta : X^{\log} \longrightarrow X_{\mathrm{ket}}$$

such that $(\eta_*\mathcal{F})(U) = \mathcal{F}(U^{\log})$ for any sheaf $\mathcal{F}$ on $X^{\log}$. When the log structure of X is trivial this is but the map defined above.

Theorem 5.9 (**[K-N], 2.6**). — *With X and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ as above, for any $\mathcal{G} \in D_c^+(X_{\mathrm{ket}}, \Lambda)$ (the full subcategory of $D_c^+(X_{\mathrm{ket}}, \Lambda)$ consisting of complexes with bounded below, constructible cohomology* (4.8)*), the adjunction map*

$$\alpha : \mathcal{G} \longrightarrow R\eta_*\eta^*\mathcal{G}$$

is an isomorphism (and consequently the natural map

$$H^q(X_{\mathrm{ket}}, \mathcal{G}) \longrightarrow H^q(X^{\log}, \eta^*\mathcal{G})$$

is an isomorphism).

The proof is formal from 5.2, 5.7 and Artin-Grothendieck's comparison theorem quoted above. Here's a sketch of the main steps.

(a) By looking at the stalks at a log geometric point of X, one sees that it is enough to prove that the map deduced from α by applying $R\varepsilon_*$ is an isomorphism.

(b) One may assume that $\mathcal{G}$ is a single constructible sheaf and even further that $\mathcal{G}$ is of form $\varepsilon^*\mathcal{F}$ with $\mathcal{F}$ constructible on X_{cl} (this follows from general facts on Kummer étale constructible sheaves, namely that any such sheaf is Zariski locally the cokernel of a map $\Lambda_{U,X} \to \Lambda_{V,X}$ with U and V Kummer étale of finite presentation over X).

(c) Applying Artin-Grothendieck's comparison theorem $K \xrightarrow{\sim} R\eta_*\eta^*K$ to $K = R\varepsilon_*\varepsilon^*\mathcal{F}$, the statement that $R\varepsilon_*(\alpha)$ is an isomorphism boils down to the fact that the base change map

$$\eta^* R\varepsilon_*\mathcal{G} \longrightarrow R\tau_*\eta^*\mathcal{G}$$

for $\mathcal{G} = \varepsilon^*\mathcal{F}$, associated to the commutative square

$$\begin{tikzcd} X^{\log} \arrow[r, "\eta"] \arrow[d, "\tau"'] & X_{\mathrm{ket}} \arrow[d, "\varepsilon"] \\ X_{\mathrm{an}} \arrow[r, "\eta"] & X_{\mathrm{cl}}, \end{tikzcd}$$

is an isomorphism. By the projection formula one is reduced to the case $\mathcal{G} = \Lambda$, *i.e.* to showing that

$$\eta^* R^q\varepsilon_*\Lambda \longrightarrow R^q\tau_*\Lambda$$

is an isomorphism for all q. By 5.2 and 5.7 it is enough to check it for $q = 1$, and this follows from the compatibility between the classes c defined by the Kummer and exponential sequences: for any integer $n \geqslant 1$, $\eta^{-1}(M^{\mathrm{gp}}_{X_{\mathrm{ket}}})$ (resp. $\mathcal{L}$) is n-divisible (resp. uniquely n-divisible) and one has a morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}(1) \arrow[r] \arrow[d] & \mathcal{L} \arrow[r, "\exp"] \arrow[d] & \tau^{-1}(M^{\mathrm{gp}}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mu_n \arrow[r] & \eta^{-1}(M^{\mathrm{gp}}_{X_{\mathrm{ket}}}) \arrow[r, "n"] & \eta^{-1}(M^{\mathrm{gp}}_{X_{\mathrm{ket}}}) \arrow[r] & 0 \end{tikzcd} \tag{5.9.1}$$

where $M^{\mathrm{gp}}_{X_{\mathrm{ket}}}$ is the sheaf M^{gp} on the Kummer étale site of X (4.1) and the middle vertical map is $s \mapsto$ image of $\exp(s/n)$.

6. Acyclicity of log blow-ups

6.1. Let X be an fs log scheme endowed with a global chart $X \to \operatorname{Spec} \mathbb{Z}[P]$, with P fs, P^{gp} torsionfree, and let I be an ideal of P (*i.e.* a subset of P such that $PI \subset I$. The *log blow-up*

$$f : X_I \longrightarrow X$$

of X along I is the map of fs log schemes defined as follows. Let $X' := \operatorname{Spec} \mathbb{Z}[P]$ and $\mathcal{I} \subset \mathcal{O}_{X'}$ the ideal generated by I. Let $Y' := \operatorname{Proj}(\oplus_{n\in\mathbb{N}}\mathcal{I}^n) \to X'$ be the blow-up of the scheme X' along $\mathcal{I}$. The scheme Y' is covered by affine open pieces U_a indexed by $a \in I$,

$$U_a = \operatorname{Spec} \mathbb{Z}[P_a],$$

where P_a is the submonoid of P^{gp} generated by P and the set of b/a for $b \in I$. Endow Y' with the unique log structure inducing the canonical one on each U_a. Note that this structure is not necessarily saturated. Let Y'^{sat} be the saturation of Y', which is an fs log scheme over X'. Finally, define $Y = X_I$ to be the fs pull-back of Y'^{sat} by $X \to X'$ and let $f : Y \to X$ be the resulting map. This map is *log étale* (1.5) since each U_a and hence each $(U_a)^{\mathrm{sat}}$ is so over X'. The ideal $\mathcal{J} = IM_Y \subset M_Y$ generated by I is *invertible* (*i.e.* locally monogenous), and one can show ([**Ni**]) that this property characterizes Y among the fs log schemes over X, namely for any map $g : T \to X$ such that IM_T is invertible, there exists a unique morphisme $h : T \to Y$ such that $fh = g$.

More generally, if X is any fs log scheme, a sheaf of ideals $\mathcal{J} \subset M_X$ is said to be *coherent* if étale locally it is of the form IM_X, where $X \to \operatorname{Spec} \mathbb{Z}[P]$ is a global chart as above and I an ideal of P. One defines the *log blow-up of X along $\mathcal{J}$*, by patching the local constructions above, using for example the universal property, *cf.* [**F-K**], [**Ni**].

Fujiwara-Kato prove the following striking result:

Theorem 6.2 **([F-K], 2.7).** — *As in* 6.1, *let $f : Y \to X$ be the log blow-up of X along a coherent sheaf of ideals $\mathcal{J}$ of M_X whose geometric stalks $\mathcal{J}_{\overline{x}}$ are nonempty. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$, where n is an integer $\geqslant 1$ (not necessarily invertible on X). Then, for any $\mathcal{F} \in D^+(X_{\mathrm{ket}}, \Lambda)$, the adjunction map*

$$\alpha : \mathcal{F} \longrightarrow Rf_* f^* \mathcal{F}$$

is an isomorphism.

The statement with X_{ket} replaced by X_{cl} is of course false, as the blow-up of the origin in the standard affine space $\mathbb{A}^r$ already shows.

For the proof of 6.2 we need a few preliminaries. First we need the following base change result:

Proposition 6.3. — *Let Λ be as in* 6.2, *and let*

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be an fs cartesian square of fs log schemes, where $\mathring{f}$ is proper *and g is* strict. *Then for $\mathcal{F} \in D^+(X_{\mathrm{ket}}, \Lambda)$, the base change map*

$$g^* Rf_* \mathcal{F} \longrightarrow Rf'_* h^* \mathcal{F}$$

is an isomorphism.

By looking at the stalks at a log geometric point of Y', one is reduced to the case where the log structures of Y and Y' are trivial. Factoring f into $f = f_{\mathrm{cl}} \circ \varepsilon$ and using the classical proper base change theorem ([**SGA 4**] XII 5.1) one may assume further that $f = \varepsilon$, in which case the conclusion follows from 5.2.

Nakayama ([**Na 1**], 5.1) shows that for f proper the conclusion holds under a weaker hypothesis than the strictness of g, which is satisfied for example if f or g is *exact.* However, under the sole assumption of properness of f, the conclusion can fail (loc. cit. B4), but holds again provided that one replaces the Kummer étale sites by the *full* log étale sites, as Kato has shown (see 9.5).

6.4. Next, we need to recall the (well known) structure of the (classical) étale cohomology of tori. Fix an algebraically closed field k of characteristic exponent p, and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ where n is assumed to be prime to p.

Let $\mathbb{A}^1$ and $\mathbb{G}_m$ be respectively the affine line and the multiplicative group over k, and let $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the canonical inclusion. We have

$$R^q j_*\Lambda = \begin{cases} \Lambda \text{ if } q = 0 \\ \Lambda(-1)_{\{0\}} \text{ if } q = 1 \\ 0 \text{ if } q > 1, \end{cases}$$

where $\{0\}$ is the origin in $\mathbb{A}^1$ and the canonical isomorphism for $q = 1$ comes from the Kummer sequence $0 \to \Lambda(1) \to \mathcal{O}^* \to \mathcal{O}^* \to 0$ on $\mathbb{G}_m$, by sending $1 \in \Lambda$ to $\delta(t)$ where $t \in (j_*\mathcal{O}^*)_{\{0\}}$ is the image of the standard coordinate on $\mathbb{A}^1$ and δ is the coboundary map. Since $H^q(\mathbb{G}_m, \Lambda) = H^0(\mathbb{A}^1, R^q j_*\Lambda)$, we get

$$H^q(\mathbb{G}_m, \Lambda = \begin{cases} \Lambda \text{ if } q = 0 \\ \Lambda(-1) \text{ if } q = 1 \\ 0 \text{ if } q > 1. \end{cases}$$

Let now T/k ($\cong \mathbb{G}_m^r$) be a torus, let $L = \mathrm{Hom}(T, \mathbb{G}_m)$ ($\cong \mathbb{Z}^r$) be its character group and $L^\vee = \mathrm{Hom}(L, \mathbb{Z}) = \mathrm{Hom}(\mathbb{G}_m, T)$ be its cocharacter group. We have $T = \operatorname{Spec} k[L]$. The pairing

$$\begin{aligned} H^1(T, \Lambda) \times L^\vee &\longrightarrow \Lambda(-1) \\ (x, f) &\longmapsto f^*(x) \in H^1(\mathbb{G}_m, \Lambda) = \Lambda(-1) \end{aligned}$$

defines an isomorphism

$$(*) \qquad H^1(T, \Lambda) \xrightarrow{\sim} L \otimes \Lambda(-1),$$

By Künneth ([**SGA 4 1/2**], Th. finitude, 1.11), the cup-product $\otimes^q H^1(T, \Lambda) \to H^q(T, \Lambda)$ factors through an isomorphism $\Lambda^q H^1(T, \Lambda) \xrightarrow{\sim} H^q(T, \Lambda)$, so that from $(*)$ we get an isomorphism

$$(6.4.1) \qquad H^q(T, \Lambda) \xrightarrow{\sim} \Lambda^q L \otimes \Lambda(-q)$$

for all $q \geqslant 0$. This isomorphism is functorial with respect to group-scheme homomorphisms $T' \to T$. In particular, multiplication by d on T induces multiplication by d^q on $H^q(T, \Lambda)$.

As in 4.2, let $\mathcal{N}$ be the set of prime to p integers $\geqslant 1$ ordered by divisibility, and let

$$\mathcal{T} := \mathrm{inv.\,lim}\, T_m$$

where m runs through $\mathcal{N}$, T_m is a copy of T and for $m \mid m'$, the transition map $T_{m'} \to T_m$ is the multiplication by $d = m'/m$, a classical Kummer étale cover of group $L^\vee \otimes \mathbb{Z}/d\mathbb{Z}\,(1)$. The scheme $\mathcal{T}$ over T is a prime to p universal cover of T, with group the prime to p fundamental group of T,

$$\pi_1^t(T) = L^\vee \otimes \widehat{\mathbb{Z}}'(1). \tag{6.4.2}$$

The (descent) spectral sequence

$$E_2^{ij} = H^i(\pi, H^j(\mathcal{T}, \Lambda)) \Longrightarrow H^{i+j}(T, \Lambda),$$

where $\pi = \pi_1^t(T)$, yields an isomorphism

$$H^q(T, \Lambda) \xrightarrow{\sim} H^q(\pi, \Lambda). \tag{6.4.3}$$

Indeed we have

$$H^q(\mathcal{T}, \Lambda) = \mathrm{inv.\,lim}\, H^q(T_m, \Lambda),$$

with transition map from m to dm given by multiplication by d^q, hence

$$H^q(\mathcal{I}, \Lambda) = \begin{cases} \Lambda \text{ for } q = 0 \\ 0 \text{ for } q \geqslant 1, \end{cases} \tag{6.4.4}$$

since the inductive system is essentially zero for $q \geqslant 1$. The isomorphism (6.4.3), combined with (6.4.2), gives an alternate way to derive (6.4.1).

As Ekedahl observed, the vanishing (6.4.4) is easily extended to toric varieties. This leads to a (slight) simplification of Fujiwara-Kato's original proof of 6.2. His observation is based on the following result:

***Lemma 6.5* (Ekedahl).** — *Let k and Λ be as in 6.4, let P be an fs monoid such that P^{gp} is torsionfree, and let $X = \operatorname{Spec} k[P]$, $X^* = \operatorname{Spec} k[P^*]$. Then the map $X \to X^*$ defined by the inclusion $P^* \subset P$ induces an isomorphism*

$$H^q(X^*, \Lambda) \xrightarrow{\sim} H^q(X, \Lambda)$$

for all q.

First recall the well-known homotopy lemma (*cf.* [**SGA 7**] XV 2.1.3) :

Lemma 6.6. — *Let S be a connected k-scheme of finite type, let s_0, s_1 be two rational points of S, let Y, Z be k-schemes of finite type and let $f : S \times Y \to Z$ be a k-map. Let $f_i := f \circ (s_i \times \mathrm{Id}_Y) : Y \to Z$. Then*

$$f_0^* = f_1^* : H^q(Z, \Lambda) \longrightarrow H^q(Y, \Lambda)$$

for all q.

By Künneth we have

$$H^*(S \times Y, \Lambda) = H^*(S, \Lambda) \otimes H^*(Y, \Lambda).$$

Since S is connected,

$$s_0^* = s_1^* : H^0(S, \Lambda)\ (= \Lambda) \longrightarrow H^0(\operatorname{Spec} k, \Lambda)\ (= \Lambda),$$

from which the conclusion follows.

Let us prove 6.5. Since P^{gp} is torsionfree, a splitting of the extension $0 \to P^* \to P^{\mathrm{gp}} \to \overline{P}^{\mathrm{gp}} \to 0$ gives a decomposition

$$P = P^* \times Q,$$

with Q fs, Q^{gp} torsionfree and $Q^* = \{1\}$. Therefore, by Künneth we may assume P to be *sharp*, *i.e.* $P^* = \{1\}$. We have then to prove that $H^q(X, \Lambda) = \Lambda$ for $q = 0$ and zero otherwise. To do this we apply 6.6 to $S = Y = Z = X$ (which is connected, being integral) and the *product* map $f : X \times X \to X$, corresponding to the comultiplciation $P \to k[P] \otimes k[P]$, $a \mapsto a \otimes a$. As rational points $s_i \in X(k)$, we take s_1 to be unit element $\{1\}$ in X, *i.e.* the point corresponding to the homomorphism $P \to k$ sending any element a to 1, and s_0 the vertex $\{0\}$ corresponding to the homomorphism $P \to k$ sending a to 0 for $a \neq 1$ and 1 for $a = 1$ (this map is well defined because P is sharp). Then (with the notations of 6.6) $f_1 = \mathrm{Id}_X$ while $f_0 : X \to X$ sends X to $\{0\}$, and $f_0^* = f_1^*$ implies the conclusion.

When $k = \mathbb{C}$, by Artin-Grothendieck's comparison theorem, 6.5 follows from the well known stronger fact that the topological space underlying $(\operatorname{Spec} \mathbb{C}[P])_{\mathrm{an}}$ is contractible when P is sharp ([**Fu**], 3.2).

6.7. In order to state Ededahl's generalization of (6.4.4) we need to review some standard definitions (*cf.* ([**Fu**], 1.4, [**Od**], 1.2, [**KKMS**], I). Let k be as in 6.4 and let T be a torus over k, with character group L. A *fan* in $L^\vee$ is a finite set Δ of rational strongly convex polyhedral cones σ in $L^\vee \otimes \mathbb{R}$ such that for any σ in Δ, every face of σ belongs to Δ, and for any pair (σ, σ') of elements of Δ, $\sigma \cap \sigma'$ is a face of both σ and σ'. Then for $\sigma \in \Delta$, $P_\sigma := \sigma^\vee \cap L$ is an fs submonoid of L with $P_\sigma^{\mathrm{gp}} = L$, and if τ is a face of σ, $P_\sigma \subset P_\tau$ is a *localization* map, defined by inverting a face of P_σ, namely $(P_\tau)^* \cap P_\sigma$; if $U_\sigma := \operatorname{Spec} k[P_\sigma]$, U_τ is open in U_σ, and the torus $T = U_{\{0\}}$ is equivariantly embedded in each U_σ. The *toric variety* $X(\Delta)$ associated to Δ is the union of the schemes U_σ, U_σ and $U_{\sigma'}$ being glued along $U_\sigma \cap U_{\sigma'} = U_{\sigma\cap\sigma'}$. A *toric variety* over k, with torus T, is a scheme X/k of the form $X(\Delta)$. Such a scheme X is of finite type over k, separated, integral, and normal; the torus T acts on X and equivariantly embeds into X as a dense open orbit. The fan Δ can be recovered from the action of T on X, namely, $\sigma \mapsto U_\sigma$ is a bijection from Δ to the set of open affine equivariant subschemes of X ([**KKMS**], I, th. 6), the equivariant embedding $T \subset U_\sigma$ determines P_σ and in turn $\sigma = (P_\sigma)^\vee \otimes \mathbb{R}_{\geqslant 0}$. Conversely, if X is a separated, integral, normal scheme of finite type over k endowed with an action of T and an equivariant embedding of T into X as an open orbit, then $X \cong X(\Delta)$ for a unique fan Δ ([**KKMS**], loc. cit.), [**Od**], 1.5).

Let X/k be a toric variety with torus T, associated to a fan Δ. Let d be an integer $\geqslant 1$. For each $\sigma \in \Delta$, multiplication by d on the monoid P_σ induces an endomorphism of U_σ. When σ runs through Δ, these endomorphisms glue to an endomorphism of X, still denoted d. Now let $\mathcal{N}$ be as in 4.2 and similar to the construction in 6.4 define

$$\mathfrak{X} := \operatorname{inv.lim} X_m ,$$

where m runs through $\mathcal{N}$, X_m is a copy of X and for $m \mid m'$, the transition map $X_{m'} \to X_m$ is the multiplication by $d = m'/m$. Ekedahl's generalization of (6.4.4) is the following formula, where $\Lambda = \mathbb{Z}/n\mathbb{Z}$, n prime to p:

$$H^q(\mathfrak{X}, \Lambda) = \begin{cases} \Lambda \text{ for } q = 0 \\ 0 \text{ for } q \geqslant 1. \end{cases} \tag{6.7.1}$$

Let us prove (6.7.1). We have

$$H^q(\mathfrak{X}, \Lambda) = \operatorname{ind.lim} H^q(X_m, \Lambda),$$

where m runs through $\mathcal{N}$ and the transition map from m to $m' = dm$ is $d^* : H^q(X, \Lambda) \to H^q(X, \Lambda)$. Let $\mathcal{U}$ be the open cover of X by the U_σ, $\sigma \in \Delta$. Consider the Leray spectral sequence of $\mathcal{U}$:

$$E_1^{ij} = \oplus H^j(U_{\sigma_0 \cap \cdots \cap \sigma_i}, \Lambda) \Longrightarrow H^{i+j}(X, \Lambda), \tag{6.7.2}$$

where $U_{\sigma_0 \cap \cdots \cap \sigma_i} = U_{\sigma_0} \cap \cdots \cap U_{\sigma_i}$ and $(\sigma_0, \ldots, \sigma_i)$ runs through Δ^{i+1}. For $X = X_m$, these spectral sequences form an inductive system indexed by $\mathcal{N}$. Its limit is a spectral sequence

$$E_1^{ij}(\mathfrak{X}) := \operatorname{inv.lim} E_1^{ij}(X_m) \Longrightarrow H^{i+j}(\mathfrak{X}, \Lambda). \tag{6.7.3}$$

By 6.5 the natural map

$$H^j(U^*_{\sigma_0 \cap \cdots \cap \sigma_i}, \Lambda) \longrightarrow H^j(U_{\sigma_0 \cap \cdots \cap \sigma_i}, \Lambda)$$

is an isomorphism. When X runs through X_m these isomorphisms are compatible with the transition maps d^*. Passing to the limit and applying (6.4.4), we get

$$E_1^{ij}(\mathfrak{X}) = \begin{cases} \oplus_{(\sigma_0, \ldots, \sigma_i) \in \Delta^{i+1}} \Lambda \text{ for } j = 0 \\ 0 \text{ for } j > 0 . \end{cases}$$

Therefore (6.7.3) degenerates at E_2 and gives

$$H^q(\mathfrak{X}, \Lambda) = H^q(E_1^{\cdot 0}(\mathfrak{X})) = H^q(C^\cdot(\Delta, \Lambda)),$$

where $C^\cdot(\Delta, \Lambda)$ is the Čech complex, with coefficients in Λ, of the open cover, by the complements of its vertices, of the standard simplex S spanned by Δ. Thus (6.7.1) follows from the contractibility of S.

6.8. Let X/k be a toric variety as above, with associated fan Δ and torus T. There is a unique log structure on X inducing the canonical one on each U_σ, $\sigma \in \Delta$. The resulting log scheme, still denoted X, is fs and log smooth over k. The open subset of triviality of the log structure is the torus T, and by ([**Ka 4**], 11.6), $M_X = \mathcal{O}_X \cap j_* \mathcal{O}_T^*$, where $j : T \hookrightarrow X$ is the inclusion. The key ingredient in the proof of 6.2 is the following result (which is in fact a special case of a general purity result stemming out of 6.2, see 7.5):

Proposition 6.8.1. — *For $\Lambda = \mathbb{Z}/n\mathbb{Z}$, n prime to p, the restriction map*

$$j^* : H^q(X_{\mathrm{ket}}, \Lambda) \longrightarrow H^q(T_{\mathrm{ket}}, \Lambda) = H^q(T_{\mathrm{cl}}, \Lambda)$$

is an isomorphism for all q.

Let L be the character group of T. By 3.2, for $d \in \mathcal{N}$, multiplication by $d : X \to X$ is a (log) Kummer étale cover of group $L^\vee \otimes \mathbb{Z}/d\mathbb{Z}\ (1)$. Denote by $\mathfrak{X}_{\mathrm{ket}}$ the 2-inverse limit (in the sense of ([**SGA 4**] VI) of the topoi $(X_m)_{\mathrm{ket}}$, where (X_m) is the inverse system considered in 6.7. Then $\mathfrak{X}_{\mathrm{ket}}/X_{\mathrm{ket}}$ plays the role of a prime to p universal (log) Kummer cover of X, with group the prime to p log fundamental group $\pi = L^\vee \otimes \mathbb{Z}'(1)$ of X, the same as that of T (6.4.2). We have again a descent spectral sequence

$$E_2^{ij} = H^i(\pi, H^j(\mathfrak{X}_{\mathrm{ket}}, \Lambda)) \Longrightarrow H^{i+j}(X_{\mathrm{ket}}, \Lambda), \tag{6.8.2}$$

with

$$H^q(\mathfrak{X}_{\mathrm{ket}}, \Lambda) = \mathrm{ind.lim}\, H^q((X_m)_{\mathrm{ket}}, \Lambda), \tag{6.8.3}$$

with transition map from m to dm given by the endomorphism d^* of $H^q(X_{\mathrm{ket}}, \Lambda)$. It turns out that this limit is actually the same as that obtained by working with the classical étale topology instead of the Kummer étale one:

Lemma 6.8.4. — *The restriction maps $H^q((X_m)_{\mathrm{cl}}, \Lambda) \longrightarrow H^q((X_m)_{\mathrm{ket}}, \Lambda)$ define an isomorphism*

$$\mathrm{ind.lim}\, H^q((X_m)_{\mathrm{cl}}, \Lambda) \xrightarrow{\sim} \mathrm{ind.lim}\, H^q((X_m)_{\mathrm{ket}}, \Lambda).$$

In fact, a stronger statement holds: the natural map $\mathfrak{X}_{\mathrm{ket}} \to \mathfrak{X}_{\mathrm{cl}}$ is an equivalence of topoi. Indeed, any Kummer étale map $U \to X$ of finite type becomes classically étale after an fs base change by multiplication by $d : X \to X$, for a suitable $d \in \mathcal{N}$. Hence the classical étale maps $U \to X_m$ generate the topology of $\mathfrak{X}_{\mathrm{ket}}$ defined in the standard way from those of the $(X_m)_{\mathrm{ket}}$'s.

From 6.8.4 and 6.7.1 we get

$$H^q(\mathfrak{X}_{\mathrm{ket}}, \Lambda) = \begin{cases} \Lambda \text{ for } q = 0 \\ \\ 0 \text{ for } q \geqslant 1\ . \end{cases} \tag{6.8.5}$$

Thus the spectral sequence (6.8.2) yields an isomorphism

$$H^q(X_{\mathrm{ket}}, \Lambda) \longrightarrow H^q(\pi, \Lambda). \tag{6.8.6}$$

By construction this isomorphism is compatible with (6.4.3), *i.e.* we have a commutative square

$$\begin{array}{ccc} H^q(X_{\mathrm{ket}}, \Lambda) & \xrightarrow{\sim} & H^q(\pi, \Lambda) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ H^q(T_{\mathrm{cl}}, \Lambda) & \xrightarrow{\sim} & H^q(\pi, \Lambda), \end{array}$$

with horizontal maps given by (6.8.6) and (6.8.3), which proves 6.8.1.

6.9. Let us – at last – turn to Fujiwara-Kato's proof of 6.2. We have to show that for any log geometric point $\widetilde{x}$ of X, the following holds:

$$\alpha_{\widetilde{x}} : \mathcal{F}_{\widetilde{x}} \longrightarrow (Rf_*f^*\mathcal{F})_{\widetilde{x}} \tag{$*$}_{\widetilde{x}}$$

is an isomorphism. We prove $(*)_{\widetilde{x}}$ by induction on

$$r(x) = \mathrm{rk}(\overline{M}^{\mathrm{gp}})_{\overline{x}}$$

(where x (resp. $\overline{x}$) is the point (resp. geometric point) of X image of $\widetilde{x}$). Let $r \in \mathbb{N}$. Assume $(*)_{\widetilde{x}}$ holds for all $(X, \mathcal{J}, \widetilde{x})$ such that $r(x) < r$, and let us prove it for $r(x) = r$. Applying 6.3 to the base change by the strict map $\overline{x} \to X$, we may assume that the underlying scheme of $X = x$ is $\operatorname{Spec} k$, with k an algebraically closed field of characteristic exponent p, and that X is equipped with a chart $X \to \operatorname{Spec} k[P]$, with P fs, sharp (*i.e.* $P^* = \{0\}$ (the monoid law written additively)), $P^{\mathrm{gp}} \cong \mathbb{Z}^r$ and $\mathcal{J} = IM_X$. By dévissage on $\mathcal{F}$ we reduce to the case $\mathcal{F} = \Lambda$. Now let X be the affine toric variety $\operatorname{Spec} k[P]$, with vertex $\{0\}$ corresponding to the map $P \to k$ sending a to 0 for $a \neq 0$ and 0 to 1, and torus $T = \operatorname{Spec} k[P^{\mathrm{gp}}]$, of rank r. Endow X with its canonical log structure and let $f : Y \to X$ be the log blow-up of X along I. Applying again 6.3, this time to the base change by the strict map $x \to X$ sending x to $\{0\}$, we are reduced to showing that the stalk of $\alpha : \Lambda \to Rf_*\Lambda$ at a log geometric point $\{0\}^\sim$ above $\{0\}$ is an isomorphism. Let $K \in D^+(X_{\mathrm{ket}}, \Lambda)$ be the cone of α, defined by the exact triangle

$$\Lambda \xrightarrow{\alpha} Rf_*\Lambda \longrightarrow K \longrightarrow \Lambda[1].$$

We have to show

$$K_{\{0\}^\sim} = 0\ . \tag{6.9.1}$$

By the induction hypothesis we know that

$$K_{\widetilde{x}} = 0 \tag{6.9.2}$$

for all log geometric points $\widetilde{x}$ over a point x of X distinct of $\{0\}$, since $r(x) < r$ for such a point. Consider again the inverse system $(X_m)_{m \in \mathcal{N}}$ used in 6.7 and 6.8, and the inverse system $(\{0\}_m)$ it induces on $\{0\}$ endowed with the induced log structure. By 4.4.1, we have

$$H^i(K_{\{0\}^\sim}) = \text{ind.}\lim\ H^i((\{0\}_m)_{\text{ket}}, K).$$

Therefore, to prove (6.9.1) it suffices to prove that

$$R\Gamma(\{0\}_{\text{ket}}, K) = 0\ .$$

But by (6.9.2) we have

$$R\Gamma(\{0\}_{\text{ket}}, K) = R\Gamma(X_{\text{ket}}, K)\ .$$

Hence it suffices to show that the inverse image map

$$f^* : H^q(X_{\text{ket}}, \Lambda) \longrightarrow H^q(Y_{\text{ket}}, \Lambda)$$

is an isomorphism for all q.

Suppose first that n is prime to p ($\Lambda = \mathbb{Z}/n\mathbb{Z}$). The log-blow up Y of X along I is a toric variety, with torus T, covered by the open affine equivariant subschemes $(U_a)^{\text{sat}} = \operatorname{Spec} k[(P_a)^{\text{sat}}]$ considered in 6.1. The projection $f : Y \to X$ is T-equivariant and induces an isomorphism on T. That f^* is an isomorphism then follows from 6.8.1.

That concludes the proof of 6.2 in the case n prime to p. In the general case the proof proceeds in the same way, except that to dispose of the case where n is a power of p, it is necessary to replace the affine toric variety $X = \operatorname{Spec} k[P]$ by a T-equivariant compactification $\overline{X}$, that is a toric variety with torus T, containing X as an equivariant dense open subset, and which is *proper* over k. It is a standard fact that such compactifications exist: if Δ is the fan of X (the set of faces of $P^\vee$), it suffices to choose a complete fan $\overline{\Delta}$ containing Δ, and $X(\overline{\Delta})$ is a T-equivariant compactification of X ([**Od**], 1.12). If Z is the closed subscheme of X defined by I, the schematic closure $\overline{Z}$ of Z in $\overline{X}$ is defined by a T-equivariant sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_{\overline{X}}$ and even by a coherent sheaf of ideals $\mathcal{J}$ of $M_{\overline{X}}$. Let $\overline{f} : \overline{Y} \to \overline{X}$ be the log blow-up of $\overline{X}$ along $\mathcal{J}$. Then $\overline{Y}$ is a toric variety with torus T, and $\overline{f}$ is equivariant and induces an isomorphism on T. Again, one has to show (6.9.1), where K is the cone of $\alpha : \Lambda \to R\overline{f}_*\Lambda$. This time, the induction hypothesis only gives us that there exists a finite number of closed points x_i ($1 \leqslant i \leqslant N$) of $\overline{X}$ (including $\{0\}$) such that

$$K_{\widetilde{x}} = 0 \tag{6.9.3}$$

for all log geometric points $\widetilde{x}$ over a point x of X not belonging to the set of $x_i's$. As above, we have

$$\bigoplus_{1\leqslant i\leqslant N} H^q(K_{(\widetilde{x}_i)}) = \bigoplus_{1\leqslant i\leqslant N} \text{ind.}\lim H^q(((x_i)_m)_{\text{ket}}, \Lambda),$$

and by (6.9.3),

$$\bigoplus_{1\leqslant i\leqslant N} R\Gamma((x_i)_{\text{ket}}, K) = R\Gamma(\overline{X}_{\text{ket}}, K),$$

so that again it is enough to show that

$$f^* : H^q(\overline{X}_{\text{ket}}, \Lambda) \longrightarrow H^q(\overline{Y}_{\text{ket}}, \Lambda)$$

is an isomorphism for all q. Writing $\Lambda = \mathbb{Z}/p^h\mathbb{Z} \times \mathbb{Z}/n'\mathbb{Z}$ with n' prime to p, and using the case treated above, we may assume $\Lambda = \mathbb{Z}/p^h\mathbb{Z}$. The result is then a consequence of the following lemma:

Lemma 6.9.4. — *Let X/k be a* proper *toric variety, with k algebraically closed of characteristic $p > 0$, and let $\Lambda = \mathbb{Z}/p^h\mathbb{Z}$, $h \geqslant 1$. Then we have*

$$H^q(X_{\mathrm{ket}}, \Lambda) = \begin{cases} \Lambda \text{ if } q = 0 \\ 0 \text{ if } q > 0 \ . \end{cases}$$

By dévissage we reduce to $\Lambda = \mathbb{Z}/p\mathbb{Z}$. Since X is proper we have

$$H^q(X, \mathcal{O}_X) = \begin{cases} k \text{ for } q = 0 \\ 0 \text{ for } q > 0 \ . \end{cases}$$

(([**KKMS**], p. 44) or ([**Od**], 2.8) in the case $k = \mathbb{C}$). Hence, by Artin-Schreier,

$$H^q(X_{\mathrm{cl}}, \Lambda) = \begin{cases} \Lambda \text{ if } q = 0 \\ 0 \text{ if } q > 0 \ . \end{cases}$$

By 6.8.4 (where n needs not be assumed to be prime to p), we get again (6.8.5) and (6.8.6), and since π is of pro-order prime to p, the conclusion follows. This concludes Fujiwara-Kato's proof of 6.2.

Fujiwara-Kato ([**F-K**] 2.4) also prove a noncommutative variant of 6.2 :

Theorem 6.10. — *Let $f : Y \to X$ be a log blow-up as in* 6.2. *Then for any log geometric point $\widetilde{y}$ of Y, with image $\widetilde{x}$ by f, the natural map*

$$\pi_1^{\log}(Y, \widetilde{y}) \longrightarrow \pi_1^{\log}(X, \widetilde{x})$$

is an isomorphism.

We will not attempt to explain their proof. Let us just say that the key point is to replace 6.8.1 by a Zariski-Nagata type purity result, namely that in the situation of 6.8.1, restriction to T defines an isomorphism between $\pi_1^{\log}(X)$ and the quotient of $\pi_1(T)$ classifying the étale covers tamely ramified at the generic points of $X - T$.

7. Purity

Recall Grothendieck's absolute purity conjecture (([**SGA 4**] XVI 3.10 b), ([**SGA 5**] I 3.1.4)), now a theorem of Gabber [**Ga**]:

Theorem 7.1. — *Let d be a positive integer, X a regular locally noetherian scheme, $i : Y \to X$ a regular closed subscheme, everywhere of codimension d, and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on X. Then*

$$R^q i^!(\Lambda) = 0 \text{ for } q \neq 2d \ ,$$

and the cohomology class of Y ([**SGA 4 1/2**], Cycle 2.2) *defines an isomorphism*

$$R^{2d}i^!(\Lambda) \cong \Lambda(-d)\ .$$

This statement implies – and in fact is equivalent to – the following one:

Theorem 7.2. — *Let* X *be a regular locally noetherian scheme and let* $D = \sum_{1 \leqslant m \leqslant N} D_m$ *be a divisor with normal crossings on* X*, with* D_m *regular for all* i*. Let* $j_m : U_m = X - D_m \hookrightarrow X$*,* $j : U = X - D \hookrightarrow X$ *be the inclusions, and let* $\Lambda = \mathbb{Z}/n\mathbb{Z}$ *with* n *invertible on* X*. Then:*

(a) $R^q j_{m*}\Lambda = 0$ *for* $q > 1$, $j_{m*}\Lambda = \Lambda$, *and the Kummer sequence defines an isomorphism* $R^1 j_{m*}\Lambda \cong \Lambda_{D_m}(-1)$;

(b) *the natural map*

$$\overset{L}{\bigotimes_{1 \leqslant m \leqslant n}} Rj_{m*}\Lambda \longrightarrow Rj_*\Lambda$$

is an isomorphism, which is equivalent to the conjunction of (i) and (ii):

(i) *the restriction map*

$$\bigoplus_{1 \leqslant m \leqslant n} R^1 j_{m*}\Lambda \longrightarrow R^1 j_*\Lambda$$

is an isomorphism;

(ii) *for all* $q \geqslant 0$, *the cup-product map* $\otimes^q R^1 j_*\Lambda \to R^q j_*\Lambda$ *factors through an isomorphism*

$$\Lambda^q R^1 j_*\Lambda \xrightarrow{\sim} R^q j_*\Lambda\ .$$

To see that 7.1 and 7.2 are equivalent one can argue as follows: from 7.1 one gets 7.2 by induction on N; conversely, one can assume X strictly local, one can factor i into $i_1 \cdots i_d$, where $i_m : Y_m \to Y_{m-1}$ is of codimension 1, Y_m is regular, $Y_d = Y$, $Y_0 = X$; it then suffices to apply 6.2 repeatedly to the regular divisors i_m's.

7.3. Using 6.2, Fujiwara-Kato prove a Kummer étale variant (and generalization) of 7.2 for log regular log schemes.

Let X be an fs log scheme. One says that X is *log regular* (*cf.* ([**Ka 4**], 2.1) if the underlying scheme $\mathring{X}$ is locally noetherian and the following condition holds:

(7.3.1) for any (classical) geometric point $\overline{x}$ of X above x, if $J_{\overline{x}}$ is the ideal of $\mathcal{O}_{X,\overline{x}}$ generated by the image of $M_{X,\overline{x}} - \mathcal{O}^*_{X,\overline{x}}$ by α, $\mathcal{O}_{X,\overline{x}}/J_{\overline{x}}$ is regular and

$$\dim \mathcal{O}_{X,\overline{x}}/J_{\overline{x}} + \operatorname{rk} \overline{M}^{\mathrm{gp}}_{X,\overline{x}} = \dim \mathcal{O}_{X,\overline{x}}.$$

Log regularity was first defined by Kato ([**Ka 4**], 2.1) for schemes equipped with log structures defined for the Zariski topology and satisfying "condition (S)" (similar to fs). The definition we make here is adapted to the context in which we are working. For the links between the two notions, see [**Ni**].

Here are some examples and properties which follow easily from ([**Ka 4**], 2.1).

(a) If X has the trivial log structure, X is log regular if and only if $\overset{\circ}{X}$ is regular.

(b) If $\overset{\circ}{X}$ is regular and X has the canonical log structure given by a divisor with normal crossings $D \subset X$, then X is log regular. In this case, $\overline{M}_{X,\overline{x}} \cong \mathbb{N}^{r(x)}$ at all geometric points $\overline{x}$, where $r(x) = \operatorname{rk} \overline{M}^{\mathrm{gp}}_{X,\overline{x}}$. Conversely, if X is log regular and if at some geometric point $\overline{x}$, $\overline{M}_{X,\overline{x}} \cong \mathbb{N}^{r(x)}$, then étale locally around $\overline{x}$, $\overset{\circ}{X}$ is regular and the log structure of X is given by a normal crossings divisor (this can be seen by taking a chart $X \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^{r(x)}]$ around $\overline{x}$, and using (7.3.1) together with ([**EGA IV**] 17.1.7).

(c) If Y is log smooth over X and X is log regular, then Y is log regular. If X is an fs log scheme separated and of finite type over the spectrum of a perfect field k endowed with the trivial log structure, then X is log regular if and only if X is log smooth over k.

(d) If X is log regular and $j : U \hookrightarrow X$ is the open subset of triviality of the log structure, then U is dense and the log structure of X is the direct image of that of U, *i.e.* $M_X = \mathcal{O}_X \cap j_* \mathcal{O}_U^*$ ([**Ka 4**], 11.6).

(e) If X is log regular, $\overset{\circ}{X}$ is Cohen-Macaulay and normal.

Theorem 7.4 **([F-K], 3.1).** — *Let X be a log regular fs log scheme, let $j : U \hookrightarrow X$ be the open subset of triviality of the log structure, and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on X. Then the adjunction map*

$$\Lambda \longrightarrow Rj_{\mathrm{ket}*}\Lambda$$

is an isomorphism.

First we treat the case where $\overset{\circ}{X}$ is regular and the log structure of X is given by a normal crossings divisor D. We may assume that D has simple normal crossings, *i.e.* $D = \sum_{1 \leqslant m \leqslant N} D_m$, with the D_m regular. The proof exploits the analogy between the formulas for $R^q\varepsilon_*\Lambda$ (5.2) and $R^q j_*\Lambda$ (7.2). More precisely, consider the commutative square

$$\begin{tikzcd} U_{\mathrm{ket}} \arrow[r, "j_{\mathrm{ket}}"] \arrow[d, "\varepsilon"', "\wr"] & X_{\mathrm{ket}} \arrow[d, "\varepsilon"] \\ U_{\mathrm{cl}} \arrow[r, "j_{\mathrm{cl}}"] & X_{\mathrm{cl}} \end{tikzcd} .$$

In the same way as in step (a) of the proof of 5.9 we see by looking at the stalks at log geometric points that to prove that the adjunction map $\alpha : \Lambda \to Rj_{\mathrm{ket}*}\Lambda$ is an isomorphism, it is enough to prove that the map

$$\beta = R\varepsilon_*(\alpha) : R\varepsilon_*\Lambda \longrightarrow Rj_{cl*}\Lambda$$

is an isomorphism (we have identified U_{ket} to U_{cl} by ε). Since $U = X - D$ and $M = M_X = \mathcal{O}_X \cap j_*\mathcal{O}_U^*$, we have

$$\overline{M}^{\mathrm{gp}} = \bigoplus_{1 \leqslant m \leqslant N} \mathbb{Z}_{D_m} .$$

Consider the square

$$\begin{array}{ccc} \overline{M}^{\mathrm{gp}} \otimes \Lambda(-1) & \xrightarrow{\ \sim\ } & R^1\varepsilon_*\Lambda \\ {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ \bigoplus_{1 \leqslant m \leqslant N} \Lambda_{D_m}(-1) & \xrightarrow{\ \sim\ } & R^1 j_{cl*}\Lambda , \end{array}$$

where the top horizontal map is the isomorphism given by 5.2, and the bottom one is the isomorphism given by 7.2 (a), (b) (i). This square commutes because both horizontal maps are given by a Kummer sequence. Therefore

$$\beta : R^q\varepsilon_*\Lambda \longrightarrow R^q j_{cl*}\Lambda$$

is an isomorphism for $q = 1$, hence for all q by 5.2 and 7.2 (b) (ii).

In the general case, working classically étale locally on X, we may assume, thanks to Kato's resolution of toric singularities ([**Ka 4**], 10.4), that there exists a log blow-up $f : Y \to X$ of X along some coherent sheaf of ideals of M_X such that $\overline{M}_{Y,\overline{y}} \cong \mathbb{N}^{r(y)}$ for all geometric points $\overline{y}$ of Y. Since f is log étale, Y is log regular (7.3 (c)), hence by 7.3 (b) $\overset{\circ}{Y}$ is regular and the log structure of Y is given by a divisor D with normal crossings. Moreover f induces an isomorphism from the open subset of triviality $V = Y - D$ for the log structure of Y to U. Let $h : V \to Y$ be the inclusion. By the particular case treated above the adjunction map $\Lambda \to Rh_{\mathrm{ket}*}\Lambda$ is an isomorphism. Applying $Rf_{\mathrm{ket}*}$ and taking into account that the adjunction map $\Lambda \to Rf_{\mathrm{ket}*}\Lambda$ is an isomorphism by Fujiwara-Kato's basic theorem 6.2, we get the conclusion.

Corollary 7.5. — *Under the assumptions of* 7.4, *the restriction map*

$$\beta : R\varepsilon_*\Lambda \longrightarrow Rj_{cl*}\Lambda$$

(deduced from the adjunction map of 7.4 *by application of $R\varepsilon_*$) is an isomorphism, and consequently the restriction map*

$$H^q(X^{\mathrm{ket}}, \Lambda) \longrightarrow H^q(U_{\mathrm{cl}}, \Lambda)$$

is an isomorphism for all q.

Fujiwara-Kato also establish a noncommutative, Zariski-Nagata type variant of 7.4, which is a generalization of the key lemma used in their proof of 6.10, namely:

Theorem 7.6 (**[F-K], 3.1**). — *Let X be a log regular fs log scheme, and let $U \subset X$ be the open subset of triviality of the log structure. Then the functor*

$$\mathrm{Kcov}(X) \longrightarrow \mathrm{Etcov}(U),\ Y \longrightarrow Y \times_X U$$

from the category of finite Kummer étale covers of X (3.1) *to the category of finite classical étale covers of* U *induces an equivalence between* $\mathrm{Kcov}(X)$ *and the full subcategory of* $\mathrm{Etcov}(U)$ *consisting of covers* $V \to U$ *which are tamely ramified along* $X - U$, i.e. *such that if* Z *is the normalization of* X *in* V, *at all points* $x \in X - U$ *with* $\dim \mathcal{O}_{X,x} = 1$, *the restriction of* Z *to* $\mathrm{Spec}\, \mathcal{O}_{X,x}$ *is tamely ramified.*

There are analogues of 7.4 and 7.6 for log Betti cohomology. The main result is the following one, whose proof is elementary:

***Theorem** 7.7* (**[Og], 5.12, [K-N], 1.5.1**). — *Let* X *be a log smooth, fs log scheme over* $\mathbb{C}$, *let* $j : U \hookrightarrow X$ *be the open subset of triviality of the log structure, and* $j^{\log} : U_{\mathrm{an}} = U^{\log} \hookrightarrow X^{\log}$ *be the corresponding inclusion* (5.5). *Then any point* $x \in X^{\log}$ *has a fundamental system of neighborhoods* V *such that* $V \cap U_{\mathrm{an}}$ *is contractible. In particular:*

(a) *the adjunction map* $\mathbb{Z} \to Rj^{\log}_* \mathbb{Z}$ *is an isomorphism;*

(b) *the restriction to* $U_{\mathrm{an}} = U^{\log}$ *defines an equivalence between the category of locally constant sheaves on* $X^{\log}$ *and the corresponding category on* U_{an}.

8. Nearby cycles

8.1. Let $S = \mathrm{Spec}\, A$, where A is a henselian discrete valuation ring with fraction field K and residue field k with characteristic exponent p. Let $s = \mathrm{Spec}\, k$, $\eta = \mathrm{Spec}\, K$. We fix an algebraic closure $\overline{K}$ of K and denote by $\overline{A}$ the normalization of A in $\overline{K}$, a valuation ring whose residue field $\overline{k}$ is an algebraic closure of k. We put $\overline{S} = \mathrm{Spec}\, \overline{A}$, $\overline{s} = \mathrm{Spec}\, \overline{k}$, $\overline{\eta} = \mathrm{Spec}\, \overline{K}$.

As in 4.7 (b), we endow S (resp. $\overline{S}$) with the natural log structure associated to the inclusion $A - \{0\} \to A$ (resp. $\overline{A} - \{0\} \to \overline{A}$). We denote by S (resp. $\widetilde{S}$) the resulting log scheme, and similarly by s (resp. $\widetilde{s}$) the point s (resp. $\overline{s}$) with the induced log structure. We will keep the notations of 4.7 (b) for the various fundamental groups associated to these data. In particular, the full Galois group $G = \mathrm{Gal}(\overline{K}/K)$ acts on $\widetilde{S}$ and $\widetilde{s}$ (by transportation of structure). A basic fact is that G *acts on* $\widetilde{s}$ *through its tame quotient*

$$G^t = \pi_1^{\log}(S, \widetilde{s}) = \pi_1^{\log}(s, \widetilde{s}) = G/P\ .$$

In order to see this it is convenient to consider the log scheme $S^t = \mathrm{Spec}(A^t)$ (with the log structure associated to the inclusion $A^t - \{0\} \to A^t$), where A^t is the union of the normalizations of A in the tamely ramified extensions of K contained in $\overline{K}$. Let s^t be the closed point of S^t with its induced log structure. Then $S^t \to S$ (resp. $s^t \to s$) is a log universal cover of S (resp. s), with (opposite) automorphism group G^t. On the other hand, the canonical projection $\widetilde{s} \to s^t$ is a limit of fs universal Kummer homeomorphisms (*cf.* 2.8), and therefore s-automorphisms of s^t extend uniquely to

s-automorphisms of $\widetilde{s}$ ([**Vi 1**], A4), ([**Na 1**], 3.1.3)). The tame inertia

$$I^t = I^{\log}(s, \widetilde{s}) \subset G^t$$

acts on s^t (hence on $\widetilde{s}$) through the tame character

$$t : I^t \xrightarrow{\sim} \widehat{\mathbb{Z}}'(1)(\overline{k})$$

as explained in 4.7 (a).

8.2. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$, with n prime to p. Let X be an fs log scheme over S. We define $\widetilde{X}$ and $X_{\widetilde{s}}$ by the fibered products in the category of integral and saturated log schemes:

$$\widetilde{X} = X \times_S \widetilde{S},\ X_{\widetilde{s}} = \widetilde{X} \times_{\widetilde{S}} \widetilde{s} = X_s \times_s \widetilde{s}\ .$$

Note that though by definition $\overline{S}$ and $\widetilde{S}$ have the same underlying schemes, the underlying scheme of $\widetilde{X}$ is in general different from the pull-back $\overline{X}$ of X by $\overline{S} \to S$ in the category of schemes. A similar remark applies to $X_{\widetilde{s}}$ and $X_{\overline{s}}$. We have cartesian squares (in the category of integral and saturated log schemes)

$$\begin{tikzcd} X_{\widetilde{s}} \arrow[r, "\widetilde{i}"] \arrow[d] & \widetilde{X} \arrow[d] & X_{\overline{\eta}} \arrow[l, "\widetilde{j}"'] \arrow[d] \\ \widetilde{s} \arrow[r] & \widetilde{S} & \overline{\eta}\ . \arrow[l] \end{tikzcd}$$

For $\mathcal{L} \in D^+(X_\eta^{\text{ket}}, \Lambda)$ we define the complex of *log nearby cycles* $R\Psi^{\log}\mathcal{L}$ by

$$R\Psi^{\log}(\mathcal{L}) = \widetilde{i}^{\text{ket}*} R\widetilde{j}^{\text{ket}}_*(\mathcal{L}|X_{\overline{\eta}}) \tag{8.2.1}$$

where the superscript "ket" means that the inverse and direct images are taken with respect to the Kummer étale topologies (defined as limits since $\widetilde{X}$ is an inverse limit of fs log schemes). The functor $R\Psi^{\log}$ *a priori* goes from $D^+(X_\eta^{\text{ket}}, \Lambda)$ to $D^+(X_{\widetilde{s}}^{\text{ket}}, \Lambda)$, but as usual, with a little more care, one can define it as going from $D^+(X_\eta^{\text{ket}}, \Lambda)$ to $D^+(X_s^{\text{ket}} \times_s \eta, \Lambda)$, where, using Deligne's notation in ([**SGA 7**] XIII), $X_s^{\text{ket}} \times_s \eta$ denotes the topos of (Kummer étale) sheaves on $X_{\widetilde{s}}$ endowed with a continuous action of G compatible with that of G on $X_{\widetilde{s}}$.

Log nearby cycles are related to the classical ones as follows. Let $\overline{i}$, $\overline{j}$ be defined by the cartesian squares of schemes

$$\begin{tikzcd} X_{\overline{s}} \arrow[r, "\overline{i}"] \arrow[d] & \overline{X} \arrow[d] & X_{\overline{\eta}} \arrow[l, "\overline{j}"'] \arrow[d] \\ \overline{s} \arrow[r] & \overline{S} & \overline{\eta}\ . \arrow[l] \end{tikzcd}$$

Denote by $R\Psi^{\text{cl}}$ the classical nearby cycles functor ([**SGA 7**] XIII), defined by

$$R\Psi^{\text{cl}}(\mathcal{F}) = \overline{i}^* R\overline{j}_*(\mathcal{F}|X_{\overline{\eta}}) \tag{8.2.2}$$

for $\mathcal{F} \in D^+(X_\eta, \Lambda)$ (this is a functor from $D^+(X_\eta, \Lambda)$ to $D^+(X_s \times_s \eta, \Lambda)$). Consider the commutative diagram

$$\begin{tikzcd}
X^{\mathrm{ket}}_{\bar{s}} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{\varepsilon}"'] & \widetilde{X}^{\mathrm{ket}} \arrow[d, "\tilde{\varepsilon}"'] & X^{\mathrm{ket}}_{\bar{\eta}} \arrow[l, "\tilde{j}"'] \arrow[d, "\varepsilon"'] \\
X^{\mathrm{cl}}_{\bar{s}} \arrow[r, "\bar{i}"] & \overline{X}^{\mathrm{cl}} & X_{\bar{\eta}}\,, \arrow[l, "\bar{j}"']
\end{tikzcd}$$

where for brevity we write $X^{\mathrm{cl}}_{\bar{s}}$ for $X_s \times_s \eta$, etc. and the middle (resp. left) vertical map is the canonical projection, *i.e.* the composition

$$\widetilde{X}^{\mathrm{ket}} \xrightarrow{\varepsilon} \widetilde{X}^{\mathrm{cl}} \longrightarrow \overline{X}^{\mathrm{cl}}$$

(resp. $X^{\mathrm{ket}}_{\bar{s}} \xrightarrow{\varepsilon} X^{\mathrm{cl}}_{\bar{s}} \to X^{\mathrm{cl}}_{\bar{s}}$). By 6.3 applied to the left square (strict base change by $\bar{i}$) and a standard limit argument we have, for $\mathcal{L} \in D^+(X^{\mathrm{log}}_\eta, \Lambda)$,

$$\bar{i}^* R\tilde{\varepsilon}_*(R\bar{j}^{\mathrm{ket}}_*(\mathcal{L}|X_{\bar{\eta}})) = R\tilde{\varepsilon}_*\tilde{i}^*(R\bar{j}^{\mathrm{ket}}_*(\mathcal{L}|X_{\bar{\eta}})),$$

and therefore,

$$R\Psi^{\mathrm{cl}} R\varepsilon_*\mathcal{L} \cong R\tilde{\varepsilon}_* R\Psi^{\mathrm{log}}\mathcal{L}\ . \tag{8.2.3}$$

In particular, when the log structure of X is *vertical*, by which we mean that the log structure of X_η is trivial (*i.e.* X_η coincides with the open subset of triviality of the log structure of X), we have $X^{\mathrm{cl}}_\eta = X^{\mathrm{ket}}_\eta$ and

$$R\Psi^{\mathrm{cl}}\mathcal{L} \cong R\tilde{\varepsilon}_* R\Psi^{\mathrm{log}}\mathcal{L}\ .$$

By analogy with the classical case, for $\mathcal{L} \in D^+(X^{\mathrm{log}}, \Lambda)$, one can also define the complex of *vanishing cycles* $R\Phi^{\mathrm{log}}\mathcal{L} \in D^+(X^{\mathrm{ket}}_s \times_s \eta, \Lambda)$, which sits in an exact triangle (of $D^+(X^{\mathrm{ket}}_s \times_s \eta, \Lambda)$)

$$i^*\mathcal{L} \longrightarrow R\Psi^{\mathrm{log}}\mathcal{L} \longrightarrow R\Phi^{\mathrm{log}}\mathcal{L} \xrightarrow{+1} \tag{8.2.4}$$

and is related to the classical one by

$$R\Phi^{\mathrm{cl}} R\varepsilon_*\mathcal{L} \cong R\tilde{\varepsilon}_* R\Phi^{\mathrm{log}}\mathcal{L}\ . \tag{8.2.5}$$

When $\mathring{X}$ is *proper* over S, then by 6.3 applied to the strict base change by $\tilde{s} \to \widetilde{S}$, we get, for $\mathcal{F} \in D^+(X^{\mathrm{ket}}_\eta, \Lambda)$, a canonical isomorphism

$$R\Gamma(X^{\mathrm{ket}}_{\bar{s}}, R\Psi^{\mathrm{log}}\mathcal{F}) \cong R\Gamma(X^{\mathrm{ket}}_{\bar{\eta}}, \mathcal{F}), \tag{8.2.6}$$

from which by (8.2.4) we derive, for $\mathcal{L} \in D^+(X^{\mathrm{ket}}, \Lambda)$, a *specialization map*

$$\mathrm{sp} : R\Gamma(X^{\mathrm{ket}}_{\bar{s}}, \bar{i}^*\mathcal{L}) \longrightarrow R\Gamma(X^{\mathrm{ket}}_{\bar{\eta}}, \bar{j}^*\mathcal{L}), \tag{8.2.7}$$

which is an isomorphism if and only if $R\Gamma(X^{\mathrm{ket}}_{\bar{s}}, R\Phi^{\mathrm{log}}\mathcal{L}) = 0$.

Concerning $R\Psi^{\mathrm{log}}$, the main result so far is the following theorem of Nakayama:

Theorem 8.3 (**[Na 2], 3.2**). — *Let X be a log smooth fs log scheme over S. Then*

$$R\Phi^{\log}\Lambda = 0\ .$$

This is a simple application of the purity theorem 7.4. Indeed, let $u : U \hookrightarrow X_\eta$ be the open subset of triviality of the log structure of X_η (which is also that of X). Let S' be the normalization of S in a finite extension K' of K contained in $\overline{K}$, s' its closed point, and put the standard log structures on S' (resp. s'). Let

$$X_{s'} \xrightarrow{i'} X' \xleftarrow{j'} X_{K'} \xleftarrow{u'} U'$$

be the diagram deduced by fs base change by $S' \to S$ from the given diagram

$$X_s \xrightarrow{i} X \xleftarrow{j} X_K \xleftarrow{u} U\ .$$

Because U' is the open subset of triviality of the log structures of both $X_{K'}$ and X', we have, by 7.4,

$$R(j'u')^{\mathrm{ket}}_*\Lambda = \Lambda_{X'}\ ,\ \ Ru'^{\mathrm{ket}}_*\Lambda = \Lambda_{X_{K'}},$$

hence

$$(*) \qquad Rj'^{\mathrm{ket}}_*\Lambda = \Lambda_{X'},$$

and therefore

$$i'^{\mathrm{ket}*}Rj'^{\mathrm{ket}}_*\Lambda = \Lambda_{X_{s'}}.$$

By definition we have

$$R^q\Psi^{\log}(\Lambda) = \mathrm{ind.lim}\, i'^{\mathrm{ket}*}R^q j'^{\mathrm{ket}}_*\Lambda\ ,$$

where K' runs through the finite extensions of K contained in $\overline{K}$, and the conclusion follows.

In [**Na 2**] Nakayama gave a proof of $(*)$ independent of Gabber's purity theorem (which is the key ingredient in 7.4). Kato gave an alternate proof as a corollary of a log smooth base change theorem (see 9.6).

Corollary 8.4. — *Let X be a log smooth fs log scheme over S. Then we have a canonical isomorphism in $D^+(S_s \times_s \eta, \Lambda)$:*

$$(8.4.1) \qquad R\Psi^{\mathrm{cl}}(R\varepsilon_*\Lambda) = R\widetilde{\varepsilon}_*(\Lambda|X_{\widetilde{s}}),$$

and in particular, if the log structure of X is vertical (8.2.3),

$$R\Psi^{\mathrm{cl}}\Lambda = R\widetilde{\varepsilon}_*(\Lambda|X_{\widetilde{s}}).$$

A striking consequence of 8.4 is that, when X is log smooth over S (resp. log smooth and vertical over S), $R\Psi^{\mathrm{cl}}R\varepsilon_*\Lambda$ (resp. $R\Psi^{\mathrm{cl}}\Lambda$), *as an object of $D^+(X_s\times_s\eta, \Lambda)$, depends only on X_s endowed with its log structure.* It has been shown by Kisin [**Ki 2**] (and independently by Vidal) that if the log structure of X is vertical, the log structure of X_s depends only on some infinitesimal neighbourhood of $\mathring{X}_s$ in $\mathring{X}$, in the following sense: if X_1, X_2 are fs log schemes over S which are log smooth and vertical, there exists an integer $m \geqslant 1$ such that if $\overset{\circ}{X}_1 \otimes A/\pi^{m+1}$ and $\overset{\circ}{X}_2 \otimes A/\pi^{m+1}$ are isomorphic (as schemes over A/π^{m+1}) then so are $(X_1)_s$ and $(X_2)_s$ (as log schemes over s); here π is a uniformizing parameter of A. In the case of semistable reduction, $m = 1$ suffices (*cf.* ([**Na 2**], A4)).

Since the wild inertia P acts trivially on $X_{\overline{s}}$ (8.1), it also acts trivially on $R\Psi^{\mathrm{cl}}(R\varepsilon_*\Lambda) = R\widetilde{\varepsilon}_*(\Lambda|X_{\overline{s}})$ (resp. $R\Psi^{\mathrm{cl}}(\Lambda)$), so we have

$$\begin{aligned} R\Psi^{\mathrm{cl}}(R\varepsilon_*\Lambda) &= R\Gamma(P, R\Psi^{\mathrm{cl}}(R\varepsilon_*\Lambda)), \\ R^q\Psi^{\mathrm{cl}}(R\varepsilon_*\Lambda) &= \Gamma(P, R^q\Psi^{\mathrm{cl}}(R\varepsilon_*\Lambda)) \end{aligned} \tag{8.4.2}$$

(note $\Gamma(P, -)$ is exact on the category of $\Lambda[P]$-modules on $X_{\overline{s}}$). Therefore, when the log structure of X is vertical, we have

$$\begin{aligned} R\Psi^{\mathrm{cl}}(\Lambda) &= R\Gamma(P, R\Psi^{\mathrm{cl}}(\Lambda)) \\ R^q\Psi^{\mathrm{cl}}(\Lambda) &= \Gamma(P, R^q\Psi^{\mathrm{cl}}(\Lambda)). \end{aligned}$$

The tameness of the $R^q\Psi\Lambda$ had been proven by Rapoport-Zink [**R-Z**] in the case X is étale locally of the form $S[x_1, \ldots, x_n]/(x_1^{a_1} \cdots x_r^{a_r} - \pi)$, with π a prime element in A and a_i prime to p for all i. By (8.4.1), this tameness holds even if some a_i are divisible by p but $\gcd(a_1, \ldots, a_r, p) = 1$. When $a_1 = \cdots = a_r = 1$ – the case of semistable reduction – the whole inertia I acts trivially on the $R^q\Psi\Lambda$ (loc. cit.), a result which also follows from 8.4 as we shall see later.

Combining (8.4.2) with (8.2.6) and 7.5 we get:

Corollary 8.4.3. — *Let X be a proper and log smooth fs log scheme over S, and let $U \hookrightarrow X_\eta \hookrightarrow X$ be the open subset of triviality of the log structure of X. Then the wild inertia P acts trivially on $H^q(U_{\overline{\eta}}^{\mathrm{cl}}, \Lambda)$ and the specialization map* (8.2.7)

$$\mathrm{sp} : H^q(X_{\overline{s}}^{\mathrm{cl}}, R\widetilde{\varepsilon}_*\Lambda) \longrightarrow H^q(U_{\overline{\eta}}^{\mathrm{cl}}, \Lambda)$$

is a (Galois equivariant) isomorphism for all q.

Unraveling (8.4.1) yields the following description of the sheaves of nearby cycles $R^q\Psi^{\mathrm{cl}}(R\varepsilon_*\Lambda)$ ($= R^q\Psi^{\mathrm{cl}}\Lambda$ when X is vertical):

***Corollary 8.4.4* ([Na 2], 3.5)**. — *Let $f : X \to S$ be a log smooth fs log scheme over S, and let $\mathcal{F} := R\varepsilon_*\Lambda = Rj_*^{\mathrm{cl}}\Lambda \in D^+(X_\eta, \Lambda)$ (where $j : U \hookrightarrow X_\eta \hookrightarrow X$ is the open subset of triviality of the log structure of X). Let C be the (classically) locally constant sheaf of finitely generated abelian groups on X_s defined by*

$$C := \mathrm{Coker}(f^*(\overline{M}_s^{\mathrm{gp}}) \xrightarrow{\phi} (\overline{M}_{X_s}^{\mathrm{gp}}))/\mathrm{torsion},$$

where φ is the canonical map. Then:

(a) *There is a natural, Galois equivariant isomorphism*

$$R^0\Psi^{\mathrm{cl}}\mathcal{F} \otimes \Lambda^q(C_{\overline{s}} \otimes \Lambda(-1)) \longrightarrow R^q\Psi^{\mathrm{cl}}\mathcal{F}$$

for all $q > 0$.

(b) *The stalk of* $R^0\Psi^{\mathrm{cl}}\mathcal{F}$ *at a (classical) geometric point* $\overline{x}$ *over* $\overline{s}$ *is (noncanonically) isomorphic to* $\Lambda[E_{\overline{x}}]$, *where* $E_{\overline{x}} := \operatorname{Coker} f_* : I^{\log}_{\overline{x}} \to I^{\log}_{\overline{s}}$ *(a finite abelian group, isomorphic to* $\operatorname{Coker}\operatorname{Hom}(\Phi_{\overline{x}}, \widehat{\mathbb{Z}}'(1))$ *with* ϕ *as above). The tame inertia* $I^t \cong \widehat{\mathbb{Z}}'(1))$ *acts on* $(R^0\Psi^{\mathrm{cl}}\mathcal{F})_{\overline{x}}$ *by the regular representation,* i.e. *through the composite map* $I^t \cong I^{\log}_{\overline{s}} \hookrightarrow \Lambda[I^{\log}_{\overline{s}}] \to \Lambda[E_{\overline{x}}]$.

When X has semistable reduction over S with special fiber $X_s = D$ a divisor with simple normal crossings ΣD_i, then $U = X_\eta$, $\mathcal{F} = \Lambda_X$, $C = \operatorname{Coker}\mathbb{Z}_D \to \oplus\mathbb{Z}_{D_i}$, $R^0\Psi^{\mathrm{cl}}\Lambda = \Lambda$, and one recovers the classical global formula for $R^q\Psi\Lambda$ (*cf.* [**Na 2**] and [**I2**]). The *global* structure of the constructible sheaf $R^0\Psi^{\mathrm{cl}}\mathcal{F}$ is controlled by the projection

$$\alpha : \mathring{X}_{\widetilde{s}} \longrightarrow \mathring{X}_{\overline{s}} ;$$

it can be proved that α is a *finite* map, which is the composition of a closed surjective immersion with a finite map with separable residual extensions, such that $R^0\Psi^{\mathrm{cl}}\mathcal{F}$ is just the direct image by α of the constant sheaf Λ (see [**Ka 7**], [**Vi 3**]). In particular, with the notation above, if we set $d_x = \operatorname{card} E_{\overline{x}}$, then $d_x = \sum_{\alpha(y)=x}[k(y) : k(x)]$. If $N = \sup d_x$, then $T^N = 1$ on $R^q\Psi^{\mathrm{cl}}\mathcal{F}$ for any $T \in I$, and therefore, in the situation of 8.4.3

$$(T^N - 1)^{q'} | H^q(U^{\mathrm{cl}}_{\overline{\eta}}, \Lambda) = 0 \tag{8.4.4.1}$$

where q' is the minimum of $q+1$ and the rank of $\overline{M}^{\mathrm{gp}}_X$ ($= \sup \operatorname{rk}(\overline{M}^{\mathrm{gp}}_X)_{\overline{x}}$, $\overline{x}$ running through the geometric points of X).

The proof of 8.4.4 from (8.4.1) is an application of 5.2, using either one of the two factorizations of $\widetilde{\varepsilon} : X^{\mathrm{ket}}_{\widetilde{s}} \to X^{\mathrm{cl}}_{\widetilde{s}}$ in 8.2 as $\widetilde{\varepsilon} = \alpha \circ \varepsilon = \varepsilon \circ \beta$:

$$\begin{tikzcd}
X^{\mathrm{ket}}_{\widetilde{s}} \arrow[d, "\varepsilon"'] & X^{\mathrm{ket}}_{\overline{s}} \arrow[l, "\beta"'] \arrow[d, "\varepsilon"] \\
X^{\mathrm{cl}}_{\widetilde{s}} & X^{\mathrm{ket}}_{\overline{s}} \arrow[l, "\alpha"']
\end{tikzcd}$$

(see([**Na 2**], 3.5) and [**Vi 3**]).

8.5. Let X be a scheme over S, which is proper and has semistable reduction with special fiber $X_s = D$ a divisor with simple normal crossings $D = \sum D_i$. Then there is defined the *Rapoport-Zink-Steenbrink double complex* realization $A^{\cdot\cdot}$ of $R\Psi^{\mathrm{cl}}\Lambda$ and its associated *weight spectral sequence* [**R-Z**]

$$E_1^{-r,n+r} \Longrightarrow H^n(X_{\overline{\eta}}, \Lambda), \tag{8.5.1}$$

where

$$E_1^{-r,n+r} = \bigoplus_{\substack{q\geqslant 0 \\ r+q\geqslant 0}} H^{n-r-2q}(D^{(r+1+2q)} \otimes \overline{k}, \Lambda)(-r-q)$$

and $D^{(r+1+2q)}$ is the disjoint sum of the intersections $r+1+2q$ by $r+1+2q$ of the components D_i of the divisor D.

It follows from (8.4.1) that these can be constructed purely in terms of X_s as a log scheme with the log structure induced from the canonical one on X (associated to D). Indeed the whole construction stems out from the choice of a complex of $\Lambda[I^t]$-modules on $X_{\bar{s}}$ with nonnegative degrees, representing $R\Psi^{\mathrm{cl}}\Lambda$; the point is that the inertia acts trivially on the cohomology sheaves $R^q\Psi^{\mathrm{cl}}\Lambda$.

By letting $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$, ℓ prime invertible on S, taking the inverse limit when ν varies, and tensoring with $\mathbb{Q}$, one deduces from (8.5.1) a spectral sequence

$$E_1^{-r,n+r} \Longrightarrow H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell) \tag{8.5.2}$$

where

$$E_1^{-r,n+r} = \bigoplus_{\substack{q\geqslant 0\\ r+q\geqslant 0}} H^{n-r-2q}(D^{(r+1+2q)} \otimes \bar{k}, \mathbb{Q}_\ell)(-r-q).$$

When k is a finite field, the Weil conjecture [**D1**] implies that the initial term of (8.5.2) is pure of weight r, and consequently (8.5.2) degenerates at E_2. This degeneration holds even if k is not finite. Indeed, because (8.5.1) can be defined purely in termes of the log scheme X_s, a specialization argument due to Nakayama [**Na 3**] enables one to reduce to the finite field case. When $k = \mathbb{C}$, using the standard comparison theorems between Betti and classical étale cohomology, one recovers the degeneration results of Steenbrink ([**Ste 1**], [**Ste 2**]).

8.6. In the situation of 8.5, the complex $R\Psi^{\mathrm{cl}}\mathbb{Q}_\ell[d]$, where $d = \dim X/S$, is a perverse sheaf, of which the *logarithm of the monodromy* $N = \log T \otimes \check{T}$ (where T is a topological generator of $\mathbb{Z}_\ell(1)$) is a nilpotent endomorphism. Up to a shift, the weight spectral sequence (8.5.2) is associated to the monodromy filtration of this perverse sheaf (*cf.* [**I2**]). It is conjectured that if X_s is projective, then the abutment filtration of (8.5.2) coincides with the monodromy filtration. This is the so-called *monodromy-weight conjecture*. This has been proven by Deligne in equal characteristic $p > 0$ (more precisely, for S equal to the henselization at a closed point of a smooth curve over a finite field), and by Rapoport-Zink in mixed characteristic for $d \leqslant 2$.

8.7. The perversity of $R\Psi^{\mathrm{cl}}\mathbb{Q}_\ell[d]$ holds for any scheme X flat over S, generically smooth, of pure relative dimension d. As Gabber observed, it is not hard to show, using de Jong's theorem over a discrete valuation ring ([**dJ**], 6.5), that the monodromy of this perverse sheaf is quasi-unipotent, so that one can define its logarithm N and the corresponding spectral sequence of monodromy filtration associated to N. Again, for X/S proper, because of Gabber's results (see [**Bry**] and [**I2**]) one expects its degeneration at E_2, and for X_s projective, the coincidence of the abutment filtration with the monodromy filtration. As for the construction of the spectral sequence and its degeneration at E_2, one can hope that the case where X underlies a vertical, log smooth log scheme over S is within reach. At least, the problem can be reduced to one over a standard log point (1.3). The monodromy-weight conjecture in the same case for X/S projective and in equal characteristic $p > 0$ should also be accessible, though not being a trivial consequence of Deligne's results in Weil II [**D1**] (the problem is to pass from the henselization at a closed point of a smooth curve over a finite field to an arbitrary henselian trait of char. p).

8.8. There is a more or less classical relation between the monodromy-weight conjecture and the *local invariant cycle* problem. Let X be a proper, flat scheme over S, of relative dimension d, generically smooth, with special fiber X_s projective over s. Assume that the quasi-unipotence of the monodromy of the perverse sheaf $\Psi := R\Psi^{\mathrm{cl}}\mathbb{Q}_\ell[d]$ has been established. Consider its logarithm N and the corresponding *kernel filtration* $K_i = \operatorname{Ker} N^{i+1} : \Psi \to \Psi$. This filtration defines a quasi-filtration of Ψ in $D^b(X_{\overline{s}}, \mathbb{Q}_\ell)$, in the sense of ([**Sa**], 5.2.17), and hence a spectral sequence ((loc. cit.), see also ([**D2**], Appendice)):

$$E_1^{ij} = H^{i+j}(X_{\overline{s}}, \mathrm{gr}^i_K \Psi) \Longrightarrow H^{i+j}(X_{\overline{s}}, \Psi) = H^{i+j+d}(X_{\overline{\eta}}, \mathbb{Q}_\ell). \tag{8.8.1}$$

Assume furthermore that X satisfies the monodromy-weight conjecture (8.6). Then, by analogy with Saito-Zucker's results over $\mathbb{C}$ ([**Sa-Z**], 0.4), one can expect that (8.8.1) degenerates at E_2 and that its abutment filtration is the kernel filtration $K_i = \operatorname{Ker} N^{i+1}$ on $H^*(X_{\overline{\eta}}, \mathbb{Q}_\ell)$. By an elementary lemma of homological algebra ([**Sa-Z**], 1.4.1), it would indeed be enough for this to establish the degeneration at E_2 of an auxiliary spectral sequence, namely the spectral sequence of hypercohomology of $X_{\overline{s}}$ with coefficients in $\mathrm{gr}_K \Psi$, filtered by the filtration M induced by the monodromy filtration of Ψ:

$$E_1^{ij} = H^{i+j}(X_s, \mathrm{gr}^i_M \mathrm{gr}_K \Psi) \Longrightarrow H^{i+j}(X_{\overline{s}}, \mathrm{gr}_K \Psi). \tag{8.8.2}$$

If the expectation above about (8.8.1) is fulfilled, then we get an exact sequence

$$H^m(X_{\overline{s}}, \mathrm{gr}^K_0 \Psi[-d]) \longrightarrow H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell) \longrightarrow H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell). \tag{8.8.3}$$

This exact sequence is to be compared with the *local invariant cycle property*, which asserts that the *specialization map*

$$\mathrm{sp} : H^m(X_{\overline{s}}, \mathbb{Q}_\ell) \longrightarrow H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell)^I \tag{8.8.4}$$

is *surjective.* Assume that X has strict semistable reduction (*i.e.* semistable reduction and X_s is a divisor with strict normal crossings in X). Then the Rapoport-Zink calculation of Ψ by a Steenbrink-type double complex [**R-Z**] (see also ([**I2**], 3.11) shows that the kernel filtration on $\Psi[-d]$ coincides with the *canonical* filtration defined by the canonical truncations $\tau_{\leqslant i}$, so that (8.8.1), after a suitable shift and renumbering, coincides with the usual *spectral sequence of vanishing cycles*

$$E_2^{ij} = H^i(X_{\overline{s}}, R^j\Psi\mathbb{Q}_\ell) \Longrightarrow H^{i+j}(X_{\overline{\eta}}, \mathbb{Q}_\ell). \tag{8.8.5}$$

Moreover, at least when s is the spectrum of a finite field, it follows from Rapoport-Zink's description of Ψ and of the Weil conjectures **[D1]** that (8.8.2) degenerates at E_2. Therefore when the monodromy-weight conjecture holds, (8.8.5) degenerates at E_3 and has the kernel filtration as abutment filtration, (8.8.3) holds and can be rewritten

$$H^m(X_{\overline{s}}, \mathbb{Q}_\ell) \longrightarrow H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell) \xrightarrow{N} H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell). \tag{8.8.6}$$

Finally, since I acts unipotently on $H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ in this case, we have

$$\mathrm{Ker}(N : H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell) \longrightarrow H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell)) = H^m(X_{\overline{\eta}}, \mathbb{Q}_\ell)^I, \tag{8.8.7}$$

and (8.8.6) is but the local invariant cycle property (8.8.4). One can perhaps expect that the results of this discussion still hold when X underlies a vertical, log smooth log scheme over S instead of having strict semistable reduction. In general, however, (8.8.7) does not hold, the kernel filtration on Ψ is not the canonical filtration, and the local invariant cycle property may be true while the exactness of (8.8.6) fails. To illustrate this, consider the following example, which was communicated to me by Gabber.

Let k be an algebraically closed field of characteristic exponent p, let d be an integer $\geqslant 3$, prime to p, let S be the henselization at the origin of the affine line over k, with local parameter t, and let $X \subset \mathbb{P}^2_S$ be the hypersurface defined by the equation

$$tx_0^d + x_1^d + x_2^d = 0 \ ,$$

where (x_0, x_1, x_2) are homogeneous coordinates on $\mathbb{P}^2$. Then X is smooth over k, and is a relative (twisted) Fermat curve over S. After base change by $t \mapsto t^d$, X becomes constant over η, so the inertia I acts on $H^1(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ through μ_d (by the standard action of μ_d on the Fermat curve $x_0^d + x_1^d + x_2^d = 0$ given by $a(x_0, x_1, x_2) = (ax_0, x_1, x_2)$). Hence (as is well known and elementary)

$$H^1(X_{\overline{\eta}}, \mathbb{Q}_\ell)^I = 0 \ .$$

On the other hand, the special fiber X_s consists of the bunch of lines $x_1^d + x_2^d = 0$, and in particular is reduced; since X/S is flat, we thus have $R^0\Psi\mathbb{Q}_\ell = \mathbb{Q}_\ell$, hence

$$H^1(X_s, R^0\Psi\mathbb{Q}_\ell) = H^1(X_s, \mathbb{Q}_\ell) = 0 \ .$$

Therefore we cannot have the degeneration of (8.8.5) at E_3 with kernel filtration as abutment, since (8.8.6) fails, N being zero, and $H^1(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ of dimension $(d-1)(d-2) > 0$. However, the local invariant cycle property holds, which agrees with Deligne's general result (**[D1]**, Weil II, 3.6.11). Note that X, equipped with the log structure given by the special fiber is *not log smooth* over S; indeed X is not log regular, since the rank of $\overline{M}^{\mathrm{gp}}$ at the singular point of X_s is $d \geqslant 3$.

8.9. Let X/S and U be as in 8.4.3 and assume $X_{\overline{\eta}}$ connected. Let $\pi_1^{(p')}(U_{\overline{\eta}})$ denote the maximal prime to p quotient of the fundamental group of $U_{\overline{\eta}}$ (relative to some geometric base point). Then $G = \mathrm{Gal}(\overline{K}/K)$ acts on $\pi_1^{(p')}(U_{\overline{\eta}})$ by outer automorphisms: we have a representation

$$\rho : G \longrightarrow \mathrm{Out}(\pi_1^{(p')}(U_{\overline{\eta}})).$$

Using Fujiwara-Kato's purity result 7.6, Kisin has proven the following theorem, which complements 8.4.3:

Theorem 8.10 **([Ki 1], 1.16).** — *The homomorphism ρ above factors through the tame quotient G/P, and depends only on the log scheme X_s (with log structure induced by that of X).*

9. Full log étale topology and cohomology

9.1. Let X be an fs log scheme. The *(full) log étale site* of X

$$X_{\mathrm{loget}}$$

(abbreviated to X_{et} when no confusion can arise) is defined as follows. The objects of X_{loget} are log étale maps of fs log schemes $T \to X$ (1.5). Morphisms are X-maps $T' \to T$ (such a map is automatically log étale (1.5)). Covering families are maps $f : T' \to T$ of X_{loget} such that f is *universally surjective*, which means that after any base change by a map $S \to T$ of fs log schemes, the underlying map of schemes $\overset{\circ}{f}{}'$ of $f' = f \times_T S$ (base change taken in the category of fs log schemes) is surjective. These covering families form a pretopology whose associated topology, called the *log étale topology*, defines the log étale site of X.

Here are two typical examples of covering families:

(i) a Kummer log étale map $f : T' \to T$ (1.6) such that $\overset{\circ}{f}$ is surjective (then f is necessarily universally surjective, as observed in 2.1);

(ii) a log blow-up $f : T_I \to T$ (6.1); note that a Zariski open subset U of T_I can surject to T but, since $f \times_T T_I : T_I \times_T T_I \to T_I$ is an isomorphism by 3.3, it universally surjects to T if and only if $U = T_I$.

Actually it is not hard to show (Nakayama) that coverings of type (i) and (ii) generate the log étale topology: if $f : T' \to T$ is a log étale map such that T admits a chart $T \to \mathrm{Spec}\,\mathbb{Z}[P]$ and $\overset{\circ}{T}{}'$ is quasi-compact, then there exists an ideal I in P such that $f \times_T T_I$ is Kummer.

Because log blow-ups are covering families, it may happen, in contrast with 2.6, that, for a map $Y \to X$ of fs log schemes, the functor $\mathrm{Hom}_X(-, Y)$ on X_{loget} is not a sheaf. For this functor to be a sheaf it is necessary and sufficient that $Y(U) \xrightarrow{\sim} Y(U_I)$ for any log blow-up $U_I \to U$ with U log étale over X.

Nakayama shows that the topos associated to X_{et} admits enough points: as a conservative family, one can take the functors $\mathcal{F} \mapsto \mathcal{F}_x := \text{ind.}\lim \mathcal{F}(X')$, where x is a log geometric point of X (4.1 (b)) such that $\overline{M}_x$ is *valuative* and X' runs through the log étale, x-pointed fs log schemes X' over X (a monoid P is called valuative if it is saturated and for any $a \in P^{\text{gp}}$, either a or a^{-1} is in P; Kato has developed a vast theory of valuative log schemes, see [**Ka 2**] (and [**I1**] for a survey)). Another way – perhaps more conceptual but essentially equivalent – of constructing points of X_{et} is the following. For x a geometric point of X, choose a chart $U \to \operatorname{Spec} \mathbb{Z}[P]$ of X in a (classical) étale neighborhood of x. For each finitely generated and nonempty ideal I of P, let U_I be the log blow-up of U along I (6.1). These U_I form an inverse system indexed by the set $\mathcal{I}(P)$ of finitely generated and nonempty ideals I, partially ordered by divisibility. Choose a compatible system $(x(\cdot) = (x(I))$ of log geometric points $x(I)$ of U_I above x. If $\mathcal{F}$ is a sheaf on X_{et}, define $\mathcal{F}_{x(\cdot)} = \text{ind.}\lim \mathcal{F}_{x(I)}$ where I runs through $\mathcal{I}(P)$ and $\mathcal{F}_{x(I)}$ is defined as in 4.3. Then $\mathcal{F} \mapsto \mathcal{F}_{x(I)}$ is a point of X_{et}, and such points form a conservative system.

9.2. Let $f : X \to Y$ be a map of fs log schemes. As in 2.3, base-changing by f in the category of fs log schemes defines an inverse image functor

$$f^{-1} : Y_{\text{et}} \longrightarrow X_{\text{et}},$$

which is continuous and commutes with finite inverse limits, hence defines a morphism of sites (resp. topoi)

$$f_{\text{et}} : X_{\text{et}} \longrightarrow Y_{\text{et}}.$$

These morphisms satisfy the usual transitivity isomorphism for a composition.

9.3. Let X be an fs log scheme. As the log étale topology is finer than the Kummer étale one, we have a natural map of sites (resp. topoi)

$$\kappa : X_{\text{et}} \longrightarrow X_{\text{ket}}.$$

When the log structure of X has the property that $\overline{M}^{\text{gp}}$ is of rank $\leqslant 1$ at each geometric point of X, then κ is an equivalence. This is the case in particular when X is a trait (with its canonical log structure) or a standard log point (1.3).

Since the full log étale topology is obtained from the Kummer one by adding the log blow-ups, the cohomological properties of κ are controlled by Fujiwara-Kato's theorems 6.2, 6.10. As Nakayama showed (private communication), it is easy to derive from them the following results:

Theorem 9.4. — *Let X be an fs log scheme, and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$, $n \geqslant 1$.*

(a) *The inverse image functor κ^* (9.3) induces an equivalence from the category of finite locally constant sheaves on X_{ket} to the category of finite locally constant sheaves on X_{et}.*

(b) *For any* $\mathcal{F} \in D^+(X_{\mathrm{ket}}, \Lambda)$, *the adjunction map*

$$\mathcal{F} \longrightarrow R\kappa_*\kappa^*\mathcal{F}$$

is an isomorphism (and therefore the inverse image map

$$\kappa^* : H^q(X_{\mathrm{ket}}, \mathcal{F}) \longrightarrow H^q(X_{\mathrm{et}}, \kappa^*\mathcal{F})$$

is an isomorphism for all q*).*

Part (a) of 9.4 can be reformulated by saying that for every valuative log geometric point x of X_{et} (9.1), if one defines the fundamental group $\pi_1(X_{\mathrm{et}}, x)$ of X_{et} at x, in the usual way, as the (profinite) automorphism group of the fiber functor $\mathcal{F} \mapsto \mathcal{F}_x$ on the category of finite locally constant sheaves on X_{et}, then the natural map (4.6)

$$\pi_1(X_{\mathrm{et}}, x) \longrightarrow \pi_1(X_{\mathrm{ket}}, x) = \pi_1^{\log}(X, x)$$

is an isomorphism. It is not known whether the statement analogous to (a) with "finite" removed holds.

As we mentioned after 6.3, proper base change for the Kummer étale cohomology requires extra assumptions on the morphisms. These restrictions can be removed if one works with the full log étale cohomology, as was shown by Kato:

Theorem 9.5 (**[Ka 5]**). — *Let* Λ *be as in* 9.4. *Let*

(9.5.1)
$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be an fs cartesian square of fs log schemes, where $\mathring{f}$ *is proper. Then for* $\mathcal{F} \in D^+(X_{\mathrm{et}}, \Lambda)$, *the base change map*

$$g^*Rf_*\mathcal{F} \longrightarrow Rf'_*h^*\mathcal{F}$$

is an isomorphism.

The proof consists of a rather long reduction to the proper base change theorem for classical étale cohomology ([**SGA 4**] XII 5.1), using 9.4 (b) as a key ingredient.

Nakayama's result on vanishing cycles 8.3 is a *local acyclicity* statement for a log smooth map over a trait. In classical étale cohomology any smooth map is locally acyclic and this in turn implies a smooth base change theorem and (combined with the finiteness theorem) a specialization theorem for proper smooth maps ([**SGA 4**], XV, XVI). Unfortunately, no such generalizations hold for Kummer étale cohomology ([**Na 2**], 4.3, B1). In this respect, however, the full log étale cohomology behaves much better. Indeed, Kato proved the following log smooth base change theorem:

Theorem 9.6 (**[Ka 5]**). — *Consider a cartesian square* (9.5.1), *where g is log smooth and $\mathring{f}$ is quasi-compact and quasi-separated. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on Y. Then for $\mathcal{F} \in D^+(X_{\text{et}}, \Lambda)$, the base change map*

$$g^* Rf_* \mathcal{F} \longrightarrow Rf'_* h^* \mathcal{F}$$

is an isomorphism.

The crucial case to which Kato reduces the proof of 9.6 is the case where g is the affine line over Y, with its standard log structure. This requires a delicate global argument inspired from Nakayama's proof of 8.3 in [**Na 2**].

Note that 8.3 is an immediate corollary of 9.6. Indeed, thanks to 9.4 (b), 9.6 reduces the verification of 8.3 to the trivial case where $X = S$.

9.7. A map $f : X \to Y$ of fs log schemes is said to be *compactifiable* if $\mathring{Y}$ is quasi-compact and quasi-separated and f can be factored into $f = g \circ i$ where $\mathring{g}$ is proper and $\mathring{i}$ is an open immersion (in other words, if $\mathring{f}$ is $\mathring{Y}$-compactifiable in the sense of ([**SGA 4**] XVII 3.2). By the proper base change theorem (9.5), the composition $Rg_* \circ i_! : D^+(X_{\text{et}}, \Lambda) \to D^+(Y_{\text{et}}, \Lambda)$ does not depend, up to a transitive system of isomorphisms, on the factorization $f = g \circ i$ (here $i_!$ is defined as $i''_! Ri'_*$ when i is factored as $i''i'$ where i'' is a strict open immersion and $\mathring{i'}$ is the identity). As usual, one defines

$$Rf_! : D^+(X_{\text{et}}, \Lambda) \longrightarrow D^+(Y_{\text{et}}, \Lambda) \tag{9.7.1}$$

as the corresponding limit. The proper base change theorem then implies that $Rf_!$ commutes with any base change.

A theory of $Rf_!$ was first developed by Nakayama in the Kummer étale framework [**Na 1**]. Working with the full log étale sites has great advantages as regard to *finiteness theorems.* For $f : X \to Y$ proper and n invertible on X, it is not true that if $\mathcal{F}$ is a constructible sheaf of Λ-modules on X, $R^i f_* \mathcal{F}$ is constructible, unless f satisfies the additional hypothesis of being *log injective* (*i.e.* for all geometric point $\overline{x}$ of X with image $\overline{y}$ in Y, $\overline{M}_{\overline{y}} \to \overline{M}_{\overline{x}}$ is injective) ([**Na 1**], 5.5.2, B3 (i)). This restriction can be lifted in the context of full log étale cohomology. Namely, we have the following theorem of Kato:

Theorem 9.8 (**[Ka 6]**). — *Let $f : X \to Y$ be a compactifiable* (9.7) *map of fs log schemes, with Y locally noetherian, and let Λ be as in 9.4. Then for $\mathcal{F} \in D^+_c(X_{\text{et}}, \Lambda)$, we have $Rf_! \mathcal{F} \in D^+_c(Y_{\text{et}}, \Lambda)$, where $Rf_!$ is the functor defined in* (9.7.1).

Here a sheaf $\mathcal{G}$ of Λ-modules on X_{et} is called *constructible* if, locally for the log étale topology on X, there exists a finite decomposition of X into locally closed subsets X_i such that the restriction of $\mathcal{G}$ to X_i is locally constant with fibers of finite type over Λ, and $D^+_c(-\Lambda)$ denotes the full subcategory of $D^+(-, \Lambda)$ consisting of complexes $\mathcal{F}$ such that $\mathcal{H}^i(\mathcal{F})$ is constructible for all i.

The key point in the proof of 9.8 is that f may be rendered exact (1.6) by log étale localization on Y. Standard dévissages using the proper base change theorem then reduce to the case where f is exact, $\overset{\circ}{f}$ is the identity, $\mathcal{F}$ consists of a constant and constructible sheaf, and $n = \ell^\nu$, with ℓ prime, either invertible or zero on X. This case is treated by a direct argument using the comparison theorems 5.2 and 9.4 (b).

As in the case of classical étale cohomology ([**SGA 4**] XIV), combining the smooth base change theorem 9.6 with the finiteness theorem 9.8, one gets a specialization (or proper smooth base change) theorem:

Theorem 9.9. — *Let $f : X \to Y$ be a proper and log smooth map of fs log schemes, with Y locally noetherian. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on Y. Let $\mathcal{F}$ be a locally constant and constructible sheaf of Λ-modules on X_{et}. Then $R^q f_* \mathcal{F}$ is locally constant and constructible for all q.*

In view of 9.4, 9.9 implies the same result for X_{et} replaced by X_{ket} (a particular case of this consequence had been previously proven by Nakayama ([**Na 2**], 4.3).

Proceeding as in ([**SGA 4**] XVII), one can define a partial right adjoint $Rf^!$ to $Rf_!$, giving rise to a global duality formula of the form $Rf_* R\underline{\mathrm{Hom}}(\mathcal{K}, Rf^! \mathcal{L}) \cong R\underline{\mathrm{Hom}}(Rf_! , \mathcal{K}, \mathcal{L})$ (see [**Na 1**] in the Kummer étale case). The problem, however, is to calculate $Rf^!$. Some cases have been treated by Nakayama. For example, we have the following result, reminiscent of the classical Poincaré duality theorem ([**SGA 4**] XVIII 3.2.5):

***Theorem 9.10* (Nakayama).** — *Let $f : X \to S$ be a compactifiable, log smooth, vertical* (8.2.3) *map, with S as in* 8.3, *and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible on S. Then*

$$Rf^! \Lambda_{S_{\mathrm{ket}}} = \Lambda_{X_{\mathrm{ket}}}(d)[2d],$$

where d is the relative dimension of f.

A similar result holds with S replaced by a trivial or standard log point (1.3). See ([**Na 1**], 7.5), ([**Na 2**], 4.4) for more general statements.

9.11. Unfortunately, if $f : X \to Y$ is a map of fs log schemes over a trivial log point s (1.3), with $\overset{\circ}{X}$ and $\overset{\circ}{Y}$ separated and of finite type over s, and $\Lambda = \mathbb{Z}/n\mathbb{Z}$, n invertible on s, it is not true in general that Rf_* carries $D^+_c(X_{\mathrm{et}}, \Lambda)$ into $D^+_c(Y_{\mathrm{et}}, \Lambda)$, as one would expect, by analogy with Deligne's finiteness theorem in ([**SGA 4 1/2**], Th. finitude). Nakayama gives the following counterexample: let S be a standard log point over s (1.3), $Y = \mathbb{A}^1_S = S \times_s \mathbb{A}^1_s$ the affine line over S, with the log structure pulled back from the standard log structure of $\mathbb{A}^1_s$, and $f : X \to Y$ the inclusion (a strict open immersion) of the complement of the zero section. Then $f_* \Lambda$ is not constructible: there is no log blow-up Y' of Y at the origin over which $f_* \Lambda$ becomes classically (or Kummer) constructible; indeed, taking again one blow-up Y'' of Y' and a point y of Y_{et} above the origin in Y', but factoring through a geometric point of the exceptional divisor of Y'' distinct from the origin (see the construction of points of the log étale topos at the end of 9.1), we would find that the stalk of $f_*\Lambda$ at y is zero, while the stalk of $f_*\Lambda$ at the trivial point of Y_{et} located at the origin (corresponding to a constant inverse system of log points, in the language of loc. cit.) is Λ.

Finding a more flexible definition of constructibility giving rise to a formalism of six operations à la Grothendieck ($\overset{L}{\otimes}, R\underline{\mathrm{Hom}}, Rf_*, f^*, Rf_!, Rf^!$) in suitable categories $D^b_c((-)_{\mathrm{et}}, \Lambda)$ on fs log schemes separated and of finite type over a trivial log point (or a standard log point, or a trait with its canonical log structure) looks like a difficult – perhaps intractable – problem.

Log étale cohomology is still largely an unknown territory. Transposing into it standard topics of classical étale cohomology such as cycles classes, perverse sheaves, trace and Euler-Poincaré formulas raises more or less difficult questions, which have not yet been taken up. Their study should hopefully cast a new light onto old problems, as the results in section 8 have already illustrated.

Acknowledgements. — I wish to thank K. Kato and C. Nakayama for helpful observations on an earlier draft of [**I3**], which was a starting basis for these notes. I am indebted to T. Ekedahl for his argument in the proof of Fujiwara-Kato's theorem in section 6 as well as for many useful discussions. I am grateful to the listeners of the lectures for their stimulating questions and especially to O. Gabber for several illuminating comments. I thank I. Vidal for her assistance in the understanding of delicate points of Kummer étale topology. I heartily thank O. Gabber and the referee for their careful reading of the manuscript and their suggestions for modifications and corrections. Finally, I wish to thank Mme Bonnardel for her neat typing of the manuscript.

References

[Br-M] C. Breuil and W. Messing, *Torsion étale cohomology and semi-stable reduction*, these Proceedings.

[Bry] J.-L. Brylinski, *Cohomologie d'intersection et faisceaux pervers*, Séminaire Bourbaki 1981/82, n° 585, 129-157, Astérisque 92-93, 1982.

[D1] P. Deligne, *La conjecture de Weil, I*, Publ. Math. IHES 43 (1974), 273-308; *La conjecture de Weil, II*, Publ. Math. IHES 52 (1980), 137-252.

[D2] P. Deligne, *Décompositions dans la catégorie dérivée*, Proc. of Symposia in Pure Math. 55 (1994), Part I, 115-128, AMS 1994.

[dJ] A. J. de Jong, *Smoothness, Semi-stability and Alterations*, Pub. Math. IHES 83 (1996), 51-93.

[F] K. Fujiwara, *Etale topology and the philosophy of log*, Algebraic Geometry Symposium (Kinosaki), 1990, 116-123 (in Japanese).

[F-K] K. Fujiwara and K. Kato, *Logarithmic étale topology theory*, (incomplete) preprint, 1995.

[Fu] W. Fulton, *Introduction to toric varieties*, Ann. of Math. Studies 131, Princeton Univ. Press, 1993.

[Ga] O. GABBER, *Approaches to absolute cohomological purity*, lecture in Toulouse, July 1, 1994.

[G-M] A. GROTHENDIECK and J. P. MURRE, *The tame fundamental group of a formal neighborhood of a divisor with normal crossings on a scheme*, SLN 208, Springer-Verlag, 1971.

[I1] L. ILLUSIE, *Logarithmic spaces (according to K. Kato)*, in Barsotti Symposium in Algebraic Geometry (V. Cristante and W. Messing, Eds.), Perspectives in Math. 15, 183-203, Academic Press, 1994.

[I2] L. ILLUSIE, *Autour du théorème de monodromie locale*, in Périodes p-adiques, Séminaire de Bures, 1988 (Ed. J.-M. Fontaine), 9-57, Astérisque 223, 1994.

[I3] L. ILLUSIE, *Logarithmic smoothness and vanishing cycles*, Prépublication univ. Paris-Sud 95-83, 1995.

[Ka 1] K. KATO, *Logarithmic structures of Fontaine-Illusie*, in Algebraic Analysis, Geometry and Number Theory, The Johns Hopkins University Press (1989), 191-224.

[Ka 2] K. KATO, *Logarithmic degeneration and Dieudonné theory*, preprint, 1989.

[Ka 3] K. KATO, *Logarithmic structures of Fontaine-Illusie* II, *Logarithmic flat topology*, (incomplete) preprint, 1991.

[Ka 4] K. KATO, *Toric singularities*, Amer. J. Math. 116 (1994), 1073-1099.

[Ka 5] K. KATO, *Log proper base change theorem and log smooth base change theorem*, preliminary notes, 1995.

[Ka 6] K. KATO, *Constructibility, proper smooth base change*, preliminary notes, 1995.

[Ka 7] K. KATO, *Description of $R\Psi$*, preliminary notes, 1995.

[KKMS] G. KEMPF, F. KNUDSEN, D. MUMFORD, B. SAINT-DONAT, *Toroidal embeddings* I, SLN 339, Springer-Verlag, 1973.

[K-N] K. KATO and C. NAKAYAMA, *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over* $\mathbb{C}$, Kodai Math. J. 22 (1999), 161-186.

[Kaw-Nam] Y. KAWAMATA and Y. NAMIKAWA, *Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties*, Inv. Math. 118 (1994), 395-409.

[Ki 1] M. KISIN, *Prime to p fundamental groups and tame Galois actions*, Ann. Inst. Fourier (Grenoble) 50, 4 (2000), 1099-1126.

[Ki 2] M. KISIN, *Endomorphisms of logarithmic schemes*, preprint, 1999.

[Na 1] C. NAKAYAMA, *Logarithmic étale cohomology*, Math. Ann. 308 (1997), 365-404.

[Na 2] C. NAKAYAMA, *Nearby cycles for log smooth families*, Comp. Math. 112 (1998), 45-75.

[Na 3] C. NAKAYAMA, *Degeneration of ℓ-adic weight spectral sequences*, Amer. J. Math. 122 (2000), 721-733.

[Ni] W. NIZIOL, *Toric singularities, log blow-ups and global resolutions*, preprint, 1999.

[Od] T. ODA, *Convex Bodies and Algebraic Geometry, An Introduction to the Theory of Toric Varieties*, Erg. der Math. Bd 15, Springer-Verlag, 1988.

[Og] A. OGUS, *Logarithmic de Rham cohomology*, preprint, 1997.

[R-Z] M. RAPOPORT and T. ZINK, *Ueber die lokale Zetafunktion von Shimuravarietäten, Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik*, Inv. Math. 68 (1982), 21-101.

[Sa] M. SAITO, *Modules de Hodge polarisables*, Pub. RIMS, Kyoto Univ. 24 (1988), 849-995.

[Sa-Z] M. SAITO and S. ZUCKER, *The kernel spectral sequence of vanishing cycles*, Duke Math. J. 61 (1990), 329-339.

[S-T] J.-P. SERRE and J. TATE, *Good reduction of abelian varieties*, Ann. of Math. 88 (1968), 492-517.

[Ste 1] J. STEENBRINK, *Limits of Hodge structures*, Inv. Math. 31 (1976), 229-257.

[Ste 2] J. STEENBRINK, *Logarithmic embeddings of varieties with normal crossings and mixed Hodge structures*, Math. Ann. 301 (1995), 105-118.

[Tsu] T. TSUJI, *Semi-stable conjecture of Fontaine-Jannsen: a survey*, these Proceedings.

[Vi 1] I. VIDAL, *Morphismes log étales et descente par homéomorphismes universels*, preprint, 1999.

[Vi 2] I. VIDAL, *Descente de Kummer log étale, log régularité et groupe fondamental logarithmique*, preliminary notes, 1999.

[Vi 3] I. VIDAL, *On the local monodromy and zeta functions of log schemes*, work in preparation.

[Vi 4] I. VIDAL, *Morphismes log étales et descente par homéomorphismes universels*, C. R. Acad. Sci. Paris, t. 332, Série I, 239-244, 2001.

[EGA IV] A. GROTHENDIECK (avec la collaboration de J. Dieudonné), *Éléments de géométrie algébrique*, IV Étude locale des schémas et des morphismes de schémas, Pub. Math. IHÉS 20 (1964), 24 (1965), 28 (1966), 32 (1967).

[SGA 1] *Revêtements étales et groupe fondamental*, Séminaire de géométrie algébrique du Bois-Marie 1960-61, dirigé par A. Grothendieck, SLN 224, Springer-Verlag, 1971.

[SGA 4] *Théorie des topos et cohomologie étale des schémas*, Séminaire de géométrie algébrique du Bois-Marie 1963-64, dirigé par M. Artin, A. Grothendieck, J.-L. Verdier, SLN 269, 270, 305, Springer-Verlag, 1972, 1973.

[SGA 5] *Cohomologie ℓ-adique et fonctions L*, Séminaire de géométrie algébrique du Bois-Marie 1965-66, dirigé par A. Grothendieck, SLN 589, Springer-Verlag, 1977.

[SGA 7] *Groupes de monodromie en géométrie algébrique*, Séminaire de géométrie algébrique du Bois-Marie 1967-69, I, dirigé par A. Grothendieck, II par P. Deligne et N. Katz, SLN 288, 340, Springer-Verlag, 1972, 1973.

[SGA 4 1/2] *Cohomologie étale*, par P. Deligne, SLN 569, Springer-Verlag, 1977.

L. ILLUSIE, Université de Paris-Sud, Mathématique, Bât. 425, UMR 8628 du CNRS, 91405 Orsay Cédex, France • *E-mail :* `Luc.Illusie@math.u-psud.fr`