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AN INTRODUCTION TO p-ADIC TEICHMÜLLER THEORY

by

Shinichi Mochizuki

Abstract. — In this article, we survey a theory, developed by the author, concerning the *uniformization of p-adic hyperbolic curves and their moduli.* On the one hand, this theory generalizes the Fuchsian and Bers uniformizations of complex hyperbolic curves and their moduli to nonarchimedean places. It is for this reason that we shall often refer to this theory as *p-adic Teichmüller theory*, for short. On the other hand, this theory may be regarded as a fairly precise hyperbolic analogue of the Serre-Tate theory of ordinary abelian varieties and their moduli.

The central object of p-adic Teichmüller theory is the *moduli stack of nilcurves.* This moduli stack forms a finite flat covering of the moduli stack of hyperbolic curves in positive characteristic. It parametrizes hyperbolic curves equipped with auxiliary "uniformization data in positive characteristic." The geometry of this moduli stack may be *analyzed combinatorially* locally near infinity. On the other hand, a global analysis of its geometry gives rise to a *proof of the irreducibility of the moduli stack of hyperbolic curves using positive characteristic methods.* Various portions of this stack of nilcurves admit *canonical p-adic liftings*, over which one obtains *canonical coordinates and canonical p-adic Galois representations.* These canonical coordinates form the analogue for hyperbolic curves of the canonical coordinates of Serre-Tate theory and the p-adic analogue of the Bers coordinates of Teichmüller theory. Moreover, the resulting Galois representations shed new light on the outer action of the Galois group of a local field on the profinite completion of the Teichmüller group.

1. From the Complex Theory to the "Classical Ordinary" p-adic Theory

In this §, we attempt to bridge the gap for the reader between the classical uniformization of a hyperbolic Riemann surface that one studies in an undergraduate complex analysis course and the point of view espoused in [**21, 22**].

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1.1. The Fuchsian Uniformization. — Let X be a *hyperbolic* algebraic curve over $\mathbb{C}$, the field of complex numbers. By this, we mean that X is obtained by removing r points from a smooth, proper, connected algebraic curve of genus g (over $\mathbb{C}$), where $2g-2+r>0$. We shall refer to (g,r) as the *type* of X. Then it is well-known that to X, one can associate in a natural way a Riemann surface $\mathbf{X}$ whose underlying point set is $X(\mathbb{C})$. We shall refer to Riemann surfaces $\mathbf{X}$ obtained in this way as "hyperbolic of finite type."

Now perhaps the most fundamental *arithmetic* – read "arithmetic at the infinite prime" – fact known about the *algebraic* curve X is that $\mathbf{X}$ admits a uniformization by the upper half plane $\mathbf{H}$:

$$\mathbf{H} \to \mathbf{X}$$

For convenience, we shall refer to this uniformization of $\mathbf{X}$ in the following as the *Fuchsian uniformization of* $\mathbf{X}$. Put another way, the uniformization theorem quoted above asserts that the universal covering space $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ (which itself has the natural structure of a Riemann surface) is holomorphically isomorphic to the upper half plane $\mathbf{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. This fact was "familiar" to many mathematicians as early as the last quarter of the nineteenth century, but was only proven rigorously much later by Koebe.

> *The fundamental thrust of* [**21, 22**] *is to generalize the Fuchsian uniformization to the p-adic context.*

At this point, the reader might be moved to interject: But hasn't this already been achieved decades ago by Mumford in [**25**]? In fact, however, Mumford's construction gives rise to a p-adic analogue *not of the Fuchsian uniformization, but rather of the Schottky uniformization* of a complex hyperbolic curve. Even in the complex case, the Schottky uniformization is an entirely different sort of uniformization – both geometrically and arithmetically – from the Fuchsian uniformization: for instance, its periods are holomorphic, whereas the periods that occur for the Fuchsian uniformization are only real analytic. This phenomenon manifests itself in the nonarchimedean context in the fact that the construction of [**25**] really has nothing to do with a fixed prime number "p," and in fact, takes place entirely in the formal analytic category. In particular, the theory of [**25**] has nothing to do with "Frobenius." By contrast, the theory of [**21, 22**] depends very much on the choice of a prime "p," and makes essential use of the "action of Frobenius." Another difference between the theory of [**25**] and the theory of [**21, 22**] is that [**25**] only addresses the case of curves whose "reduction modulo p" is totally degenerate, whereas the theory of [**21, 22**] applies to curves whose reduction modulo p is only assumed to be "sufficiently generic." Thus, at any rate, the theory of [**21, 22**] is entirely different from and has little directly to do with the theory of [**25**].

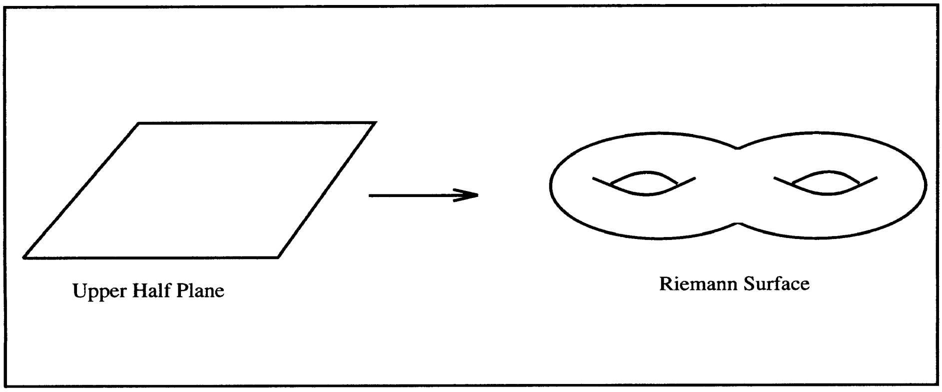

FIGURE 1. The Fuchsian Uniformization

1.2. Reformulation in Terms of Metrics. — Unfortunately, if one sets about trying to generalize the Fuchsian uniformization $\mathbf{H} \rightarrow \mathbf{X}$ to the p-adic case in any sort of naive, literal sense, one immediately sees that one runs into a multitude of apparently insurmountable difficulties. Thus, it is natural to attempt to recast the Fuchsian uniformization in a more universal form, a form more amenable to relocation from the archimedean to the nonarchimedean world.

One natural candidate that arises in this context is the notion of a metric – more precisely, the notion of a *real analytic Kähler metric*. For instance, the upper half plane admits a natural such metric, namely, the metric given by

$$\frac{dx^2 + dy^2}{y^2}$$

(where $z = x + iy$ is the standard coordinate on $\mathbf{H}$). Since this metric is invariant with respect to all holomorphic automorphisms of $\mathbf{H}$, it induces a natural metric on $\widetilde{\mathbf{X}} \cong \mathbf{H}$ which is independent of the choice of isomorphism $\widetilde{\mathbf{X}} \cong \mathbf{H}$ and which descends to a metric $\mu_{\mathbf{X}}$ on $\mathbf{X}$.

Having constructed the canonical metric $\mu_{\mathbf{X}}$ on $\mathbf{X}$, we first make the following observation:

> *There is a general theory of canonical coordinates associated to a real analytic Kähler metric on a complex manifold.*

(See, e.g., [**21**], Introduction, § 2, for more technical details.) Moreover, the canonical coordinate associated to the metric $\mu_{\mathbf{X}}$ is precisely the coordinate obtained by pulling back the standard coordinate "z" on the unit disc via any holomorphic isomorphism of $\widetilde{\mathbf{X}} \cong \mathbf{H}$ with the unit disc. Thus, in other words, passing from $\mathbf{H} \rightarrow \widetilde{\mathbf{X}}$ to $\mu_{\mathbf{X}}$ is a "faithful operation," i.e., one doesn't really lose any information.

Next, let us make the following observation: Let $\mathcal{M}_{g,r}$ denote the moduli stack of smooth r-pointed algebraic curves of genus g over $\mathbb{C}$. If we order the points that were removed from the compactification of X to form X, then we see that X defines a point $[X] \in \mathcal{M}_{g,r}(\mathbb{C})$. Moreover, it is elementary and well-known that the cotangent space to $\mathcal{M}_{g,r}$ at $[X]$ can be written in terms of square differentials on X. Indeed, if, for simplicity, we restrict ourselves to the case $r = 0$, then this cotangent space is naturally isomorphic to $Q \stackrel{\text{def}}{=} H^0(X, \omega_{X/\mathbb{C}}^{\otimes 2})$ (where $\omega_{X/\mathbb{C}}$ is the algebraic coherent sheaf of differentials on X). Then the observation we would like to make is the following: Reformulating the Fuchsian uniformization in terms of the metric $\mu_{\mathbf{X}}$ allows us to "push-forward" $\mu_{\mathbf{X}}$ to obtain a canonical real analytic Kähler metric $\mu_{\mathbf{M}}$ on the complex analytic stack $\mathbf{M_{g,r}}$ associated to $\mathcal{M}_{g,r}$ by the following formula: if $\theta, \psi \in Q$, then

$$\langle \theta, \psi \rangle \stackrel{\text{def}}{=} \int_{\mathbf{X}} \frac{\theta \cdot \overline{\psi}}{\mu_{\mathbf{X}}}$$

(Here, $\overline{\psi}$ is the complex conjugate differential to ψ, and the integral is well-defined because the integrand is the quotient of a $(2,2)$-form by a $(1,1)$-form, i.e., the integrand is itself a $(1,1)$-form.)

This metric on $\mathbf{M_{g,r}}$ is called the *Weil-Petersson metric.* It is known that

> *The canonical coordinates associated to the Weil-Petersson metric coincide with the so-called Bers coordinates on* $\widetilde{\mathbf{M}}_{g,r}$ *(the universal covering space of* $\mathbf{M_{g,r}}$*).*

The Bers coordinates define an anti-holomorphic embedding of $\widetilde{\mathbf{M}}_{g,r}$ into the complex affine space associated to Q. We refer to the Introduction of [**21**] for more details on this circle of ideas.

At any rate, in summary, we see that much that is useful can be obtained from this reformulation in terms of metrics. However, although we shall see later that the reformulation in terms of metrics is not entirely irrelevant to the theory that one ultimately obtains in the p-adic case, nevertheless this reformulation is still not sufficient to allow one to effect the desired translation of the Fuchsian uniformization into an analogous p-adic theory.

1.3. Reformulation in Terms of Indigenous Bundles. — It turns out that the "missing link" necessary to translate the Fuchsian uniformization into an analogous p-adic theory was provided by Gunning ([**13**]) in the form of the notion of an *indigenous bundle.* The basic idea is as follows: First recall that the group $\mathrm{Aut}(\mathbf{H})$ of holomorphic automorphisms of the upper half plane may be identified (by thinking about linear fractional transformations) with $\mathrm{PSL}_2(\mathbb{R})^0$ (where the superscripted "0" denotes the connected component of the identity). Moreover, $\mathrm{PSL}_2(\mathbb{R})^0$ is naturally contained inside $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. Let $\Pi_{\mathbf{X}}$ denote the (topological) fundamental group of $\mathbf{X}$ (where we ignore the issue of choosing a base-point since this will be irrelevant for what we do). Then since $\Pi_{\mathbf{X}}$ acts naturally on $\widetilde{\mathbf{X}} \cong \mathbf{H}$, we get a natural representation

$$\rho_{\mathbf{X}} : \Pi_{\mathbf{X}} \to \mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$$

which is well-defined up to conjugation by an element of $\mathrm{Aut}(\mathbf{H}) \subseteq \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. We shall henceforth refer to $\rho_{\mathbf{X}}$ as the *canonical representation associated to* $\mathbf{X}$. Thus, $\rho_{\mathbf{X}}$ gives us an action of $\Pi_{\mathbf{X}}$ on $\mathbb{P}^1_{\mathbb{C}}$, hence a diagonal action on $\widetilde{\mathbf{X}} \times \mathbb{P}^1_{\mathbb{C}}$. If we form the quotient of this action of $\Pi_{\mathbf{X}}$ on $\widetilde{\mathbf{X}} \times \mathbb{P}^1_{\mathbb{C}}$, we obtain a $\mathbb{P}^1$-bundle over $\widetilde{\mathbf{X}}/\Pi_{\mathbf{X}} = \mathbf{X}$ which automatically algebraizes to an algebraic $\mathbb{P}^1$-bundle $P \to X$ over X. (For simplicity, think of the case $r = 0$!)

In fact, $P \to X$ comes equipped with more structure. First of all, note that the trivial $\mathbb{P}^1$-bundle $\widetilde{\mathbf{X}} \times \mathbb{P}^1_{\mathbb{C}} \to \widetilde{\mathbf{X}}$ is equipped with the trivial connection. (Note: here we use the "Grothendieck definition" of the notion of a connection on a $\mathbb{P}^1$-bundle: i.e., an isomorphism of the two pull-backs of the $\mathbb{P}^1$-bundle to the first infinitesimal neighborhood of the diagonal in $\widetilde{\mathbf{X}} \times \widetilde{\mathbf{X}}$ which restricts to the identity on the diagonal $\widetilde{\mathbf{X}} \subseteq \widetilde{\mathbf{X}} \times \widetilde{\mathbf{X}}$.) Moreover, this trivial connection is clearly fixed by the action of $\Pi_{\mathbf{X}}$, hence descends and algebraizes to a connection ∇_P on $P \to X$. Finally, let us observe that we also have a section $\sigma : X \to P$ given by descending and algebraizing the section $\widetilde{\mathbf{X}} \to \widetilde{\mathbf{X}} \times \mathbb{P}^1_{\mathbb{C}}$ whose projection to the second factor is given by $\widetilde{\mathbf{X}} \cong \mathbf{H} \subseteq \mathbb{P}^1_{\mathbb{C}}$. This section is referred to as the *Hodge section*. If we differentiate σ by means of ∇_P, we obtain a *Kodaira-Spencer morphism* $\tau_{X/\mathbb{C}} \to \sigma^*\tau_{P/X}$ (where "$\tau_{A/B}$" denotes the relative tangent bundle of A over B). It is easy to see that this Kodaira-Spencer morphism is necessarily an isomorphism.

This triple of data $(P \to X, \nabla_P, \sigma)$ is the prototype of what Gunning refers to as an *indigenous bundle*. We shall refer to this specific $(P \to X, \nabla_P)$ (one doesn't need to specify σ since σ is uniquely determined by the property that its Kodaira-Spencer morphism is an isomorphism) as the *canonical indigenous bundle*. More generally, an *indigenous bundle on* X (at least in the case $r = 0$) is any $\mathbb{P}^1$-bundle $P \to X$ with connection ∇_P such that $P \to X$ admits a section (necessarily unique) whose Kodaira-Spencer morphism is an isomorphism. (In the case $r > 0$, it is natural to introduce log structures in order to make a precise definition.)

Note that the notion of an indigenous bundle has the virtue of being *entirely algebraic* in the sense that at least as an object, the canonical indigenous bundle $(P \to X, \nabla_P)$ exists in the algebraic category. In fact, the space of indigenous bundles forms a torsor over the vector space Q of quadratic differentials on X (at least for $r = 0$). Thus,

> *The issue of which point in this affine space of indigenous bundles on X corresponds to the canonical indigenous bundle is a deep arithmetic issue, but the affine space itself can be defined entirely algebraically.*

One aspect of the fact that the notion of an indigenous bundle is entirely algebraic is that indigenous bundles can, in fact, be defined over $\mathbb{Z}[\frac{1}{2}]$, and in particular, over $\mathbb{Z}_p$ (for p odd). In **[21]**, Chapter I, a fairly complete theory of indigenous bundles in the p-adic case (analogous to the complex theory of [**13**]) is worked out. To summarize, indigenous bundles are closely related to projective structures and Schwarzian

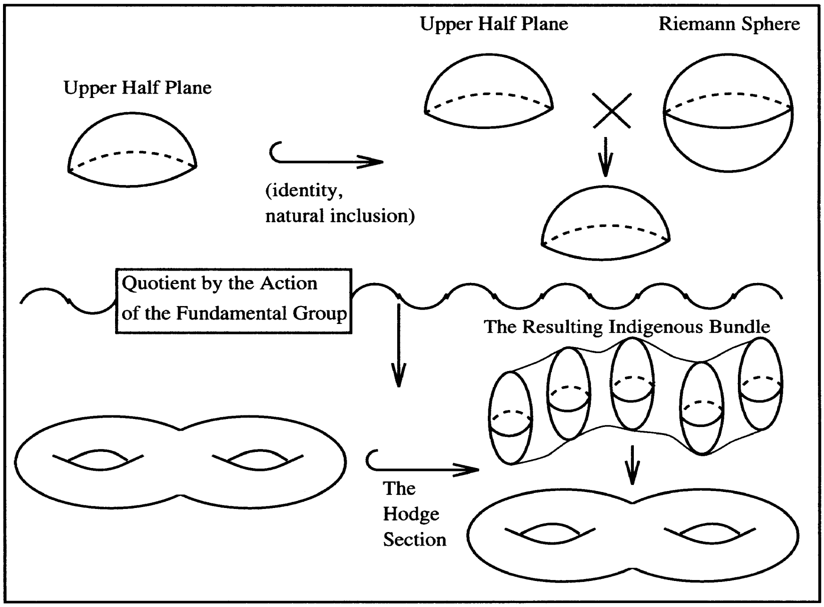

FIGURE 2. The Construction of the Canonical Indigenous Bundle

derivatives on X. Moreover, the underlying $\mathbb{P}^1$-bundle $P \to X$ is always the same (for all indigenous bundles on X), i.e., the choice of connection ∇_P determines the isomorphism class of the indigenous bundle. We refer the reader to [**21**], Chapter I, for more details. (Note: Although the detailed theory of [**21**], Chapter I, is philosophically very relevant to the theory of [**22**], most of this theory is technically and logically unnecessary for reading [**22**].)

At any rate, to summarize, the introduction of indigenous bundles allows one to consider the Fuchsian uniformization as being embodied by an object – the canonical indigenous bundle – which exists in the algebraic category, but which, compared to other indigenous bundles, is somehow "special." In the following, we would like to analyze the sense in which the canonical indigenous bundle is special, and to show how this sense can be translated immediately into the p-adic context. Thus, we see that

> *The search for a p-adic theory analogous to the theory of the Fuchsian uniformization can be reinterpreted as the search for a notion of "canonical p-adic indigenous bundle" which is special in a sense precisely analogous to the sense in which the canonical indigenous bundle arising from the Fuchsian uniformization is special.*

1.4. Frobenius Invariance and Integrality. — In this subsection, we explore in greater detail the issue of what precisely makes the canonical indigenous bundle (in the complex case) so special, and note in particular that a properly phrased characterization of the canonical indigenous bundle (in the complex case) translates very naturally into the p-adic case.

First, let us observe that in global discussions of motives over a number field, it is natural to think of the operation of complex conjugation as a sort of "Frobenius at the infinite prime." In fact, in such discussions, complex conjugation is often denoted by "Fr_∞." Next, let us observe that one special property of the canonical indigenous bundle is that its monodromy representation (i.e., the "canonical representation" $\rho_{\mathbf{X}} : \Pi_{\mathbf{X}} \to \mathrm{PGL}_2(\mathbb{C})$) is *real-valued*, i.e., takes its values in $\mathrm{PGL}_2(\mathbb{R})$. Another way to put this is to say that the canonical indigenous bundle is Fr_∞-invariant, i.e.,

> *The canonical indigenous bundle on a hyperbolic curve is invariant with respect to the Frobenius at the infinite prime.*

Unfortunately, as is observed in [**5**], this property of having real monodromy is not sufficient to characterize the canonical indigenous bundle completely. That is to say, the indigenous bundles with real monodromy form a discrete subset of the space of indigenous bundles on the given curve X, but this discrete subset consists (in general) of more than one element.

Let us introduce some notation. Let $\mathcal{M}_{g,r}$ be the stack of r-pointed smooth curves of genus g over $\mathbb{C}$. Let $\mathcal{S}_{g,r}$ be the stack of such curves equipped with an indigenous bundle. Then there is a natural projection morphism $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ (given by forgetting the indigenous bundle) which exhibits $\mathcal{S}_{g,r}$ as an affine torsor on $\mathcal{M}_{g,r}$ over the vector bundle $\Omega_{\mathcal{M}_{g,r}/\mathbb{C}}$ of differentials on $\mathcal{M}_{g,r}$. We shall refer to this torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ as the *Schwarz torsor.*

Let us write $\mathcal{S}_X$ for the restriction of the Schwarz torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ to the point $[X] \in \mathcal{M}_{g,r}(\mathbb{C})$ defined by X. Thus, $\mathcal{S}_X$ is an affine complex space of dimension $3g-3+r$. Let $\mathcal{R}_X \subseteq \mathcal{S}_X$ be the set of indigenous bundles with real monodromy. As observed in [**5**], $\mathcal{R}_X$ is a discrete subset of $\mathcal{S}_X$. Now let $\mathcal{S}'_X \subseteq \mathcal{S}_X$ be the subset of indigenous bundles $(P \to X, \nabla_P)$ with the following property:

> (*) The associated monodromy representation $\rho : \Pi_{\mathbf{X}} \to \mathrm{PGL}_2(\mathbb{C})$ is injective and its image Γ is a *quasi-Fuchsian* group. Moreover, if $\Omega \subseteq \mathbb{P}^1(\mathbb{C})$ is the domain of discontinuity of Γ, then Ω/Γ is a disjoint union of two Riemann surfaces of type (g, r).

(Roughly speaking, a "quasi-Fuchsian group" is a discrete subgroup of $\mathrm{PGL}_2(\mathbb{C})$ whose domain of discontinuity Ω (i.e., the set of points of $\mathbb{P}^1(\mathbb{C})$ at which Γ acts discontinuously) is a disjoint union of two topological open discs, separated by a topological circle. We refer to [**10, 27**] for more details on the theory of quasi-Fuchsian groups.)

It is known that $\mathcal{S}'_X$ is a bounded ([**10**], p. 99, Lemma 6), open (cf. the discussion of § 5 of [**27**]) subset of $\mathcal{S}_X$ (in the complex analytic topology). Moreover, since a quasi-Fuchsian group with real monodromy acts discretely on the upper half plane (see, e.g., [**26**], Chapter I, Proposition 1.8), it follows immediately that such a quasi-Fuchsian group is Fuchsian. Put another way, we have that:

> *The intersection $\mathcal{R}_X \bigcap \mathcal{S}'_X \subseteq \mathcal{S}_X$ is the set consisting of the single point corresponding to the canonical indigenous bundle.*

It is this characterization of the canonical indigenous bundle that we will seek to translate into the p-adic case.

To translate the above characterization, let us first recall the point of view of Arakelov theory which states, in effect, that $\mathbb{Z}_p$-integral structures (on say, an affine space over $\mathbb{Q}_p$) correspond to closures of bounded open subsets (of, say, an affine space over $\mathbb{C}$). Thus, from this point of view, one may think of $\mathcal{S}'_X$ as defining a natural integral structure (in the sense of Arakelov theory) on the complex affine space $\mathcal{S}_X$. Thus, from this point of view, one arrives at the following characterization of the canonical indigenous bundle:

> *The canonical indigenous bundle is the unique indigenous bundle which is integral (in the Arakelov sense) and Frobenius invariant (i.e., has monodromy which is invariant with respect to complex conjugation).*

This gives us at last an answer to the question posed earlier: How can one characterize the canonical indigenous bundle in the complex case in such a way that the characterization carries over word for word to the p-adic context? In particular, it gives rise to the following conclusion:

> *The proper p-adic analogue of the theory of the Fuchsian and Bers uniformizations should be a theory of $\mathbb{Z}_p$-integral indigenous bundles that are invariant with respect to some natural action of the Frobenius at the prime p.*

This conclusion constitutes the fundamental philosophical basis underlying the theory of [**22**]. In [**21**], this philosophy was partially realized in the sense that *certain* $\mathbb{Z}_p$-integral Frobenius indigenous bundles were constructed. The theory of [**21**] will be reviewed later (in § 1.6). The goal of [**22**], by contrast, is to lay the foundations for a general theory of *all* $\mathbb{Z}_p$-integral Frobenius indigenous bundles and to say as much as is possible in as much generality as is possible concerning such bundles.

1.5. The Canonical Real Analytic Trivialization of the Schwarz Torsor

In this subsection, we would like to take a closer look at the *Schwarz torsor* $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$. For general g and r, this affine torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ does not admit any algebraic or holomorphic sections. Indeed, this affine torsor defines a class in $H^1(\mathcal{M}_{g,r}, \Omega_{\mathcal{M}_{g,r}/\mathbb{C}})$ which is the Hodge-theoretic first Chern class of a certain ample line bundle $\mathcal{L}$ on $\mathcal{M}_{g,r}$. (See [**21**], Chapter I, § 3, especially Theorem 3.4, for more details on this Hodge-theoretic Chern class and Chapter III, Proposition 2.2, of [**22**] for a proof of ampleness.) Put another way, $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ is the torsor of (algebraic) connections on the line bundle $\mathcal{L}$. However, the map that assigns to X the canonical indigenous bundle on X defines a *real analytic* section

$$s_{\mathbf{H}} : \mathcal{M}_{g,r}(\mathbb{C}) \to \mathcal{S}_{g,r}(\mathbb{C})$$

of this torsor.

The first and most important goal of the present subsection is to remark that

> *The single object $s_{\mathbf{H}}$ essentially embodies the entire uniformization theory of complex hyperbolic curves and their moduli.*

Indeed, $s_{\mathbf{H}}$ by its very definition contains the data of "which indigenous bundle is canonical," hence already may be said to embody the Fuchsian uniformization. Next, we observe that $\overline{\partial} s_{\mathbf{H}}$ is equal to the Weil-Petersson metric on $\mathcal{M}_{g,r}$ (see [**21**], Introduction, Theorem 2.3 for more details). Moreover, (as is remarked in Example 2 following Definition 2.1 in [**21**], Introduction, § 2) since the canonical coordinates associated to a real analytic Kähler metric are obtained by essentially integrating (in the "sense of anti-$\overline{\partial}$-ing") the metric, it follows that (a certain appropriate restriction of) $s_{\mathbf{H}}$ "is" essentially the Bers uniformization of Teichmüller space. Thus, as advertised above, the single object $s_{\mathbf{H}}$ stands at the very center of the uniformization theory of complex hyperbolic curves and their moduli.

In particular, it follows that we can once again reinterpret the fundamental issue of trying to find a p-adic analogue of the Fuchsian uniformization as the issue of *trying to find a p-adic analogue of the section* $s_{\mathbf{H}}$. That is to say, the torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ is, in fact, defined over $\mathbb{Z}[\frac{1}{2}]$, hence over $\mathbb{Z}_p$ (for p odd). Thus, forgetting for the moment that it is not clear precisely what p-adic category of functions corresponds to the real analytic category at the infinite prime, one sees that

> *One way to regard the search for a p-adic Fuchsian uniformization is to regard it as the search for some sort of canonical p-adic analytic section of the torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$.*

In this context, it is thus natural to refer to $s_{\mathbf{H}}$ as the *canonical arithmetic trivialization of the torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ at the infinite prime.*

Finally, let us observe that this situation of a torsor corresponding to the Hodge-theoretic first Chern class of an ample line bundle, equipped with a canonical real analytic section occurs not only over $\mathcal{M}_{g,r}$, but over any individual hyperbolic curve X (say, over $\mathbb{C}$), as well. Indeed, let $(P \to X, \nabla_P)$ be the canonical indigenous bundle on X. Let $\sigma : X \to P$ be its Hodge section. Then by [**21**], Chapter I, Proposition 2.5, it follows that the $T \stackrel{\mathrm{def}}{=} P - \sigma(X)$ has the structure of an $\omega_{X/\mathbb{C}}$-torsor over X. In fact, one can say more: namely, this torsor is the Hodge-theoretic first Chern class corresponding to the ample line bundle $\omega_{X/\mathbb{C}}$. Moreover, if we compose the morphism $\widetilde{\mathbf{X}} \cong \mathbf{H} \subseteq \mathbb{P}^1_{\mathbb{C}}$ used to define σ with the standard complex conjugation morphism on $\mathbb{P}^1_{\mathbb{C}}$, we obtain a new $\Pi_{\mathbf{X}}$-equivariant $\widetilde{\mathbf{X}} \to \mathbb{P}^1_{\mathbb{C}}$ which descends to a real analytic section $s_{\mathbf{X}} : X(\mathbb{C}) \to T(\mathbb{C})$. Just as in the case of $\mathcal{M}_{g,r}$, it is easy to compute (cf. the argument of [**21**], Introduction, Theorem 2.3) that $\overline{\partial} s_{\mathbf{X}}$ is equal to the canonical hyperbolic metric $\mu_{\mathbf{X}}$. Thus, just as in the case of the real analytic section $s_{\mathbf{H}}$ of the Schwarz torsor over $\mathcal{M}_{g,r}$, $s_{\mathbf{X}}$ *essentially "is" the Fuchsian uniformization of* $\mathbf{X}$.

1.6. The Classical Ordinary Theory. — As stated earlier, the purpose of [**22**] is to study all integral Frobenius invariant indigenous bundles. On the other hand, in [**21**], a very important special type of Frobenius invariant indigenous bundle was constructed. This type of bundle will henceforth be referred to as *classical ordinary.* (Such bundles were called "ordinary" in [**21**]. Here we use the term "classical ordinary" to refer to objects called "ordinary" in [**21**] in order to avoid confusion with the more general notions of ordinarity discussed in [**22**].) Before discussing the theory of the [**22**] (which is the goal of § 2), it is thus natural to review the classical ordinary theory. In this subsection, we let p be an odd prime.

If one is to construct p-adic Frobenius invariant indigenous bundles for arbitrary hyperbolic curves, the first order of business is to make precise the notion of Frobenius invariance that one is to use. For this, it is useful to have a prototype. The prototype that gave rise to the classical ordinary theory is the following:

> Let $\mathcal{M} \stackrel{\text{def}}{=} (\mathcal{M}_{1,0})_{\mathbb{Z}_p}$ be the moduli stack of elliptic curves over $\mathbb{Z}_p$. Let $\mathcal{G} \to \mathcal{M}$ be the universal elliptic curve. Let $\mathcal{E}$ be its first de Rham cohomology module. Thus, $\mathcal{E}$ is a rank two vector bundle on $\mathcal{M}$, equipped with a Hodge subbundle $\mathcal{F} \subseteq \mathcal{E}$, and a connection $\nabla_{\mathcal{E}}$ (i.e., the "Gauss-Manin connection"). Taking the projectivization of $\mathcal{E}$ defines a $\mathbb{P}^1$-bundle with connection $(P \to \mathcal{M}, \nabla_P)$, together with a Hodge section $\sigma : \mathcal{M} \to P$. It turns out that (the natural extension over the compactification of $\mathcal{M}$ obtained by using log structures of) the bundle (P, ∇_P) is an *indigenous bundle* on $\mathcal{M}$. In particular, (P, ∇_P) defines a crystal in $\mathbb{P}^1$-bundles on $\mathrm{Crys}(\mathcal{M} \otimes \mathbf{F}_p/\mathbb{Z}_p)$. Thus, one can form the pull-back $\Phi^*(P, \nabla_P)$ via the Frobenius morphism of this crystal. If one then adjusts the integral structure of $\Phi^*(P, \nabla_P)$ (cf. Definition 1.18 of Chapter VI of [**22**]; [**21**], Chapter III, Definition 2.4), one obtains the *renormalized Frobenius pull-back* $\mathbb{F}^*(P, \nabla_P)$. Then (P, ∇_P) is Frobenius invariant in the sense that $(P, \nabla_P) \cong \mathbb{F}^*(P, \nabla_P)$.

Thus, *the basic idea behind* [**21**] *was to consider to what extent one could construct indigenous bundles on arbitrary hyperbolic curves that are equal to their own renormalized Frobenius pull-backs, i.e., satisfying*

$$\mathbb{F}^*(P, \nabla_P) \cong (P, \nabla_P)$$

In particular, it is natural to try to consider moduli of indigenous bundles satisfying this condition. Since it is not at all obvious how to do this over $\mathbb{Z}_p$, a natural first step was to make the following *key* observation:

> *If (P, ∇_P) is an indigenous bundle over $\mathbb{Z}_p$ preserved by $\mathbb{F}^*$, then the reduction modulo p of (P, ∇_P) has square nilpotent p-curvature.*

(The "p-curvature" of an indigenous bundle in characteristic p is a natural invariant of such a bundle. We refer to [**21**], Chapter II, as well as § 1 of Chapter II of [**22**] for more details.) Thus, if $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is the *stack of r-pointed stable curves of genus g* (as in [**4, 20**]) in characteristic p, one can define the stack $\overline{\mathcal{N}}_{g,r}$ of such curves equipped with a "nilpotent" indigenous bundle. (Here, "nilpotent" means that its p-curvature is square nilpotent.) In the following, we shall often find it convenient to refer to pointed stable curves equipped with nilpotent indigenous bundles as *nilcurves*, for short. Thus, $\overline{\mathcal{N}}_{g,r}$ is the moduli stack of nilcurves. We would like to emphasize that

> *The above observation – which led to the notion of "nilcurves" – is the key technical breakthrough that led to the development of the "p-adic Teichmüller theory" of* [**21, 22**].

The first major result of [**21**] is the following (cf. [**22**], Chapter II, Proposition 1.7; [**21**], Chapter II, Theorem 2.3):

***Theorem 1.1* (Stack of Nilcurves).** — *The natural morphism $\overline{\mathcal{N}}_{g,r} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is a finite, flat, local complete intersection morphism of degree p^{3g-3+r}.*

> *In particular, up to "isogeny" (i.e., up to the fact that $p^{3g-3+r} \neq 1$), the stack of nilcurves $\overline{\mathcal{N}}_{g,r} \subseteq \overline{\mathcal{S}}_{g,r}$ defines a canonical section of the Schwarz torsor $\overline{\mathcal{S}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ in characteristic p.*

Thus, relative to our discussion of complex Teichmüller theory – which we saw could be regarded as the study of a certain canonical real analytic section of the Schwarz torsor – it is natural that "p-adic Teichmüller theory" should revolve around the study of $\overline{\mathcal{N}}_{g,r}$.

Although the structure of $\overline{\mathcal{N}}_{g,r}$ is now been much better understood, at the time of writing of [**21**] (Spring of 1994), it was not so well understood, and so it was natural to do the following: Let $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r} \subseteq \overline{\mathcal{N}}_{g,r}$ be the open substack where $\overline{\mathcal{N}}_{g,r}$ is *étale* over $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$. This open substack will be referred to as the *(classical) ordinary locus of* $\overline{\mathcal{N}}_{g,r}$. If one sets up the theory (as is done in [**21, 22**]) using *stable curves* (as we do here), rather than just smooth curves, and applies the theory of *log structures* (as in [**18**]), then it is easy to show that the ordinary locus of $\overline{\mathcal{N}}_{g,r}$ is *nonempty*.

It is worth pausing here to note the following: The reason for the use of the term "ordinary" is that it is standard general practice to refer to as "ordinary" situations where Frobenius acts on a linear space equipped with a "Hodge subspace" in such a way that it acts with slope zero on a subspace of the same rank as the rank of the

Hodge subspace. Thus, we use the term "ordinary" here because the Frobenius action on the cohomology of an ordinary nilcurve satisfies just such a condition. In other words, ordinary nilcurves are ordinary in their capacity as nilcurves. However, it is important to remember that:

> *The issue of whether or not a nilcurve is ordinary is entirely different from the issue of whether or not the Jacobian of the underlying curve is ordinary (in the usual sense). That is to say, there exist examples of ordinary nilcurves whose underlying curves have nonordinary Jacobians as well as examples of nonordinary nilcurves whose underlying curves have ordinary Jacobians.*

Later, we shall comment further on the issue of the incompatibility of the theory of [**21**] with Serre-Tate theory relative to the operation of passing to the Jacobian.

At any rate, since $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r}$ is étale over $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$, it lifts naturally to a p-adic formal stack $\mathcal{N}$ which is étale over $(\overline{\mathcal{M}}_{g,r})_{\mathbb{Z}_p}$. Let $\mathcal{C} \to \mathcal{N}$ denote the tautological stable curve over $\mathcal{N}$. Then the main result (Theorem 0.1 of the Introduction of [**21**]) of the theory of [**21**] is the following:

***Theorem 1.2* (Canonical Frobenius Lifting)**

There exists a unique pair $(\Phi_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}; (P, \nabla_P))$ satisfying the following:

(1) *The reduction modulo p of the morphism $\Phi_{\mathcal{N}}$ is the Frobenius morphism on $\mathcal{N}$, i.e., $\Phi_{\mathcal{N}}$ is a* Frobenius lifting.

(2) *(P, ∇_P) is an indigenous bundle on $\mathcal{C}$ such that the renormalized Frobenius pull-back of $\Phi^*_{\mathcal{N}}(P, \nabla_P)$ is isomorphic to (P, ∇_P), i.e., (P, ∇_P) is* Frobenius invariant with respect to $\Phi_{\mathcal{N}}$.

Moreover, this pair also gives rise in a natural way to a Frobenius lifting $\Phi_{\mathcal{C}} : \mathcal{C}^{\mathrm{ord}} \to \mathcal{C}^{\mathrm{ord}}$ on a certain formal p-adic open substack $\mathcal{C}^{\mathrm{ord}}$ of $\mathcal{C}$ (which will be referred to as the ordinary locus of $\mathcal{C}$).

Thus, this Theorem is a partial realization of the goal of constructing a canonical integral Frobenius invariant bundle on the universal stable curve.

Again, we observe that

> *This canonical Frobenius lifting $\Phi_{\mathcal{N}}$ is by no means compatible (relative to the operation of passing to the Jacobian) with the canonical Frobenius lifting $\Phi_{\mathcal{A}}$ (on the p-adic stack of ordinary principally polarized abelian varieties) arising from Serre-Tate theory (cf., e.g.,* [**22**], *§0.7, for more details).*

At first glance, the reader may find this fact to be extremely disappointing and unnatural. In fact, however, when understood properly, this incompatibility is something which is to be expected. Indeed, relative to the analogy between Frobenius liftings and Kähler metrics implicit in the discussion of §1.1 ∼ 1.5 (cf., e.g., [**22**], §0.8, for more details) such a compatibility would be the p-adic analogue of a compatibility between the Weil-Petersson metric on $(\mathcal{M}_{g,r})_{\mathbb{C}}$ and the Siegel upper half plane metric on $(\mathcal{A}_g)_{\mathbb{C}}$. On the other hand, it is easy to see in the complex case that these two metrics are far from compatible. (Indeed, if they were compatible, then the Torelli map $(\mathcal{M}_g)_{\mathbb{C}} \to (\mathcal{A}_g)_{\mathbb{C}}$ would be unramified, but one knows that it is ramified at hyperelliptic curves of high genus.)

Another important difference between $\Phi_{\mathcal{N}}$ and $\Phi_{\mathcal{A}}$ is that in the case of $\Phi_{\mathcal{A}}$, by taking the union of $\Phi_{\mathcal{A}}$ and its transpose, one can compactify $\Phi_{\mathcal{A}}$ into an entirely algebraic (i.e., not just p-adic analytic) object, namely a Hecke correspondence on $\mathcal{A}_g$. In the case of $\Phi_{\mathcal{N}}$, however, such a compactification into a correspondence is impossible. We refer to **[23]** for a detailed discussion of this phenomenon.

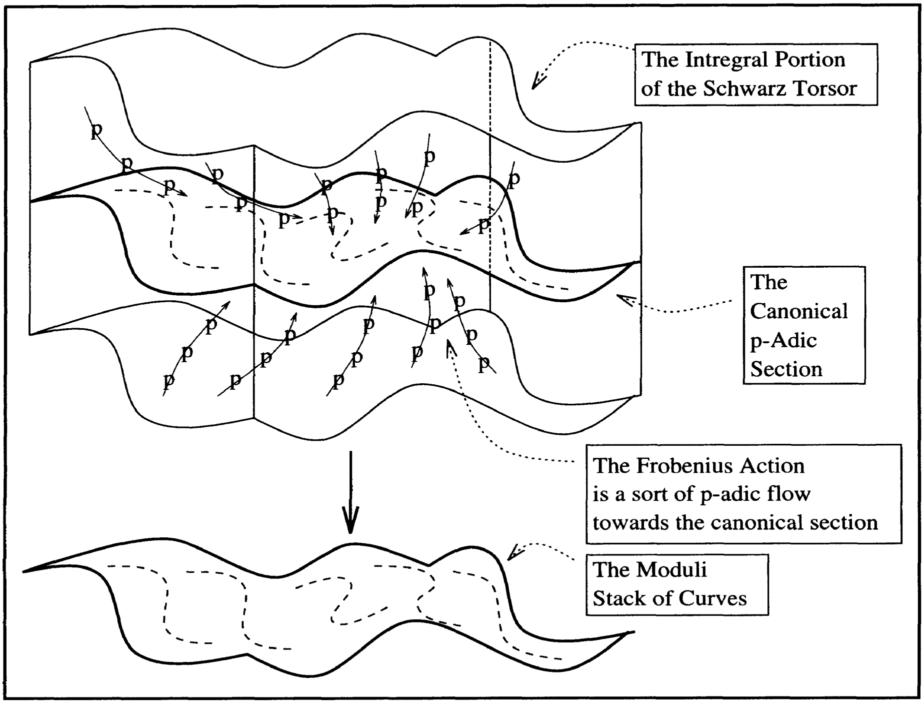

FIGURE 3. The Canonical Frobenius Action Underlying Theorem 1.2

So far, we have been discussing the differences between $\Phi_{\mathcal{N}}$ and $\Phi_{\mathcal{A}}$. In fact, however, in one very important respect, they are very similar objects. Namely, they are both *(classical) ordinary Frobenius liftings*. A (classical) ordinary Frobenius lifting is defined as follows: Let k be a perfect field of characteristic p. Let $A \stackrel{\text{def}}{=} W(k)$ (the Witt vectors over k). Let S be a formal p-adic scheme which is formally smooth over A. Let $\Phi_S : S \to S$ be a morphism whose reduction modulo p is the Frobenius morphism. Then differentiating Φ_S defines a morphism $d\Phi_S : \Phi_S^*\Omega_{S/A} \to \Omega_{S/A}$ which is zero in characteristic p. Thus, we may form a morphism

$$\Omega_\Phi : \Phi_S^*\Omega_{S/A} \to \Omega_{S/A}$$

by dividing $d\Phi_S$ by p. Then Φ_S is called a *(classical) ordinary Frobenius lifting* if Ω_Φ is an isomorphism. Just as there is a general theory of canonical coordinates associated to real analytic Kähler metrics, there is a general theory of canonical coordinates associated to ordinary Frobenius liftings. This theory is discussed in detail in § 1 of Chapter III of [**21**]. The main result is as follows (cf. § 1 of [**21**], Chapter III):

Theorem 1.3 **(Ordinary Frobenius Liftings)**. — *Let $\Phi_S : S \to S$ be a (classical) ordinary Frobenius lifting. Then taking the invariants of $\Omega_{S/A}$ with respect to Ω_Φ gives rise to an étale local system $\Omega_\Phi^{\mathrm{et}}$ on S of free $\mathbb{Z}_p$-modules of rank equal to* $\dim_A(S)$.

Let $z \in S(\overline{k})$ be a point valued in the algebraic closure of k. Then $\Omega_z \stackrel{\mathrm{def}}{=} \Omega_\Phi^{\mathrm{et}}|_z$ may be thought of as a free $\mathbb{Z}_p$-module of rank $\dim_A(S)$; write Θ_z for the $\mathbb{Z}_p$-dual of Ω_z. Let S_z be the completion of S at z. Let $\widehat{\mathbf{G}}_{\mathrm{m}}$ be the completion of the multiplicative group scheme $\mathbf{G}_{\mathrm{m}}$ over $W(\overline{k})$ at 1. Then there is a unique isomorphism

$$\Gamma_z : S_z \cong \widehat{\mathbf{G}}_{\mathrm{m}} \otimes_{\mathbb{Z}_p}^{\mathrm{gp}} \Theta_z$$

such that:

(i) the derivative of Γ_z induces the natural inclusion $\Omega_z \hookrightarrow \Omega_{S/A}|_{S_z}$;

(ii) the action of Φ_S on S_z corresponds to multiplication by p on $\widehat{\mathbf{G}}_{\mathrm{m}} \otimes_{\mathbb{Z}_p}^{\mathrm{gp}} \Theta_z$.

Here, by "$\widehat{\mathbf{G}}_{\mathrm{m}} \otimes_{\mathbb{Z}_p}^{\mathrm{gp}} \Theta_z$," we mean the tensor product in the sense of (formal) group schemes. Thus, $\widehat{\mathbf{G}}_{\mathrm{m}} \otimes_{\mathbb{Z}_p}^{\mathrm{gp}} \Theta_z$ is noncanonically isomorphic to the product of $\dim_A(S) = \mathrm{rank}_{\mathbb{Z}_p}(\Theta_z)$ *copies of* $\widehat{\mathbf{G}}_{\mathrm{m}}$.

Thus, we obtain canonical multiplicative parameters on $\mathcal{N}$ and $\mathcal{C}^{\mathrm{ord}}$ (from $\Phi_{\mathcal{N}}$ and $\Phi_{\mathcal{C}}$, respectively). If we apply Theorem 1.3 to the canonical lifting $\Phi_{\mathcal{A}}$ of Serre-Tate theory (cf., e.g., [**22**], § 0.7), we obtain the Serre-Tate parameters. Moreover, note that in Theorem 1.3, the identity element "1" of the formal group scheme $\widehat{\mathbf{G}}_{\mathrm{m}} \otimes_{\mathbb{Z}_p} \Omega_z$ corresponds under Γ_z to some point $\alpha_z \in S(W(\overline{k}))$ that lifts z. That is to say,

> *Theorem 1.3 also gives rise to a notion of canonical liftings of points in characteristic p.*

In the case of $\Phi_{\mathcal{A}}$, this notion coincides with the well-known notion of the Serre-Tate canonical lifting of an ordinary abelian variety. In the case of $\Phi_{\mathcal{N}}$, the theory of canonically lifted curves is discussed in detail in Chapter IV of [**21**]. In [**22**], however, the theory of canonical curves in the style of Chapter IV of [**21**] does not play a very important role.

Remark. — Certain special cases of Theorem 1.3 already appear in the work of Ihara ([**14, 15, 16, 17**]). In fact, more generally, the work of Ihara ([**14, 15, 16, 17**]) on the Schwarzian equations of Shimura curves and the possibility of constructing an analogue of Serre-Tate theory for more general hyperbolic curves anticipates, at least at a philosophical level, many aspects of the theory of [**21, 22**].

Thus, in summary, although the classical ordinary theory of [**21**] is not compatible with Serre-Tate theory relative to the Torelli map, it is in many respects deeply structurally analogous to Serre-Tate theory. Moreover, this close structural affinity arises from the fact that in both cases,

> *The ordinary locus with which the theory deals is defined by the condition that some canonical Frobenius action have slope zero.*

Thus, although some readers may feel unhappy about the use of the term "ordinary" to describe the theory of [**21**] (i.e., despite the fact that this theory is incompatible with Serre-Tate theory), we feel that this close structural affinity arising from the common condition of a slope zero Frobenius action justifies and even renders natural the use of this terminology.

Finally, just as in the complex case, where the various indigenous bundles involved gave rise to monodromy representations of the fundamental group of the hyperbolic curve involved, in the p-adic case as well, the canonical indigenous bundle of Theorem 1.2 gives rise to a canonical Galois representation, as follows. We continue with the notation of Theorem 1.2. Let $\mathcal{N}' \to \mathcal{N}$ be the morphism $\Phi_{\mathcal{N}}$, which we think of as a covering of $\mathcal{N}$; let $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C} \otimes_{\mathcal{N}} \mathcal{N}'$. Note that $\mathcal{C}$ and $\mathcal{N}$ have natural log structures (obtained by pulling back the natural log structures on $\overline{\mathcal{M}}_{g,r}$ and its tautological curve, respectively). Thus, we obtain $\mathcal{C}^{\log}$, $\mathcal{N}^{\log}$. Let

$$\Pi_{\mathcal{N}} \stackrel{\text{def}}{=} \pi_1(\mathcal{N}^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p); \quad \Pi_{\mathcal{C}} \stackrel{\text{def}}{=} \pi_1(\mathcal{C}^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

Similarly, we have $\Pi_{\mathcal{N}'}$; $\Pi_{\mathcal{C}'}$. Then the main result is the following (Theorem 0.4 of [**21**], Introduction):

***Theorem 1.4* (Canonical Galois Representation).** — *There is a natural $\mathbb{Z}_p$-flat, p-adically complete "ring of additive periods" $\mathcal{D}^{\mathrm{Gal}}_{\mathcal{N}}$ on which $\Pi_{\mathcal{N}'}$ (hence also $\Pi_{\mathcal{C}'}$ via the natural projection $\Pi_{\mathcal{C}'} \to \Pi_{\mathcal{N}'}$) acts continuously, together with a twisted homomorphism*

$$\rho : \Pi_{\mathcal{C}'} \to \mathrm{PGL}_2(\mathcal{D}^{\mathrm{Gal}}_{\mathcal{N}})$$

where "twisted" means with respect to the action of $\Pi_{\mathcal{C}'}$ on $\mathcal{D}^{\mathrm{Gal}}_{\mathcal{N}}$. This representation is obtained by taking Frobenius invariants of (P, ∇_P), using a technical tool known as crystalline induction.

Thus, in summary, the theory of [**21**] gives one a fairly good understanding of what happens over the ordinary locus $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r}$, complete with analogues of various objects (monodromy representations, canonical modular coordinates, etc.) that appeared in the complex case. On the other hand, it begs the following questions:

(1) What does the nonordinary part of $\overline{\mathcal{N}}_{g,r}$ look like? What sorts of nonordinary nilcurves can occur? In particular, what does the p-curvature of such nonordinary nilcurves look like?

(2) Does this "classical ordinary theory" admit any sort of compactification? One sees from [**23**] that it does not admit any sort of compactification via correspondences. Still, since the condition of being ordinary is an "open condition," it is natural to ask what happens to this classical ordinary theory as one goes to the boundary.

The theory of [**22**] answers these two questions to a large extent, not by adding on a few new pieces to [**21**], but by starting afresh and developing from new foundations a general theory of integral Frobenius invariant indigenous bundles. The theory of [**22**] will be discussed in § 2.

2. Beyond the "Classical Ordinary" Theory

2.1. Atoms, Molecules, and Nilcurves. — Let p be an odd prime. Let g and r be nonnegative integers such that $2g - 2 + r \geq 1$. Let $\overline{\mathcal{N}}_{g,r}$ be the stack of nilcurves in characteristic p. We denote by $\mathcal{N}_{g,r} \subseteq \overline{\mathcal{N}}_{g,r}$ the open substack consisting of *smooth nilcurves*, i.e., nilcurves whose underlying curve is smooth. Then the first step in our analysis of $\overline{\mathcal{N}}_{g,r}$ is the introduction of the following notions (cf. Definitions 1.1 and 3.1 of [**22**], Chapter II):

Definition 2.1. — We shall call a nilcurve *dormant* if its p-curvature (i.e., the p-curvature of its underlying indigenous bundle) is identically zero. Let d be a nonnegative integer. Then we shall call a smooth nilcurve *spiked of strength d* if the zero locus of its p-curvature forms a divisor of degree d.

If d is a nonnegative integer (respectively, the symbol ∞), then we shall denote by

$$\mathcal{N}_{g,r}[d] \subseteq \mathcal{N}_{g,r}$$

the locally closed substack of nilcurves that are spiked of strength d (respectively, dormant). It is immediate that there does indeed exist such a locally closed substack, and that if k is an algebraically closed field of characteristic p, then

$$\mathcal{N}_{g,r}(k) = \coprod_{d=0}^{\infty} \mathcal{N}_{g,r}[d]\ (k)$$

Moreover, we have the following result (cf. [**22**], Chapter II, Theorems 1.12, 2.8, and 3.9):

Theorem 2.2 **(Stratification of $\mathcal{N}_{g,r}$).** — *Any two irreducible components of $\overline{\mathcal{N}}_{g,r}$ intersect. Moreover, for $d = 0, 1, \ldots, \infty$, the stack $\mathcal{N}_{g,r}[d]$ is smooth over $\mathbf{F}_p$ of dimension $3g - 3 + r$ (if it is nonempty). Finally, $\mathcal{N}_{g,r}[\infty]$ is irreducible, and its closure in $\overline{\mathcal{N}}_{g,r}$ is smooth over $\mathbf{F}_p$.*

Thus, in summary, we see that

> *The classification of nilcurves by the size of the zero locus of their p-curvatures induces a natural decomposition of $\mathcal{N}_{g,r}$ into smooth (locally closed) strata.*

Unfortunately, however, Theorem 2.2 still only gives us a very rough idea of the structure of $\mathcal{N}_{g,r}$. For instance, it tells us nothing of the degree of each $\mathcal{N}_{g,r}[d]$ over $\mathcal{M}_{g,r}$.

Remark. — Some people may object to the use of the term "stratification" here for the reason that in certain contexts (e.g., the Ekedahl-Oort stratification of the moduli stack of principally polarized abelian varieties – cf. [**11**], § 2), this term is only used for decompositions into locally closed subschemes whose closures satisfy certain (rather stringent) axioms. Here, we do not mean to imply that we can prove any nontrivial results concerning the closures of the $\mathcal{N}_{g,r}[d]$'s. That is to say, in [**22**], we use the term "stratification" only in the weak sense (i.e., that $\mathcal{N}_{g,r}$ is the union of the $\mathcal{N}_{g,r}[d]$). This usage conforms to the usage of Lecture 8 of [**24**], where "flattening stratifications" are discussed.

In order to understand things more explicitly, it is natural to attempt to do the following:

(1) *Understand the structure – especially, what the p-curvature looks like – of all* molecules *(i.e., nilcurves whose underlying curve is totally degenerate).*

(2) *Understand how each molecule* deforms, *i.e., given a molecule, one can consider its formal neighborhood $\mathcal{N}$ in $\overline{\mathcal{N}}_{g,r}$. Then one wants to know the degree of each $\mathcal{N} \bigcap \mathcal{N}_{g,r}[d]$ (for all d) over the corresponding formal neighborhood $\mathcal{M}$ in $\overline{\mathcal{M}}_{g,r}$.*

Obtaining a complete answer to these two questions is the topic of [**22**], Chapters IV and V.

First, we consider the problem of understanding the structure of *molecules.* Since the underlying curve of a molecule is a totally degenerate curve – i.e., a stable curve obtained by gluing together $\mathbb{P}^1$'s with three nodal/marked points – it is natural to restrict the given nilpotent indigenous bundle on the whole curve to each of these $\mathbb{P}^1$'s with three marked points. Thus, for each irreducible component of the original curve, we obtain a $\mathbb{P}^1$ with three marked points equipped with something very close to a nilpotent indigenous bundle. The only difference between this bundle and an indigenous bundle is that its monodromy at some of the marked points (i.e., those marked points that correspond to nodes on the original curve) might not be nilpotent.

In general, a bundle (with connection) satisfying all the conditions that an indigenous bundle satisfies except that its monodromy at the marked points might not be nilpotent is called a *torally indigenous bundle* (cf. [**22**], Chapter I, Definition 4.1). (When there is fear of confusion, indigenous bundles in the strict sense (as in [**21**], Chapter I) will be called *classical indigenous*.) For simplicity, we shall refer to any pointed stable curve (respectively, totally degenerate pointed stable curve) equipped with a nilpotent *torally indigenous* bundle as a *nilcurve* (respectively, *molecule*) (cf. § 0 of [**22**], Chapter V). Thus, when it is necessary to avoid confusion with the toral case, we shall say that "$\overline{\mathcal{N}}_{g,r}$ is the stack of classical nilcurves." Finally, we shall refer to a (possibly toral) nilcurve whose underlying curve is $\mathbb{P}^1$ with three marked points as an *atom*.

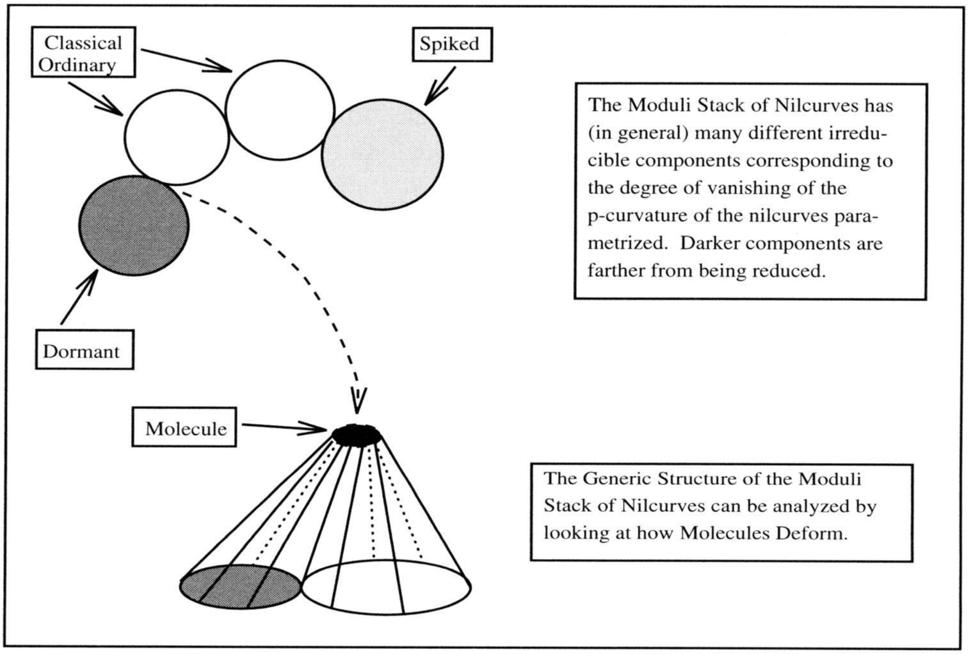

FIGURE 4. The Structure of $\overline{\mathcal{N}}_{g,r}$

At any rate, to summarize, *a molecule may be regarded as being made up of atoms*. It turns out that the monodromy at each marked point of an atom (or, in fact, more generally any nilcurve) has an invariant called the *radius*. The radius is, strictly speaking, an element of $\mathbf{F}_p/\{\pm\}$ (cf. Proposition 1.5 of [**22**], Chapter II) – i.e., the quotient set of $\mathbf{F}_p$ by the action of ± 1 – but, by abuse of notation, we shall often speak of the radius ρ as an element of $\mathbf{F}_p$. Then we have the following answer to (1) above (cf. § 1 of [**22**], Chapter V):

***Theorem 2.3* (The Structure of Atoms and Molecules).** — *The structure theory of atoms (over any field of characteristic p) may be summarized as follows:*

(1) *The three radii of an atom define a bijection of the set of isomorphism classes of atoms with the set of ordered triples of elements of $\mathbf{F}_p/\{\pm 1\}$.*

(2) *For any triple of elements $\rho_\alpha, \rho_\beta, \rho_\gamma \in \mathbf{F}_p$, there exist integers $a, b, c \in [0, p-1]$ such that (i) $a \equiv \pm 2\rho_\alpha$, $b \equiv \pm 2\rho_\beta$, $c \equiv \pm 2\rho_\gamma$; (ii) $a+b+c$ is odd and $< 2p$. Moreover, the atom of radii $\rho_\alpha, \rho_\beta, \rho_\gamma$ is dormant if and only if the following three inequalities are satisfied simultaneously: $a+b>c$, $a+c>b$, $b+c>a$.*

(3) *Suppose that the atom of radii $\rho_\alpha, \rho_\beta, \rho_\gamma$ is nondormant. Let $v_\alpha, v_\beta, v_\gamma$ be the degrees of the zero loci of the p-curvature at the three marked points. Then the non-negative integers $v_\alpha, v_\beta, v_\gamma$ are uniquely determined by the following two conditions: (i) $v_\alpha + v_\beta + v_\gamma$ is odd and $< p$; (ii) $v_\alpha \equiv \pm 2\rho_\alpha$, $v_\beta \equiv \pm 2\rho_\beta$, $v_\gamma \equiv \pm 2\rho_\gamma$.*

Molecules are obtained precisely by gluing together atoms at their marked points in such a way that the radii at marked points that are glued together coincide (as elements of $\mathbf{F}_p/\{\pm 1\}$).

In the last sentence of the theorem, we use the phrase "obtained precisely" to mean that all molecules are obtained in that way, and, moreover, any result of gluing together atoms in that fashion forms a molecule. Thus,

> *Theorem 2.3 reduces the structure theory of atoms and molecules to a matter of combinatorics.*

Theorem 2.3 follows from the theory of [**22**], Chapter IV.

Before proceeding, we would like to note the analogy with the theory of "pants" (see [**1**] for an exposition) in the complex case. In the complex case, the term "pants" is used to describe a Riemann surface which is topologically isomorphic to a Riemann sphere minus three points. The *holomorphic* isomorphism class of such a Riemann surface is given precisely by specifying three radii, i.e., the size of its three holes. Moreover, any hyperbolic Riemann surface can be analyzed by decomposing it into a union of pants, glued together at the boundaries. Thus, there is a certain analogy between the theory of pants and the structure theory of atoms and molecules given in Theorem 2.3.

Next, we turn to the issue of understanding how molecules deform. Let M be a nondormant *classical molecule* (i.e., it has nilpotent monodromy at all the marked points). Let us write $n_{\rm tor}$ for the number of "toral nodes" (i.e., nodes at which the monodromy is not nilpotent) of M. Let us write $n_{\rm dor}$ for the number of dormant atoms in M. To describe the deformation theory of M, it is useful to choose a *plot* Π for M. A plot is an ordering (satisfying certain conditions) of a certain subset of the nodes of M (see § 1 of [**22**], Chapter V for more details). This ordering describes the order in which one deforms the nodes of M. (Despite the similarity in notation, plots have nothing to do with the "VF-patterns" discussed below.) Once the plot is fixed, one can contemplate the various *scenarios* that may occur. Roughly speaking, a scenario is an assignment (satisfying certain conditions) of one of the three symbols $\{0, +, -\}$ to each of the branches of each of the nodes of M (see § 1 of [**22**], Chapter V for more details). There are $2^{n_{\mathrm{dor}}}$ possible scenarios. The point of all this terminology is the following:

> *One wants to deform the nodes of M in a such a way that one can always keep track of how the p-curvature deforms as one deforms the nilcurve.*

If one deforms the nodes in the fashion stipulated by the plot and scenario, then each deformation that occurs is one the following four types: *classical ordinary*, *grafted*, *philial*, *aphilial*.

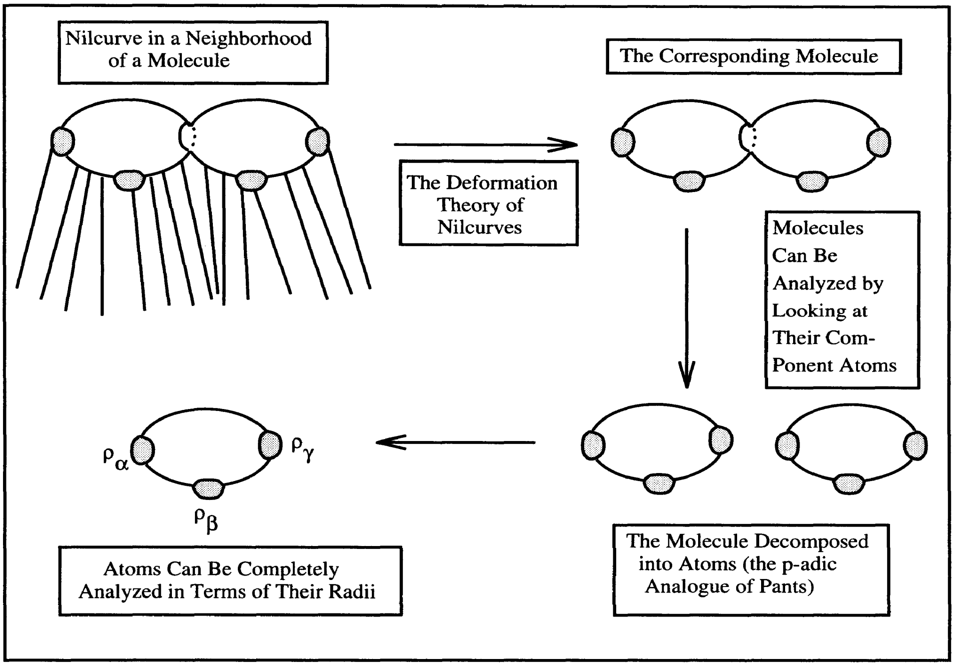

FIGURE 5. The Step Used to Analyze the Structure of $\overline{\mathcal{N}}_{g,r}$

The *classical ordinary* case is the case where the monodromy (at the node in question) is nilpotent. It is also by far the most technically simple and is already discussed implicitly in [**21**]. The *grafted case* is the case where a dormant atom is grafted on to (what after previous deformations is) a nondormant smooth nilcurve. This is the case where the consequent deformation of the p-curvature is the most technically difficult to analyze and is the reason for the introduction of "plots" and "scenarios." In order to understand how the p-curvature deforms in this case, one must introduce a certain technical tool called the *virtual p-curvature.* The theory of virtual p-curvatures is discussed in § 2.2 of [**22**], Chapter V. The *philial case* (respectively, *aphilial case*) is the case where one glues on a nondormant atom to (what after previous deformations is) a nondormant smooth nilcurve, and the parities (i.e., whether the number is even or odd) of the vanishing orders of the p-curvature at the two branches of the node are *opposite* to one another (respectively, the *same*). In the philial case (respectively, aphilial case), deformation gives rise to a spike (respectively, no spike). An illustration of these four fundamental types of deformation is given in Fig. 6. The signs in this illustration are the signs that are assigned to the branches of the nodes by the "scenario." When the p-curvature is not identically zero (i.e., on the light-colored areas), this sign is the parity (i.e., plus for even, minus for odd) of the vanishing order of the p-curvature. For a given scenario Σ, we denote by $n_{\mathrm{phl}}(\Sigma)$ (respectively, $n_{\mathrm{aph}}(\Sigma)$) the number of philial (respectively, aphilial) nodes that occur when the molecule is deformed according to that scenario.

If $U = \mathrm{Spec}(A)$ is a connected noetherian scheme of dimension 0, then we shall refer to the length of the artinian ring A as the *padding degree of U*. Then the theory just discussed gives rise to the following answer to (2) above (cf. Theorem 1.1 of [**22**], Chapter V):

Theorem 2.4 **(Deformation Theory of Molecules).** — *Let M be a classical molecule over an algebraically closed field k of characteristic p. Let $\mathcal{N}$ be the completion of $\overline{\mathcal{N}}_{g,r}$ at M. Let $\mathcal{M}$ be the completion of $(\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ at the point defined by the curve underlying M. Let $\overline{\eta}$ be the strict henselization of the generic point of $\mathcal{M}$. Then the natural morphism $\mathcal{N} \to \mathcal{M}$ is finite and flat of degree $2^{n_{\mathrm{tor}}}$. Moreover:*

(1) *If M is* dormant, *then $\mathcal{N}_{\mathrm{red}} \cong \mathcal{M}$, and $\mathcal{N}_{\overline{\eta}}$ has padding degree 2^{3g-3+r}.*

(2) *If M is* nondormant, *fix a plot Π for M. Then for each of the $2^{n_{\mathrm{dor}}}$ scenarios associated to Π, there exists a natural open substack $\mathcal{N}_{\Sigma} \subseteq \mathcal{N}_{\overline{\eta}} \overset{\mathrm{def}}{=} \mathcal{N} \times_{\mathcal{M}} \overline{\eta}$ such that: (i.) $\mathcal{N}_{\overline{\eta}}$ is the disjoint union of the $\mathcal{N}_{\Sigma}$ (as Σ ranges over all the scenarios); (ii.) every residue field of $\mathcal{N}_{\Sigma}$ is separable over (hence equal to) $k(\overline{\eta})$; (iii) the degree of $(\mathcal{N}_{\Sigma})_{\mathrm{red}}$ over $\overline{\eta}$ is $2^{n_{\mathrm{aph}}(\Sigma)}$; (iv) each connected component of $\mathcal{N}_{\Sigma}$ has padding degree $2^{n_{\mathrm{phl}}(\Sigma)}$; (v) the smooth nilcurve represented by any point of $(\mathcal{N}_{\Sigma})_{\mathrm{red}}$ is spiked of strength $p \cdot n_{\mathrm{phl}}(\Sigma)$.*

In particular, this Theorem reduces the computation of the degree of any $\mathcal{N}_{g,r}[d]$ over $(\mathcal{M}_{g,r})_{\mathbf{F}_p}$ to a matter of combinatorics.

For instance, let us denote by $n^{\mathrm{ord}}_{g,r,p}$ the degree of $\mathcal{N}^{\mathrm{ord}}_{g,r}$ (which – as a consequence of Theorem 2.4! (cf. Corollary 1.2 of [**22**], Chapter V) – is open and dense in $\mathcal{N}_{g,r}[0]$) over $(\mathcal{M}_{g,r})_{\mathbf{F}_p}$. Then following the algorithm implicit in Theorem 2.4, $n^{\mathrm{ord}}_{g,r,p}$ is computed explicitly for low g and r in Corollary 1.3 of [**22**], Chapter V (e.g., $n^{\mathrm{ord}}_{1,1,p} = n^{\mathrm{ord}}_{0,4,p} = p$;

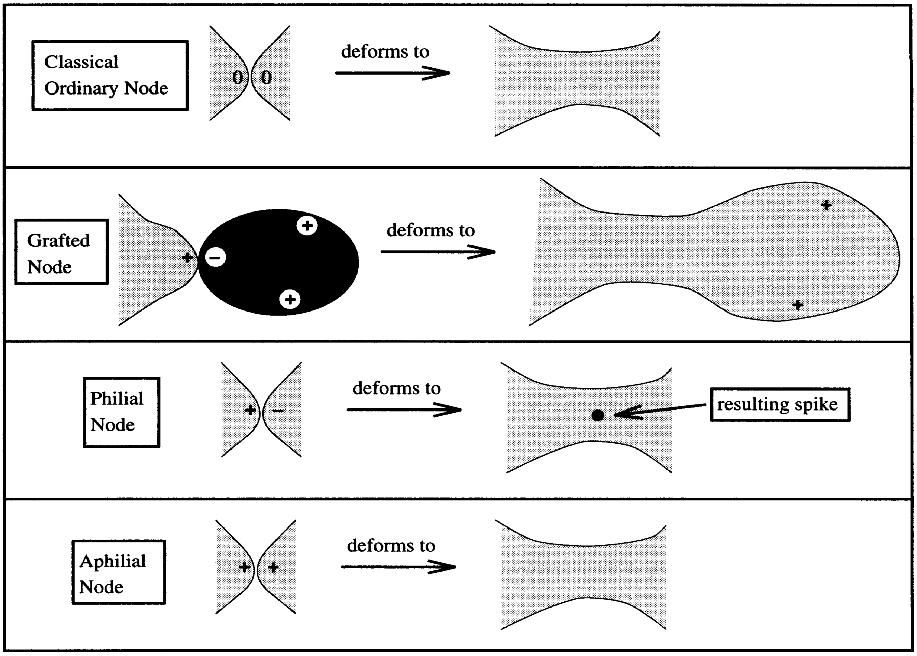

FIGURE 6. The Four Types of Nodal Deformation

$n^{\rm ord}_{0,5,p} = \frac{1}{2}(p^2+1)$; etc.). Moreover, we note the following two interesting phenomena:

(1) *Degrees such as* $n^{\rm ord}_{g,r,p}$ *tend to be well-behaved – even polynomial, with coefficients equal to various integrals over Euclidean space – as functions of* p. Thus, for instance, the limit, as p goes to infinity, of $n^{\rm ord}_{0,r,p}/p^{r-3}$ exists and is equal to the volume of a certain polyhedron in $(r-3)$-dimensional Euclidean space. See Corollary 1.3 of [**22**], Chapter V for more details.

(2) Theorem 2.4 gives, for every choice of totally degenerate r-pointed stable curve of genus g, an (a priori) distinct algorithm for computing $n^{\rm ord}_{g,r,p}$. Since $n^{\rm ord}_{g,r,p}$, of course, does not depend on the choice of underlying totally degenerate curve, we thus obtain equalities between the various sums that occur (to compute $n^{\rm ord}_{g,r,p}$) in each case. If one writes out these equalities, *one thus obtains various combinatorial identities.* Although the author has yet to achieve a systematic understanding of these combinatorial identities, already in the cases that have been computed (for low g and r), these identities reduce to such nontrivial combinatorial facts as Lemmas 3.5 and 3.6 of [**22**], Chapter V.

Although the author does not have even a conjectural theoretical understanding of these two phenomena, he nonetheless feels that they are very interesting and deserve further study.

2.2. The $\mathcal{MF}^\nabla$-Object Point of View. — Before discussing the general theory of canonical liftings of nilpotent indigenous bundles, it is worth stopping to examine the general conceptual context in which this theory will be developed. To do this, let us first recall the theory of *$\mathcal{MF}^\nabla$-objects* developed in §2 of [**6**]. Let p be a prime number, and let S be a smooth $\mathbb{Z}_p$-scheme. Then in *loc. cit.*, a certain category $\mathcal{MF}^\nabla(S)$ is defined. Objects of this category $\mathcal{MF}^\nabla(S)$ consist of: (1) *a vector bundle $\mathcal{E}$ on S equipped with an integrable connection $\nabla_\mathcal{E}$* (one may equivalently regard the pair $(\mathcal{E}, \nabla_\mathcal{E})$ as a crystal on the crystalline site Crys$(S \otimes_{\mathbb{Z}_p} \mathbf{F}_p/\mathbb{Z}_p)$ *valued in the category of vector bundles*); (2) *a filtration $F^\cdot(\mathcal{E}) \subseteq \mathcal{E}$ (called the Hodge filtration) of subbundles of $\mathcal{E}$*; (3) *a Frobenius action $\Phi_\mathcal{E}$ on the crystal* $(\mathcal{E}, \nabla_\mathcal{E})$. Moreover, these objects satisfy certain conditions, which we omit here.

Let Π_S be the fundamental group of $S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (for some choice of base-point). In *loc. cit.*, for each $\mathcal{MF}^\nabla(S)$-object $(\mathcal{E}, \nabla_\mathcal{E}, F^\cdot(\mathcal{E}), \Phi_\mathcal{E})$, a certain natural Π_S-module V is constructed by taking invariants of $(\mathcal{E}, \nabla_\mathcal{E})$ with respect to its Frobenius action $\Phi_\mathcal{E}$. If $\mathcal{E}$ is of rank r, then V is a free $\mathbb{Z}_p$-module of rank r. On typical example of such an $\mathcal{MF}^\nabla(S)$-object is the following:

> If $X \to S$ is the tautological abelian variety over the moduli stack of principally polarized abelian varieties, then the relative *first de Rham cohomology module* of $X \to S$ forms an $\mathcal{MF}^\nabla(S)$-module whose restriction to the ordinary locus of S is (by Serre-Tate theory) intimately connected to the "p-adic uniformization theory" of S.

In the context of [**22**], we would like to consider the case where $S = (\mathcal{M}_{g,r})_{\mathbb{Z}_p}$. Moreover, just as the first de Rham cohomology module of the universal abelian variety gives rise to a "fundamental uniformizing $\mathcal{MF}^\nabla(S)$-module" on the moduli stack of principally polarized abelian varieties, we would like to define and study a corresponding "fundamental uniformizing $\mathcal{MF}^\nabla$-object" on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$. Unfortunately, as long as one sticks to the conventional definition of $\mathcal{MF}^\nabla$-object given in [**6**], it appears that such a natural "fundamental uniformizing $\mathcal{MF}^\nabla$-object" simply does not exist on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$. This is not so surprising in view of the nonlinear nature of the Teichmüller group (i.e., the fundamental group of $(\mathcal{M}_{g,r})_\mathbb{C}$). In order to obtain a natural "fundamental uniformizing $\mathcal{MF}^\nabla$-object" on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$, one must generalize the "classical" linear notion of [**6**] as follows: Instead of considering crystals (equipped with filtrations and Frobenius actions) valued in the category of *vector bundles*, one must consider crystals (still equipped with filtrations and Frobenius actions in some appropriate sense) valued in the category of *schemes* (or more generally, *algebraic spaces*). Thus,

> *One philosophical point of view from which to view* [**22**] *is that it is devoted to the study of a certain canonical uniformizing $\mathcal{MF}^\nabla$-object on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$ valued in the category of algebraic spaces.*

Just as in the case of abelian varieties, this canonical uniformizing $\mathcal{MF}^{\nabla}$-object will be obtained by taking some sort of de Rham cohomology of the universal curve over $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$. The rest of this subsection is devoted to describing this $\mathcal{MF}^{\nabla}$-object in more detail.

Now let S be the spectrum of an algebraically closed field (of characteristic not equal to 2), and let X be a smooth, proper, geometrically curve over S of genus ≥ 2. Let $P \to X$ be a $\mathbb{P}^1$-bundle equipped with a connection ∇_P. If $\sigma : X \to P$ is a section of this $\mathbb{P}^1$-bundle, then we shall refer to the number $\frac{1}{2}\mathrm{deg}(\sigma^* \tau_{P/X})$ (where $\tau_{P/X}$ is the relative tangent bundle of P over X) as the *canonical height of* σ. Moreover, note that by differentiating σ by means of ∇_P, one obtains a morphism $\tau_{X/S} \to \sigma^* \tau_{P/X}$. We shall say that σ is *horizontal* if this morphism is identically zero.

(Roughly speaking) we shall call (P, ∇_P) *crys-stable* if it does not admit any horizontal sections of canonical height ≤ 0 (see Definition 1.2 of [**22**], Chapter I for a precise definition). (Roughly speaking) we shall call (P, ∇_P) *crys-stable of level* 0 (or just *stable*) if it does not admit any sections of canonical height ≤ 0 (see Definition 3.2 of [**22**], Chapter I for a precise definition). Let l be a positive half-integer (i.e., a positive element of $\frac{1}{2}\mathbb{Z}$). We shall call (P, ∇_P) *crys-stable of level* l if it admits a section of canonical height $-l$. If it does admit such a section, then this section is the unique section of $P \to X$ of negative canonical height. This section will be referred to as the *Hodge section* (see Definition 3.2 of [**22**], Chapter I for more details). For instance, if $\mathcal{E}$ is a vector bundle of rank two on X such that $\mathrm{Ad}(\mathcal{E})$ is a stable vector bundle on X (of rank three), and $P \to X$ is the projective bundle associated to $\mathcal{E}$, then (P, ∇_P) will be crys-stable of level 0 (regardless of the choice of ∇_P). On the other hand, an indigenous bundle on X will be crys-stable of level $g - 1$. More generally, these definitions generalize to the case when X is a family of pointed stable curves over an arbitrary base (on which 2 is invertible).

The nonlinear $\mathcal{MF}^{\nabla}$-object on $(\mathcal{M}_{g,r})_{\mathbb{Z}_p}$ (where p is odd) that is the topic of [**22**] is (roughly speaking) the crystal in algebraic spaces given by the considering the *fine moduli space* $\mathcal{Y} \to (\mathcal{M}_{g,r})_{\mathbb{Z}_p}$ *of crys-stable bundles* on the universal curve (cf. Theorem 2.7, Proposition 3.1 of [**22**], Chapter I for more details). Put another way, this crystal is a sort of de Rham-theoretic H^1 with coefficients in PGL_2 of the universal curve over $\mathcal{M}_{g,r}$. The nonlinear analogue of the Hodge filtration on an $\mathcal{MF}^{\nabla}$-object is the collection of subspaces given by the *fine moduli spaces* $\mathcal{Y}^l$ *of crys-stable bundles of level* l (for various l) – cf. [**22**], Chapter I, Proposition 3.3, Lemmas 3.4 and 3.8, and Theorem 3.10 for more details.

Remark. — This collection of subspaces is reminiscent of the stratification (on the moduli stack of smooth nilcurves) of §2.1. This is by no means a mere coincidence. In fact, in some sense, the stratification of $\mathcal{N}_{g,r}$ which was discussed in § 2.1 is the Frobenius conjugate of the Hodge structure mentioned above. That is to say, *the relationship between these two collections of subspaces is the nonlinear analogue of the* *relationship between the filtration on the de Rham cohomology of a variety in positive characteristic induced by the "conjugate spectral sequence" and the Hodge filtration on the cohomology.* (That is to say, the former filtration is the Frobenius conjugate of the latter filtration.)

Thus, to summarize, relative to the analogy between the nonlinear objects dealt with in this paper and the "classical" $\mathcal{MF}^{\nabla}$-objects of [**6**], the only other piece of data that we need is a *Frobenius action.* It is this issue of defining a natural Frobenius action which occupies the bulk of [**22**].

2.3. The Generalized Notion of a Frobenius Invariant Indigenous Bundle. — In this subsection, we would like to take up the task of describing the Frobenius action on crys-stable bundles. Just as in the case of the linear $\mathcal{MF}^{\nabla}$-objects of [**6**], and as motivated by comparison with the complex case (see the discussion of § 1), we are interested in *Frobenius invariant sections of the $\mathcal{MF}^{\nabla}$-object*, i.e., Frobenius invariant bundles. Moreover, since ultimately we are interested in *uniformization theory,* instead of studying general Frobenius invariant crys-stable bundles, we will only consider *Frobenius invariant indigenous bundles.* The reason that we must nonetheless introduce crys-stable bundles is that in order to obtain canonical lifting theories that are valid at generic points of $\mathcal{N}_{g,r}$ parametrizing *dormant* or *spiked* nilcurves, it is necessary to consider indigenous bundles that are fixed not (necessarily) after *one* application of Frobenius, but after *several* applications of Frobenius. As one applies Frobenius over and over again, the bundles that appear at intermediate stages need not be indigenous. They will, however, be crys-stable. This is why we must introduce crys-stable bundles.

In order to keep track of how the bundle transforms after various applications of Frobenius, it is necessary to introduce a certain combinatorial device called a *VF-pattern* (where "VF" stands for "Verschiebung/Frobenius"). VF-patterns may be described as follows. Fix nonnegative integers g, r such that $2g - 2 + r > 0$. Let $\chi \stackrel{\text{def}}{=} \frac{1}{2}(2g - 2 + r)$. Let $\mathcal{L}ev$ be the set of $l \in \frac{1}{2}\mathbb{Z}$ satisfying $0 \leq l \leq \chi$. We shall call $\mathcal{L}ev$ the *set of levels.* (That is, $\mathcal{L}ev$ is the set of possible levels of crys-stable bundles.) Let $\Pi : \mathbb{Z} \to \mathcal{L}ev$ be a map of sets, and let ϖ be a positive integer. Then we make the following definitions:

(i) We shall call (Π, ϖ) a *VF-pattern* if $\Pi(n + \varpi) = \Pi(n)$ for all $n \in \mathbb{Z}$; $\Pi(0) = \chi$; $\Pi(i) - \Pi(j) \in \mathbb{Z}$ for all $i, j \in \mathbb{Z}$ (cf. Definition 1.1 of [**22**], Chapter III).

(ii) A VF-pattern (Π, ϖ) will be called *pre-home* if $\Pi(\mathbb{Z}) = \{\chi\}$. A VF-pattern (Π, ϖ) will be called the *home VF-pattern* if it is pre-home and $\varpi = 1$.

(iii) A VF-pattern (Π, ϖ) will be called *binary* if $\Pi(\mathbb{Z}) \subseteq \{0, \chi\}$. A VF-pattern (Π, ϖ) will be called the *VF-pattern of pure tone ϖ* if $\Pi(n) = 0$ for all $n \in \mathbb{Z}$ not divisible by ϖ.

(iv) Let (Π, ϖ) be a VF-pattern. Then $i \in \mathbb{Z}$ will be called *indigenous (respectively, active; dormant) for this VF-pattern* if $\Pi(i) = \chi$ (respectively, $\Pi(i) \neq 0$; $\Pi(i) = 0$). If $i, j \in \mathbb{Z}$, and $i < j$, then (i, j) will be called *ind-adjacent for this VF-pattern* if $\Pi(i) = \Pi(j) = \chi$ and $\Pi(n) \neq \chi$ for all $n \in \mathbb{Z}$ such that $i < n < j$.

At the present time, all of this terminology may seem rather abstruse, but eventually, we shall see that it corresponds in a natural and evident way to the p-adic geometry defined by indigenous bundles that are Frobenius invariant in the fashion described by the VF-pattern in question. Finally, we remark that often, in order to simplify notation, we shall just write Π for the VF-pattern (even though, strictly speaking, a VF-pattern is a pair (Π, ϖ)).

Now fix an odd prime p. Let (Π, ϖ) be a *VF-pattern*. Let S be a perfect scheme of characteristic p. Let $X \to S$ be a smooth, proper, geometrically connected curve of genus $g \geq 2$. (Naturally, the theory goes through for arbitrary pointed stable curves, but for simplicity, we assume in the present discussion that the curve is smooth and without marked points.) Write $W(S)$ for the (ind-)scheme of Witt vectors with coefficients in S. Let $\mathcal{P}$ be a crystal on $\mathrm{Crys}(X/W(S))$ valued in the category of $\mathbb{P}^1$-bundles. Thus, the restriction $\mathcal{P}|_X$ of $\mathcal{P}$ to $\mathrm{Crys}(X/S)$ may be thought of as a $\mathbb{P}^1$-bundle with connection on the curve $X \to S$. Let us assume that $\mathcal{P}|_X$ defines an *indigenous bundle on* X. Now we consider the following procedure (cf. Fig. 7):

> Using the Hodge section of $\mathcal{P}|_X$, one can form the *renormalized Frobenius pull-back* $\mathcal{P}_1 \stackrel{\text{def}}{=} \mathbb{F}^*(\mathcal{P})$ of $\mathcal{P}$. Thus, $\mathbb{F}^*(\mathcal{P})$ will be a crystal valued in the category of $\mathbb{P}^1$-bundles on $\mathrm{Crys}(X/W(S))$. Assume that $\mathcal{P}_1|_X$ is crys-stable of level $\Pi(1)$. Then there are two possibilities: either $\Pi(1)$ is zero or nonzero. If $\Pi(1) = 0$, then let $\mathcal{P}_2$ be the usual (i.e., non-renormalized) Frobenius pull-back $\Phi^*\mathcal{P}_1$ of the crystal $\mathcal{P}_1$. If $\Pi(1) \neq 0$, then $\mathcal{P}_1|_X$ is crys-stable of positive level, hence admits a *Hodge section*; thus, using the Hodge section of $\mathcal{P}_1|_X$, we may form the renormalized Frobenius pull-back $\mathcal{P}_2 \stackrel{\text{def}}{=} \mathbb{F}^*(\mathcal{P}_1)$ of $\mathcal{P}_1$. Continuing inductively in this fashion – i.e., always assuming $\mathcal{P}_i|_X$ to be crys-stable of level $\Pi(i)$, and forming $\mathcal{P}_{i+1}$ by taking the renormalized (respectively, usual) Frobenius pull-back of $\mathcal{P}_i$ if $\Pi(i) \neq 0$ (respectively, $\Pi(i) = 0$), we obtain a sequence $\mathcal{P}_i$ of crystals on $\mathrm{Crys}(X/W(S))$ valued in the category of $\mathbb{P}^1$-bundles.

Then we make the following

Definition 2.5. — We shall refer to $\mathcal{P}$ as Π-*indigenous* (on X) if all the assumptions (on the $\mathcal{P}_i$) necessary to carry out the above procedure are satisfied, and, moreover, $\mathcal{P}_\varpi \cong \mathcal{P}$.

Thus, to say that $\mathcal{P}$ is Π-indigenous (more properly, (Π, ϖ)-indigenous) is to say that it is Frobenius invariant in the fashion specified by the combinatorial data (Π, ϖ).

Now we are ready to define a certain stack that is of central importance in [**22**]. The stack $\mathcal{Q}^{\Pi}$ – also called the *stack of quasi-analytic self-isogenies of type* (Π, ϖ) – is defined as follows:

> *To a perfect scheme S, $\mathcal{Q}^{\Pi}(S)$ assigns the category of pairs $(X \to S, \mathcal{P})$, where $X \to S$ is a curve as above and $\mathcal{P}$ is a Π-indigenous bundle on X.*

Thus, $\mathcal{Q}^{\Pi}$ *is may be regarded as the moduli stack of indigenous bundles that are Frobenius invariant in the fashion specified by the VF-pattern* Π.

We remark that in fact, more generally, one can define $\mathcal{Q}^{\Pi}$ on the category of *epiperfect schemes S*. (Whereas a perfect scheme is a scheme on which the Frobenius morphism is an isomorphism, an epiperfect scheme is one on which the Frobenius morphism is a closed immersion.) Then instead of using $W(S)$, one works over $B(S)$ – i.e., the "universal PD-thickening of S." For instance, the well-known ring B_{crys} introduced by Fontaine (and generalized to the higher-dimensional case in [**6**]) is a special case of $B(S)$. *The point is that one needs the base spaces that one works with to be $\mathbb{Z}_p$-flat and equipped with a natural Frobenius action.* The advantage of working with arbitrary $B(S)$ (for S epiperfect) is that the theory of crystalline representations (and the fact that B_{crys} is a special case of $B(S)$) suggest that $B(S)$ is likely to be the most general natural type of space having these two properties – i.e., $\mathbb{Z}_p$-flatness and being equipped with a natural Frobenius action. The disadvantage of working with arbitrary $B(S)$ (as opposed to just $W(S)$ for perfect S) is that many properties of $\mathcal{Q}^{\Pi}$ are technically more difficult or (at the present time impossible) to prove in the epiperfect case. For the sake of simplicity, in this Introduction, we shall only consider the perfect case. For more details, we refer to [**22**], Chapter VI.

Now, we are ready to discuss the main results concerning $\mathcal{Q}^{\Pi}$. The general theory of $\mathcal{Q}^{\Pi}$ is the topic of [**22**], Chapter VI. We begin with the following result (cf. Theorem 2.2 of [**22**], Chapter VI):

***Theorem 2.6* (Representability and Affineness).** — *The stack $\mathcal{Q}^{\Pi}$ is representable by a perfect algebraic stack whose associated coarse moduli space (as in* [**7**]*, Chapter 1, Theorem 4.10) is quasi-affine. If* Π *is pre-home, then this coarse moduli space is even affine.*

Thus, in the pre-home case, $\mathcal{Q}^{\Pi}$ is *perfect and affine*. In particular, any sort of de Rham/crystalline-type cohomology on $\mathcal{Q}^{\Pi}$ must vanish. It is for this reason that we say (in the pre-home case) that $\mathcal{Q}^{\Pi}$ is *crystalline contractible* (cf. Fig. 8). Moreover, (cf. Theorem 2.12 of [**22**], Chapter III),

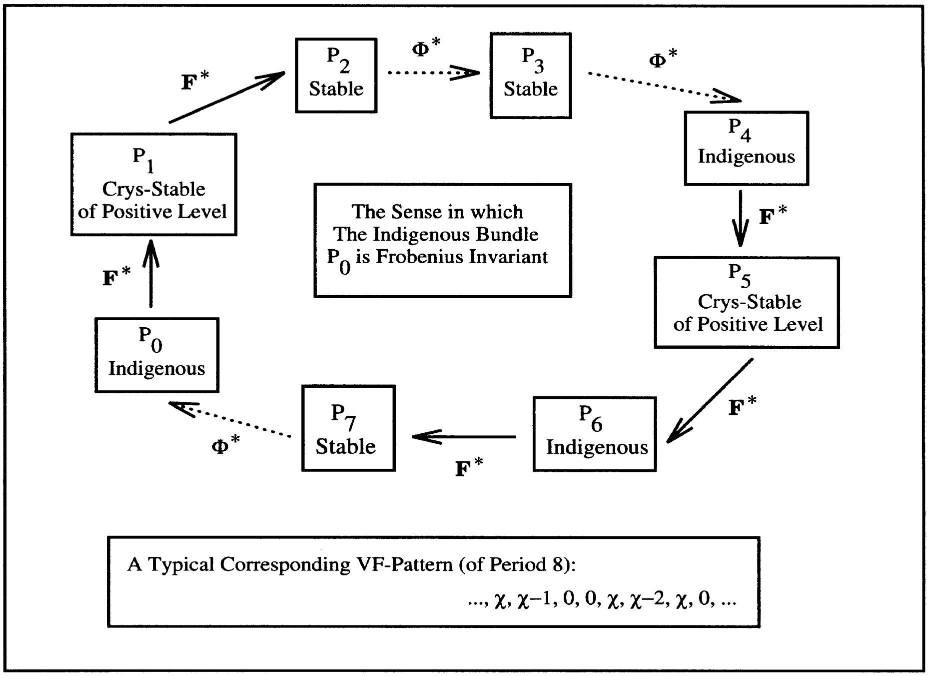

FIGURE 7. The Sense of Frobenius Invariance Specified by a VF-Pattern

***Corollary 2.7* (Irreducibility of Moduli).** — *(The fact that $\mathcal{Q}^{\Pi}$ is crystalline contractible for the home VF-pattern is intimately connected with the fact that) $\mathcal{M}_{g,r}$ is irreducible.*

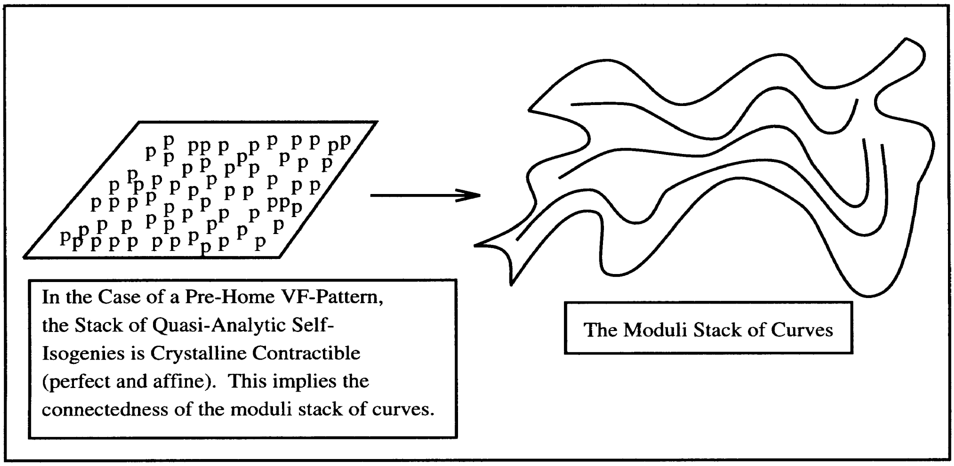

FIGURE 8. Crystalline Contractibility in the Pre-Home Case

The basic idea here is the following: By induction on g, it suffices to prove that $\mathcal{M}_{g,r}$ does not admit any *proper connected components*. But if it did admit such a component J, then one can apply the following analysis to $\mathcal{N}_J \stackrel{\text{def}}{=} \mathcal{N}_{g,r} \times_{\mathcal{M}_{g,r}} J$: First of all, by Theorem 1.1, $\mathcal{N}_J$ is finite and flat of degree p^{3g-3+r} over J. Now let I be an irreducible component of $\mathcal{N}_J$ for which the vanishing locus of the p-curvature of the nilcurve parametrized by the generic point of I is *maximal* (in other words, an irreducible component whose generic point lies in $\mathcal{N}_{g,r}[d]$, for d maximal). It is then a formal consequence of Theorems 1.1 and 2.2 that I is *smooth and proper* over $\mathbf{F}_p$, and that the whole of I (i.e., not just the generic point) lies in some $\mathcal{N}_{g,r}[d]$. Now we apply the fact that $\mathcal{N}_{g,r}[0]$ is *affine* (a fact which belongs to the same circle of ideas as Theorem 2.6). This implies (since I is proper and of positive dimension) that the d such that $I \subseteq \mathcal{N}_{g,r}[d]$ is *nonzero*. Thus, since (by **[21]**, Chapter II, Corollary 2.16) $\mathcal{N}_{g,r}$ is *nonreduced* at the generic point of $\mathcal{N}_{g,r}[d]$, it follows that the degree of I over J is $< p^{3g-3+r}$. On the other hand, by using the fact that the Schwarz torsor may also be interpreted as the Hodge-theoretic first Chern class of a certain ample line bundle (cf. **[21]**, Chapter I, § 3), it is a formal consequence (of basic facts concerning Chern classes in crystalline cohomology) that $\deg(I/J)$ (which is a positive integer) is divisible by p^{3g-3+r}. This contradiction (i.e., that $\deg(I/J)$ is a positive integer $< p^{3g-3+r}$ which is nevertheless divisible by p^{3g-3+r}) concludes the proof.

As remarked earlier, this derivation of the irreducibility of the moduli of $\mathcal{M}_{g,r}$ from the basic theorems of p-adic Teichmüller theory is reminiscent of the proof of the irreducibility of $\mathcal{M}_{g,r}$ given by using complex Teichmüller theory to show that Teichmüller space is contractible (cf., e.g., **[2, 4]**). Moreover, it is also interesting in that it suggests that perhaps at some future date the theory (or some extension of the theory) of **[22]** may be used to compute other cohomology groups of $\mathcal{M}_{g,r}$. Other proofs of the irreducibility of $\mathcal{M}_{g,r}$ include those of **[8, 9]**, but (at least as far the author knows) the proof given here is *the first that relies on essentially characteristic p methods* (i.e., "Frobenius").

Before proceeding, we must introduce some more notation. If Z is a smooth stack over $\mathbb{Z}_p$, let us write Z_W for the stack on the category of perfect schemes of characteristic p that assigns to a perfect S the category $Z(W(S))$. We shall refer to Z_W as the *infinite Weil restriction of* Z. It is easy to see that Z_W is representable by a perfect stack (Proposition 1.13 of **[22]**, Chapter VI). Moreover, this construction generalizes immediately to the logarithmic category. Write $\mathcal{M}_W$ (respectively, $\mathcal{S}_W$) for $((\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{Z}_p})_W$ (respectively, $((\overline{\mathcal{S}}^{\log}_{g,r})_{\mathbb{Z}_p})_W$). (Here $\overline{\mathcal{S}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ is the Schwarz torsor over $\overline{\mathcal{M}}_{g,r}$; we equip it with the log structure obtained by pulling back the log structure of $\overline{\mathcal{M}}^{\log}_{g,r}$.) Now if $\mathcal{P}$ is Π-indigenous on X, it follows immediately from the elementary theory of indigenous bundles that there exists a unique curve $X_W \to W(S)$ whose restriction to $S \subseteq W(S)$ is $X \to S$ and such that the restriction of the crystal $\mathcal{P}$ to X_W defines an indigenous bundle on X_W. The assignment $\mathcal{P} \mapsto (X_W \to W(S), \mathcal{P}|_{X_W})$

(respectively, $\mathcal{P} \mapsto \{X_W \to W(S)\}$) thus defines a natural morphism $\mathcal{Q}^{\Pi} \to \mathcal{S}_W$ (respectively, $\mathcal{Q}^{\Pi} \to \mathcal{M}_W$). Now we have the following results (cf. Propositions 2.3, 2.9; Corollaries 2.6 and 2.13 of [**22**], Chapter VI):

***Theorem 2.8* (Immersions).** — *The natural morphism $\mathcal{Q}^{\Pi} \to \mathcal{S}_W$ is an immersion in general, and a closed immersion if the VF-pattern is pre-home or of pure tone. The morphism $\mathcal{Q}^{\Pi} \to \mathcal{M}_W$ is a closed immersion if the VF-pattern is the home VF-pattern.*

***Theorem 2.9* (Isolatedness in the Pre-Home Case).** — *In the pre-home case, $\mathcal{Q}^{\Pi}$ is closed inside $\mathcal{S}_W$ and disjoint from the closure of any non-pre-home $\mathcal{Q}^{\Pi'}$'s.*

We remark that in both of these cases, much more general theorems are proved in [**22**]. Here, for the sake of simplicity, we just selected representative special cases of the main theorems in [**22**] so as to give the reader a general sense of the sorts of results proved in [**22**].

The reason that Theorem 2.9 is interesting (or perhaps a bit surprising) is the following: The reduction modulo p of a Π-indigenous bundle (in the pre-home case) is an *admissible* nilpotent indigenous bundle. (Here, the term "admissible" means that the p-curvature has no zeroes.) Moreover, the admissible locus $\overline{\mathcal{N}}_{g,r}^{\mathrm{adm}}$ of $\overline{\mathcal{N}}_{g,r}$ is by no means closed in $\overline{\mathcal{N}}_{g,r}$, nor is its closure disjoint (in general) from the closure of the dormant or spiked loci of $\overline{\mathcal{N}}_{g,r}$. On the other hand, the reductions modulo p of Π'-indigenous bundles (for non-pre-home Π') may, in general, be dormant or spiked nilpotent indigenous bundles. Thus,

> *Theorem 2.9 states that considering $\mathbb{Z}_p$-flat Frobenius invariant liftings of indigenous bundles (as opposed to just nilpotent indigenous bundles in characteristic p) has the effect of "blowing up" $\overline{\mathcal{N}}_{g,r}$ in such a way that the genericization/specialization relations that hold in $\overline{\mathcal{N}}_{g,r}$ do not imply such relations among the various $\mathcal{Q}$'s.*

We shall come back to this phenomenon again in the following subsection (cf. Fig. 9).

2.4. The Generalized Ordinary Theory. — In this subsection, we maintain the notations of the preceding subsection. Unfortunately, it is difficult to say much more about the explicit structure of the stacks $\mathcal{Q}^{\Pi}$ without making more assumptions. Thus, just as in the classical ordinary case (reviewed in § 1.6), it is natural to define an open substack – the *ordinary locus of* $\mathcal{Q}^{\Pi}$ – and to see if more explicit things can be said concerning this open substack. This is the topic of [**22**], Chapter VII. We shall see below that in fact much that is interesting can be said concerning this ordinary locus.

We begin with the definition of the ordinary locus. First of all, we observe that there is a natural algebraic stack

$$\overline{\mathcal{N}}_{g,r}^{\Pi,s}$$

(of finite type over $\mathbf{F}_p$) that parametrizes "data modulo p for $\mathcal{Q}^{\Pi}$" (Definition 1.11 of [**22**], Chapter III). That is to say, roughly speaking, $\overline{\mathcal{N}}_{g,r}^{\Pi,s}$ parametrizes the reductions modulo p of the $\mathcal{P}_i$ appearing in the discussion preceding Definition 2.5. We refer to [**22**], Chapter III for a precise definition of this stack. At any rate, by reducing modulo p the data parametrized by $\mathcal{Q}^{\Pi}$, we obtain a natural morphism of stacks

$$\mathcal{Q}^{\Pi} \to \overline{\mathcal{N}}_{g,r}^{\Pi,s}$$

On the other hand, since $\overline{\mathcal{N}}_{g,r}^{\Pi,s}$ parametrizes curves equipped with certain bundles, there is a natural morphism $\overline{\mathcal{N}}_{g,r}^{\Pi,s} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$. Let $\mathcal{N}^{\mathrm{ord}} \subseteq \overline{\mathcal{N}}_{g,r}^{\Pi,s}$ denote the open substack over which the morphism $\overline{\mathcal{N}}_{g,r}^{\Pi,s} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is *étale*. Let $\mathcal{Q}^{\mathrm{ord}} \subseteq \mathcal{Q}^{\Pi}$ denote the open substack which is the inverse image of $\mathcal{N}^{\mathrm{ord}} \subseteq \overline{\mathcal{N}}_{g,r}^{\Pi,s}$.

Definition 2.10. — We shall refer to $\mathcal{Q}^{\mathrm{ord}}$ as the *(Π-)ordinary locus of* $\mathcal{Q}^{\Pi}$.

Just as in the classical ordinary case, there is an equivalent definition of Π-ordinarity given by looking at the action of Frobenius on the first de Rham cohomology modules of the $\mathcal{P}_i$ (cf. Lemma 1.4 of [**22**], Chapter VII). Incidentally, the classical ordinary theory corresponds to the Π-ordinary theory in the case of the home VF-pattern. (In particular, $\mathcal{N}^{\mathrm{ord}}$ is simply the ordinary locus $\overline{\mathcal{N}}_{g,r}^{\mathrm{ord}}$ of $\overline{\mathcal{N}}_{g,r}$.) Thus, in some sense, the theory of [**21**] is a special case of the generalized ordinary theory.

Our first result is the following (cf. Theorem 1.6 of [**22**], Chapter VII):

***Theorem 2.11* (Basic Structure of the Ordinary Locus)**. — $\mathcal{Q}^{\mathrm{ord}}$ *is naturally isomorphic to the perfection of* $\mathcal{N}^{\mathrm{ord}}$.

Thus, already one has a much more explicit understanding of the structure of $\mathcal{Q}^{\mathrm{ord}}$ than of the whole of $\mathcal{Q}^{\Pi}$. That is to say, Theorem 2.11 already tells us that $\mathcal{Q}^{\mathrm{ord}}$ is the perfection of a smooth algebraic stack of finite type over $\mathbf{F}_p$.

Our next result – which is somewhat deeper than Theorem 2.11, and is, in fact, one of the main results of [**22**] – is the following (cf. Theorem 2.11 of [**22**], Chapter VII):

***Theorem 2.12* (ω-Closedness of the Ordinary Locus)**. — *If* Π *is* binary, *then* $\mathcal{Q}^{\mathrm{ord}}$ *is* ω-closed *(roughly speaking, "closed as far as the differentials are concerned" – cf.* [**22**], *Chapter VII, § 0, § 2.3 for more details) in* $\mathcal{Q}^{\Pi}$. *In particular,*

(1) *If* $3g-3+r=1$, *then* $\mathcal{Q}^{\mathrm{ord}}$ *is actually* closed *in* $\mathcal{Q}^{\Pi}$.

(2) *If* $\mathcal{R} \subseteq \mathcal{Q}^{\Pi}$ *is a subobject containing* $\mathcal{Q}^{\mathrm{ord}}$ *and which is "pro" (cf.* [**22**], *Chapter VI, Definition 1.9) of a fine algebraic log stack which is locally of finite type over* $\mathbf{F}_p$, *then* $\mathcal{Q}^{\mathrm{ord}}$ *is* closed *in* $\mathcal{R}$.

In other words, at least among perfections of fine algebraic log stacks which are locally of finite type over $\mathbf{F}_p$, $\mathcal{Q}^{\mathrm{ord}}$ *is already "complete" inside* $\mathcal{Q}^{\Pi}$.

Thus, if Π *is pre-home or of pure tone, then* $\mathcal{Q}^{\mathrm{ord}}$ *is an* ω*-closed substack of* $\mathcal{S}_W$*. If the VF-pattern in question is the home pattern, then* $\mathcal{Q}^{\mathrm{ord}}$ *is an* ω*-closed substack of* $\mathcal{M}_W$*.*

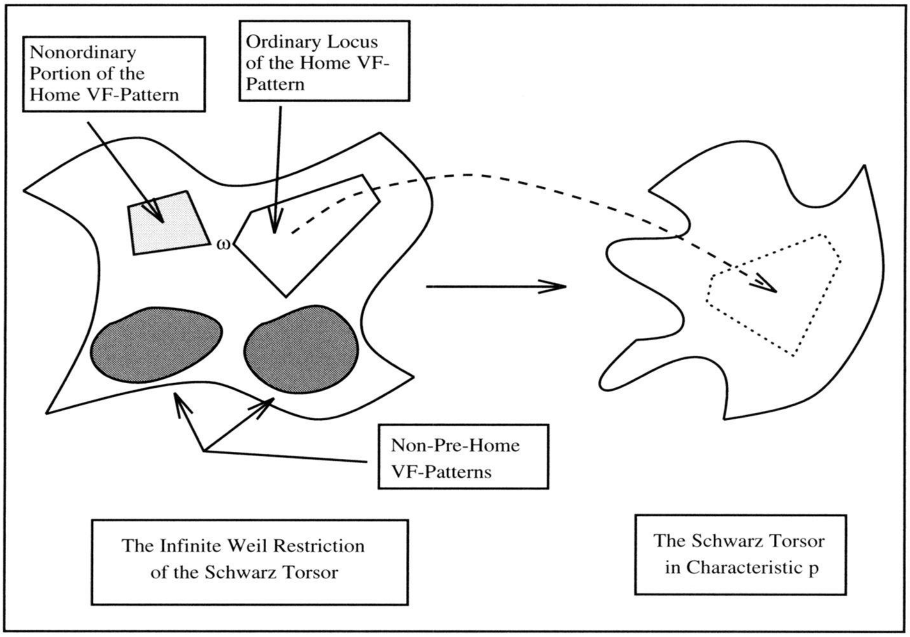

FIGURE 9. The ω-Closedness and Isolatedness of the Classical Ordinary Theory

This is a rather surprising result in that the definition of $\mathcal{Q}^{\mathrm{ord}}$ was such that $\mathcal{Q}^{\mathrm{ord}}$ is naturally an open substack of $\mathcal{Q}^{\Pi}$ which has no a priori reason to be closed (in any sense!) inside $\mathcal{Q}^{\Pi}$. Moreover, $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r}$ is most definitely *not* closed in $\overline{\mathcal{N}}_{g,r}$. Indeed, one of the original motivations for trying to generalize the theory of [**21**] was to try to compactify it. Thus, Theorem 2.12 states that if, instead of just considering ordinary nilpotent indigenous bundles modulo p, one considers $\mathbb{Z}_p$-flat Frobenius invariant indigenous bundles, the theory of [**21**] is, in some sense, already compact! Put another way, if one thinks in terms of the Witt vectors parametrizing such $\mathbb{Z}_p$-flat Frobenius invariant indigenous bundles, then although the scheme defined by the *first* component of the Witt vector is not "compact," if one considers *all* the components of the Witt vector, the resulting scheme is, in some sense, "compact" (i.e., ω-closed in the space $\mathcal{S}_W$ of all indigenous bundles over the Witt vectors). This phenomenon is similar to the phenomenon observed in Theorem 2.9. In fact, if one combines Theorem 2.9 with Theorem 2.12, one obtains that:

In the home (i.e., classical ordinary) case, the stack $\mathcal{Q}^{\mathrm{ord}}$ is ω-closed in $\mathcal{S}_W$ and disjoint from the closures of all $\mathcal{Q}^{\Pi'}$ for all non-pre-home Π'. Moreover, $\mathcal{Q}^{\mathrm{ord}}$ is naturally an ω-closed substack of $\mathcal{Q}^{\Pi'}$ for all pre-home Π'.

This fact is rendered in pictorial form in Fig. 9; cf. also the discussion of § 3 below.

The next main result of the generalized ordinary theory is the generalized ordinary version of Theorem 1.2. First, let us observe that since the natural morphism $\mathcal{N}^{\mathrm{ord}} \to (\overline{\mathcal{M}}_{g,r})_{\mathbf{F}_p}$ is *étale*, it admits a unique lifting to an étale morphism

$$\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p} \to (\overline{\mathcal{M}}_{g,r})_{\mathbb{Z}_p}$$

of smooth p-adic formal stacks over $\mathbb{Z}_p$. Unlike in the classical ordinary case, however, where one obtains a single canonical modular Frobenius lifting, in the generalized case, one obtains a whole *system of Frobenius liftings* (cf. Theorem 1.8 of [**22**], Chapter VII) on $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$:

***Theorem 2.13* (Canonical System of Frobenius Liftings).** — *Over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$, there is a canonical system of Frobenius liftings and indigenous bundles: i.e., for each indigenous i (i.e., such that $\Pi(i) = \chi$), a lifting*

$$\Phi_i^{log} : \mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p} \to \mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$$

of a certain power of the Frobenius morphism, together with a collection of indigenous bundles $\mathcal{P}_i$ on the tautological curve (pulled back from $(\overline{\mathcal{M}}_{g,r})_{\mathbb{Z}_p}$) over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. Moreover, these Frobenius liftings and bundles are compatible, *in a natural sense (Definition 1.7 of* [**22**], *Chapter VII).*

See Fig. 10 for an illustration of the system of Frobenius liftings obtained for the VF-pattern illustrated in Fig. 7.

At this point, one very important question arises:

To what extent are the stacks $\mathcal{N}^{\mathrm{ord}}$ nonempty?

Needless to say, this is a very important issue, for if the $\mathcal{N}^{\mathrm{ord}}$ are empty most of the time, then the above theory is meaningless. In the classical ordinary case, it was rather trivial to show the nonemptiness of $\overline{\mathcal{N}}^{\mathrm{ord}}_{g,r}$. In the present generalized ordinary setting, however, it is much more difficult to show the nonemptiness of $\mathcal{N}^{\mathrm{ord}}$. In particular, one needs to make use of the extensive theory of [**22**], Chapters II and IV. Fortunately, however, one can show the nonemptiness of $\mathcal{N}^{\mathrm{ord}}$ in a fairly wide variety of cases (Theorems 3.1 and 3.7 of [**22**], Chapter VII):

***Theorem 2.14* (Binary Existence Result).** — *Suppose that $g \geq 2$; $r = 0$; and $p > 4^{3g-3}$. Then for any* binary *VF-pattern (i.e., VF-pattern such that $\Pi(\mathbb{Z}) \subseteq \{0, \chi\}$), the stack $\mathcal{N}^{\mathrm{ord}}$ is nonempty.*

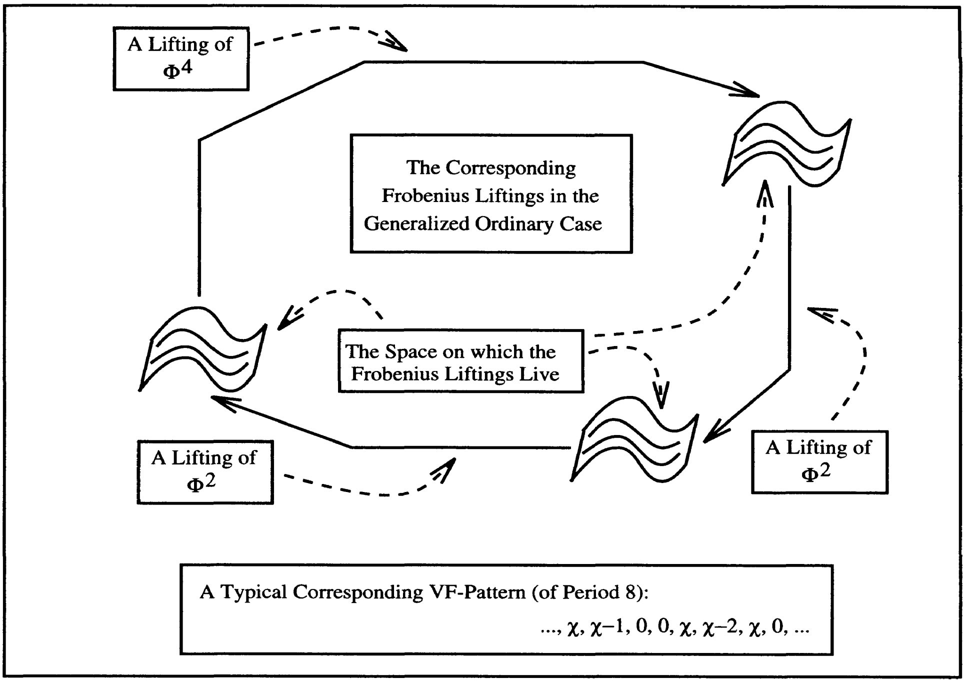

FIGURE 10. The Canonical System of Modular Frobenius Liftings

***Theorem 2.15* (Spiked Existence Result).** — *Suppose that $2g - 2 + r \geq 3$ and $p \geq 5$. Then there exists a "spiked VF-pattern" of period 2 (i.e., $\varpi = 2$ and $0 < \Pi(1) < \chi$) for which $\mathcal{N}^{\mathrm{ord}}$ is nonempty.*

In fact, there is an open substack of $\mathcal{N}^{\mathrm{ord}}$ called the very ordinary locus *(defined by more stringent conditions than ordinarity); moreover, one can choose the spiked VF-pattern so that not only $\mathcal{N}^{\mathrm{ord}}$, but also the "very ordinary locus of $\mathcal{N}^{\mathrm{ord}}$" is nonempty.*

These cases are "fairly representative" in the following sense: In general, in the binary case, the reduction modulo p of a Π-indigenous bundle will be *dormant.* In the spiked case (of Theorem 2.15), the reduction modulo p of a Π-indigenous bundle will be *spiked.* Thus, in other words,

> *Roughly speaking, these two existence results show that for each type (admissible, dormant, spiked) of nilcurve, there exists a theory (in fact, many theories) of canonical liftings involving that type of nilcurve.*

Showing the existence of such a theory of canonical liftings for *each generic point of* $\overline{\mathcal{N}}_{g,r}$ was one of the original motivations for the development of the theory of **[22]**.

Next, we observe that just as in Theorem 1.2 (the classical ordinary case), *In the cases discussed in Theorems 2.14 and 2.15, one can also construct canonical systems of Frobenius liftings on certain "ordinary loci" of the universal curve over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. Moreover, these systems of canonical Frobenius lifting lie over the canonical system of modular Frobenius liftings of Theorem 2.13.*

We refer to Theorem 3.2 of [**22**], Chapter VIII and Theorem 3.4 of [**22**], Chapter IX for more details.

We end this subsection with a certain philosophical observation. In [**22**], Chapter VI,

The stack $\mathcal{Q}^{\Pi}$ is referred to as the stack of quasi-analytic self-isogenies.

That is to say, in some sense it is natural to regard the Frobenius invariant indigenous bundles parametrized by $\mathcal{Q}^{\Pi}$ as *isogenies* of the curve (on which the bundles are defined) onto itself. Indeed, this is suggested by the fact that over the ordinary locus (i.e., relative to the Frobenius invariant indigenous bundle in question) of the curve, the bundle actually does define a literal morphism, i.e., a Frobenius lifting (as discussed in the preceding paragraph). Thus, one may regard a Frobenius invariant indigenous bundle as the appropriate way of compactifying such a self-isogeny to an object defined over the whole curve. This is why we use the adjective "quasi-analytic" in describing the self-isogenies. (Of course, such self-isogenies can never be p-adic analytic over the whole curve, for if they were, they would be algebraic, which, by the Riemann-Hurwitz formula, is absurd.) Note that this point of view is in harmony with the situation in the parabolic case ($g = 1$, $r = 0$), where there is an algebraically defined canonical choice of indigenous bundle, and having a Frobenius invariant indigenous bundle really does correspond to having a lifting of Frobenius (hence a self-isogeny of the curve in question).

Moreover, note that in the case where the VF-pattern has several $\chi = \frac{1}{2}(2g-2+r)$'s in a period, so that there are various indigenous $\mathcal{P}_i$'s in addition to the original Frobenius invariant indigenous bundle $\mathcal{P}$, one may regard the situation as follows. Suppose that $\mathcal{P}$ is indigenous over a curve $X \to W(S)$, whereas $\mathcal{P}_i$ is indigenous over $X_i \to W(S)$. Then one can regard the "quasi-analytic self-isogeny" $\mathcal{P} : X \to X$ as the composite of various quasi-analytic isogenies $\mathcal{P}_i : X_i \to X_j$ (where i and j are "ind-adjacent" integers). Note that this point of view is consistent with what literally occurs over the ordinary locus (cf. Theorem 3.2 of [**22**], Chapter VIII). Finally, we observe that

The idea that $\mathcal{Q}^{\Pi}$ is a moduli space of some sort of p-adic self-isogeny which is "quasi-analytic" is also compatible with the analogy between $\mathcal{Q}^{\Pi}$ and Teichmüller space (cf. the discussion of Corollary 2.7) in that Teichmüller space may be regarded as a moduli space of quasiconformal maps (cf., e.g., [**2**]*).*

2.5. Geometrization. — In the classical ordinary case, once one knows the existence of the canonical modular Frobenius lifting (Theorem 1.2), one can apply a general result on ordinary Frobenius liftings (Theorem 1.3) to conclude the existence of canonical multiplicative coordinates on $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. We shall refer to this process of passing (as in Theorem 1.3) from a certain type of Frobenius lifting to a local uniformization/canonical local coordinates associated to the Frobenius lifting as the *geometrization of the Frobenius lifting.* In the generalized ordinary context, Theorem 2.13 shows the existence of a canonical *system of Frobenius liftings* on the $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$ associated to a VF-pattern (Π, ϖ). Thus, the following question naturally arises:

> *Can one geometrize the sort of system of Frobenius liftings that one obtains in Theorem 2.13 in a fashion analogous to the way in which ordinary Frobenius liftings were geometrized in Theorem 1.3?*

Unfortunately, we are not able to answer this question in general. Nevertheless, in the cases discussed in Theorems 2.14 and 2.15, i.e., the binary and very ordinary spiked cases, we succeed (in [**22**], Chapters VIII and IX) in geometrizing the canonical system of modular Frobenius liftings. The result is *uniformizations/geometries based not on* $\widehat{\mathbf{G}}_{\mathrm{m}}$ *as in the classical ordinary case, but rather on more general types of Lubin-Tate groups, twisted products of Lubin-Tate groups, and fibrations whose bases are Lubin-Tate groups and whose fibers are such twisted products.* In the rest of this subsection, we would like to try to give the reader an idea of what sorts of geometries occur in the two cases studied.

In the following, we let k be a perfect field of characteristic p, A its ring of Witt vectors $W(k)$, and S a smooth p-adic formal scheme over A. Let λ be a positive integer, and let $\mathcal{O}_\lambda \stackrel{\mathrm{def}}{=} W(\mathbf{F}_{p^\lambda})$. For simplicity, we assume that $\mathcal{O}_\lambda \subseteq A$. Let $\mathcal{G}_\lambda$ be the *Lubin-Tate formal group associated to* $\mathcal{O}_\lambda$. (See [**3**] for a discussion of Lubin-Tate formal groups.) Then $\mathcal{G}_\lambda$ is a formal group over $\mathcal{O}_\lambda$, equipped with a natural action by $\mathcal{O}_\lambda$ (i.e., a ring morphism $\mathcal{O}_\lambda \hookrightarrow \mathrm{End}_{\mathcal{O}_\lambda}(G_\lambda)$). Moreover, it is known that the space of invariant differentials on $\mathcal{G}_\lambda$ is canonically isomorphic to $\mathcal{O}_\lambda$. Thus, in the following, we shall identify this space of differentials with $\mathcal{O}_\lambda$.

We begin with the simplest case, namely, that of a *Lubin-Tate Frobenius lifting.* Let $\Phi : S \to S$ be a morphism whose reduction modulo p is the λ^{th} power of the Frobenius morphism. Then differentiating Φ_S defines a morphism $\mathrm{d}\Phi_S : \Phi_S^*\Omega_{S/A} \to \Omega_{S/A}$ which is zero in characteristic p. Thus, we may form a morphism

$$\Omega_\Phi : \Phi_S^*\Omega_{S/A} \to \Omega_{S/A}$$

by dividing $\mathrm{d}\Phi_S$ by p. Then Φ_S is called a *Lubin-Tate Frobenius lifting (of order λ)* if Ω_Φ is an isomorphism. If Φ_S is a Lubin-Tate Frobenius lifting, then it induces a "Lubin-Tate geometry" – i.e., a geometry based on $\mathcal{G}_\lambda$ – on S. That is to say, one has the following analogue of Theorem 1.3 (cf. Theorem 2.17 of [**22**], Chapter VIII):

***Theorem 2.16* (Lubin-Tate Frobenius Liftings).** — *Let $\Phi_S : S \to S$ be a Lubin-Tate Frobenius lifting of order λ. Then taking the invariants of $\Omega_{S/A}$ with respect to Ω_Φ gives rise to an étale local system $\Omega_\Phi^{\mathrm{et}}$ on S of free $\mathcal{O}_\lambda$-modules of rank equal to* $\dim_A(S)$.

Let $z \in S(\overline{k})$ be a point valued in the algebraic closure of k. Then $\Omega_z \stackrel{\mathrm{def}}{=} \Omega_\Phi^{\mathrm{et}}|_z$ may be thought of as a free $\mathcal{O}_\lambda$-module of rank $\dim_A(S)$; write Θ_z for the $\mathcal{O}_\lambda$-dual of Ω_z. Let S_z be the completion of S at z. Then there is a unique isomorphism

$$\Gamma_z : S_z \cong \mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$$

such that:

(i) *the derivative of Γ_z induces the natural inclusion $\Omega_z \hookrightarrow \Omega_{S/A}|_{S_z}$;*

(ii) *the action of Φ_S on S_z corresponds to multiplication by p on $\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$.*

Here, by "$\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$," we mean the tensor product over $\mathcal{O}_\lambda$ of (formal) group schemes with $\mathcal{O}_\lambda$-action. Thus, $\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$ is noncanonically isomorphic to the product of $\dim_A(S) = \mathrm{rank}_{\mathcal{O}_\lambda}(\Theta_z)$ *copies of $\mathcal{G}_\lambda$.*

Of course, this result has nothing to do with the moduli of curves. In terms of VF-patterns, Theorem 2.13 gives rise to a Lubin-Tate Frobenius lifting of order ϖ when the VF-pattern is of pure tone ϖ.

The next simplest case is the case of an *anabelian system of Frobenius liftings.* Let n be a positive integer. Then an *anabelian system of Frobenius liftings of length n and order λ* is a collection of n Lubin-Tate Frobenius liftings

$$\Phi_1, \ldots, \Phi_n : S \to S$$

each of order λ. Of course, in general such Frobenius liftings will not commute with one another. In fact, it can be shown that two Lubin-Tate Frobenius liftings of order λ commute with each other if and only if they are equal (Lemma 2.24 of [**22**], Chapter VIII). This is the reason for the term "anabelian." Historically, this term has been used mainly in connection with Grothendieck's Conjecture of Anabelian Geometry ([**12**]). The reason why we thought it appropriate to use the term here (despite the fact that anabelian geometries as discussed here have nothing to do with the Grothendieck Conjecture) is the following: (Just as for the *noncommutative* fundamental groups of Grothendieck's anabelian geometry) *the sort of noncommutativity that occurs among the Φ_i's* (at least in the modular case – cf. Theorem 2.13) *arises precisely as a result of the hyperbolicity of the curves* on whose moduli the Φ_i's act.

Let $\delta_i \stackrel{\mathrm{def}}{=} \frac{1}{p}\mathrm{d}\Phi_i$. Let $\Delta \stackrel{\mathrm{def}}{=} \delta_n \circ \cdots \circ \delta_1$. Then taking invariants of $\Omega_{S/A}$ with respect to Δ gives rise to an étale local system $\Omega_\Phi^{\mathrm{et}}$ on S in free $\mathcal{O}_{n\lambda}$-modules of rank $\dim_A(S)$. Next let S_{PD} denote the p-adic completion of the PD-envelope of the diagonal in the product (over A) of n copies of S; let S_{FM} denote the p-adic completion of the completion at the diagonal of the product (over A) of n copies of S. Thus, we have a natural morphism

$$S_{\mathrm{PD}} \to S_{\mathrm{FM}}$$

Moreover, one may think of S_{PD} as a sort of *localization* of S_{FM}. Write $\Phi_{\mathrm{PD}} : S_{\mathrm{PD}} \to S_{\mathrm{PD}}$ for the morphism induced by sending

$$(s_1, \ldots, s_n) \mapsto (\Phi_1(s_2), \Phi_2(s_3), \ldots, \Phi_n(s_1))$$

(where $(s_1, \ldots, s_n)$ represents a point in the product of n copies of S). Then we have the following result (cf. Theorem 2.17 of [**22**], Chapter VIII):

***Theorem 2.17* (Anabelian System of Frobenius Liftings).** — *Let $\Phi_1, \ldots, \Phi_n : S \to S$ be a system of anabelian Frobenius liftings of length n and order λ. Let $z \in S(\overline{k})$ be a point valued in the algebraic closure of k. Then $\Omega_z \stackrel{\mathrm{def}}{=} \Omega^{\mathrm{et}}_{\Phi}|_z$ may be thought of as a free $\mathcal{O}_{n\lambda}$-module of rank* $\dim_A(S)$*; write Θ_z for the $\mathcal{O}_{n\lambda}$-dual of Ω_z. Let $(S_{\mathrm{PD}})_z$ be the completion of S_{PD} at z. Then there is a unique morphism*

$$\Gamma_z : (S_{\mathrm{PD}})_z \to \mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$$

such that:

(i) *the derivative of Γ_z induces a certain (see Theorem 2.15 of* [**22**]*, Chapter VIII for more details) natural inclusion of Ω_z into the restriction to $(S_{\mathrm{PD}})|_z$ of the differentials of $\prod_{i=1}^n S$ over A;*

(ii) *the action of Φ_{PD} on $(S_{\mathrm{PD}})_z$ is compatible with multiplication by p on $\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$.*

Here, by "$\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$," we mean the tensor product over $\mathcal{O}_\lambda$ of (formal) group schemes with $\mathcal{O}_\lambda$-action. Thus, $\mathcal{G}_\lambda \otimes^{\mathrm{gp}}_{\mathcal{O}_\lambda} \Theta_z$ is noncanonically isomorphic to the product of $n \cdot \dim_A(S) = \mathrm{rank}_{\mathcal{O}_\lambda}(\Theta_z)$ *copies of $\mathcal{G}_\lambda$.*

Moreover, in general, Γ_z does not descend to $(S_{\mathrm{FM}})_z$ (cf. [**22**], Chapter VIII, § 2.6, 3.1).

One way to envision anabelian geometries is as follows: The various Φ_i's induce various linear Lubin-Tate geometries on the space S that (in general) do not commute with one another. Thus, *the anabelian geometry consists of various linear geometries on S all tangled up inside each other.* If one localizes in a sufficiently drastic fashion – i.e., all the way to $(S_{\mathrm{PD}})_z$ – then one can untangle these tangled up linear geometries into a single $\mathcal{O}_{n\lambda}$-linear geometry (via Γ_z). However, the order λ Lubin-Tate geometries are so tangled up that even localization to a relatively localized object such as $(S_{\mathrm{FM}})_z$ is not sufficient to untangle these geometries.

Finally, to make the connection with Theorem 2.13, we remark that the system of Theorem 2.13 gives rise to an anabelian system of length n and order λ in the case of a VF-pattern (Π, ϖ) for which $\varpi = n \cdot \lambda$, and $\Pi(i) = \chi$ (respectively, $\Pi(i) = 0$) if and only if i is divisible (respectively, not divisible) by λ.

In fact, both Lubin-Tate geometries and anabelian geometries are special cases of *binary ordinary geometries* (the sorts of geometries that occur for *binary* VF-patterns, i.e., Π whose image $\subseteq \{0, \chi\}$). A general geometrization result for binary ordinary geometries is given in Theorem 2.17 of [**22**], Chapter VIII. Here, we chose to concentrate on the Lubin-Tate and anabelian cases (in fact, of course, Lubin-Tate geometries are a special case of anabelian geometries) since they are relatively representative and relatively easy to envision.

The other main type of geometry that is studied in [**22**] is the geometry associated to a *very ordinary spiked Frobenius lifting* $\Phi : S \to S$. Such a Frobenius lifting reduces modulo p to the square of the Frobenius morphism and satisfies various other properties which we omit here (see Definition 1.1 of [**22**], Chapter IX for more details). In particular, such a Frobenius lifting comes equipped with an invariant called the *colevel*. The colevel is a nonnegative integer c. Roughly speaking,

> *A very ordinary spiked Frobenius lifting is a Frobenius lifting which is "part Lubin-Tate of order* 2*" and "part anabelian of length* 2 *and order* 1*."*

The colevel c is the number of dimensions of S on which Φ is Lubin-Tate of order 2. The main geometrization theorem (roughly stated) on this sort of Frobenius lifting is as follows (cf. Theorems 1.5 and 2.3 of [**22**], Chapter IX):

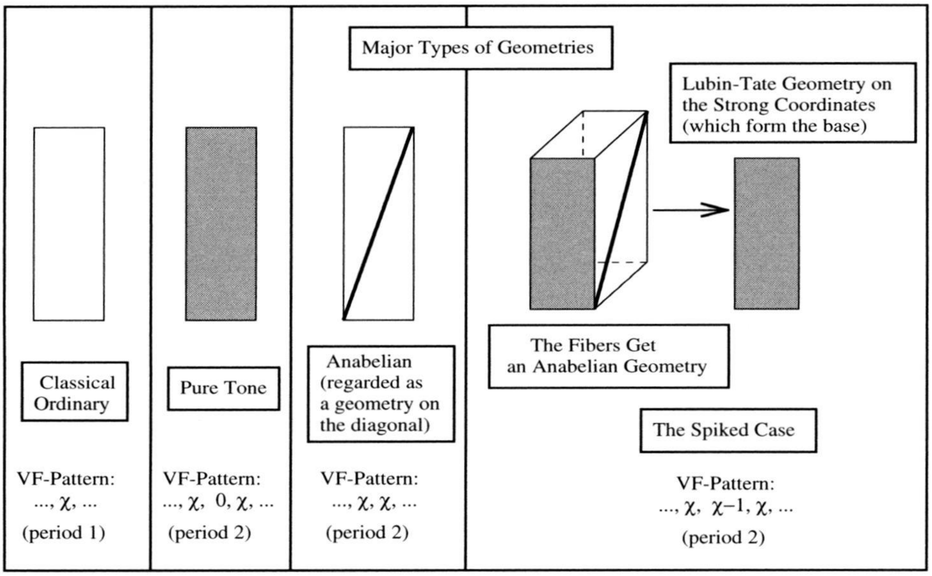

FIGURE 11. Major Types of p-adic Geometries

***Theorem 2.18* (Very Ordinary Spiked Frobenius Liftings).** — *Let $\Phi : S \to S$ be a very ordinary spiked Frobenius lifting of colevel c. Then Φ defines an étale local system $\Omega^{\mathrm{st}}_{\Phi}$ on S of free $\mathcal{O}_2$-modules of rank c equipped with a natural inclusion $\Omega^{\mathrm{st}}_{\Phi} \hookrightarrow \Omega_{S/A}$.*

Let $z \in S(\overline{k})$ be a point valued in the algebraic closure of k. Then $\Omega^{\mathrm{st}}_z \stackrel{\mathrm{def}}{=} \Omega^{\mathrm{st}}_{\Phi}|_z$ may be thought of as a free $\mathcal{O}_2$-module of rank c; write Θ^{st}_z for the $\mathcal{O}_2$-dual of Ω^{st}_z. Let S_z be the completion of S at z. Then there is a unique morphism

$$\Gamma_z : S_z \to \mathcal{G}_2 \otimes^{\mathrm{gp}}_{\mathcal{O}_2} \Theta^{\mathrm{st}}_z$$

such that:

(i) *the derivative of Γ_z induces the natural inclusion of Ω^{st}_z into $\Omega_{S/A}$;*

(ii) *the action of Φ on S_z is compatible with multiplication by p on $\mathcal{G}_2 \otimes^{\mathrm{gp}}_{\mathcal{O}_2} \Theta^{\mathrm{st}}_z$.*

Here, the variables on S_z obtained by pull-back via Γ_z carry a Lubin-Tate geometry of order 2, and are called the strong variables *on S_z. Finally, the fiber of Γ_z over the identity element of the group object $\mathcal{G}_2 \otimes^{\mathrm{gp}}_{\mathcal{O}_2} \Theta^{\mathrm{st}}_z$ admits an anabelian geometry of length 2 and order 1 determined by Φ (plus a "Hodge subspace" for Φ – cf.* **[22]***, Chapter IX, § 1.5, for more details). The variables in these fibers are called the* weak variables.

Thus, in summary, Φ defines a *virtual fibration* on S to a base space (of dimension c) naturally equipped with a Lubin-Tate geometry of order 2; moreover, (roughly speaking) the fibers of this fibration are naturally equipped with an anabelian geometry of length 2 and order 1. In terms of VF-patterns, this sort of Frobenius lifting occurs in the case $\varpi = 2$, $\Pi(1) \neq 0$ (cf. Theorem 2.15). The colevel is then given by $2(\chi - \Pi(1))$.

Next, we note that as remarked toward the end of § 2.4, in the binary ordinary and very ordinary spiked cases one obtains geometrizable systems of Frobenius liftings not only over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$ (which is étale over $(\overline{\mathcal{M}}_{g,r})_{\mathbb{Z}_p}$) but also on the ordinary locus of the universal curve over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. (More precisely, in the very ordinary spiked case, one must replace $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$ by the formal open substack defined by the very ordinary locus.) Thus, in particular,

> *In the binary ordinary and very ordinary spiked cases, one obtains geometries as discussed in the above theorems not only on the moduli of the curves in question, but also on the ordinary loci of the universal curves themselves.*

See Fig. 11 for a pictorial representation of the major types of geometries discussed.

Finally, we observe that one way to understand these generalized ordinary geometries is the following:

> *The "Lubin-Tate-ness" of the resulting geometry on the moduli stack is a reflection of the extent to which the p-curvature (of the indigenous bundles that the moduli stack parametrizes) vanishes.*

That is to say, the more the p-curvature vanishes, the more Lubin-Tate the resulting geometry becomes. For instance, in the case of a Lubin-Tate geometry, the order of the Lubin-Tate geometry (cf. Theorem 2.16) corresponds precisely to the number of dormant crys-stable bundles in a period (minus one). In the case of a spiked geometry, the number of "Lubin-Tate dimensions" is measured by the colevel. Moreover, this colevel is proportional to the degree of vanishing of the p-curvature of the indigenous bundle in question.

2.6. The Canonical Galois Representation. — Finally, since we have been considering Frobenius invariant indigenous bundles,

> *We would like to construct representations of the fundamental group of the curve in question into* PGL_2 *by looking at the Frobenius invariant sections of these indigenous bundles.*

Such representations will then be the p-adic analogue of the canonical representation in the complex case of the topological fundamental group of a hyperbolic Riemann surface into $\mathrm{PSL}_2(\mathbb{R}) \subseteq \mathrm{PGL}_2(\mathbb{C})$ (cf. the discussion at the beginning of § 1.3). Unfortunately, things are not so easy in the p-adic (generalized ordinary) case because *a priori* the canonical indigenous bundles constructed in Theorem 2.13 only have connections and Frobenius actions with respect to the *relative* coordinates of the tautological curve over $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$. This means, in particular, that we cannot immediately apply the theory of [**6**], § 2, to pass to representations of the fundamental group. To overcome this difficulty, we must employ the technique of *crystalline induction* developed in [**21**]. Unfortunately, in order to carry out crystalline induction, one needs to introduce an object called the *Galois mantle* which can only be constructed when the system of Frobenius liftings on $\mathcal{N}^{\mathrm{ord}}_{\mathbb{Z}_p}$ is geometrizable. Thus, in particular, we succeed (in [**22**], Chapter X) in constructing representations of the sort desired only in the binary ordinary and very ordinary spiked cases.

First, we sketch what we mean by the *Galois mantle.* The Galois mantle can be constructed for any geometrizable system of Frobenius liftings (e.g., any of the types discussed in § 2.5). In particular, the notion of the Galois mantle has nothing to do with curves or their moduli. For simplicity, we describe the Galois mantle in the *classical ordinary case.* Thus, let S and A be as in § 2.5. Let Π_S be the fundamental group of $S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (for some choice of base-point). Let Φ be a *classical ordinary Frobenius lifting* (in other words, Lubin-Tate of order 1) on S. Then by taking Frobenius invariant sections of the tangent bundle, one obtains an étale local system $\Theta^{\mathrm{et}}_{\Phi}$ on S of free $\mathbb{Z}_p$-modules of rank $\dim_A(S)$. Moreover, Φ defines a natural exact sequence of continuous Π_S-modules

$$0 \to \Theta^{\mathrm{et}}_{\Phi}(1) \to E_{\Phi} \to \mathbb{Z}_p \to 0$$

where the "(1)" denotes a Tate twist, and "$\mathbb{Z}_p$" is equipped with the trivial Π_S-action. Roughly speaking, this extension of Π_S-modules is given by taking the p^{th} power roots of the canonical multiplicative coordinates of Theorem 1.3 (cf. § 2.2 of [**22**], Chapter VIII for a detailed discussion of the p-divisible group whose Tate module may be identified with E_Φ). Let $\mathcal{B}'$ be the affine space of dimension $\dim_A(S)$ over $\mathbb{Z}_p$ parametrizing splittings of the above exact sequence. Then the action of Π_S on the above exact sequence induces a natural action of Π_S on $\mathcal{B}'$. Roughly speaking, the *Galois mantle $\mathcal{B}$ associated to Φ* is the p-adic completion of a certain kind of p-adic localization of $\mathcal{B}$.

> *More generally, to any geometrizable system of Frobenius liftings (as in § 2.5) on S, one can associate a natural p-adic space $\mathcal{B}$* – the Galois mantle associated to the system of Frobenius liftings – *with a continuous Π_S-action. In the binary ordinary case, $\mathcal{B}$ will have a natural affine structure over some finite étale extension of $\mathbb{Z}_p$. In the very ordinary spiked case, $\mathcal{B}$ will be fibred over an affine space over $\mathcal{O}_2$ with fibers that are also equipped with an affine structure over $\mathcal{O}_2$.*

In fact, to be more precise, $\mathcal{B}$ is only equipped with an action by a certain open subgroup of Π_S, but we shall ignore this issue here since it is rather technical and not so important. We refer to § 2.3 and § 2.5 of [**22**], Chapter IX for more details on the Galois mantle. So far, for simplicity, we have been ignoring the logarithmic case, but everything is compatible with log structures.

We are now ready to state the main result on the canonical Galois representation in the generalized ordinary case, i.e., the generalized ordinary analogue of Theorem 1.4 (cf. Theorems 1.2 and 2.2 of [**22**], Chapter X). See Fig. 12 for a graphic depiction of this theorem.

***Theorem 2.19* (Canonical Galois Representation).** — *Let p be an odd prime. Let g and r be nonnegative integers such that $2g - 2 + r \geq 1$. Fix a VF-pattern (Π, ϖ) which is either binary ordinary or spiked of order 2. Let $S \stackrel{\text{def}}{=} \mathcal{N}^{\text{ord}}_{\mathbb{Z}_p}$ in the binary ordinary case, and let S be the very ordinary locus of $\mathcal{N}^{\text{ord}}_{\mathbb{Z}_p}$ in the spiked case. Let $Z \to S$ be a certain appropriate finite covering which is log étale in characteristic zero (cf. the discussion preceding Theorems 1.2 and 2.2 of* [**22**]*, Chapter X for more details). Let $X^{\log}_Z \to Z^{\log}$ be the tautological log-curve over $Z^{\log}$. Let Π_{X_Z} (respectively, Π_Z) be the fundamental group of $X^{\log}_Z \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (respectively, $Z^{\log} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$) for some choice of base-point. (Of course, despite the similarity in notation, these fundamental groups have no direct relation to the VF-pattern "Π.") Thus, there is a natural morphism $\Pi_{X_Z} \to \Pi_Z$. Let $\mathcal{B}$ be the* Galois mantle *associated to the canonical system of Frobenius liftings of Theorem 2.13. The morphism $\Pi_{X_Z} \to \Pi_Z$ allows us to regard $\mathcal{B}$ as being equipped with a Π_{X_Z}-action.*

Let $\mathcal{P}$ be the tautological Π-indigenous bundle on X. Then by taking Frobenius invariants of $\mathcal{P}$, one obtains a $\mathbb{P}^1$-bundle

$$\mathbb{P}_{\mathcal{B}} \to \mathcal{B}$$

equipped with a natural continuous Π_{X_Z}-action compatible with the above-mentioned action of Π_{X_Z} on the Galois mantle $\mathcal{B}$.

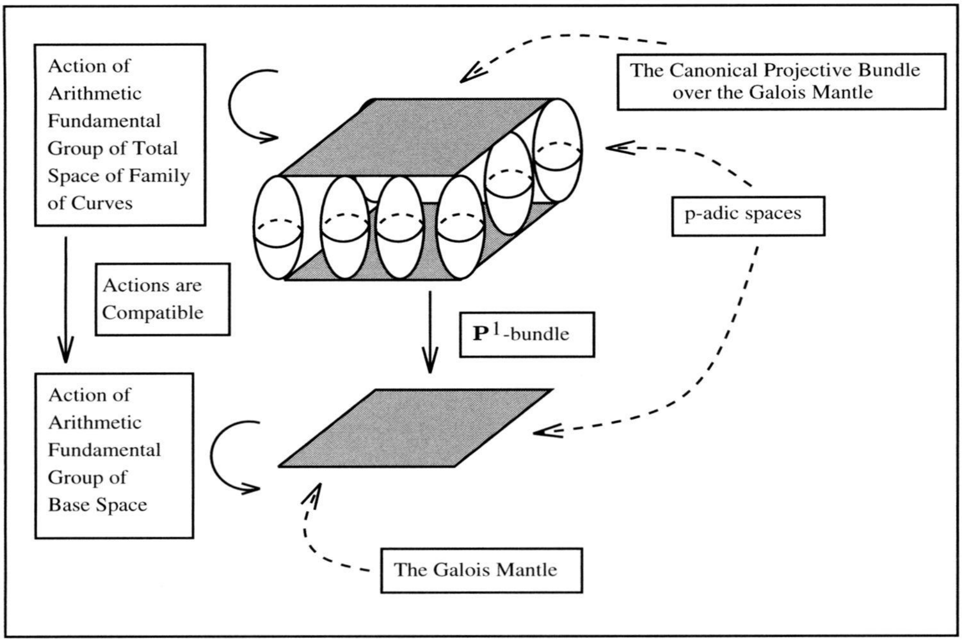

FIGURE 12. The Canonical Galois Representation

Put another way, one obtains a twisted homomorphism of Π_{X_Z} into PGL_2 of the functions on $\mathcal{B}$. (Here, "twisted" refers to the fact that the multiplication rule obeyed by the homomorphism takes into account the action of Π_{X_Z} on the functions on $\mathcal{B}$.) Finally, note that for any point of $Z \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (at which the log structure is trivial), one also obtains similar representations by restriction. This gives one canonical Galois representations even in the non-universal case.

Finally, in [**22**], Chapter X, § 1.4, 2.3, we show that:

> *The Galois representation of Theorem 2.19 allows one to relate the various p-adic analytic structures constructed throughout* [**22**] *(i.e., canonical Frobenius liftings, canonical Frobenius invariant indigenous bundles, etc.) to the* algebraic/arithmetic *Galois action on the* profinite Teichmüller group (cf. [**22**], Chapter X, Theorems 1.4, 2.3).

More precisely: By iterating the canonical Frobenius liftings on $\mathcal{N} \stackrel{\text{def}}{=} \mathcal{N}^{\text{ord}}_{\mathbb{Z}_p}$, we obtain a certain natural infinite covering

$$\mathcal{N}[\infty] \to \mathcal{N}$$

(i.e., projective limit of finite coverings which are log étale in characteristic zero). On the other hand, if we denote by

$$\mathcal{C}_{\mathbb{Z}_p} \to \mathcal{M}_{\mathbb{Z}_p} \stackrel{\text{def}}{=} (\overline{\mathcal{M}}^{\log}_{g,r})_{\mathbb{Z}_p}$$

the *universal log-curve over the moduli stack*, and by $\mathcal{C}_{\overline{\eta}}$ the geometric generic fiber of this morphism, then the *natural outer action* $\pi_1(\mathcal{M}_{\mathbb{Q}_p})$ (i.e., action on a group defined modulo inner automorphisms of the group) on $\pi_1(\mathcal{C}_{\overline{\eta}})$ defines an action of $\pi_1(\mathcal{M}_{\mathbb{Q}_p})$ on

$$\mathbf{Rep}_{\mathbb{Q}_p} \stackrel{\text{def}}{=} \mathrm{Rep}(\pi_1^{\mathrm{top}}(\mathcal{X}), PGL_2(\mathcal{O}_{\varpi}))$$

(where $\mathcal{O}_{\varpi}$ is defined to be the ring of Witt vectors with coefficients in the finite field of p^{ϖ} elements, and "Rep" denotes the set of isomorphism classes of homomorphisms $\pi_1^{\mathrm{top}}(\mathcal{X}) \to PGL_2(\mathcal{O}_{\varpi})$; two such homomorphisms are regarded as isomorphic if they differ by composition with an inner automorphism of $PGL_2(\mathcal{O}_{\varpi})$). Moreover, this action defines (by the "definition of π_1") an infinite étale covering $\mathcal{R}_{\mathbb{Q}_p} \to \mathcal{M}_{\mathbb{Q}_p}$. We denote the normalization of $\mathcal{M}_{\mathbb{Z}_p}$ in $\mathcal{R}_{\mathbb{Q}_p}$ by $\mathcal{R}_{\mathbb{Z}_p}$. Let $\widehat{\mathcal{M}}$ be the *p-adic completion of* $\mathcal{M}_{\mathbb{Z}_p}$, and $\widehat{\mathcal{R}} \stackrel{\text{def}}{=} \mathcal{R}_{\mathbb{Z}_p} \times_{\mathcal{M}_{\mathbb{Z}_p}} \widehat{\mathcal{M}}_{\mathbb{Z}_p}$. Then the main results on this topic (i.e., [**22**], Chapter X, Theorems 1.4, 2.3) state that the Galois of representation of Theorem 2.19 induces a commutative diagram

$$\begin{array}{ccc} \mathcal{N}[\infty] & \xrightarrow{\widehat{\kappa}} & \widehat{\mathcal{R}} \\ \downarrow & & \downarrow \\ \mathcal{N} & \longrightarrow & \widehat{\mathcal{M}} \end{array}$$

in which the horizontal morphism (which is denoted $\widehat{\kappa}$ in [**22**], Chapter X) on top is an *open immersion.*

The proof that $\widehat{\kappa}$ is an open immersion divides naturally into *three parts*, corresponding to the three "layers" of the morphism $\mathcal{N}[\infty] \to \widehat{\mathcal{M}}$. The *first layer* is the quasi-finite (but not necessarily finite) étale morphism $\mathcal{N} \to \widehat{\mathcal{M}}$. Because the morphism $\mathcal{N} \to \widehat{\mathcal{M}}$ is étale even in characteristic p, this layer is rather easy to understand. The *second layer* corresponds to the finite covering $Z \to S$ of Theorem 2.19. Together, the first and second layers correspond to the "mod p portion" of the Galois representation of Theorem 2.19 – i.e., the first layer corresponds to the "slope zero portion" of this representation modulo p, while the second layer corresponds to the "positive slope portion" of this representation modulo p. From the point of view of the "$\mathcal{MF}^{\nabla}$-objects" over $B(\mathcal{N})$ (cf. the discussion following Definition 2.5 in § 2.3) corresponding to the representation of Theorem 2.19, this *slope zero portion* (i.e., the first layer) parametrizes the isomorphism class of these $\mathcal{MF}^{\nabla}$-objects over $(B(\mathcal{N})_{\mathbf{F}_p})_{\mathrm{red}}$, while the *positive slope portion* (i.e., the second layer) parametrizes the isomorphism class of the *deformations* of these $\mathcal{MF}^{\nabla}$-objects from bundles on curves over $(B(\mathcal{N})_{\mathbf{F}_p})_{\mathrm{red}}$ to bundles on curves over $B(\mathcal{N})_{\mathbf{F}_p}$.

Finally, the *third layer* of the covering is what remains between $\mathcal{N}[\infty]$ and the "Z" of Theorem 2.19. This portion is the *analytic* portion of the covering (i.e., the portion of the covering equipped with a natural "analytic structure"). Put another way, this portion is the portion of the covering which is dealt with by the technique of *crystalline induction* (which is concerned precisely with equipping this portion of the covering with a natural "crystalline" analytic structure – cf. [**22**], Chapter IX, § 2.3 – especially the Remark following Theorem 2.11 – for more details).

Thus, the fact that the morphism $\widehat{\kappa}$ "does not omit any information" at all three layers is essentially a tautological consequence of the various aspects of the extensive theory developed throughout [**22**]. From another point of view, by analyzing this morphism $\widehat{\kappa}$, we obtain a rather detailed understanding of a certain portion of the *canonical tower of coverings of* $\mathcal{M}_{\mathbb{Q}_p} \stackrel{\text{def}}{=} (\overline{\mathcal{M}}_{g,r}^{\log})_{\mathbb{Q}_p}$ given by

$$\mathcal{R}_{\mathbb{Q}_p} \to \mathcal{M}_{\mathbb{Q}_p}$$

analogous to the analysis given in [**19**] of coverings of the moduli stack of elliptic curves over $\mathbb{Z}_p$ obtained by considering p-power torsion points (cf. the Remark following [**22**], Chapter X, Theorem 1.4, for more details).

Thus, in summary, Theorem 2.19 concludes our discussion of "p-adic Teichmüller theory" as exposed in [**22**] by constructing a p-adic analogue of the canonical representation discussed at the beginning of § 1.3, that is to say, a p-adic analogue of something very close to the Fuchsian uniformization itself – which was where our discussion began (§ 1.1).

3. Conclusion

Finally, we pause to take a look at what we have achieved. Just as in § 1, we would like to *describe the p-adic theory by comparing it to the classical theory at the infinite prime.* Thus, let us write

$$\mathcal{C}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}} \stackrel{\text{def}}{=} (\overline{\mathcal{M}}_{g,r}^{\log})_{\mathbb{C}}$$

for the *universal log-curve over the moduli stack* $(\overline{\mathcal{M}}_{g,r}^{\log})_{\mathbb{C}}$ of r-pointed stable log-curves of genus g over the complex numbers. Let us fix a "base-point" (say, in the interior – i.e., the open substack parametrizing smooth curves – of $\mathcal{M}_{\mathbb{C}}$) $[X] \in \mathcal{M}_{\mathbb{C}}(\mathbb{C})$ corresponding to some hyperbolic algebraic curve X over $\mathbb{C}$. Let us write $\mathcal{X} \stackrel{\text{def}}{=} X(\mathbb{C})$ for the corresponding *hyperbolic Riemann surface.* Next, let us consider the space

$$\mathbf{Rep}_{\mathbb{C}} \stackrel{\text{def}}{=} \mathrm{Rep}(\pi_1^{\mathrm{top}}(\mathcal{X}), PGL_2(\mathbb{C}))$$

of isomorphism classes of representations of the *topological fundamental group* $\pi_1^{\mathrm{top}}(\mathcal{X})$ into $PGL_2(\mathbb{C})$. This space has the structure of an *algebraic variety* over $\mathbb{C}$, induced by the algebraic structure of $PGL_2(\mathbb{C})$ by choosing generators of $\pi_1^{\mathrm{top}}(\mathcal{X})$. Note, moreover, that as $[X]$ varies, the resulting spaces $\mathrm{Rep}(\pi_1(\mathcal{X}), PGL_2(\mathbb{C}))$ form a *local* *system* on $\mathcal{M}_{\mathbb{C}}$ (valued in the category of algebraic varieties over $\mathbb{C}$) which we denote by

$$\mathcal{R}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}$$

One can also think of $\mathcal{R}_{\mathbb{C}}$ as the local system defined by the natural action of $\pi_1^{\mathrm{top}}(\mathcal{M}_{\mathbb{C}}(\mathbb{C}))$ on $\mathbf{Rep}_{\mathbb{C}} \stackrel{\mathrm{def}}{=} \mathrm{Rep}(\pi_1^{\mathrm{top}}(\mathcal{X}), PGL_2(\mathbb{C}))$ which is induced by the *natural outer action* of $\pi_1^{\mathrm{top}}(\mathcal{M}_{\mathbb{C}}(\mathbb{C}))$ on $\pi_1^{\mathrm{top}}(\mathcal{X})$ – cf. the discussion of the p-adic case at the end of § 2.6 above (for more details, see [**22**], Chapter X, § 1.4, 2.3).

Next, let us denote by

$$\mathcal{QF} \subseteq \mathcal{R}_{\mathbb{C}}$$

the subspace whose fiber over a point $[Y] \in \mathcal{M}_{\mathbb{C}}(\mathbb{C})$ is given by the representations $\pi_1^{\mathrm{top}}(\mathcal{Y}) \to PGL_2(\mathbb{C})$ that define *quasi-Fuchsian groups* (cf. § 1.4), i.e., simultaneous uniformizations of pairs of Riemann surfaces (of the same type (g, r)), for which one (say, the "first" one) of the pair of Riemann surfaces uniformized is the Riemann surface $\mathcal{Y}$ corresponding to $[Y]$. Thus, whereas the fibers of $\mathcal{R}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}$ are of *dimension* $2(3g - 3 + r)$ over $\mathbb{C}$, the fibers of $\mathcal{QF} \to \mathcal{M}_{\mathbb{C}}$ are of *dimension* $3g - 3 + r$ over $\mathbb{C}$.

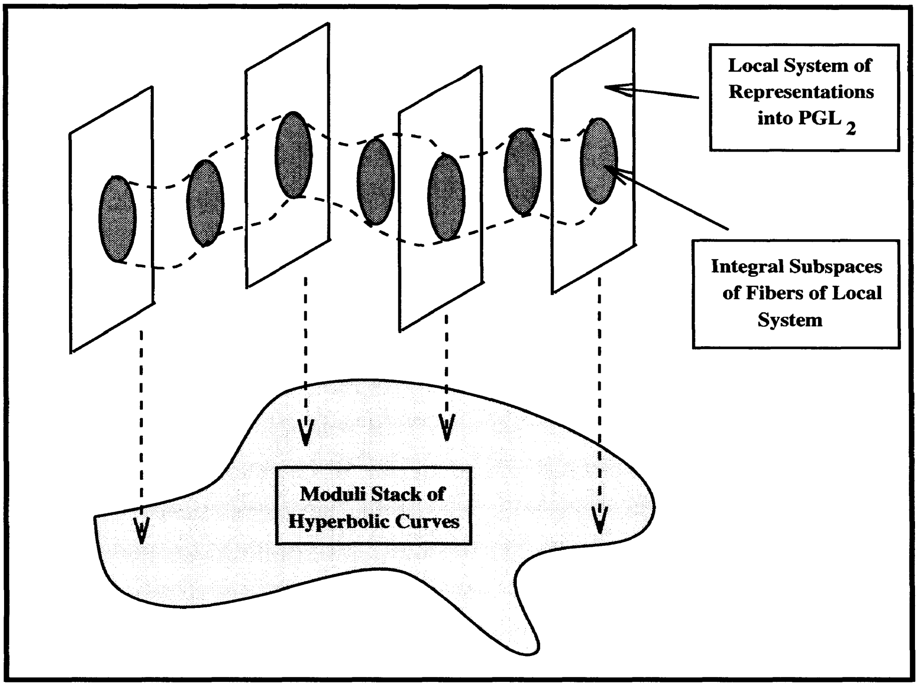

FIGURE 13. Integral Subspaces of the Local System of Representations

Then, relative to the notation of [**22**], Chapter X, § 1.4, 2.3, the *analogy between the complex and p-adic cases* may be summarized by the following diagram:

$$\begin{array}{ccc} \mathcal{QF} & \Longleftrightarrow & \mathcal{N}[\infty] \\ \cap & & \cap \\ (\mathcal{R}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}) & \Longleftrightarrow & (\widehat{\mathcal{R}} \to \widehat{\mathcal{M}}) \end{array}$$

(where the vertical inclusion on the left is the natural one; and the vertical inclusion on the right is the morphism $\widehat{\kappa}$ of [**22**], Chapter X, Theorems 1.4, 2.3). We also give an *illustration* (Fig. 13) of this sort of situation. Relative to this illustration, the "integral (or bounded) subspaces" of the local system are $\mathcal{QF}$ and $\mathcal{N}[\infty]$ (cf. § 1.4 for an explanation of the term "integral"). Note that just as in the complex case, the fibers of the covering $\mathcal{N}[\infty] \to \widehat{\mathcal{M}}$ have, so to speak, "Galois dimension" $3g-3+r$ over $\mathcal{O}_{\varpi}$ (cf. the crystalline induction portion of the proof of [**22**], Chapter X, Theorem 1.4), whereas the fibers of the covering $\widehat{\mathcal{R}} \to \widehat{\mathcal{M}}$ are of "Galois dimension" $2(3g-3+r)$ over $\mathcal{O}_{\varpi}$. In the p-adic case, $\mathcal{N}[\infty]$ denotes the "*crystalline*" or "*Frobenius invariant indigenous bundle*" locus of $\widehat{\mathcal{R}}$ – cf. the discussion of § 1.4.

In the complex case, the "Frobenius" (i.e., complex conjugation) invariant portion of $\mathcal{QF}$ is the space of *Fuchsian groups*, hence defines the *Bers uniformization* of Teichmüller space (cf. § 1.5). On the other hand, in the p-adic case, the covering $\mathcal{N}[\infty] \to \widehat{\mathcal{M}}$ is "made up of" composites of Frobenius liftings, by *forgetting* that these Frobenius liftings are morphisms from a single space to itself, and just thinking of them as *coverings*. If one then invokes the structure of Frobenius liftings as *morphisms from a single space to itself*, one so-to-speak recovers the original Frobenius liftings, which (by the theory of [**22**], Chapters VIII and IX) define *p-adic uniformizations of* $(\overline{\mathcal{M}}_{g,r}^{\log})_{\mathbb{Z}_p}$.

In the complex case, the space of quasi-Fuchsian groups $\mathcal{QF}$ may also be interpreted in terms of *quasi-conformal maps*. Similarly, in the p-adic case, one may interpret integral Frobenius invariant indigenous bundles as *quasi-analytic self-isogenies* of hyperbolic curves (cf. the end of § 2.4).

Finally, in the complex case, although $\mathcal{QF}$ is *not closed* in $\mathcal{R}_{\mathbb{C}}$, the space $\mathcal{QF}$ (when regarded as a space of representations) is *complete* relative to the condition that the representations always define *indigenous bundles* for some conformal structures on the *two* surfaces being uniformized. Note that one may think of these two surfaces as *reflections* of another, i.e., translates of one another by some action of *Frobenius* at the infinite prime (i.e., complex conjugation). Similarly, although $\mathcal{N}[\infty]$ is *not closed* in $\widehat{\mathcal{R}}$, it is *complete* (at least for binary VF-patterns Π) in the sense discussed at the end of [**22**], Chapter X, § 1.4, i.e., relative to the condition that the representation always defines an indigenous bundle on the universal thickening $B^+(-)$ of the base. Note that this thickening $B^+(-)$ is in some sense the *minimal thickening of* (the normalization of the maximal log étale in characteristic zero extension of) "$(-)$" *that* *admits an action of Frobenius* (cf. the theory of [**22**], Chapter VI, § 1; $B^+(-)$ is the PD-completion of the rings $B(-)$ discussed in [**22**], Chapter VI, § 1; in fact, instead of using $B^+(-)$ here, it would also be quite sufficient to use the rings $B(-)$ of [**22**], Chapter VI, § 1). In other words, just as in the complex case,

> $\mathcal{N}[\infty]$ *is already complete* relative *to the condition that the representations it parametrizes always define indigenous bundles on the given curve* and all of its Frobenius conjugates.

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