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CRACKTIP IS A GLOBAL MUMFORD-SHAH MINIMIZER

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CRACKTIP IS A GLOBAL MUMFORD-SHAH MINIMIZER

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Abstract. — We show that the pair (u, K) given by $K = (-\infty, 0] \subset \mathbb{R}^2$ and

$$u(r\cos\theta, r\sin\theta) = \sqrt{2/\pi}\, r^{1/2}\sin(\theta/2) \text{ for } r > 0 \text{ and } -\pi < \theta < \pi$$

is a global Mumford-Shah minimizer. This means that if $\widetilde{K}$ is another closed set in the plane with locally finite Hausdorff measure, $\widetilde{u}$ is a function on $\mathbb{R}^2 \setminus \widetilde{K}$ with a derivative in $L^2_{\text{loc}}(\mathbb{R}^2 \setminus \widetilde{K})$, and the pair $(\widetilde{u}, \widetilde{K})$ coincides with (u, K) out of some disk B, then

$$H^1(K \cap B) + \int_{B \setminus K} |\nabla u|^2 \leqslant H^1(\widetilde{K} \cap B) + \int_{B \setminus G} |\nabla \widetilde{u}|^2,$$

where H^1 denotes Hausdorff measure.

We shall also show that every global Mumford-Shah minimizer (u', K') that is sufficiently close to (u, K) near infinity must be equivalent to it. This is the case, for instance, if some blow-in limit of (u', K') equals (u, K).

The proofs will be based on a detailed study of the harmonic function v' conjugated to u', and its level sets. We shall also use blow-up techniques and the monotonicity of an energy integral.

***Résumé* (Cracktip est un minimum de Mumford-Shah global)**

Le résultat principal de ce texte est que le couple (u, K) défini par $K =]-\infty, 0] \subset \mathbb{R}^2$ et

$$u(r\cos\theta, r\sin\theta) = \sqrt{2/\pi}\, r^{1/2} \sin(\theta/2) \text{ pour } r > 0 \text{ et } -\pi < \theta < \pi$$

est un minimum global de la fonctionnelle de Mumford-Shah. Ceci signifie que si $\widetilde{K}$ est un fermé du plan de mesure de Hausdorff de dimension 1 localement finie, $\widetilde{u}$ est une fonction définie sur $\mathbb{R}^2 \setminus \widetilde{K}$ dont la dérivée est dans $L^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \widetilde{K})$, et si le couple $(\widetilde{u}, \widetilde{K})$ coïncide avec (u, K) hors d'un disque B, on a

$$H^1(K \cap B) + \int_{B \setminus K} |\nabla u|^2 \leqslant H^1(\widetilde{K} \cap B) + \int_{B \setminus G} |\nabla \widetilde{u}|^2 ,$$

où l'on note H^1 la mesure de Hausdorff.

On montrera aussi que tout minimum global (u', K') de la fonctionnelle de Mumford-Shah qui est suffisament proche de (u, K) à l'infini lui est équivalent. C'est le cas par exemple si l'une des limites de (u', K') par implosions est égale à (u, K).

La démonstration est basée sur une étude détaillée de la fonction harmonique conjuguée de u' et de ses ensembles de niveau. On utilise aussi des techniques d'explosion et la monotonie d'une intégrale d'énergie.

CONTENTS

CHAPTER A

GENERAL INTRODUCTION

1. Introduction

The main goal of this paper is to verify that cracktips (as defined below) are global minimizers of the Mumford-Shah functional. This gives a positive answer to a question of E. De Giorgi [DG]. We shall also prove that all the global minimizers of the Mumford-Shah functional that are close enough to cracktips (in ways that will soon be made precise) are cracktips.

The global version of the Mumford-Shah functional that we consider here is morally given by

$$J(u,K) = \int_{\mathbb{R}^2 \setminus K} |\nabla u|^2 + H^1(K), \tag{1.1}$$

where $H^1(K)$ denotes the one-dimensional Hausdorff measure of the closed set K [see for instance [Fa] or [Ma] for definitions]. This is only a moral definition, because $J(u,K) = +\infty$ for all interesting competitors (u,K), and so we shall have to give a more local definition of competitors and minimizers.

Let us first define the set U_0 of *admissible pairs* (or competitors). These are the pairs (u,K), where K is a closed subset of the plane, $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \setminus K)$ is a function defined on $\mathbb{R}^2 \setminus K$ and whose distributional gradient ∇u lies in $L^2_{\text{loc}}\big(\mathbb{R}^2 \setminus K\big)$, and which satisfy the additional requirement that

$$H^1(K \cap B_R) + \int_{B_R \setminus K} |\nabla u|^2 < +\infty \quad \text{for all} \quad R > 0, \tag{1.2}$$

where we denote by $B_R = B(0,R)$ the open disk centered at the origin and with radius R.

Next let $(u,K) \in U_0$ be given. A *competitor for* (u,K) is an admissible pair $(v,G) \in U_0$ which satisfies the following properties for some (large) $R > 0$:

$$G \setminus B_R = K \setminus B_R, \tag{1.3}$$

$$v(x) = u(x) \ \text{ on } \ \mathbb{R}^2 \setminus (K \cup B_R), \tag{1.4}$$

and

(1.5) if $x, y \in \mathbb{R}^2 \setminus (K \cup B_R)$ are separated by K, then G also separates them.

[We say that K separates x from y when x, y lie in different connected components of $\mathbb{R}^2 \setminus K$.]

Note that if (1.3)-(1.5) hold for $R > 0$, they also hold for all $R' > R$.

Definition 1.6. — A *global minimizer* (for the Mumford-Shah functional) is an admissible pair $(u, K) \in U_0$ such that

$$H^1(K \cap B_R) + \int_{B_R \setminus K} |\nabla u|^2 \leqslant H^1(G \cap B_R) + \int_{B_R \setminus G} |\nabla v|^2 \tag{1.7}$$

for all competitors (v, G) for (u, K) and $R > 0$ such that (1.3)-(1.5) hold.

Note that we do not need to be too specific about R: if (1.7) holds for some R such that (1.3)-(1.5) hold, it stays true for all such R (by (1.3) and (1.4)).

This class of global minimizers was introduced in [Bo], where it is shown that if (u_0, K_0) is a minimizer for the (usual) Mumford-Shah functional

$$J(u, K) = \int_{\Omega \setminus K} |u - g|^2 + H^1(K) + \int_{\Omega \setminus K} |\nabla u|^2 \tag{1.8}$$

(on a bounded domain Ω, and with a given bounded function g on Ω), then all limits of (u, K) under blow-ups are global minimizers as in Definition 1.6. Our topological condition (1.5) actually comes from this in a natural way; see Section 12 for details about this.

Note that if (u, K) is a global minimizer, we can always add any closed set of H^1-measure zero to K, and this gives another global minimizer equivalent to (u, K). We shall say that the global minimizer (u, K) is "*reduced*" if there is no proper closed subset $\widetilde{K}$ of K such that u extends to a function $\widetilde{u} \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \setminus \widetilde{K})$ and $(\widetilde{u}, \widetilde{K})$ is a competitor for (u, K). [We add this last constraint because of the topological condition (1.5); we want to avoid opening holes that would change the true nature of (u, K).]

It is not too hard to check that for each global minimizer (u, K) there is a reduced global minimizer $(\widetilde{u}, \widetilde{K})$ which is equivalent to (u, K), that is, such that $\widetilde{K} \subset K$, $\widetilde{u}$ is an extension of u, and $(\widetilde{u}, \widetilde{K})$ is a competitor for (u, K). Because of this, we shall always assume that all our global minimizers are reduced. We don't loose any generality, and this will allow us to give more pleasant descriptions of K.

The natural analogue in the present context of the celebrated Mumford-Shah conjecture in [MuSh] is that all (reduced) global minimizers belong to the following short list:

$$K = \varnothing \ \text{ and } u \text{ is constant;} \tag{1.9}$$

$$\begin{cases} K \text{ is a line and } u \text{ is constant on each} \\ \text{of the two connected components of } \mathbb{R}^2 \setminus K \ ; \end{cases} \tag{1.10}$$

$$\begin{cases} K \text{ is a propeller (see the definition below) and} \\ u \text{ is constant on each of the 3 components of } \mathbb{R}^2 \setminus K \ ; \end{cases} \tag{1.11}$$

$$(u, K) \text{ is a cracktip (also see below).} \tag{1.12}$$

We call propeller a union of three half-lines with a common endpoint (called center) and that make 120° angles with each other.

We call cracktips the pairs (u, K) such that, after a suitable change of coordinates in the plane,

$$K = \{(x, 0) \ ; \ x \leqslant 0\} \tag{1.13}$$

(a half-line) and u is given by

$$u\,(r\cos\theta, r\sin\theta) = \pm\sqrt{2/\pi}\; r^{1/2}\sin\frac{\theta}{2} + C \tag{1.14}$$

for $r > 0$ and $-\pi < \theta < \pi$. The (constant) sign $\pm$ and the value of the constant C obviously do not matter.

If the conjecture above where true, then the Mumford-Shah conjecture would quite probably follow, using the arguments in [Bo] (we did not check all the details). The converse is less clear a priori (there could be global minimizers that do not show up as blow-ups of Mumford-Shah minimizers).

Recall that if (u, K) is a global minimizer and K is connected, then (u, K) belongs to the short list above. This is one of the main points of [Bo], where it is used to prove that isolated components of K_0 for Mumford-Shah minimizers are finite unions of C^1-curves.

The pairs (u, K) in (1.9), (1.10), and (1.11) are easily seen to be global minimizers. Note that the topological condition (1.5) is needed for this. The main result of this paper is that

(1.15) Cracktips (as defined by (1.13) and (1.14)) are global minimizers.

Note that in the case of cracktips, the topological condition (1.5) on competitors for (u, K) is void, because K does not separate any pair of points. Thus cracktips are also minimizers for De Giorgi's definition, even though [DG] does not mention (1.5).

Our proof of (1.15) will also give that all global minimizers that are sufficiently close to a cracktip are actually cracktips. The precise meaning of "sufficiently close" may vary a little. Here is an example of sufficient condition.

Theorem 1.16. — *Let (u, K) be a global minimizer. Suppose that there is a connected component K_0 of K such that $K \setminus K_0$ is bounded. Then (u, K) is one of the minimizers of the short list* (1.9)-(1.12).

As we shall see later, the only new case is when K_0 looks like a half line at infinity, and then (u, K) is a cracktip. The other cases correspond to simpler global minimizers as in (1.9)-(1.11).

Another result that we shall prove is that if (u, K) is a global minimizer and at least one of its blow-ins is a cracktip, then (u, K) is a cracktip. Since we want to avoid giving the definition of blow-ins now, the precise statement will only be given later. See Theorem 40.4.

The two statements above can be seen as progress in the direction of the Mumford-Shah conjecture. We can also hope that the techniques of this paper will be useful to get further information on Mumford-Shah and global minimizers. On the other hand, the results obtained here can be seen as one more perturbation result of the type "global minimizers that are close enough to a minimizer of the short list (1.9)-(1.12) are in the list".

It seems very plausible that there exist simple domains Ω (like the unit disk $B(0, 1)$), functions $g \in L^\infty(\Omega)$ (like the restriction to $B(0, 1)$ of the cracktip function u in (1.14)), and minimizers (u, K) of the Mumford-Shah functional in (1.8), such that K is a C^1 curve with an open end in Ω (like $K = (-1, 0]$ in $B(0, 1)$). The results of this paper do not seem to imply this directly.

The proof of (1.15) will be long and technical, but the general scheme is not too hard to describe. To make this paper less unpleasant to read, we give in the next section a reasonably precise description of the argument; this description would almost be a proof if we knew already that our minimizers are sufficiently smooth (for instance, finite unions of C^1 curves). After this, we give the proof in all its gory details. Probably the reader will not want to see some of them (like the proof of existence of minimizers for a functional which is very close to to the usual one); we shall try to help by cutting the proof in fairly small steps with explicit titles, so that the reader can skip some sections and see where he stands in the general scheme.

Our proof contains a few partial results that also hold for global minimizers, and may have independent interest. We also reprove some known results, because we need to extend them to our special minimizers. In particular we give (near Section 9) a slightly more detailed version of a proof of existence of Mumford-Shah minimizers from Dal Maso, Morel, and Solimini [DMS]. Note that this will be done also in [MaSo], without the slightly unpleasant worry about the extra boundary $(-\infty, -1]$, and with an argument that works in higher dimensions.

We also check with a little more detail than in [Bo] the fact that limits of global minimizers are global minimizers (Section 12)(also see [Lé3]), and that if (v, G) is a reduced global minimizer, $\mathbb{R}^2 \setminus G$ has no bounded connected component (Section 15). In fact, every component of $\mathbb{R}^2 \setminus G$ contains disks of arbitrarily large radii (Section 17), and is even a John domain with center at infinity (see Lemma 20.1 and Remark 20.5). We also have to settle the relatively easy case when v is locally constant somewhere (Section 18).

The construction of a harmonic function w conjugated to v, as well as the description of its boundary behaviour near G and the structure of almost all its level sets Γ_m (as in Sections 25-32) could be useful in later developments. See in particular Lemma 22.3, Propositions 25.1, 26.2, 28.2, 30.1, and the first lines of Section 32.

We should mention here that since this manuscript was written, it has been proved [DaLé] that for reduced global minimizers, $\mathbb{R}^2 \setminus G$ is connected (unless G is a line or a propeller). Some additional local regularity properties of G are deduced from this, but although these results can simplify some of the arguments below, they do not seem to allow enormous shortcuts.

The second author wishes to mention the fact that the scheme of the proof is due to the first author. He is ready to admit that he is responsible for many of the technical details, though. Both authors wish to thank Marie-Claude Vergne for typing the manuscript.

2. A description of the proof

We shall denote by (u_0, K_0) the cracktip with

$$K_0 = \{(x, 0) \ ; \ x \leqslant 0\} \tag{2.1}$$

and

$$u_0 (r\cos\theta, r\sin\theta) = \sqrt{2/\pi}\; r^{1/2} \sin\frac{\theta}{2} \tag{2.2}$$

for $r > 0$ and $-\pi < \theta < \pi$, which corresponds to the sign $+$ and $C = 0$ in (1.14).

We shall assume that (u_0, K_0) is not a global minimizer, and our first task will be to find a minimizer $(v, G) \neq (u_0, K_0)$ which looks a lot like (u_0, K_0) at infinity. Unfortunately we shall not be able to do this with exactly the same functional J as above. The point is that if (u_1, K_1) is a competitor for (u_0, K_0) which is strictly better, it is too easy to find even better competitors by just dilating (u_1, K_1). To avoid this lack of compactness, we shall introduce slightly different functionals J_R (where the parameter R is a large radius, and our competitors will be forced to equal (u_0, K_0) out of $\overline{B}_R$). In the definition of J_R we shall replace $H^1(K)$ with a nonlinear function of $H^1(K)$. The usual theory of Mumford-Shah minimizers will still give us minimizers (u_R, K_R) of J_R, and the effect of the nonlinear function will be to force K_R to be a compact perturbation of K_0 (with the symmetric difference $K_R \Delta K_0$ contained in a ball that does not depend on R), without having to put special constraints to that effect. The desired minimizer (v, G) (of a slightly different functional) will then be obtained as a limit of the pairs (u_R, K_R).

The rest of the argument consists in showing that any (reduced) minimizer (v, G) (either for the initial Mumford-Shah problem or the slightly modified functional) that is close enough to (u_0, K_0) at infinity must be a cracktip. This will then produce the desired contradiction.

We shall have to use many of the known results on global minimizer (and their straightforward extension to our modified functional). In particular, we shall constantly use the fact that for H^1-almost every point z_0 of G there is a small radius $r > 0$ such that $G \cap B(z_0, r)$ is a C^1-curve that crosses $B(z_0, r)$. [See [Da] or [AFP].] We shall call such points z_0 regular points of G.

Since v minimizes $\displaystyle\int |\nabla v|^2$ locally, it is harmonic on $\mathbb{R}^2 \setminus G$. We shall check that v has a harmonic conjugate, that is, a harmonic function w on $\mathbb{R}^2 \setminus G$ such that $v + iw$ is holomorphic on $\mathbb{R}^2 \setminus G \approx \mathbb{C} \setminus G$. This is a (probably quite classical) consequence of the fact that v satisfies the Neumann condition $\partial v / \partial n = 0$ on the boundary G. See Section 22.

Next we want to check that w has a continuous extension to $\mathbb{R}^2$, and that this extension is constant on each connected component of G. This is not surprising, and would even be very easy if we knew that G is sufficiently smooth. The point is that at the locations where G is a C^1-curve, v and w have C^1 boundary values (one from each side of the curve). [See Section 14.] From the Neumann condition $\partial v / \partial n = 0$ and the definition of w we deduce that the tangential derivative $\partial w / \partial \tau = \partial v / \partial n$ vanishes, so that the boundary values of w near regular points are locally constant. This essentially gives the result; the technical part of the proof consists in checking that w does not have weird jumps near the (possibly infinitely many) non regular points of G. See Sections 25-27.

We shall need to know that for each regular point z_0 of G,

$$\text{the jump of } v \text{ at } z_0 \text{ is } \neq 0. \tag{2.3}$$

By jump of v we mean the difference between the boundary values of v from both sides of G; we do not need to be precise about the sign of the jump here. This will be fairly easy to check: if v had no jump at z_0, we would be able to produce a competitor strictly better that (v, G) by removing a small arc $G \cap B(z_0, r)$ near z_0 from G, and modifying v in $B(z_0, 2r)$ so that the values in $B(z_0, r)$ patch nicely. By choosing r small enough (and because v has no jump at zero) it would be possible to do this with a much smaller loss in energy $\left(\displaystyle\int |\nabla v|^2\right)$ than $H^1(G \cap B(z_0, r))$. Note that removing a piece from G is allowed by our topological condition (1.5), because $\mathbb{R}^2 \setminus G$ has only one unbounded component. Thus we would get a contradiction, and (2.3) holds. See Section 16 for details.

A consequence of (2.3) is that

$$\mathbb{R}^2 \setminus G \text{ is connected} \tag{2.4}$$

In other words, $\mathbb{R}^2 \setminus G$ has no bounded connected component. Indeed, if Ω was a bounded component of $\mathbb{R}^2 \setminus G$, we would be able to find a point $z_0 \in \partial\Omega$ which is a regular point of G. Note that we can add any constant to v in Ω without affecting the minimizing property of (v, G). By choosing the constant correctly, we can arrange a contradiction with (2.3). This proves (2.4). [See Section 15, and Section 17 for a more precise result.]

An important part of the argument is the study of the level sets

$$\Gamma_m = \{z \in \mathbb{R}^2 \; ; \; w(z) = m\}.$$

Let us normalize w (by adding a constant to it) so that $w(z) = 0$ on the unbounded component of G. We shall prove that $\Gamma_m = \varnothing$ for $m > 0$ and Γ_m is a rectifiable Jordan curve through infinity for almost all $m < 0$. Moreover, the function v is strictly monotonous on each such Jordan curve.

If G had only countably many connected components, this would be fairly easy. Let us describe the argument in that case. Because w is constant on each component of G, only countably many sets Γ_m can meet G. For the other values of m (and if $\Gamma_m \neq \varnothing$), Γ_m is a level set of the harmonic function w in $\mathbb{R}^2 \setminus G$. Note that w is not locally constant anywhere in $\mathbb{R}^2 \setminus G$, because $\mathbb{R}^2 \setminus G$ is connected and this would imply that v is constant on $\mathbb{R}^2 \setminus G$. [This is not possible if (v, G) looks like (u_0, K_0) at infinity.] Thus Γ_m is composed of analytic arcs, which may possibly meet in a starlike way at critical points of w. These critical points are zeroes of the holomorphic function $(v + iw)'$, so they are isolated point in $\mathbb{R}^2 \setminus G$. They are also isolated points in $\Gamma_m \subset \mathbb{R}^2 \setminus G$. Because (v, G) is close to (u_0, K_0) near infinity, it will be easy to check that w has no critical point in a neighborhood of infinity. Altogether, Γ_m contains only finitely many critical points. Thus Γ_m has a simple structure: finitely many analytic arcs that connect critical points or go to infinity.

Next Γ_m does not contain any (closed) loops. Indeed, suppose γ is a loop in Γ_m, and denote by Ω the bounded domain with boundary γ. We may assume that Ω is minimal (among such domains), because Γ_m is composed of finitely many arcs. Then Γ_m does not meet Ω. (This also uses the fact that arcs of Γ_m have to end at a critical point or escape at infinity). So $w - m$ does not change signs on Ω. Assume for definiteness that $w(z) > m$ on Ω. Except at (a finite number of) critical points, the normal derivative $\partial w/\partial n$ (with a choice of normal pointing into Ω) is positive on $\partial\Omega$, and so the tangential derivative $\partial v/\partial \tau = \partial w/\partial n$ is also positive (if we choose the trigonometric orientation on $\gamma = \partial\Omega$). Thus v is strictly increasing along γ, which is of course impossible since γ is a loop.

Thus Γ_m has no loops. Then each connected component of Γ_m is a tree, with finite arcs connecting critical points and infinite arcs escaping to infinity. Because (v, G) is close to (u_0, K_0) near infinity, we can easily check that there are at most 2 branches of Γ_m going to infinity. [We shall find lots of large radii R for which $\partial B_R \cap \Gamma_m$ has at most 2 points.] Then the only options are $\Gamma_m = \varnothing$ (which happens for $m > 0$) and Γ_m is a Jordan curve through infinity (which happens for $m < 0$).

The fact that v is strictly monotonous on Γ_m when Γ_m is a Jordan curve as above is easy, because we now know that there is no critical point on Γ_m (they would create too many branches) and $\partial v/\partial \tau = \partial w/\partial n$ does not change signs on Γ_m.

In the general case when G has uncountably many connected components, we can exclude first the countably many values of m for which Γ_m meets a nontrivial connected component of G. [There is at most countably many such components, since each of them has positive measure.] Nonetheless we are left with a bad subset G' of G (the set of points $x \in G$ such that $\{x\}$ is the component of x in G). Of course $H^1(G') = 0$ because G' contains no regular point of G, but nonetheless G' could a priori meet many (or even most) Γ_m. We shall have to check that G' has a small effect on the properties of Γ_m described above, at least for almost every value of m. See Sections 28-32 for details.

Our study of the level sets Γ_m will allow us to give a good description of the variations of the boundary values of v on G when you turn around a bounded connected component G_0 of G. The only interesting case is when G_0 is not reduced to a point. Then almost every point of G_0 is a regular point, and v has boundary values at those points (from both accesses).

We shall prove that when you turn around G_0 and restrict to regular points, the variations of (the boundary values of) v are as simple as they can be: v is strictly increasing and then strictly decreasing. Let us only give a fairly loose argument here, based on Figures 2.1 and 2.1 bis below, just to convince the reader that this is coherent with what we know from the behavior of v on the level sets Γ_m. Denote by m_0 the constant value of w on G_0 (so that $G_0 \subset \Gamma_{m_0}$). We can find sequences of numbers m tending to m_0 from above and from below, such that each Γ_m is a Jordan curve through infinity and v is strictly monotonous on Γ_m. Note that these curves do not meet each other, and all the points of G_0 are limits of points of such curves. [Indeed, otherwise w would have a local minimum somewhere, and since almost all its level lines contain no loop, this would force w, and then v, to be locally constant somewhere.] We shall see in the proof that this forces a behavior of v on G_0 like the one suggested by Figure 2.1. and Figure 2.1 bis. See Section 34.

To continue our argument we set $m_0 = \inf\{w(z)\ ;\ z \in G\}$. Note that $m_0 > -\infty$ because G has only one unbounded component G_∞, and $G \setminus G_\infty$ is bounded. Next set

$$\Omega_0 = \left\{ z \in \mathbb{R}^2\ ;\ w(z) < m_0 \right\}. \tag{2.5}$$

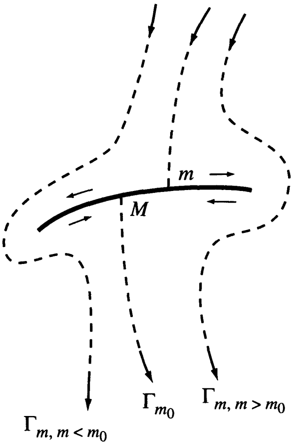


FIGURE 2.1

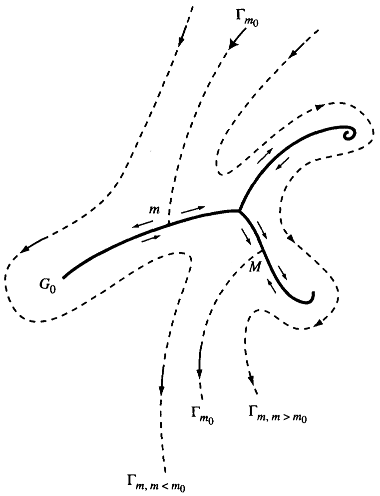


FIGURE 2.1 BIS

FIGURES 2.1 AND 2.1 BIS. The arrows indicate the direction where v increases. The minimum of v is at m and the maximum at M.

It is not too hard to show that

$$\partial\Omega_0 \text{ is connected.} \tag{2.6}$$

Actually, we shall not really need to check (2.6), it will be enough to know that $\partial\Omega_0$ is the limit of Jordan curves Γ_m, $m < m_0$.

For each fixed point $x \in \overline{\Omega}_0$, set

$$\Phi(r) = \frac{1}{r}\int_{\Omega_0\cap B(x,r)} |\nabla v|^2 \tag{2.7}$$

for $r > 0$. Because of (2.6), we can apply the monotonicity argument of [Bo] to get that

$$\Phi \text{ is nondecreasing.} \tag{2.8}$$

Indeed, the argument in [Bo] only needs (2.6), plus the fact that v is harmonic and $\partial v/\partial n = 0$ on $\partial\Omega_0$. This last property holds on $\partial\Omega_0 \cap G$ by the Neumann condition on v, and on $\partial\Omega_0 \setminus G \subset \Gamma_{m_0} \cap \big(\mathbb{R}^2 \setminus G\big)$ because $\partial v/\partial n = \partial w/\partial\tau = 0$ there. See Section 35.

Set $\ell(x) = \lim_{r\to 0} \Phi(r)$. We can use (2.8) and the fact that (v, G) looks like (u_0, K_0) at infinity to show that $\ell(x) \leqslant \lim_{r\to\infty} \Phi(r) = 1$.

Because the only case when $\Phi(r)$ can be locally constant in the monotonicity argument of [Bo] is the case of a multiple of a cracktip, we even get that $\ell(x) < 1$.

It is also possible to get rid of the case when $0 < \ell(x) < 1$. The rough idea is that in this case (v, G) would have blow-ups at x for which the analogue of Φ is constant. These blow-ups would have to be multiples of cracktips (as above), and they also would have to be global minimizers (as are all blow-ups of (v, G)). This last would force $\ell \geqslant 1$ (essentially, by direct computation on a cracktip).

The argument in Sections 36-37 is a little more complicated than what was just implied, because we have to take care of the difference between $\Omega_0 \cap B(x, r)$ and $B(x, r) \setminus G$ in (2.7).

So we finally get to know that $\ell(x) = 0$ for all $x \in \overline{\Omega}_0$. Modulo the difference between $\Omega_0 \cap B(x, r)$ and $B(x, r) \setminus G$, this means that all points of $\overline{\Omega}_0$ are "low energy points". We are only interested in points $x \in G \cap \overline{\Omega}_0 = G \cap \partial\Omega_0$ and (like in the standard Mumford-Shah theory for low energy points) we shall be able to prove that all points of $G \cap \partial\Omega_0$ are either regular points of G (as defined above) or "spider points". This last means that for r small enough, $G \cap B(x, r)$ is the union of three disjoint C^1 arcs that connect x to $\partial B(x, r)$, and make $120°$ angles with each other at their common endpoint x.

We are now fairly close to the final contradiction. Pick a point $x_0 \in \partial\Omega_0 \cap G$. Note that G is not connected, because otherwise the argument in [Bo] would show that (v, G) is a cracktip. Thus we can choose x_0 out of the unbounded connected component G_∞ of G. [This uses the fact that $w = 0$ on G_∞ and $w \leqslant 0$ everywhere; actually we even show that $m_0 < 0$, so that $\partial\Omega_0$ does not touch G_∞.] Finally denote by G_0 the connected component of x_0 in G. Since x_0 is a regular or a spider point of G, G_0 is not reduced to one point.

We now use our description of the variations of (the boundary values of) v when we turn around G_0. We shall use a parameterization $z : \mathbb{S}^1 \to G_0$ of G_0, which we shall call the "tour of G_0", and which corresponds to turning around G_0 (once) in the trigonometric sense. For almost-every $t \in \mathbb{S}^1$, $z(t)$ is a regular point of G, and $z'(t)$ exists and is nonzero. For such t, v has a limit at $z(t)$, where we only consider the access from $\mathbb{R}^2 \setminus G$ from the right of G (which makes sense because $z'(t) \neq 0$). Call this limit $u(t)$. It turns out that u has a continuous extension to $\mathbb{S}^1$ and that, as was suggested above, $\mathbb{S}^1$ splits into two intervals I_1 and I_2 with disjoint interiors, where u is respectively strictly increasing and decreasing.

When $t \in I_1$ and $z(t)$ is a regular point of G, the part of $\mathbb{R}^2 \setminus G$ on the right of G near $z(t)$ lies in Ω_0, because $\partial w/\partial n = -\partial v/\partial\tau < 0$ at $z(t)$. Hence $z(t) \in \partial\Omega_0$, and this stays true for all $t \in I_1$ by continuity.

So $z(I_1)$ is entirely composed of regular and spider points of G. Since spider points are isolated by definition, and G_0 is compact, there are only finitely many of them. Hence $z(I_1)$ is composed of finitely many C^1 arcs, which may only meet at spider points. See Figure 2.2; we do not know how ugly G can be on $z(I_2)$, but we shall not care.

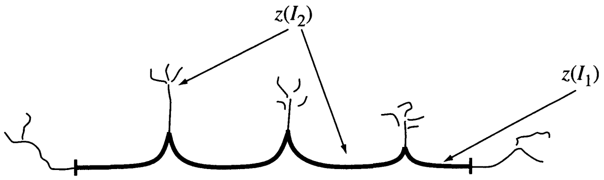


FIGURE 2.2. Symbolic picture of $z(I_1)$.

Note that the restriction of z to I_1 is injective. This comes from (2.4), which forbids the existence of any loop in G, and our description of $z(I_1)$ as a finite union of C^1 curves with no endpoint in I_1 (which prevents $z(.)$ from arriving at the end of an arc, and then returning on the other side, as one imagines happens for a cracktip).

Let $t \in I_1$ be such that $z(t)$ is a regular point. Then there is another point $t^* \in \mathbb{S}^1 \setminus \{t\}$ such that $z(t^*) = z(t)$, and we know that $t^* \in I_2$ because $t \in I_1$. Set $u^*(t) = u(t^*)$. One can check that t^* is a (locally) decreasing function of t (by plane topology and because z turns around G in the trigonometric sense), hence u^* is increasing on I_1 (because u is decreasing on I_2). This is true near points $t \in I_1$ such that $z(t)$ is regular, but when $z(t)$ is a spider point, t^* and $u^*(t)$ may have jumps, which are respectively negative (by topology again) and positive.

Altogether, u is continuous and increasing on I_1 (the continuity comes from the fact that all points of $z(t)$ are regular or spider points), and u^* is increasing also, with a finite number of positive jumps that come from spider points. Also, $u(t) \neq u^*(t)$ on I_1, because of (2.3). [When $z(t)$ is a spider point, $u^*(t)$ takes two values, and both are different from $u(t)$.]

Here comes at last the contradiction. Call a the initial endpoint of I_1. Because u increases on I_1 and decreases on I_2, $u(a) = \inf\{u(t); t \in \mathbb{S}^1\}$. Hence $u(a) \leqslant u^*(a)$, with obvious adaptation to the case when $z(a)$ is a spider point. Since $u^*(t) \neq u(t)$ on I_1, u is continuous, and u^* has only positive jumps, we get that $u^*(t) > u(t)$ on I_1. In particular, if b denotes the endpoint of I_1, we get that $u^*(b) > u(b)$. But $u^*(b) \leqslant u(b)$, by the same argument that led to $u(a) \leqslant u^*(a)$. This contradiction will complete our proof.

For the rest of this paper we shall try to give proofs that are as detailed as seems reasonably possible. Hopefully the reader will be able to skip rapidly through some sections; we shall try to make this easier by giving explicit names to them. In some occasions, we shall ask the reader to believe that the usual proof that Mumford-Shah minimizers have some property extends to the situation of this paper, instead of repeating the (long) proof; we shall try to limit these occasions to a small number.

CHAPTER B

EXISTENCE AND REGULARITY RESULTS FOR A MODIFIED FUNCTIONAL

3. The functional J_R, $R > 1$

As we said in the introduction, our general plan for proving (1.15) is to assume that the cracktip defined by (2.1) and (2.2) is not a global minimizer, try to use this to define an other, exotic minimizer, and eventually derive a contradiction.

Our first stage will be to define, for each $R > 1$, a functional J_R which is fairly close to the global Mumford-Shah functional of the introduction, and which admits minimizers that are not the cracktip. For J_R we shall force our competitors to coincide with (u_0, K_0) outside of $\overline{B}_R = \overline{B}(0, R)$, but later on we shall let R tend to $+\infty$, and then undesirable local boundary effects will disappear.

Let us first define our set U_R of acceptable competitors. Set

$$
\begin{aligned}
U_R = \; & \{(v, G) \in U_0 : G \text{ contains } \mathrm{L} = (-\infty, -1],\ G \setminus \overline{B}_R = K_0 \setminus \overline{B}_R, \\
& \text{and } v(x) = u_0(x) \text{ for all } x \in \mathbb{R}^2 \setminus (\overline{B}_R \cup K_0)\},
\end{aligned} \tag{3.1}
$$

where U_0 is the set of admissible pairs in the introduction. Next set

$$
J_R(v, G) = h\left(H^1(G \setminus L)\right) + \int_{B_R \setminus G} |\nabla v|^2 , \tag{3.2}
$$

where h is a nice increasing function with the following properties. First

$$
h(t) = t \text{ for } 0 \leqslant t \leqslant t_1, \tag{3.3}
$$

where $t_1 \geqslant 2$ will be chosen soon. The main point will be to make sure that (u_0, K_0) does not minimize J_R. Next we want that

$$
h(t) = At \text{ for } t \text{ large enough}, \tag{3.4}
$$

with a (large) constant A that will be chosen later. The point here is that if A is large enough, it will not be worth taking competitors with large values of $H^1(G \setminus L)$. This in turn will help us prove that for minimizers (v_R, G_R) of J_R, $G_R \setminus L$ stays bounded (independently of R). We also require that

$$h \text{ be increasing, convex, of class } C^1, \tag{3.5}$$

and that

$$h'(Bt) \leqslant 2h'(t) \quad \text{for all} \quad t > 0, \tag{3.6}$$

with a large constant B to be chosen later. [So h' only increases very slowly; this will make our life easier.]

Of course the constraints on h are not optimal; we only want a reasonable choice of h to work with.

We shall now choose t_1; the other constants A and B will be chosen later in the proof. Since we have assumed that (u_0, K_0) is not a global minimizer, there is a competitor (u_1, K_1) for (u_0, K_0) which does strictly better. Thus (u_1, K_1) coincides with (u_0, K_0) (as in (1.3) and (1.4)) out of some ball $B_1 = B(0, R_1)$, and

$$H^1(K_1 \cap B_1) + \int_{B_1 \setminus K_1} |\nabla u_1|^2 < H^1(K_0 \cap B_1) + \int_{B_1 \setminus K_0} |\nabla u_0|^2 . \tag{3.7}$$

[See (1.7) and the remark that follows; also note that (1.5) is void here, because $\mathbb{R}^2 \setminus K_0$ is connected.]

A small computation using the homogeneity of u_0 shows that $K_2 = R_1^{-1} K_1$ and $u_2(x) = R_1^{-1/2} u_1(R_1 x)$ also have the properties above, but with the radius $R_2 = 1$. Thus we may assume that (u_1, K_1) was already chosen with $R_1 = 1$. Then $(u_1, K_1) \in U_R$ for all $R \geqslant 1$. We now choose $t_1 \geqslant 2$ so that $H^1(K_1 \setminus L) \leqslant t_1$. With this choice of t_1,

$$\begin{aligned} J_R(u_1, K_1) &= H^1(K_1 \setminus L) + \int_{B_R \setminus K_1} |\nabla u_1|^2 \\ &< H^1(K_0 \setminus L) + \int_{B_R \setminus K_0} |\nabla u_0|^2 = J_R(u_0, K_0), \end{aligned} \tag{3.8}$$

by (3.7), and because the contributions from $\overline{B}_R \setminus B_1$ and $H^1\left(L \cap \overline{B}_R\right)$ are equal. Set

$$\eta(R) = \inf\{J_R(u, K)\ ;\ (u, K) \in U_R\}. \tag{3.9}$$

We have just seen that

$$\eta(R) < J_R(u_0, K_0). \tag{3.10}$$

In the next few sections, we want to prove the existence of minimizers for the functional J_R on U_R. At the same time, we want to derive some information on the minimizers. All this will be very close to standard proofs in the Mumford-Shah theory. We shall give the details anyway, both for completeness and because we need to convince the reader that some of our estimates do not depend on A.

Our proof of existence will use minimizing sequences and uniform estimates (like the concentration property of [DMS]), instead of the compactness properties of BV or SBV. Because of this, it will be useful to consider also the restriction J_R^k of J_R to the smaller class of candidates

$$U_R^k = \{(u,K) \in U_R : K \text{ has at most } k \text{ connected components}\}. \tag{3.11}$$

The advantage of this class, as we shall see, is that J_R^k automatically reaches its minimum on U_R^k.

4. Local Ahlfors-regularity for minimizers of J_R

Let $R > 1$ be fixed, and let (v,G) be a minimizer for J_R (on U_R) or for J_R^k (on U_R^k).

We shall systematically assume that (v,G) is "reduced", which means that there is no closed subset G' of G, with $G' \neq G$ and $L \subset G'$, and such that v has an extension in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus G')$. In the case of J_R^k, we only consider sets G' with at most k components. In other words, we require that G be minimal with the given v. It is fairly easy to replace any minimizer for J_R (or J_R^k) with a reduced minimizer with (essentially) the same v. Set

$$G^- = G \setminus L = G \setminus (-\infty, -1]. \tag{4.1}$$

The main purpose of this section is the prove that G^- is locally Ahlfors-regular, like in the elimination lemma of [DMS]. We start with simpler properties.

Lemma 4.2. — *For $x \in \mathbb{R}^2$ and $r > 0$,*

$$H^1\left(G^- \cap B(x,r)\right) \leqslant 2\pi r. \tag{4.3}$$

To prove this, simply compare (v,G) with the competitor (v_1,G_1) where

$$G_1 = G \cup \partial\left(B_R \cap B(x,r)\right) \setminus \left(G^- \cap B(x,r)\right), \tag{4.4}$$

and v_1 is the same as v except on $B_R \cap B(x,r)$, where we take v_1 to be any constant. It is clear that $(v_1,G_1) \in U_R$. If $(v,G) \in U_R^k$ and $G \cap B(x,r) \neq \varnothing$, then G_1 does not have more connected components than G, and $(v_1,G_1) \in U_R^k$ as well. [Otherwise, (4.3) is true trivially and we don't need to worry.] Obviously

$$\int_{B_R \setminus G_1} |\nabla v_1|^2 = \int_{B_R \setminus G} |\nabla v|^2 - \int_{B_R \cap B(x,r) \setminus G} |\nabla v|^2 \leqslant \int_{B_R \setminus G} |\nabla v|^2, \tag{4.5}$$

and since

$$H^1\left(G_1^-\right) \leqslant H^1\left(G^-\right) - H^1\left(G^- \cap B(x,r)\right) + 2\pi r \tag{4.6}$$

by (4.4), we get (4.3) by minimality of (v,G). □

The same argument also tells us something about

$$a = \int_{B_R \cap B(x,r) \setminus G} |\nabla v|^2 .$$

Let us assume that

$$G \cap B(x,r) \neq \varnothing, \tag{4.7}$$

so that G_1 above does not have more connected components than G and $(v_1, G_1) \in U_R^k$ if $(v, G) \in U_R^k$. Since (v, G) is a minimizer,

$$a \leqslant h\left(H^1\left(G_1^-\right)\right) - h\left(H^1\left(G^-\right)\right) \tag{4.8}$$

by (4.5) and (3.2). This will be easier to use if r is not too large. Set

$$r_G = B + BH^1\left(G^-\right), \tag{4.9}$$

where B is as in (3.6), and

$$\lambda = \lambda_G = h'\left(H^1\left(G^-\right)\right). \tag{4.10}$$

Let us check that

$$h\left(H^1\left(G^-\right) + t\right) \leqslant h\left(H^1\left(G^-\right)\right) + 4\lambda t \text{ for } 0 \leqslant t \leqslant 10\, r_G. \tag{4.11}$$

If $t \leqslant \left(B^2 - 1\right) H^1\left(G^-\right)$, this is an immediate consequence of (3.6) (because h' is nondecreasing). Otherwise $H^1\left(G^-\right) \leqslant t/(B^2 - 1) \leqslant 10 r_G/(B^2 - 1)$ and, if B is large enough, a comparison with (4.9) gives that $H^1\left(G^-\right) \leqslant 1$. In this case $\lambda = 1$, $t \leqslant 10 r_G \leqslant 20B$, and (4.11) follows again from (3.6).

We may now return to (4.8). Note that $H^1\left(G_1^-\right) \leqslant H^1\left(G^-\right) + 2\pi r$ by (4.6). Hence, if $r \leqslant r_G$ we may apply (4.11) with $t = 2\pi r$, and (4.8) says that $a \leqslant 8\pi r\lambda$. Let us summarize.

Lemma 4.12. — *If (v, G) minimizes J_R or J_R^k, $x \in \mathbb{R}^2$, $0 \leqslant r \leqslant r_G$, and if (4.7) holds, then*

$$\int_{B_R \cap B(x,r) \setminus G} |\nabla v|^2 \leqslant 8\pi r\lambda. \tag{4.13}$$

We now come to the more delicate part of this section, the local Ahlfors-regularity of G^-. The next result is only a minor modification of the "elimination lemma" in [DMS]. Our proof will be more like the one in [MoSo] or [DaSe1]; we shall repeat the main lines because we need to be careful about the dependence on constants.

Proposition 4.14. — *There is a constant $C_1 \leqslant 1$, that does not depend on A, B, or R, such that for all $R > 1$, all minimizers (v, G) for J_R or for some J_R^k, all $x \in G^-$ and all radii $r > 0$ such that*

$$r \leqslant r_G \tag{4.15}$$

and

$$B(x,r) \subset B_R \setminus \{-1\}, \tag{4.16}$$

we have that

$$H^1\left(G^- \cap B(x,r)\right) \geqslant C_1^{-1} r. \tag{4.17}$$

We want to prove the proposition by contradiction, and so we assume that we can find x, r as in the statement, but such that (4.17) does not hold. We want to produce a contradiction (if C_1 is large enough).

First observe that since (v, G) is a *reduced* minimizer, $H^1(G \cap B(x,\rho)) > 0$ for all $\rho > 0$. This uses the fact that closed sets with vanishing one-dimensional Hausdorff measure are removable for bounded functions in $W^{1,2}_{\text{loc}}$. In more concrete terms, if $H^1(G \cap B(x,\rho)) = 0$ for some $\rho > 0$, we could extend v through $G \cap B(x,\rho)$ to get a function in $W^{1,2}_{\text{loc}}$ near x. The argument is classical (see already [MuSh]), and so we do not elaborate.

Since $x \in G^- = G \setminus L$ we also get that $H^1(G^- \cap B(x,\rho)) > 0$ for all $\rho > 0$.

By a minor variant of the Lebesgue differentiation theorem (see for instance [Ma], page 86), we know that for H^1-almost all $y \in G^-$,

$$\limsup_{\rho \to 0} \rho^{-1} H^1\left(G^- \cap B(y,\rho)\right) \geqslant 1. \tag{4.18}$$

In particular, we can find points $y \in G^- \cap B(x, r/10)$ such that (4.18) holds. For such points, the disk $B(y, r/10)$ still satisfies the constraints (4.15) and (4.16), and does not satisfy the conclusion (4.17) either, except for the fact that we have to replace C_1 with $10C_1$. Thus we may as well assume that our first choice of x satisfies (4.18).

All the disks $B\left(x, 10^{-j} r\right)$, $j \geqslant 0$, satisfy the hypotheses (4.15) and (4.16). Because x satisfies (4.18) (and if $C_1 > 10$, say), many of these disks satisfy (4.17). Thus we may replace our initial choice of r with a new one (of the type $10^{-j} r$, but which we'll call r immediately), which still satisfies the negation of (4.17), i.e.,

$$H^1\left(G^- \cap B(x,r)\right) < C_1^{-1} r, \tag{4.19}$$

but for which $r/10$ satisfies (4.17), i.e.,

$$H^1\left(G^- \cap B(x, r/10)\right) \geqslant (10C_1)^{-1} r. \tag{4.20}$$

We want to show that this is impossible if C_1 is large enough. We shall start with the most delicate case when

$$B(x, r/8) \text{ meets } L. \tag{4.21}$$

Denote by x_1 the point of L closest to x. Obviously

$$|x_1 - x| < \frac{r}{8}. \tag{4.22}$$

We want to construct a competitor (v_1, G_1) that should be better than (v, G), by removing a good part of $G^- \cap B(x,r)$. We first choose a disk $B_1 = B(x_1, r_1)$ with the following properties:

$$\frac{2r}{3} \leqslant r_1 \leqslant \frac{3r}{4}, \tag{4.23}$$

$$\partial B_1 \cap G^- = \varnothing, \tag{4.24}$$

$$\int_{\partial B_1 \setminus G} |\nabla v|^2 \leqslant 10^3 \lambda, \tag{4.25}$$

and

$$\int_{G^- \cap B(x,r)} \operatorname{dist}(z, \partial B_1)^{-1/2} \, dH^1(z) \leqslant I, \tag{4.26}$$

where we set

$$I = \frac{100}{r} \int_{t=2r/3}^{3r/4} \int_{G^- \cap B(x,r)} \operatorname{dist}(z, \partial B(x_1, t))^{-1/2} \, dH^1(z) dt. \tag{4.27}$$

The precise reasons for our last constraint (4.26) will become clear later. Let us first observe that we can choose B_1 as above. Indeed the set of radii r_1 that satisfy (4.23) but not (4.24) has measure $\leqslant C_1^{-1} r$ because of (4.19). [This set is contained in the image under the 1-Lipschitz mapping $z \to |z - x_1|$ of $G^- \cap B(x,r)$.] So most of the choices of r_1 in (4.23) satisfy (4.24). Similarly, (4.25) holds for most choices of r_1 because of (4.13), and we can also add the final constraint (4.26), by Chebyshov and the definition of I.

So we can choose B_1 with the properties (4.23)-(4.26). Note that since $x_1 \in L$, $|x_1 - x| \leqslant r/8$ (by (4.22)), and $B(x,r) \subset B_R \setminus \{-1\}$ (by (4.16)), we know that the half-line L contains a diameter of B_1. Let us first modify G and v in the upper half

$$B_1^+ = \{(a,b) \in B_1 \; ; \; b > 0\} \tag{4.28}$$

of B_1. We take

$$G_1 = G \setminus B_1^+, \tag{4.29}$$

and let v_1 be the function defined on $\mathbb{R}^2 \setminus G_1$, equal to v on $\mathbb{R}^2 \setminus \left(B_1^+ \cup G\right)$, continuous on $\overline{B}_1^+ \setminus L$, harmonic on B_1^+, and which minimizes

$$E_1 = \int_{B_1^+} |\nabla v_1|^2 . \tag{4.30}$$

Note that the half-circle $\partial B_1^+ \setminus L$ does not meet G because of (4.24); hence the values of v_1 on B_1^+ can be computed from the values of v on $\partial B_1^+ \setminus L$ by first extending these to ∂B_1 by symmetry, and then taking the restriction to B_1^+ of the Dirichlet extension. We shall give more details about this construction after (4.53); for the moment, let us just observe that $(v_1, G_1) \in U_R$. We also have that $(v_1, G_1) \in U_R^k$

if $(v, G) \in U_R^k$. The point is that the piece of G that we have removed could not be essential to connect together pieces of G_1: the remaining piece $L \cap B_1$ can be used at least as efficiently.

Now we want to compare our two candidates, and in particular estimate the cost in energy that we have to pay for removing $G \cap B_1^+$, namely,

$$\Delta E = E_1 - E = E_1 - \int_{B_1^+ \setminus G^-} |\nabla v|^2 . \tag{4.31}$$

In the computations that follow, and in order to get more rapidly to the central part of the argument, we shall integrate by parts on the slightly irregular domain $\Omega = B_1^+ \setminus G^-$ without much justification. These integrations by parts are the same (in fact, a little simpler because there is no initial image g here) as in [MoSo], where they are carefully justified.

For the sake of completeness, let us also describe an (essentially equivalent) way to deal with this issue. For each small $\varepsilon > 0$, we can surround $G^- \cap B_1^+$ by a finite collection Γ_ε of piecewise C^1 curves. In fact, we can even take Γ_ε to be composed of line segments and arcs of circles, to stay within ε of $G^- \cap B_1^+$, and to be contained in $B_1^+ \cup (L \cap B_1)$. Then consider the domains Ω_ε bounded by $\partial B_1^+ \cup \Gamma_\varepsilon$ and contained in Ω. On these domains we define harmonic functions v_ε, with the same boundary values on $\partial B_1^+ \setminus L$ as v and v_1, and which minimize the energy $E_\varepsilon = \int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2$.

On the domain Ω_ε we can integrate by parts, and the argument below will give good estimates on $E_1 - E_\varepsilon$ that do not depend on ε. It is easy to check that (after taking a subsequence if you wish) the functions v_ε converge to v, and that E_ε tends to E. We then get the desired estimates on ΔE with a limiting argument (and without integrating by parts on irregular domains). Let us not give more details here (and refer to [MoSo]), but only mention that the construction of the curves Γ_ε and domains Ω_ε will be done later, although in a slightly different situation (see Section 23), and the limiting arguments concerning v_ε and E_ε also (see Section 24).

So let us proceed to our estimate of

$$\Delta E = \int_\Omega \left\{ |\nabla v_1|^2 - |\nabla v|^2 \right\} . \tag{4.32}$$

Write $|\nabla v_1|^2 - |\nabla v|^2 = \nabla (v_1 - v) \cdot \nabla (v_1 + v)$; then Green says that

$$\begin{aligned} \Delta E &= - \int_\Omega (v_1 - v) \, \Delta (v_1 + v) + \int_{\partial\Omega} (v_1 - v) \frac{\partial (v_1 + v)}{\partial n} \, dH^1 \\ &= \int_{\partial\Omega} (v_1 - v) \frac{\partial (v_1 + v)}{\partial n} \, dH^1, \end{aligned} \tag{4.33}$$

with the choice of unit normal n to $\partial\Omega$ that makes the signs in our formula right, and because v and v_1 are both harmonic on Ω.

Decompose $\partial\Omega$ into three disjoint pieces ∂_1, ∂_2, ∂_3, with

$$\partial_1 = \partial B_1^+ \setminus L, \quad \partial_2 = \partial B_1^+ \cap L, \text{ and } \quad \partial_3 = G^- \cap B_1^+. \tag{4.34}$$

By definition of v_1 (and also (4.24)), $v_1 - v$ vanishes on ∂_1. So we may forget about ∂_1 in (4.33). Next $\partial v_1/\partial n = 0$ on ∂_2; this is the (classical) Neumann condition on v_1 that comes from the minimality of E_1 (through the Euler-Lagrange condition). Similarly, $\partial v/\partial n = 0$ on $\partial_2 \cup \partial_3$. Thus we can also forget about ∂_2 in (4.33), and (4.33) simplifies to

$$\Delta E = \int_{\partial_3} (v_1 - v)\, \frac{\partial\,(v_1 + v)}{\partial n}\, dH^1 = \int_{G^- \cap B_1^+} (v_1 - v)\, \frac{\partial v_1}{\partial n}. \tag{4.35}$$

Note that we just committed a slight abuse of notation, because in the last integral, we still mean an integral on ∂_3. Thus it is implied here that a given point of $G^- \cap B_1^+$ may be counted more than once in the integral (in general, twice and with opposite orientations of the unit normal).

We can also get rid of $v_1 \partial v_1/\partial n$ in (4.35). First apply Green's formula on B_1^+ to get that

$$\int_{B_1^+} |\nabla v_1|^2 = -\int_{B_1^+} v_1 \Delta v_1 + \int_{\partial B_1^+} v_1 \frac{\partial v_1}{\partial n} = \int_{\partial B_1^+} v_1 \frac{\partial v_1}{\partial n} \tag{4.36}$$

because v_1 is harmonic on B_1^+. The same computation with the domain $\Omega = B_1^+ \setminus G^-$ yields

$$\int_{B_1^+} |\nabla v_1|^2 = \int_\Omega |\nabla v_1|^2 = \int_{\partial B_1^+} v_1 \frac{\partial v_1}{\partial n} + \int_{G^- \cap B_1^+} v_1 \frac{\partial v_1}{\partial n}, \tag{4.37}$$

with just one additional term that comes from the inside boundary, and with the same convention (or abuse of notation) concerning integration on $G^- \cap B_1^+$ as in (4.35). From (4.36) and (4.37) we get that

$$\int_{G^- \cap B_1^+} v_1 \frac{\partial v_1}{\partial n} = 0, \tag{4.38}$$

and then (4.35) yields

$$\Delta E = -\int_{G^- \cap B_1^+} v \frac{\partial v_1}{\partial n}. \tag{4.39}$$

This is the usual formula about jumps (except that we managed to get to it without talking about jumps). Indeed it turns out that almost every point of $G^- \cap B_1^+$ is counted twice in (4.39) (one for each access from Ω), with opposite choices of unit normals. Note also that v_1 is differentiable on $G^- \cap B_1^+$ because it is harmonic on B_1^+. We may choose to regroup the two occurrences of almost each point of $G^- \cap B_1^+$, and then we would be integrating $\mathrm{Jump}(v)\partial v_1/\partial n$ in (4.39). We decided to avoid this way of presenting things, because it makes it more clear that our computations also work in approximating domains Ω_ε (as above), and almost allows us not to mention that G^- is rectifiable.

To estimate ΔE we shall use a localization argument. Let D be any disk centered on $\partial_3 = B_1^+ \cap G^-$ and with radius $10C_1^{-1}r$. Note that $2D \subset B(x,r)$ by (4.22) and (4.23). Hence (4.19) and the same argument as for (4.24) (with the image of $2D \cap G^-$ by a Lipschitz mapping) allow us to find a new disk D' with the same center as D, such that $D \subset D' \subset 2D$, and

$$\partial D' \cap G^- = \varnothing. \tag{4.40}$$

Also, because of Lemma 4.12 and our hypothesis (4.15), we can even take D' so that

$$\int_{\partial D' \setminus L} |\nabla v|^2 \leqslant 10^3 \lambda. \tag{4.41}$$

[Here (4.7) holds because D is centered on G^-, and we use Chebyshov as for (4.25).]

Set $D'' = D' \cap B_1^+$, call D''' the connected component of $D'' \setminus G$ that touches $\partial D'' \setminus G$, and denote by

$$\operatorname{osc}(v\ ;\ D''') = \sup_{D'''} v - \inf_{D'''} v \tag{4.42}$$

the oscillation of v on D'''. We want to check that

$$\operatorname{osc}(v\ ;\ D''') \leqslant \operatorname{osc}(v\ ;\ \partial D'' \setminus G), \tag{4.43}$$

and even that

$$\sup_{D'''} v \leqslant \sup_{\partial D'' \setminus G} v, \tag{4.44}$$

and the similar (but opposite) inequality for the infimum. This is just saying that the values of v on D''' are controlled by its values on the part of $\partial(D'' \setminus G)$ which is not contained in G.

At this point we should rapidly discuss the uniqueness of v. We know that v minimizes the energy $I = \int_{D'' \setminus L} |\nabla v|^2$, with the given boundary values on $\partial D'' \setminus G$. If $v' \in W^{1,2}(D'' \setminus G)$ has the same boundary values as v on $\partial D'' \setminus G$, and $\int_{D'' \setminus L} |\nabla v'|^2 = I$, then, by the strict convexity of the L^2-norm, $\frac{1}{2}(v+v')$ does strictly better, unless $|\nabla v'| = |\nabla v|$ almost everywhere on $D'' \setminus G$. Thus v may not be unique, but $|\nabla v|$ is. In the situation above, we can take $v' = \operatorname{Min}\{v(z), \sup_{\partial D'' \setminus G} v\}$ and get that $|\nabla(v' - v)| = 0$ on $D'' \setminus G$. This implies that $v' = v$ on D''', and (4.44) and (4.43) follow.

If W is a connected component of $D'' \setminus G$ that does not touch $\partial D'' \setminus G$, we only know that is v is constant on W, but the value of the constant could be anything, and will not matter.

Note that $\partial D'' \setminus G = \partial D'' \setminus L$ (because of (4.24) and (4.40)), and that it is connected. Thus

$$\begin{aligned}(4.45)\qquad \operatorname{osc}(v\,;\,\partial D''\setminus G) &\leqslant \int_{\partial D''\setminus L} |\nabla v|\, dH^1 \\ &\leqslant H^1(\partial D''\setminus L)^{1/2}\left\{\int_{\partial D''\setminus L}|\nabla v|^2\right\}^{1/2} \\ &\leqslant \left(40\pi C_1^{-1} r\right)^{1/2}\left(2.10^3\lambda\right)^{1/2} \leqslant 10^3 C_1^{-1/2} r^{1/2}\lambda^{1/2}\end{aligned}$$

by Cauchy-Schwarz, because $\partial D'' \setminus L \subset (\partial B_1 \setminus G) \cup (\partial D' \setminus L)$, and by (4.25) and (4.41). From (4.43) and (4.45) we deduce that there is a constant C_D such that

$$(4.46)\qquad |v(z) - C_D| \leqslant 10^3 C_1^{-1/2} r^{1/2} \lambda^{1/2} \text{ for } z \in D'''.$$

We are almost ready to plug this in (4.39), but let us first check that

$$(4.47)\qquad \int_{\partial_3 \cap D''} \frac{\partial v_1}{\partial n} = 0,$$

where $\partial_3 = B_1^+ \cap G^-$ as usual. Indeed,

$$(4.48)\qquad \int_{D''} \Delta v_1 = \int_{D' \cap B_1^+} \Delta v_1 = \int_{\partial D''} \frac{\partial v_1}{\partial n}$$

by Green, and the same computation on the almost identical domain $D'' \setminus G$ gives

$$(4.49)\qquad \int_{D''} \Delta v_1 = \int_{\partial(D''\setminus G)} \frac{\partial v_1}{\partial n} = \int_{\partial D''} \frac{\partial v_1}{\partial n} + \int_{D''\cap G^-} \frac{\partial v_1}{\partial n}$$

(with the same abuse of notation as usual, and because L does not meet B_1^+ (and hence $D'' \cap G = D'' \cap G^-$)). This proves (4.47).

Note that (4.47) stays true if we replace D'' with another regular domain contained in D'' and whose boundary does not meet G^-. Actually we shall use (4.47) with domains of the type $D'' \setminus \bigcup_{j=1}^m D_j''$, where the D_j'' are constructed just like D''.

Cover ∂_3 by disks D_j, $j \geqslant 1$, centered on ∂_3 and with the same radius $10C_1^{-1}r$, in such a way that the $2D_j$, $j \geqslant 1$, have bounded overlap. For each D_j, select D_j' as above, and set $D_j'' = D_j' \cap B_1^+$. Obviously the D_j'' still cover ∂_3, because $\partial_3 \subset B_1^+$. Finally set $D_j^* = D_j'' \setminus \bigcup_{k<j} D_k''$. Then

$$(4.50)\qquad \Delta E = -\int_{\partial_3} v \frac{\partial v_1}{\partial n} = -\sum_j \int_{\partial_3 \cap D_j^*} v \frac{\partial v_1}{\partial n}$$

by (4.39), and because the D_j^* are disjoint and cover ∂_3. We can apply (4.47) to D_j^*, because ∂D_j^* is composed of pieces of $\partial D_k''$, $k \leqslant j$, and $\partial D_k''$ does not meet G^- (by (4.24) and (4.40)).

To compute $\int_{\partial_3 \cap D_j^*} v \partial v_1 / \partial n$, we would like to use (4.46), but it only tells us about the values of v in D_j''', the component of $D_j'' \setminus G$ that touches $\partial D_j'' \setminus G$. We also need to control v in the other components of $D_j'' \setminus G$. Let W be such a component; then W is also a component of $\mathbb{R}^2 \setminus G$. Also,

$$\text{(4.51)} \qquad \operatorname{diam} W \leqslant H^1(\partial W \setminus L) = H^1(\partial W \cap G^-) \leqslant H^1(G^- \cap D_j'') \leqslant C_1^{-1} r,$$

by (4.19). Hence W does not meet any $\partial D_k''$ (because $\partial D_k'' \setminus L$ is connected, contained in $\mathbb{R}^2 \setminus G$, and its diameter is at least $10 C_1^{-1} r$). Thus W is either contained in D_j^*, or does not meet it. If we add or subtract a constant to v in W, this does not change $\int_{\partial_3 \cap D_j^*} v \partial v_1 / \partial n$, because either $\partial W \cap \partial_3 \cap D_j^* = \varnothing$, or else $\partial W \subset (\partial_3 \cap D_j^*) \cup L$, and then $\int_{\partial_3 \cap \partial W \cap D_j^*} \partial v_1 / \partial n = \int_{\partial W} \partial v_1 / \partial n = 0$ as well, by the proof of (4.47). This is not too surprising, because adding a constant to v in W does not change ΔE either.

So let us replace v by the same constant C_{D_j} as in (4.46) in all the components W, so that now (4.46) holds with D_j''' replaced with $D_j'' \setminus G$. Then

$$\text{(4.52)} \quad \left| \int_{\partial_3 \cap D_j^*} v \frac{\partial v_1}{\partial n} \right| = \left| \int_{\partial_3 \cap D_j^*} (v - C_{D_j}) \frac{\partial v_1}{\partial n} \right| \leqslant 10^3 C_1^{-1/2} r^{1/2} \lambda^{1/2} \int_{\partial_3 \cap D_j^*} \left| \frac{\partial v_1}{\partial n} \right|$$

because (4.47) allows us to remove a constant, and by (4.46). Then (4.50) yields

$$\text{(4.53)} \quad \Delta E \leqslant \sum_j 10^3 C_1^{-1/2} r^{1/2} \lambda^{1/2} \int_{\partial_3 \cap D_j^*} \left| \frac{\partial v_1}{\partial n} \right| \leqslant 10^3 C_1^{-1/2} r^{1/2} \lambda^{1/2} \int_{\partial_3} \left| \frac{\partial v_1}{\partial n} \right|.$$

Next we want to estimate the derivative of v_1 on B_1^+. We first return to the way we can compute v_1 from its boundary values on $\partial B_1^+ \setminus L$. Denote by w the function defined on ∂B_1 which is equal to v on $\partial B_1^+ \setminus L$ and which is symmetric with respect to L. [In other words, take $w(a,-b) = w(a,b) = v(a,b)$ for $(a,b) \in \partial B_1^+ \setminus L$.] Note that w is C^1 on $\partial B_1 \setminus L$ (because v is harmonic away from G and ∂B_1 does not meet G^-), and it is even continuous across the two points of $\partial B_1 \cap L$, because it is symmetric and has limits at those points. This last comes from (4.25), which implies that $\int_{\partial B_1 \setminus L} |\nabla v| < +\infty$. Denote by v_1^* the harmonic extension of w to B_1. It is also the (unique) continuous function on $\overline{B}_1$ which coincides with w on ∂B_1 and minimizes $E_1^* = \displaystyle\int_{B_1} |\nabla v_1^*|^2$. [Uniqueness follows from the convexity of the problem, or from uniqueness for the Dirichlet problem.]

Since v_1^* is symmetric, $E_1^* = 2 \displaystyle\int_{B_1^+} |\nabla v_1^*|^2$. Thus, by definition of v_1 and E_1, we have that $\frac{1}{2} E_1^* \geqslant E_1$ (because the restriction of v_1^* to B_1^+ is a competitor in the definition of E_1).

Also, it is not too hard to check that the symmetric extension of v_1 to B_1 (which we shall still denote by v_1) lies in $W^{1,2}(B_1)$, coincides with w on ∂B_1, and that $\displaystyle\int_{B_1} |\nabla v_1|^2 = 2 \int_{B_1^+} |\nabla v_1|^2 = 2E_1$. Then $2E_1 \geqslant E_1^*$ by definition of E_1^*. Thus $E_1^* = 2E_1$, and $v_1 = v_1^*$ by uniqueness of v_1 (or v_1^*).

Altogether, v_1 is equal on B_1^+ to the Poisson extension of w, and $|\nabla v_1|$ is easy to estimate. In fact, with (4.25) and a few brutal estimates using the Poisson kernel, one can get that

$$|\nabla v_1(z)| \leqslant C\lambda^{1/2}\mathrm{dist}\,(z, \partial B_1)^{-1/2} \text{ for } z \in B_1^+. \tag{4.54}$$

Here C is an absolute constant. We may now plug this in (4.53) and use (4.26) to get that

$$\Delta E \leqslant CC_1^{-1/2}\lambda r^{1/2} \int_{\partial_3} \mathrm{dist}\,(z, \partial B_1)^{-1/2}\, dH^1(z) \leqslant CC_1^{-1/2}\lambda r^{1/2} I \tag{4.55}$$

because $\partial_3 = G^- \cap B_1^+ \subset G^- \cap B(x,r)$.

We still need to estimate I. By Fubini-Tonnelli,

$$\begin{aligned} I &= \frac{100}{r}\int_{G^-\cap B(x,r)} \left\{ \int_{t=2r/3}^{3r/4} ||z-x_1|-t|^{-1/2}\, dt \right\} dH^1(z) \\ &\leqslant Cr^{-1/2}\int_{G^-\cap B(x,r)} dH^1(z) \leqslant Cr^{-1/2} H^1\left(G^-\cap B(x,r)\right) \\ &\leqslant Cr^{-1/2}H^1\left(G^-\cap B_1\right) \end{aligned} \tag{4.56}$$

by (4.19) and (4.20), and because $B\,(x, r/10) \subset B_1$ (by (4.22) and (4.23)). Hence

$$\Delta E \leqslant CC_1^{-1/2}\lambda H^1\left(G^-\cap B_1\right), \tag{4.57}$$

where C is some absolute constant.

Recall from (4.31) and (4.30) that ΔE is the cost in the energy term of (3.2) that you have to pay for replacing our minimizer (v, G) with the competitor (v_1, G_1). Since (v, G) is a minimizer for J_R or J_R^k, this cost must at least compensate what you win in the length term by removing $G \cap B_1^+ = G^- \cap B_1^+$ from G, as in (4.29). Thus

$$h\left(H^1\left(G^-\right)\right) - h\left(H^1\left(G_1^-\right)\right) \leqslant \Delta E. \tag{4.58}$$

Let us check that

$$h(b) - h(a) \geqslant \frac{1}{4}h'(b)(b-a) \text{ for } 0 \leqslant a \leqslant b\,. \tag{4.59}$$

Note that $h'(a) \geqslant h'(b)/2$ for $a \geqslant b/2$, by (3.5) and (3.6). This gives (4.59) (with a twice better constant) for $a \geqslant b/2$. For $a < b/2$, we simply notice that $h(b) - h(a) \geqslant h(b) - h(b/2) \geqslant bh'(b)/4 \geqslant (b-a)h'(b)/4$, by the previous case; (4.59) follows.

Let us apply (4.59) with $b = H^1\left(G^-\right)$, $a = H^1\left(G_1^-\right) = H^1\left(G^-\right) - H^1\left(G^-\cap B_1^+\right) = b - H^1\left(G^-\cap B_1^+\right)$ (because $G_1^- = G^- \setminus B_1^+$ by (4.29)) and $h'(b) = h'\left(H^1\left(G^-\right)\right) = \lambda$ (by (4.10)). Then compare with (4.58) and (4.57). We get that

$$H^1\left(G^-\cap B_1^+\right) = b - a \leqslant \frac{4}{\lambda}\left(h(b) - h(a)\right) \leqslant \frac{4}{\lambda}\Delta E \leqslant 4CC_1^{-1/2}H^1\left(G^-\cap B_1\right). \tag{4.60}$$

We have obtained (4.60) by comparing the minimizer (v, G) with a candidate (v_1, G_1) obtained by modifying (v, G) on the upper half B_1^+ of B_1. The same construction on the lower half B_1^- gives that

$$H^1\left(G^- \cap B_1^-\right) \leqslant 4CC_1^{-1/2} H^1\left(G^- \cap B_1\right). \tag{4.61}$$

Thus

$$H^1\left(G^- \cap B_1\right) = H^1\left(G^- \cap B_1^+\right) + H^1\left(G^- \cap B_1^-\right) \leqslant 8CC_1^{-1/2} H^1\left(G^- \cap B_1\right), \tag{4.62}$$

with the same absolute constant C as in (4.57). If C_1 is chosen larger than $64C^2$, we get the desired contradiction.

This settles our most delicate case when $B(x, r/8)$ meets L (as in (4.21)). When $B(x, r/8)$ does not meet L, we can use a similar but simpler argument. We choose $B_1 = B(x, r_1)$, with $1/10 < r_1 < 1/8$, and otherwise the same properties (4.24)-(4.26) as above (the proof of existence is the same). Then we take $G_1 = G \setminus B_1$ and choose v_1 equal to v on the complement of B_1, continuous on $\overline{B}_1$, and harmonic in B_1. The estimates are the same as above, except that we don't have to care about L, and they give a contradiction in this case as well. This completes our proof of Proposition 4.14. □

5. G^- stays in a fixed ball

We continue to assume that (v, G) is a reduced minimizer for J_R or J_R^k and want to prove that $G^- = G \setminus L$ stays in a fixed ball B_{R_0} (independent of R).

Proposition 5.1. — *If the constants A and B in* (3.4) *and* (3.6) *are chosen large enough, there exist constants C_0 and R_0, independent of R, such that if (v, G) is a minimizer for J_R or J_R^k,*

$$G^- := G \setminus L \subset B_{R_0} \tag{5.2}$$

and

$$H^1\left(G^-\right) \leqslant C_0. \tag{5.3}$$

Remark. — In view of our main application, the fact that cracktips are global minimizers of the Mumford-Shah functional, we shall be happy to fix A and B once and for all, with the simple constraints (5.11), (5.59), and (5.83) that come from Proposition 5.1, and similar constraints that will make our proof of existence for minimizers of J_R easier. Then the fact that C_0 and R_0 may depend on this choice of A and B will not disturb us. It is very important that they do not depend on R, though.

Proposition 5.1 will rely mostly on the local Ahlfors-regularity property of Proposition 4.14 (and specifically the fact that the constant C_1 there does not depend on

R, A, or B). Before we come to that, we need to estimate the maximal amount of energy we can hope to save by adding length to G in a ball. Let us first compute

$$E_0 = \int_{B_R \setminus K_0} |\nabla u_0|^2, \tag{5.4}$$

where (u_0, K_0) still denotes the reference cracktip, as in (2.1) and (2.2).

Let us continue to use polar coordinates (r, θ) as in (2.2). Then

$$\frac{\partial u_0}{\partial r} = \frac{1}{2}\sqrt{2/\pi} r^{-1/2} \sin\frac{\theta}{2}, \tag{5.5}$$

$$\frac{\partial u_0}{\partial \theta} = \frac{1}{2}\sqrt{2/\pi} r^{1/2} \cos\frac{\theta}{2}, \tag{5.6}$$

$$|\nabla u_0|^2 = \left(\frac{\partial u_0}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u_0}{\partial \theta}\right)^2 = \frac{1}{4}\cdot\frac{2}{\pi}\cdot r^{-1}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) = \frac{1}{2\pi r}, \tag{5.7}$$

and

$$E_0 = \int_{B_R} \frac{1}{2\pi r} = \int_{\pi}^{-\pi}\int_0^R \frac{r dr d\theta}{2\pi r} = R. \tag{5.8}$$

Since $(u_0, K_0) \in U_R^k$ for all R, a brutal comparison yields

$$h\left(H^1\left(G^-\right)\right) \leqslant J_R(v, G) \leqslant J_R\left(u_0, K_0\right) = 1 + R \tag{5.9}$$

because we continue to assume that (v, G) minimizes J_R or some J_R^k, and by the definition (3.2) of J_R.

Note that the conclusions of Proposition 5.1 are automatically satisfied when R is small. Indeed (5.2) holds trivially for $R \leqslant R_0$ (because $G^- \subset B_R$ by (3.1)), and (5.3) follows from (5.9) for $R \leqslant C_0 - 1$. So we may assume that R is as large as we want.

Since $h(t) = At$ for t large (by (3.4)), (5.9) implies that

$$H^1\left(G^-\right) \leqslant \frac{1+R}{A} \tag{5.10}$$

if R is large enough.

Let $x \in G^-$ be given, and let us try to apply Proposition 4.14 with the radius $r = 2C_1 H^1\left(G^-\right)$. This is doomed to failure, because we chose r so that (4.17) does not hold. So one of the hypotheses (4.15) or (4.16) is violated. We shall assume that

$$B \geqslant 60 C_1 \quad\text{and}\quad A \geqslant 10 C_1. \tag{5.11}$$

Note that this make sense because C_1 does not depend or A or B in Proposition 4.14. Then

$$r = 2C_1 H^1\left(G^-\right) \leqslant \frac{B}{30} H^1\left(G^-\right) < r_G \tag{5.12}$$

by (5.11) and (4.9). Hence (4.15) holds and (4.16) must be false. One option is that $-1 \in B(x,r)$, in which case

$$|x| \leqslant 1+r = 1+2C_1H^1\left(G^-\right) \leqslant 2+\frac{2C_1R}{A} < \frac{R}{4} \tag{5.13}$$

by (5.10) and (5.11), and if R is large enough. The other option is that

$$|x| \geqslant R-r > \frac{3R}{4} \tag{5.14}$$

(for the same reasons).

Our next task is to exclude this second option, i.e., show that all $x \in G^-$ satisfy (5.13). The proof will be similar to Proposition 4.14, but we shall have to deal with the different boundary conditions on ∂B_R. The general principle is the same: because of (5.10), $G^- \setminus B_{R/4}$ is too small to be really useful. Set

$$G_1 = \left(G \cap B\left(0,R/4\right)\right) \cup L, \tag{5.15}$$

and define v_1 as follows. Observe that

$$G_1 \cap B\left(0,3R/4\right) = G \cap B\left(0,3R/4\right) \tag{5.16}$$

because

$$G \cap B\left(0,3R/4\right) \setminus B\left(0,R/4\right) \subset L \tag{5.17}$$

(see (5.13) and (5.14)). We decide to keep

$$v_1(z) = v(z) \text{ for } z \in \overline{B}\left(0,R/2\right) \setminus G_1 \tag{5.18}$$

(which makes sense because of (5.16)) and

$$v_1(z) = v(z) = u_0(z) \text{ for } z \in \mathbb{R}^2 \setminus \left(B_R \cup L\right), \tag{5.19}$$

which is natural if we want (v_1, G_1) to lie in U_R. Let

$$H = B_R \setminus \left(\overline{B}\left(0,R/2\right) \cup L\right) \tag{5.20}$$

denote the remaining domain. We take v_1 to be harmonic on H, have a continuous extension to $\overline{B}_R \setminus \left(B\left(0,R/2\right) \cup L\right)$ which coincides with v on $\partial B\left(0,R/2\right) \setminus L$ and with u_0 on $\partial B_R \setminus L$, and to minimize the energy

$$E_1 = \int_H |\nabla v_1|^2 . \tag{5.21}$$

Note that the boundary values that we gave ourselves on $\partial B\left(0,R/2\right)$ and $\partial B_R \setminus R$ are C^1 (and even, as we shall see soon, with bounded derivatives), so there is no difficulty with the existence of v_1. Also, it is easy to check that $(v_1, G_1) \in U_R$, i.e., that $v_1 \in W^{1,2}_{\text{loc}}\left(\mathbb{R}^2 \setminus G_1\right)$. Finally, if $(v,G) \in U^k_R$ for some k, then G_1 has at most k connected components also (by (5.17)), and so $(v_1, G_1) \in U^k_R$. We shall need estimates on v_1.

Lemma 5.22. — *There is a universal constant $C > 0$ such that*

(5.23) $$|\nabla v_1| \leqslant CR^{-1/2} \text{ on } H.$$

Let us first check that

(5.24) $$|\nabla v| \leqslant CR^{-1/2} \text{ on } \partial B(0, R/2) \setminus L.$$

To do this, observe that we may assume that

(5.25) $$\sup\{|v(x)| \ ; \ x \in B_R \setminus G\} \leqslant \sup\{|u_0(x)| \ ; \ x \in \partial B_R \setminus L\} = \sqrt{2/\pi}\ R^{1/2}.$$

This is not necessarily true without modification, because $B_R \setminus G$ may a priori have connected components that do not reach ∂B_R, but then v is constant on each of these components, and we may modify the values of the corresponding constants (to make them all equal to 0, for instance) without changing ∇v or (5.24). The main reason for (5.25) is that if we set

(5.26) $$\widetilde{v}(x) = \operatorname{Max}\left\{-\sqrt{2/\pi}R^{1/2}, \operatorname{Min}\left(\sqrt{2/\pi}R^{1/2}, v(x)\right)\right\}$$

on $B_R \setminus G$ and $\widetilde{v}(x) = u_0(x)$ on $\mathbb{R}^2 \setminus (B_R \cup L)$, then $(\widetilde{v}, G) \in U_R$ (and also U_R^k if $(v, G) \in U_R^k$) as well, and $\int_{B_R \setminus G} |\nabla \widetilde{v}|^2 \leqslant \int_{B_R \setminus G} |\nabla v|^2$.

Note that once G has been fixed, the problem of minimizing $\int_{B_R \setminus G} |\nabla v|^2$ with the constraint that $v = u_0$ on $\partial B_R \setminus L$ is convex: if v and $\widetilde{v}$ both satisfy the constraint, then $(v + \widetilde{v})/2$ also, and if in addition v and $\widetilde{v}$ both minimize the energy integral above, then $\nabla v = \nabla \widetilde{v}$ in L^2 because otherwise $(v + \widetilde{v})/2$ would give a strictly smaller energy integral. Thus v may not be unique (because of the connected components that do not touch ∂B_R), but ∇v is. Our observation (5.25) follows from this.

We now return to the proof of (5.24). For $x \in \partial B(0, R/2)$ such that $\operatorname{dist}(x, L) > R/100$, we use (5.25) and the fact that v is harmonic on $B(x, R/100)$ to get that

(5.27) $$|\nabla v(x)| \leqslant CR^{-1} \sup\{|v(z)| \ ; \ z \in B(x, R/100)\} \leqslant CR^{-1/2}.$$

So we only have to bound $|\nabla v|$ on $\partial B(0, R/2) \cap \overline{B} \setminus L$, where B is the disk of radius $R/100$ centered on the point of $L \cap \partial B(0, R/2)$.

By the same symmetry argument as above (see between (4.53) and (4.54)), the values of v on each of the two half-disks $2B^{\pm}$ that compose $2B \setminus L$ can be obtained from the values of v on the corresponding half-circle $\partial(2B^{\pm}) \setminus L$ by symmetric extension to $\partial(2B)$ and then Poisson extension. In this situation also we can use (5.25) and the same estimate as in (5.27) to get that $|\nabla v(x)| \leqslant CR^{-1/2}$ on $\partial B(0, R/2) \cap \overline{B} \setminus L$. This proves (5.24).

Because of (5.24) and the corresponding estimate on the restriction of u_0 to ∂B_R, Lemma 5.22 will be a direct consequence of the following.

Lemma 5.28. — *Let H be as in* (5.20), *and let w be defined and differentiable on $\partial H \setminus L$. Denote by $\partial w/\partial\tau$ the derivative of w on the two arcs of circles that compose $\partial H \setminus H$. Suppose that*

$$|w| \leqslant R^{1/2} \text{ and } \left|\frac{\partial w}{\partial \tau}\right| \leqslant R^{-1/2} \text{ on } \partial H \setminus L. \tag{5.29}$$

Denote by v_1 the harmonic function on H which admits boundary values on $\partial H \setminus L$ equal to w and which minimizes $\int_H |\nabla v_1|^2$. Then

$$|\nabla v_1| \leqslant CR^{-1/2} \quad \text{on } H. \tag{5.30}$$

In this statement the constant C does not depend on R, and actually the lemma will follow from the special case when $R = 2$ (and with the same constant C) because its statement is invariant under dilations.

The existence and uniqueness of v_1 in the statement of Lemma 5.28 is classical, but let us just say a few words about it because it is an issue that will come out often in this text. It is well-known and fairly easy to prove that functions in $W^{1,2}(H)$ have boundary values radially almost everywhere on $\partial H \setminus L$. Our boundary condition on $\partial H \setminus L$, taken in the radial almost everywhere sense, defines a nonempty, closed affine subspace of $W^{1,2}(H)$, and v_1 is the point of that subspace that minimizes the norm. It is then easy to check that v_1 is harmonic on H and extends continuously to $\partial H \setminus L$. Note that this fits with our earlier declarations concerning the definition of the competitor v_1, around (5.21).

Now let w and v_1 be as in the statement of Lemma 5.28, with $R = 2$. We first want to check that

$$|\nabla v_1| \leqslant CR^{-1/2} = C \quad \text{on} \quad H_1, \tag{5.31}$$

where $H_1 = \{re^{i\theta}\ ;\ 1 \leqslant r \leqslant 2 \text{ and } |\theta| < 2\pi/3\}$. This is a little easier because we shall not be bothered by L. Clearly $\int_H |\nabla v_1|^2 \leqslant C$ (because it is very easy to find extensions of w that satisfy this), so we can find θ_0 such that $3\pi/4 \leqslant \theta_0 \leqslant 4\pi/5$ and

$$\int_1^2 \left\{\left|\nabla v_1\left(re^{i\theta_0}\right)\right|^2 + \left|\nabla v_1\left(re^{-i\theta_0}\right)\right|^2\right\} dr \leqslant C. \tag{5.32}$$

The values of v_1 in H_1 are obtained from the values of v_1 on the line segments $\{re^{i\theta}\ ;\ 1 \leqslant r \leqslant 2 \text{ and } \theta = \pm\theta_0\}$ and the values of $v_1 = w$ on the arcs of circles $\{re^{i\theta}\ ;\ r = 1 \text{ or } 2 \text{ and } |\theta| \leqslant \theta_0\}$ by application of an appropriate Poisson kernel. A fairly brutal estimate then gives (5.31). [Note that the corners of the domain $\{re^{i\theta}\ ;\ 1 \leqslant r \leqslant 2 \text{ and } |\theta| \leqslant \theta_0\}$ are far from H_1; suitable estimates on the Poisson kernel could always be obtained by mapping this domain conformally to a disk.]

We need also to estimate $|\nabla v_1|$ on the two remaining pieces of $H \setminus H_1$. For each of them we can apply the usual reflection argument to reduce our mixed Neumann-Dirichlet problem to a pure Dirichlet problem on a similar domain with twice the

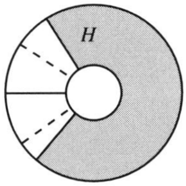


FIGURE 5.1

aperture. The estimates are then the same as for (5.31). [We can even make them a tiny bit simpler because (5.31) gives a slightly better estimate than (5.32) on the radial part of the boundary.] This is probably more than enough details about the proof of Lemma 5.28. □

As was mentioned before, Lemma 5.22 follows from Lemma 5.28. □

We are now ready to start estimating

$$\Delta E = \int_{B_R \setminus G_1} |\nabla v_1|^2 - \int_{B_R \setminus G} |\nabla v|^2 = \int_{H \setminus G} \left\{ |\nabla v_1|^2 - |\nabla v|^2 \right\} \tag{5.33}$$

(because the pairs (v_1, G_1) and (v, G) coincide on $B(0, R/2)$; see (5.18) in particular).

Set $\Omega = H \setminus G = B_R \setminus \left(\overline{B}(0, R/2) \cup G\right)$. The same integration by parts as for (4.33) yields

$$\Delta E = \int_{\partial\Omega} (v_1 - v) \frac{\partial (v_1 + v)}{\partial n} dH^1. \tag{5.34}$$

On $\partial B(0, R/2) \setminus L$ and on $\partial B_R \setminus G$, the functions v_1 and v coincide, and so the contribution of these sets is null. On the set L, the normal derivatives $\partial v_1 / \partial n$ and $\partial v / \partial n$ vanish, and so we can forget about L as well. So we are only left with the set $G^- \cap \overline{H} = G^- \setminus B(0, R/2)$. Since $\partial v / \partial n = 0$ on this set, we get that

$$\Delta E = \int_{G^- \setminus B(0,R/2)} (v_1 - v) \frac{\partial v_1}{\partial n}, \tag{5.35}$$

with the usual abuse of notation that a given point of G^- may be counted twice, one for each access from Ω.

We want to "localize" this integral, like in the proof of Proposition 4.14. Set $\partial = G^- \setminus B(0, R/2)$ and let D be any disk centered on ∂ and with radius $r = 2H^1(G^-)$. We want to estimate the oscillation of $v_1 - v$ on D (or on the two pieces of $D \setminus L$, when D meets L). We first choose a disk D' with the same center as D, such that $D \subset D' \subset 2D$,

$$G^- \cap \partial D' = \varnothing, \tag{5.36}$$

and

$$\int_{\partial D' \cap B_R \setminus L} |\nabla v|^2 \leqslant 100\lambda. \tag{5.37}$$

It is easy to get (5.36) because the set of radii for D' such that $G^- \cap \partial D' \neq \varnothing$ has measure at most $H^1(G^-) \leqslant r/2$. The second constraint is also easy to get, because $\int_{(B_R \cap 2D)\setminus G} |\nabla v|^2 \leqslant 16\pi r\lambda$. This last comes from lemma 4.12, and the fact that $2r = 4H^1(G^-) \leqslant r_G$, by (4.9).

So let D' be as above, and denote by D'' any connected component of $D' \cap \overline{B}_R \setminus L$. Most of the time, D' does not meet L and $D'' = D' \cap \overline{B}_R$, but when L meets D' there may be two possibilities for D''. In the computations that follow, the oscillation of v_1 on D'' will not disturb, because

$$\operatorname{osc}(v_1\ ;\ D'') := \sup_{D''} v_1 - \inf_{D''} v_1 \leqslant CrR^{-1/2} \tag{5.38}$$

by (5.23).

Next we want to estimate the oscillation of v on $D'' \setminus G$, and let us start with the case when D' does not meet ∂B_R or, equivalently, when

$$D' \subset B_R. \tag{5.39}$$

Because v minimizes $\int_{B_R \setminus G} |\nabla v|^2$, we may as well assume that

$$\operatorname{osc}(v\ ;\ D'' \setminus G) \leqslant \operatorname{osc}(v\ ;\ \partial D'' \setminus L)\,. \tag{5.40}$$

[Note that the right-hand side makes sense because $\partial D''$ does not meet G^-.] As for (5.25), the "may as well assume" comes from the fact that we may need first to modify the constant values of v on the connected components of $B_R \setminus G$ that do not reach all the way to ∂B_R. [Such modifications clearly do not change ΔE, and we shall see soon that they do not change our computations.] The reason for (5.40) is the same as for (5.25): otherwise we could reduce $\int |\nabla v|^2$ by replacing $v(x)$ on D'' with

$$\widetilde{v}(x) = \operatorname{Max}\left\{\inf_{\partial D'' \setminus L} v,\ \operatorname{Min}\left(v(x),\ \sup_{\partial D'' \setminus L} v\right)\right\}. \tag{5.41}$$

Note that by (5.39), $\partial D'' \setminus L$ is a single (connected!) arc of the circle $\partial D'$. It does not meet G by (5.36), and so we may use (5.37) and Cauchy-Schwarz to get that

$$\operatorname{osc}(v\ ;\ \partial D'' \setminus L) \leqslant \int_{\partial D' \setminus L} |\nabla v| \leqslant Cr^{1/2}\lambda^{1/2}. \tag{5.42}$$

Altogether

$$\operatorname{osc}(v_1 - v\ ;\ D'' \setminus G) \leqslant Cr^{1/2}\lambda^{1/2} \tag{5.43}$$

by (5.38), (5.40), (5.42), and because $rR^{-1/2} \leqslant r^{1/2} \leqslant r^{1/2}\lambda^{1/2}$ [since $r = 2H^1(G^-) \leqslant R$ (by (5.10)) and $\lambda \geqslant 1$ (by (4.10), (3.5), and (3.3))].

Observe that

$$\int_{G^- \cap D''} \frac{\partial v_1}{\partial n} = 0. \tag{5.44}$$

Indeed $\int_{D''} \Delta v_1 = \int_{D''\setminus G^-} \Delta v_1 = 0$, and when we apply Green to these domains (as in (4.48) and (4.49) above) we get that

$$\int_{\partial D''} \frac{\partial v_1}{\partial n} = \int_{\partial(D''\setminus G^-)} \frac{\partial v_1}{\partial n}. \tag{5.45}$$

The difference between the two boundaries is $G^- \cap D''$ (with the usual abuse of notation concerning multiplicities), and (5.44) follows.

Because of (5.44) we can subtract a constant from $v_1 - v$ in the integral just below, and get that

$$\begin{aligned}\left|\int_{G^-\cap D''} (v_1 - v)\frac{\partial v_1}{\partial n}\right| &\leqslant \operatorname{osc}\left(v_1 - v\ ;\ D''\setminus G^-\right)\int_{G^-\cap D''}\left|\frac{\partial v_1}{\partial n}\right| \\ &\leqslant Cr^{1/2}\lambda^{1/2}R^{-1/2}H^1\left(G^-\cap D''\right),\end{aligned} \tag{5.46}$$

because of (5.43) and (5.23). Here we used the fact that when we add constants to v on components of $B_R \setminus K$ that do not touch $\partial D''$ (to get (5.40)-(5.43)), we do not change the left-hand side of (5.46). The argument is the same as for (4.53). Also, we used (5.17) implicitly, to show that $D'' \subset H$.

Our estimate (5.46) also holds if we replace D'' with $D''\setminus\bigcup_{i=1}^m D_i''$, where the D_i'' are other pieces of disks like D''; the only thing that matters is that we still have (5.44) for this smaller domain, for the same reason.

Let us continue our localization of the main piece of (5.35). Set

$$G_0 = \left\{x \in G^- \setminus B(0, R/2)\ ;\ \operatorname{dist}(x, \partial B_R) \geqslant 5r\right\}. \tag{5.47}$$

Cover G_0 by disks D_j, $j \geqslant 1$, centered on G_0, with the same radius r, and such that the $2D_j$ have bounded overlap. Construct the slightly larger disks D_j', and then define the pieces D_j'', as above. Re-enumerate the D_j'', if needed, to account for the fact that some D_j' decompose into two pieces D_j''. Finally set $D_j^* = D_j''\setminus\bigcup_{i=1}^{j-1} D_i''$, and $G_0^* = G^-\cap\left(\bigcup_j D_j^*\right)$. By construction, G_0^* is the disjoint union of the $G^-\cap D_j^*$, and

$$\begin{aligned}\left|\int_{G_0^*}(v_1 - v)\frac{\partial v_1}{\partial n}\right| &= \left|\sum_j\int_{G^-\cap D_j^*}(v_1 - v)\frac{\partial v_1}{\partial n}\right| \\ &\leqslant Cr^{1/2}\lambda^{1/2}R^{-1/2}\sum_j H^1\left(G^-\cap D_j^*\right) \\ &\leqslant Cr^{1/2}\lambda^{1/2}R^{-1/2}H^1\left(G_0^*\right),\end{aligned} \tag{5.48}$$

because (5.46) also holds for the D_j^*.

We still need to estimate the integral on the remaining set $\partial_1 = G^-\setminus\left(B(0,R/2)\cup G_0^*\right)$. Let us check that

$$|v_1(x) - v(x)| \leqslant Cr^{1/2}\lambda^{1/2} \text{ for } x \in \partial_1. \tag{5.49}$$

To be fair, we should mention that we are again abusing notations: we should note that $v(x)$ stands for the boundary value of $v(x)$ (from the access which corresponds to the "point" of $\partial\Omega = \partial(H \setminus G)$ that x stands for). We shall actually prove the inequality in (5.49) for $x \in \Omega$ in a neighborhood of ∂_1. This is of course enough to control the boundary values that we just referred to; the proof will also give enough control on $v - v_1$ to allow uniform estimates in the limiting arguments described after (4.31), if we want to avoid integrating by parts directly on nonsmooth domains.

So let $x \in \partial_1$ be given, and set $D = B(x, 10r)$. Since G_0^* contains G_0 by construction, $\operatorname{dist}(x, \partial B_R) < 5r$ and D meets ∂B_R.

Let $\widetilde{x}$ denote the point of ∂B_R which is closest to x, and choose a disk $D' = B(\widetilde{x}, r')$ such that $15r < r' < 30r$, and which satisfies (5.36) and (5.37). The existence of D' can be proved as above (note that $30r = 60H^1(G^-) \leqslant r_G$ by (4.9)). We decided to center D' on ∂B_R to make the picture below simpler. Note that $D \subset D'$, though.

Let D'' denote the connected component of $D' \cap \overline{B}_R \setminus L$ that contains x and $\widetilde{x}$. [See Figure 5.2.]

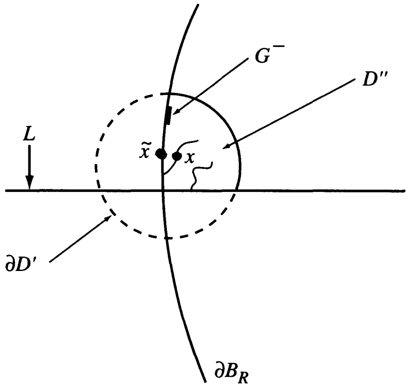


FIGURE 5.2

By the same truncature argument as before, we may assume that

$$\operatorname{osc}(v\,;\, D'' \setminus G) \leqslant \operatorname{osc}(v\,;\, \partial D'' \setminus G). \tag{5.50}$$

The relevant part $\partial D'' \setminus G$ of $\partial D''$ is itself composed of two pieces, $\delta_1 = \partial B_R \cap \partial D'' \setminus G$ and $\delta_2 = B_R \cap \partial D'' \setminus G$. On δ_1 the function v coincides with u_0 (by definition of U_R) and so

$$\operatorname{osc}(v, \delta_1) = \operatorname{osc}(u_0, \delta_1) \leqslant CrR^{-1/2} \leqslant Cr^{1/2}. \tag{5.51}$$

Because of (5.36), $\delta_2 = B_R \cap \partial D'' \setminus L$, which is an arc of the circle $\partial D'$. To be fair, this would not be true if we had chosen our radius r' just a tiny bit larger than the distance from $\widetilde{x}$ to L (as in Figure 5.3), but we have enough latitude on our choice of r' to avoid this stupid case. [Recall also that $r = H^1(G^-) \ll R$ by (5.10).]

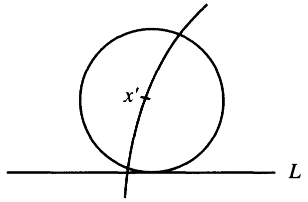


FIGURE 5.3. A bad case (easy to avoid).

At least one of the two extremities of the arc δ_2 lies in $\partial B_R \setminus L$. Call it z_0. Then $z_0 \in \delta_1$ (it does not lies in G^- because of (5.36)). Hence

$$v(z_0) = v_1(z_0) = u_0(z_0). \tag{5.52}$$

Since δ_2 is connected and contained in $\partial D' \cap B_R \setminus L$,

$$\operatorname{osc}(v\,;\,\delta_2) \leqslant \int_{\delta_2} |\nabla v| \leqslant Cr^{1/2}\lambda^{1/2} \tag{5.53}$$

by (5.37) and Cauchy-Schwarz. Since δ_2 ends up in z_0, we get that

$$|v(z) - v(z_0)| \leqslant Cr^{1/2}\lambda^{1/2} \tag{5.54}$$

for $z \in \delta_2$. Because of (5.51), and since $z_0 \in \delta_1$, (5.54) also holds for $z \in \delta_1$. Altogether, (5.54) holds on $\partial D'' \setminus G = \delta_1 \cup \delta_2$, and then on $D'' \setminus G$ by (5.50). On the other hand,

$$|v_1(z) - v(z_0)| = |v_1(z) - v_1(z_0)| \leqslant CrR^{-1/2} \leqslant Cr^{1/2} \tag{5.55}$$

for $z \in D'' \setminus G$, because of (5.52) and (5.23). This and (5.54) yield

$$|v_1(z) - v(z)| \leqslant Cr^{1/2}\lambda^{1/2} \text{ for } z \in D'' \setminus G. \tag{5.56}$$

Note that D'' contains a neighborhood of our initial point $x \in \partial_1$, and so (5.49) follows from (5.56).

Now

$$\left|\int_{\partial_1} (v_1 - v)\frac{\partial v_1}{\partial n}\right| \leqslant Cr^{1/2}\lambda^{1/2}\int_{\partial_1}\left|\frac{\partial v_1}{\partial n}\right| \leqslant Cr^{1/2}\lambda^{1/2}R^{-1/2}H^1(\partial_1) \tag{5.57}$$

by (5.49) and (5.23). By definition of ∂_1 (just above (5.49)), $G^- \setminus B(0, R/2)$ is the disjoint union of G_0^* and ∂_1, and so

$$\begin{aligned}\Delta E = \int_{G_0^* \cup \partial_1} (v_1 - v)\frac{\partial v_1}{\partial n} &\leqslant Cr^{1/2}\lambda^{1/2}R^{-1/2}\left\{H^1(G_0^*) + H^1(\partial_1)\right\} \\ &= Cr^{1/2}\lambda^{1/2}R^{-1/2}H^1\left(G^- \setminus B(0, R/2)\right)\end{aligned} \tag{5.58}$$

by (5.35), (5.48), and (5.57).

Recall that $r = 2H^1(G^-) \leqslant (2R+2)/A$ (if R is large enough), by (5.10). Let us assume that, say,

$$A \geqslant 10^3 C^2 \tag{5.59}$$

where C is the same absolute constant as in (5.58). Then (5.58) yields

$$\Delta E \leqslant 10^{-1}\lambda^{1/2} H^1\left(G^- \setminus B(0,R/2)\right). \tag{5.60}$$

Let us summarize the situation. We have implicitly assumed here that $G^- \setminus B(0,R/2)$ is not empty, and we have constructed a new competitor (v_1, G_1). [See (5.15)-(5.21).] Then we estimated the extra energy ΔE that we had to pay for removing $G^- \setminus B(0,R/2)$ and we arrived to (5.60). Since (v,G) is a minimizer for J_R or J_R^k, we must have that

$$\Delta h \leqslant \Delta E \leqslant 10^{-1}\lambda^{1/2} H^1\left(G^- \setminus B(0,R/2)\right), \tag{5.61}$$

where

$$\Delta h = h\left(H^1(G^-)\right) - h\left(H^1(G_1^-)\right) \tag{5.62}$$

is what we gained in the length term of (3.2). Set $a = H^1(G_1^-)$ and $b = H^1(G^-)$ Then

$$b - a = H^1\left(G^- \setminus G_1^-\right) = H^1\left(G^- \setminus B(0,R/4)\right) = H^1\left(G^- \setminus B(0,R/2)\right), \tag{5.63}$$

by (5.15) and (5.17). Then $h'(b) = \lambda$ by (4.10), and (4.59) says that

$$\Delta h = h(b) - h(a) \geqslant \frac{\lambda}{4}(b-a) = \frac{\lambda}{4} H^1\left(G^- \setminus B(0,R/2)\right). \tag{5.64}$$

Recall that $\lambda \geqslant 1$ (by (3.3), (3.5), and (4.10)), and so (5.64) contradicts (5.61) unless $G^- \subset B(0,R/2)$.

Recall from the discussion that led to (5.13)-(5.14) that every point of G^- has to satisfy (5.13) or (5.14). We finally managed to show that $G^- \subset B(0,R/2)$, which excludes (5.14). Set

$$R^+ = \sup\left\{|z| \ ; \ z \in G^-\right\}. \tag{5.65}$$

We have just proved that

$$R^+ \leqslant 1 + 2C_1 H^1\left(G^-\right) \tag{5.66}$$

where C_1 is still as in Proposition 4.14. [Compare with (5.13).]

To complete our proof of Proposition 5.1 (i.e., show in particular that R^+ is less than an absolute constant) we need an estimate of $H^1(G^-)$ in terms of R^+. This will be obtained by improving the energy estimate that led to (5.9), taking into account that R^+ may be much smaller than R.

Define a new competitor (v^+, G^+) as follows. Take

$$G^+ = L \cup \partial B_{R^+}, \tag{5.67}$$

$$v^+(x) = u_0(x) \quad \text{on} \quad \mathbb{R}^2 \setminus (B_R \cup L)\,, \tag{5.68}$$

$$v^+(x) = 0 \quad \text{on} \quad B_{R^+} \setminus L, \tag{5.69}$$

and

$$v^+(r\cos\theta, r\sin\theta) = \alpha r^{1/2} \sin\frac{\theta}{2} + \alpha r^{-1/2} R^+ \sin\frac{\theta}{2} \tag{5.70}$$

for $R^+ < r \leqslant R$ and $-\pi < \theta < \pi$, and where

$$\alpha = \sqrt{2/\pi}\ \frac{R^{1/2}}{R^{1/2} + R^{-1/2}R^+}\ . \tag{5.71}$$

We chose the constant α so that v^+ would be continuous across $\partial B_R \setminus L$ (compare (5.70) with (2.2) when $r = R$). Thus (v^+, G^+) is an acceptable competitor for J_R, and even $(v^+, G^+) \in U_R^k$ for all k.

Of course the formula (5.70) was not chosen at random. Let us check that v^+ is the harmonic function on $B_R \setminus (B_{R^+} \cup L)$ that coincides with u_0 on $\partial B_R \setminus L$ and minimizes

$$E^+ = \int_{B_R \setminus (B_{R^+} \cup L)} |\nabla v^+|^2 . \tag{5.72}$$

We already know that $v^+ = u_0$ on $\partial B_R \setminus L$. To prove what we just said, it will be enough to check that v^+ is harmonic on $B_R \setminus (B_{R^+} \cup L)$ and that its normal derivative on $\partial B_{R^+} \cup L$ vanishes. This is the usual characterization of energy minimizers in terms of Dirichlet and Neumann conditions. We avoided to use it so far, but now it is slightly more convenient. [The reader that would not know about this characterization can retrieve it from our formulae giving ΔE after integrations by parts.]

Set $z = re^{i\theta}$ and $z^{1/2} = r^{1/2}e^{i\theta/2}$ with the notations above. Then $v^+(z) = \alpha \,\mathrm{Im}\{z^{1/2} - R^+ z^{-1/2}\}$, which proves that v^+ is harmonic (because $z^{1/2}$ and $z^{-1/2}$ are holomorphic).

The radial derivative is

$$\frac{\partial v^+}{\partial r} = \frac{\alpha}{2} r^{-1/2} \sin\frac{\theta}{2} - \frac{\alpha}{2} r^{-3/2} R^+ \sin\frac{\theta}{2}, \tag{5.73}$$

which vanishes when $r = R^+$. This takes care of the normal derivative on ∂B_{R^+}. The normal derivative along L is also null, because

$$\frac{\partial v^+}{\partial \theta} = \frac{\alpha}{2} r^{1/2} \cos\frac{\theta}{2} + \frac{\alpha}{2} r^{-1/2} R^+ \cos\frac{\theta}{2}, \tag{5.74}$$

which vanishes when $\theta = \pm\pi$. So v^+ minimizes the energy, as promised.

By definition of R^+, G^- is contained in $\overline{B}_{R^+}$, and hence

$$\int_{B_R \setminus G} |\nabla v|^2 \geqslant \int_{B_R \setminus (B_{R^+} \cup L)} |\nabla v|^2 = E^+, \tag{5.75}$$

by the energy-minimizing property of v^+. Next compute

$$\Delta E^+ = \int_{B_R \setminus K_0} |\nabla u_0|^2 - \int_{B_R \setminus (B_{R^+} \cup L)} |\nabla v^+|^2 . \tag{5.76}$$

We could do this brutally with (5.73) and (5.74), but let us use the analogue of (4.39) in our situation, i.e.,

$$\Delta E^+ = \int_{\partial B_{R^+}} v^+ \frac{\partial u_0}{\partial r} dH^1 . \tag{5.77}$$

The access to ∂B_{R^+} from the inside of B_{R^+} does not contribute to the integral (by (5.69)). Also,

$$\frac{\partial u_0}{\partial r} = \sqrt{2/\pi} \cdot \frac{1}{2}\, r^{-1/2} \sin\frac{\theta}{2}, \tag{5.78}$$

while (5.70) yields

$$v^+ = 2\alpha r^{1/2} \sin\frac{\theta}{2} \quad \text{when} \quad r = R^+ . \tag{5.79}$$

Thus

$$\begin{aligned} \Delta E^+ &= \sqrt{2/\pi}\, \alpha \int_{\partial B_{R^+}} \sin^2\frac{\theta}{2}\, dH^1 = \sqrt{2/\pi}\pi R^+ \alpha \\ &= 2\frac{R^{1/2}}{R^{1/2} + R^{-1/2}R^+}\, R^+ \leqslant 2R^+ . \end{aligned} \tag{5.80}$$

Altogether

$$\begin{aligned} h\left(H^1\left(G^-\right)\right) &= J_R(v,G) - \int_{B_R \setminus G} |\nabla v|^2 \\ &\leqslant J_R(v,G) - E^+ \leqslant J_R\left(u_0, K_0\right) - E^+ \\ &= h\left(H^1\left(K_0 \setminus L\right)\right) + \int_{B_R \setminus K_0} |\nabla u_0|^2 - E^+ \\ &= 1 + \Delta E^+ \leqslant 1 + 2R^+ \end{aligned} \tag{5.81}$$

by (3.2), (5.75), because (v, G) is a minimizer and (u_0, K_0) a candidate, by (3.2) again, and then (3.3) and (5.80).

We compare this with (5.66) and get that

$$h\left(H^1\left(G^-\right)\right) \leqslant 3 + 4C_1 H^1\left(G^-\right) . \tag{5.82}$$

Let us decide to choose

$$A \geqslant 5C_1 . \tag{5.83}$$

Since $h(t) = At$ for t large, (5.82) gives an upper bound on $H^1\left(G^-\right)$, which proves (5.3). The other conclusion (5.2) follows from (5.66). This completes our proof of Proposition 5.1. □

6. Existence of minimizers: general strategy

So far we introduced the functionals J_R and proved nice properties for their minimizers. We intend to take such minimizers, find a sequence of radii R that tends to $+\infty$ such that the corresponding minimizers converge to a pair (v, G), and then prove that (v, G) minimizes a modest variant of the global Mumford-Shah functional. After this, we shall prove that (v, G) must be a cracktip and eventually get the desired contradiction. Before we do all this, we shall have to check the existence of minimizers for J_R, at least for R large.

The verification of existence will take some time, probably much too long compared to the amount of new information contained in the proof. We feel compelled to give an argument for self-containedness and because the situation of J_R is slightly different from the usual one, but this argument contains no surprise, and we would not be shocked if the reader decided not to read it.

Our proof of existence will be rather constructive and will avoid compactness properties of BV. Instead we shall rely on the approach of Dal Maso, Morel, and Solimini [DMS] and use their "concentration property". Our excuse for this is that we had to prove reasonably carefully the local Ahlfors-regularity property of Proposition 4.14, and we can use these estimates again for the existence.

Our general scheme for the proof of existence will be the same as in [DMS] or [MoSo], but there will be a few differences because we don't want to use BV or SBV. Parts of the arguments will use ideas from [DiKo] (for the "property of projection") or [DaSe2], or their presentation in the survey [DaSe3]. Also see [MaSo] for a new existence argument based on these ideas.

To prove the existence of minimizers for J_R, we want to start from a minimizing sequence (v_n, G_n) and try to get the minimizer as a limit. Of course this cannot work with any sequence, because even if the sets G_n converge nicely (for the Hausdorff metric on compact subsets of $\overline{B}_R$, say) to a set G, we do not have that

$$H^1(G \setminus L) \leqslant \liminf_{n\to\infty} H^1(G_n \setminus L). \tag{6.1}$$

The point of our argument will be that we can modify our sequence (v_n, G_n) in such a way that G_n satisfies a nice regularity property (the uniform concentration property of [DMS]), so that (6.1) holds.

One way to ensure the concentration property (and then (6.1)) will be to demand that (v_n, G_n) be a minimizer for some J_R^k. If we want to do this and get minimizing sequences, we need to show that

$$\eta(R) = \lim_{k\to\infty} \eta_k(R), \tag{6.2}$$

where

$$\eta(R) = \inf\{J_R(v, G)\ ;\ (v, G) \in U_R\} \tag{6.3}$$

and, for $k \geqslant 1$,

$$\eta_k(R) = \inf \left\{ J_R(v,G) \ ; \ (v,G) \in U_R^k \right\}. \tag{6.4}$$

Of course there are other ways to get minimizing sequences with the uniform concentration property without actually proving (6.2), but going trough (6.2) will be just as convenient. Either way, we need to take a pair $(v, G) \in U_R$ that almost minimizes J_R and improve it in various way. This is what we shall do in the next two sections. In Section 9 we shall see how to deduce the existence of minimizers for J_R from (6.2).

7. First cleaning of a competitor for J_R

In this section we give ourselves a competitor $(v, G) \in U_R$ that nearly minimizes J_R, and we show that G is not far from being locally Ahlfors-regular and rectifiable.

Let us remind the reader that if G is a closed subset of $\mathbb{R}^2$ with sigma-finite H^1-measure, then G has an essentially (up to sets of vanishing H^1-measure) unique decomposition as

$$G = G_{\text{rec}} \cup G_{\text{irr}}, \tag{7.1}$$

where G_{rec} is rectifiable, which means that we can find a countable family $\{\Gamma_i\}$ of simple curves of class C^1 such that

$$H^1\left(G_{\text{rec}} \setminus \bigcup_i \Gamma_i\right) = 0, \tag{7.2}$$

while G_{irr} is irregular, or totally unrectifiable, which means that

$$H^1\left(G_{\text{irr}} \cap \Gamma\right) = 0 \ \text{ for each } C^1\text{−curve } \Gamma. \tag{7.3}$$

See for instance [Ma] (or [Fe], or other sources on geometric measure theory) for this and other standard facts on rectifiability that will be used in this section.

Proposition 7.4. — *Let $(v, G) \in U_R$ be such that*

$$J_R(v,G) \leqslant \eta(R) + \varepsilon \tag{7.5}$$

for some $\varepsilon > 0$. Then

$$H^1\left(G_{\text{irr}}\right) \leqslant C\varepsilon. \tag{7.6}$$

See (6.3) and the beginning of Section 3 for the definitions. The constant C will depend on various geometric constructions, but not on v or G. The dependence on R does not really interest us here, but C does not depend on R either, if we add the constraint that ε be small enough (depending on R).

Also note that other information on G will come along the way (concerning almost Ahlfors-regularity, etc...); (7.6) is just an example.

To prove the proposition we can assume that ε is as small as we want, because otherwise a brutal estimate gives

$$\text{(7.7)}\qquad \begin{aligned} H^1(G_{\text{irr}}) &\leqslant H^1(G^-) \leqslant h\left(H^1(G^-)\right) \leqslant J_R(v,G) \\ &\leqslant J_R(u_0,K_0)+\varepsilon \leqslant 1+R+\varepsilon \leqslant C\varepsilon, \end{aligned}$$

by (7.5), (5.8) and (5.4).

From now on we let $(v,G) \in U_R$ be such that (7.5) holds, and we also assume that $\varepsilon \leqslant 1$. We shall proceed by a series of little steps where we shall prove that, except on fairly small sets, G has increasingly good properties. The various steps will look like each other, if only by the systematic use of the same covering argument. We start with a very simple property.

Lemma 7.8. — *Set*

$$\text{(7.9)}\qquad Z_1 = \left\{x \in G^- \ ; \ \text{there is an } r>0 \text{ such that } H^1\left(G^- \cap B(x,r)\right) \geqslant 3\pi r\right\}.$$

Then

$$\text{(7.10)}\qquad H^1(Z_1) \leqslant 15\varepsilon.$$

For each $x \in Z_1$, set

$$\text{(7.11)}\qquad r(x) = \sup\left\{r>0 \ ; \ H^1\left(G^- \cap B(x,r)\right) \geqslant 3\pi r\right\}.$$

We still have that

$$\text{(7.12)}\qquad H^1\left(G^- \cap B(x,r(x))\right) \geqslant 3\pi r(x)$$

(by a trivial limiting argument). Also, $r(x) \leqslant (3\pi)^{-1}H^1(G^-)$, so the radii $r(x)$ are bounded. Thus we can use the standard $\frac{1}{5}$-covering lemma of Vitali (see the first pages of [St]). We get a finite or countable set $X \subset Z_1$ such that the $\overline{B}(x,r(x))$, $x \in X$, are disjoint, but the $B(x,5r(x))$, $x \in X$, cover Z_1.

Let X_0 be any finite subset of X. We construct a competitor $(v_1,G_1) \in U_R$ as follows. Set

$$\text{(7.13)}\qquad G_1 = \left[G \setminus \bigcup_{x\in X_0} B_x\right] \cup \left[\bigcup_{x\in X_0} \partial B_x\right] \cup L,$$

where $B_x = B_R \cap B(x,r(x))$. It is easy to check that G_1 is closed, and it contains L. Keep

$$\text{(7.14)}\qquad v_1(x) = v(x) \text{ out of } G_1 \cup \left[\bigcup_{x\in X_0} B_x\right],$$

and set

$$\text{(7.15)}\qquad v_1(x) = 0 \quad \text{on} \quad \bigcup_{x\in X_0} B_x.$$

Then $(v_1, G_1) \in U_R$, and (7.5) says that

$$J(v,G) \leqslant J(v_1,G_1) + \varepsilon. \tag{7.16}$$

Let us now estimate how much we won in the length term. Observe that

$$G_1^- \subset \left[G^- \setminus \bigcup_{x \in X_0} B(x, r(x)) \right] \cup \left[\bigcup_{x \in X_0} \partial B_x \right]. \tag{7.17}$$

This would be obvious if we had not replaced the B_x coming from (7.13) with the larger $B(x, r(x))$. So we may have lost the sets $G^- \cap B(x,r(x)) \setminus B_x = G^- \cap B(x,r(x)) \cap \partial B_R$. This is not the case: these sets are contained in the corresponding ∂B_x anyway.

From (7.17) and the disjointness of the $B(x, r(x))$ we get that

$$\begin{aligned} H^1(G_1^-) &\leqslant H^1(G^-) - \sum_{x\in X_0} H^1(G^- \cap B(x,r(x))) + \sum_{x \in X_0} H^1(\partial B_x) \\ &\leqslant H^1(G^-) - \pi \sum_{x \in X_0} r(x), \end{aligned} \tag{7.18}$$

by (7.12) and because $H^1(\partial B_x) \leqslant 2\pi r(x)$.

Because of (7.14) and (7.15) we have not increased the energy term. So

$$\begin{aligned} J(v,G) - J(v_1,G_1) &\geqslant h(H^1(G^-)) - h(H^1(G_1^-)) \\ &\geqslant H^1(G^-) - H^1(G_1^-) \geqslant \pi \sum_{x\in X_0} r(x) \end{aligned} \tag{7.19}$$

by (3.2), because $h'(t) \geqslant 1$ (by (3.3) and (3.5)), and by (7.18). Now (7.16) says that $\pi \sum_{x\in X_0} r(x) \leqslant \varepsilon$ and, since X_0 was any finite subset of X, $\pi \sum_{x \in X} r(x) \leqslant \varepsilon$ as well. Now we use the fact that the $B(x, 5r(x))$ cover Z_1 to get that

$$H^1(Z_1) \leqslant \sum_{x\in X} H^1(G^- \cap B(x, 5r(x))) \leqslant 3\pi \sum_{x \in X} 5r(x) \leqslant 15\varepsilon, \tag{7.20}$$

where we used the definition (7.11) of $r(x)$ for the second inequality. This completes our proof of Lemma 7.8. □

Next we want to control $|\nabla v|^2$, like in Lemma 4.12.

Lemma 7.21. — *Set*

$$Z_2 = \left\{ x \in G^- \ ; \ \int_{B(x,r)\cap B_R \setminus G} |\nabla v|^2 \geqslant 5\pi r \lambda \ \text{ for some } \ 0 < r \leqslant 1 \right\}, \tag{7.22}$$

where $\lambda = h'(H^1(G^-))$ *as usual. Then*

$$H^1(Z_2) \leqslant 30\varepsilon. \tag{7.23}$$

Because of Lemma 7.8, it is enough to control $Z_2 \setminus Z_1$. For each $x \in Z_2 \setminus Z_1$ choose $0 < r(x) \leqslant 1$ such that

$$\int_{B(x,r(x))\cap B_R\setminus G} |\nabla v|^2 \geqslant 5\pi r(x)\lambda. \tag{7.24}$$

Then let X be a finite or countable subset of $Z_2 \setminus Z_1$ such that the balls $B(x, r(x))$, $x \in X$, are disjoint but the $B(x, 5r(x))$, $x \in X$, cover $Z_2 \setminus Z_1$. Let X_0 be any finite subset of X, and let (v_1, G_1) be the same competitor as in Lemma 7.8. [The construction is the same but the accounting will be different.] Obviously

$$H^1\left(G_1^-\right) \leqslant H^1\left(G^-\right) + 2\pi \sum_{x\in X_0} r(x) \tag{7.25}$$

by (7.13). Let us assume that

$$\sum_{x\in X_0} r(x) \leqslant 5 \tag{7.26}$$

to simplify our estimates. Then (3.3), (3.5) and (3.6) imply that $h'(t) \leqslant 2\lambda$ for $t \leqslant H^1\left(G^-\right) + 2\pi \sum_{x\in X_0} r(x)$ (recall that we already assumed that $B \geqslant 60$). Thus

$$\begin{aligned} h\left(H^1\left(G_1^-\right)\right) - h\left(H^1\left(G^-\right)\right) &\leqslant h\left[H^1\left(G^-\right) + 2\pi \sum_{x\in X_0} r(x)\right] - h\left(H^1\left(G^-\right)\right) \\ &\leqslant 4\pi\lambda \sum_{x\in X_0} r(x). \end{aligned} \tag{7.27}$$

On the other hand

$$\begin{aligned} \int_{B_R\setminus G_1} |\nabla v_1|^2 &= \int_{B_R\setminus G} |\nabla v|^2 - \sum_{x\in X_0} \int_{B_R\cap B(x,r(x))\setminus G} |\nabla v|^2 \\ &\leqslant \int_{B_R\setminus G_1} |\nabla v|^2 - 5\pi\lambda \sum_{x\in X_0} r(x), \end{aligned} \tag{7.28}$$

because the $B(x, r(x))$ are disjoint and by (7.24)). Altogether

$$J\left(v_1, G_1\right) \leqslant J(v, G) - \pi\lambda \sum_{x\in X_0} r(x) \tag{7.29}$$

(by (3.2), (7.27) and (7.28)), and (7.5) yields

$$\pi\lambda \sum_{x\in X_0} r(x) \leqslant \varepsilon. \tag{7.30}$$

So far we only proved this for finite subsets of X that satisfy (7.26). On the other hand, all $r(x)$ are $\leqslant 1$, $\lambda \geqslant 1$, and $\varepsilon \leqslant 1$, so we can get (7.30) for all finite subsets of

X by adding elements one by one (so that (7.26) stays true). Then (7.30) is also true with $X_0 = X$, and

$$(7.31)\qquad \begin{aligned} H^1(Z_2 \setminus Z_1) &\leqslant \sum_{x\in X} H^1\left((Z_2 \setminus Z_1) \cap B(x, 5r(x))\right) \\ &\leqslant \sum_{x\in X} H^1\left(G^- \cap B(x, 5r(x))\right) \\ &\leqslant 15\pi \sum_{x\in X} r(x) \leqslant 15\varepsilon \end{aligned}$$

because the $B(x, 5r(x))$, $x \in X$, cover $Z_2 \setminus Z_1$, the points $x \in X$ do not lie in Z_1, by (7.30), and because $\lambda \geqslant 1$.

Now (7.23) follows from this and (7.10); this completes our proof of Lemma 7.21. □

Next we want to worry about local Ahlfors-regularity (as in Proposition 4.14). Let $C_2 > 0$ be a fairly large constant, to be chosen soon. [It will be an analogue of C_1 in Proposition 4.14.]

Lemma 7.32. — *Set*

$$(7.33)\qquad Z_3 = \big\{x \in G^- \setminus \partial B_R\ ;\ \text{there exists } 0 < r \leqslant 1 \text{ such that } B(x,r) \subset B_R \setminus L \text{ and } H^1\left(G^- \cap B(x,r)\right) \leqslant C_2^{-1} r\big\}.$$

Then, if C_2 is large enough (depending on nothing),

$$(7.34)\qquad H^1(Z_3) \leqslant C\,\varepsilon.$$

Note that we decided not to worry about points of L or ∂B_R because we won't need to and this would complicate the argument (as in Sections 4 and 5).

To prove the lemma, first observe that since Z_3 is a Borel set with finite H^1-measure, the Lebesgue differentiation theorem says that for almost all $x \in Z_3$,

$$(7.35)\qquad \limsup_{r\to 0} r^{-1} H^1\left(G^- \cap B(x,r)\right) \geqslant 1.$$

See for instance [Ma], Theorem 6.2 p.86.

Let Z denote the set of points of $Z_3 \setminus (Z_1 \cup Z_2)$ that satisfy (7.35). Of course it will be enough to control Z, since

$$(7.36)\qquad H^1(Z_3) \leqslant H^1(Z_1) + H^1(Z_2) + H^1(Z)\,.$$

For each $x \in Z$ choose a first radius $r(x)$ such that

$$(7.37)\qquad 0 < r(x) \leqslant 1,$$

$$(7.38)\qquad B(x, r(x)) \subset B_R \setminus L,$$

$$(7.39)\qquad H^1\left(G^- \cap B(x, r(x)\right) \leqslant C_2^{-1} r(x),$$

but $r(x)/8$ does not satisfy (7.39), i.e.,

$$H^1\left(G^- \cap B\left(x, r(x)/8\right)\right) > \left(8C_2\right)^{-1} r(x). \tag{7.40}$$

The definition of Z_3 gives a radius that satisfies (7.37)-(7.39). It is then easy to get (7.40) as well by replacing $r(x)$ with $r(x)/8$, or $r(x)/64$, etc., as needed. The process converges because of (7.35) (and if C_2 is large enough).

Note incidentally that since $B(x, r(x))$ does not meet L we can replace G^- with G in (7.39) and (7.40).

Choose again a subset X of Z such that

$$\text{the disks } B\left(x, r(x)/5\right), x \in X, \quad \text{are disjoint} \tag{7.41}$$

and

$$Z \subset \bigcup_{x \in X} B(x, r(x)). \tag{7.42}$$

Let X_0 be a finite subset of X, and let us construct a competitor (v_1, G_1) like the one used for Proposition 4.14. For each $x \in X_0$ choose a disk $B_x = B\left(x, r_x\right)$ with the properties

$$\frac{r(x)}{8} < r_x < \frac{r(x)}{5}, \tag{7.43}$$

$$\partial B_x \cap G = \varnothing, \tag{7.44}$$

$$\int_{\partial B_x} |\nabla v|^2 \leqslant 10^3 \lambda, \tag{7.45}$$

and

$$\int_{G \cap B(x, r(x))} \operatorname{dist}\left(z, \partial B_x\right)^{-1/2} dH^1(z) \leqslant I_x, \tag{7.46}$$

where

$$I_x = 10^3 r(x)^{-1} \int_{t=r(x)/8}^{r(x)/5} \int_{G \cap B(x, r(x))} \operatorname{dist}\left(z, \partial B(x, t)\right)^{-1/2} dH^1(z) dt. \tag{7.47}$$

[Compare with (4.23)-(4.27).] We can find such a radius r_x because of (7.39) (to get (7.44)), because $x \notin Z_2$ (to get (7.45)), and of course by Fubini and Chebyshov.

We still need to cut X_0 into two subsets. Set

(7.48)
$$X_0' = \left\{ x \in X_0 \; ; \int_{B_x \cap B(y, 20 C_2^{-1} r(x)) \setminus G} |\nabla v|^2 \leqslant 10^3 \lambda C_2^{-1} r(x) \text{ for all } y \in G \cap B_x \right\},$$

and $X_0'' = X_0 \setminus X_0'$. The computations that follow are the same as in Section 4, only simplified slightly because we do not have to worry about L or ∂B_R. Our new competitor is given by

$$G_1 = G \setminus \Big(\bigcup_{x\in X_0'} B_x\Big), \tag{7.49}$$

$$v_1(x) = v(x) \quad \text{on} \quad \mathbb{R}^2 \setminus \Big[G_1 \cup \Big(\bigcup_{x\in X_0'} B_x\Big)\Big], \tag{7.50}$$

while on each B_x we require v_1 to be harmonic and have a continuous extension to $\overline{B}_x$ that coincides with v on ∂B_x. This makes sense, because ∂B_x does not meet G, by (7.44). Note also that the disks B_x are disjoint (by (7.41) and (7.43)), so our modifications are independent from each other.

It is easy to check that $(v_1, G_1) \in U_R$ (i.e., it is an acceptable competitor for J_R). The same computations as in Section 4 (but simpler, because we do not need to reflect on L) give the analogue of (4.55), i.e.,

$$\Delta E_x = \int_{B_x} |\nabla v_1|^2 - \int_{B_x\setminus G} |\nabla v|^2 \leqslant C C_2^{-1/2}\, \lambda\, r(x)^{1/2} I_x \ . \tag{7.51}$$

[This is where we need the condition in (7.48).]

The estimate for I_x can be carried as in (4.56): Fubini gives that

$$I_x \leqslant C\, r(x)^{-1/2}\, H^1\left(G \cap B(x, r(x))\right), \tag{7.52}$$

and then we can use (7.39) and (7.40) to see that this is $\leqslant 8C\, r(x)^{-1/2}\, H^1\left(G\cap B_x\right)$. Thus

$$\Delta E_x \leqslant C C_2^{-1/2}\, \lambda\, H^1\left(G\cap B_x\right) = C C_2^{-1/2}\, \lambda\, H^1\left(G^- \cap B_x\right), \tag{7.53}$$

with a constant C that does not depend on C_2. We choose C_2 so large that $CC_2^{-1/2} < 1/8$ in (7.53), and then sum over $x \in X_0'$ to get that

$$\Delta E = \int_{B_R\setminus G_1} |\nabla v_1|^2 - \int_{B_R\setminus G} |\nabla v|^2 = \sum_{x\in X_0'} \Delta E_x \leqslant \frac{\lambda}{8} \sum_{x\in X_0'} H^1\left(G^-\cap B_x\right). \tag{7.54}$$

On the other hand we may apply (4.59) with $a = H^1\left(G_1^-\right)$, $b = H^1\left(G^-\right)$, and $h'(b) = \lambda$ (by (4.10)). We get that

$$h\left(H^1\left(G^-\right)\right) - h\left(H^1\left(G_1^-\right)\right) = b - a \geqslant \frac{\lambda}{4}(b-a) = \frac{\lambda}{4} \sum_{x\in X_0'} H^1\left(G^-\cap B_x\right), \tag{7.55}$$

by (7.49). Now we can sum and get that

$$\begin{aligned} J(v,G) - J\left(v_1, G_1\right) &= h\left(H^1\left(G^-\right)\right) - h\left(H^1\left(G_1^-\right)\right) - \Delta E \\ &\geqslant \frac{\lambda}{8} \sum_{x\in X_0'} H^1\left(G^-\cap B_x\right), \end{aligned} \tag{7.56}$$

by (3.2), (7.55) and (7.54). Since $J(v,G) \leqslant J(v_1,G_1)+\varepsilon$ by (7.5), we get that

$$\lambda \sum_{x\in X_0'} H^1(G\cap B_x) \leqslant 8\varepsilon. \tag{7.57}$$

We still need to control the similar sum for $X_0'' = X_0 \setminus X_0'$. For each $x \in X_0''$, select $y \in G\cap B_x$ such that

$$\int_{D(x)\setminus G} |\nabla v|^2 \geqslant 10^3 \lambda C_2^{-1} r(x), \tag{7.58}$$

where we set $D(x) = B_x \cap B(y, 20C_2^{-1}r(x))$. We construct a new competitor $(v_1,G_1) \in U_R$ by taking

$$G_1 = G \cup \Big(\bigcup_{x\in X_0''} \partial D(x)\Big), \tag{7.59}$$

$v_1 = v$ out of $\bigcup_{x\in X_0''} D(x)$, and $v_1 = 0$ on $\bigcup_{x\in X_0''} D(x)\setminus G$. Then

$$\int_{B_R\setminus G} |\nabla v|^2 - \int_{B_R\setminus G_1} |\nabla v_1|^2 = \sum_{x\in X_0''} \int_{D(x)\setminus G} |\nabla v|^2 \geqslant 10^3\, \lambda\, C_2^{-1} \sum_{x\in X_0''} r(x)\ , \tag{7.60}$$

by (7.58) and because the $D(x), x\in X_0''$, are disjoint (by (7.41) and (7.43)). On the other hand,

$$H^1(G_1^-) - H^1(G^-) \leqslant \sum_{x\in X_0''} H^1(\partial D(x)) \leqslant 100\ C_2^{-1} \sum_{x\in X_0''} r(x)\ , \tag{7.61}$$

hence

$$h(H^1(G_1^-)) - h(H^1(G^-)) \leqslant 200\ \lambda\ C_2^{-1} \sum_{x\in X_0''} r(x)\ , \tag{7.62}$$

at least if

$$100\ \lambda\ C_2^{-1} \sum_{x\in X_0''} r(x) \leqslant 1\ . \tag{7.63}$$

In this case, we can use (7.5) and the definition (3.2) to get that

$$100\ \lambda\ C_2^{-1} \sum_{x\in X_0''} r(x) \leqslant \varepsilon\ , \tag{7.64}$$

by (7.60) and (7.62). Thus (7.64) holds when we have (7.63), but we can add points of X_0'' one by one, and get (7.64) and (7.63) in all cases; the argument is the same as before.

We deduce from (7.57), (7.40), (7.43) and (7.64) that

$$\lambda \sum_{x\in X_0} H^1(G^-\cap B_x) \leqslant 8\ \varepsilon + \lambda \sum_{x\in X_0''} H^1(G^-\cap B_x) \leqslant 8\ \varepsilon + 8\ C_2\lambda \sum_{x\in X_0''} r(x) \leqslant C\varepsilon. \tag{7.65}$$

Since this inequality holds for all finite subsets X_0 of X, it also holds for X, and then

$$(7.66)\quad H^1(Z) \leqslant \sum_{x\in X} H^1(Z\cap B(x,r(x))) \leqslant \sum_{x\in X} H^1(G^-\cap B(x,r(x)))$$
$$\leqslant 8\sum_{x\in X} H^1(G^-\cap B(x,r(x)/8)) \leqslant 8\sum_{x\in X} H^1(G^-\cap B_x) \leqslant C\varepsilon$$

by (7.42), (7.39), (7.40), (7.43), and (7.65).

Lemma 7.32 follows from this, (7.36), and the two previous lemmas (on Z_1 and Z_2). □

Now we want to prove Proposition 7.4 itself. To control G_{irr}, we want to use the equivalent here of the "property of projections" introduced by F. Dibos and G. Koepfler ([DiKo], [Di]). Our presentation will be closer to the one in [Lé1] or [DaSe3]. Set $Z_4 = G_{\mathrm{irr}} \setminus (Z_1 \cup Z_2 \cup Z_3)$. For H^1-almost every $x \in Z_4$, we have that

$$(7.67)\quad x \notin L \cup \partial B_R$$

by (7.3),

$$(7.68)\quad \limsup_{r\to 0} \frac{1}{r} H^1(Z_4\cap B(x,r)) \geqslant 1$$

by the same density theorem as for (7.35), and

$$(7.69)\quad \limsup_{r\to 0} \frac{1}{r} H^1((G\setminus Z_4)\cap B(x,r)) = 0,$$

again by a standard density theorem. [See for instance Theorem 6.2 on page 86 of [Ma].]

Denote by Z the set of $x \in Z_4$ that satisfy (7.67)-(7.69). Because of the previous remarks and the three lemmas above, (7.6) and Proposition 7.4 will follow as soon as we prove that

$$(7.70)\quad H^1(Z) \leqslant C\varepsilon.$$

For each $x \in Z$, choose a radius $r(x)$ such that

$$(7.71)\quad 0 < r(x) \leqslant 1,$$

$$(7.72)\quad B(x,r(x)) \subset B_R \setminus L,$$

$$(7.73)\quad H^1(Z_4\cap B(x,r(x)/5)) \geqslant \frac{r(x)}{6},$$

and

$$(7.74)\quad H^1[(G\setminus Z_4)\cap B(x,r(x))] \leqslant \tau r(x),$$

where the small constant τ will be chosen soon.

As usual, choose a family $X \subset Z$ such that

$$\text{the disks } B(x,r(x)),\ x \in X, \text{ are disjoint,} \tag{7.75}$$

but

$$Z \subset \bigcup_{x\in X} B(x,5r(x)). \tag{7.76}$$

Let X_0 be a finite subset of X; we want to modify (v,G) on the disks $B(x,r(x))$, $x \in X_0$, to construct a better competitor (v_1,G_1).

For the moment, fix $x \in X_0$. For each direction $\theta \in \mathbb{S}^1$, denote by π_θ the orthogonal projection from $\mathbb{R}^2$ onto the line of direction θ through the origin. Also call π_θ^* the projection in the orthogonal direction. A theorem of Besicovitch tells us that

$$H^1\left(\pi_\theta\left(G_{\mathrm{irr}}\right)\right) = 0 \text{ for almost all } \theta \in \mathbb{S}^1. \tag{7.77}$$

See for instance [Ma], Theorem 18.1 on page 241. The same thing obviously holds for π_θ^* and, since $Z_4 \subset G_{\mathrm{irr}}$, we can choose $\theta \in S^1$ such that

$$H^1\left(\pi_\theta\left(Z_4\right)\right) + H^1\left(\pi_\theta^*\left(Z_4\right)\right) = 0. \tag{7.78}$$

Then (7.74) yields

$$H^1\left(\pi_\theta(G\cap B(x,r(x))\right) + H^1\left(\pi_\theta^*(G\cap B(x,r(x))\right) \leqslant 2\tau r(x). \tag{7.79}$$

We want to use this information to choose a square Q_x such that $\partial Q_x \cap G = \varnothing$, and then remove $Q_x \cap G$ from G, but before we do this we need some estimates on the derivative of v.

Fix $p \in (1,2)$, for instance $p = 3/2$. For all $y \in G \cap B\left(x,r(x)/5\right)$ and $0 < t < r(x)/5$, set

$$\omega_p(y,t) = t^{1-4/p}\left\{\iint_{B(y,t)\smallsetminus G} |\nabla v|^p\right\}^{2/p} \tag{7.80}$$

(where we integrate against the Lebesgue measure). The exponent $1-4/p$ was just chosen so that $\omega_p(y,t)$ be a dimensionless number. If $y \in Z_4$ (so that in particular $y \notin Z_2$), we can easily get that $\omega_p(y,t) \leqslant C\lambda$ by Hölder (see (7.22)); the following lemma says that much more is true on average.

Lemma 7.81. — *There is a constant $C_p > 0$ such that*

$$\int_{y\in Z\cap B(x,r(x)/5)}\int_{0<t<r(x)/2} \omega_p(y,t)\frac{dH^1(y)dt}{t} \leqslant C_p\,\lambda\, r(x). \tag{7.82}$$

This is actually proved in [DaSe2], Proposition 4.5 page 311, modulo slightly different hypotheses that do not really matter. In order not to confuse the reader (and also not to force on them the definition of Carleson measures) we shall give a rapid proof here.

Set $K = Z \cap B(x, r(x))$ and, for $z \in \mathbb{R}^2$, $d(z) = \mathrm{dist}(z, K)$. Let $2 < \sigma < +\infty$ be such that $\frac{1}{\sigma} + \frac{p}{2} = 1$, and also choose $b \in (0, 1/\sigma)$. Write

$$|\nabla u(z)|^p = \left\{ |\nabla u(z)|^2 \left(\frac{d(z)}{t}\right)^{2b/p} \right\}^{p/2} \left\{ \frac{d(z)}{t} \right\}^{-b} \tag{7.83}$$

and apply Hölder. We get that

$$\omega_p(y,t) \leqslant t^{1-4/p}\, I(y,t)\, J(y,t)^{2/p\sigma}, \tag{7.84}$$

where

$$I(y,t) = \iint_{B(y,t)\smallsetminus G} |\nabla u(z)|^2 \left(\frac{d(z)}{t}\right)^{2b/p} \tag{7.85}$$

and

$$J(y,t) = \iint_{B(y,t)\smallsetminus G} \left(\frac{d(z)}{t}\right)^{-b\sigma}. \tag{7.86}$$

To estimate $J(y,t)$ we shall need to know that for $0 < \rho \leqslant 1$,

$$|\{z \in B(y,t)\ ;\ \mathrm{dist}(z,K) \leqslant \rho t\}| \leqslant C\rho t^2. \tag{7.87}$$

To prove this we may assume that $\rho < 1/3$ (the other case is trivial). Cover the set in (7.87) by balls $B(\omega, 2\rho t)$ centered on $K \cap B(y, t+2\rho t)$ and such that the $B(\omega, \rho t/5)$ are disjoint. Since Z does not meet Z_3 (because $Z \subset Z_4 = G_{\mathrm{irr}} \smallsetminus (Z_1 \cup Z_2 \cup Z_3)$) we have that

$$H^1\left(G^- \cap B(w, \rho t/5)\right) \geqslant (5C_2)^{-1} \rho t \tag{7.88}$$

(compare with (7.33)). Since we are only interested in points $y \in Z$ (see (7.82)), we may assume that $y \in G^- \smallsetminus Z_1$, and hence $H^1(G^- \cap B(y, 2t)) \leqslant 6\pi t$. [Compare with (7.9).] Since the $B(w, \rho/5)$ are disjoint and contained in $B(y, 2t)$, (7.88) implies that there are at most $C\rho^{-1}$ such balls; (7.87) follows.

To estimate $J(y,t)$ we decompose $B(y,t) \smallsetminus G$ into regions B_k where $d(z)/t \sim 2^{-k}$, $k \geqslant 0$. The contribution of B_k to the integral in (7.86) is $\leqslant C 2^{kb\sigma} |B_k| \leqslant C 2^{kb\sigma} 2^{-k} t^2$ by (7.87). The series converges because we chose $b < \sigma^{-1}$, and the sum is less than Ct^2. So

$$J(y,t)^{2/p\sigma} \leqslant C t^{4/p\sigma}, \tag{7.89}$$

and

$$\omega_p(y,t) \leqslant C t^{1-\frac{4}{p}+\frac{4}{p\sigma}} I(y,t) = C t^{-1} I(y,t) \tag{7.90}$$

by (7.84) and because $\frac{1}{\sigma} + \frac{p}{2} = 1$. Thus the left-hand side of (7.82) is less than C times

$$I = \iiint |\nabla u(z)|^2 \left(\frac{d(z)}{t}\right)^{2b/p} \frac{dz\,dH^1(y)\,dt}{t^2}, \tag{7.91}$$

where we integrate with the constraints $y \in Z \cap B(x, r(x)/5)$, $0 < t < r(x)/2$, and $z \in \mathbb{R}^2 \setminus G$ lies at distance $< t$ from y. In particular $z \in B(x, 7r(x)/10)$. By Fubini,

$$I \leqslant \int_{z \in B(x,r(x)) \setminus G} a(z) \left|\nabla u(z)\right|^2 dz, \tag{7.92}$$

where

$$a(z) = \int_{0<t<r(x)/2} \left(\frac{d(z)}{t}\right)^{2b/p} \left\{\int_{y \in K \cap B(z,t)} dH^1(y)\right\} \frac{dt}{t^2} \tag{7.93}$$

(recall that $K = Z \cap B(x, r(x))$). Let us check that

$$\int_{K \cap B(z,t)} dH^1(y) \leqslant Ct. \tag{7.94}$$

Suppose $K \cap B(z,t)$ is not empty, and let ξ denote one of this points. Then $K \cap B(z,t) \subset G^- \cap B(\xi, 2t)$. [Recall that $K \subset Z$, which does not meet L by (7.67).] Since $\xi \in K$, it lies in $G^- \setminus Z_1$ and $H^1(G^- \cap B(\xi, 2t)) \leqslant 6\pi t$ (see (7.9)); hence (7.94) holds with $C = 6\pi$.

Note also that the left-hand side of (7.94) vanishes unless $d(z) = \operatorname{dist}(z, K) \leqslant t$. Hence

$$a(z) \leqslant C \int_{d(z) \leqslant t < r(x)/2} \left(\frac{d(z)}{t}\right)^{2b/p} \frac{dt}{t} \leqslant C, \tag{7.95}$$

and then

$$I \leqslant C \int_{B(x,r(x)) \setminus G} \left|\nabla u(z)\right|^2 \leqslant C\lambda r(x) \tag{7.96}$$

by (7.92), and because $x \in Z \subset G^- \setminus Z_2$. [See (7.22).]

This completes our proof of Lemma 7.81. □

Next we want to deduce from the lemma that for each (small) $\alpha > 0$ there is a $C(\alpha) > 0$ such that for all $x \in X_0$ as above, we can find $y \in Z \cap B(x, r(x)/5)$ and $t \in \left[C(\alpha)^{-1} r(x), r(x)/2\right]$ such that

$$\omega_p(y,t) < \alpha\lambda. \tag{7.97}$$

Suppose not. Call J the left-hand side of (7.82). We would have that

$$\begin{aligned} J &\geqslant \int_{y \in Z \cap B(x,r(x)/5)} \int_{C(\alpha)^{-1} r(x)}^{r(x)/2} \alpha \lambda \, dH^1(y) \frac{dt}{t} \\ &\geqslant \alpha\lambda H^1\left(Z \cap B(x, r(x)/5)\right) \operatorname{Log} \frac{C(\alpha)}{2} \geqslant \alpha\lambda \frac{r(x)}{6} \operatorname{Log} \frac{C(\alpha)}{2} \end{aligned} \tag{7.98}$$

by (7.73), and because almost all of Z_4 lies in Z. [See around (7.67)-(7.69).] Of course this would contradict (7.82) if $C(\alpha)$ is large enough, whence the existence of the pair (y, t).

Now choose τ so small, depending on α (which will be chosen soon), so that $\tau < (100C(\alpha))^{-1}$. Then (7.79) tells us that we can find a square Q_x centered at y, with sides parallel to θ and the orthogonal direction, and such that

$$B(y,t/10) \subset Q_x \subset B(y,t) \tag{7.99}$$

and

$$\partial Q_x \cap G = \varnothing. \tag{7.100}$$

Indeed, if ℓ denotes half the sidelength of Q_x, (7.99) allows all choices of ℓ between $t/10$ and $t/2$, while (7.79) (and the fact that $B(y,t) \subset B(x,r(x))$) only forbid a set of values of ℓ with measure $\leqslant 4\tau r(x)$. [The extra factor 2 comes from the fact that both ℓ and $-\ell$ have to avoid the projections in (7.79).] Since $4\tau r(x) < \frac{4r(x)}{100C(\alpha)} \leqslant \frac{4t}{100}$, this leaves a lot of room for the choice of ℓ and Q_x.

In fact, there is even enough room left for the choice of ℓ to allow the extra requirement that

$$\begin{aligned}\int_{\partial Q_x} |\nabla v|^p \, dH^1 \leqslant \frac{100}{t} \iint_{B(y,t)\setminus G} |\nabla v|^p = \frac{100}{t}\left\{t^{\frac{4}{p}-1}\omega_p(y,t)\right\}^{p/2} \\ = 100t^{1-p/2}\omega_p(y,t)^{p/2} \leqslant 100\alpha^{p/2}\lambda^{p/2}t^{1-p/2},\end{aligned} \tag{7.101}$$

by (7.80) and (7.97).

Denote by v_1 the harmonic extension on Q_x of the values of v on ∂Q_x (which are of class C^1 by (7.100)). It is well-known (and not hard to check with the Poisson kernel) that functions on a circle which have a derivative in L^p have an extension to the disk with finite energy $\iint |\nabla v|^2$. This fact, plus the bilipschitz equivalence of squares to disks and a small homogeneity argument, yields

$$\iint_{Q_x} |\nabla v_1|^2 \leqslant C\, t^{2-2/p} \left\{\int_{\partial Q_x} |\nabla v|^p \, dH^1\right\}^{2/p} \leqslant C\,\alpha\,\lambda\, t\,, \tag{7.102}$$

by (7.101).

Now we can choose α so small that $C\alpha < (100C_2)^{-1}$, where C_2 is the same constant as in the definition (7.33) of Z_3. Then

$$\begin{aligned}\iint_{Q_x} |\nabla v_1|^2 \leqslant (100C_2)^{-1}\lambda t \leqslant 10^{-1}\lambda H^1\left(G^- \cap B(y,t/10)\right) \\ \leqslant 10^{-1}\lambda H^1\left(G^- \cap Q_x\right)\end{aligned} \tag{7.103}$$

because $y \in Z$, $Z \subset G^- \setminus Z_3$, and by (7.99).

We are now ready to conclude. For each $x \in X_0$ (our finite subset of X, chosen just after (7.76)) we choose y, t, and Q_x as above. Then we take

$$G_1 = G \setminus \Big(\bigcup_{x\in X_0} Q_x\Big). \tag{7.104}$$

We still have that $G_1 \supset L$ and $G_1 \subset \overline{B}_R$, because $Q_x \subset B(y,t) \subset B(x,r(x)) \subset B_R \setminus L$ (by (7.72) in particular). Also, the Q_x are disjoint because the $B(x,r(x))$ are disjoint (by (7.75)), and so we may define v_1 on $\mathbb{R}^2 \setminus G_1$ by keeping $v_1 = v$ out of the squares Q_x and defining v_1 on each Q_x as above. Then

$$\int_{B_R\setminus G_1} |\nabla v_1|^2 - \int_{B_R\setminus G} |\nabla v|^2 \leqslant \sum_{x\in X_0} \int_{Q_x} |\nabla v_1|^2 \leqslant 10^{-1}\lambda \sum_{x\in X_0} H^1\left(G^- \cap Q_x\right) \leqslant 10^{-1}\lambda \left[H^1\left(G^-\right) \setminus H^1\left(G_1^-\right)\right] \tag{7.105}$$

because the Q_x are disjoint, by (7.103), and by (7.104).

On the other hand,

$$h\left(H^1\left(G^-\right)\right) - h\left(H^1\left(G_1^-\right)\right) \geqslant \frac{\lambda}{4}\left[H^1\left(G^-\right) - H^1\left(G_1^-\right)\right] \tag{7.106}$$

by (4.59), and so

$$J(v,G) - J\left(v_1,G_1\right) \geqslant \left(\frac{\lambda}{4} - \frac{\lambda}{10}\right)\left[H^1\left(G^-\right) - H^1\left(G_1^-\right)\right] = \frac{6\lambda}{40}\sum_{x\in X_0} H^1\left(G^- \cap Q_x\right) \geqslant \frac{6\lambda}{40}\sum_{x\in X_0} C_2^{-1}\frac{r(x)}{10C(\alpha)} \tag{7.107}$$

because for each x, Q_x contains $B\left(y,t/10\right)$, which is centered at $y \in Z \subset G^- \setminus Z_3$, and $t \geqslant r(x)/C(\alpha)$. See (7.99), the definition of y and t before (7.97), and (7.33).

We also know from (7.5) that $J(v,G) \leqslant J\left(v_1,G_1\right) + \varepsilon$, and hence

$$\lambda \sum_{x\in X_0} r(x) \leqslant C\varepsilon. \tag{7.108}$$

Of course (7.108) still holds with X, because it holds for all its finite subsets X_0. Finally

$$H^1(Z) \leqslant \sum_{x\in X} H^1\left(Z \cap B(x,5r(x))\right) \leqslant 15\pi \sum_{x\in X} r(x) \leqslant C\varepsilon \tag{7.109}$$

by (7.76), because $Z \subset G^-$, because all points $x \in X$ lie in $G^- \setminus Z_1$, by definition (7.9) of Z_1, and because $\lambda \geqslant 1$.

This completes our proof of (7.70) and, as explained just before (7.70), of Proposition 7.4. □

8. The J_R^k approximate J_R

In this section we complete the proof of (6.2), i.e. the fact that $\eta(R)$, the infimum of J_R, is the limit of the corresponding infima $\eta_k(R)$ of the J_R^k.

Note that U_R^k increases with k, so $\eta_k(R)$ is a nonincreasing function of k and $\lim\limits_{R\to\infty} \eta_k(R) \geqslant \eta(R)$. To complete the proof we give ourselves a small $\varepsilon > 0$ and a competitor $(v, G) \in U_R$ that satisfies (7.5) (i.e., for which $J_R(v, G)$ is almost minimal). We want to use (v, G) to construct a competitor (u, K) in some U_R^k such that $J(u, K) \leqslant \eta(R) + C\varepsilon$. This will give a k for which $\eta_k(R) \leqslant \eta(R) + C\varepsilon$, and (6.2) will follow because $\eta_k(R)$ is a nonincreasing function of k.

So let $\varepsilon > 0$ and (v, G) be given, and assume that (7.5) holds (just like in the previous section). Let us look more closely at the rectifiable part $G_{\rm rec}$ of G. The definition (7.2) says that $G_{\rm rec}$ is covered, except perhaps for a subset of vanishing H^1-measure, by a countable collection $\{\Gamma_i\}$ of simple curves of class C^1. We may as well assume that $\Gamma_1 = L$. Then $G \setminus \Gamma_1$ has finite H^1-measure, and we can find a finite subcollection $\Gamma_1, \ldots, \Gamma_m$ such that

$$H^1\Big(G_{\rm rec} \setminus \bigcup_{i=1}^m \Gamma_i\Big) \leqslant \varepsilon. \tag{8.1}$$

Set $E_j = (G_{\rm rec} \cap \Gamma_j) \setminus \bigcup_{i<j} E_i$. Then $E_j \subset \Gamma_j \cap G_{\rm rec}$, the E_j are disjoint, and $H^1\Big(G_{\rm rec} \setminus \bigcup_j E_j\Big) \leqslant \varepsilon$. The Lebesgue (or Hausdorff) measure on each Γ_j is regular, so we can find an open set $\mathcal{O}_j \subset \Gamma_j$ which contains E_j and such that $H^1\left(\mathcal{O}_j \setminus E_j\right) \leqslant \varepsilon/m$. The open set $\mathcal{O}_j$ is a union of at most countably many intervals of Γ_j, and we can choose a finite union $\mathcal{O}'_j$ of these intervals so that $H^1\left(\mathcal{O}_j \setminus \mathcal{O}'_j\right) \leqslant \varepsilon/m$. For $j = 1$ we keep $\mathcal{O}'_j = \mathcal{O}_j = L$. Also, for $j > 1$, me may have taken $\Gamma_j \subset \overline{B}_R$, and then $\mathcal{O}'_j \subset \overline{B}_R$ as well.

Finally denote by K_0 the union of all the closures $\overline{\mathcal{O}}'_j$ of these intervals. Then

$$K_0 \text{ is closed, contains } L, \text{ and is contained in } L \cup \overline{B}_R, \tag{8.2}$$

$$K_0 \text{ has finitely many connected components}, \tag{8.3}$$

$$H^1\left(G_{\rm rec} \setminus K_0\right) \leqslant 2\varepsilon, \tag{8.4}$$

$$H^1\left(K_0 \setminus G_{\rm rec}\right) \leqslant \varepsilon. \tag{8.5}$$

Because of (7.6) we also have that

$$H^1\left(G \setminus K_0\right) \leqslant C\varepsilon. \tag{8.6}$$

We need to find a way to get rid of $G \setminus K_0$, without paying too much in terms of J_R, and most importantly without adding infinitely many connected components to K_0.

As usual, we start with a covering of $G \setminus K_0$. For each $x \in G \setminus K_0$, set

$$r_1(x) = \sup\Big\{r \in [0,1] \ ; \ \int_{B(x,r)\cap B_R \setminus G} |\nabla v|^2 \geqslant 5\pi r\lambda\Big\} \tag{8.7}$$

if $x \in Z_2$ (see (7.22)), and $r_1(x) = 0$ otherwise. Thus $r_1(x) > 0$ if $x \in Z_2$. Also $r_1(x) < 1/10$ (if ε is small enough), because otherwise we could add some $\partial(B(x,r) \cap B_R)$ to G, replace v with a constant on $B(x,r) \cap B_R$, and reduce $J_R(v,G)$ by much more than ε. See the proof of Lemma 7.21 for more detail. Of course,

$$\int_{B(x,r)\cap B_R \setminus G} |\nabla v|^2 \leqslant 5\pi r\lambda \quad \text{for} \quad r_1(x) \leqslant r \leqslant 1, \tag{8.8}$$

by definition of $r_1(x)$ and a small limiting argument (to get $r = r_1(x)$ also). Next set

$$r_2(x) = \sup\left\{r \geqslant 0 \ ; \ H^1\left(B(x,r) \cap G \setminus K_0\right) \geqslant r/4\right\} \tag{8.9}$$

for $x \in G \setminus K_0$. By the same density theorem as for (4.18) or (7.35), we have that $r_2(x) > 0$ for H^1-almost every $x \in G \setminus K_0$. Because of (8.6), $r_2(x) < 1/10$ everywhere. Set

$$r_3(x) = \operatorname{Max}\left(r_1(x), r_2(x)\right), \tag{8.10}$$

$$Z_5 = \{x \in G \setminus K_0 : r_3(x) > 0\}, \tag{8.11}$$

and

$$Z_6 = G \setminus (K_0 \cup Z_5). \tag{8.12}$$

Choose a set $X_5 \subset Z_5$ such that the balls $B\left(x, r_3(x)\right)$, $x \in X_5$, are disjoint, but the $B\left(x, 5r_3(x)\right)$, $x \in X_5$, cover Z_5.

Denote by X_5' the set of $x \in X_5$ such that $r_3(x) = r_2(x)$. Then

$$\begin{aligned} \sum_{x\in X_5'} r_3(x) = \sum_{x\in X_5'} r_2(x) &\leqslant 4 \sum_{x\in X_5'} H^1\left(B\left(x, r_2(x)\right) \cap G \setminus K_0\right) \\ &\leqslant 4H^1\left(G \setminus K_0\right) \leqslant C\varepsilon \end{aligned} \tag{8.13}$$

by definition of $r_2(x)$, and then the disjointness of the $B\left(x, r_3(x)\right)$ and (8.6).

Also set $X_5'' = X_5 \setminus X_5'$. Then $X_5'' \subset Z_2$, because $r_3(x) = r_1(x) > 0$ for $x \in X_5''$ (and $x \in G^-$ because $L \subset K_0$). The same argument as in Lemma 7.21 tells us that

$$\pi\lambda \sum_{x\in X_5''} r_1(x) \leqslant \varepsilon. \tag{8.14}$$

[See in particular (7.30) and the comment that follows it; the additional information that the points of the set X (in the proof of Lemma 7.21) all lie in $G^- \setminus Z_1$ was not used for (7.30), but only in (7.31) to deduce an estimate on $H^1\left(Z_2 \setminus Z_1\right)$ from (7.30).] Since $r_3(x) = r_1(x)$ on X_5'', we can group (8.13) with (8.14) and get that

$$\sum_{x\in X_5} r_3(x) \leqslant C\varepsilon. \tag{8.15}$$

We also want to cover Z_6. Since $r_2(x) > 0$ almost-everywhere on $G \setminus K_0$, $H^1\left(Z_6\right) = 0$. Thus we can cover Z_6 by disks $B\left(x, r_4(x)\right)$, $x \in X_6'$, with $\sum_{x\in X_6'} r_4(x) \leqslant \varepsilon$. We may even assume that $X_6' \subset Z_6$ (otherwise, change the centers and double the radii). Let us use the Vitali type covering lemma again to find a subset X_6 of X_6' such that the disks $B(x, r_4(x))$, $x \in X_6$, are disjoint, and the $B(x, 5r_4(x))$, $x \in X_6$, cover Z_6.

To make our notations more uniform, set $X = X_5 \cup X_6$, and also $r_4(x) = r_3(x)$ for $x \in X_5$. By the preceding discussion,

$$G \setminus K_0 = Z_5 \cup Z_6 \subset \bigcup_{x \in X} B(x, 5r_4(x)), \tag{8.16}$$

and

$$\sum_{x \in X} r_4(x) \leqslant C\varepsilon. \tag{8.17}$$

Denote by $x_1, x_2, \dots x_i, \dots$ the elements of X. We can choose this description of X in such a way that the sequence of radii $\{r_4(x_i)\}$ be nonincreasing. [This is because of (8.17).] For $i \geqslant 1$, we shall choose a last radius $r_i = r(x_i)$ with the following properties. First,

$$5r_4(x_i) < r_i < 10r_4(x_i) \tag{8.18}$$

and, if we set $B_i = B(x_i, r_i)$,

$$\partial B_i \cap G \setminus K_0 = \varnothing. \tag{8.19}$$

So far, this is easy to obtain, because $r_4(x_i) \geqslant r_2(x_i)$, and by the definition (8.9). We also require that

$$\int_{\partial B_i \cap B_R \setminus G} |\nabla v|^2 \leqslant C\lambda. \tag{8.20}$$

This is also easy to get because $r_4(x_i) \geqslant r_1(x_i)$, $10r_4(x_i) < 1$ (by (8.17)), and by (8.7). We add a last constraint on the choice of r_i. Set

$$J(i) = \{j < i : B(x_j, 11r_4(x_j)) \text{ meets } B(x_i, 11r_4(x_i))\}. \tag{8.21}$$

Note that $J(i)$ has at most C elements, because $r_4(x_j) \geqslant r_4(x_i)$ for $j \in J(i)$, the disks $B(x, r_4(x))$, $x \in X_5$, are disjoint, and so are the $B(x, r_4(x))$, $x \in X_6$. We also require that for all $j \in J(i)$ either

$$\operatorname{dist}(\partial B_i, \partial B_j) \geqslant \theta \operatorname{Min}(r_i, r_j) \tag{8.22a}$$

or else

$$\partial B_i \text{ and } \partial B_j \text{ intersect with an angle } \geqslant \theta, \tag{8.22b}$$

where $\theta > 0$ is a small positive constant.

Finally we require that (8.22a) or (8.22b) hold also with ∂B_R instead of ∂B_j (and R instead of r_j).

If we choose θ small enough, then we can add the new constraints (8.22) to the previous ones (8.18)-(8.20). This is because for each $j \in J(i)$, the constraint (8.22) only forbids a set of measure $\leqslant C\theta r_4(x_i)$ of potential choices for r_i, and there are at most C such constraints. [The final constraint on transversality with ∂B_R is treated the same way.]

So we choose the radii r_i with the properties above. Note that our condition (8.22) is actually satisfied for all $j < i$ (because (8.22a) holds trivially when $j \notin J(i)$), and even for all $j \neq i$ (because (8.22) is symmetric).

Let us modify a last time our list of disks B_i. For each $i \geqslant 1$, set

$$D_i = B_i \cap B_R. \tag{8.23}$$

Remove from our list of disks B_i all those for which D_i is not maximal, i.e., those for which D_i is strictly contained in some other D_j. If two or more disks D_i coincide, just keep one of them and remove all the other ones. Of course these modifications do not change $\bigcup_i (B_i \cap \overline{B}_R)$, so we still have that

$$G \setminus K_0 \subset \bigcup_i B_i \tag{8.24}$$

after this modification, and now

$$D_i \text{ is not contained in } D_j \text{ when } i \neq j. \tag{8.25}$$

Let us also use the opportunity to re-number the B_i. Choose a new ordering such that the sequence $\{r_i\}$ (with the new ordering) is non-increasing.

We are now ready to start the construction of a new competitor $(u, K) \in U_R$. This competitor will be the limit of a sequence $\{(v_i, G_i)\}$ which we want to construct now. We shall define at the same time a nondecreasing sequence of closed sets $K_i \subset G_i$; the main property of K_i will be that it does not have more connected components than K_0.

We start with K_0 as above, $G_0 = G \cup K_0$, and for v_0 the restriction of v to $\mathbb{R}^2 \setminus G_0$. [This may look artificial, but we want to have $K_0 \subset G_0$.] Note that $(v_0, G_0) \in U_R$ because of (8.2) in particular.

Now suppose that $i \geqslant 1$ and that we already defined the v_j, G_j and K_j, $0 \leqslant j \leqslant i-1$. Our construction will be simpler if

$$\partial D_i \text{ meets } K_{i-1} \tag{8.26}$$

(with D_i as in (8.23)); we shall then say that $i \in I_1$. In this case we take

$$K_i = K_{i-1} \cup \partial D_i, \tag{8.27}$$

$$G_i = K_i \cup [G_{i-1} \setminus D_i], \tag{8.28}$$

and we define v_i by

$$\begin{cases} v_i = v_{i-1} & \text{out of } \quad D_i \\ v_i = 0 & \text{on } \quad D_i. \end{cases} \tag{8.29}$$

It is easy to check that $K_i \subset G_i$, K_i does not have more connected components than K_{i-1} (by (8.26)), G_i is closed, and $(v_i, G_i) \in U_R$ (if we already have that $(v_{i-1}, G_{i-1}) \in U_R$). Note also that

$$K_i, G_i, \text{ and } v_i \text{ coincide with } K_{i-1},\ G_{i-1}, \text{ and } v_{i-1} \text{ out of } \overline{D}_i. \tag{8.30}$$

Now consider the more delicate case when $i \in I_2$, where

$$I_2 = \{i \geqslant 1\ ;\ \partial D_i \text{ does not meet } K_{i-1}\}. \tag{8.31}$$

In this case we do not want to add ∂D_i to our boundary (because this would increase the number of components of K_i), and instead we shall try to get rid of the part of G_{i-1} inside B_i. More precisely, set

$$D_i^* = \overline{D}_i \setminus \bigcup_{j<i} \overline{B}_j, \tag{8.32}$$

and then take

$$K_i = K_{i-1} \tag{8.33}$$

and

$$G_i = K_i \cup (G_{i-1} \setminus D_i^*). \tag{8.34}$$

We still have that $K_i \subset G_i$ and, since $K_{i-1} \subset G_{i-1}$ by our (implicit) induction hypothesis, G_i coincides with G_{i-1} out of D_i^*. Before we are able to define v_i, we shall need a lot of information on the sets D_i and D_i^*. Let us first check that

$$\overline{D}_i \cap \overline{D}_j = \varnothing \quad \text{when} \quad i \in I_2, \quad j \in I_1, \quad \text{and } j < i. \tag{8.35}$$

Assume instead that $\overline{D}_i$ meets $\overline{D}_j$. Because of (8.25), ∂D_i meets ∂D_j (because otherwise we would have that $D_i \subset D_j$ or $D_j \subset D_i$). This is not possible under the conditions of (8.35), because $\partial D_j \subset K_j \subset K_{i-1}$ and ∂D_i does not meet K_{i-1}. So (8.35) holds.

For most of the definition of v_i we shall have to wait a little more, but we can already decide to keep

$$v_i(x) = v_{i-1}(x) \text{ out of } D_i^*. \tag{8.36}$$

This seems reasonable because we did not modify G_{i-1} out of D_i^*. If we keep with this resolution, we will have that

$$K_i, G_i, \text{ and } v_i \text{ coincide with } K_{i-1}, G_{i-1} \text{ and } v_{i-1} \text{ out ot } D_i^*. \tag{8.37}$$

For the rest of the definition of v_i, we shall need to know more about D_i^* before we can do the necessary gluing. Note that the slightly degenerate case when D_i^* is empty may happen. This will not disturb us, in this case we simply keep $(v_i, G_i) = (v_{i-1}, G_{i-1})$. So let us assume now that D_i^* is not empty, and let us try to describe it better.

Lemma 8.38. — *The boundary* ∂D_i^* *is composed of a bounded number of arcs of circles of radii* $\geqslant r_i$, *and* $\operatorname{int}(D_i^*)$ *is composed of a bounded number of disjoint simply connected domains that are all bilipschitz-equivalent to disks of radius* r_i.

We shall even have a uniform control on the bilipschitz constants for the components of $\operatorname{int}(D_i^*)$ that depends only on the constant $\theta > 0$ in (8.22).

To prove the lemma, first note that ∂D_i^* is composed of a finite number of arcs of circles. This is clear from the definitions (8.32) and (8.23). The corresponding circles are either ∂B_R, or various ∂B_j, $j \leqslant i$, and they all have radii $\geqslant r_i$ because r_j is a monotone function of j. Moreover, (8.22) and its analogue for ∂B_R tell us that the circles are at distances $\geqslant \theta r_i$ from each other or meet with angles $\geqslant \theta$. Thus there are at most C such circles (they all meet $\overline{B}_i$), and there are at most C' arcs of circles.

The open set $\operatorname{int}(D_i^*)$ is obtained from $D_i = B_i \cap B_R$ by removing its intersection with various disks $\overline{B}_j$. To prove that each connected component of $\operatorname{int}(D_i^*)$ is simply connected we can proceed by induction on the number of $\overline{B}_j$ that were removed from D_i. The point is that if U is a simply connected domain contained in D_i and B_j is a disk which is not contained in D_i, then all the connected components of $U \setminus \overline{B}_j$ are simply connected. A well known fact from two-dimensional topology is that this follows from the connectedness of the complement of $U \setminus \overline{B}_j$. This last comes from the fact that all points of $\overline{B}_j$ are connected through $\overline{B}_j$ to the exterior of U. Thus $\operatorname{int}(D_i^*)$ has simply connected components. The fact that they are all bilipschitz equivalent to disks of radius r_i comes from our estimate on the number of arcs of circles (of radii $\geqslant r_i$) that compose ∂D_i^*, their angles $\geqslant \theta$ when they meet, and the fact that $D_i^* \subset D_i$. This completes our proof of Lemma 8.38. □

Next we want information on $\partial_i = \partial D_i^* \cap B_R$ and its closure $\overline{\partial}_i$.

Lemma 8.39. — *We have that*

$$\overline{\partial}_i \cap G_j = \varnothing \quad \text{for} \quad 0 \leqslant j \leqslant i-1, \tag{8.40}$$

$$\overline{\partial}_i \cap \overline{D}_j = \varnothing \quad \text{for} \quad j \in I_1, \quad j < i, \tag{8.41}$$

and

$$\overline{\partial}_i \cap \operatorname{int}\left(D_j^*\right) = \varnothing \quad \text{for} \quad j \in I_2, \quad j < i. \tag{8.42}$$

To prove the lemma, first recall that ∂_i is composed of a finite number of arcs of circles ∂B_j, $j \leqslant i$. Then $\overline{\partial}_i$ is the same as ∂_i, except that we also add the endpoints of these arcs that lie in ∂B_R. Let us first check that

$$\overline{\partial}_i \text{ does not meet } G_0 = G \cup K_0. \tag{8.43}$$

Let γ be one of the arcs that compose ∂_i, and let $j \leqslant i$ denote the index such that $\gamma \subset \partial B_j$. Since $\gamma \subset \overline{D}_j \cap \overline{D}_i$, (8.35) says that $j \in I_2$ if $j < i$; this is also true if $j = i$, trivially. Note that

$$\partial B_j \cap \overline{B}_R \cap G_0 = \varnothing \quad \text{for} \quad j \in I_2. \tag{8.44}$$

Indeed ∂D_j does not meet $K_0 \subset K_{j-1}$ (by definition of I_2, see (8.31)), hence $\partial B_j \cap \overline{B}_R \subset \partial D_j$ does not meet K_0 either, and finally ∂B_j does not meet $G_0 \setminus K_0 = G \setminus K_0$ (by 8.19)).

So the arc $\overline{\gamma}$ cannot meet G_0, because it is contained in $\partial B_j \cap \overline{B}_R$. This proves (8.43).

The rest of (8.40), i.e., the case when $j > 0$, will follow from (8.43) as soon as we prove (8.41). This is because $G_j \subset G_{j-1} \cup \partial D_j$ when $j \in I_1$ (by (8.27), (8.28), and the fact that $K_{j-1} \subset G_{j-1}$), and $G_j \subset G_{j-1}$ when $j \in I_2$ (by (8.33), (8.34), and because $K_{j-1} \subset G_{j-1}$).

To prove (8.41) simply observe that $\overline{\partial}_i \cap \overline{D}_j \subset \overline{D}_i \cap \overline{D}_j$, which is empty when $j \in I_1$, $j < i$, because of (8.35).

Finally (8.42) holds because $\overline{\partial}_i \subset \partial D_i^*$, which cannot meet $\operatorname{int}(D_j^*)$ for $j \in I_2$, $j < i$, because $\operatorname{int} D_j^* \subset B_j$ and by (8.32). This completes our proof of Lemma 8.39. □

We are now ready to define the function v_i on D_i^*. [Recall that the rest of $\mathbb{R}^2 \setminus G_i$ was taken care of by (8.36) already. We want to take

$$v_i(x) = v(x) \quad \text{on} \quad \partial_i, \tag{8.45}$$

$$v_i(x) = u_0(x) \quad \text{on} \quad \partial B_R \cap \partial D_i^*, \tag{8.46}$$

and then define v_i on $\operatorname{int}(D_i^*)$ by the condition that

$$v_i \text{ is continuous on } \overline{D}_i^* \text{ and harmonic on } \operatorname{int}(D_i^*). \tag{8.47}$$

We want to check that this is coherent, and we shall proceed by induction. Thus we assume that all the functions v_j, $j < i$, $j \in I_2$, have been chosen according to these rules, and we shall check that v_i also can be defined through (8.45)-(8.47).

Because of (8.40), all the function v_j, $j \leqslant i-1$ are defined on $\overline{\partial}_i$. Let us check that

$$v_j(x) = v(x) \text{ for } x \in \overline{\partial}_i \text{ and } j \leqslant i-1. \tag{8.48}$$

This is true for $j = 0$, because v_0 is the restriction of v to $\mathbb{R}^2 \setminus G_0$. If $1 \leqslant j \leqslant i-1$ and (8.48) holds for $j-1$, there are two options. If $j \in I_1$, then (8.48) holds because of (8.41) and (8.30). If $j \in I_2$ and $x \in \overline{\partial}_i \setminus D_j^*$, then $v_j(x) = v_{j-1}(x) = v(x)$ by (8.36) and induction hypothesis. Since $\overline{\partial}_i$ does not meet $\operatorname{int}(D_j^*)$ by (8.42), we are left with the case when $j \in I_2$ and $x \in \overline{\partial}_i \cap \partial D_j^*$. Then either $x \in \partial_j = \partial D_j^* \cap B_R$, and $v_j(x) = v(x)$ by (8.45) (and induction hypothesis) or else $x \in \partial B_R \cap \partial D_j^*$ and $v_j(x) = u_0(x) = v(x)$ by (8.46). This proves (8.48).

Because of (8.48) for $j = i-1$, the two definitions that we have given of v_i on ∂_i (in (8.36) and (8.45)) coincide. The situation on $\partial B_R \cap \partial D_i^*$ is similar; we chose $v_i = u_0$ in

(8.46), and (8.36) also forced $v_i = u_0$ on $\partial B_R \cap \partial D_i^* \setminus G_{i-1}$ because $(v_{i-1}, G_{i-1}) \in U_R$ (by induction hypothesis). Here we may have defined v_i (in (8.46)) also on a piece of $\partial B_R \cap G_i$, but this does not matter. So (8.45) and (8.46) are compatible with (8.36).

Since $\overline{\partial}_i \cap G_0 = \varnothing$ (by (8.40)), the restriction of v to ∂_i has limits at the points of $\overline{\partial}_i \cap \partial B_R$ which coincide with the values of u_0 at these points. [This is because $v(x) = u_0(x)$ on $\partial B_R \setminus G_0$, since $(v, G) \in U_R$.] Thus

(8.49) (8.45) and (8.46) define a continuous function v_i on ∂D_i^*.

Because of this (and also of Lemma 8.38) our definition of v_i on $\mathrm{int}(D_i^*)$ (that is, by (8.47)) makes sense.

Thus we can define v_i as in (8.36) and (8.45)-(8.47). Moreover, v_i is continuous (and even piecewise C^1) across ∂D_i^*. It is easy to check that $(v_i, G_i) \in U_R$, because of this, (8.36), and (8.34).

This completes our definition of v_i, G_i, and K_i by induction. We need some estimates on ∇v_i before we let i tend to $+\infty$. Because of Lemma 8.38, we can estimate the Dirichlet integral of v_i on D_i^* like on a bounded collection of disks of radius r_i, and

$$\int_{D_i^*} |\nabla v_i|^2 \leqslant C r_i \int_{\partial D_i^*} \left| \frac{\partial v_i}{\partial \tau} \right|^2 dH^1, \tag{8.50}$$

where $\partial v_i / \partial \tau$ denotes the (tangential) derivative of the restriction of v_i to ∂D_i^*. Obviously

$$\int_{\partial D_i^* \cap \partial B_R} \left| \frac{\partial v_i}{\partial \tau} \right|^2 \leqslant C r_i R^{-1} \tag{8.51}$$

by (8.46). Since $\partial D_i^* \setminus \partial B_R = \partial D_i^* \cap B_R = \partial_i$ is composed of finitely many arcs of circle ∂B_j, $j \leqslant i$, we may use (8.43) and (8.20) to get that

$$\int_{\partial D_i^* \setminus \partial B_R} \left| \frac{\partial v_i}{\partial \tau} \right|^2 \leqslant C\lambda. \tag{8.52}$$

Thus $\int_{D_i^*} |\nabla v_i|^2 \leqslant C r_i \lambda$, by (8.50)-(8.52), and then

$$\int_{B_R \setminus G_i} |\nabla v_i|^2 \leqslant \int_{B_R \setminus G_{i-1}} |\nabla v_{i-1}|^2 + C r_i \lambda \tag{8.53}$$

because of (8.36).

We are now ready to go to the limit. Set

$$K = \Big(\bigcup_{i=1}^{\infty} K_i \Big)^{-}. \tag{8.54}$$

Let us first check that K does not have more connected components than K_0. For each connected component A_ℓ of K_0, choose an origin $z_\ell \in A_\ell$. For all $i \geqslant 1$, every point of K_i can be connected to one of the z_ℓ by a path contained in K_i. [This is because the only times when we modified the sets K_i were when $i \in I_1$, and then we added a piece ∂D_i connected to K_{i-1} (by (8.26)).] Consequently, if A_ℓ^* denotes the connected component of z_ℓ in $\bigcup_i K_i$, the union of the sets A_ℓ^* contains all the sets K_i (and hence is dense in K). The desired result follows, because the closure (in K) of each A_ℓ^* is connected.

Next we check that

$$K \subset K_0 \cup G \cup \Big(\bigcup_{i \in I_1} \partial D_i\Big). \tag{8.55}$$

Since by (8.27) and (8.33) all the K_i are contained in the right-hand side of (8.55), the only interesting points are those of $K \setminus \bigcup_i K_i$. Let x be such a point, and let $\{x_n\}$ be a sequence of points of $\bigcup_i K_i$ which converges to x. If $x_n \in G \cup K_0$ for infinitely many values of n, then $x \in G \cup K_0$ (which is closed by (8.2)) and we are happy. If x_n lies in a single ∂D_i, $i \in I_1$, for infinitely many values of n, then $x \in \partial D_i$; this does not happen because we assumed that $x \notin K_i$. We are left with the case when $x_n \in \partial D_{i_n}$ for n large enough, with an index i_n that tends to $+\infty$. Since $\partial D_{i_n} \subset \overline{B}_{i_n}$ and B_{i_n} is centered on G, we have that $\operatorname{dist}(x_n, G) \leqslant r_{i_n}$. Also r_{i_n} tends to 0 (by (8.17) and (8.18)), and so $x \in G$. This proves (8.55).

Lemma 8.56. — *For all $x \in \mathbb{R}^2 \setminus K$, there is a neighborhood W of x and an index $j \geqslant 1$ such that, for all $i \geqslant j$,*

$$W \subset \mathbb{R}^2 \setminus G_i \tag{8.57}$$

and

$$v_i(z) = v_j(z) \quad \text{on} \quad W. \tag{8.58}$$

In other words, every point of $\mathbb{R}^2 \setminus K$ has a neighborhood W on which v_i is defined for i large and $\{v_i\}$ is stationary.

Let us first check that for all $x \in \mathbb{R}^2 \setminus K$ there is a neighborhood W_0 of x that meets only finitely many sets $\overline{D}_i$, $i \in I_1$.

Suppose not. Then there is a sequence $\{i_n\}_{n \geqslant 1}$ of indices $i_n \in I_1$ such that i_n tends to $+\infty$ and $d_n = \operatorname{dist}(x, D_{i_n})$ tends to 0. Note that $\operatorname{dist}(x, K) \leqslant d_n + \operatorname{diam} D_{i_n} \leqslant d_n + 2r_{i_n}$ because $\partial D_{i_n} \subset K_n \subset K$. Since r_{i_n} tends also to 0, we get that $x \in K$, a contradiction. This give the existence of W_0.

Let us first prove the conclusion of the lemma when

$$x \in D = \bigcup_{i \in I_1} D_i. \tag{8.59}$$

Let j denote the largest integer of I_1 such that $x \in D_j$; j is well-defined because $x \in W_0$ and W_0 meets only finitely many D_i, $i \in I_1$. Note that $x \notin \partial D_i$ for any

$i \in I_1$, because $\partial D_i \subset K$ and $x \notin K$. Thus

$$\operatorname{dist}\big(x, \overline{D}_i\big) > 0 \quad \text{for all} \quad i \in I_1, i > j. \tag{8.60}$$

Only finitely many $\overline{D}_i$, $i \in I_1$, can ever get close to x (because $x \in W_0$), so (8.60) implies that we can find a neighborhood W of x that does not meet any $\overline{D}_i$, $i \in I_1$ and $i > j$. We may as well modify W (by reducing it) so that W be open,

$$W \subset D_j \cap W_0, \text{ and } W \cap K = \varnothing \tag{8.61}$$

(because $x \notin K$).

Note that by (8.35) $\overline{D}_j$ does not meet any $\overline{D}_i$, $i \in I_2$ and $i > j$. Then it does not meet the corresponding D_i^* either. We already know that W does not meet any of the $\overline{D}_i$, $i \in I_1$ and $i > j$. Altogether, (8.30) and (8.37) tell us that all the K_i, G_i, and v_i, $i \geqslant j$, coincide with K_j, G_j, and v_j on W. Thus (8.57) and (8.58) will follow as soon as we prove that W does not meet G_j.

By (8.61), W does not meet $K_j \subset K$. Since $G_j \cap D_j = K_j \cap D_j$ by (8.28), we get that W does not meet G_j (by (8.61)), as needed. This completes our discussion of the case when $x \in D$ (as in (8.59)).

Now suppose that $x \notin D$. Then

$$\operatorname{dist}(x, D) = \operatorname{dist}\Big(x, \bigcup_{i \in I_1} \partial D_i\Big) \geqslant \operatorname{dist}(x, K) > 0\ , \tag{8.62}$$

because K contains all the ∂D_i, $i \in I_1$. Let us now assume that

$$x \in B_j \quad \text{for some} \quad j \in I_2. \tag{8.63}$$

Let W be a neighborhood of x which is contained in B_j and does not meet K or $\overline{D}$. [Such a W exists, by (8.62).] Then W does not meet any of the $\overline{D}_i$, $i \in I_1$. It does not meet the D_i^*, $i \in I_2$ and $i > j$ either (by (8.32)); hence the K_i, G_i, and v_i stay the same on W for $i > j$ as for $i = j$, by (8.30) and (8.37). Once again, it will be enough to show that W does not meet G_j.

Since $K_j \subset K$ and W does not meet K, it will be enough to check that

$$G_j \cap \overline{B}_j \subset K_j \quad \text{for} \quad j \in I_2. \tag{8.64}$$

(because $W \subset B_j$). To prove (8.64) we may forget about L (which is contained in K_0); thus it is enough to control $G_j \cap \overline{B}_j \cap \overline{B}_R = G_j \cap \overline{D}_j$.

The only situations where we ever add something to G_i are when $i \in I_1$ and we add ∂D_i. This never affects $\overline{D}_j$, because $\overline{D}_i \cap \overline{D}_j = \varnothing$ for such i, $i < j$ (by (8.35)). Thus all the points of $G_j \cap \overline{D}_j$ already lie in G_0. Let z be such a point. Let j_0 denote the smallest index of I_2 such that $z \in \overline{D}_{j_0}$; then $j_0 \leqslant j$ because $z \in \overline{D}_j$. If $z \notin K_j$ (the only interesting case in view of (8.64)), $z \notin K_{j_0}$ either. By definition of j_0, $z \in D_{j_0}^*$ (see (8.32), use (8.35) again, and note that $z \in \overline{D}_\ell$ if it lies in some $\overline{B}_\ell$). By (8.34),

$z \notin G_{j_0}$, and since z was never put back in G_j (by the remark about the $\overline{D}_i$, $i \in I_1$), $z \notin G_j$. This proves (8.64).

We are left with the case when (8.59) and (8.63) both fail. Let us first check that there is a neighborhood W_1 of x that only meets finitely many sets $\overline{D}_i$, $i \in I_1 \cup I_2$.

Suppose not. Then there is a sequence i_n that tends to $+\infty$ and such that $\mathrm{dist}\left(x, \overline{D}_{i_n}\right)$ tends to 0. Since the B_{i_n} are centered on G, $\mathrm{dist}(x, G) \leqslant \mathrm{dist}\left(x, \overline{D}_{i_n}\right) + r_{i_n}$, which tends to 0 because r_{i_n} also tend to 0. Thus $x \in G$. Since it does not lie in K, $x \in G \setminus K_0$. By (8.24), x lies in some B_i. Obviously $i \in I_1$, since (8.63) does not hold. We do not have $x \in D_i$, because (8.59) fails. Thus $x \in B_i \setminus D_i$, and there are only two options: $x \in \partial D_i$ or $x \in L$. Both cases are impossible because $L \cup \partial D_i \subset K$, and $x \in \mathbb{R}^2 \setminus K$, by the only hypothesis of Lemma 8.56. This proves the existence of W_1.

Now let j be so large that no D_i, $i > j$, meets W_1. Then for all $i > j$ the restrictions to W_1 of K_i, G_i, and v_i are the same as for $i = j$. Thus it is enough to find a neighborhood W of x such that $W \cap G_j = \varnothing$ and $W \subset W_1$.

Simply choose W connected, contained in W_1, and disjoint from K. Suppose, to get a contradiction, that $W \cap G_j$ contains some point z. Note that $G_j \subset G \cup K$ (because $G_0 = G \cup K_0$ and all the pieces that we ever added to the G_i were also added to some K_i, hence lie in K). Since W does not meet K, z lies in G. We even have that $z \in G \setminus K_0$ because $K_0 \subset K$. In particular $z \in \overline{B}_R$ (because $L \subset K_0$), and also (8.24) says that $z \in B_{i_0}$ for some $i_0 \geqslant 1$.

We claim that W does not meet any $\overline{D}_i$, $i \in I_1$. Indeed if $i \in I_1$, ∂D_i does not meet W (because $\partial D_i \subset K$) and $x \notin D_i$ (because we are still in the case when (8.59) fails). Since W is connected, it cannot meet D_i. This proves the claim, and the consequence is that $i_0 \in I_2$ (since $z \in B_{i_0} \cap \overline{B}_R \subset \overline{D}_{i_0}$).

We know from (8.64) that $G_{i_0} \cap B_{i_0} \subset K_{i_0}$. Hence $z \notin G_{i_0}$, since $z \in W$ and W does not meet K. Since $z \in W \cap G_j$ (by definition of z) and all $W \cap G_i$, $i > j$, coincide with $W \cap G_j$ (because $W \subset W_1$), we must have $i_0 < j$. This means that z has been added to G_j some time between i_0 and j. This is impossible because the only points that we ever add to the sets G_i lie in some ∂D_i, $i \in I_1$, and so lie in K.

This contradiction proves that $W \cap G_j = \varnothing$, and completes our proof of (8.57) and (8.58) in the last of our three cases. Lemma 8.56 follows. □

Because of Lemma 8.56, the functions v_i converge, uniformly on compact subsets of $\mathbb{R}^2 \setminus K$, to a function u. The convergence is excellent: the sequence is stationary on each compact subset of $\mathbb{R}^2 \setminus K$. Hence $u \in W^{1,2}(B_R \setminus K)$, and

$$\int_{B_R \setminus K} |\nabla u|^2 \leqslant \liminf_{i \to +\infty} \int_{B_R \setminus G_i} |\nabla v_i|^2 \leqslant \int_{B_R \setminus G} |\nabla v|^2 + C\lambda \sum_{i=1}^{\infty} r_i \tag{8.65}$$

by Fatou and (8.53). On the other hand,

$$(8.66)\quad H^1(K^-) := H^1(K \setminus L) \leqslant H^1(G^-) + H^1(K_0 \setminus G) + \sum_{i\in I_1} H^1(\partial D_i) \leqslant H^1(G^-) + \varepsilon + 2\pi \sum_{i=1}^{\infty} r_i$$

by (8.55), (8.5), and the trivial fact that $G_{\mathrm{rec}} \subset G$. Since

$$(8.67)\quad \sum_{i=1}^{\infty} r_i \leqslant C\varepsilon$$

by (8.17) and (8.18),

$$(8.68)\quad J_R(u,K) \leqslant J_R(u,G) + C\lambda\varepsilon$$

by (3.2), (8.65)-(8.67), and also the definition (4.10) of λ, the basic properties (3.3), (3.5) and (3.6) of h, and our assumption that $\varepsilon \leqslant 1$. Note that

$$(8.69)\quad (u,K) \in U_R^k \text{ for some } k,$$

The fact that $(u,K) \in U_R$ comes from our construction of K and u, and in particular the fact that $u \in W^{1,2}(B_R \setminus K)$; then $(u,K) \in U_R^k$ for some k by the observation that follows (8.54).

We are now ready to conclude. For each $\varepsilon > 0$ (small enough) we can choose a competitor $(v,G) \in U_R$ that satisfies (7.5). We have just shown that we can use (v,G) to construct a new competitor (u,K) in some U_R^k, that satisfies (8.68). Thus $\eta_k(R) \leqslant \eta(R) + C\lambda\varepsilon$ for that k; the desired estimate (6.2) follows because $\eta_k(R)$ is a nonincreasing function of k, as was explained at the beginning of this section. [Recall from (3.4), (3.5), and (4.10) that $\lambda \leqslant A$.]

This completes our proof of (6.2). □

9. Existence of minimizers (the concentration property)

In this section we complete the program announced in Section 6 and prove the following result.

Proposition 9.1. — *For $R > 1$ large enough we can find $(v_R, G_R) \in U_R$ such that*

$$(9.2)\quad J_R(v_R, G_R) = \eta(R),$$

where $\eta(R) = \inf\{J_R(v,G); (v,G) \in U_R\}$ as in (6.3).

To prove the proposition we shall use (6.2) and a classical argument based on the concentration lemma of [DMS]. See [MoSo].

Our first observation is that for each $k \geqslant 1$ there is a pair $(v_R, G_R) \in U_R^k$ such that

$$(9.3)\quad J_R(v_R, G_R) = \eta_k(R).$$

To find a minimizer for J_R^k we start from a sequence $\{(v_{k,\ell}, G_{k,\ell})\}_\ell$ in U_R^k such that

$$\lim_{\ell\to\infty} J_R(v_{k,\ell}, G_{k,\ell}) = \eta_k(R). \tag{9.4}$$

Modulo extracting a subsequence, we may assume that the closed sets $G_{k,\ell}$ converge, for the Hausdorff metric, to a limit G_k. We can also assume that for each ℓ, $v_{k,\ell}$ has been chosen so as to minimize

$$E_\ell = \int_{B_R\setminus G_{k,\ell}} |\nabla v_{k,\ell}|^2 \tag{9.5}$$

with $G_{k,\ell}$ fixed, and under the usual constraint that $v = u_0$ on $\partial B_R \setminus G_{k,\ell}$. Then $v_{k,\ell}$ is harmonic. This is good, because we have an upper bound for E_ℓ that does not depend on ℓ, and we can use this to prove uniform estimates on $\nabla v_{k,\ell}$ away from G_k. Then Montel allows us to extract a new subsequence so that (after extraction) the sequence $\{v_{k,\ell}\}_\ell$ converges on $\mathbb{R}^2 \setminus G_k$ to a function v_k, uniformly on compact subsets of $B_R \setminus G_k$. Then v_k is continuous on $\mathbb{R}^2 \setminus G_k$, harmonic on $B_R \setminus G_k$, and $(v_k, G_k) \in U_R$.

Also, G_k has at most k connected components, because Hausdorff limits of connected compact sets are connected, and so $(v_k, G_k) \in U_R^k$. By Fatou,

$$\int_{B_R\setminus G_k} |\nabla v_k|^2 \leqslant \liminf_{\ell\to+\infty} \int_{B_R\setminus G_{k,\ell}} |\nabla v_{k,\ell}|^2 . \tag{9.6}$$

So far, the argument is quite general, and would work with any minimizing sequence in U_R. The point now is that the restriction of the Hausdorff measure H^1 to compact *connected* sets of finite length is lower semicontinuous, and hence the same thing works with sets with at most k components. [You may always extract subsequences so that each component converges.] See for instance [MoSo], p.125. The conclusion is that, in the present case

$$H^1\left(G_k^-\right) \leqslant \liminf_{\ell\to+\infty} H^1\left(G_{k,\ell}^-\right) \tag{9.7}$$

and hence

$$J_R(v_k, G_k) \leqslant \liminf_{\ell\to+\infty} J_R(v_{k,\ell}, G_{k,\ell}) = \eta_k(R). \tag{9.8}$$

This completes our rapid proof of existence of minimizers for each J_R^k. See [MoSo] for additional details.

Now we turn to the existence of minimizers for J_R. We start from the sequence $\{(v_k, G_k)\}_k$ of minimizers for J_R^k that we just obtained. Note that this is a minimizing sequence for J_R, because of (6.2). Follow the same argument as above; we get a limit $(v, G) \in U_R$ as before, and we even get the analogue of (9.6), i.e., that

$$\int_{B_R\setminus G} |\nabla v|^2 \leqslant \liminf_{k\to+\infty} \int_{B_R\setminus G_k} |\nabla v_k|^2 . \tag{9.9}$$

For the next stage (i.e., the analogue of (9.7)) we cannot use the same argument as before, since we have no uniform bound on the number of components of the G_k. Of course, the analogue of (9.7) is false in general (that is, without information on the sets G_k other that the fact that they converge to G). [Think about powder-like sets that converge to a thick limit G.]

In order to get the analogue of (9.7) anyway, we shall use the fact that (v_k, G_k) minimizes J_R^k to get an additional regularity property of the G_k (the uniform concentration property of [DMS]).

We shall say that $\{G_k\}$ satisfies the uniform concentration property if for all $\varepsilon > 0$ we can find a constant $C(\varepsilon)$ (independent of k) with the following property. For each disk $B = B(x, r)$ centered on G_k^-, with radius $r \leqslant 1$, and that does not meet L, we can find a disk $B(y, t)$ centered on G_k^-, contained in B, and such that

$$t \geqslant C(\varepsilon)^{-1} r \tag{9.10}$$

and

$$H^1\left(G_k^- \cap B(y,t)\right) \geqslant (2-\varepsilon)t. \tag{9.11}$$

Our definition is slightly different from the one in [DMS] or [MoSo]. This is mostly to simplify the statement and accommodate the special status of L here, but the difference does not really matter.

We shall sketch a proof of this property soon, but let us first see how to use it to get the analogue of (9.7) and conclude.

Consider the compact set $A = G \cap \overline{B}_R$, which is the Hausdorff limit of the sets $A_k = G_k \cap \overline{B}_R$. We want to apply Proposition 10.10 on page 123 of [MoSo], and so we need a Vitali covering $\widehat{B}$ of A. We pick a countable, dense subset of A, and take for $\widehat{B}$ the collection of all disks centered on the dense subset and with rational radii $\leqslant 1$. Obviously $\widehat{B}$ is countable, and it is a Vitali covering of A, as defined in the first lines of Subsection 2 on page 82 of [MoSo].

To apply Proposition 10.10 of [MoSo] we have to check that for each $\varepsilon > 0$ we can find a constant C_ε such that for all $B \in \widehat{B}$ there is a $k_0 \geqslant 1$ such that for $k \geqslant k_0$ we can find $B_k \in \widehat{B}$ with $B_k \subset B$,

$$\mathrm{diam} B_k \geqslant C_\varepsilon \mathrm{diam} B \tag{9.12}$$

and

$$H^1\left(A_k \cap B_k\right) \geqslant (1-\varepsilon)\mathrm{diam} B_k. \tag{9.13}$$

[Compare with (10.11) in [MoSo] (with $\alpha = 1$), and with the relevant definition (8.1) in [MoSo].]

This is easy to check. When $B = B(x, r)$ is a disk of $\widehat{B}$ such that $B(x, r/2)$ meets L, then we can find a disk $B' \in \widehat{B}$ with comparable diameter, contained in B, centered on $L \cap \overline{B}_R$ (or at least very close to $L \cap \overline{B}_R$ if we forgot to include a dense subset of $L \cap \overline{B}_R$ in the dense set in A that was used to define $\widehat{B}$), and which is completely crossed by $L \cap \overline{B}_R$. Then we can take $B_k = B'$ for all k, and (9.12),(9.13) are easily checked.

Now suppose that $B = B(x,r) \in \widehat{B}$ and $B(x,r/2)$ does not meet L. For k large enough, we can find $x_k \in A_k$ such that

$$|x - x_k| \leqslant r/10. \tag{9.14}$$

Moreover, $x_k \in G_k^-$ because L is too far. We apply the uniform concentration property to the set G_k^- and the disk $B(x_k, r/3)$ and get a disk $B(y_k, t_k)$ centered on G_k^-. If k is large enough (depending only on r and ε) we can find points of A very close to y_k, and hence there is a disk $B_k = B(z_k, \rho_k)$ in $\widehat{B}$ such that

$$|z_k - y_k| \leqslant [10C(\varepsilon)]^{-1}\, \varepsilon r \tag{9.15}$$

and ρ_k is larger than, but extremely close to, $t_k + [10C(\varepsilon)]^{-1}\, \varepsilon r$.

Clearly $B_k \subset B$ (by (9.14), because $B(y,t_k) \subset B(x_k, r/3)$, and because B_k is very close to $B(y,t_k)$). Also, (9.12) holds with $C_\varepsilon = 3C(\varepsilon)$, by (9.10). Finally, (9.13) follows from (9.11), the fact that $B(y,t_k) \subset B_k$ by construction, and because ρ_k is close enough to t_k.

We are almost in position to apply Proposition 10.10 in [MoSo]. The only detail that still needs to be addressed is that sets in $\widehat{B}$ are required to be compact in [MoSo]. This is easily fixed: take closed disks instead of open ones in the definition of $\widehat{B}$ above.

So we get that $\{A_k\}_k$ has a "uniformly concentrated subsequence", in the sense of [MoSo] this time. We then apply Theorem 10.14 in [MoSo] and get that

$$H^1(A) \leqslant \liminf_{k\to\infty} H^1(A_k), \tag{9.16}$$

maybe after extracting a new subsequence (but a posteriori we know that this is not needed). After removing the constant contribution of $L \cap \overline{B}_R$, we get the desired analogue of (9.7), that is,

$$H^1(G^-) \leqslant \liminf_{k\to\infty} H^1(G_k^-). \tag{9.17}$$

The argument can then be completed as in the existence of minimizers for J_R^k above: (9.9) and (9.17) yield

$$J_R(v,G) \leqslant \liminf_{k\to\infty} J_R(v_k, G_k) \leqslant \eta(R) \tag{9.18}$$

because $\{(v_k, G_k)\}$ is an extracted subsequence of a minimizing sequence for J_R.

To complete our proof of Proposition 9.1, we still need to say how to get the uniform concentration property for minimizers of J_R^k.

In our definition of that property, we restricted ourselves (on purpose) to disks $B(x,r)$ that do not meet L and have radii $r \leqslant 1$. Also these disks never get close to ∂B_R, because of Proposition 5.1 (and if R is large enough). Finally the presence of the function h in the definition (3.2) is not a special worry either, because $r \leqslant 1$ and none of the modifications involved in the proof will ever need adding more than Cr in length to G. For all these reasons, the proof of the uniform concentration property for (reduced) minimizers of J_R^k is the same as the usual proofs given in [DMS], [DaSe3], or [MoSo].

Because we already gave in Section 6 a big part of the argument, let us rapidly review how it goes for the convenience of the reader.

Let $\varepsilon > 0$ be given, and let $B = B(x,r)$ be a disk of radius $r \leqslant 1$ centered on G_k^- and that does not meet L, as in the definition of the uniform concentration property. Let $\alpha > 0$ be a small constant (to be chosen later, depending on ε). The same argument as for (7.97), but simpler because here we have a true minimizer (instead of a candidate that satisfies (7.5)) and hence we can use the mass estimates from Section 4 (instead of their ersatz with the sets Z_i), says that we can find another constant $C(\alpha) > 0$ such that, in the situation above, there is a disk $B(y,t)$ centered on $G^- \cap B(x, r/5)$, with radius $t \geqslant C(\alpha)^{-1} r$, such that $B(y,t) \subset B(x,r)$, and for which (7.97) holds. By the definition (7.80), this means that

$$\left\{ \iint_{B(y,t) \smallsetminus G_k} |\nabla v_k|^p \right\}^{2/p} \leqslant \alpha \lambda t^{\frac{4}{p}-1}. \tag{9.19}$$

We want to prove that this ball $B(y,t)$ satisfies the required properties in the definition of uniform concentration, at least if α is chosen small enough. Since (9.10) is obvious, we only need to check (9.11). Suppose that this is not the case, that

$$H^1 (G_k \cap B(y,t)) < (2-\varepsilon) t. \tag{9.20}$$

[Note that $G_k^- \cap B(y,t) = G_k \cap B(y,t)$, because $B(x,r)$ does not meet L.] Set

$$E = \{\rho \in (0,t)\ ;\ G_k \cap \partial B(y,\rho) \text{ has at most one point}\}. \tag{9.21}$$

We claim that

$$|E| \geqslant \frac{\varepsilon t}{2}. \tag{9.22}$$

This is not hard to prove, especially if we allow ourselves to use the rectifiability of G_k. The point is that the radial projection $z \to |z-y|$ is 1-Lipschitz, so that

$$H^1 \left(\pi^{-1}\left((0,t) \smallsetminus E\right)\right) \geqslant 2\,|(0,t) \smallsetminus E| = 2t - 2\,|E|, \tag{9.23}$$

where the factor 2 comes from the multiplicity. Since (9.23) is fairly intuitive, and anyway we can refer to other proofs of the uniform concentration property, we omit the details. Of course (9.22) follows from (9.23) and (9.20).

From (9.22) and (9.19) we deduce that we can find $\rho \in E$ such that $\rho \geqslant \varepsilon t/4$ and

$$\int_{\partial B(y,\rho) \smallsetminus G_k} |\nabla v_k|^p \leqslant C(\varepsilon, \lambda) \alpha^{p/2}\, t^{1-p/2}, \tag{9.24}$$

where $C(\varepsilon,\lambda)$ depends on ε and λ, but we won't care.

From (9.24) and Hölder we get a very good control on the jump of the restriction of v_k to $\partial B(y,\rho)$ at the only point of $\partial B(y,\rho) \cap G_k$. [If $\partial B(y,\rho)$ does not meet G_k, the argument is even simpler.]

We can use this to construct a better competitor than (v_k, G_k), as follows. We remove $G_k \cap B(y,\rho)$ and replace it with an arc γ of $\partial B(y,\rho)$ centered at the point of $\partial B(y,\rho) \cap G_k$, and with length $\frac{1}{2}H^1\left(G_k \cap B(y,\rho)\right) \geqslant C^{-1}\rho$ (by the local Ahlfors-regularity of G_k). This way, we shall save at least $(2C)^{-1}\rho$ in the length term.

The interest of γ is that we can extend the restriction of v_k to $\partial B(y,\rho) \setminus \gamma$ linearly on γ, and get a continuous function w on $\partial B(y,\rho)$ such that

$$\int_{\partial B(y,\rho)} |\nabla w|^p \leqslant C'(\varepsilon,\lambda)\alpha^{p/2}\, t^{1-p/2}. \tag{9.25}$$

[The estimate is not hard to get, but we shall omit the details.] Then we can extend w to the whole disk $B(y,\rho)$, with

$$\iint_{B(y,\rho)} |\nabla w|^2 \leqslant C''(\varepsilon,\lambda)\alpha t. \tag{9.26}$$

We keep the function v_k out of $B(y,\rho)$, and replace it with (the extension of) w in $B(y,\rho)$; (9.26) is an upper bound for what we have to pay in the energy term of J_R.

Altogether we can choose α so small that the loss in the energy term is the less than the gain in length; this gives a contradiction with the minimality of (v_k, G_k), and proves (9.11) by contradiction.

This completes our rapid review of how to get the uniform concentration property for G_k, and at the same time the proof of Proposition 9.1. □

10. Limit in R of the minimizers for J_R

So far we defined for each large enough R a functional J_R on a set U_R, and we proved the existence of minimizers (v_R, G_R) for J_R. All this will be used to produce a minimizer for a special (global) functional. In this section we choose an increasing sequence of radii for which the corresponding pairs (v_R, G_R) converge to a limit (v, G). We also give a few estimates on the rapidity of convergence. These estimates will be used later, like in Section 11 to prove that (v, G) is a minimizer for our special functional.

We start with estimates on $v_R - u_0$. For each (fixed) $R > 1$ and $(v, G) \in U_R$, set

$$E(v) = \int_{B_R \setminus G} |\nabla v|^2. \tag{10.1}$$

Let R_0 denote, as in Proposition 5.1, a large radius such that

$$G_R^- = G_R \setminus L \subset B\left(0, R_0\right) \tag{10.2}$$

when (v_R, G_R) is a (reduced) minimizer for J_R. To simplify the exposition, we shall only consider parameters $R \geqslant 2R_0$.

Denote by v_R^+ the continuous function on $\overline{B}_R \setminus (\overline{B}_{R_0} \cup L)$ which coincides with u_0 on $\partial B_R \setminus L$, is harmonic on $B_R \setminus (\overline{B}_{R_0} \cap L)$, and minimizes

$$E^+ = \int_{B_R \setminus (B_{R_0} \cup L)} |\nabla v_R^+|^2. \tag{10.3}$$

We can always extend v_R^+ by taking $v_R^+ \equiv 0$ on B_{R_0} and $v_R^+ \equiv u_0$ out of B_R. Then $(v_R^+, L \cup \partial B_{R_0}) \in U_R$ and $E^+ = E(v_R^+)$. The pair $(v_R^+, L \cup \partial B_{R_0})$ is the same as the pair (v^+, G^+) in (5.67)-(5.71), except that now R^+ is called R_0. At any rate the estimate (5.80) on $\Delta E^+ = E(u_0) - E^+$ is still valid, and writes

$$E(u_0) \leqslant E^+ + 2R_0. \tag{10.4}$$

[See (5.76) for the definition of ΔE^+.] From (10.2) and the definition of v_R^+ we deduce that

$$E(v_R) \geqslant E^+ \geqslant E(u_0) - 2R_0 \tag{10.5}$$

when (v_R, G_R) minimizes J_R.

We also need an upper bound for $E(v_R)$. Denote by v_R^- the harmonic function on $B_R \setminus L$ associated to the choice of $G = L$. [In other words, v_R^- is also continuous on $\overline{B}_R \setminus L$, coincides with u_0 on $\partial B_R \setminus L$, and minimizes $E(v_R^-)$ with these constraints.] Note that

$$\Delta E^- := E(v_R^-) - E(u_0) \tag{10.6}$$

is a decreasing function of R, because for $R' > R$ the function v_R^- is an acceptable candidate for the minimization of $\int_{B_{R'} \setminus L} |\nabla v|^2$ (as in the definition of $v_{R'}^-$).

Denote by C_3 the value of ΔE^- for $R = 2$. Then

$$E(v_R) \leqslant E(v_R^-) \leqslant E(u_0) + C_3 \quad \text{for} \quad R \geqslant 2. \tag{10.7}$$

We want to use (10.5) and (10.7) to get estimates for $v_R - u_0$. We will do this with Hilbert spaces and the Pythagoras theorem.

Denote by H the set of functions on $D = B_R \setminus (\overline{B}_{R_0} \cup L)$ that are defined modulo an additive constant and have a derivative in $L^2(D)$. With the natural norm

$$\|w\|_H = \left\{ \int_D |\nabla w|^2 \right\}^{1/2}, \tag{10.8}$$

H is a Hilbert space.

All the function v_R^+, v_R^-, u_0, v_R mentioned so far have restrictions in H, and even in

$$V = \{w \in H \; ; \; w \text{ is harmonic on } D \text{ and has a continuous extension to } \overline{B}_R \setminus (B_{R_0} \cup L) \text{ that coincides with } u_0 \text{ on } \partial B_R \setminus L\}. \tag{10.9}$$

Note that V is a closed affine subspace of H (because $\|w\|_H$ controls moduli of continuity of w, including up to $\partial B_R \setminus L$, for functions of V). Also, the boundary constraint in (10.9) cancels the indetermination on functions: functions in V are no longer defined modulo a constant.

Among all element of V, v_R^+ is the one that minimizes the norm $\| \ \|_H$. This is just the definition of v_R^+. In other words, v_R^+ is the orthogonal projection of the origin on V. Then

$$\|w\|_H^2 = \|v_R^+\|_H^2 + \|w - v_R^+\|_H^2 \quad \text{for} \quad w \in H \tag{10.10}$$

(by Pythagoras), and in particular if we take $w = u_0$ or $w = v_R$ we get that

$$\|w\|_H^2 \leqslant E(w) \leqslant E(u_0) + C_3 \leqslant E^+ + 2R_0 + C_3 = \|v_R^+\|_H^2 + 2R_0 + C_3 \tag{10.11}$$

by (10.7), (10.5), and (10.3). Hence

$$\|w - v_R^+\|_H^2 = \|w\|_H^2 - \|v_R^+\|_H^2 \leqslant 2R_0 + C_3 \tag{10.12}$$

for $w = u_0$ and $w = v_R$ (by (10.10)), and the triangle inequality yields

$$\|u_0 - v_R\|_H \leqslant 2(2R_0 + C_3)^{1/2} \quad \text{for} \quad R \geqslant 2. \tag{10.13}$$

Lemma 10.14. — *For all $x \in \mathbb{R}^2 \setminus L$ such that $|x| \geqslant 2R_0$ and all $R > 2|x|$,*

$$|\nabla v_R(x) - \nabla u_0(x)| \leqslant C|x|^{-1}. \tag{10.15}$$

Here as above, v_R comes from a minimizer (v_R, G_R) of J_R. To prove the lemma, consider $w = u_0 - v_R$. Then w is harmonic on $D = B_R \setminus (\overline{B}_{R_0} \cup L)$, satisfies the usual Neumann condition $\partial w/\partial n = 0$ on L, and (10.13) says that $\|w\|_H \leqslant C$.

If x and R are as in the statement and in addition $\operatorname{dist}(x, L) \geqslant |x|/10$, we simply use the harmonicity of w on $B = B(x, |x|/10)$ to get that

$$|\nabla w(x)| \leqslant \frac{1}{|B|}\int_B |\nabla w| \leqslant C|x|^{-1}\left\{\int_B |\nabla w|^2\right\}^{1/2} \leqslant C|x|^{-1}, \tag{10.16}$$

by the mean value property for ∇w and Cauchy-Schwarz.

When $\operatorname{dist}(x, L) < |x|/10$ we take for B a disk centered on L, with radius $\leqslant |x|/3$, and such that $x \in \frac{1}{2}B$. By Fubini and Chebychev, we can choose B so that

$$\int_{\partial B} |\nabla w|^2 \leqslant C|x|^{-1}\|w\|_H^2 \leqslant C|x|^{-1}. \tag{10.17}$$

The values of w on the half-disk B^+ that contains x are obtained by symmetrizing the values of w on $\partial B^+ \setminus L \simeq \partial B^+ \cap \partial B$ with respect to L and then taking the Poisson integral. In this case we easily deduce (10.15) from (10.17). This proves the lemma. □

We want to use Lemma 10.14 to estimate $v_R - u_0$, and for this it will be good to control the mean value of $v_R - u_0$ on some circles. We claim that

$$\int_{\partial B_r \setminus L} v_R = \int_{\partial B_R \setminus L} v_R = \int_{\partial B_R \setminus L} u_0 = 0 \quad \text{for } R_0 < r \leqslant R. \tag{10.18}$$

The last equality is true by definition of u_0, and the previous one because $(v_R, G_R) \in U_R$ and G_R does not meet $\partial B_R \setminus L$ (by 10.2)), so that $v_R = u_0$ on $\partial B_R \setminus L$. To prove the first one we differentiate with respect to r. Set

$$a(r) = \frac{1}{r} \int_{\partial B_r \setminus L} v_R = \int_{\partial B_1 \setminus \{-1\}} v_R(r\theta) d\theta. \tag{10.19}$$

Then

$$\begin{aligned} a'(r) &= \int_{\partial B_1 \setminus \{-1\}} \frac{\partial v_R}{\partial r}(r\theta) d\theta = \frac{1}{r} \int_{\partial B_r \setminus L} \frac{\partial v_R}{\partial r} \\ &= \frac{1}{r} \int_{\partial (B_r \setminus G_R)} \frac{\partial v_R}{\partial n} = \frac{1}{r} \int_{B_r \setminus G_R} \Delta v_R = 0 \ , \end{aligned} \tag{10.20}$$

because $\partial v_R / \partial n$ on the piece of boundary $[\partial (B_r \setminus G_R)] \setminus [\partial B_r \setminus L]$ that we had to add to $\partial B_r \setminus L$ (by the Neumann condition that comes from the minimality of v_R). The last equalities in (10.20) use Green and the harmonicity of v_R; the reader that would (still) be worried about the validity of our integration by parts can consult [MoSo], or go through the limiting argument suggested in Section 4. This completes our proof of the claim (10.18).

Let $x \in \mathbb{R}^2 \setminus L$ be such that $2R_0 \leqslant |x| \leqslant R/2$. Since the mean value of $v_R - u_0$ on the arc of circle $\partial B_r \setminus L$, $r = |x|$, is zero, we can integrate (10.15) on $\partial B_r \setminus L$ and get that

$$|v_R(x) - u_0(x)| \leqslant C \ \text{ for } \ 2R_0 \leqslant |x| \leqslant \frac{R}{2}. \tag{10.21}$$

We now have as much information on $v_R - u_0$ as we'll ever need, and we can start to think about taking limits (in R).

First choose a sequence $\{R_m\}_{m \geqslant 1}$ of radii $R_m > 2R_0$ such that R_m tends to $+\infty$ and the compact sets $\overline{B}_{R_0} \cap G_{R_m}$ converge (for the usual Hausdorff metric) to a limit G'. Set $G = G' \cup L$.

Note that the sets G_{R_m} have the uniform concentration property discussed in Section 9. The situation is a little different here, but the proof is the same. Thus we can apply Proposition 10.10 and Theorem 10.14 in [MoSo] as in the last section and get that

$$H^1 (G \setminus L) \leqslant \liminf_{m \to +\infty} H^1 (G_{R_m} \setminus L) \,. \tag{10.22}$$

Now we want to take a subsequence so that the v_{R_m} converge as well. Let us first check that for each $r > 2R_0$ there is a constant $C(r)$ such that

$$|v_{R_m}(x)| \leqslant C(r) \quad \text{on} \quad B_r \setminus G_{R_m}. \tag{10.23}$$

at least when $R_m \geqslant 2r$. Of course it is important that $C(r)$ does not depend on m.

To prove (10.23), it is enough to restrict to $x \in \partial B_r \setminus L$ because, by the usual truncature argument and the minimality of v_{R_m}, the values of v_{R_m} on $\partial B_r \setminus L$ control the values of v_{R_m} inside. We do not need to worry about the components of $B_R \setminus G$ that do not touch ∂B_R, because be can set $v_{R_m} = 0$ there. Now (10.23) follows from (10.21) (applied with $|x| = r$); we can take $C(r) = \sup_{\partial B_r \setminus L} |u_0(x)| + C = \sqrt{2/\pi}\, r^{1/2} + C$, where C is as in (10.21).

Because the function v_{R_m} are harmonic in $B_{R_m} \setminus G_{R_m}$, we can use the Montel property, the estimates (10.23), and the fact that G_{R_m} converges to G to extract a subsequence of our sequence $\{R_m\}$ (which we'll still call $\{R_m\}$) such that

$$\begin{cases} \{v_{R_m}\} \text{ converges on } \mathbb{R}^2 \setminus G \text{ to a limit } v, \\ \text{uniformly on every compact subset of } \mathbb{R}^2 \setminus G. \end{cases} \tag{10.24}$$

Also, v is harmonic on $\mathbb{R}^2 \setminus G$, since the v_{R_m} are harmonic. We can say a little more than (10.24) when we stay away from B_{R_0}: from (10.24) and the estimate for ∇v_{R_m} that follows from (10.15), we can easily deduce that

$$\{v_{R_m}\} \text{ converges to } v \text{ uniformly on every } B_r \setminus (L \cup B_{2R_0}), \ r > 2R_0. \tag{10.25}$$

This is slightly better than (10.24), because it means that we don't have to worry about losing uniform convergence when we get close to L (outside of B_{2R_0}).

For $1 \leqslant p \leqslant +\infty$ and $r > R_0$, denote by $W^{1,p}(\partial B_r \setminus L)$ the set of (continuous) functions on $\partial B_r \setminus L$ with a derivative in L^p. We claim that we can always extract a new subsequence so that (after extraction)

$$\{v_{R_m}\} \text{ converges to } v \text{ in } W^{1,2}(\partial B_r \setminus L) \tag{10.26}$$

for every rational radius $r \geqslant 2R_0$.

This is easy. We restricted to rational radii to simplify the argument; with this restriction we only have to be able to get (10.26) for a single radius. Then one uses (10.15) and compactness properties of $W^{1,+\infty}(\partial B_r \setminus L)$ in $W^{1,2}(\partial B_r \setminus L)$. Of course any other value of $p \in [1, +\infty)$ (instead of $p = 2$) would work as well. We omit the (classical) details.

We have enough information now on the convergence of the pairs (v_{R_m}, G_{R_m}) to (v, G) to prove minimizing properties of (v, G). This will be done in the next section, but let us first record an easy estimate on v, to be used much later.

Lemma 10.27. — *For every $x \in \mathbb{R}^2$,*

$$\limsup_{r \to +\infty} \frac{1}{r} \int_{B(x,r) \setminus G} |\nabla v|^2 \leqslant 1. \tag{10.28}$$

In fact the limit exists and is equal to 1, but we only need (10.28). Of course it is enough to prove (10.28) for $x = 0$, because $B(x,r) \subset B(0, |x| + r)$ and $r^{-1}(|x| + r)$ tends to 1. The contribution of $B(0, R_0)$ does not change the limsup either, so it will be enough to estimate $\int_S |\nabla v|^2$, where $S = B_r \setminus (B_{R_0} \cup L)$. First,

$$\left\{ \int_S |\nabla v|^2 \right\}^{1/2} \leqslant \left\{ \int_S |\nabla u_0|^2 \right\}^{1/2} + \left\{ \int_S (|\nabla u_0| - |\nabla v|)^2 \right\}^{1/2}, \tag{10.29}$$

by the triangular inequality. Next

$$\int_S |\nabla u_0|^2 \leqslant \int_{B_r \setminus K_0} |\nabla u_0|^2 = r \tag{10.30}$$

by (5.8) and the notation (5.4), while

$$\begin{aligned} \int_S (|\nabla u_0| - |\nabla v|)^2 &\leqslant \int_S |\nabla u_0 - \nabla v|^2 \leqslant \limsup_{m \to +\infty} \int_S |\nabla u_0 - \nabla v_{R_m}|^2 \\ &\leqslant \limsup_{m \to +\infty} \int_{B_{R_m} \setminus (B_{R_0} \cup L)} |\nabla u_0 - \nabla v_{R_m}|^2 \leqslant C \end{aligned} \tag{10.31}$$

by Fatou, (10.13), and the definition (10.8).

Finally, (10.29) yields

$$\left\{ \int_S |\nabla v|^2 \right\}^{1/2} \leqslant r^{1/2} + C, \tag{10.32}$$

and Lemma 10.27 follows. □

11. Our modified functional and why (v, G) minimizes it

Let us first define our modified (global) functional. Denote by U (our set of competitors) the set of pairs (v, G), where $G \subset \mathbb{R}^2$ is a closed set that contains $L = (-\infty, -1]$, $G^- := G \setminus L$ is bounded, $H^1(G^-) < +\infty$, and v is a function defined on $\mathbb{R}^2 \setminus G$ with a derivative in $L^2_{\text{loc}}(\mathbb{R}^2 \setminus G)$ and even such that $\int_{B \setminus G} |\nabla v|^2 < +\infty$ for every ball B. The restriction that G^- be bounded is not really needed here, but will not disturb either.

Let $(v, G) \in U$ be given. A *competitor for* (v, G) is a pair $(u, K) \in U$ such that for R large enough,

$$G^- \text{ and } K^- = K \setminus L \text{ are contained in } B_R \tag{11.1}$$

and more importantly

$$u(x) = v(x) \text{ for } x \in \mathbb{R}^2 \setminus (B_R \cup L). \tag{11.2}$$

Here we don't have to worry about an analogue of the topological condition (1.5), because $\mathbb{R}^2 \setminus (B_R \cup G) = \mathbb{R}^2 \setminus (B_R \cup L)$ for R large, and $\mathbb{R}^2 \setminus (B_R \cup L)$ is connected.

Definition 11.3. — A *minimizer of the modified functional* is a pair $(v, G) \in U$ such that

$$h\left(H^1\left(G^-\right)\right) + \int_{B_R \setminus G} |\nabla v|^2 \leqslant h\left(H^1\left(K^-\right)\right) + \int_{B_R \setminus K} |\nabla u|^2 \tag{11.4}$$

for each competitor (u, K) for (v, G) and all large enough R.

Here h is the same function as in Section 3, with A and B chosen so that the results of Section 4-9 are valid. Note also that as long as (11.1) and (11.2) hold, the inequality (11.4) does not depend on R. The main goal of this section is to prove the following.

Propositions 11.5. — *The pair (v, G) introduced in the last section is a minimizer of the modified functional.*

The proof will be very similar to arguments in [Bo]; the situation is a little simpler here because we won't have to worry about the number of connected components at infinity. We give the proof for the convenience of the reader.

We have already checked in the last section that $(v, G) \in U$. See in particular (10.2), (5.3), (10.22), and (10.28). Let us argue by contradiction and assume that we can find a strictly better competitor (u, K) for (v, G). Let R be such that (11.1) and (11.2) hold, but (11.4) fails. We can easily assume that R is rational (to be able to use (10.26)) and that $R > 10R_0$, because otherwise we can replace R with a larger (rational) radius. Of course we want to use (u, K) to construct a competitor for J_{R_m} that is better than (v_{R_m}, G_{R_m}) for some (large) m.

For the moment, let m be any integer such that $R_m > R$, and set $G_m = G_{R_m}$ and $v_m = v_{R_m}$ to simplify notations. We want to construct a pair $(\widetilde{v}, \widetilde{G}) \in U_{R_m}$. Let us keep

$$\widetilde{G} = K \tag{11.6}$$

and

$$\widetilde{v} = v_m \text{ out of } B_R. \tag{11.7}$$

This is possible, because anyway all our singular sets G, G_m, K are reduced to L outside B_R. Since $(v_m, G_m) \in U_{R_m}$ and $R_m > R$, (11.6) and (11.7) imply that

$$(\widetilde{v}, \widetilde{G}) = (u_0, K_0) \text{ out of } \overline{B}_{R_m}. \tag{11.8}$$

Next we want to define $\widetilde{v}$ on $B_R \setminus \widetilde{G} = B_R \setminus K$. We shall take $\widetilde{v}$ of the type

$$\widetilde{v} = u + w \text{ on } B_R \setminus \widetilde{G}, \tag{11.9}$$

where w will actually be defined on the whole $\overline{B}_R \setminus L$. Since we want $\widetilde{v}$ to be continuous across $\partial B_R \setminus L$, we have to require that

$$w = v_m - u \quad \text{on} \quad \partial B_R \setminus L \tag{11.10}$$

(because of (11.7)). Note that this is equivalent to

$$w = v_m - v \quad \text{on} \quad \partial B_R \setminus L, \tag{11.11}$$

by (11.2).

We choose for w any reasonable extension to $B_R \setminus L$ of the values of $v_m - v$ on $\partial B_R \setminus L$. We can take the harmonic extension with the usual Neumann condition on L as the reader probably expected, but even cruder choices will work as well. For instance we could interpolate radially between the boundary values on $\partial B_R \setminus L$ and the mean value β of $v_m - v$ on $\partial B_R \setminus L$ using a formula like

$$w\left(re^{i\theta}\right) = \frac{2r - R}{R}\left\{v_m\left(Re^{i\theta}\right) - v\left(Re^{i\theta}\right)\right\} + \frac{2R - 2r}{R}\beta \tag{11.12}$$

for $R/2 \leqslant r \leqslant R$ and $-\pi < \theta < \pi$, and

$$w\left(re^{i\theta}\right) = \beta \quad \text{for} \quad r \leqslant \frac{R}{2}. \tag{11.13}$$

The only constraint on the choice of w is to have

$$\int_{B_R \setminus L} |\nabla w|^2 \leqslant C_R \int_{\partial B_R \setminus L} |\nabla (v_m - v)|^2, \tag{11.14}$$

where we shall not even care about the way the constant C_R depends on R. Of course (11.14) is easy to get.

The right-hand side of (11.14) is equal to $C_R \|v_m - v\|^2_{W^{1,2}(\partial B_R \setminus L)}$ and (10.26) tells us that it tends to 0 when m tends to $+\infty$. Let $\varepsilon > 0$ be given, to be chosen later. Then (11.14) says that

$$\int_{B_R \setminus L} |\nabla w|^2 \leqslant \varepsilon^2 \quad \text{for } m \text{ large enough.} \tag{11.15}$$

Now we want to compare $(\widetilde{v}, \widetilde{G})$ with (v_m, G_m). First notice that $(\widetilde{v}, \widetilde{G}) \in U_{R_m}$, by (11.7), (11.9), (11.15), and because $\widetilde{v}$ is continuous across $\partial B_R \setminus L$. Moreover

$$\begin{aligned} J_{R_m}(\widetilde{v}, \widetilde{G}) &= h(H^1(\widetilde{G} \setminus L)) + \int_{B_{R_m} \setminus \widetilde{G}} |\nabla \widetilde{v}|^2 \\ &= h(H^1(K \setminus L)) + \int_{B_{R_m} \setminus (B_R \cup L)} |\nabla v_m|^2 + \int_{B_R \setminus \widetilde{G}} |\nabla (u + w)|^2 \end{aligned} \tag{11.16}$$

by (3.2), because $\widetilde{G} = K$ (by (11.6)), by (11.7) (and the fact that $K = \widetilde{G} = L$ out of B_R), and by (11.9).

Set $a^2 = \int_{B_R \setminus K} |\nabla u|^2$ and $b^2 = \int_{B_R \setminus K} |\nabla w|^2$. We know from (11.15) that $b \leqslant \varepsilon$ if m is large enough; hence

$$\int_{B_R \setminus \widetilde{G}} |\nabla(u+w)|^2 \leqslant a^2 + 2ab + b^2 \leqslant a^2 + 2a\varepsilon + \varepsilon^2, \tag{11.17}$$

because $\widetilde{G} = K$. Now set

$$\Delta = J_{R_m}(\widetilde{v}, \widetilde{G}) - J_{R_m}(v_m, G_m). \tag{11.18}$$

Then

$$\begin{aligned} \Delta &= J_{R_m}(\widetilde{v}, \widetilde{G}) - h(H^1(G_m^-)) - \int_{B_{R_m} \setminus G_m} |\nabla v_m|^2 \\ &= h(H^1(K^-)) - h(H^1(G_m^-)) + \int_{B_R \setminus \widetilde{G}} |\nabla(u+w)|^2 - \int_{B_R \setminus G_m} |\nabla v_m|^2 \end{aligned} \tag{11.19}$$

by (11.16) and because the middle integrals cancel. Then (11.17) yields

$$\Delta \leqslant \Delta h + \int_{B_R \setminus K} |\nabla u|^2 + 2a\varepsilon + \varepsilon^2 - \int_{B_R \setminus G_m} |\nabla v_m|^2, \tag{11.20}$$

with $\Delta h = h\left(H^1\left(K^-\right)\right) - h\left(H^1\left(G_m^-\right)\right)$.

Now we want to use our assumption that (11.4) fails. Denote by δ the left-hand side of (11.4), minus the right-hand side. [Thus $\delta > 0$.] By (11.20),

$$\begin{aligned} \Delta + \delta \leqslant &\left\{h\left(H^1\left(G^-\right)\right) - h\left(H^1\left(G_m^-\right)\right)\right\} \\ &+ \int_{B_R \setminus G} |\nabla v|^2 - \int_{B_R \setminus G_m} |\nabla v_m|^2 + 2a\varepsilon + \varepsilon^2. \end{aligned} \tag{11.21}$$

Now

$$\limsup_{m \to +\infty} h\left(H^1\left(G^-\right)\right) - h\left(H^1\left(G_m^-\right)\right) \leqslant 0 \tag{11.22}$$

by (10.22), and

$$\int_{B_R \setminus G} |\nabla v|^2 \leqslant \limsup_{m \to +\infty} \int_{B_R \setminus G_m} |\nabla v_m|^2 \tag{11.23}$$

by Fatou and because the ∇v_m converge to ∇v, in fact uniformly on every compact subset of $B_R \setminus G$. Of course $2a\varepsilon + \varepsilon^2$ is as small as we wish (because a is a fixed number that does not depend on m in particular). By choosing ε small enough, we can force the right-hand side of (11.21) to be $\leqslant \delta/2$ for m large enough. This gives $\Delta < 0$, a contradiction with the minimality of $(v_m, G_m) = (v_{R_m}, G_{R_m})$. [See (11.18) for the definition of Δ.]

So we were wrong to assume that (11.4) could fail, and (v, G) is a minimizer of the modified functional. Proposition 11.5 follows. □

CHAPTER C

REVIEWS ON BLOW-UP LIMITS AND C^1 PIECES

12. Blow up

The main goal of this section is to prove that limits of our minimizer (v, G) by blow-up procedures are essentially global minimizers of the Mumford-Shah functional, as in Definition 1.6. The word "essentially" means that we'll have to multiply the length terms in (1.7) by $\lambda = h'\left(H^1\left(G^-\right)\right)$.

First we'll need a few definitions; we start with the appropriate extension of the notion of global minimizer.

We keep the same class U_0 of admissible pairs as in Section 1 (a little before (1.2)), and also the same notion of competitors for an admissible pair $(u, K) \in U_0$ (see (1.3)-(1.5)).

Definition 12.1. — Let $\lambda > 0$ be given. A global λ-minimizer (for the Mumford-Shah functional) is a pair $(u, K) \in U_0$ such that

$$\lambda H^1\left(K \cap B_R\right) + \int_{B_R \setminus K} |\nabla u|^2 \leqslant \lambda H^1\left(G \cap B_R\right) + \int_{B_R \setminus G} |\nabla v|^2 \tag{12.2}$$

for all competitors (v, G) for (u, K) and all large enough radii R.

This definition is mostly a matter of convenience, because there is not much difference between global minimizers (with $\lambda = 1$, as in Section 1) and global λ-minimizers: it is easy to see that (u, K) is a global λ-minimizer if and only if $\left(\lambda^{-1/2} u, K\right)$ is a global 1-minimizer.

Next we want to give a few definitions about limits. We start with limits of closed subsets of the plane.

If G, K are nonempty closed subsets of the plane and $R > 0$, set

$$d_R(G, K) = \sup\left\{\operatorname{dist}(x, K)\ ;\ x \in G \cap \overline{B}_R\right\} + \sup\left\{\operatorname{dist}(x, G)\ ;\ x \in K \cap \overline{B}_R\right\}. \tag{12.3}$$

For small values of R it may be that $G \cap \overline{B}_R$ or $K \cap \overline{B}_R$ is empty. We shall then define the corresponding supremum to be zero.

If $\{G_n\}$ is a sequence of nonempty closed sets in the plane and G is a nonempty closed set, we say that $\{G_n\}$ converges to G if $d_R(G_n, G)$ tends to 0 for each $R > 0$. Of course it is enough to check this on integer values of R, since $d_R(G, K)$ is a nondecreasing function of R.

Next we define convergence of pairs (v, G). Our presentation will be slightly closer to the one in [Lé2] than the original definition of [Bo], but the notion is the same. We are interested in pairs (v, G), where $G \subset \mathbb{R}^2$ is closed and nonempty, and where v is a C^1-function defined on $\mathbb{R}^2 \setminus G$. Let us call U_1 the set of such pairs. Note that in the present paper the C^1 requirement is easily checked; our functions v will always be harmonic on $\mathbb{R}^2 \setminus G$.

Let $\{(v_n, G_n)\}_{n \geqslant 1}$ be a sequence in U_1 and $(v, G) \in U_1$. We say that $\{(v_n, G_n)\}$ converges to (v, G) if $\{G_n\}$ converges to G (as above) and if for each compact subset A of $\mathbb{R}^2 \setminus G$, the derivatives ∇v_n converges to ∇v uniformly on A.

Note that for n large enough, G_n does not meet A, which allows us to talk about the convergence of ∇v_n on A. Also, this definition only determines v up to an additive constant on each connected component of $\mathbb{R}^2 \setminus G$. We can remove this ambiguity by choosing an origin in each component of $\mathbb{R}^2 \setminus G$, and fixing the values of v there. The issue is the same as in the following.

Lemma 12.4. — *Suppose that $\{G_n\}$ converges to G and that $\{\nabla v_n\}$ converges uniformly on each compact subset of $\mathbb{R}^2 \setminus G$. Suppose also that for each connected component $\mathcal{O}$ of $\mathbb{R}^2 \setminus G$ there is a point $z \in \mathcal{O}$ such that $\{v_n(z)\}$ converges. Then $\{v_n\}$ converges to a limit v on $\mathbb{R}^2 \setminus G$, and the convergence is uniform on every compact subset of $\mathbb{R}^2 \setminus G$.*

This is easy. The point is that if A is a compact subset of $\mathbb{R}^2 \setminus G$, then A meets only finitely many components of $\mathbb{R}^2 \setminus G$. If $\mathcal{O}$ is one of them, then $A \cap \mathcal{O}$ can be connected to the point z of the statement by a compact connected set contained in $\mathcal{O}$. The rest of the proof is routine.

Note also that it is always easy to find converging subsequences, as in the following lemma.

Lemma 12.5. — *Let $\{(v_n, G_n)\}$ be a sequence in U_1, suppose that for each n, v_n is harmonic on $\mathbb{R}^2 \setminus G_n$, and also that for each $R > 0$ there is a constant C_R such that*

$$\int_{B_R \setminus G_n} |\nabla v_n|^2 \leqslant C_R \quad \text{for all} \quad n. \tag{12.6}$$

Then there is a subsequence $\{(v_{n_k}, G_{n_k})\}$ that converges to some (v, G).

This is also standard. We can extract a first subsequence so that the G_{n_k} converge to some G. Our statement was a little imprudent, because we have to allow also the case when $G = \varnothing$, but this makes no real difference. Then we use (12.6) and the harmonicity of the v_n to get uniform bounds on the second derivatives of the v_{n_k}, k large, on every compact subset of $\mathbb{R}^2 \setminus G$. We can then extract repeated subsequences, using Montel, an exhaustion of $\mathbb{R}^2 \setminus G$ by compact sets, and the same sort of arguments as in Lemma 12.4. We omit the easy details.

Let us now describe blow-ups for our minimizer (v, G) of the modified functional. We select a sequence of points $y_n \in G$ (the most interesting case) and a sequence of radii $t_n > 0$. Then we consider the pairs $(v_n, G_n) \in U_1$ given by

$$G_n = t_n^{-1} (G - y_n) \tag{12.7}$$

and

$$v_n(x) = t_n^{-1/2} v (t_n x + y_n) \quad \text{for} \quad x \in \mathbb{R}^2 \setminus G_n. \tag{12.8}$$

[Note that $B(y_n, t_n)$ for G becomes the unit disk for G_n.] We shall first restrict to situations where

$$\lim_{n\to\infty} t_n = 0 \tag{12.9}$$

(which corresponds to what we mean by blow-up) and

$$\lim_{n\to\infty} t_n^{-1} \operatorname{dist}(y_n, L) = +\infty, \tag{12.10}$$

which will make boundary effects due to L disappear.

Note that we are in the situation of Lemma 12.5. Indeed

$$\int_{B_R \setminus G_n} |\nabla v_n|^2 = t_n^{-1} \int_{B(y_n, t_n R) \setminus G} |\nabla v|^2 \leqslant 4\pi\lambda R \tag{12.11}$$

as soon as $t_n R \leqslant 1$, say. The inequality comes from the fact that (v, G) is a minimizer of the modified functional (try a competitor with $K = G \cup \partial B(y_n, t_n R)$ and $u = 0$ in $B(y_n, t_n R)$). We could also get it from Lemma 4.12 and Fatou, because ∇v is the limit of a sequence ∇v_{R_m} . We easily deduce (12.6) from (12.11), because for each fixed R we do not care about the first values of n.

Because of this, we can always extract subsequences so that the pairs (v_{n_k}, G_{n_k}) converge (in the sense described above).

Proposition 12.12. — *Let (v, G) be a minimizer of the modified functional. Let $\{t_n\}$, $\{y_n\}$ be as above, and in particular satisfy* (12.9) *and* (12.10). *Suppose the sequence $\{(v_n, G_n)\}$ converges to a limit (u, K). Then (u, K) is a global λ-minimizer, with $\lambda = h'\left(H^1(G \setminus L)\right)$.*

The proof of Proposition 12.12 is almost the same as in [Bo], Section 2.2 (iii). We sketch it here anyway, mostly because of the slight differences in definitions.

Let (v, G), $\{(v_n, G_n)\}$, and (u, K) be as in the proposition, and suppose (u, K) is not a global minimizer. We want to proceed as in Section 11 and find a better competitor than (v, G). Let us first check that $(u, K) \in U_0$, the set of acceptable pairs defined in Section 1. For this and the rest of Proposition 12.12, the main point is that for all $R > 1$,

$$H^1(K \cap B_R) \leqslant \liminf_{n \to +\infty} H^1(G_n \cap B_R) \leqslant 2\pi R. \tag{12.13}$$

The last inequality is easy to obtain, with the same proof as in Lemma 4.2. Note that the argument involves replacing $G_n \cap B(y_n, Rt_n)$ with $\partial B(y_n, Rt_n)$, and that $B(y_n, Rt_n)$ does not meet L for n large enough, by (12.10).

The first inequality is proved like its counterpart (10.22), using the uniform concentration property for the sets $G_n \cap B_R$ (or equivalently, the sets $G \cap B(y_n, Rt_n)$). The uniform concentration property can be proved exactly as (suggested) for the sets G_R in Section 9, using the minimality of (v, G). The details are the same as in Section 9, [Bo], or [DMS].

The functions v_n are harmonic on their domain of definition $\mathbb{R}^2 \setminus G_n$, and so u is harmonic on $\mathbb{R}^2 \setminus K$. Also,

$$\int_{B_R \setminus K} |\nabla u|^2 \leqslant \liminf_{n \to +\infty} \int_{B_R \setminus G_n} |\nabla v_n|^2 \leqslant 4\pi\lambda R \tag{12.14}$$

by Fatou and (12.11). Altogether, (1.2) holds and $(u, K) \in U_0$.

Since we have assumed that (u, K) is not a global λ-minimizer, there is a competitor $(\widetilde{u}, \widetilde{K})$ for (u, K) such that (12.2) fails for R large enough. [Recall that the failure of (12.2) does not depend on R, provided that R is large enough, because of (1.3) and (1.4)]. Choose a (large) radius R such that (1.3), (1.4), and (1.5) hold, (12.2) fails, and also such that

$$K \cap \partial B_R \text{ has at most 10 points.} \tag{12.15}$$

[This last is easy to get, by (12.13) and a standard argument using radial projections and Chebychev.]

Let $\varepsilon > 0$ be small, to be chosen later. Set

$$Z = \{z \in \partial B_R\ ;\ \operatorname{dist}(z, K \cap \partial B_R) \leqslant \varepsilon\}. \tag{12.16}$$

Note that $\overline{\partial B_R \setminus Z}$ is a compact subset of $\mathbb{R}^2 \setminus K$. For n large enough, it does not meet G_n. We shall restrict to these large values of n and set

$$\widetilde{G}_n = [(G_n \cup Z) \setminus B_R] \cup \left[\widetilde{K} \cap B_R\right]. \tag{12.17}$$

Note that $\widetilde{G}_n$ is closed because $\widetilde{K} \cap \partial B_R = K \cap \partial B_R$ and $G_n \cap \partial B_R$ are contained in Z.

On the complement of $B_R \cup \widetilde{G}_n$, we keep $\widetilde{v}_n(x) = v_n(x)$. In $B_R \setminus \widetilde{G}_n$, we want to set

$$\widetilde{v}_n(x) = \widetilde{u}(x) + w_n(x), \tag{12.18}$$

where we'll need to choose w_n such that

$$w_n(x) = v_n(x) - \widetilde{u}(x) \text{ on } \partial B_R \setminus Z, \tag{12.19}$$

so that our definitions of $\widetilde{v}_n$ on B_R and its complement agree nicely on $\partial B_R \setminus Z$.

Lemma 12.20. — *For n large enough we can find a C^1-function w_n on $\overline{B}_R \setminus (Z \cup \widetilde{G}_n)$ such that* (12.19) *holds and*

$$\int_{B_R \setminus \widetilde{G}_n} |\nabla w_n|^2 \leqslant \varepsilon. \tag{12.21}$$

To prove this lemma, we need to construct functions w_n on $B_R \setminus \widetilde{G}_n$ with prescribed boundary values on $\partial B_R \setminus Z$, and obviously we can do the construction separately (and independently) on each connected component of $\overline{B}_R \setminus (Z \cup \widetilde{G}_n)$. So let V denote a connected component of $\overline{B}_R \setminus (Z \cup \widetilde{G}_n) = \overline{B}_R \setminus (Z \cup \widetilde{K})$ (by (12.17)). If V does not meet ∂B_R, then we can take $w_n \equiv 0$ on V, because there is no boundary condition coming from (12.19). So suppose that V meets ∂B_R. The various points of $V \cap \partial B_R$ are not separated by $\widetilde{K}$ (by definition of V), hence (1.5) tells us that they are not separated by K either. [Recall that $\widetilde{K}$ plays the role of G in (1.5).] Thus

$$V \cap \partial B_R \text{ is contained in some connected component } \mathcal{O} \text{ of } \mathbb{R}^2 \setminus K. \tag{12.22}$$

We need to know more about the values of $v_n - \widetilde{u}$ on the intersection of such an $\mathcal{O}$ with ∂B_R. For each $\mathcal{O}$ of $\mathbb{R}^2 \setminus K$ that meets ∂B_R, set

$$\mathcal{O}^* = \left\{ z \in \mathcal{O} \cap \partial B_R \ ; \ \operatorname{dist}(z, \partial B_R \cap K) \geqslant \frac{\varepsilon}{2} \right\}. \tag{12.23}$$

Obviously $\mathcal{O}^*$ is a compact subset of $\mathcal{O}$. Choose a point $z_{\mathcal{O}}$ in $\mathcal{O}^*$, and set $a_n = v_n(z_{\mathcal{O}}) - \widetilde{u}(z_{\mathcal{O}}) = v_n(z_{\mathcal{O}}) - u(z_{\mathcal{O}})$ (because $\widetilde{u}$ and u coincide on ∂B_R, by the analogue here of (1.4)). Then

$$\{\nabla v_n\} \text{ converges to } \nabla u \text{ uniformly on } \mathcal{O}^*, \tag{12.24}$$

just because (u, K) is the limit of the sequence $\{(v_n, G_n)\}$ and $\mathcal{O}^*$ is compact in $\mathbb{R}^2 \setminus K$, but also

$$\{v_n - a_n\} \text{ converges to } u \text{ uniformly on } \mathcal{O}^*, \tag{12.25}$$

by definition of a_n and the argument of Lemma 12.4. [Some argument was needed here, because $\mathcal{O}^*$ could be composed of more than one interval of ∂B_R, and we need to use the fact that these intervals are connected by paths in $\mathbb{R}^2 \setminus K$.].

From (12.24) and (12.25) we easily deduce that for n large enough we can find a function h_n defined on the whole ∂B_R which coincides with $v_n - a_n - u$ on the slightly smaller set

$$\mathcal{O}_1^* = \{ z \in \mathcal{O} \cap \partial B_R \ ; \ \operatorname{dist}(z, \partial B_R \cap K) \geqslant \varepsilon \}, \tag{12.26}$$

and which satisfies

$$\int_{\partial B_R} |\nabla h_n|^2 \leqslant 10^{-10} \varepsilon R^{-1}. \tag{12.27}$$

Let us still denote by h_n the harmonic extension of h_n to $\overline{B}_R$. Then

$$\int_{B_R} |\nabla h_n|^2 \leqslant \frac{\varepsilon}{10}. \tag{12.28}$$

We are now ready to define w_n on each connected component V of $\overline{B}_R \setminus (Z \cup \widetilde{G}_n)$. By (12.22) and the definitions (12.16) and (12.26), $V \cap \partial B_R$ is contained in $\mathcal{O}_1^*$ for some $\mathcal{O}$ as above. We take $w_n = h_n + a_n$ on V. Obviously

$$\int_V |\nabla w_n|^2 \leqslant \frac{\varepsilon}{10} \tag{12.29}$$

by (12.28), and $w_n = v_n - u = v_n - \widetilde{u}$ on $V \cap \partial B_R$ by definition of h_n and because $u = \widetilde{u}$ on $\partial B_R \setminus K$ (by (1.4)). The function w_n that we have defined this way satisfies the requirements for Lemma 12.20; in particular (12.21) follows from (12.29), because there are at most 10 components V that touch ∂B_R, by (12.15). The lemma follows. □

Now choose w_n, n large enough, as in Lemma 12.20. This gives a pair $(\widetilde{v}_n, \widetilde{G}_n)$ that coincides with (v_n, G_n) out of $\overline{B}_R$. Set

$$G_n^* = t_n \widetilde{G}_n + y_n \tag{12.30}$$

and

$$v_n^*(x) = t_n^{1/2}\, \widetilde{v}_n \left(\frac{x - y_n}{t_n} \right) \quad \text{on } \mathbb{R}^2 \setminus G_n^*. \tag{12.31}$$

[This is the inverse transform to the one in (12.7) and (12.8).] We get a competitor (v_n^*, G_n^*) for (v, G).

We still have to do the comparison of (v_n^*, G_n^*) with (v, G) and get a contradiction (if ε is chosen small enough and n is large enough). The computations from now on are essentially the same as in Section 11. First

$$\begin{aligned} H^1(G_n^* \setminus L) - H^1(G^-) &= t_n \left[H^1\left(\widetilde{G}_n \cap \overline{B}_R\right) - H^1(G_n \cap \overline{B}_R) \right] \\ &\leqslant t_n \left[H^1\left(\widetilde{K} \cap B_R\right) - H^1(G_n \cap \overline{B}_R) + 21\varepsilon \right] \end{aligned} \tag{12.32}$$

by (12.30), (12.7), (12.10), (12.17), (12.16), and (12.15). Since $H^1(K \cap B_R) \leqslant \liminf_{n \to +\infty} H^1(G_n \cap B_R)$ by (12.13), we get that

$$H^1(G_n^* \setminus L) - H^1(G^-) \leqslant t_n \left[H^1\left(\widetilde{K} \cap B_R\right) - H^1(K \cap B_R) + 22\varepsilon \right] \tag{12.33}$$

for n large enough, and then

$$h\left(H^1(G_n^* \setminus L)\right) - h\left(H^1(G^-)\right) \leqslant \lambda t_n \left[H^1\left(\widetilde{K} \cap B_R\right) - H^1(K \cap B_R) + 23\varepsilon \right] \tag{12.34}$$

for n large enough, because $\lambda = h'\left(H^1\left(G^-\right)\right)$ and t_n tends to 0. Next

$$\begin{aligned}(12.35)\qquad \Delta E &:= \int_{B(y_n,t_nR)\smallsetminus G_n^*} |\nabla v_n^*|^2 - \int_{B(y_n,t_nR)\smallsetminus G} |\nabla v|^2 \\ &= t_n \int_{B_R\smallsetminus \widetilde{G}_n} |\nabla \widetilde{v}_n|^2 - t_n \int_{B_R\smallsetminus G_n} |\nabla v_n|^2\end{aligned}$$

by (12.31) and (12.8). We can use (12.18) and the fact that $a = \left\{\int_{B_R\smallsetminus \widetilde{G}_n} |\nabla \widetilde{u}|^2\right\}^{1/2}$ is finite to get that

$$\begin{aligned}(12.36)\qquad \int_{B_R\smallsetminus \widetilde{G}_n} |\nabla \widetilde{v}_n|^2 &\leqslant \left[a + \left\{\int_{B_R\smallsetminus \widetilde{G}_n} |\nabla w_n|^2\right\}^{1/2}\right]^2 \\ &\leqslant a^2 + 2a\varepsilon^{1/2} + \varepsilon \leqslant a^2 + 3a\varepsilon^{1/2} = \int_{B_R\smallsetminus \widetilde{K}} |\nabla \widetilde{u}|^2 + 3a\varepsilon^{1/2}\end{aligned}$$

by (12.21), if ε is small enough, and because $B_R \cap \widetilde{G}_n = B_R \cap \widetilde{K}$ (by (12.17)). We also have that

$$(12.37)\qquad \int_{B_R\smallsetminus K} |\nabla u|^2 \leqslant \int_{B_R\smallsetminus G_n} |\nabla v_n|^2 + \varepsilon$$

for n large enough, by (12.14) (or Fatou and the convergence of ∇v_n to ∇u). Altogether,

$$(12.38)\qquad t_n^{-1}\,\Delta E \leqslant \int_{B_R\smallsetminus \widetilde{K}} |\nabla \widetilde{u}|^2 - \int_{B_R\smallsetminus K} |\nabla u|^2 + 3a\varepsilon^{1/2} + \varepsilon$$

for n large enough, by (12.35), (12.36), and (12.37).

It is now time to use our hypothesis that (12.2) does not hold to get the desired contradiction: if ε is small enough, (12.34) and (12.38) contradict the minimality of (v, G). This completes our proof of Proposition 12.12. □

We shall also need variants of Proposition 12.12 where the centers y_n in the blow-up procedure do not satisfy (12.10), but instead lie on the half-line L. The limits of such blow-up sequences will then be "global λ-minimizers in $\mathbb{R}^2 \smallsetminus \mathbb{R}$" (when y_n stays away from -1), or "global λ-minimizers in $\mathbb{R}^2 \smallsetminus (-\infty, 0]$" (when $y_n = -1$ for all n). Let us first say what this means. Set

$$(12.39)\qquad U_2 = \{(u, K) \in U_0\ ;\ K \text{ contains } \mathbb{R}\}$$

and

$$(12.40)\qquad U_3 = \{(u, K) \in U_0\ ;\ K \text{ contains } (-\infty, 0]\}\,.$$

Definition 12.41. — A global λ-minimizer in $\mathbb{R}^2 \smallsetminus \mathbb{R}$ (respectively, in $\mathbb{R}^2 \smallsetminus (-\infty, 0]$) is a pair (u, K) in U_2 (respectively, in U_3) such that (12.2) holds for all competitors (v, G) for (u, K) that lie in U_2 (respectively, in U_3) and all large enough radii R.

In other words, the definition is the same as for global λ-minimizers, except that we add the constraint that K contains the line $\mathbb{R}$ or the half-line $(-\infty, 0]$, both on the admissible pairs and on the competitors. In the case of $\mathbb{R}^2 \setminus \mathbb{R}$, a global λ-minimizer on $\mathbb{R}^2 \setminus \mathbb{R}$ is nothing more than the juxtaposition of two independent global λ-minimizers on the two half-planes that compose $\mathbb{R}^2 \setminus \mathbb{R}$.

Proposition 12.42. — *Let (v, G) be a minimizer for the modified functional. Let $\{t_n\}$ be a sequence of positive numbers tending to 0 (as in (12.9)) and $\{y_n\}$ a sequence of points of L. Define (v_n, G_n) as in (12.7) and (12.8), and suppose that the sequence $\{(v_n, G_n)\}$ converges to a limit (u, K) (with the notion of convergence defined at the beginning of this section). Set $\lambda = h'\left(H^1\left(G \setminus L\right)\right)$ as usual. If furthermore*

$$\lim_{n\to+\infty} t_n^{-1} \operatorname{dist}(y_n, -1) = +\infty, \tag{12.43}$$

then (u, K) is a global λ-minimizer in $\mathbb{R}^2 \setminus \mathbb{R}$. If instead $y_n = -1$ for all n, then (u, K) is a global λ-minimizer in $\mathbb{R}^2 \setminus (-\infty, 0]$.

The statement was long, but the proof is the same as above (except for a few minor details that we leave to the reader).

The situation for limits of global λ-minimizers is even a little simpler, because we don't need to worry about (12.9) and the derivative of h.

Proposition 12.44. — *Let $\{(u_n, K_n)\}$ be a sequence of global λ-minimizers (for some fixed $\lambda > 0$). Suppose that $\{(u_n, K_n)\}$ converges to a limit (u, K), and that the origin lies in K (a brutal way to require that $K \neq \varnothing$). Then (u, K) is also a global λ-minimizer.*

The proof is still the same as in [Bo] and Proposition 12.12. There is an additional verification (compared with the proof of Proposition 12.12) that we need to do in this case. When we construct our competitor (v_n^*, G_n^*) for (v, G) (as in (12.30) and (12.31)), we also need to check the topological condition (1.5) before we do the comparisons. Let us sketch the argument, even though it is very close to its analogue in [Bo]. After a dilation, we are reduced to showing that for n large enough, $\widetilde{G}_n$ satisfies the analogue of (1.5) with respect to G_n. That is, we need to prove that if $y, z \in \mathbb{R}^2 \setminus \left(\overline{B}_R \cup G_n\right)$ lie in different components of $\mathbb{R}^2 \setminus G_n$, then $\widetilde{G}_n$ also separates them.

Suppose this is not the case. Then we can find a path $\gamma \subset \mathbb{R}^2 \setminus \widetilde{G}_n$ that goes from y to z. If γ does not meet B_R, then it does not meet G_n either, because $G_n \setminus B_R \subset \widetilde{G}_n$ (by (12.17)). So γ meets B_R. We can even find a subarc γ' of γ which lies in $B_R \setminus \widetilde{G}_n$, and has its two endpoints z_1 and z_2 in $\partial B_R \setminus Z$, and in two distinct components of $\mathbb{R}^2 \setminus G_n$. [Consider successive points of $\gamma \cap \partial B_R$, and recall that y, z lie in different components of $\mathbb{R}^2 \setminus G_n$]. Call I_1 and I_2 the components of z_1 and z_2 in $\partial B_R \setminus Z$. Thus I_1 and I_2 are separated by G_n in $\mathbb{R}^2$, but they are not separated by $\widetilde{G}_n \cap \overline{B}_R$ in $\overline{B}_R$. Note that $\widetilde{G}_n \cap \overline{B}_R \subset \widetilde{K} \cap \overline{B}_R$, by (12.17), so $\widetilde{K} \cap \overline{B}_R$ does not separate I_1 from

I_2 in $\overline{B}_R$, and even less in $\mathbb{R}^2$. Recall that by construction $(\widetilde{u}, \widetilde{K})$ was a competitor of (u, K), and so $\widetilde{K}$ satisfies the analogue of (1.5) with respect to K. Hence K does not separate I_1 from I_2 in $\mathbb{R}^2$.

For each given pair I_1, I_2 of components of $\partial B_R \setminus Z$, if K does not separate I_1 from I_2, then for n large G_n does not separate them either, because $\{G_n\}$ converges to K. [Think about a path from I_1 to I_2 that does not meet K.] Note that there are only finitely many pairs I_1, I_2 to consider, and so for n large enough, we cannot find points $y, z \in \mathbb{R}^2 \setminus (\overline{B}_R \cup G_n)$ that lie in different components of $\mathbb{R}^2 \setminus G_n$, but which $\widetilde{G}_n$ does not separate. This completes our sketch of verification for (1.5) and Proposition 12.44. □

Let us add that there are also results like the ones in Proposition 12.42, where limits of sequences of global λ-minimizers in $\mathbb{R}^2 \setminus \mathbb{R}$ and $\mathbb{R}^2 \setminus (-\infty, 0]$ (corresponding to blow up sequences or not) are shown to be global λ-minimizers in $\mathbb{R}^2$, $\mathbb{R}^2 \setminus \mathbb{R}$, or $\mathbb{R}^2 \setminus (-\infty, 0]$, depending on the situation. We shall only be more precise when we need such results.

13. C^1 curves and spiders

For most of the rest of this text, we shall give ourselves a pair (v, G) and assume that either

(13.1) $\qquad (v, G)$ is a (reduced) minimizer of the modified functional

(as in Definition 11.3), or

(13.2) $\qquad (v, G)$ is a (reduced) global $\lambda-$minimizer

(as in Definition 12.1).

In this section we want to recapitulate some of the standard regularity properties of (ordinary) Mumford-Shah minimizers that are satisfied by (v, G), with essentially the same proofs. Thus this section will not contain much in terms of proofs.

The following convention will help us treat (13.1) and (13.2) at the same time.

Definition 13.3. — When (13.1) holds, we say that a disk $B(x, r)$ centered on G is *acceptable* if $r \leqslant 1$ and $B(x, r)$ does not meet L. When (13.2) holds, an acceptable disk is just any disk centered on G.

We start our list of regularity properties with an Ahlfors-regularity result: there is a constant C such that

$$C^{-1} r \leqslant H^1(G \cap B(x, r)) \leqslant Cr \tag{13.4}$$

for all $x \in G$ and $r > 0$, with the constraint that $r \leqslant 1$ when (13.1) holds. Here C depends on λ in a simple way, but we don't really care. The second inequality is proved as in Lemma 4.2. The proof of the first inequality is the same as in Proposition

4.14 when $B(x, r/2)$ is acceptable; otherwise (13.1) holds and $B(x, r/2)$ meets L, and the desired inequality holds because G contains L. Next

$$\int_{B(x,r)\setminus G} |\nabla v|^2 \leqslant 4\pi r\lambda \tag{13.5}$$

for all disks when (13.2) holds, and for all disks of radius $r \leqslant 1$ when (13.1) holds. Otherwise, we could add $\partial B(x,r)$ to G and replace v with a constant in $B(x,r)$. The constraint on r comes from the function h, and the proof is the same as for Lemma 4.12.

Now we want to describe results from [Da] or [AFP].

Definition 13.6. — A *regular point* of G is a point $x \in G$ such that there is a disk $B = B(x,r)$ for which

(13.7) $G \cap \overline{B}$ is a simple C^1 curve that contains x and crosses $\overline{B}$

(so that in particular $G \cap \partial B$ is composed of exactly 2 points, and $G \cap \overline{B}$ is a curve that joins them), and furthermore

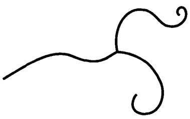
(13.8)
the curve $G \cap \overline{B}$ is the image under a rotation of some 10^{-2}–Lipschitz graph.

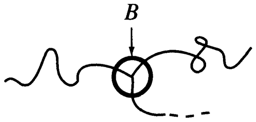
A disk with the properties (13.7) *and* (13.8) is called a *disk of regularity* for G.

Of course there is something a little arbitrary in the definition, but it will be useful to have (13.8) in addition to the more natural (13.7), and this will not cost us anything. Note that, thanks to (13.8), $B(x,r')$ is a disk of regularity for all $r' \leqslant r$ when $B(x,r)$ is a disk of regularity. We shall also need the analogue of Definition 13.6 with spiders instead of curves.

Definition 13.9. — A *spider* centered at x is a union of three simple curves of class C^1 that all start from x_0, make 120^o angles with each other at x_0, and do not intersect except at x_0. [See Figure 13.1.]. The three curves will also be called the *legs* of the spider. We say that the spider is *flat* when each of its legs is the image under some rotation of a 10^{-2}-Lipschitz graph.

Definition 13.10. — A *spider point* of G is a point $x \in G$ such that for some disk $B = B(x,r)$, $G \cap \overline{B}$ is a flat spider centered at x, and $G \cap \partial B$ has exactly three points (one in each leg of the spider). Such a disk B will be called a *spider disk.* [See Figure 13.2.]





FIGURES 13.1 AND 13.2.

Proposition 13.11. — *H^1-almost every point of G is a regular point of G.*

Let us start with the contribution of L in the special case when (13.1) holds. We want to prove that in this case,

(13.12) almost-every point of L is a regular point of G.

To prove this, it is enough to check that for all (large) m, almost every point of $L_m = \left[-m, -1 - \frac{1}{m}\right]$ is a regular point of G. Let $\varepsilon > 0$ be given, and denote by X_m the set of points of L_m that are not regular. Cover X_m with disks $B(x, 5\varepsilon)$, $x \in X_m^0$, where $X_m^0 \subset X_m$ is chosen so that the disk $B(x, \varepsilon)$, $x \in X_m^0$, are disjoint. Since $x \in X_m^0$ is not a point of regularity, $B(x, \varepsilon/2)$ meets $G \setminus L$. By the proof of Proposition 4.14 (applied to a point of $B(x, \varepsilon/2) \cap G \setminus L$),

(13.13) $$H^1(G \cap B(x, \varepsilon) \setminus L) \geqslant C^{-1}\varepsilon \quad \text{for} \quad x \in X_m^0.$$

Hence

(13.14) $$\begin{aligned} H^1(X_m) = H^1(L \cap X_m) &\leqslant \sum_{x \in X_m^0} H^1(L \cap B(x, 5\varepsilon)) \\ &\leqslant 10\, C \sum_{x \in X_m^0} H^1(G \cap B(x, \varepsilon) \setminus L) \leqslant 10\, C\, H^1(Z_\varepsilon), \end{aligned}$$

where $Z_\varepsilon = \{y \in G \setminus L \ ; \ \operatorname{dist}(y, L_m) \leqslant \varepsilon\}$. Since the intersection of the Z_ε is empty and each Z_ε has finite H^1-measure, we get that

(13.15) $$\lim_{\varepsilon \to 0} H^1(Z_\varepsilon) = 0,$$

and hence $H^1(X_m) = 0$. This proves (13.12) when (13.1) holds.

The rest of the proof of Proposition 13.11 can be imported, essentially without modification, from [Da] and probably also from [AFP]. Let us be more specific. Because of (13.12), we can stay away from L and avoid all complications related to L. The second difference with the situation of [Da] and [AFP] is the absence of the third term in the usual Mumford-Shah functional (the term with the initial image g). Of course this is only good for us, and would allow considerable simplifications in the proofs of [Da] if we were to copy them down.

The third difference comes from the composition of the term $H^1(G^-)$ with the slightly non-linear function h. This never causes a real problem, for the same reasons as we have seen in the previous sections (like Section 4). The situation is even simpler here in that respect, because we can restrict to very small disks (and hence systematically stay in the region where $\lambda/2 \leqslant h'(t) \leqslant 2\lambda$).

The last difference with the situation of [Da] and [AFP] comes from the topological condition (1.5) on the competitors for (v, G) when we have (13.2). It turns out that in all the arguments used in [Da], the competitors that are used to compare with (v, G) and get estimates satisfy the additional constraint (1.5), except perhaps in the situation that we shall rapidly describe below. This is probably also the case in [AFP], but we did not check this.

Let us describe the situation in [Da] where we compare (v, G) with pairs $(\widetilde{v}, \widetilde{G})$ for which (1.5) does not necessarily hold. This shows up in Lemma 3.19 on page 790 of [Da], where we assume that $G \cap \partial B(x,t)$ is contained in the union of two small intervals I_1, I_2 of $\partial B(x,t)$, and that, for some $p \in (1,2)$,

$$h_p(x,t) = t^{\frac{p}{2}-1} \int_{\partial B(x,t) \setminus G} |\nabla v|^p \tag{13.16}$$

is not too large, and we want to get a lower bound on the variation of v on $\partial B(x,t) \setminus (I_1 \cup I_2)$.

To prove that lemma, one compares (v, G) with a pair $(\widetilde{v}, \widetilde{G})$, where $\widetilde{G}$ is obtained from G by adding two small intervals I_1', I_2' of $\partial B(x,t)$ that contain I_1 and I_2, and removing $G \cap B(x,t)$. We get a problem with (1.5) when the two connected components of $\partial B(x,t) \setminus (I_1 \cup I_2)$ lie in different components of $\mathbb{R}^2 \setminus G$, because $\widetilde{G}$ no longer separates them. On the other hand, if this is the case, we can add a large constant to v in one of the connected components of $\mathbb{R}^2 \setminus G$; this does not change (v, G) in any essential way, but it makes the variation of v on $\partial B(x,t) \setminus (I_1 \cup I_2)$ as large as we want. In other words, whenever we cannot apply the proof of [Da] because (1.5) would be violated, we can still get the desired estimate cheaply by modifying v.

The reader should not be shocked by this. In fact the lower bounds on jumps of v that we get from Lemma 3.19 in [Da] are only used to control the sizes of holes in curves that connect I_1 to I_2 above, i.e., the length of sets that we would have to add to $G \cap B(x,t)$ to get a curve in $B(x,t)$ that connects I_1 to I_2. If we cannot use the argument of [Da] because of (1.5), then $G \cap \overline{B}(x,t)$ disconnects the two pieces of $\partial B(x,t) \setminus (I_1 \cup I_2)$ and there is simply no hole to fill.

Thus we can use [Da]; this completes our proof of Proposition 13.11. □

In addition to Proposition 13.11, we shall need some of the partial results from [Da] that were used to prove it. We start with the basic way to find disks of regularity.

Lemma 13.17. — *There is a constant $\varepsilon > 0$ such that if $B(x,r)$ is an acceptable disk (see Definition 13.3), if*

$$\int_{B(x,r) \setminus G} |\nabla v| \leqslant \varepsilon r^{3/2}, \tag{13.18}$$

and if there is a line D such that

$$\operatorname{dist}(z, D) \leqslant \varepsilon r \quad \text{for all} \quad z \in G \cap B(x,r) \tag{13.19}$$

and

$$\operatorname{dist}(z, G) \leqslant \varepsilon r \quad \text{for all} \quad z \in D \cap B(x,r), \tag{13.20}$$

then $B(x, r/2)$ is a disk of regularity.

See Theorem 4.8 and the relevant definitions (2.5) and (2.11) in [Da]. Lemma 13.17 can be seen as a perturbation result. If, in the disk $B(x,r)$, (u,K) looks enough like the line minimizer from (1.10) (that is, a pair (u',K') where K' is a line and u' is locally constant), then $K \cap B(x,r/2)$ is a C^1 curve. A consequence of this is that if (u,K) is a (reduced) global minimizer for which some blow-in sequence converges to a line minimizer, then (u,K) itself is a line minimizer. [See Corollary 18.26 below.] Note also that line minimizers are the tangent objects to minimizers at regular points.

We shall also need the analogue of Lemma 13.17 with curves replaced with spiders and lines replaced by propellers (the corresponding tangent objects). Recall that we call "propeller" a union of three half-lines that start from a same point (the center of the propeller) and make 120° angles at this point. [So propellers are very flat unbounded spiders.]

Lemma 13.21. — *There is a constant $\varepsilon > 0$ such that if $B(x,r)$ is an acceptable disk, if* (13.18) *holds, and if there is a propeller P centered at x such that*

$$\operatorname{dist}(z,P) \leqslant \varepsilon r \quad \text{for all} \quad z \in G \cap B(x,r) \tag{13.22}$$

and

$$\operatorname{dist}(z,G) \leqslant \varepsilon r \quad \text{for all} \quad z \in P \cap B(x,r), \tag{13.23}$$

then there exists $x_0 \in G \cap B(x, 10\varepsilon r)$ such that $B(x_0, r/2)$ is a spider disk.

This follows from Theorem 10.7 on page 876 of [Da], modulo a couple of minor technical differences that we discuss now. The hypotheses (10.1)-(10.3) of that theorem were slightly different from the ones we have in Lemma 13.21. Here we do not care about (10.1), which was only coming from the additional g-term in the standard Mumford-Shah functional. Next (10.2) in [Da] is the same as (13.22) here, so that we only need to worry about the hypothesis (10.3) on the jumps of v across a neighborhood of $\partial B(x,r) \cap P$.

We cannot deduce (10.3) in [Da] directly from the hypotheses of our Lemma 13.21, because (10.3) asks for a very large jump. However we can use an argument similar to the one in Lemma 3.19 of [Da] to show that (10.3) in [Da] holds for some (significantly smaller) $B(x, r/C)$.

More precisely, we can first use (13.18) and Fubini to replace r with a slightly smaller r' such the variation of v on each of the three main pieces of $\partial B(x,r') \setminus G$ is extremely small. Call J_1, J_2, J_3 these main pieces, and α_1, α_2, α_3 the mean values of v on J_1, J_2, J_3 respectively.

If α_1, α_2, α_3 are all very close to each other, we can get a contradiction roughly as follows. From (13.18), (13.5), and Hölder, we get a very good control of $\int_{B(x,r)\setminus G} |\nabla v|^p$, say, for $p = 3/2$. Because of this, we can assume that r' has also been chosen so that $h_p(x,r')$ is very small (where $h_p(x,r')$ is as in (13.16)). Then we can use the fact that α_1, α_2, α_3 are very close to each other to modify v on a small neighborhood of P in $\partial B(x,r')$ and get a function $\widetilde{v}$ on $\partial B(x,r')$ such that

$$r^{\frac{p}{2}-1}\int_{\partial B(x,r')}|\nabla\widetilde{v}|^p \quad \text{is very small} \tag{13.24}$$

(in fact, as small as we want). [Compare with (13.16).] Then we can extend $\widetilde{v}$ harmonically in $B(x,r')$, in such a way that

$$r^{-1}\int_{B(x,r')}|\nabla\widetilde{v}|^2 \quad \text{is very small,} \tag{13.25}$$

and replace G with $\widetilde{G}=[G\setminus B(x,r')]\cup Z$, where Z is a small neighborhood of P in $\partial B(x,r')$. We get a contradiction, at least if $\left(\widetilde{v},\widetilde{G}\right)$ is an acceptable competitor. The other case is when (13.2) holds and $\widetilde{G}$ does not satisfy (1.5). In this case we can modify v on a component of $\mathbb{R}^2\setminus G$ so that α_1, α_2, α_3 are not all close to each other. [See the discussion in the proof of Proposition 13.11.] The construction is almost the same as in Lemma 3.19 of [Da], so we omit the details.

We also need to exclude the situation where α_1 and α_2 are very close to each other, but reasonably far from α_3. In this case we first use (13.18) and the co-area formula to show that we can find a closed subset H of $\overline{B}(x,r')$ such that

$$H^1(H\setminus G)\leqslant C\varepsilon\, r \tag{13.26}$$

and H disconnects completely J_3 from $J_1\cup J_2$ in $\overline{B}(x,r')$. The argument is very close to Lemma 3.39 in [Da], so we omit the details.

We can easily assume that H is contained in an εr-neighborhood of P. [If it is not, we can always project it on such a neighborhood, and this only makes (13.26) slightly worse, because of (13.22).] Call P_3 the branch of P that goes from the center x to the point of $P\cap\partial B(x,r')$ near the junction between J_1 and J_2. Set

$$H_1=\{z\in H\ ;\ \operatorname{dist}(z,P_3)\geqslant\varepsilon r\}\cup\partial B(x,2\varepsilon r)\,. \tag{13.27}$$

Then H_1 still separates J_3 from J_1 and J_2 in $\overline{B}(x,r')$. [See Figure 13.3]

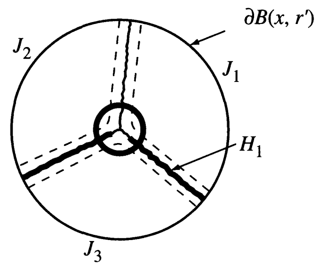


FIGURE 13.3

Note that

$$H^1(H_1) \leqslant H^1\left(G \cap \overline{B}(x,r')\right) - \frac{r}{C} \tag{13.28}$$

by (13.26), and because a good portion of $G \cap B(x,r')$ has been removed in (13.27), by (13.23) and (13.4). Set

$$\widetilde{G} = \left[G \setminus \overline{B}(x,r')\right] \cup H_1 \cup Z, \tag{13.29}$$

where Z is the union of three arcs of $\partial B(x,r')$ centered at the points of $P \cap \partial B(x,r')$ and with length $r/6C$, C as in (13.28). Thus

$$H^1(\widetilde{G} \cap \overline{B}(x,r')) \leqslant H^1(G \cap \overline{B}(x,r')) - \frac{r}{2C}. \tag{13.30}$$

The rest of the argument goes as above (or as in Lemma 3.19) of [Da]). We can define a function $\widetilde{v}$ on the circle $\partial B(x,r')$ that coincides with v on $J_1 \setminus Z$ and $J_2 \setminus Z$ (but not on J_3), and which satisfies (13.24). [This is possible because α_1 is very close to α_2.] Then we can extend $\widetilde{v}$ on $B(x,r')$, so that (13.25) holds. Because of the separation property of H_1, we can keep $\widetilde{v} = v$ on the complement of $B(x,r')$ and on the component of $\overline{B}(x,r') \setminus \widetilde{G}$ that contains Z_3. This gives a pair $(\widetilde{v},\widetilde{G})$ that strictly improves (v,G), because of (13.30) and (13.25). If $(\widetilde{v},\widetilde{G})$ is an acceptable competitor for (v,G), we hold the desired contradiction. Otherwise we are in the situation of (13.2), $\widetilde{G}$ does not satisfy (1.5), and hence G separates J_1 from J_2. In this case we can modify v on the component of J_2 in $\mathbb{R}^2 \setminus G$ so that (after modification) α_1, α_2, α_3 become fairly distant from each other.

At this point of the discussion, we know that we can assume that α_1, α_2, α_3 are all fairly different from each other. Of course they are not hugely different, as would be required if we wanted to apply Theorem 10.7 of [Da] directly to $B(x,r')$. On the other hand, we can use (13.18), (13.22), and the harmonicity of v to get that v is nearly constant on each of the three components of

$$\{z \in B(x,r')\ ;\ \operatorname{dist}(z,P) \geqslant 2\varepsilon r\}. \tag{13.31}$$

Consider the small disk $B_1 = B(x,c_1 r)$, where c_1 will be chosen soon. If ε is small enough (depending on c_1 in particular), the jumps associated to B_1 as in (10.3) of [Da] are all $\geqslant C^{-1}r^{1/2}$ (because $|\alpha_i - \alpha_j| \geqslant 2C^{-1}r^{1/2}$). If c_1 is chosen small enough, this is $\geqslant \varepsilon_4^{-1}(c_1 r)^{1/2}$, as required in (10.3) of [Da].

Thus Theorem 10.7 in [Da] applies to B_1, and we get that $G \cap B\left(x, \frac{1}{2}c_1 r\right)$ is a nice spider. In the statement of that theorem, one insisted more on the chord-arc property of the spider, but the proof also gives that $G \cap B(x,c_2 r)$ is a flat spider for some (perhaps smaller) constant c_2.

This does not give yet Lemma 13.21, because we also need to control G outside of $B(x,c_2 r)$. This control is easy to obtain (if ε is small enough), because we can apply Lemma 13.17 above to a fairly dense collection of points of $G \cap B(x,3r/4)$. We get that $G \cap B(x,3r/4)$ is a spider by gluing the various curves obtained in this way.

Then we get the desired control on the variation of the angle of the tangent to each leg of the spider from the flatness of the curves given by Lemma 13.17 and the fact that $G \cap B(x,r)$ stays εr-close to P (by (13.22)). To be honest, to get the precise constant 10^{-2} in the definition of a flat spider, we need Lemma 13.17 to give us a slightly better constant than 10^{-2} in (13.8). Of course there is no problem with this.

This completes our sketch of proof for Lemma 13.21. □

The following consequence of Lemmas 13.17 and 13.21 will be useful.

Corollary 13.32. — *Let $y \in G$ and a sequence $\{t_n\}$ of positive numbers be given. Suppose that the sets $G_n = t_n^{-1}(G - y)$ converge to a limit K. If we are in the situation of* (13.1), *suppose in addition that $y \in G \setminus L$ and $\{t_n\}$ tends to* 0.

If K is a line, then y is a regular point of G. If K is a propeller, then y is a spider point of G.

Here the notion of convergence of sets is the same as in the beginning of Section 12, i.e. convergence for the Hausdorff metric on every disk of the plane. Regular and spider points are defined at the beginning of this section.

Let y and $\{t_n\}$ be as in the statement of Corollary 13.32. Because of the remark just before Proposition 12.12 (or the corresponding observation for global λ-minimizers), we can always extract a subsequence so that the pair (v_n, G_n) converges to some limit (u, K), where we define v_n by (12.8) with $y_n = y$. Let us assume that we already extracted such a subsequence.

By Proposition 12.12 or its analogue Proposition 12.44 for global λ-minimizers, (u, K) is a global λ-minimizer.

If K is a line or a propeller, the associated function u must be constant on each component of $\mathbb{R}^2 \setminus K$. This is well-known, and fairly easy to prove. For instance, if K is a line through the origin, we can use the fact that for all $R > 0$,

$$\int_{B_R \setminus K} |\nabla u|^2 \leqslant C R \tag{13.33}$$

(by (13.5)) to get that for $r < R$,

$$\int_{B(0,r) \setminus K} |\nabla u|^2 \leqslant C R^{-1} r \tag{13.34}$$

by easy estimates on the solution of the Dirichlet-Neumann problem on $B(0,R) \setminus K$ (i.e., eventually, by scale invariance and even easier estimates on the Poisson kernel on the disk). Once we have (13.34), we get that $\nabla u = 0$ on $\mathbb{R}^2 \setminus K$ by letting R tend to $+\infty$ with r fixed. The case when K is propeller can be treated in the same way; the estimates are even more favorable.

So u is constant on each component of $\mathbb{R}^2 \setminus K$, and hence ∇v_n tends to 0 uniformly on each compact subset of $\mathbb{R}^2 \setminus K$, by convergence of $\{v_n\}$ to u, and harmonicity of all our functions. Let us check that we even have that

$$\lim_{n\to+\infty}\int_{B(0,R)\setminus(K\cup G_n)}|\nabla v_n|=0 \tag{13.35}$$

for every $R>0$. For each small $\eta>0$, decompose $B(0,R)\setminus(K\cup G_n)$ into $\Omega_1^n\cup\Omega_2^n$, where

$$\Omega_1^n=\{x\in B(0,R)\setminus(K\cup G_n)\ ;\ |\nabla v_n(x)|\leqslant\eta\} \tag{13.36}$$

and $\Omega_2^n=B(0,R)\setminus(K\cup G_n\cup\Omega_1^n)$. Then

$$\int_{\Omega_1^n}|\nabla v_n|\leqslant\eta\,|B(0,R)|=\pi\eta R^2 \tag{13.37}$$

trivially, while

$$\begin{aligned}\int_{\Omega_2^n}|\nabla v_n|&\leqslant\Big\{\int_{\Omega_2^n}|\nabla v_n|^2\Big\}^{1/2}|\Omega_2^n|^{1/2}\\&\leqslant\Big\{t_n^{-1}\int_{B(y,t_nR)\setminus G}|\nabla v|^2\Big\}^{1/2}|\Omega_2^n|^{1/2}\leqslant\{4\pi R\lambda\}^{1/2}|\Omega_2^n|^{1/2}\end{aligned} \tag{13.38}$$

by Cauchy-Schwarz, (12.8) and a change of variable, and (13.5). Because $|\nabla v_n|$ converges to 0 uniformly on compact subsets of $B(0,R)\setminus K$, the right-hand side of (13.38) tends to 0 (for each fixed η), and (13.35) follows easily.

If K is a line, it follows from (13.35), the convergence of $\{G_n\}$ to K, and the formulae (12.7) and (12.8) (with $y_n=y$) that $B(y,t_n)$ satisfies the hypotheses of Lemma 13.17 for n large enough. In this case y is a point of regularity.

If K is a propeller, then for n large enough $B(y,t_n)$ satisfies the hypotheses of Lemma 13.21 and y is a spider point of G.

This completes the proof of Corollary 13.32. □

Here is another application of Lemma 13.17.

Lemma 13.39. — *For each constant $C_1>0$ there is a constant $C_2>0$ such that for every acceptable disk $B(x,r)$ and every measurable set $E\subset G\cap B(x,r/2)$ such that $H^1(E)\geqslant C_1^{-1}r$, we can find a disk of regularity $B(y,t)$ centered on E and with radius $t\in\big[C_2^{-1}r,\frac14 r\big]$.*

See Definitions 13.3 and 13.6 for acceptable disks and disks of regularity.

As the reader may have guessed, we want to find $y\in E$ and $t\in\big[2C_2^{-1}r,\frac12 r\big]$ that satisfy the hypotheses of Lemma 13.17. So let us fix our acceptable disk $B(x,r)$ and call $\mathcal{A}$ the set of pairs $(y,t)\in E\times\big[2C_2^{-1}r,\frac12 r\big]$ that do not satisfy the hypotheses of Lemma 13.17. Decompose $\mathcal{A}$ into $\mathcal{A}_1\cup\mathcal{A}_2$, where

$$\mathcal{A}_1=\{(y,t)\in\mathcal{A}\ ;\ (13.18)\text{ fails}\} \tag{13.40}$$

and $\mathcal{A}_2=\mathcal{A}\setminus\mathcal{A}_1$.

Let us first take care of $\mathcal{A}_1$. Fix $1 < p < 2$, and set

$$\omega_p(y,t) = t^{1-4/p}\left\{\iint_{B(y,t)\setminus G} |\nabla v|^p\right\}^{2/p}, \tag{13.41}$$

just like in (7.80). For each $(y,t) \in \mathcal{A}_1$,

$$\begin{aligned}\omega_p(y,t) &\geqslant t^{1-4/p}\left\{\iint_{B(y,t)\setminus G} |\nabla v|\right\}^2 |B(y,t)|^{\frac{2}{p}-2} \\ &= \pi^{\frac{2}{p}-2}t^{-3}\left\{\iint_{B(y,t)\setminus G} |\nabla v|\right\}^2 \geqslant \pi^{\frac{2}{p}-2}\varepsilon^2,\end{aligned} \tag{13.42}$$

by Hölder and since (13.18) fails. On the other hand,

$$\int_{y\in G\cap B(x,r/2)}\int_{0<t\leqslant r/2} \omega_p(y,t)\frac{dH^1(y)dt}{t} \leqslant C\,\lambda\, r. \tag{13.43}$$

This is still Proposition 4.5 on page 311 of [DaSe2], and the proof is the same (but a little simpler) as for Lemma 7.81 above. From (13.42) and (13.43) we deduce that

(13.44)

$$\iint_{\mathcal{A}_1} \frac{dH^1(y)dt}{t} \leqslant \pi^{2-\frac{2}{p}}\varepsilon^{-2}\int_{y\in G\cap B(x,r/2)}\int_{0<t\leqslant r/2} \omega_p(y,t)\frac{dH^1(y)dt}{t} \leqslant C\,\varepsilon^{-2}\,\lambda\, r,$$

where we do not really care about the dependence in λ, but it is important that C does not depend on C_2.

To estimate the size of $\mathcal{A}_2$ we use the local uniform rectifiability of G.

Lemma 13.45. — *For each $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that, for every acceptable disk $B(x,r)$,*

$$\iint_{\mathcal{A}_3(x,r)} \frac{dH^1(y)dt}{t} \leqslant C(\varepsilon)\, r, \tag{13.46}$$

where $\mathcal{A}_3(x,r)$ denotes the set of pairs $(y,t) \in G\cap B(x,r/2) \times (0,r/2]$ for which there is no line D with the properties (13.19) *and* (13.20) *(with (x,r) replaced by (y,t)).*

This lemma is a consequence of the local uniform rectifiability of G, and more precisely of the fact that for every acceptable disk $B(x,r)$, there is an Ahlfors-regular curve Γ with constant $\leqslant C$ that contains $G \cap B(x,2r/3)$, say. Here we can forget about the precise definition of Ahlfors-regular curves (in terms of parameterizations) and just recall that Γ is a connected set such that

$$H^1(\Gamma \cap B(z,s)) \leqslant Cs \text{ for all disks } B(z,s). \tag{13.47}$$

The constant C, just like $C(\varepsilon)$ in our lemma, is allowed to depend on λ.

The existence of the Ahlfors-regular curves Γ is proved in [DaSe2]. (See Theorem 2.8 on page 302 there). Of course the same proof works here also.

Now Lemma 13.45 follows from the fact that uniformly rectifiable sets satisfy the "bilateral weak geometric lemma". The justification is the same as for Corollary 6.3 on page 319 of [DaSe2]; see in particular the definition (6.5). For a general discussion about uniform rectifiability including the bilateral weak geometric lemma, we refer to [DaSe1].

We may now return to the proof of Lemma 13.39. It is clear from the definitions of $\mathcal{A}_2$ and $\mathcal{A}_3(x,r)$ that $\mathcal{A}_2 \subset \mathcal{A}_3(x,r)$. Hence Lemma 13.45 and (13.44) imply that

$$\iint_{\mathcal{A}} \frac{dH^1(y)dt}{t} \leqslant C'(\varepsilon)\, r. \tag{13.48}$$

On the other hand

$$\iint_{E\times\left[2C_2^{-1}r,\frac{1}{2}r\right]} \frac{dH^1(y)dt}{t} = H^1(E)\operatorname{Log}\left(\frac{C_2}{4}\right) \geqslant C_1^{-1}\, r\operatorname{Log}\frac{C_2}{4} \tag{13.49}$$

by our assumption on E, and it is now easy to choose C_2 so large, depending on C_1 and a fixed choice of ε (the one that makes Lemma 13.17 work), that the right-hand side of (13.49) is strictly larger than the right-hand side of (13.48). Then $E\times\left[C_2^{-1}r,\frac{1}{2}r\right]$ is not contained in $\mathcal{A}$, Lemma 13.17 applies to some pair (y,t), and this proves Lemma 13.39. □

In view of the statement of Lemma 13.39, the reader may be worried about the situation when (13.1) holds and $B(x,r)$ meets L. The following lemma can help reduce to the situation of Lemma 13.39.

Lemma 13.50. — *There is a constant C_3 such that if (13.1) holds, $x\in L$, $B(x,r)$ does not contain -1 and $G\cap B(x,r/2)$ is not contained in L, then*

$$H^1\left(\{y\in G\cap B(x,r)\ ;\ \operatorname{dist}(y,L)\geqslant C_3^{-1}r\}\right)\geqslant C_3^{-1}r. \tag{13.51}$$

Suppose not. Let us first check that

$$G\cap B\big(x,2r/3\big)\subset\{y\ ;\ \operatorname{dist}(y,L)\leqslant \alpha\, r\}, \tag{13.52}$$

where α is as small as we want (if we choose C_3 accordingly large). This comes from the local Ahlfors-regularity of G far from L (as in (13.4)): if $G\cap B\,(x,2r/3)$ contained a point z at distance $\geqslant \alpha r$ from L, then the disk $B\,(z,\alpha\, r/2)$ would be contained in the left-hand side of (13.51) (if $C_3>2\alpha^{-1}$) but have a mass $\geqslant C^{-1}\alpha\, r > C_3^{-1}r$ (again if C_3 is large enough). This proves (13.52).

We want to use (13.52) to construct a better competitor for (v,G). The idea will be to apply a deformation of a disk of radius $r/6$ that collapses a good part of G onto L, so as to win a nontrivial amount of length. First choose a point $z\in L\cap B\,(x,r/2)$ which is essentially as close as possible to $G^-=G\setminus L$. Then

$$\operatorname{dist}\big(z,G^-\big)\leqslant 2\,\alpha\, r, \tag{13.53}$$

by (13.52) and because $B(x, r/2)$ contains points of G^- by assumption. Now there is a C^1 mapping Ψ of $\mathbb{R}^2$ to itself such that

$$\Psi(y) = y \text{ on } L \text{ and out of } B = B(z, r/6)\,, \tag{13.54}$$

$$\Psi(y) \in L \text{ for all } y \in B(z, r/12) \text{ such that } \operatorname{dist}(y, L) \leqslant \alpha\, r, \tag{13.55}$$

$$|D\Psi(y)| \leqslant 1 + C\alpha \text{ everywhere}, \tag{13.56}$$

and also such that there is an open set U of $B \setminus L$ that contains the set $\{z \in B \setminus L\,;\ \operatorname{dist}(z, L) > \alpha r \text{ or } z \notin B(z, r/10)\}$, such that

$$\text{the restriction of } \Psi \text{ to } U \text{ is a diffeomorphism onto its image } B \setminus L, \tag{13.57}$$

and finally

$$|D\Psi(y) - \mathrm{Id}| \leqslant C\alpha \text{ on } U. \tag{13.58}$$

The simplest way to construct Ψ is to choose a nice pair of curves Γ_+, Γ_- as suggested by Figure 13.4, and collapse the region between Γ_+ and Γ_- onto L. The domain U is the part of B that does not lie between Γ_+ and Γ_-.

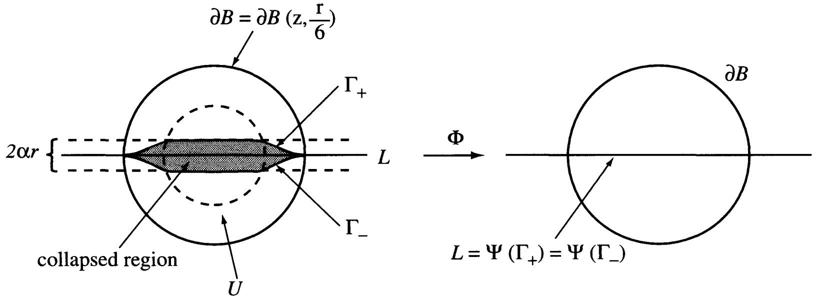


FIGURE 13.4

Now we take $\widetilde{G} = \Psi(G)$. Note that $\widetilde{G}$ still contains L, by (13.54). Next

$$H^1(\widetilde{G}^-) := H^1(\widetilde{G} \setminus L) \leqslant H^1(G^-) + C\alpha H^1(G^- \cap B) - H^1(G^- \cap B(z, r/12)), \tag{13.59}$$

where we loose no more than $C\alpha H^1\,(G^- \cap B)$ because only $G^- \cap B$ moves (by (13.54)), and by (13.56), and we win $H^1\,(G^- \cap B\,(z, r/12))$ because all these points are sent to L (by (13.55)).

Because G^- is locally Ahlfors-regular (which can be proved as in Section 4), this gives that

$$H^1(\widetilde{G}^-) \leqslant H^1(G^-) - C^{-1} r, \tag{13.60}$$

at least if α is small enough.

Now set $\widetilde{v}(\xi) = v\left(\Psi^{-1}(\xi)\right)$ on $\mathbb{R}^2 \setminus \widetilde{G}$. This is well defined because $L \subset \widetilde{G}$, by (13.57), and because we can take $\Psi^{-1}(\xi) = \xi$ out of B. Because of (13.58),

$$|\nabla\widetilde{v}(\xi)| \leqslant (1+C\alpha)\left|\nabla v\left(\Psi^{-1}(\xi)\right)\right|, \tag{13.61}$$

and a change of variable yields

$$\int_{B\setminus\widetilde{G}} |\nabla\widetilde{v}|^2 \leqslant (1+C\alpha)\int_{B\setminus G} |\nabla v|^2 . \tag{13.62}$$

Thus the difference is as small as we want (compared to r), and because of (13.60) $(\widetilde{v},\widetilde{G})$ is a strictly better competitor than (v,G). This contradiction finishes the proof of Lemma 13.50. □

14. C^1-regularity of v near regular and spider points

In this section we take care of a minor technical issue: the fact that near regular and spider points, v has boundary values on G (from both sides) that are of class C^1. Let us start with regular points.

Lemma 14.1. — *Let $B(x,r)$ be a disk of regularity for G, and denote by Ω_1, Ω_2 the two connected components of $B(x,r)\setminus G$. Then each of the restrictions of v to Ω_i has a C^1-extension to $\overline{\Omega}_i \cap \overline{B}(x,r/2)$.*

See Definition 13.6 for disks of regularity. Also, do not pay attention to the radius $r/2$, which is here mostly for convenience.

The lemma is standard. Here we can use the conformal invariance of energy integrals to give a soft proof. Obviously it is enough to prove the result for Ω_1. First choose a domain D_1 such that

$$\Omega_1 \cap B(x,2r/3) \subset D_1 \subset \Omega_1, \tag{14.2}$$

and whose boundary is a Jordan curve composed of two $C^{1+\varepsilon}$ arcs, one contained in G and one contained in Ω_1, and which meet with right angles. [See Figure 14.1]. This is possible because we know that $G\cap B(x,r)$ is actually of class $C^{1+\varepsilon}$ for some $\varepsilon > 0$ (and even $C^{1,1}$ by [Bo]). This extra regularity on G at the places where it is C^1 is easy to get (and actually comes for free in [Da]), and will allow a simpler control on the conformal mapping below.

Next let D_2 be a half disk, and map conformally D_1 onto D_2. We can do this so that the conformal mapping $\psi : D_1 \to D_2$ sends the arc $G\cap\partial D_1$ to the straight part of ∂D_2 and the other piece $\Omega_1\cap\partial D_1$ to the circular part of ∂D_2. This makes sense because Ψ has a continuous extension to the boundary, and it is possible because we can find conformal mappings of the unit disk $\mathbb{D}$ that send two given distinct points of $\partial\mathbb{D}$ to any two given distinct points of $\partial\mathbb{D}$.

Because D_1 and D_2 are piecewise $C^{1+\varepsilon}$, the extension of ψ to $\overline{D}_1$ is also C^1, except perhaps at the two corners. This comes from the fact that conformal mappings

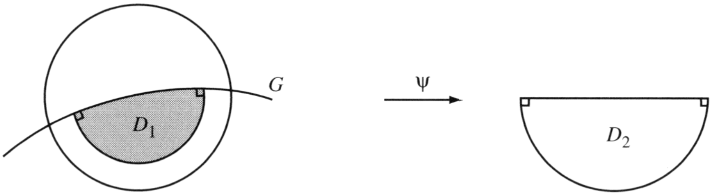


FIGURE 14.1

between $C^{1+\varepsilon}$ domains are C^1 up to the boundary. [See for instance [Po], page 48]. Here, if we want to apply such theorems directly, we can always compose with conformal mappings like $z \to (z - z_0)^2$ near the corners, to get rid of the angles. Our mapping is also fairly regular near the corners, because of the choice of right angles in the definition of D_1, but this does not really matter.

Consider $\widetilde{v} = v \circ \psi^{-1}$, which is defined on $\psi\left(\overline{D}_1 \setminus G\right)$ (i.e., on the union of D_2 and the circular part ∂ of its boundary). The restriction of $\widetilde{v}$ to ∂ is also C^1, with perhaps a singularity when we approach the two endpoints. This singularity is fairly mild, because v and $\widetilde{v}$ are bounded anyway.

Also, $\widetilde{v}$ is the function on D_2 which minimizes the energy with the given boundary data on ∂. This comes from the conformal invariance of energy integrals. By the same symmetry argument as in Section 4 (see after (4.53)), the values of $\widetilde{v}$ on D_2 are obtained from its values on ∂ by symmetrization and integration against the Poisson kernel. [This also uses some of the mildness of $\widetilde{v}$ on ∂ mentioned above.] Thus $\widetilde{v}$ extends in a C^1 way to $\overline{D}_2$, except perhaps at the two corners.

Since $v = \widetilde{v} \circ \psi$, this gives the desired properties on v as well. [Note that $\psi\left(\overline{B}\,(x, r/2) \cap \overline{\Omega}_1\right)$ lies in $\overline{D}_2$, far from the corners.] This proves Lemma 14.1. □

Here is an analogue of Lemma 14.1 for spider disks.

Lemma 14.3. — *Let $B(x, r)$ be a spider disk for G, and denote by Ω_i, $i = 1, 2, 3$, the connected components of $B(x, r) \setminus G$. Then the restriction of v to each Ω_i has a C^1 extension to $\overline{\Omega}_i \cap \overline{B}\,(x, r/2)$.*

The proof is almost the same as for Lemma 14.1. Let us use a first conformal mapping Φ to get rid of the angle at x. Set $\Phi(z) = (z - x)^{3/2}$ on Ω_i, where the choice of holomorphic square root does not really matter.

If D_1 is a domain like the one used in Lemma 14.1 (but with one extra corner at x), then $D_1' = \Phi\,(D_1)$ has only two corners left. We then map D_1' onto a half-disk D_2 by a conformal mapping ψ as above. Once again, $\widetilde{v} = v \circ \Phi^{-1} \circ \psi^{-1}$ is C^1 up to the boundary, except perhaps at the two corners. Finally, $v = \widetilde{v} \circ \psi \circ \Phi$ has a C^1 extension to $\overline{\Omega}_i \cap \overline{B}\,(x, r/2)$ because Φ has a (zero) derivative at x. [See the picture.] The lemma follows. □

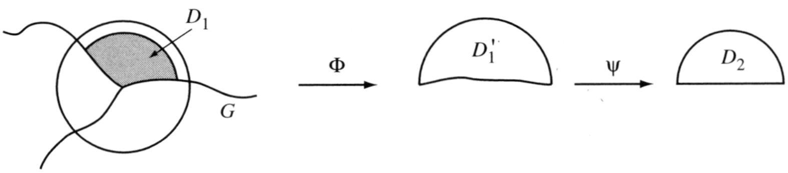


FIGURE 14.2

CHAPTER D

MISCELLANEOUS PROPERTIES OF MINIMIZERS

15. $\Omega = \mathbb{R}^2 \setminus G$ has no bounded component

We continue with our assumption that (v, G) is a minimizer as in (13.1) or (13.2), and set $\Omega = \mathbb{R}^2 \setminus G$.

Lemma 15.1. — *The open set Ω has no bounded component.*

In the special case of (13.1), we know from the definition at the beginning of Section 11 that $G^- = G \setminus L$ is bounded, and so Ω has only one unbounded component. Thus Lemma 15.1 will show that

$$\Omega \text{ is connected when (13.1) holds.} \tag{15.2}$$

Actually Lemma 15.1 is proved in [Bo], modulo minor details, but we continue with our tradition of giving a proof for the convenience of the reader.

Let us proceed by contradiction and assume that Ω has a bounded component Ω_0. Obviously $\partial\Omega_0 \subset G$ and $H^1(\partial\Omega_0) > 0$. If we are in the situation of (13.1), $\partial\Omega_0$ cannot be contained in L, even up to a set of H^1-measure 0, and so $H^1(\partial\Omega_0 \cap G^-) > 0$. By Proposition 13.11, we can find a regular point $x \in G \cap \partial\Omega_0$, and if (13.1) holds we can even take x in G^-.

Let $B = B(x, r)$ be a disk of regularity centered at x. Since $B(x, r')$ is also a disk of regularity for all $r' < r$ (see Definition 13.6 or the remark that follows it), we can always assume that B is an acceptable disk (i.e., is not too large and does not meet L) if (13.1) holds.

By Lemma 14.1, ∇v has continuous extensions to the closures of both components of $B(x, r/2) \setminus G$, and hence $|\nabla v|$ stays bounded near x. Thus we can assume that

$$\int_{B(x,r)\setminus G} |\nabla v|^2 \leqslant \varepsilon r, \tag{15.3}$$

where ε is as small as we want, because otherwise we could replace r with a smaller radius.

By Fubini and Chebychev, we can find ρ such that $r/4 < \rho < r/2$ and

$$\int_{\partial D \setminus G} |\nabla v|^2 \leqslant 10\varepsilon, \tag{15.4}$$

where we set $D = B(x, \rho)$

Since $x \in \partial\Omega_0$, one of the two components of Ω that touch x must be Ω_0. Let us call Ω_1 the other one. Our intention is to modify v slightly in $\Omega_1 \cap D$ and a little more in Ω_0 to find a better competitor than (v, G) and get a contradiction. The idea will be to first modify v on Ω_0 (by adding a constant) to get a good contact near the point x, and then do some gluing in D and eliminate a good portion of $G \cap D$.

Let us try to make our notations more transparent by calling v_i the restriction of v to Ω_i, $i = 0, 1$. Also denote by m_i the mean value of v_i on the arc of circle $\partial D \cap \Omega_i$. [See Figure 15.1, and recall that the geometry of the situation is simple because $B(x, r)$ is a disk of regularity.]

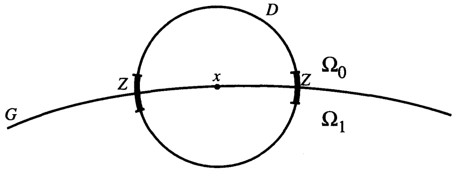


FIGURE 15.1

Let Z denote the union of two small arcs of ∂D centered at the points of $\partial D \cap G$ and with lengths equal to $r/100$. Denote by $\widetilde{v}$ the function on ∂D which is equal to v_1 on $\partial D \cap \Omega_1 \setminus Z$, to $v_0 + m_1 - m_0$ on $\partial D \cap \Omega_0 \setminus Z$, and which interpolates in the obvious linear way on Z. Then

$$\int_{\partial D} |\nabla \widetilde{v}|^2 \leqslant C\,\varepsilon \tag{15.5}$$

by (15.4) (and because $\widetilde{v}$ has essentially the same averages on the two pieces of $\partial D \setminus Z$).

The harmonic extension of $\widetilde{v}$ to D satisfies

$$\int_{D} |\nabla \widetilde{v}|^2 \leqslant C\,\varepsilon\, r. \tag{15.6}$$

We define $\widetilde{v}$ on $\Omega_0 \setminus D$ by $\widetilde{v}(z) = v_0(z) + m_1 - m_0 = v(z) + m_1 - m_0$, and on the rest of $\mathbb{R}^2 \setminus (G \cup D)$ by $\widetilde{v}(z) = v(z)$. We also set

$$\widetilde{G} = (G \setminus D) \cup Z. \tag{15.7}$$

Note that $(\widetilde{v}, \widetilde{G})$ is a compact perturbation of (v, G) (i.e., is equal to it outside a large ball) because Ω_0 is bounded. Also, $\widetilde{v} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \widetilde{G})$. If (13.1) holds, $(\widetilde{v}, \widetilde{G})$ is a competitor for (v, G) because we did not touch L (recall that $B(x, r)$ is an acceptable disk). If (13.2) holds also $(\widetilde{v}, \widetilde{G})$ is an acceptable competitor for (v, G); the topological condition (1.5) is not violated because we only connected Ω_1 to a bounded component Ω_0.

The comparison is easy to do. We did not modify ∇v out of D, and in D we lost at most $C\varepsilon r$ in the energy term, by (15.6). On the other hand (15.7) says that we saved $H^1(G \cap D) - H^1(Z) \geqslant 2\rho - r/50$ in terms of length, so $(\widetilde{v}, \widetilde{G})$ is strictly better than (v, G) if ε is small enough.

This gives the desired contradiction and proves Lemma 15.1. □

Remark 15.8. — Lemma 15.1 also holds for global minimizers in $\mathbb{R}^2 \setminus \mathbb{R}$ and $\mathbb{R}^2 \setminus (-\infty, 0]$, with essentially the same proof. The only point is that, as in the situation of (13.1), we can find regular points of $\partial\Omega_0$ that lie away from $\mathbb{R}$ or $(-\infty, 0]$.

16. v really jumps at regular and spider points

Lemma 16.1. — *Let x_0 be a regular point of G, $B_0 = B(x_0, r_0)$ a disk of regularity, and denote by Ω_1 and Ω_2 the two connected components of $B_0 \setminus G$. If* (13.1) *holds, assume that $x_0 \notin L$. If* (13.2) *holds, assume that Ω_1 and Ω_2 are contained in the same connected component of $\Omega = \mathbb{R}^2 \setminus G$. Then*

$$\lim_{\substack{x \to x_0 \\ x \in \Omega_1}} v(x) \neq \lim_{\substack{x \to x_0 \\ x \in \Omega_2}} v(x). \tag{16.2}$$

Note that the limits in (16.2) exist by Lemma 14.1. The proof of Lemma 16.1 is almost the same as above. Because of Lemma 14.1, the restrictions of v to Ω_1 and Ω_2 have C^1 extensions up to the boundary in some neighborhood of x_0, and so we can find a radius $r_1 < r_0$ such that

$$\int_{B(x_0, r_1) \setminus G} |\nabla v|^2 \leqslant \varepsilon r_1, \tag{16.3}$$

where ε will be chosen soon. We can also choose r_1 so small that for $i = 1, 2$,

$$\sup\{v(x) \; ; \; x \in \Omega_i \cap B(x_0, r_1)\} - \inf\{v(x) \; ; \; x \in \Omega_i \cap B(x_0, r_1)\} \leqslant \varepsilon r_1^{1/2}. \tag{16.4}$$

If (13.1) holds, let us also choose r_1 so small that $B(x_0, r_1)$ does not meet L. Next we choose a disk $D = B(x_0, r)$ such that $r_1/4 < r < r_1/2$ and

$$\int_{\partial D \setminus G} |\nabla v|^2 \leqslant 10\varepsilon. \tag{16.5}$$

This is possible, by (16.3) and Chebychev. We want to define a competitor $(\widetilde{v}, \widetilde{G})$ with

$$\widetilde{G} = (G \setminus D) \cup Z, \tag{16.6}$$

where Z is the union of two small arcs of ∂D centered at the points of $\partial D \cap G$ and with lengths equal to $r/100$. Figure 15.1 is still an acceptable picture for this, except that the Ω_i have different indices and the center is now called x_0. We want to keep

$$\widetilde{v}(x) = v(x) \text{ on } \mathbb{R}^2 \setminus (G \cup D \cup Z), \tag{16.7}$$

and then extend $\widetilde{v}$ in a natural way to D. Still denote by $\widetilde{v}$ the function on ∂D that extends $\widetilde{v}$ linearly on each of the two arcs of Z. We claim that if (16.2) does not hold,

$$\int_{\partial D} |\nabla \widetilde{v}|^2 \leqslant C\,\varepsilon. \tag{16.8}$$

Indeed denote by m_i, $i = 1, 2$, the mean value of v on the arc $\partial D \cap \Omega_i \setminus Z$, and by m_0 the common value of the limits in (16.2). Then

$$|m_1 - m_2| \leqslant |m_1 - m_0| + |m_2 - m_0| \leqslant 2\varepsilon\, r_1^{1/2} \tag{16.9}$$

by (16.4), and (16.8) follows from this and (16.5), because the arcs of Z are not too short.

Now define $\widetilde{v}$ in D to be the harmonic extension of its values on ∂D; thus

$$\int_{D} |\nabla \widetilde{v}|^2 \leqslant C\,\varepsilon\, r, \tag{16.10}$$

by (16.8). The accounting will then be easy; when we replace (v, G) with $(\widetilde{v}, \widetilde{G})$, we lose at most $C\,\varepsilon\, r$ in energy (by (16.10)), and we win at least $H^1(G \cap D) - H^1(Z) \geqslant r_1/4$ in length. So we'll get the desired contradiction as soon as we prove that $(\widetilde{v}, \widetilde{G})$ is a legitimate competitor for (v, G).

The fact that $\widetilde{v} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \widetilde{G})$ is easy to check, as usual. If (13.1) holds, we simply have to check that we did not remove a piece of L accidentally. This is the case, because we made sure that $B(x_0, r_1)$ would not meet L. If (13.2) holds, we have to check that $\widetilde{G}$ satisfies (1.5), i.e., that for x, y out of some ball $B(0, R)$, $\widetilde{G}$ separates x from y whenever G separates them. Because we only modified G inside D, there can only be a problem when x and y are connected to ∂D in $\mathbb{R}^2 \setminus G$. But then x and y lie in the same component of $\mathbb{R}^2 \setminus G$, by assumption on Ω_1 and Ω_2.

Thus $(\widetilde{v}, \widetilde{G})$ was an authorized competitor, and our assumption that (16.2) fails leads to a contradiction. This proves the lemma. □

We'll also need a version of Lemma 16.1 for spider points.

Lemma 16.11. — *Let $B(x_0, r_0)$ be a spider disk, and denote by Ω_1, Ω_2, Ω_3 the connected components of $B(x_0, r_0) \setminus G$. If* (13.1) *holds, assume that $x_0 \neq -1$. If* (13.2) *holds, assume that Ω_1 and Ω_2 are contained in the same connected component of Ω. Then* (16.2) *holds.*

The proof is the same as above, except that we now work in $D \cap \left(\overline{\Omega_1 \cup \Omega_2}\right)$ instead of D, and we only remove the branch of the spider $D \cap G$ that separates Ω_1 from

Ω_2. See the picture. Note also that the case when $x_0 \in L$, $x_0 \neq -1$ is impossible by definition of a spider point. We leave the details.

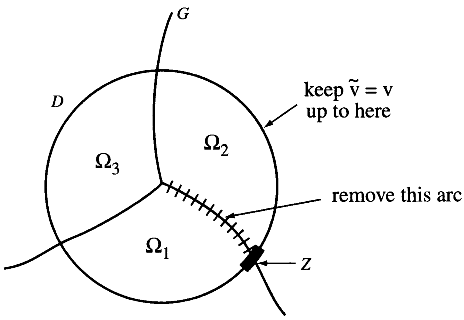


FIGURE 16.1

17. Ω has no thin connected component

In this section we only consider global minimizers, as opposed to minimizers of the modified functional for which the question of thin connected components does not arise because Ω is connected (by (15.2)).

So we assume that (u, K) is a (reduced) global minimizer, either in the whole plane (as in (13.2)), or in $\mathbb{R}^2 \setminus \mathbb{R}$ or $\mathbb{R}^2 \setminus (-\infty, 0]$ (as in Definition 12.41). Let us still denote by Ω the complement of K in $\mathbb{R}^2$, $\mathbb{R}^2 \setminus \mathbb{R}$, or $\mathbb{R}^2 \setminus (-\infty, 0]$. Here is what we mean by "not thin".

Lemma 17.1. — *Every connected component of Ω contains disks of arbitrarily large radii.*

We already know from Lemma 15.1 and Remark 15.8 that Ω has no bounded component, so we only need to worry about infinite (but thin) components.

So let Ω_0 be a connected component of Ω, and assume to get a contradiction that for some $d > 0$, all points of Ω_0 lie at distance $\leqslant d$ from $\partial\Omega_0$.

First pick an origin $z_0 \in \partial\Omega_0$. If (u, K) is a global minimizer in $\mathbb{R}^2 \setminus \mathbb{R}$ (respectively, in $\mathbb{R}^2 \setminus (-\infty, 0]$), choose z_0 out of $\mathbb{R}$ (respectively, out of $(-\infty, 0]$).

Then choose $x_0 \in \Omega_0$ and x_1 in some other component Ω_1 of Ω, both very close to z_0. In particular, require that $|x_i - z_0| < d/2$ for $i = 0, 1$, and that Ω_1 lies in the same connected component of $\mathbb{R}^2 \setminus \mathbb{R}$ as Ω_0 when (u, K) is a global minimizer in $\mathbb{R}^2 \setminus \mathbb{R}$.

Let us first complete the proof in the simpler case when (u, K) is a global minimizer in $\mathbb{R}^2$. Set $D_r = B(z_0, r)$ for all $r \geqslant d$. Because Ω_0 and Ω_1 meet D_r and are both unbounded (by Lemma 15.1), ∂D_r meets both Ω_0 and Ω_1, and so it meets $\partial\Omega_0$ (in fact, at least twice). Because of this,

$$H^1(\partial\Omega_0 \cap D_R \setminus D_d) \geqslant R - d \quad \text{for all} \quad R > d. \tag{17.2}$$

Take R very large, and apply Lemma 13.39 to the disk $B(z_0, R)$, with $C_1 = 2$, and with $E = \partial\Omega_0 \cap D_R \setminus D_{R/2}$. We get a disk of regularity $B(y,t)$ centered on E and with radius $t \in [C_2^{-1}R, R/2]$. We also get a contradiction because one of the two components of $B(y,t) \setminus G$ must be contained in Ω_0 (otherwise, y would not lie in $\partial\Omega_0$), and both components of $B(y,t) \setminus G$ contain disks of radius $> d$ if $C_2^{-1}R \geqslant 2d$. This proves Lemma 17.1 when (u,K) is a global minimizer in $\mathbb{R}^2$.

Now suppose that (u,K) is a global minimizer in $\mathbb{R}^2 \setminus \mathbb{R}$. The same argument as above works if we can choose $z_0 \in \partial\Omega_0$ (at the beginning of the construction) so that $\operatorname{dist}(z_0, \mathbb{R}) > 2R$, where $R = 2C_2\, d$ and C_2 comes from Lemma 13.39 applied with $C_1 = 2$. So we are left with the case suggested by Figure 17.1 when $\partial\Omega_0$, and hence also Ω_0, is contained in a strip

$$S_R = \{z \in \mathbb{R}^2 \ ; \ \operatorname{dist}(z, \mathbb{R}) \leqslant 2R\}. \tag{17.3}$$

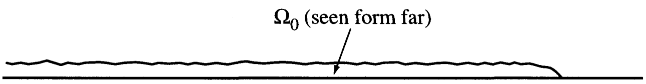


FIGURE 17.1

This situation is impossible, by the same argument as for Lemma 13.50. That is, we can modify (u,K) in a very large disk $D = B(z_0, C\,R)$ (with any choice of z_0 as above) to get a strictly better competitor. The idea is to collapse $\Omega_0 \cap \frac{1}{2}D$ and $\partial\Omega_0 \cap \frac{1}{2}D$ onto the line $\mathbb{R}$, and so save about $H^1\left(\frac{1}{2}D \cap \partial\Omega_0 \setminus \mathbb{R}\right) \geqslant C\,R/3$ in length. For this we use a function Ψ like the one in (13.54)-(13.58) and Figure 13.4. The estimates are the same as in the end of Section 13 (starting with the construction of Ψ a few lines before (13.53); we do not need the measure theory before because we know that $H^1\left(\frac{1}{2}D \cap \partial\Omega_0 \setminus R\right) \geqslant C\,R/3$). We leave the details.

The case when (u,K) is a global minimizer in $\mathbb{R}^2 \setminus (-\infty, 0]$ is treated in the same way. This time the only case when the argument above does not apply is when Ω_0 stays at distance $\leqslant 2R$ from $(-\infty, 0]$, and we can apply the deformation argument of Lemma 13.50 to a disk of radius $C\,R$, C very large, centered close to $(-\infty, 0]$ and very far to the left.

This completes our proof of Lemma 17.1. □

18. The case when u is locally constant somewhere

Lemma 18.1. — *Let (u, K) be a global minimizer. If u is constant on one of the connected components of $\Omega = \mathbb{R}^2 \setminus K$, then K is empty, or a line, or a propeller, and u is constant on every component of Ω.*

Recall that a propeller is a union of three half-lines ending at the same point, and making 120° angles with each other at this point.

Of course, if u is constant near some point $x \in \Omega$, it is constant in the component of x, because it is harmonic.

Remark 18.2. — If (v, G) is a minimizer of the modified functional (as in Definition 11.3) and v is constant near some point of $\mathbb{R}^2 \setminus G$, then v is constant on $\mathbb{R}^2 \setminus G$, by the simple observation above and because Ω is connected (by (15.2)). Hence the only option allowed by Definition 11.3 is that $G = L$ and $v = C^{te}$. [This option will often be ruled out later by conditions on the behavior of v at infinity, but it is still allowed here.] This is the reason why we do not need to study minimizers of the modified functional in (the rest of) this section.

Let (u, K) be as in the lemma, and let Ω_0 be a connected component of Ω where u is constant. We can assume that K is not empty, since otherwise the result is trivial.

Lemma 18.3. — *Ω_0 is convex.*

Suppose not. Then we can find a line segment I whose endpoints lie in Ω_0 but which is not contained in Ω_0. Since we can move the endpoints a little bit, we can even assume that I is not contained in the closure $\overline{\Omega}_0$. This is easy to check, especially because we can use (13.4) and Proposition 13.11, for instance. We can still move I a tiny bit in the direction orthogonal to I and keep these properties. We use this observation to require two additional properties that will simplify our construction. First we demand that

(18.4) every point of $I \cap K$ be a regular point of K.

This is easy to arrange, because H^1-almost every point of K is regular (by Proposition 13.11), and so the projection of the remaining part of K in the direction orthogonal to I has measure zero. We just have to move I so that its projection avoids this set of measure zero.

By Fubini, and the fact that $H^1(K \cap B_R) < +\infty$ for all R, we can also require that

(18.5) $I \cap K$ be finite,

and even that

(18.6) K meet I transversally at each point of $I \cap K$, because the projection in the direction orthogonal to I of the set of regular points of K with a tangent parallel to I has zero measure, by Sard's theorem. [Note that here the fact that we restrict to regular points helps (because we can reduce to the case of finitely many C^1-curves), and we can get (18.5) and (18.6) by very elementary means.]

So let I have all the properties above, and then denote by J the closure of any nonempty connected component of $I \setminus \overline{\Omega}_0$. Thus J is a closed interval, its endpoints lie in $\partial\Omega_0$, but the rest of J lies in $\mathbb{R}^2 \setminus \overline{\Omega}_0$.

Call a and b the endpoints of J. Since a, b lie in $\partial\Omega_0$ and are regular point of K, we can find a path γ in Ω_0 (except for its two endpoints) and which connects a to b. [See Figure 18.1; we start with two little segments orthogonal to K that start at a and b and point into Ω_0, and then use the connectedness of Ω_0 to join their endpoints.] We may even require that γ be of class C^1 and that it be simple.

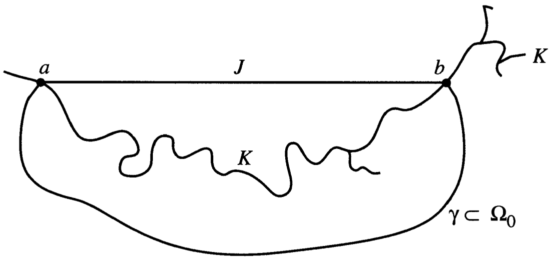


FIGURE 18.1

If we add the segment J to γ, we get a Jordan arc $\widehat{\gamma}$. This is because J only meets γ at a and b, since the rest of J does not meet $\overline{\Omega}_0$.

Call V the bounded component of $\mathbb{R}^2 \setminus \widehat{\gamma}$. We replace K with

$$\widetilde{K} = (K \setminus V) \cup J, \tag{18.7}$$

and then define $\widetilde{u}$ on $\mathbb{R}^2 \setminus \widetilde{K}$ by

$$\widetilde{u} = u \text{ on } \mathbb{R}^2 \setminus (K \cup J \cup V) \tag{18.8}$$

and

$$\widetilde{u} = c_0 \text{ on } V, \tag{18.9}$$

where c_0 is the constant value of u on Ω_0. [See Figure 18.2.]

Note that the definition is complete: $\mathbb{R}^2 \setminus \widetilde{K}$ is the disjoint union of $\mathbb{R}^2 \setminus (K \cup J \cup V)$ and V. Now we want to show that $(\widetilde{u}, \widetilde{K})$ is a better competitor than (u, K).

First we need to check that $\widetilde{u} \in W^{1,2}_{\text{loc}}(\mathbb{R}^2 \setminus \widetilde{K})$. The only potential trouble is with the interface between the two definition of $\widetilde{u}$, i.e., with the regularity of $\widetilde{u}$ across ∂V. There is no difficulty with J, since we put it in $\widetilde{K}$. There is no difficulty with $\gamma \setminus \{a, b\}$

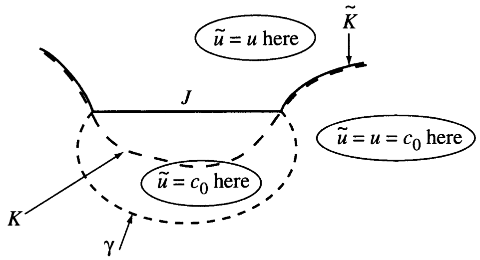


FIGURE 18.2

either, because $\widetilde{u}$ is constant and equal to c_0 in a neighborhood of $\gamma \setminus \{a, b\}$. [Recall that this arc is contained in Ω_0.]

To prove that $(\widetilde{u}, \widetilde{K})$ is a competitor for (u, K), we also need to check that $(\widetilde{u}, \widetilde{K})$ coincides with (u, K) out of some large disk B_R, and that the topological constraint (1.5) is satisfied. The first condition is of course fulfilled if $\overline{V} \subset B_R$, and so it is enough to check (1.5).

So suppose that $x, y \in \mathbb{R}^2 \setminus (K \cup B_R)$ are separated by K, but not by $\widetilde{K}$, and let us find a contradiction. Let ξ denote a path from x to y that does not meet $\widetilde{K}$. Since K separates x from y, ξ must meet K, and this can only happen in V (because the rest of K is contained in $\widetilde{K}$).

Call α and β the first and last points of $\xi \cap \partial V$ when we run along ξ from x to y. These points exist because x, y lie out of $\overline{V}$ (we implicitly assumed that $\overline{V} \subset B_R$). Since ξ does not meet $\widetilde{K}$ and $J \subset \widetilde{K}$, the points α and β must lie in $\gamma \setminus \{a, b\}$. We can replace the arc of ξ between α an β by the arc of γ between α and β; we get a new arc $\widetilde{\xi}$ which still joins x to y, but no longer meets K (by definition of α and β, and because $\gamma \setminus \{a, b\} \subset \Omega_0$). This contradicts our assumption that K separates x from y, and proves that $(\widetilde{u}, \widetilde{K})$ is a competitor for (u, K).

The accounting will be easy. We did not lose anything in the energy term, since $\widetilde{u}$ is constant wherever it differs from u. In the length term, we won at least

$$\Delta L = H^1(K \cap V) - H^1(J). \tag{18.10}$$

Let us check that $\Delta L > 0$. For each point $x \in J \setminus \{a, b\}$, denote by L_x the half-line with extremity x which is perpendicular to J and starts (from x) in the direction of V. Obviously L_x eventually crosses γ, and it must meet $K \cap V$ before it does so for the first time, because $x \in \mathbb{R}^2 \setminus \overline{\Omega}_0$ and $\gamma \subset \Omega_0$. See Figure 18.3.

Denote by π the orthogonal projection onto the line that contains J. We have just checked that $\pi(K \cap V)$ contains $J \setminus \{a, b\}$. Thus $\Delta L \geqslant 0$. To prove the strict inequality, we can use the fact that a is a regular point of K and the tangent to K at a is transverse to I (by (18.4) and (18.6)). This shows that $H^1\left[\pi^{-1}(A) \cap K \cap V\right] > H^1(A)$ for some small subinterval A of J (close to a); see Figure 18.3 again. This proves the strict inequality.

Thus $(\widetilde{u}, \widetilde{K})$ was a better competitor than (u, K), and this contradiction shows that I did not exist, i.e., that Ω_0 is convex. $\square$

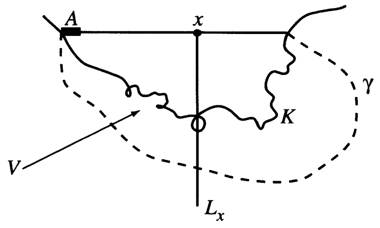


FIGURE 18.3

Next we want to say that Ω_0 looks like a cone at infinity. Fix an origin $x_0 \in \Omega_0$. Since our problem is invariant, we may as well assume that $x_0 = 0$. Set

$$J = \{\theta \in \mathbb{S}^1 \text{ ; there is a sequence } \{y_n\} \text{ of points of } \Omega_0 \text{ such that } |y_n| \text{ tends to } +\infty \text{ and } \frac{y_n}{|y_n|} \text{ tends to } \theta\}. \tag{18.11}$$

Note that since Ω_0 is not bounded (by Lemma 15.1) and $\mathbb{S}^1$ is compact, J is not empty. Let us check that

$$\text{for all } \theta \in J, \text{ the half line } D_\theta = \{r\theta \text{ ; } r \geqslant 0\} \text{ is contained in } \Omega_0. \tag{18.12}$$

First we show that $D_\theta \subset \overline{\Omega}_0$. Let $r \geqslant 0$ be given, and let $\{y_n\}$ be a sequence in Ω_0 such that $|y_n|$ tends to $+\infty$ and $|y_n|^{-1} y_n$ tends to θ. As soon as $|y_n| > r$, the point $r\,|y_n|^{-1} y_n$ lies in Ω_0 (by convexity and because $0 \in \Omega_0$), and so $r\theta \in \overline{\Omega}_0$, as needed.

To complete the proof of (18.12), observe that Ω_0 contains a small disk B_ρ centered at 0. Let $r\theta$ be any point of D_θ. If $r = 0$, then $r\theta = 0 \in \Omega_0$, as needed. Otherwise, $2r\theta \in \overline{\Omega}_0$ by the argument above, and so we can find $z \in \Omega_0 \cap B\,(2r\theta, \rho/10)$. Since $B_\rho \subset \Omega_0$ and Ω_0 is convex, all points $\frac{1}{2}(z+u)$, $|u| < \rho$ lie in Ω_0. In other words, Ω_0 contains $B\,(z/2, \rho/2)$ and in particular $r\theta \in \Omega_0$. This proves (18.12).

Lemma 18.13. — *There is a positive (universal) constant δ such that* $\operatorname{diam} J \geqslant \delta$.

Let us choose $\theta_0 \in J$, and set $D_0 = D_{\theta_0}$. [Recall that J is not empty.] Since our problem is invariant under rotations, we may as well assume that $\theta_0 = 1$. We want to proceed by contradiction and so we assume that $\operatorname{diam} J \leqslant \delta$, where $0 < \delta < 1/100$ will be chosen soon.

Since J does not contain any point at distance $\geqslant \delta$ from 1, we can find a radius $R_1 > 1$ such that

$$\left| \frac{y}{|y|} - 1 \right| \leqslant 2\delta \quad \text{for all } \; y \in \Omega_0 \setminus B\,(0, R_1)\,. \tag{18.14}$$

Let us now check that

$$H^1\left(\partial\Omega_0 \cap B(0,R) \setminus B\left(0,9R/10\right)\right) \geqslant \frac{R}{5} \tag{18.15}$$

for all $R \geqslant 2R_1$. Indeed, $D_0 \subset \Omega_0$ by (18.12), while Ω_0 does not meet

$$D^* = \{(0,t)\ ;\ |t| \geqslant R_1\} \tag{18.16}$$

(i.e., most of the vertical axis). Consequently every circle $\partial B(0,r)$, $9R/10 \leqslant r \leqslant R$, must meet $\partial\Omega_0$ at least twice (once above D_0 and once below), and (18.16) follows. See Figure 18.4.

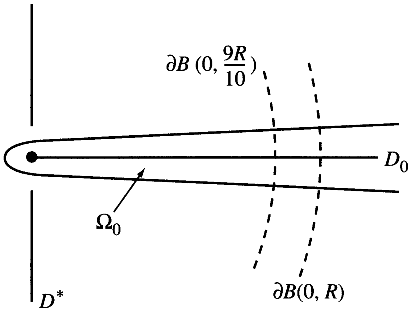


FIGURE 18.4

Now let us choose a point $x \in \partial\Omega_0 \cap \partial B(0,R)$ and apply Lemma 13.39 with $r = R/3$. We take for E the same set $\partial\Omega_0 \cap B(0,R) \setminus B\left(0,9R/10\right)$ as in (18.15). Note that $E \subset B(x,r)$ because of (18.14). We can apply Lemma 13.39 with $C_1 = 5/3$, and we get a disk of regularity $B(y,t)$ centered on E and such that $C_2^{-1}r \leqslant t \leqslant r/2$.

Since $y \in \partial\Omega_0$, one of the two components of $B(y,t) \setminus K$ is contained in Ω_0. Call it $\mathcal{O}$. Note that

$$\mathcal{O} \subset B(0,2R) \setminus B\left(0,R/2\right), \tag{18.17}$$

because $t \leqslant r/2 = R/6$ and $\operatorname{dist}\left(y, \partial B(0,R)\right) \leqslant R/10$. Since $R > 2R_1$, $\mathcal{O}$ is contained in the thin cone of (18.14). On the other hand, $\mathcal{O}$ contains a disk of radius $t/10 \geqslant R/30C_2$. This is clearly incompatible with staying in the thin cone, at least if we choose δ small enough.

Thus we get the desired contradiction; this completes our proof of Lemma 18.13. □

Remark 18.18. — The same argument as in Lemma 18.13 shows that J cannot be reduced to a set composed of two opposite points in $\mathbb{S}^1$. Indeed, if this were the case, there would be a radius R_1 such that all the points of $\Omega_0 \setminus B\left(0,R_1\right)$ lie in a thin cone around some line D (instead of half a line as in (8.14)). We could still apply the same argument as above to get a contradiction.

Claim 18.19. — *Let $\theta_1, \theta_2 \in J$ be such that $\theta_1 \neq -\theta_2$. Then the shortest interval of S^1 that contains θ_1 and θ_2 is contained in J.*

Indeed (18.12) says that D_{θ_1} and D_{θ_2} are contained in Ω_0, and so the convex cone generated by D_{θ_1} and D_{θ_2} is also contained in Ω_0. The claim follows, by definition of J.

We are now ready to prove that

$$\begin{cases} J \text{ is a closed interval of } \mathbb{S}^1, \text{ with diameter} \\ \text{at least } \delta > 0 \text{ and length at most } \pi. \end{cases} \tag{18.20}$$

First, J is closed (by (18.11) and (18.12), say) and its diameter is at least δ. Then J contains at least two points, as in Claim 18.19 (see Remark 18.18); hence J contains some nontrivial interval J_0 of $\mathbb{S}^1$. For each other point of J, we can apply Claim 18.19 with some point of J_0; hence J is an interval of $\mathbb{S}^1$. Now the length of J cannot be more than π, because otherwise J would be the whole circle and Ω_0 would be the whole plane (by Claim 18.19 and (18.12)). This proves (18.20).

Our information on J will be easier to use after a blow-in. Pick a (fixed) point $y \in K$ and a sequence $\{t_n\}$ that tends to $+\infty$. As was observed just before Proposition 12.12, we can always extract a subsequence so that the pairs (u_n, K_n) defined from (u, K) as in (12.7) and (12.8) converge to some limit (u^1, K^1). Proposition 12.44 tells us that (u^1, K^1) is a global minimizer.

From (18.12) and the definition (18.11) of J it follows that $\{t_n^{-1}\overline{\Omega}_0\}$ converges to the closed cone

$$\mathcal{C}(J) = \{r\theta\ ;\ r \geqslant 0 \text{ and } \theta \in J\}. \tag{18.21}$$

For the same reason, $\{t_n^{-1}(\partial\Omega_0)\}$ converges to $\partial\mathcal{C}(J)$, the boundary of that cone. In particular, K^1 contains $\partial\mathcal{C}(J)$, and $\mathrm{int}(\mathcal{C}(J))$ is one of the connected components of $\mathbb{R}^2 \setminus K^1$. By the convergence of $\{u_n\}$ to u^1, we also know that u^1 is constant on $\mathrm{int}(\mathcal{C}(J))$.

By Proposition 13.11, almost every point of $\partial\mathcal{C}(J)$ is a regular point of K^1. Let x be such a point and $B(x,r)$ a disk of regularity centered at x. Thus $K^1 \cap B(x,r)$ is just a diameter of $B(x,r)$, and coincides with $\partial\mathcal{C}(J) \cap B(x,r)$.

Denote by Ω_1 the connected component of $\mathbb{R}^2 \setminus K^1$ that contains $B(x,r) \setminus \mathcal{C}(J)$. We claim that

$$u^1 \text{ is constant on } \Omega_1. \tag{18.22}$$

By Lemma 14.1, u^1 has a C^1 extension on $\overline{\Omega}_1 \cap \overline{B}(x, r/2)$, which we still denote by u^1. Note that (18.22) will follow if we prove that

$$\text{the boundary values on } \partial\Omega_1 \text{ of } \nabla u^1 \text{ vanish in a neighborhood of } x. \tag{18.23}$$

This can be seen as a consequence of a theorem of Riesz but here, since u^1 is C^1, we can also use the Schwarz reflection principle and analytic continuation.

The shortest way to get to (18.23) is to use the classical formula that gives the jump of $|\nabla u^1|^2$ in terms of the curvature of K^1 when K^1 is a C^1-curve (as here). [See [MuSh].] Let us nonetheless sketch a direct proof (which is of course equivalent). Let $w \in \mathbb{R}^2$ denote the unit vector perpendicular to $K^1 \cap B(x,r)$ and pointing in the direction of Ω_1 (as seen from x). Let φ be a smooth bump function supported on $B(x,r)$ and equal to 1 on $B(x,r/2)$, and set $\Phi(z) = z + t\varphi(z)w$, where t is a small parameter. We replace (u^1, K^1), with $(\widetilde{u}, \widetilde{K})$, where $\widetilde{K} = \Phi(K^1)$, we keep $\widetilde{u} = u^1$ out of $\Phi(\mathcal{C}(J))$, and take $\widetilde{u}(z) \equiv c_0$ on $\Phi(\mathrm{int}(\mathcal{C}(J)))$, where c_0 is the constant value of u^1 on $\mathrm{int}(\mathcal{C}(J))$. The point of the modification is that we save all the integral

$$\Delta E = \int_{\Phi(\mathcal{C}(J)) \smallsetminus \mathcal{C}(J)} |\nabla u^1|^2 \tag{18.24}$$

in the energy term, and $\Delta E \geqslant C^{-1}t$ (for t small) if ∇u^1 does not vanish on $\partial\Omega_1 \cap B(x,r/2)$, while the length that we have to add is at most Ct^2. This completes our proof of (18.23)); (18.22) follows, as we said before.

Now we found a second connected component Ω_1 of $\mathbb{R}^2 \smallsetminus K^1$ (in addition to $\mathrm{Int}(\mathcal{C}(J))$, where u^1 is constant. To this new component we can associate an interval J_1 of $\mathbb{S}^1$, defined as in (18.11) but with Ω_1 instead of Ω_0. Note incidentally that J (or J_1) does not depend on the choice of origin, and neither does the description in (18.20). So J_1 also fits the description (18.20).

It is easy to see that J and J_1 have disjoint interiors, simply because Ω_1 does not meet $\mathcal{C}(J)$.

Let us take a second blow-in. Denote by (u^2, K^2) the limit of some blow-in sequence constructed with (u^1, K^1), just like (u^1, K^1) was constructed above. Then (u^2, K^2) is also a global minimizer. The same argument as above shows that $\partial\mathcal{C}(J)$ and $\partial\mathcal{C}(J_1)$ are contained in K^2, and $\mathrm{Int}(\mathcal{C}(J))$ and $\mathrm{Int}(\mathcal{C}(J_1))$ are two components of $\mathbb{R}^2 \smallsetminus K^2$ on which u^2 is constant.

If $\mathbb{R}^2 = \mathcal{C}(J) \cup \mathcal{C}(J_1)$, then K_2 is a line, and we shall see later how to conclude from this.

Otherwise, we can choose a regular point of K^2 in $\partial(\mathcal{C}(J) \cup \mathcal{C}(J_1))$, and follow the same argument as above. We find a new connected component Ω_2 of $\mathbb{R}^2 \smallsetminus K^2$, which is contained in $\mathbb{R}^2 \smallsetminus (\mathcal{C}(J) \cup \mathcal{C}(J_1))$, and on which u^2 is constant. We associate to Ω_2 a third interval J_2 as in (18.11), and it is easy to see that J_2 is contiguous to $J \cup J_1$, but its interior does not meet $J \cup J_1$.

We continue this construction until we get a union $J \cup J_1 \cdots \cup J_k$ that covers the circle, or equivalently

$$\mathcal{C}(J) \cup \mathcal{C}(J_1) \cup \cdots \cup \mathcal{C}(J_k) = \mathbb{R}^2. \tag{18.25}$$

This has to happen eventually, because the intervals J_ℓ have diameters $\geqslant \delta$ and disjoint interiors.

When (18.25) happens, the new blow-in limit (u^{k+1}, K^{k+1}) is such that $K^{k+1} = \partial\mathcal{C}(J) \cup \cdots \cup \partial\mathcal{C}(J_k)$ and u^{k+1} is constant on $\operatorname{int}(\mathcal{C}(J))$ and its successors. It is then easy to see that the only options are either $k=1$ and K^2 is a line, or $k=2$ and K^3 is a propeller.

At this point we were able to show that some iterated blow-in of (u,K) is a line or a propeller (each time with a function that is constant on every component of the complement). Thus Lemma 18.1 will follow from the next lemma.

Lemma 18.26. — *Let (u,K) be a global minimizer and suppose that there is a sequence $\{t_n\}$ that tends to $+\infty$ and for which $\{t_n^{-1}K\}$ tends to a line or a propeller. Then K is a line or a propeller, and u is constant on each component of $\mathbb{R}^2 \setminus K$.*

This is not new; see for instance [Bo]. Let us sketch the proof anyway. First we need to know that if K is a line or a propeller, then u is constant on each component of $\mathbb{R}^2 \setminus K$. Suppose for instance that K is a line through the origin. We know that

$$\int_{B_R \setminus K} |\nabla u|^2 \leqslant CR \tag{18.27}$$

(see (13.5)), which allows us to choose radii R as large as we want such that

$$\int_{\partial B_R \setminus K} |\nabla u|^2 \leqslant C. \tag{18.28}$$

Then we use the fact that we can compute the values of u inside each half of $B_R \setminus K$ by integration of the Poisson kernel against the symmetrized restrictions of u to $\partial B_R \setminus K$. This yields

$$\int_{B_r \setminus K} |\nabla u|^2 \leqslant C\frac{r^2}{R}\int_{\partial B_R \setminus K} |\nabla u|^2 \leqslant C\frac{r^2}{R} \tag{18.29}$$

for $r < R/2$, say. Now we can fix r and let R tend to $+\infty$, and we get that $\nabla u = 0$ everywhere.

The case of a propeller (through the origin, say) is treated in the same way. The point is that if V is one of the components of $B_R \setminus K$, we can still compute the values of u on V from its values on $\partial V \setminus K$: we first use the mapping $z \to z^{3/2}$ to change variables and reduce to a half-disk, and then compute as above. This gives an estimate which is even more favorable than (18.29); the rest of the argument is the same.

Now let (u,K) be as in the lemma. Because the hypothesis of the lemma is invariant under translations, we can assume that the origin lies in K. [This is just a minor issue, due to the fact that we defined blow-up sequences only with points $y_n \in K$.]

Because of the remark before Proposition 12.12, we may as well assume that the blow-in sequence $\{(u_n, K_n)\}$ associated to $y_n = 0$ and t_n as in (12.7), (12.8) converges to a limit (u_∞, K_∞). [Otherwise, extract a subsequence.] We already know about the convergence of $\{K_n\}$, and so K_∞ is a line or a propeller. Since (u_∞, K_∞) is a global minimizer by Proposition 12.44, we get that u_∞ is locally constant. In other words, the functions ∇u_n converge to 0, uniformly on every compact subset of $\mathbb{R}^2 \setminus K_\infty$. [See our definition of convergence, a little before Lemma 12.4.]

Now we want to apply Lemma 13.17 or Lemma 13.21 to the unit disk and the global minimizer (u_n, K_n) (if n is large enough). We get the hypotheses (13.19) and (13.20) (for a line) or (13.22) and (13.23) (for a propeller) by convergence of K_n to a line or a propeller. So we only have to check (13.18), i.e., the fact that

$$\int_{B(0,1)\setminus K_n} |\nabla u_n| \leqslant \varepsilon \tag{18.30}$$

for n large enough. This is an easy consequence of the uniform convergence of ∇u_n to 0 on compact subsets of $B(0,1) \setminus K_\infty$, Hölder, and the fact that

$$\int_{B(0,1)\setminus K_n} |\nabla u_n|^2 \leqslant C. \tag{18.31}$$

[See (13.35)-(13.38) for the same argument, almost in the same context.]

Thus we can apply Lemma 13.17 or Lemma 13.21, and we get that for n large enough, $K_n \cap B(0, 1/2)$ is a C^1-curve or spider. In other words, $K \cap B(0, t_n/2)$ is a C^1 curve or spider for n large.

We can now deduce that K is a line or a propeller. The shortest way in terms of amounts of estimates is probably to observe that our conclusion that $K_n \cap B(0, 1/2)$ is a C^1 curve or spider comes with uniform estimates. [In fact, we even have uniform $C^{1+\alpha}$-estimates for some $\alpha > 0$ with the same proofs.] These uniform estimates, when scaled down to K, give better and better approximations of $K \cap B_r$ for any fixed r, and the conclusion follows.

Since we do not want to rely on estimates that were not stated, let us also give a brutal (but implacable) argument: from the fact that $K \cap B(0, t_n/2)$ is (arcwise) connected for a sequence of t_n that tends to $+\infty$, we can deduce that K is connected, and then K is a line or a propeller, by [Bo].

This completes our proof of Lemma 18.26; Lemma 18.1 follows as well. □

CHAPTER E

THE JOHN CONDITION

19. Connectedness and rectifiable curves

In this section we address a minor technical point: the existence of rectifiable curves that connect points in a connected set with finite H^1-measure. We consider a subset G_0 of the plane, such that

(19.1) $\quad G_0$ is closed, connected, and $H^1(G_0 \cap B) < +\infty$ for every disk B,

and we want to study the arcwise connectedness of G_0.

Lemma 19.2. — *If G_0 satisfies* (19.1) *then for every choice of $x, y \in G_0$ there is a path $\gamma_{x,y}$ which is rectifiable (i.e., with finite length), supported in G_0, and which connects x to y.*

This is of course a slight modification of the classical result that says that if G_0 is connected, closed, and if $H^1(G_0) < +\infty$, then G_0 is arcwise connected. The little additional difficulty here is that G_0 may not be bounded.

So let G_0 satisfy (19.1), and let us assume that

(19.3) $\quad G_0$ is not bounded;

otherwise, we may always use the classical result mentioned above (see for instance [Fa]), or modify slightly the proof below. First we want to check that for all $x \in G_0$ and $R > |x|$,

(19.4) $\quad$ there is a rectifiable arc supported in G_0 and which connects x to ∂B_R.

The proof is standard; let us only sketch the argument. For every (small) $\varepsilon > 0$, call "ε-chain" a finite sequence $\{x_0, \dots, x_m\}$ of points in G_0 such that $|x_{i+1} - x_i| \leqslant \varepsilon$ for $0 \leqslant i < m$. Since G_0 is connected, for all $y \in G_0$ and all $\varepsilon > 0$, there is an ε-chain that goes from x to y (that is, such that $x_0 = x$ and $x_m = y$). Let us choose any point $y \in G_0 \setminus B_R$, and then only keep the beginning of the ε-chain (before we leave

B_R for the first time). This gives an ε-chain of points in $G_0 \cap B_R$ that goes from x to some point of $G_0 \cap B_R \setminus B_{R-\varepsilon}$.

For each (small) ε, let $\{x_0, \dots, x_m\}$ be such an ε-chain. Let us choose this chain so that m is minimal. Then $|x_i - x_j| > \varepsilon$ for $|i - j| > 1$, because otherwise we could take a shortcut. Hence the disks $B(x_j, \varepsilon/2)$ have a covering number $\leqslant 2$ (i.e., no point of the plane lies in 3 of them). Since $H^1(G_0 \cap B(x_i, \varepsilon/2)) \geqslant \varepsilon/2$ by (19.3) and the connectedness of G_0, we see that

$$m \leqslant 4\varepsilon^{-1} H^1(G_0 \cap B_R) \tag{19.5}$$

because $x_0, \dots, x_{m-1}$ all lie in $B_{R-\varepsilon}$ by minimality.

Denote by γ_ε the polygonal arc obtained by connecting each point x_i, $0 \leqslant i < m$, to x_{i+1} by a line segment. Then

$$\text{length}(\gamma_\varepsilon) \leqslant 4H^1(G_0 \cap B_R) + \varepsilon \tag{19.6}$$

by (19.5). This allows us to define a parameterization $z_\varepsilon : [0,1] \to \mathbb{R}^2$ of γ_ε, which is C-Lipschitz for some C that does not depend on ε. [For instance, we can take $C = 4H^1(G_0 \cap B_R) + 1$]. By Montel, we can find a sequence $\{\varepsilon_n\}$ that tends to 0 such that z_{ε_n} converge uniformly to some C-Lipschitz function z. It is easy to see that z parameterizes the arc that we want for (19.4).

Now return to Lemma 19.2. For each $x \in G_0$, denote by $G(x)$ the set of points $y \in G_0$ that can be connected to x by an arc of finite length supported in G_0. We want to show that $G(x) = G_0$.

Let us first check that

$$H^1(\mathcal{R}) = 0, \tag{19.7}$$

where

$$\mathcal{R} = \{R > 0 \ ; \ G_0 \cap \partial B_R \text{ is infinite}\} \tag{19.8}$$

Clearly it is enough to show that $H^1(\mathcal{R} \cap I) = 0$ for all intervals $I = [0, N)$. Let μ denote the image by $x \to |x|$ of the restriction of H^1 to $G_0 \cap B_N$. Thus

$$\mu(E) = H^1(\{x \in G_0 \ ; \ |x| \in E\}) \tag{19.9}$$

for measurable subsets of I. By (19.1), μ is a finite measure. By the Hardy-Littlewood maximal theorem, or the existence of a Lebesgue density for μ (Lebesgue-)almost-everywhere, the upper density

$$d^+(t) = \limsup_{\varepsilon \to 0^+} \frac{\mu([t-\varepsilon, t+\varepsilon])}{2\varepsilon} \tag{19.10}$$

is finite almost everywhere on I.

On the other hand, if $t \in I$ in such that $G_0 \cap \partial B_t$ has at least k point, then $d^+(t) \geqslant k/2$. This uses the connectedness of G_0. The estimate (19.7) follows.

Note that (19.7) would still hold with the same proof if G_0 was a countable union of connected sets, provided of course that we keep the condition $H^1(G_0 \cap B) < +\infty$ for all B.

Now let R (large) be such that $G_0 \cap \partial B_R$ is finite, and denote by $x_1, \dots, x_\ell$ the points of $G_0 \cap \partial B_R$. For $1 \leqslant j \leqslant \ell$, set $E_j = \overline{B}_R \cap G(x_j)$. By (19.4),

$$G_0 \cap \overline{B}_R = \bigcup_j E_j. \tag{19.11}$$

Let us check that

$$E_j \text{ is closed.} \tag{19.12}$$

Let $\{z_n\}$ be a sequence in E_j that converges to some $z \in G_0 \cap \overline{B}_R$. For each n the proof of (19.4) gives an arc γ_n, supported in $G_0 \cap \overline{B}_R$, with length at most $C = 4H^1(G_0 \cap B_R) + 1$, and that connects z_n to some point $y_n \in G_0 \cap \partial B_R$. Since $z_n \in E_j$, we also have that $y_n \in E_j$ (by definition of E_j).

Modulo extracting a subsequence, we can assume that the C-Lipschitz parameterizations of the arcs γ_n by $[0,1]$ given by the proof of (19.4) converge uniformly to the C-Lipschitz parameterization of some arc γ. Then γ has finite length $\leqslant C$, is supported in $G_0 \cap \overline{B}_R$, and connects z to some point $y \in G_0 \cap \partial B_R$. Since $G_0 \cap \partial B_R$ is finite and y is the limit of $\{y_n\}$, we see that $y = y_n$ for n large enough, and so $y \in E_j$. Then $z \in E_j$ as well (because of γ); this proves (19.12).

Note that for $x, y \in G_0 \cap \overline{B}_R$, $y \in G(x)$ if and only if x and y lie in a same E_j. Then (19.12) implies that $G(x) \cap \overline{B}_R$ is closed. Since we can take R as large as we want, we get that

$$G(x) \text{ is closed.} \tag{19.13}$$

By definition, the sets $G(x), x \in G_0$, are either equal or disjoint. Since for every R as above, B_R only meets finitely many such $G(x)$ (because each of them coincides with a union of sets E_j in $\overline{B}_R$), we get that $G(x)$ is open as well.

Thus all $G(x)$ are both closed and open. Since they are not empty and G_0 is connected, we get that $G(x) = G_0$ for all x, as need for Lemma 19.2. □

We shall need the following slight improvement of Lemma 19.2.

Lemma 19.14. — *If G_0 satisfies* (19.1), *then for every choice of $x, y \in G_0$ there is a simple, rectifiable arc $\gamma_{x,y}$ which is supported on G_0 and connects x to y.*

To prove this we just need to take the rectifiable arc $\gamma^0_{x,y}$ given by Lemma 19.2, and make it simple. The standard way to do this is by removing the unnecessary loops, and it is described in [Fa]. Let us point out a slightly different method. Set

$$L = \inf\{\text{length}(\gamma)\ ;\ \gamma \text{ is a rectifiable arc in } G_0 \text{ that connects } x \text{ to } y\}. \tag{19.15}$$

Lemma 19.2 tell us that $L < +\infty$, and now we claim that we can find $\gamma_{x,y}$ as above so that length $(\gamma_{x,y}) = L$.

This is fairly easy. We start from a minimizing sequence of curves γ_n, where $L_n = \text{length}\,(\gamma_n)$ tends to L. We can parameterize γ_n in a L_n-Lipschitz way from the unit interval. As before, we can use Montel to find a subsequence of parameterizations that converges uniformly on $[0,1]$ to some L-Lipschitz function. That function is a parameterization of the desired curve $\gamma_{x,y}$.

It is also easy to see that our minimizing curve $\gamma_{x,y}$ is simple (because if it had a loop, it could be made shorter); Lemma 19.14 follows. □

Lemma 19.16. — *Suppose G_0 satisfies* (19.1) *and is not bounded. Then for all $x \in G_0$ there is an injective Lipschitz path $\gamma_x : [0,+\infty) \to G_0$ such that $\gamma_x(0) = x$ and $\lim_{t\to+\infty} |\gamma_x(t)| = +\infty$*

Let $x \in G_0$ be given, and choose an increasing sequence $\{R_n\}$ that tends to $+\infty$ and such that

$$\partial_n = G_0 \cap \partial B_{R_n} \text{ is finite} \tag{19.17}$$

for every n. Such a sequence exists because of (19.7).

For each n, (19.4) tells us that we can find an arc γ_n from x to some point of ∂_n. [From now on, all arcs will have finite length and be supported in G_0, by convention.]

Denote by z_n the last point of $\gamma_n \cap \partial_1$, when you run along γ_n from x to ∂_n. Since ∂_1 is finite, we can find $x_1 \in \partial_1$ such that $z_n = x_1$ for infinitely many values of n. Thus there is an arc from x to x_1, and x_1 is connected to ∂_n by an arc in $G_0 \setminus B_{R_1}$ for infinitely many values of n. [Note that this also means "for n large enough", since you must cross ∂_m to go from ∂_1 to ∂_n when $1 \leqslant m \leqslant n$.]

Let us continue our construction. We can get a sequence $\{x_m\}$ of points $x_m \in \partial_m$ such that for all $m > 1$,

$$x_{m-1} \text{ is connected to } x_m \text{ by an arc supported in } G_0 \setminus B_{R_{m-1}}, \tag{19.18}$$

and

$$x_m \text{ can be connected to } \partial_n \text{ by an arc supported in } G_0 \setminus B_{R_m} \text{ for infinitely many values of } n > m. \tag{19.19}$$

The verification is easy; we proceed as above with repeated applications of the pigeon hole principle.

We construct a first path $\gamma^{(0)}$ as follows. We glue together a first arc from x to x_1, then an arc supported in $G_0 \setminus B_{R_1}$ from x_1 to x_2, then an arc supported in $G_0 \setminus B_{R_2}$ from x_2 to x_3, and so on. Since our path $\gamma^{(0)}$ eventually leaves every B_{R_m} (and R_m tends to $+\infty$), (the obvious Lipschitz parameterization of) $\gamma^{(0)}$ satisfies all the desired properties, except perhaps for injectivity.

Let us see now how to modify our initial path $\gamma^{(0)}$ to make it simple. We want to construct a sequence of paths $\gamma^{(n)}$, starting with $\gamma^{(0)}$.

To construct $\gamma^{(1)}$ from $\gamma^{(0)}$ we take the last point z_1 of $\gamma^{(0)} \cap \partial_1$ and replace the subarc of $\gamma^{(0)}$ between x and z_1 with a simple arc (in G_0) from x to z_1.

Now let us construct $\gamma^{(2)}$. Denote by z_2 the first point of $\gamma^{(1)} \cap \partial_2$ and by y_1 the first point of $\gamma^{(1)} \cap \partial_1$ that can be connected to z_2 by an arc in $G_0 \setminus G_{R_1}$. Such a point exists, because ∂_1 is finite. We simply replace the arc of $\gamma^{(1)}$ between y_1 and z_2 with a shortest arc in $G_0 \setminus G_{R_1}$ from y_1 to z_2. Such a shortest arc exists by the same argument as above, and it is automatically simple. It does not meet the arc of $\gamma^{(1)}$ between x and y_1 (except at y_1), by definition of y_1. Thus the subarc of $\gamma^{(2)}$ between x and z_2 is simple.

We proceed in the same way to construct $\gamma^{(n)}$ from $\gamma^{(n-1)}$. We denote by z_n the first point of $\gamma^{(n-1)} \cap \partial_n$ and by y_{n-1} the first point $\gamma^{(n-1)} \cap \partial_{n-1}$ that can be connected to z_n by an arc in $G_0 \setminus B_{R_{n-1}}$. We then replace the arc of $\gamma^{(n-1)}$ between y_{n-1} and z_n with a shortest arc in $G_0 \setminus B_{R_{n-1}}$ that goes from y_{n-1} to z_n. This gives a curve $\gamma^{(n)}$. The arc of $\gamma^{(n)}$ between y_{n-1} and z_n is simple (by minimality), and it does not meet the arc of $\gamma^{(n-1)}$ (or $\gamma^{(n)}$) between x and y_{n-1} (by definition of y_{n-1}). Thus the arc of $\gamma^{(n)}$ between x and z_n is simple.

This completes our construction of the sequence $\{\gamma^{(n)}\}$. Note that for $m > n$ we shall never modify the arc of $\gamma^{(n)}$ between x and the first point ξ_n of $\gamma^{(n)} \cap \partial_n$.

Denote by $\gamma_x^{(n)} : [0, T_n] \to G_0$ the parameterization by arclength of the arc of $\gamma^{(n)}$ between x and ξ_n. Since R_n tends to $+\infty$, T_n tends to $+\infty$ as well. Also, we just said that the restriction of $\gamma_x^{(m)}$ to $[0, T_n]$ coincides with $\gamma_x^{(n)}$ for $m > n$. Thus $\{\gamma_x^{(n)}\}_n$ converges to a limit $\gamma_x : [0, +\infty) \to G_0$.

The function γ_x is injective, because each $\gamma_x^{(n)}$ is. Then $\gamma_x(t)$ eventually leaves every B_{R_n}, because it can only cross ∂B_{R_n} finitely many times (by (19.17)) and $\gamma_x(T_m)$ lies out of B_{R_n} for infinitely many (in fact, all) values of $m > n$.

Thus γ_x satisfies all the properties required in Lemma 19.16; this proves the lemma. □

20. Paths of escape to infinity in Ω

We shall need later to estimate differences $|v(x) - v(y)|$ in terms of L^2-averages of $|\nabla v|$ on Ω. For this, some uniform control on the way Ω is connected will be useful. In effect, we shall show that Ω (or each of its components when (v, G) is a global minimizer) is a John domain with center at infinity, at least locally.

In this section again, we suppose that (v, G) satisfies (13.1) or (13.2). Our main result will be the existence for each $x \in \Omega = \mathbb{R}^2 \setminus G$ of an escape path to infinity, as follows.

Lemma 20.1. — *There is a constant $C_1 > 0$ such that, for every $z \in \Omega = \mathbb{R}^2 \setminus G$, there is an escape path $\gamma_z : [0,1] \to \Omega$ such that*

$$\gamma_z(0) = z, \tag{20.2}$$

$$\gamma_z \text{ is } C_1\text{-lipschitz}, \tag{20.3}$$

and

$$\operatorname{dist}(\gamma_z(t), G) \geqslant C_1^{-1}\{t + \operatorname{dist}(z,G)\} \text{ for all } t \in [0,1]. \tag{20.4}$$

Remark 20.5. — In the case of global minimizers (i.e., when (13.2) holds), we can even define γ_z on $[0,+\infty)$. This is not surprising (in view of the invariance of (13.2) under dilations), and it will easily follow from the proof.

We start the proof with a more local escape lemma.

Lemma 20.6. — *There is a constant $C_2 > 0$ such that, for each disk $B(x,r)$ of radius $r \leqslant C_2^{-1}$ and each $z \in B(x,r)$ such that*

$$\operatorname{dist}(z,G) \geqslant 10^{-2}r, \tag{20.7}$$

we can find an arc γ in Ω such that

$$\gamma \subset B(x, C_2 r) \setminus G, \tag{20.8}$$

$$\operatorname{dist}(\gamma, G) \geqslant C_2^{-1}r, \tag{20.9}$$

and

$$\gamma \text{ connects } z \text{ to a point } w \text{ such that } \operatorname{dist}(w,G) \geqslant r. \tag{20.10}$$

We want to prove this by compactness. Let us restrict to the case of minimizers of the modified functional (as in (13.1)); the case of global minimizers would be similar, but simpler. So let us assume that the lemma is false for minimizers of the modified functional. Then for each integer $j \geqslant 1$, we can find a minimizer (v_j, G_j), a disk $B_j = B(x_j, r_j)$ of radius $r_j \leqslant 2^{-j}$, and a point $z_j \in B_j$ such that

$$\operatorname{dist}(z_j, G_j) \geqslant 10^{-2}r_j, \tag{20.11}$$

for which there is no path $\gamma \subset \Omega_j$ such that the analogue of (20.8)-(20.10) with $C_2 = 2^j$ hold.

We are not exactly in the same situation as in Section 12 to take a blow-up sequence, because the minimizer (v_j, G_j) depends on j, but this will not matter. Also, we shall need to distinguish cases, depending on the position of x_j relative to L and -1.

First observe that $\operatorname{dist}(z_j, G_j) \leqslant r_j$, because otherwise the trivial path form z_j to itself would have worked. So we can choose $y_j \in G_j$ such that $|y_j - z_j| \leqslant r_j$. Set

$$G_j^* = r_j^{-1}(G_j - y_j) \tag{20.12}$$

and

$$v_j^*(x) = r_j^{-1/2} v_j\left(r_j x + y_j\right) \text{ on } \mathbb{R}^2 \setminus G_j^*; \tag{20.13}$$

these are the analogues of G_n and v_n in (12.7) and (12.8), except that here G_j and v_j are changing with j.

Our first case is when

$$\lim_{j\to+\infty} r_j^{-1} \operatorname{dist}\left(y_j, L\right) = +\infty. \tag{20.14}$$

The remark before Proposition 12.12 and Proposition 12.12 itself are still valid in this case (with (20.14) and the fact that $r_j \leqslant 2^{-j}$ tends to 0 playing the role of (12.10) and (12.9)). So we can extract a subsequence so that (after extraction) $\left\{\left(v_j^*, G_j^*\right)\right\}$ converges to a global minimizer (u, K). Modulo an additional extraction, we can also assume that $z_j^* = r_j^{-1}\left(z_j - y_j\right)$ tends to a limit z.

By (20.11), $\operatorname{dist}\left(z_j^*, G_j^*\right) \geqslant 10^{-2}$ and hence $\operatorname{dist}(z, K) \geqslant 10^{-2}$. In particular, $z \in \mathbb{R}^2 \setminus K$, and Lemma 17.1 says that its connected component in $\mathbb{R}^2 \setminus K$ contains disks of arbitrarily large radii. So we can find a path γ in $\mathbb{R}^2 \setminus K$ that connects z to some point w such that $\operatorname{dist}(w, K) \geqslant 2$.

Let D denote the diameter of γ and d its distance to K. Call γ_j^* the path obtained from γ by adding to it a line segment from z_j^* to z at the beginning. For j large enough, γ_j^* stays at distance $\geqslant d/2$ from G_j^*, and connects z_j^* to w. Also, $\operatorname{dist}\left(w, G_j^*\right) \geqslant 1$ and γ_j^* is contained in $B\left(z_j^*, D+1\right)$. It is easy to check that for j large enough, $\gamma_j = r_j\, \gamma_j^* + y_j$ satisfies the analogue of (20.8)-(20.10) with $C_2 = 2^j$, in contradiction with the definition of (v_j, G_j).

This settles the case when (20.14) holds. The proof of Lemma 20.6 when (v, G) satisfies (13.2) and without the restriction that $r \leqslant C_2^{-1}$ also follows from this argument.

If (20.14) does not hold, then we can extract a subsequence so that

$$\operatorname{dist}\left(y_j, L\right) \leqslant A, \tag{20.15}$$

where A is some fixed number (that depends on our sequence (v_j, G_j), but this does not matter). Denote by y_j' the point of L that is closest to y_j. Our second case is when

$$\lim_{j\to+\infty} r_j^{-1} \operatorname{dist}\left(y_j', -1\right) = +\infty. \tag{20.16}$$

Then we define v_j^*, G_j^*, z_j^* as above, but with y_j' instead of y_j. We can apply Proposition 12.42 and, modulo extraction of a new subsequence, we get that $\left\{\left(v_j^*, G_j^*\right)\right\}$ converges to a global minimizer on $\mathbb{R}^2 \setminus \mathbb{R}$.

We can continue the argument just as before, since Lemma 17.1 also applies to global minimizers on $\mathbb{R}^2 \setminus \mathbb{R}$. This settles our second case when (20.16) holds.

In the remaining case when (20.16) fails, we can extract a new subsequence for which $r_j^{-1}\left(y_j'+1\right)$ is bounded. This time we have that for some $A'>0$,

$$\operatorname{dist}(z_j,-1)\leqslant A'\,r_j$$

(by (20.15) and because $|y_j-z_j|\leqslant r_j$).

In this case we do the blow up relative to the fixed origin -1, and Proposition 12.42 allows us to extract a subsequence of the corresponding $\left\{\left(v_j^*,G_j^*\right)\right\}$ that converges to a global minimizer on $\mathbb{R}^2\smallsetminus(-\infty,0]$. The rest of the argument goes as in the two preceding cases.

This completes our proof of Lemma 20.6 by contradiction and compactness. □

We can now proceed with the proof of Lemma 20.1.

Let $z\in\Omega$ be given. We want to construct a sequence of paths γ_i, which we shall glue to each other to obtain γ_z.

We start from $z_0=z$, and distinguish between two cases. Set $r_0=10\operatorname{dist}(z_0,G)$. If $r_0>C_2^{-1}$, we can simply take for γ_x a parameterization with constant speed of any line segment $[z_0,z_1]$, where $|z_1-z_2|=\frac{1}{2}\operatorname{dist}(z_0,G)$.

So assume that $r_0\leqslant C_2^{-1}$. In this case we can apply Lemma 20.6 to the disk $B(z_0,r_0)$. We get a first arc $\gamma_0\subset B(z_0,C_2r_0)\smallsetminus G$, with $\operatorname{dist}(\gamma_0,G)\geqslant C_2^{-1}r_0$, and which connects z_0 to some point z_1 such that $\operatorname{dist}(z_1,G)\geqslant r_0$. We may as well assume that $\operatorname{length}(\gamma_0)\leqslant C_3\,r_0$, because otherwise we may replace γ_0 with a piecewise linear arc that satisfies this, maybe at the expense of multiplying C_2 by 2.

We want to continue this and construct a sequence of arcs γ_m, with the following properties. First, γ_m starts form a point $z_m\in\Omega$ (the endpoint of the previous arc if $m\geqslant 1$). Next,

$$\operatorname{dist}(\gamma_m,G)\geqslant C_2^{-1}r_m,\tag{20.17}$$

where $r_m=10\operatorname{dist}(z_m,G)$. Also,

$$\operatorname{length}(\gamma_m)\leqslant C_3\,r_m,\tag{20.18}$$

and finally γ_m ends at a point z_{m+1} such that

$$\operatorname{dist}(z_{m+1},G)\geqslant r_m=10\operatorname{dist}(z_m,G).\tag{20.19}$$

We continue this construction as long as we can apply Lemma 20.6, i.e., as long as $r_m\leqslant C_2^{-1}$. When this finally fails, we can complete our construction with a last segment $\gamma_m=[z_m,z_{m+1}]$ of length $\frac{1}{2}\operatorname{dist}(z_m,G)$.

We get an arc γ by concatenating the arcs γ_m defined above, and then the Lipschitz parameterization γ_z required for Lemma 20.1 is obtained by parameterizing γ with constant speed. Note that

$$10r_m\leqslant r_{m+1}\leqslant(10C_3+1)\,r_m\tag{20.20}$$

by (20.19) and (20.18). Hence the radii r_m increase geometrically, the total length of γ is comparable to the last radius (and hence, to 1), and the constant speed referred to above is bounded. This proves (20.3). For the same reasons,

$$C^{-1} r_{m-1} \leqslant t \leqslant C r_{m-1} \tag{20.21}$$

when $\gamma_z(t)$ lies in γ_m, $m \geqslant 1$; (20.4) easily follows from this and (20.17). This completes our proof of Lemma 20.1. □

In the special case of global minimizers (i.e., when (13.2) holds), we do not have the constraint that $r \leqslant C_2^{-1}$ in Lemma 20.6, and so we can construct our curves γ_m indefinitely. This proves Remark 20.5. □

21. Consequences on the moduli of continuity of functions

In this section we continue to assume that (v, G) satisfies (13.1) or (13.2), and we give ourselves a C^1-function f on $\Omega = \mathbb{R}^2 \setminus G$ and a disk $B = B(x_0, r_0)$. We assume that

$$|\nabla f(x)| \leqslant C_0\, r_0^{-1/2} + C_0 \operatorname{dist}(x, G)^{-1/2} \text{ for } x \in \Omega, \tag{21.1}$$

and we want to estimate $|f(x) - f(y)|$ when

$$x \text{ and } y \text{ lie in a same connected component of } B \setminus G. \tag{21.2}$$

If (13.1) holds, let us also assume that $r_0 \leqslant 1$.

Lemma 21.3. — *There is an absolute constant C_4 such that, with the notations and assumptions above,*

$$|f(x) - f(y)| \leqslant C_0\, C_4\, r_0^{1/2}. \tag{21.4}$$

Remark. — Although the authors are quite proud of the proof below, it is only fair to say that this is not the first occurrence of Hölder estimates in a John domain. See [HaKo] for similar (and anterior) arguments.

To prove the lemma, let B, x, y be as above. Set $B_1 = B(x_0, (1 + C_1)\, r_0)$, where C_1 is the same constant as in Lemma 20.1, and choose a finite subset E of $B_1 \setminus G$ which is $(10C_1)^{-1}\, r_0$-dense in $B_1 \setminus G$. We can choose E with less than C elements, where C depends only on C_1.

Let us also choose a continuous path $\xi : [0,1] \to B \setminus G$ such that $\xi(0) = x$ and $\xi(1) = y$.

For each $t \in [0,1]$, denote by $E(t)$ the set of points $z \in E$ which can be connected to $\xi(t)$ by a path $\gamma = \gamma_{t,z} : [0, r_0] \to \Omega$ with the properties

$$\gamma(0) = \xi(t)\ ,\ \gamma(r_0) = z, \tag{21.5}$$

$$\gamma \text{ is } 2C_1\text{-Lipschitz}, \tag{21.6}$$

and

$$\text{dist}\,(\gamma(s),G) \geqslant (10C_1)^{-1}\,\{s+\text{dist}\,(\xi(t),G)\} \ \text{ for } 0\leqslant s\leqslant r_0. \tag{21.7}$$

[Note the similarity with the conclusions of Lemma 20.1.] Let us first check that

$$E(t) \text{ is not empty.} \tag{21.8}$$

Indeed Lemma 20.1 gives us a path $\gamma_{\xi(t)}$ which starts from $\xi(t)$, is C_1-Lipschitz, and satisfies (20.4) (which is 10 times better than (21.7). Note that, by Remark 20.5 in particular, $\gamma_{\xi(t)}$ is defined on an interval at least as large as $[0,r_0]$. Its restriction to $[0,r_0]$ is almost what we need, except that its endpoints $z'=\gamma_{\xi(t)}(r_0)$ may not lie in E. On the other hand, (20.3) tells us that $z'\in B_1\setminus G$, and hence we can pick $z\in E$ such that $|z'-z|\leqslant(10C_1)^{-1}r_0$. Note that

$$\text{dist}\,([z',z],G) \geqslant \text{dist}(z',G)-|z'-z| \geqslant \frac{9}{10}\,C_1^{-1}\,r_0, \tag{21.9}$$

by (20.4). We now set

$$\gamma(s)=\gamma_{\xi(t)}\left(\frac{3s}{2}\right) \ \text{ for } 0\leqslant s\leqslant\frac{2r_0}{3}, \tag{21.10}$$

and

$$\gamma(s)=z'+\frac{3s-2r_0}{r_0}\,(z-z') \ \text{ for } \frac{2r_0}{3}<s\leqslant r_0. \tag{21.11}$$

It is easy to see that γ satisfies (21.5)-(21.7), because of the analogous properties of $\gamma_{\xi(t)}$ and (21.9). This proves (21.8).

Note that we kept some free room in our proof of (21.8). Because of this, the proof also shows that for all $t\in[0,1]$, we can find a point $z(t)\in E(t)$ such that

$$z(t)\in E(t') \text{ for all } t' \text{ in some neighborhood of } t \text{ in } [0,1]. \tag{21.12}$$

Indeed, we just have to modify γ above to make it start from $\xi(t')$ instead of $\xi(t)$. Note that we do not want to get any precise lower bound on the size of the neighborhood of t in (21.12).

Now let $f\in C^1(\Omega)$ be as in the beginning of this section, and let us check that

$$|f(\xi(t))-f(z)|\leqslant C\,C_0\,r_0^{1/2} \ \text{ for } z\in E(t). \tag{21.13}$$

Simply choose γ as in (21.5)-(21.7) and note that

$$\begin{aligned}
|f(\xi(t))-f(z)| &\leqslant \int_0^{r_0} |\nabla f(\gamma(s))|\,|\gamma'(s)|\,ds \\
&\leqslant 2C_1\,C_0\int_0^{r_0}\left\{r_0^{-1/2}+\ \text{dist}\,(\gamma(s),G)^{-1/2}\right\}ds \\
&\leqslant C\,C_0\left\{r_0^{1/2}+\int_0^{r_0} s^{-1/2}ds\right\}\leqslant C\,C_0\,r_0^{1/2}\,,
\end{aligned} \tag{21.14}$$

by (21.6), (21.1), and (21.7). Next set

$$E_1 = \{z \in E \ ; \ z \in E(t) \text{ for some } t \in [0,1]\}\,, \tag{21.15}$$

and then

$$t(z) = \sup\{t \ ; \ z \in E(t)\} \tag{21.16}$$

for $z \in E_1$. We still have that

$$|f(\xi(t(z))) - f(z)| \leqslant C\, C_0\, r_0^{1/2}, \tag{21.17}$$

by (21.13) and the continuity of $f \circ \xi$.

Our goal is to use (21.13) for diverse values of $t \in [0,1]$, to get an estimate on $|f(x) - f(y)| = |f(\xi(0)) - f(\xi(1))|$. We shall just need to be a little careful about the way z varies in such formulae.

Set $t_0 = 0$ and $z_0 = z\,(t_0)$, where $z\,(t_0)$ is as in (21.12). In particular $z_0 \in E\,(t_0)$ and (21.13) says that

$$|f(x) - f\,(z_0)| = |f\,(\xi\,(t_0)) - f\,(z_0)| \leqslant C\, C_0\, r_0^{1/2}. \tag{21.18}$$

Set $t_1 = t\,(z_0)$. Then $t_1 > t_0$, by definition of $z_0 = z\,(t_0)$ and by (21.12). Also,

$$|f\,(\xi\,(t_1)) - f\,(z_0)| = |f\,(\xi\,(t\,(z_0))) - f\,(z_0)| \leqslant C\, C_0\, r_0^{1/2} \tag{21.19}$$

by (21.17)

If $t_1 = 1$ we can stop here, because then $\xi\,(t_1) = \xi(1) = y$ and (21.4) follows from (21.18) and (21.19). So let us assume that $t_1 < 1$, and set $z_1 = z\,(t_1)$. Since $z_1 \in E\,(t_1)$, (21.13) says that

$$|f\,(\xi\,(t_1)) - f\,(z_1)| \leqslant C\, C_0\, r_0^{1/2}. \tag{21.20}$$

Note that $z_1 \neq z_0$, because by definition (21.12) of $z_1 = z\,(t_1)$, $z_1 \in E(t')$ for values of $t' > t_1$. This would not be possible if $z_1 = z_0$, by definition of $t_1 = t\,(z_0)$.

We can continue the construction in the same manner. We define a sequence of parameters $t_i \in [0,1]$ by

$$t_i = t\,(z_{i-1})\,, \tag{21.21}$$

and at the same time a sequence of points $z_i \in E_1$ by

$$z_i = z\,(t_i)\,. \tag{21.22}$$

We stop as soon as we get $t_k = 1$ for some k, and then we do not even define z_k.

First observe that $\{t_i\}$ is increasing, because $t_i = t\,(z_{i-1})$ and $z_{i-1} = z\,(t_{i-1})$. [This uses the definition (21.12) and the fact that $t_{i-1} < 1$ if t_i is ever defined.]

Also, each z_i, $i < k$, is different from all its predecessors. Indeed $z_i = z\,(t_i)$, hence $z_i \in E(t')$ for values of $t' > t_i$ (note that $t_i < 1$ because $i \neq k$). But for all $j < i$, $t_i \geqslant t_{j+1} = t\,(z_j)$, and hence z_j could not lie in $E(t')$ for any $t' > t_i$, as z_i does. So $z_i \neq z_j$, as needed.

Since $E_1 \subset E$ has at most C elements and all the z_i are distinct, our construction must stop for some $k \leqslant C$. In the mean time, we have the following estimates. First

$$|f(\xi(t_i)) - f(z_i)| \leqslant C\, C_0\, r_0^{1/2} \text{ for } i < k, \tag{21.23}$$

because $z_i = z(t_i) \in E(t_i)$ and by (21.13), and

$$|f(\xi(t_{i+1})) - f(z_i)| \leqslant C\, C_0\, r_0^{1/2} \text{ for } i < k, \tag{21.24}$$

because $\xi(t_{i+1}) = \xi(t(z_i))$ by (21.21), and by (21.17).

Note that $\xi(t_k) = \xi(1) = y$ and $\xi(t_0) = \xi(0) = x$. We can now add up all the inequalities (21.23) and (21.24), and get the desired estimate (21.4) . [Note that there are no more than $2C$ inequalities to add up, because $k \leqslant C$.]

This completes our proof of Lemma 21.3. □

CHAPTER F

THE CONJUGATED FUNCTION w

22. The conjugated function w

We continue with the hypotheses (13.1) or (13.2). Since v minimizes $\int_\Omega |\nabla v|^2$ locally, it is harmonic and satisfies the usual Neumann condition $\partial v/\partial n = 0$ on G. This will allow us to define a conjugated function w, such that

$$f(z) = v(z) + iw(z) \text{ is holomorphic on } \Omega. \tag{22.1}$$

The construction is very general, and only requires the Neumann condition, so it is probably very classical as well.

We want to define w separately on each component Ω_i of $\Omega = \mathbb{R}^2 \setminus G$ by

$$\omega(z) = C_i + \int_{I_\gamma} \nabla v\,(\gamma(t))\,(-i\gamma'(t))\,dt, \tag{22.2}$$

where C_i is a constant, $\gamma : I_\gamma \to \Omega_i$ is an arc of class C^1 that goes from some origin $z_i \in \Omega_i$ to z, and where $\nabla v\,(\gamma(t))\,(-i\gamma'(t))$ denotes the derivative of v at $\gamma(t)$, applied to $-i\gamma'(t)$, a vector orthogonal to the tangent to the curve γ at $\gamma(t)$. [We are abusing notation here, because we identify freely $\mathbb{R}^2$ with the complex plane.] Of course (22.2) is the standard way of recuperating a holomorphic (or harmonic) function from its derivative, except that we modified it a little to recover w to v.

Lemma 22.3. — *The formula* (22.2) *defines (for each choice of constants C_i) a harmonic function w on Ω such that* (22.1) *holds.*

What is almost obvious here is that locally (22.2) defines a harmonic function w conjugated to v. We only need to check that the definition by (22.2) is coherent, i.e., that two choices of C^1 arcs γ_1 and γ_2 as above always give the same value of $w(z)$. In fact, it is enough to prove that

$$\int_{I_{\gamma_1}} \nabla v\,(\gamma_1(t))\,(-i\gamma_1'(t))\,dt = \int_{I_{\gamma_2}} \nabla v\,(\gamma_2(t))\,(-i\gamma_2'(t))\,dt \tag{22.4}$$

for all choices of two C^1-arcs γ_1 and γ_2 supported in Ω, with the same origins and the same endpoints. We can even assume that γ_1 and γ_2 are simple and only meet at their extremities; the general case follows by a slightly painful, but neither hard nor surprising argument. We skip the details. Thus it is enough to show that

$$\int_\gamma \nabla v\,(\gamma(t))\,(-i\gamma'(t))\,dt = 0 \tag{22.5}$$

for every C^1, simple loop γ in Ω. So let γ be such a loop, and denote by W the bounded Jordan domain bounded by γ. We may apply Green in the domain $W \setminus G$ and find that

$$\begin{aligned}\int_{I_\gamma} \nabla v\,(\gamma(t))\,(-i\gamma'(t))\,dt &= \pm\int_\gamma \frac{\partial v}{\partial n} dH^1 \\ &= \pm\int_{\partial(W\setminus G)} \frac{\partial v}{\partial n} dH^1 = \pm\int_{W\setminus G} \Delta v = 0,\end{aligned} \tag{22.6}$$

where the sign $\pm$ would be easy to compute, but does not matter here.

Here we used the fact that $\partial v/\partial n = 0$ on $G \cap W$ (the missing part of $\partial(W \setminus G)$). The reader may be worried because we have again applied Green on a domain that is not too smooth. Since we feel too bad about doing this repeatedly, we shall give in Section 24 a proof of (22.6) that only uses integrations by parts on smooth domains (and a small limiting argument). See also [MoSo] for a justification of the Green formula in our context.

This completes our discussion of the proof of Lemma 22.3.

23. How to surround a compact set with curves

In this section we give ourselves a compact set G^0 in the plane and, for small values of $\varepsilon > 0$, surround G^0 with a finite collection of curves Γ_ε^j, $j \in J(\varepsilon)$, at distance about ε from G^0. We intend to apply this to various compact subsets G^0 of G (where (v, G) satisfies (13.1) or (13.2)), to justify some of our integrations by parts, but also to finesse some issues related to the existence and regularity of boundary values of v and w on G. The construction below is of course standard.

So let G^0 be a compact set in the plane, and $\varepsilon > 0$ be given. First we want to cover

$$G_\varepsilon^0 = \left\{x \in \mathbb{R}^2 \ ; \ \operatorname{dist}\left(x, G^0\right) \leqslant \varepsilon\right\} \tag{23.1}$$

with disks B_i, $i \in I(\varepsilon)$, centered on G^0. Choose points x_i, $i \in I(\varepsilon)$, on G^0, so that

(23.2) every point of G^0 lies at distance $\leqslant 10^{-2}\varepsilon$ from some $x_i, i \in I(\varepsilon)$.

Since G^0 is compact, we can do this with a finite set $I(\varepsilon)$. Let us even choose our points $x_i, i \in I(\varepsilon)$, at mutual distances $\geqslant 2 \cdot 10^{-3}\varepsilon$, say, so that

(23.3) the $B\left(x_i, 10^{-3}\varepsilon\right)$, $i \in I(\varepsilon)$, are disjoint.

This will be useful later.

For each $i \in I(\varepsilon)$, choose a radius $r_i \in [3\varepsilon/2, 2\varepsilon]$ and set $B_i = B(x_i, r_i)$. We can use our latitude in the choice of r_i to ensure that

$$\text{the circles } \partial B_i \text{ never meet tangentially,} \tag{23.4}$$

and

$$\partial B_{i_1} \cap \partial B_{i_2} \cap \partial B_{i_3} = \varnothing \text{ whenever } i_1, i_2, i_3 \in I(\varepsilon) \text{ are distinct.} \tag{23.5}$$

Next set

$$H(\varepsilon) = \bigcup_{i \in I(\varepsilon)} B_i. \tag{23.6}$$

Obviously

$$G_\varepsilon^0 \subset H(\varepsilon) \subset \left\{x \in \mathbb{R}^2 \; ; \; \operatorname{dist}\left(x, G^0\right) < 2\varepsilon\right\} \tag{23.7}$$

because of (23.2). Also denote by $U(\varepsilon)$ the unbounded connected component of $\mathbb{R}^2 \setminus \overline{H}(\varepsilon)$.

Let us first study the boundary $\partial U(\varepsilon)$. Clearly $\partial U(\varepsilon)$ is composed of arcs of circles coming from the ∂B_i, $i \in I(\varepsilon)$. There is only finitely many of these arcs, and they can only meet transversally (by (23.4)); furthermore, there is no triple point (by (23.5)).

Denote by Γ_ε^j, $j \in J$, the connected components of $\partial U(\varepsilon)$. We want to check first that

$$\text{each } \Gamma_\varepsilon^j, j \in J, \text{ is a Jordan curve.} \tag{23.8}$$

Let $z \in \Gamma_\varepsilon^j$ be given. Assume first that z only lies on one circle ∂B_i. Near z, we must have $H(\varepsilon)$ on one side of ∂B_i (the side that contains $B_i \subset H(\varepsilon)$), and $U(\varepsilon)$ on the other side (because otherwise z would not lie on $\partial U(\varepsilon)$). In this case, there is a small neighborhood of z on which $\partial U(\varepsilon)$ coincides with ∂B_i, with $U(\varepsilon)$ on one side and $H(\varepsilon)$ on the other side.

If z lies on more than one circle, we can find $i, \ell \in I(\varepsilon)$ such that $z \in \partial B_i \cap \partial B_\ell$. In this case, ∂B_i meets ∂B_ℓ transversally at z, and z does not lie on any other ∂B_j (by (23.4) and (23.5)). Then, in a small neighborhood of z, $H(\varepsilon)$ coincides with $B_i \cup B_\ell$, $U(\varepsilon)$ coincides with the complement of $\overline{B}_i \cup \overline{B}_\ell$, and $\partial U(\varepsilon)$ coincides with $\partial(B_i \cup B_\ell)$. See Figure 23.1.

Thus every point $z \in \Gamma_\varepsilon^j$ has a neighborhood where $\partial U(\varepsilon)$ is a simple, piecewise C^∞ curve; (23.8) follows easily because $\partial U(\varepsilon)$ is bounded. Note that

$$\Gamma_\varepsilon^j \subset \partial U(\varepsilon) \subset \partial H(\varepsilon) \tag{23.9}$$

by definitions; hence (23.7) and (23.1) imply that

$$\varepsilon \leqslant \operatorname{dist}\left(x, G^0\right) \leqslant 2\varepsilon \text{ for all } x \in \Gamma_\varepsilon^j, j \in J. \tag{23.10}$$

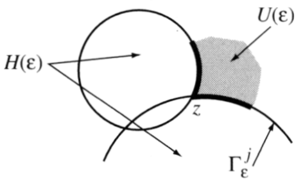


FIGURE 23.1

Next denote by Ω_ε^j, $j \in J$, the bounded connected component of $\mathbb{R}^2 \setminus \Gamma_\varepsilon^j$. [Recall from (23.8) that Γ_ε^j is a Jordan curve.] Let us now check that

(23.11) $\quad \mathbb{R}^2$ is the disjoint union of $U(\varepsilon)$, the $\Omega_\varepsilon^j, j \in J$, and the $\Gamma_\varepsilon^j, j \in J$.

Let us first check that

$$U(\varepsilon) \cap \overline{\Omega}_\varepsilon^j = \varnothing \quad \text{for } j \in J. \tag{23.12}$$

Let $z \in U(\varepsilon)$ and $j \in J$ be given. Let γ be a path from z to ∞ in $\mathbb{R}^2 \setminus \overline{H(\varepsilon)}$; such a path exists by definition of $U(\varepsilon)$. Since $\Gamma_\varepsilon^j \subset \partial H(\varepsilon)$ (see (23.9)), γ does not meet Γ_ε^j, and hence z lies in the unbounded component of $\mathbb{R}^2 \setminus \Gamma_\varepsilon^j$; (23.12) follows.

Next we want to check that for $j, k \in J$, $j \neq k$,

$$\overline{\Omega}_\varepsilon^j \cap \overline{\Omega}_\varepsilon^k = \varnothing. \tag{23.13}$$

First notice that

$$\Gamma_\varepsilon^j \cap \Omega_\varepsilon^k = \varnothing, \tag{23.14}$$

because (23.12) tells us that every point of Ω_ε^k has a neighborhood which does not meet $U(\varepsilon)$, while on the other hand $\Gamma_\varepsilon^j \subset \partial U(\varepsilon)$.

Of course Γ_ε^j does not meet Γ_ε^k (because these are distinct connected components of $\partial U(\varepsilon)$); hence $\Gamma_\varepsilon^j \cap \overline{\Omega}_\varepsilon^k = \varnothing$. Similarly, $\Gamma_\varepsilon^k \cap \overline{\Omega}_\varepsilon^j = \varnothing$. So we only need to prove that $\Omega_\varepsilon^j \cap \Omega_\varepsilon^k = \varnothing$ to establish (23.13). If this failed, Ω_ε^j (which is connected and does not meet $\partial\Omega_\varepsilon^k = \Gamma_\varepsilon^k$) would be contained in Ω_ε^k. Similarly, Ω_ε^k would be contained in Ω_ε^j, and so $\Omega_\varepsilon^j = \Omega_\varepsilon^k$ and $\Gamma_\varepsilon^j = \partial\Omega_\varepsilon^j = \partial\Omega_\varepsilon^k = \Gamma_\varepsilon^k$. This is of course impossible when $j \neq k$; hence $\Omega_\varepsilon^j \cap \Omega_\varepsilon^k = \varnothing$ and finally (23.13) holds.

So far we proved that all the sets mentioned in (23.11) are disjoint; we still need to check that their union is the whole plane.

Let $z \in \mathbb{R}^2$ be given, and let γ be a C^1-arc from z to ∞. If γ does not meet any Γ_ε^j, then it does not meet their union $\partial U(\varepsilon)$, and hence $z \in U(\varepsilon)$ (because $U(\varepsilon)$ contains a neighborhood of ∞). So let us assume that γ meets some Γ_ε^j, and denote by z_1 the first point of γ (when we leave from z) which lies on some Γ_ε^j. We may assume that $z_1 \neq z$, because otherwise $z \in \Gamma_\varepsilon^j$ and we are happy. Note that near z_1, Γ_ε^j is a piecewise smooth curve, with $U(\varepsilon)$ on one side of Γ_ε^j and consequently Ω_ε^j on the other side (because it cannot be on the same side, by (23.12)). Then z lies in $U(\varepsilon)$ or in Ω_ε^j, depending on the position relative to Γ_ε^j of points of γ just before z_1. This proves (23.11). □

Next we want to check that

$$G^0 \subset G_\varepsilon^0 \subset H(\varepsilon) \subset \bigcup_{j\in J} \Omega_\varepsilon^j. \tag{23.15}$$

where G_ε^0 and $H(\varepsilon)$ are as in (23.1) and (23.6). The first inclusions come from (23.1) and (23.7), and so it is enough to show that for each of the balls B_i, $i \in I(\varepsilon)$, that compose $H(\varepsilon)$,

$$B_i \subset \Omega_\varepsilon^j \text{ for some } j \in J. \tag{23.16}$$

By definition of $U(\varepsilon)$ (as a component of $\mathbb{R}^2 \setminus \overline{H(\varepsilon)}$), B_i does not meet $U(\varepsilon)$. Since B_i is open, it does not meet $\partial U(\varepsilon)$ either. Since $\partial U(\varepsilon)$ is the union of the Γ_ε^j, (23.11) tells us that B_i is contained in the union of the Ω_ε^j, $j \in J$. But this is a disjoint union of open sets, and B_i is connected. Hence B_i is contained in a single Ω_ε^j, as needed for (23.16) and (23.15).

Remark 23.17. — When G^0 is connected, J has only one element, i.e., $\partial U(\varepsilon)$ is connected and composed of a single Jordan curve Γ_ε.

To prove this, first observe that G^0 is connected and contained in the disjoint union of open sets Ω_ε^j (by (23.15)). Hence G^0 is contained in a single Ω_ε^j.

Let us also check that every Ω_ε^j contains some B_i. Indeed, pick any point $z \in \Gamma_\varepsilon^j = \partial\Omega_\varepsilon^j$. Then z lies on some ∂B_i, and near z we must have $U(\varepsilon)$ one one side of Γ_ε^j (because $\Gamma_\varepsilon^j \subset \partial U(\varepsilon)$). This must be the side that does not meet B_i, because $U(\varepsilon)$ does not meet $H(\varepsilon)$. We must also have Ω_ε^j on one side of Γ_ε^j, by definition of Ω_ε^j, and this cannot be on the same side as $U(\varepsilon)$, by (23.12). Hence B_i meets Ω_ε^j, and (23.16) (and the disjointness of the Ω_ε^j) says that $B_i \subset \Omega_\varepsilon^j$, as announced.

Now every B_i meets G^0 (at its center, for instance), and hence all Ω_ε^j must meet G^0. Since we know that G^0 is contained in a single Ω_ε^j, the remark follows.

It will be convenient to apply the construction above with a sequence $\{\varepsilon_n\}$ that tends to 0, and for which the corresponding domains $U(\varepsilon_n)$ are nested and converge to their natural limit Ω^0.

Lemma 23.18. — *Let G^0 be a compact subset of the plane. We can find a decreasing sequence $\{\varepsilon_n\}$ that tends to 0 and for which*

$$\overline{U(\varepsilon_n)} \subset U(\varepsilon_{n+1}) \text{ for all } n, \tag{23.19}$$

and

$$\Omega_0 = \bigcup_n U(\varepsilon_n), \tag{23.20}$$

where Ω_0 denotes the unbounded connected component of $\mathbb{R}^2 \setminus G^0$, and the $U(\varepsilon_n)$ are constructed as above.

To prove the lemma, let us first check that for all $\varepsilon > 0$,

$$\overline{U}(\varepsilon) \subset \Omega_0. \tag{23.21}$$

Since $U(\varepsilon)$ is connected (by definition), contains a neighborhood of ∞, and does not meet $\partial\Omega_0 \subset G^0$ (by (23.15) and (23.12)), we get that $U(\varepsilon) \subset \Omega_0$. Hence $\overline{U(\varepsilon)} \subset \overline{\Omega}_0$, and (23.21) will follow as soon as we show that $\partial U(\varepsilon)$ does not meet $\partial\Omega_0 \subset G^0$. But $\partial U(\varepsilon)$ is the union of the Γ_ε^j, while G^0 is contained in the union of the Ω_ε^j, by (23.15). Thus (23.21) follows from the disjointness property in (23.11) or (23.14).

Next we want to prove that

$$\begin{cases} \text{for each compact subset } T \text{ of } \Omega_0, \text{there exists} \\ \varepsilon_0 > 0 \text{ such that } T \subset U(\varepsilon) \text{ for all } \varepsilon \leqslant \varepsilon_0. \end{cases} \tag{23.22}$$

Let T be any compact subset of Ω_0, set $\tau = \operatorname{dist}\left(T, G^0\right) > 0$, and cover T with finitely many disks D_ℓ centered on T and with radius $\tau/2$. For each ℓ, choose a path γ_ℓ in Ω_0 that connects D_ℓ to ∞. Then $\tau_\ell = \operatorname{dist}\left(\gamma_\ell, G^0\right) > 0$. Set

$$\varepsilon_0 = \operatorname{Min}\left\{\frac{\tau}{6}, \frac{1}{3} \operatorname{Min}_\ell \tau_\ell\right\}. \tag{23.23}$$

Thus for every point $x \in T$ there is a path γ_x from x to ∞ that stays at distance $\geqslant 3\varepsilon_0$ from G^0.

If $\varepsilon \leqslant \varepsilon_0$ and $x \in T$, γ_x does not meet any Γ_ε^j (by (23.10) and because $\operatorname{dist}\left(\gamma_x, G^0\right) \geqslant 3\varepsilon$). Therefore x does not lie in any Ω_ε^j, $j \in J$. We already know that it does not lie on any Γ_ε^j; thus the only option left by (23.11) is that $x \in U(\varepsilon)$. This proves (23.22).

We are now ready to prove Lemma 23.18. If we choose $\{\varepsilon_n\}$ so that ε_n tends to 0, then we shall get (23.20) automatically, by (23.21) and (23.22).

The other condition (23.19) is also easy to obtain. First we can find $R > 0$ such that $\mathbb{R}^2 \setminus B_R \subset U(\varepsilon)$ when $\varepsilon \leqslant 1$, say. Then assume that ε_n has already been defined, and apply (23.22) with $T_n = \overline{U(\varepsilon_n)} \cap \overline{B}_R$. We get that (23.19) holds if we choose ε_{n+1} small enough. A simple iteration gives the lemma. □

Remark 23.24. — In the typical application of the construction above, G^0 will be a compact piece of G such that

$$\operatorname{dist}\left(G^0, G \setminus G^0\right) > 0. \tag{23.25}$$

In this case, we get that

$$\Gamma_\varepsilon^j \subset \Omega = \mathbb{R}^2 \setminus G \text{ for all } j \in J \tag{23.26}$$

as soon as ε is small enough, by (23.10).

We shall also use the construction with sets G^0 like

$$G^0 = G \cap \overline{B}, \tag{23.27}$$

where $\overline{B}$ is a closed disk such that

$$G \cap \partial B = \{x_0\}, \tag{23.28}$$

$$x_0 \text{ is a regular point of } G, \tag{23.29}$$

and

$$G \text{ is transverse to } \partial B \text{ at } x_0. \tag{23.30}$$

[A typical situation of this type is when (13.1) holds, $B = B_R$ for some very large R, and then x_0 is some point of L.] In the present situation, we do not have (23.26), but we still have a good control on the way the Γ_ε^j may meet G. From (23.10) we deduce that for ε small enough,

$$\operatorname{dist}(z, G) \geqslant \varepsilon \text{ for all } z \in \partial U(\varepsilon) \setminus B(x_0, 5\varepsilon). \tag{23.31}$$

On the other hand, a close look at the construction of $U(\varepsilon)$ from $H(\varepsilon)$ and the B_i gives that for ε small enough,

$$\begin{cases} B(x_0, 5\varepsilon) \cap \partial U(\varepsilon) \text{ is composed of a single, piecewise } C^\infty \text{ arc of} \\ \text{some } \Gamma_\varepsilon^j, \text{ which cuts } G \text{ transversally in a single point of } B(x_0, 2\varepsilon). \end{cases} \tag{23.32}$$

See Figure 23.2.

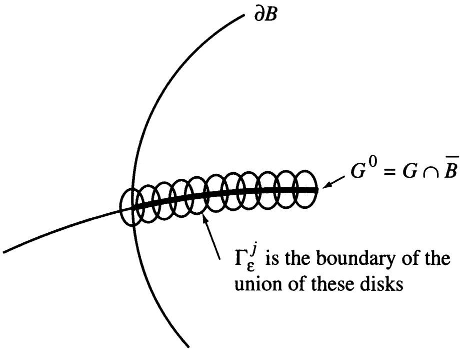


FIGURE 23.2

24. A limiting argument to integrate by parts

Let us rapidly fulfill our promise of proving (22.6) without integrating by parts on nonsmooth domains. So let γ be a C^1, simple loop in $\Omega = \mathbb{R}^2 \setminus G$ and let us try to prove that

$$\int_\gamma \frac{\partial v}{\partial n} dH^1 = 0, \tag{24.1}$$

where the unit normal is associated to the domain W bounded by γ.

Set $G^0 = G \cap W$; this is a compact subset of W. Define a sequence $\{\varepsilon_n\}$ and let $U(\varepsilon_n)$ be as in Lemma 23.18. Since $\gamma \subset \Omega_0$ (the unbounded component of $\mathbb{R}^2 \setminus G^0$), γ is contained in $U(\varepsilon_n)$ for n large. Let us only keep the indices n for which $\gamma \subset U(\varepsilon_n)$, and set

$$W_n = W \cap U(\varepsilon_n). \tag{24.2}$$

By (23.19) and (23.20), $\{W_n\}$ in an increasing sequence of piecewise smooth domains, and

$$\bigcup_n W_n = W^0, \tag{24.3}$$

where W^0 is the intersection of W with the unbounded component of $\mathbb{R}^2 \setminus G^0$.

The boundary ∂W_n consist of γ (the exterior boundary), and a finite collection of curves $\Gamma^j_{\varepsilon_n}$ (which all lie inside W, by (23.10)).

Let v_n denote the harmonic function on W_n which has boundary values on γ equal to the values of v there, and minimizes

$$E_n = \int_{W_n} |\nabla v_n|^2 \tag{24.4}$$

under these constraints. The existence of v_n can be proved with a reasonably simple convexity argument; the main point is that our constraint on the boundary values of v_n on γ defines a closed affine subspace of the Hilbert space of functions on W_n with finite energy (as in the definition of E_n); v_n is the closest point of this subspace to the origin.

Also, v_n has piecewise C^1 extensions to the curves $\Gamma^j_{\varepsilon_n}$ that compose the inner boundary of W_n, and $\partial v_n/\partial n = 0$ on those. This is also easy to prove, for instance with the conformal invariance techniques of Section 14. Thus we can integrate by parts, as follows.

Let $\widetilde{\gamma}$ be a C^1, Jordan curve in W_n; we are only interested in the case when $\widetilde{\gamma}$ is very close to γ, but this does not matter. Denote by $\widetilde{W}$ the bounded component of $\mathbb{R}^2/\widetilde{\gamma}$, and set $\widetilde{W}_n = \widetilde{W} \cap W_n = \widetilde{W} \cap U(\varepsilon_n)$ (by (24.2)). Then

$$\int_{\widetilde{\gamma}} \frac{\partial v_n}{\partial n} dH^1 = \int_{\partial \widetilde{W}_n} \frac{\partial v_n}{\partial n} dH^1 = \int_{\widetilde{W}_n} \Delta v_n = 0 \tag{24.5}$$

because $\partial v_n/\partial n = 0$ on the inside boundary of $\widetilde{W}_n$, by Green, and because v_n is harmonic on W_n.

We want to deduce (24.1) from this by taking limits. Set

$$E = \int_{W\setminus G} |\nabla v|^2 = \int_{W^0} |\nabla v|^2, \tag{24.6}$$

where W^0 is, as for (24.3), the intersection of W with the unbounded component of $\mathbb{R}^2 \setminus G^0$. In fact, $W^0 = W \setminus G$ because $\mathbb{R}^2 \setminus G$ (and hence $\mathbb{R}^2 \setminus G^0$) does not have any bounded connected component (see Section 15), but even if we did not know this, we would still have the second equality in (24.6), because v would have to be constant on the bounded components of $\mathbb{R}^2 \setminus G^0$, by the local energy minimizing property of v (and the fact that γ does not meet these bounded components).

Since W_n is an increasing sequence of domains and W^0 is its limit (by (24.3)), $\{E_n\}$ is a nondecreasing sequence and $E_n \leqslant E$ (because v is a competitor in the definition of v_n and E_n, for instance). Since E_n is uniformly bounded (by E) and v_n is harmonic on W_n, we can extract a subsequence so that (after extraction) $\{v_n\}$ converges to some limit v_∞, uniformly on every compact subset of W^0. Then

$$\int_{W^0} |\nabla v_\infty|^2 \leqslant \liminf_{n\to\infty} \int_{W_n} |\nabla v_n|^2 = \liminf_{n\to\infty} E_n \leqslant E, \tag{24.7}$$

by Fatou.

It is not difficult to check that v_∞ also has boundary values on γ equal to v. So v_∞ does at least as well as v in terms of minimizing the energy E on W^0. By uniqueness of such minimizers, $v_\infty = v$ on W^0. We can also deduce from (24.7) that

$$E = \lim_{n\to\infty} E_n, \tag{24.8}$$

but what we really wanted to know here is that $\{v_n\}$ converges to v in W^0. Because we know that these functions are harmonic, this also implies that ∇v_n converges to ∇v, uniformly on compact subsets of W^0. Then

$$\int_{\widetilde{\gamma}} \frac{\partial v}{\partial n} dH^1 = 0 \tag{24.9}$$

for all the C^1, Jordan arcs $\widetilde{\gamma}$ for which we established (24.5).

We can now deduce (24.1) from (24.9) by letting $\widetilde{\gamma}$ tend to γ. This completes our proof of (22.6). □

Note that the argument of this section is sufficiently general to be applied in the situations above where we have used integrations by parts on domain delimited by G (or similar sets coming from minimizers) to compute differences of energy. See in particular the proof of local Ahlfors-regularity (Proposition 4.14) in Section 4, but also similar computations in Sections 5 and 7.

25. v and w have limits at points where G is not connected

With this section we start an investigation of the boundary values of v and its conjugate function w. We continue to assume that (v, G) satisfies (13.1) or (13.2). We shall call "point where G is not connected" a point $x_0 \in G$ such that $\{x_0\}$ is a connected component of G.

We keep the notation $f = v + iw$, as in Section 22.

Proposition 25.1. — *If $x_0 \in G$ is a point where G is not connected, then the limit*

$$f^*(x_0) = \lim_{\substack{z \to x_0 \\ z \in \mathbb{R}^2 \setminus G}} f(x) \tag{25.2}$$

exists.

First we want to put ourselves in position to apply the construction of Section 23 and surround x_0 with curves that do not meet G. The following lemma is a little too general for the present situation (where we could take $G_0 = \{x_0\}$), but we shall use it again later.

Lemma 25.3. — *Let G_0 be a bounded connected component of G. For every $\delta > 0$, we can find a closed set G^0 such that*

$$G_0 \subset G^0 \subset \{z \in G \ ; \ \operatorname{dist}(z, G) < \delta\} \tag{25.4}$$

and

$$\operatorname{dist}\left(G^0, G \setminus G^0\right) > 0. \tag{25.5}$$

Since we did not find any simple, purely topological proof of this lemma, we shall use the rectifiability of G and the fact that $H^1(G)$ is locally finite. The following argument is a minor modification of an argument of [DaSe2], to which we also refer the reader for additional detail. First define a sort of distance to G_0 by

$$d(x) = \inf\left\{H^1(\gamma \setminus G) \ ; \ \gamma \text{ is a simple, rectifiable arc from } x \text{ to } G_0\right\}. \tag{25.6}$$

It is easy to check that $d(x)$ is a 1-Lipschitz function of x, that its restriction to G is differentiable at every regular point of G, and that the derivative of d at these points is zero. Since almost-every point of G is regular (by Proposition 13.11) and d is Lipschitz, a minor modification of Sard's theorem gives that

$$H^1(d(G)) = 0. \tag{25.7}$$

Next we check that

$$d(x) = 0 \text{ if and only if } x \in G_0. \tag{25.8}$$

It is clear that $d(x) = 0$ on G_0. Conversely, suppose that $d(x) = 0$. For every $n \geqslant 0$, choose a simple rectifiable arc γ_n from x to some point of G_0, such that

$$H^1(\gamma_n \setminus G) \leqslant 2^{-n}. \tag{25.9}$$

First assume that γ_n stays in a fixed disk B_R. Then $H^1(\gamma_n)$ stays bounded (because $H^1(G \cap B_R) < +\infty$), so we can find uniformly Lipschitz parameterizations $\xi_n : [0,1] \to \mathbb{R}^2$ of the γ_n. Then we can find a subsequence of $\{\xi_n\}$ that converges uniformly to some limit ξ. Note incidentally that

$$G_0 \text{ is closed,} \tag{25.10}$$

just like any connected component of G (because $\overline{G}_0$ would contain G_0 and still be connected). Thus the arc γ parameterized by ξ still connects x to G_0. If γ was not contained in G, we could find y on γ, at distance $d > 0$ from G, and then we would have $H^1(\gamma_n \setminus G) > d/2$ for n large enough (because γ_n would have to cross at least half of $B(y,d)$ to get close to y); this contradiction with (25.9) shows that $\gamma \subset G$. But then $\gamma \subset G_0$ (which is a connected component of G), and $x \in G_0$. This takes care of the case when the arcs γ_n stay in a fixed disk B_R.

Suppose now that infinitely many arcs γ_n leave B_R, where we choose R so large that $G_0 \cup \{x\} \subset B_{R-1}$. Of course we may as well extract a subsequence and so we can assume that all γ_n leave B_R before they reach G_0. [We can always assume that γ_n does not meet G_0 before its endpoint.] Then we can find a sub-arc γ_n' of γ_n that is contained in $\overline{B}_R$ and connects ∂B_R to G_0. The same argument as above gives a limiting arc $\gamma \subset G$ which still connects ∂B_R to G_0. This is impossible because G_0 is a connected component of G and by definition of R. Thus the γ_n were all contained in a fixed disk, as in our first case. This proves (25.8).

Choose a decreasing sequence $\{t_n\}$ of positive numbers such that $t_n \notin d(G)$ for all n and t_n tends to 0. This can be done, by (25.7). Set

$$V_n = \{x \in \mathbb{R}^2 \ ; \ d(x) \leqslant t_n\}. \tag{25.11}$$

Note that V_n is compact, a least for n large. Indeed, if R is such that $G_0 \subset B_{R-1}$, say, then $\rho = \inf_{\partial B_R} d(x)$ is positive, and $V_n \subset B_R$ as soon as $t_n < \rho$. By (25.8),

$$\bigcap_n V_n = G_0, \tag{25.12}$$

hence for every fixed δ

$$V_n \subset \{x \in \mathbb{R}^2 \ ; \ \operatorname{dist}(x, G_0) < \delta\} \tag{25.13}$$

for n large enough.

Choose $G^0 = G \cap V_n$ for such an n. Then (25.4) follows from (25.13), and we only need to check (25.5). But

$$\sup\{d(x) \ ; \ x \in G^0\} < t_n, \tag{25.14}$$

because the supremum in question is attained (since G^0 is compact and d is continuous), and because $t_n \notin d(G)$. Then for $x \in G^0$ and $y \in G \setminus G^0$,

$$|x-y| \geqslant |d(x) - d(y)| \geqslant t_n - d(x) \geqslant t_n - \sup\{d(x) \ ; \ x \in G^0\} > 0, \tag{25.15}$$

which implies (25.5). Lemma 25.3 follows. □

Remark 25.16. — We can also apply the proof of Lemma 25.3 when G is replaced with some closed subset $\widetilde{G}$ of G, and G_0 is a bounded connected component of $\widetilde{G}$. In the conclusion, we just need to replace (25.5) with

$$\operatorname{dist}\left(G^0, \widetilde{G} \setminus G^0\right) > 0. \tag{25.17}$$

Return to the proof of Proposition 25.1. Let $x_0 \in G$ be a point where G is not connected, and let $\delta > 0$ be small. Let $G^0 \subset G \cap B(x_0, \delta)$ denote the piece of G given by Lemma 25.3 applied with $G_0 = \{x_0\}$. Apply to G^0 the construction of Section 23, where we choose $\varepsilon = \varepsilon(\delta)$ so small that the curves Γ_ε^j that compose $\partial U(\varepsilon)$ do not meet G. [See Remark 23.24, and in particular (23.26).] Let us also choose ε so small that $\partial U(\varepsilon) \subset B(x_0, 2\delta)$. This is possible, by (23.10) and because $G^0 \subset B(x_0, \delta)$.

By (23.15), x_0 is contained in some Ω_ε^j, which we shall denote by $\Omega(\delta)$. Also denote by $\Gamma(\delta) = \partial\Omega(\delta)$ the corresponding curve Γ_ε^j. By construction, $\Gamma(\delta) \subset B(x_0, 2\delta)$, and hence

$$\overline{\Omega(\delta)} \subset B(x_0, 2\delta). \tag{25.18}$$

Let us now verify that

$$\operatorname{osc}\left(f\ ;\ \overline{\Omega(\delta)} \setminus G\right) := \sup\left\{|f(x) - f(y)|\ ;\ x, y \in \overline{\Omega(\delta)} \setminus G\right\} \leqslant C\,\delta^{1/2}. \tag{25.19}$$

We want to use Lemma 21.3, and so we need to check the hypothesis (21.1). First observe that

$$\int_{B(x,r)\setminus G} |\nabla f|^2 = 2 \int_{B(x,r)\setminus G} |\nabla v|^2 \leqslant 8\pi\lambda r \tag{25.20}$$

for all disks $B(x, r)$, with the only constraint that $r \leqslant 1$ when (13.1) holds. The equality follows from the definition of f; for the inequality see (13.5).

If $x \in \Omega = \mathbb{R}^2 \setminus G$ we can apply this with $r = \operatorname{dist}(x, G)$ when (13.2) holds, or $r = \operatorname{Min}(1, \operatorname{dist}(x, G))$ when (13.1) holds. Since f is holomorphic, we get that

$$\begin{aligned} |\nabla f(x)| &\leqslant |B(x,r)|^{-1} \int_{B(x,r)\setminus G} |\nabla f| \\ &\leqslant \left\{|B(x,r)|^{-1} \int_{B(x,r)\setminus G} |\nabla f|^2\right\}^{1/2} \leqslant C\,\lambda^{1/2}\, r^{-1/2} \end{aligned} \tag{25.21}$$

by Cauchy-Schwarz and (25.20). Thus

$$|\nabla f(x)| \leqslant C\,\lambda^{1/2} \operatorname{dist}(x, G)^{-1/2} \tag{25.22}$$

when (13.2) holds, and also when (13.1) holds and $\operatorname{dist}(x, G) \leqslant 1$. Otherwise

$$|\nabla f(x)| \leqslant C \text{ when } \operatorname{dist}(x, G) \geqslant 1. \tag{25.23}$$

Therefore f satisfies (21.1) in all cases (because $r_0 \leqslant 1$ when (13.1) holds). [Recall that at this point λ is a fixed constant.]

Now we want to check that

$$\overline{\Omega(\delta)} \setminus G \text{ is connected.} \tag{25.24}$$

The simple curve $\partial\Omega(\delta) = \Gamma(\delta)$ is clearly connected and does not meet G, so it is enough to show that every point $z \in \Omega(\delta) \setminus G$ can be connected to $\partial\Omega(\delta)$ by an arc in Ω. [Note that such an arc stays in $\Omega(\delta) \setminus G$ until it touches $\partial\Omega(\delta)$ for the first time.]

We simply use the escape path γ_z given by Lemma 20.1. If δ is small enough, (20.4) applied with $t = 1$ shows that

$$|\gamma_z(1) - x_0| \geqslant \operatorname{dist}(\gamma_z(1), G) \geqslant C_1^{-1} > 2\delta, \tag{25.25}$$

and hence $\gamma_z(1)$ lies outside of $\Omega(\delta)$ by (25.18). This proves that γ_z crosses $\partial\Omega(\delta)$, and (25.24) follows.

We are now ready to prove (25.19). Let $x, y \in \overline{\Omega(\delta)} \setminus G$ be given. Take $B = B(x_0, 2\delta)$; then (21.2) holds because of (25.24) and (25.18). Thus we can apply Lemma 21.3 (provided that we took $\delta \leqslant 1/2$). We get that $|f(x) - f(y)| \leqslant C\,\delta^{1/2}$; this proves (25.19).

Now we can easily check that the limit $f^*(x_0)$ in (25.2) exists. For each small enough δ, we found a neighborhood $\Omega(\delta)$ of x_0 such that (25.19) holds. The existence of $f^*(x_0)$ follows at once (using the Cauchy criterion). This completes our proof of Proposition 25.1. □

26. w has a continuous extension to $\mathbb{R}^2$

In this section we continue to assume that (v, G) satisfies (13.1) or (13.2), but we also suppose, mostly for convenience, that

$$\Omega = \mathbb{R}^2 \setminus G \text{ is connected.} \tag{26.1}$$

Note that (26.1) is always true when (13.1) holds; see (15.2).

Proposition 26.2. — *The function w has a continuous extension to $\mathbb{R}^2$, and the extension is constant on every connected component of G.*

As was said in Section 2, the proof of this would be easy if we knew that G is piecewise C^1. Then we would observe that w has piecewise C^1 boundary values on G, and the Neumann condition $\partial v/\partial n = 0$ would tell us that the tangential derivative $\partial w/\partial\tau$ equals 0. This would show that the boundary values of w on G are locally constant, and the conclusion would follow. Thus all the trouble in this section and the next one will come from the possible lack of regularity of G, which will force slightly more complicated arguments.

The most delicate part of the argument will concern the boundary values of w on connected components of G that are not reduced to one point. An important first step will be to control the boundary values at regular points, as follows.

Lemma 26.3. — *Let G_0 be a connected component of G, not reduced to one point. Then there exists $m_0 \in \mathbb{R}$ such that*

$$\lim_{\substack{z \to x \\ z \in \Omega}} w(z) = m_0 \text{ for all regular points } x \in G_0. \tag{26.4}$$

We shall first prove the lemma in the slightly simpler case when G_0 is a bounded component of G, and in this case we shall not need our connectedness assumption (26.1). The case when G_0 is unbounded will be considered in the next section, and Proposition 26.2 will be deduced from Lemma 26.3 just after that.

First we want to surround G_0 by a Jordan curve in Ω. By Lemma 25.3 (applied with $\delta = 1$) we can find a compact set G^0, with $G_0 \subset G^0 \subset G$, such that $\operatorname{dist}\left(G^0, G \setminus G^0\right) > 0$.

Next we apply the construction of Section 23 to enclose G^0 in a collection of Jordan curves Γ_ε^j. If we choose ε small enough, the curves Γ_ε^j do not meet G; see Remark 23.24 and (23.26).

We know from (23.15) and (23.11) that G_0 is contained in the disjoint union of open sets Ω_ε^j, and since G_0 is connected, it is contained in a single Ω_ε^j. [See Remark 23.17 for a similar argument.] We shall call Ω_1 the Ω_ε^j that contains G_0, and $\Gamma_1 = \partial\Omega_1$ the corresponding Γ_ε^j. Thus

$$G_0 \subset \Omega_1, \tag{26.5}$$

$$\Omega_1 \text{ is the bounded connected component of } \mathbb{R}^2 \setminus \Gamma_1, \tag{26.6}$$

and

$$\Gamma_1 \text{ is a piecewise } C^1 \text{ Jordan curve that does not meet } G. \tag{26.7}$$

So far, we only chose a domain Ω_1 where it will be convenient to work. Consider $G^1 = G \cap \Omega_1$, and apply Lemma 23.18 to G^1. We get a collection of domains $U(\varepsilon_n)$ that satisfy (23.19) and (23.20).

Denote by Ω_0 the unbounded component of $\mathbb{R}^2 \setminus G^1$ (as in (23.20)). Clearly $\Gamma_1 \subset \Omega_0$ (by (26.6) and the definition of G^1), and hence

$$\Omega_0 = \mathbb{R}^2 \setminus G^1, \tag{26.8}$$

because for every $z \in \Omega_1 \setminus G^1$, the escape path γ_z of Lemma 20.1 goes through Γ_1. The proof of this last fact in the same as for (25.24); it relies on the fact that if we chose ε in the definition of Ω_1 and Γ_1 above small enough (which of course we can easily do), then all the points of Γ_1 (and hence of Ω_1) are pretty close to G^1, by (23.10). We may also get (26.8) directly from the connectedness of Ω if we want.

From (26.8), (23.19) and (23.20), we deduce that for n large Γ_1 is contained in $U(\varepsilon_n)$. Let us remove the first few values of n where this may fail. Next set

$$W_n = \Omega_1 \cap U(\varepsilon_n)\,. \tag{26.9}$$

Thus W_n is a piecewise smooth domain whose outer boundary is Γ_1 and whose inner boundary is composed of the Jordan curves $\Gamma^j_{\varepsilon_n}$, $j \in J_n$, that also compose $\partial U(\varepsilon_n)$. We still have that

$$W_n \subset W_{n+1}, \tag{26.10}$$

by (23.19), and

$$\bigcup_n W_n = \Omega_1 \setminus G^1 = \Omega_1 \setminus G, \tag{26.11}$$

by (23.20) and (26.8).

Our intention is to deduce information on v from similar information on functions v_n defined on W_n. Before we do this, we want to modify slightly our domains W_n to make the comparison between v and v_n easier near some given points of G_0.

So let x_1, x_2 be two given distinct regular points on G_0. Such points are easy to find, because almost every point of G is regular (by Proposition 13.11) and G_0 is connected and not reduced to one point. Let us also choose two disks D_ℓ, $\ell = 1, 2$, centered at x_ℓ, such that

$$2D_\ell \text{ is a disk of regularity for } G, \tag{26.12}$$

$$\overline{D}_\ell \subset \Omega_1 \tag{26.13}$$

for $\ell = 1, 2$, and

$$\overline{D}_1 \cap \overline{D}_2 = \varnothing. \tag{26.14}$$

For $\ell = 1, 2$, choose one of the two connected components of $D_\ell \setminus G$, and call it V_ℓ. We want to study the boundary values of w near the points x_ℓ, with access from the regions V_ℓ.

First let us describe the sets $U(\varepsilon_n) \cap D_\ell$ for n large. In the construction of $U(\varepsilon)$ (in the early Section 23) we first defined a union $H(\varepsilon_n)$ of small disks B_i centered on a $10^{-2}\varepsilon$-dense subset of G^1, as in (23.6). Because $2D_\ell$ is a disk of regularity for G, $H(\varepsilon_n) \cap D_\ell$ looks a lot like the thin corrugated tube of Figure 26.1 when n is large enough.

Denote by y_ℓ the central point of the arc of circle $\overline{V}_\ell \cap \partial D_\ell$. From (26.10), (26.11), and (26.13) we deduce that

$$y_\ell \in W_n \text{ for } n \text{ large}, \tag{26.15}$$

and then that

$$V_\ell \setminus H(\varepsilon_n) \subset W_n \tag{26.16}$$

(again for n large), because $\partial U(\varepsilon_n) \subset \partial H(\varepsilon_n)$ (by (23.9)). We also get that

$$\partial W_n \cap \overline{V}_\ell = \partial U(\varepsilon_n) \cap \overline{V}_\ell = \partial H(\varepsilon_n) \cap \overline{V}_\ell \tag{26.17}$$

(in the picture, the lower part of the corrugated curve).

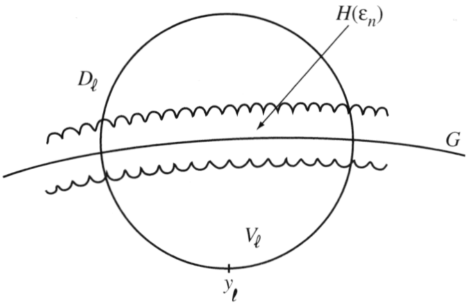


FIGURE 26.1

Note that since G_0 is connected, it is contained in a single $\Omega^j_{\varepsilon_n}$ (by the same argument as in Remark 23.17, say). We shall denote by $\Gamma_{n,0}$ the boundary of that domain $\Omega^j_{\varepsilon_n}$. Because of the description above each of the two sets $\partial W_n \cap \overline{V}_\ell$ (as in (26.17)) is contained in $\Gamma_{n,0}$, and

$$\partial W_n \cap \overline{V}_\ell = \Gamma_{n,0} \cap \overline{V}_\ell \tag{26.18}$$

for $\ell = 1, 2$ and n large enough.

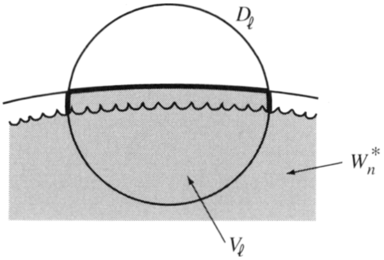


FIGURE 26.2

We decide to replace our domains W_n, n large enough, with

$$W_n^* = W_n \cup V_1 \cup V_2. \tag{26.19}$$

[See Figure 26.2; in effect we only add two thin tubular regions in D_1 and D_2.] We still have that

$$W_n^* \subset W_{n+1}^* \tag{26.20}$$

and

$$\bigcup_n W_n^* = \Omega_1 \setminus G \tag{26.21}$$

by (26.10) and (26.11), and by construction. [Note in particular that $W_n \subset W_n^* \subset \Omega_1 \setminus G$.]

By the discussion above, ∂W_n^* and ∂W_n coincide out of $\overline{D}_1 \cup \overline{D}_2$, and in $\overline{D}_\ell$ we get ∂W_n^* from ∂W_n by replacing the arc $\Gamma_{n,0} \cap V_\ell$ with a new arc with the same endpoints, and which is composed of two small arcs of the circle ∂D_ℓ and the arc $G \cap D_\ell$. In particular

(26.22) ∂W_n^* is a finite collection of piecewise C^1 Jordan curves,

and

(26.23) $G \cap D_1$ and $G \cap D_2$ are two arcs of the same connected component of ∂W_n^*

(i.e., the Jordan arc $\Gamma_{n,0}^*$ obtained from $\Gamma_{n,0}$ by the modification explained above).

Our domains W_n^* have the same exterior boundary as the W_n, i.e., $\Gamma_1 = \partial \Omega_1$.

Let us now define a function $v_n \in W^{1,2}(W_n^*)$ by requiring that it have boundary values on the exterior boundary Γ_1 which coincide with v on Γ_1 (note that v is defined and continuous on Γ_1, by (26.7)), and that it minimize the energy

$$E_n = \int_{W_n^*} |\nabla v_n|^2 \tag{26.24}$$

under this constraint.

As for v_n in Section 24, the existence and uniqueness of v_n can be proved by convexity arguments. [Also see (26.25) below if you are worried about components of W_n^* that would not touch Γ_1.] We can use (26.20), (26.21), and the local energy minimizing property of v to show that (modulo extracting a subsequence to make the argument easier), v_n converges to v uniformly on every compact subset of $\Omega_1 \setminus G$. The argument is the same as in Section 24.

Next we want to define conjugated functions w_n. Let us first check that

(26.25) W_n^* is connected.

By definition, $U(\varepsilon_n)$ is the unbounded connected component of $\mathbb{R}^2 \setminus H(\varepsilon_n)$. Thus every point of $U(\varepsilon_n)$ can be connected to ∞ by an arc in $U(\varepsilon_n)$. Then every point of $W_n = \Omega_1 \cap U(\varepsilon_n)$ can be connected to Γ_1 by an arc in W_n [take the previous arcs, and just stop when you hit $\partial\Omega_1$.] The same thing holds for W_n^*, because the points of V_ℓ that were added in (26.19) can be connected to points of $V_\ell \cap W_n$ by arcs in V_ℓ. Our claim (26.25) follows because Γ_1 clearly has a small neighborhood whose intersection with W_n is connected.

Choose any origin $A \in W_1$ (the smallest of our domains W_n) , and define a conjugated function w_n to v_n on W_n^* by

$$w_n(z) = w(A) + \int_{I_\gamma} \nabla v_n(\gamma(t))(-i\gamma'(t))dt, \tag{26.26}$$

where $\gamma : I_\gamma \to W_n^*$ is an arc of class C^1 that goes from A to z, and the conventions are the same as for (22.2) above.

For the same reasons as in Section 22 (and with simpler integrations by parts, since W_n^* is piecewise smooth), w_n is well defined on W_n^*, and

$$v_n + iw_n \text{ is holomorphic on } W_n^*. \tag{26.27}$$

Since w_n and w are both given in terms of v_n and v by a formula like (26.26), the uniform convergence of v_n to v on compact subsets of $\Omega_1 \setminus G$ and the uniform bounds on ∇v_n and ∇v given by the harmonicity of these functions and our bounds on energy imply that

$$\{w_n\} \text{ converges to } w \text{ uniformly on every compact subset of } \Omega_1 \setminus G. \tag{26.28}$$

Recall from (26.22) that ∂W_n^* is piecewise C^1. In fact it is piecewise $C^{1+\alpha}$ for some $\alpha > 0$ (by the same proof), and the same argument as in Lemmas 14.1 and 14.3 shows that v_n has boundary values on ∂W_n^*, and that theses boundary values are continuous on ∂W_n^* and C^1 on each of the $C^{1+\alpha}$-arcs that compose it. Because of the formula (26.26) we also get that

$$w_n^*(x) = \lim_{\substack{z \to x \\ z \in W_n^*}} w_n(z) \tag{26.29}$$

exists and is continuous on ∂W_n^*, and also that it is C^1 on each of the (open) $C^{1+\alpha}$ arcs that compose it.

Since the tangential derivative $\partial w_n^* / \partial \tau$ on the $C^{1+\alpha}$-arcs of $\partial W_n^* \setminus \Gamma_1$ coincides, by (26.26), with the normal derivative $\partial v_n / \partial n$, and since $\partial v_n / \partial n = 0$ by the usual Neumann condition on energy-minimizing functions, we get that

$$\frac{\partial w_n^*}{\partial \tau} = 0 \text{ on the } C^{1+\alpha} \text{ arcs of } \partial W_n^* \setminus \Gamma_1, \tag{26.30}$$

and hence

$$w_n^* \text{ is constant on } \Gamma_{n,0}^*. \tag{26.31}$$

Now we want to combine (26.31) with (26.28) to get an analogous statement on w. Let us return to our two domains $V_\ell \subset D_\ell$. Recall from (26.19) that W_n^* contains V_ℓ; this will make things a little more pleasant, because it will allow us to take limits on fixed domains.

Denote by r_ℓ the radius of D_ℓ, and set $V_\ell' = V_\ell \cap B(x_\ell, 3r_\ell/4)$ and $\mathcal{C}_\ell = V_\ell \cap \partial B(x_\ell, r_\ell/2)$. By the same argument as in Lemma 14.1, the functions v_n, v, w_n, w have C^1 extensions to $\overline{V}_1'$ and $\overline{V}_2'$.

Lemma 26.32. — *The sequences $\{v_n\}$ and $\{w_n\}$ converge to v and w uniformly on $\mathcal{C}_\ell$.*

Of course the only new information concerns the uniformity of the convergence near the extremities of $\mathcal{C}_\ell$, since we already know that there is uniform convergence on compact subsets of $\Omega_1 \setminus G$. First note that

$$\int_{V_\ell} \left\{|\nabla v_n|^2 + |\nabla v|^2\right\} \leqslant E_n + \int_{\Omega_1 \setminus G} |\nabla v|^2 \leqslant 2 \int_{\Omega_1 \setminus G} |\nabla v|^2 \tag{26.33}$$

by definitions, and also the fact that $W_n^* \subset \Omega_1 \setminus G$, by (26.21). Thus we can find radii $r = r(\ell, n)$ such that $2r_\ell/3 < r < 3r_\ell/4$ and

$$\int_{V_\ell \cap \partial B(x_\ell, r)} \left\{|\nabla v_n|^2 + |\nabla v|^2\right\} \leqslant 25 \int_{\Omega_1 \setminus G} |\nabla v|^2. \tag{26.34}$$

Next we can use the fact that v and v_n minimize the energy on $V_\ell \cap B(x_\ell, r)$ with the given boundary data on $V_\ell \cap \partial B(x_\ell, r)$. Because $V_\ell \cap B(x_\ell, r)$ is regular and by the same technique as in Lemma 14.1 (i.e., use a conformal mapping to reduce to a half-disk, and then a reflection to reduce to a simple Dirichlet problem on a disk) it is easy to deduce from (26.34) that

$$\int_{\mathcal{C}_\ell} \left\{|\nabla v_n|^2 + |\nabla v|^2\right\} \leqslant C \int_{\Omega_1 \setminus G} |\nabla v|^2 := C', \tag{26.35}$$

where the precise value of the constant C' will not matter. Set

$$\delta_n = \sup\{|v_n(x) - v(x)|\,; x \in \mathcal{C}_\ell\} \tag{26.36}$$

and, for each compact subarc T of $\mathcal{C}_\ell$;

$$\delta_n(T) = \sup\{|v_n(x) - v(x)|\,; x \in T\}. \tag{26.37}$$

Then

$$\delta_n \leqslant \delta_n(T) + \int_{\mathcal{C}_\ell \setminus T} \{|\nabla v_n| + |\nabla v|\} \leqslant \delta_n(T) + 2(C')^{1/2} H^1(\mathcal{C}_\ell \setminus T)^{1/2} \tag{26.38}$$

by (26.35) and Cauchy-Schwarz.

The second term can be made as small as we want by choosing T close to the whole $\mathcal{C}_\ell$, and then $\delta_n(T)$ is small for n large, by the uniform convergence of $\{v_n\}$ to v on compact subsets of $\Omega_1 \setminus G$ (like T). Thus δ_n tends to 0, and $\{v_n\}$ tends to v uniformly on $\mathcal{C}_\ell$. The same argument, together with the fact that $|\nabla w_n| = |\nabla v_n|$ by definition of a conjugated function, gives the uniform convergence of $\{w_n\}$ to w on $\mathcal{C}_\ell$. This completes the proof of Lemma 26.32. □

Set $m_\ell = \lim_{\substack{z \to x_\ell \\ z \in V_\ell}} w(z)$. The existence of m_ℓ was never an issue (because V_ℓ is regular), but what we want to know is that $m_1 = m_2$. Note that m_ℓ is also the value of w at the two points of $\overline{\mathcal{C}}_\ell \cap G$, again because V_ℓ is regular and by the same argument as for (26.30). Also denote by $m(n)$ the constant value of w_n^* on $\Gamma_{n,0}^*$. [See (26.31).] In particular $m(n)$ is the value of w_n^* at the endpoints of $\mathcal{C}_\ell$. Now Lemma

26.32 and the fact that w and the w_n have limits at the endpoints of $\mathcal{C}_\ell$ imply that $m_\ell = \lim_{n\to+\infty} m(n)$ for each ℓ, and hence $m_1 = m_2$.

Let us verify that Lemma 26.3 (in our special case when G_0 is bounded) follows from this. We have chosen two distinct, but otherwise arbitrary regular points in G_0, and for each one an access region V_ℓ. Then we proved that the two limits m_1 and m_2 are equal. If we apply this to the same choice of points x_1 and x_2, the same V_2, but the other choice of V_1, we also get that the two approach regions to x_1 give the same limit (by comparing). The existence of the constant m_0 as in (26.4) now follows easily.

27. The case when G_0 is not bounded

We still need to prove Lemma 26.3 when G_0 is an unbounded component of G. [In the special case of (13.1), this means that G_0 is the component of G that contains L.] We shall need to modify the argument above, but fortunately not too much.

Our first trouble comes from the construction of Ω_1 and Γ_1, at the very beginning of the argument. If G_0 is not bounded, we cannot surround it entirely by a curve that does not meet G. Instead of this we shall choose a curve Γ_1 that surrounds a big piece of G_0 and crosses G only once, in a nice transversal way.

Let again two distinct points of regularity x_1, x_2 of G_0 be given. By Lemma 19.2, there is a rectifiable curve γ in G_0 which connects x_1 to x_2.

Lemma 27.1. — *There is a piecewise C^∞ (closed) Jordan curve Γ_1 and a point of regularity x_0 of G such that*

$$\Gamma_1 \cap G = \{x_0\}, \tag{27.2}$$

(27.3) Γ_1 meets G transversally (and even perpendicularly) at x_0,

and

(27.4) $\gamma \subset \Omega_1$, the bounded connected component of $\mathbb{R}^2 \setminus \Gamma_1$.

We leave for later in this section the proof of this lemma, and first show how to use it to complete the proof of Lemma 26.3.

Just as in the bounded case, set $G^1 = G \cap \Omega_1$, and apply Lemma 23.18 to G^1. We get an increasing sequence of domains $U(\varepsilon_n)$, whose union is still $\Omega_0 = \mathbb{R}^2 \setminus G^1$ as in (26.8).

Because of the intersection at x_0, we cannot say that the whole Γ_1 is contained in $U(\varepsilon_n)$ for n large, but only that

$$\Gamma_1 \setminus D_0 \subset U(\varepsilon_n), \tag{27.5}$$

where D_0 is a small disk of regularity centered at x_0.

Inside D_0, the situation is not too complicated, and looks like the one described in the second part of Remark 23.24. The fact that Ω_1 is not a disk (as in (23.27)) does not play a serious role, and (23.28)-(23.30) come from Lemma 27.1. First, $G^1 \cap D_0$ is just one of the two halves of the arc $G \cap D_0$, which starts from ∂D_0 and ends at x_0. Then $H(\varepsilon_n) \cap D_0$ is, for n large enough, some sort of wrinkled sock around $G^1 \cap D_0$, as suggested by Figure 27.1. The points of D_0 which do not lie in the sock $H(\varepsilon_n)$ lie in $U(\varepsilon_n)$, at least if n is large enough. The arguments are the same as in Remark 23.24 (compare with (23.32) and Figure 23.2).

We set $W_n = \Omega_1 \cap U(\varepsilon_n)$ as before (see (26.9)). We still have (26.10) and (26.11); the only major difference with the situation of Section 26 is that now the outside boundary of W_n is composed of a long arc of Γ_1 (in fact, almost all of Γ_1), connected to a long part of the arc $\Gamma^j_{\varepsilon_n}$ that gets close to x_0 [See Figure 27.1.] The arc $\Gamma^j_{\varepsilon_n}$ that we just mentioned is the arc $\Gamma_{n,0}$ that surrounds the connected component of x_0 in G^1; the argument is the same as in Remark 23.17: the component of x_0 in G^1 must be contained in a single $\Gamma^j_{\varepsilon_n}$.

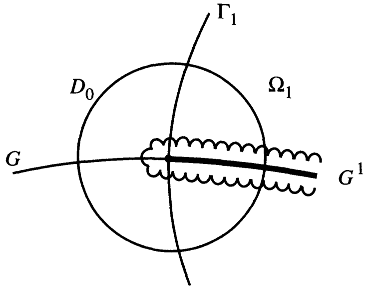


FIGURE 27.1

Also, the component of x_0 in G^1 contains γ for the following reason. We know from (19.4) (and because G_0 is not bounded) that there exist arcs in G_0 that connect γ to points of G_0 arbitrarily far away. Such arcs start in Ω_1 by (27.4), and end out of Ω_1 because Ω_1 is bounded. The only point where they are allowed to cross $\Gamma_1 = \partial\Omega_1$ is x_0, by (27.2). Our claim follows.

Let us continue our construction as above. Choose small disks of regularity D_ℓ centered at x_ℓ, $\ell = 1, 2$, with the properties (26.12)-(26.14), as well as access regions (i.e., components of $D_\ell \setminus G$) V_ℓ. Our description of $U(\varepsilon_n) \cap D_\ell$ is still valid, and in particular (26.15)-(26.18) still hold for the same reasons.

Define W_n^* as in (26.19). We still have (26.20)-(26.23), but the slight difference is that the curve $\Gamma_{n,0}$ that has been modified to construct $\Gamma^*_{n,0}$ in now part of the exterior boundary of W_n. Thus the exterior boundary of W_n^* is composed of a long subarc of Γ_1 (most of it, as before), and a long part of $\Gamma^*_{n,0}$.

We define $v_n \in W^{1,2}(W_n^*)$ as before: we demand that it have the same boundary values as v on the arc $\Gamma_1 \cap \partial W_n^*$ (i.e., most of Γ_1), and that it minimize the energy in (26.24) under this constraint. Thus we only put Dirichlet conditions on part of the exterior boundary, but this will not prevent the argument above from working.

Here also, W_n^* is connected. As before, every point of $W_n = \Omega_1 \cap U(\varepsilon_n)$ can be connected to infinity by an arc in $U(\varepsilon_n)$. The only way such an arc can leave W_n is by crossing $\partial W_n \setminus \partial U(\varepsilon_n)$, i.e., the part of Γ_1 that lies on ∂W_n. Since this arc of Γ_1 is connected and has a small neighborhood whose intersection with W_n is connected, we get that W_n is connected. The connectedness of W_n^* follows as before.

The construction of the conjugated function w_n on W_n^*, and the various limiting arguments that follow, can be repeated as above.

This completes our proof of Lemma 26.3 in the remaining case when G_0 is not bounded, modulo the proof of Lemma 27.1 which we undertake now. □

Note that in the special case of (13.1), Lemma 27.1 is trivial because we can take $\Gamma_1 = \partial B_R$ for a very large R. [Recall that when (v, G) is a minimizer of the modified functional, G is the union of the half-line L with a bounded set.] Similarly, Lemma 27.1 is fairly easy when (v, G) is a global minimizer which is asymptotic to a cracktip (or which has a blow-in sequence that converges to a cracktip). Thus the proof of Lemma 27.1 that follows is not required for the main results of this paper.

Lemma 27.6. — *Assume that $\Omega = \mathbb{R}^2 \setminus G$ is connected, and let G_0 be an unbounded connected component of G. For every regular point $x \in G_0$, $G_0 \setminus \{x\}$ has exactly two connected components.*

Let G_0 and $x \in G_0$ be as in the statement. Choose a small disk of regularity $D = B(x, r)$, and also a line segment $I = [a, b] \subset B(x, r/2)$ which is centered at x and crosses G perpendicularly at the point x.

Since Ω is connected, we can find a polygonal arc γ in Ω that connects a to b. We can easily manage to make γ simple, and disjoint from I except for its endpoints a and b. By putting together γ and the segment $[a, b]$, we get a polygonal Jordan curve $\Gamma(x)$. Denote by $\Omega(x)$ the bounded component of $\mathbb{R}^2 \setminus \Gamma(x)$, and by $\Omega^+(x)$ the unbounded component.

Set $G(x) = G_0 \cap \Omega(x)$ and $G^+(x) = G_0 \cap \Omega^+(x)$; these will be the two components of $G_0 \setminus \{x\}$ promised in Lemma 27.6. Since $\Gamma(x) \cap G = \{x\}$, it is clear that

$$G_0 \setminus \{x\} = G(x) \cup G^+(x), \tag{27.7}$$

and that (27.7) is a partition of $G_0 \setminus \{x\}$ into (relatively) open sets. The partition is nontrivial, because $G(x)$ and $G^+(x)$ both contain little arcs of G_0 near x (one on each side). Thus Lemma 27.6 will follow as soon as we prove that $G(x)$ and $G^+(x)$ are connected.

Suppose we have a partition $G(x) = F_1 \cup F_2$ into disjoint (relatively) closed subsets. Since the little arc $D \cap G(x)$ is connected, it is entirely contained in F_1 or in F_2; say it is contained in F_1. Set

$$\widetilde{F}_1 = F_1 \cup (G_0 \setminus \Omega(x)) \text{ and } \widetilde{F}_2 = F_2. \tag{27.8}$$

The set $\widetilde{F}_2$ is closed in $G(x) = G_0 \cap \Omega(x)$, and does not get close to $\partial\Omega(x) \cap G_0 = \{x\}$, hence it is closed in G_0. Similarly, $\widetilde{F}_1 \setminus \{x\}$ is closed in $G_0 \setminus \{x\}$, and $\widetilde{F}_1$ contains a whole neighborhood of x in G_0, so $\widetilde{F}_1$ is closed in G_0. Thus $\widetilde{F}_1$ and $\widetilde{F}_2$ form a partition of G_0 into closed subsets, and the connectedness of G_0 implies that $F_2 = \varnothing$ (we know that $F_1 \neq \varnothing$).

This proves that $G(x)$ is connected. The connectedness of $G^+(x)$ can be proved the same way, and Lemma 27.6 follows. □

We may now return to the proof of Lemma 27.1. Recall that we are given two points x_1, x_2 of an unbounded connected component of G_0, and a rectifiable curve γ in G_0 that connects them. We want to find a regular point $x_0 \in G_0$ such that the curve $\Gamma_1 = \Gamma(x_0)$ constructed above satisfies the conclusions of Lemma 27.1. We already have (27.2) and (27.3) automatically, by construction of $\Gamma(x_0)$, and so we only need to choose x_0 so that $\gamma \subset \Omega_1 = \Omega(x_0)$.

Because of Lemma 19.16, we can find a simple curve γ_1 in G_0 that starts from the final endpoint of γ, say, and goes to ∞. Denote by γ_2 the curve composed of γ, followed by γ_1.

Choose a radius R so large that $\gamma \subset B_R$, and also such that all points of $G_0 \cap \partial B_R$ are regular. Such an R exists, by Proposition 13.11.

Denote by x_0 the last point of $\gamma_2 \cap \overline{B}_R$, let $\overline{\gamma}_2$ be the portion of γ_2 strictly before x_0, and γ_2^+ the portion of γ_2 strictly after x_0. Both portions are clearly connected, and they do not meet because γ_1 is simple and γ does not meet γ_2^+ by definition of R.

Let D denote the small disk of regularity centered at x_0 that we already used to define $\Gamma(x_0)$. Since γ_2^+ and γ_2^- are connected, disjoint, and both end up (or start) at x_0, each of them contains one of the two arcs that compose $D \cap G \setminus \{x_0\}$.

From this and the connectedness of γ_2^+ and γ_2^-, we deduce that γ_2^+ and γ_2^- are contained in the two components $G(x_0)$ and $G^+(x_0)$ of $G_0 \setminus \{x_0\}$. Since γ_2^+ is not bounded, the only possibility is that $\gamma_2^+ \subset G^+(x_0)$, and then $\gamma_2^- \subset G(x_0)$. Since γ_2^+ does not meet γ, γ must be contained in γ_2^-, and hence in $G(x_0)$. This is exactly what we wanted, because $G(x_0) = G_0 \cap \Omega(x_0)$.

This completes our proof of Lemma 27.1 and, by the same token, of Lemma 26.3. □

Now we want to deduce Proposition 26.2 from Lemma 26.3.

Lemma 27.9. — *Let G_0 be a component of G, not reduced to a point, and let m_0 be as in Lemma 26.3. Then*

$$|w(x) - m_0| \leqslant C \mathrm{dist}\,(x, G_0)^{1/2} \tag{27.10}$$

for all $x \in \Omega$ such that $\mathrm{dist}\,(x, G_0) \leqslant 1$.

So let x be as in the lemma, and set $\delta = \mathrm{dist}\,(x, G_0)$. Let us first replace x with a point y which is reasonably far from G (instead of just G_0). Lemma 20.1 gives us an escape path γ_x from x, but let us just use the restriction of γ_x to the interval $[0, \delta]$. Set $y = \gamma_x(\delta)$. Then

$$\mathrm{dist}\,(y, G) \geqslant C_1^{-1}\delta, \tag{27.11}$$

by (20.4). Also choose $z_0 \in G_0$ such that

$$|z_0 - x| = \delta. \tag{27.12}$$

Note that by (20.3) the length of γ_x between x and y is at most $C_1\delta$. Thus this arc is contained in $B = B\,(z_0, (C_1 + 2)\,\delta)$, and x and y lie in the same component of $B \setminus G$ because γ_x does not meet G. We want to apply Lemma 21.3 to control $w(x) - w(y)$, but we may not be able to do so directly if $(C_1 + 2)\,\delta > 1$. This is not a serious issue; the simplest way to deal with it is probably to cut γ_x into less than C consecutive arcs of length $\leqslant 1$, and apply Lemma 21.3 to each of these arcs. Let us not worry about this issue and do as if we could apply Lemma 21.3 directly with the disk B in all cases.

Note that $f = v + iw$ satisfies the condition (21.1). This was checked in Section 25; see in particular (25.22) and (25.23). Here we only need to know that

$$|\nabla w(z)| \leqslant C\ \mathrm{dist}(z, G)^{-1/2} + C \text{ on } \Omega, \tag{27.13}$$

which of course follows from (25.22) and (25.23). Thus Lemma 21.3 applies, and

$$|w(y) - w(x)| \leqslant C\,\delta^{1/2}. \tag{27.14}$$

Next we want to compare $w(y)$ with the limit of w at some regular point of G_0. Choose a regular point $z_1 \in G_0$ such that $|z_1 - z_0| < \delta$; this can be done because H^1-almost all points of G_0 are regular, and $H^1\,(G_0 \cap B\,(z_0, \delta)) > 0$ (since G_0 is connected and not reduced to a point). Also choose $z \in \Omega$ such that $|z - z_1| \leqslant \delta$ and

$$|w(z) - m_0| < \delta^{1/2}, \tag{27.15}$$

where m_0 is as in (26.4) (and Lemma 27.9).

Let γ_z denote the escape path given by Lemma 20.1, and set $y_1 = \gamma_z(\delta)$. Then

$$\mathrm{dist}\,(y_1, G) \geqslant C_1^{-1}\delta \tag{27.16}$$

by (20.4),

$$|z_0 - y_1| \leqslant 2\delta + |z - y_1| \leqslant (2 + C_1)\,\delta \tag{27.17}$$

by (20.3), and the same argument as for (27.14) gives that

$$|w(y_1) - w(z)| \leqslant C\,\delta^{1/2}. \tag{27.18}$$

Now set $r = (2C_1)^{-1}\,\delta$, $D = B(y,r)$, and $D_1 = B(y_1, r)$. Note that

$$|w(t) - w(y)| \leqslant C\delta^{1/2} \text{ for } t \in D \tag{27.19}$$

and

$$|w(t) - w(y_1)| \leqslant C\delta^{1/2} \text{ for } t \in D_1, \tag{27.20}$$

by (27.13), (27.11), and (27.16).

Denote by P the line through y and orthogonal to $[y, y_1]$ and, for each $\xi \in P \cap D$, set $\ell(\xi) = [\xi, \xi + y_1 - y]$. Set

$$P_1 = \{\xi \in P \cap D\ ;\ \ell(\xi) \text{ contains some point of } G \text{ which is not regular}\}.$$

Since H^1-almost every point of G is regular (by Proposition 13.11), P_1 is contained in the projection of some set of H^1-measure zero, and hence $H^1(P_1) = 0$. Also set

$$P_2 = \{\xi \in P \cap D; \ell(\xi) \cap G \text{ is infinite}\}. \tag{27.21}$$

Since $H^1\left(G \cap \bigcup_{\xi \in P \cap D} \ell(\xi)\right) < +\infty$, it is fairly easy to check that $H^1(P_2) = 0$. This is essentially Fubini's theorem, and the fact that we can remove P_1 and only consider regular points of G makes the proof slightly easier. Nevertheless we shall omit the proof.

Set $P_3 = (P \cap D) \smallsetminus (P_1 \cup P_2)$. Thus $H^1(P_3) = 2r$. Note that all $\ell(\xi)$, $\xi \in P \cap D$, are contained in $B(z_0, (C_1+3)\,\delta)$ (by (27.17) and the analogous estimate for y). Then

$$\begin{aligned}\int_{\xi \in P_3} \left\{\int_{\ell(\xi)\smallsetminus G} |\nabla w|\, dH^1\right\} dH^1 &\leqslant \int_{B(z_0,(C_1+3)\delta)\smallsetminus G} |\nabla w| \\ &\leqslant C\,\delta \left\{\int_{B(z_0,(C_1+3)\delta)\smallsetminus G} |\nabla w|^2\right\}^{1/2} \leqslant C\,\delta^{3/2}\end{aligned} \tag{27.22}$$

by Fubini, Cauchy-Schwarz, and (25.20). In particular we can choose $\xi \in P_3$ such that

$$\int_{\ell(\xi)\smallsetminus G} |\nabla w|\, dH^1 \leqslant C\,\delta^{1/2}. \tag{27.23}$$

Since $\xi \in P_3$, $\ell(\xi)$ only meets G a finite number of times, and only at regular points of G. Of course all these points lie in connected components of G that are not reduced to one point (by definition of a regular point), and Lemma 26.3 tells us that w is continuous at all these points. Recall that $\ell(\xi) = [\xi, \xi']$, with $\xi' = \xi + y_1 - y \in D_1$. We have that

$$|w(\xi') - w(\xi)| \leqslant \int_{\ell(\xi)\smallsetminus G} |\nabla w|\, dH^1 \leqslant C\,\delta^{1/2} \tag{27.24}$$

by (27.23), and hence

$$|w(x) - m_0| \leqslant C\,\delta^{1/2} \tag{27.25}$$

by (27.14), (27.15), (27.18), (27.19), (27.20), and (27.24). This completes our proof of Lemma 27.9. □

A trivial consequence of Lemma 27.9 is that if G_0 is a component of G which is not reduced to a point and m_0 is as in Lemma 26.3, then

$$\lim_{\substack{z \to x \\ z \in \Omega}} w(z) = m_0 \text{ for all } x \in G_0. \tag{27.26}$$

Since we know from Proposition 25.1 that w also has limits at points of G where G is not connected, we can extend w to $\mathbb{R}^2$ by taking

$$w(x) = \lim_{\substack{z \to x \\ z \in \Omega}} w(z) \text{ for } x \in G. \tag{27.27}$$

To complete the proof of Proposition 26.2, we still have to check that w is continuous. At this point, this is mostly formal. The continuity on Ω is not a surprise: w is even harmonic there. Let $x \in G$ be given, and let us check that w is continuous at x. Let $\{x_n\}$ be a sequence that tends to x. For each n, choose a point $z_n \in \Omega$ such that $|z_n - x_n| \leqslant 2^{-n}$ and $|w(z_n) - w(x_n)| \leqslant 2^{-n}$. [We can take $z_n = x_n$ if $x_n \in \Omega$, and otherwise we use (27.27).] Then $w(z_n)$ tends to $w(x)$ by (27.27), and $w(x_n)$ tends to $w(x)$ as well. This proves the continuity of w.

Our proof of Proposition 26.2 is now complete. □

Remark 27.28. — Our hypothesis (26.1) that Ω be connected was probably not needed, but it was convenient. [Otherwise, we would at least have to worry about gluing together the various restrictions of w to the different components of Ω.]

CHAPTER G

THE LEVEL SETS OF w

28. Variations of v on the level sets of w

In the next few sections we want to study the level sets

$$\Gamma_m = \{z \in \mathbb{R}^2; w(z) = m\} \tag{28.1}$$

and show that, for almost-every $m \in \mathbb{R}$, Γ_m is either empty or a single, locally rectifiable Jordan curve through ∞. This will take some time; the purpose of this section is to show that for almost-every m, the variations of v along (curves in) Γ_m are given by its derivative on Ω, with no contribution from the various jumps and singularities that v may have on G.

We continue with our usual assumptions that (13.1) or (13.2) holds, and that Ω is connected (to be sure that w is continuous).

Proposition 28.2. — *For almost-every $m \in \mathbb{R}$, v has a continuous extension to $\Omega \cup \Gamma_m$, with the following property. For every simple arc $\gamma : [0,1] \to \Gamma_m$ such that*

$$\gamma' \in L^1([0,1]), \tag{28.3}$$

the function $v \circ \gamma$ also has a derivative in $L^1([0,1])$, and

$$\begin{aligned} v \circ \gamma(1) - v \circ \gamma(0) &= \int_0^1 (v \circ \gamma)'(t)dt \\ &= \int_{\{t\ ;\ \gamma(t) \in \Omega\}} \nabla v(\gamma(t)) \cdot \gamma'(t)dt. \end{aligned} \tag{28.4}$$

By (28.3) we mean not only that γ has a derivative almost-everywhere, but also that γ is the integral of γ'.

The proposition does not use the fact that the functions v and w are conjugated, and would hold with v replaced with any other function that satisfies the same estimates, i.e., (25.22) and (25.23).

Before we start to estimate, let us exclude the values of m for which v probably has jumps on Γ_m. Denote by

$$\mathcal{S} = \{x \in G\ ;\ \{x\} \text{ is a connected component of } G\} \tag{28.5}$$

the set of points where G is not connected. Since almost-every point of G is a point of regularity (and hence is not in $\mathcal{S}$),

$$H^1(\mathcal{S}) = 0. \tag{28.6}$$

The set $G \setminus \mathcal{S}$ will be easy to exclude from our discussion. Every non trivial component of G has positive measure, and so there are at most countably many such components. Since w is constant on each nontrivial component of G (by Proposition 26.2) we get that

$$\{m \in \mathbb{R} \ ; \ \Gamma_m \text{ meets } G \setminus \mathcal{S}\} \text{ is at most countable.} \tag{28.7}$$

The set $\mathcal{S}$ is small, but will create trouble. We only know that (28.6) holds, and in particular $\mathcal{S}$ has no reason to be countable. Proposition 25.1 tells us that v is continuous at each point x_0 of $\mathcal{S}$, but we expect $|\nabla v|$ to blow up like $|x - x_0|^{-1/2}$ at such a point. If inside Γ_m the set $\mathcal{S}$ is a little too fat (typically, if its dimension there is $\geqslant 1/2$), we can expect a nontrivial contribution of the singular part of ∇v. We want to show that in average (over m) this behavior does not occur. To do this we shall use a covering argument (to use (28.6)) and the co-area formula (to take averages in m).

Let $\varepsilon > 0$ be small; it will tend to zero at the end of the argument. Because of (28.6), we can find disks B_j, $j \in J$, with

$$B_j = B(x_j, r_j), \ x_j \in \mathcal{S}, \tag{28.8}$$

$$\mathcal{S} \subset \bigcup_{j \in J} B_j, \tag{28.9}$$

and

$$\sum_{j \in J} r_j < \varepsilon. \tag{28.10}$$

We may always replace r_j with $5r_j$ and apply the Vitali covering lemma in the first pages of [St], and so we can assume that

$$\text{the } B(x_j, r_j/5), j \in J, \text{ are disjoint.} \tag{28.11}$$

Let $m \in \mathbb{R}$ be given, and suppose that

$$\Gamma_m \text{ does not meet } G \setminus \mathcal{S}. \tag{28.12}$$

Let $\gamma : [0,1] \to \Gamma_m$ be a simple arc with a derivative in L^1, as in the statement of Proposition 28.2. Set

$$J(\gamma) = \{j \in J \ ; \ B_j \text{ meets } \gamma([0,1])\} \tag{28.13}$$

and then choose a finite subset $J_0(\gamma)$ of $J(\gamma)$ such that

$$G \cap \gamma([0,1]) = \mathcal{S} \cap \gamma([0,1]) \subset \bigcup_{j \in J_0(\gamma)} B_j. \tag{28.14}$$

The identity in (28.14) comes from (28.12), and $J_0(\gamma)$ exists because $G \cap \gamma([0,1])$ is compact and the B_j, $j \in J(\gamma)$, cover it (by (28.9) and the first part of (28.14)).

For each $j \in J$, denote by $\mathcal{A}(j)$ the set of connected components of $B(x_j, 2r_j) \setminus G$ that meet $\overline{B}_j$.

Lemma 28.15. — *$\mathcal{A}(j)$ has at most C elements.*

For every component $a \in \mathcal{A}(j)$, choose a point $x_a \in a \cap \overline{B}_j$, and denote by γ_a the escape path of Lemma 20.1 (applied with $z = x_a$). Set $t_j = (2C_1)^{-1} r_j$, where C_1 is as in Lemma 20.1, and $y_a = \gamma_a(t_j)$. The image of γ_a does no meet G (by (20.4)) and is contained in $B(x_j, 2r_j)$ (by (20.3) and our choice of t_j). Hence $y_a \in a$. Also $\operatorname{dist}(y_a, G) \geqslant C_1^{-1} t_j = \left(2C_1^2\right)^{-1} r_j$ by (20.4). Altogether,

$$\begin{cases} \text{for each component } a \in \mathcal{A}(j) \text{ we can find } y_a \in a \cap B(x_j, 2r_j) \\ \text{such that } B_a = B\left(y_a, C^{-1} r_j\right) \subset a, \end{cases} \tag{28.16}$$

where we set $C = 2C_1^2$. The disks B_a, $a \in \mathcal{A}(j)$ are disjoint (because the components a are disjoint), and there cannot be more than C' of them because they are all contained in $B(x_j, 2r_j)$. This proves Lemma 28.15. □

Return to our arc γ in Γ_m, and set

$$R = \left\{t \in [0,1] \ ; \ \gamma(t) \in \mathbb{R}^2 \setminus \textstyle\bigcup_{j \in J_0} B_j\right\}, \tag{28.17}$$

where $J_0 = J_0(\gamma)$ is the same finite subset of $J(\gamma)$ as in (28.14). Obviously R is compact, and by (28.14) $\gamma(R)$ does not meet G. Hence

$$\gamma(t) \in \Omega \text{ for } t \text{ in some neighborhood of } R \tag{28.18}$$

(because Ω is open and γ is continuous), and $v \circ \gamma$ has a derivative in L^1 in some neighborhood of R.

Now we want to define two sequences $\{s_n\}$ and $\{t_n\}$ in $[0,1]$. We start with $s_0 = 0$ and already distinguish between two cases. If $s_0 \in R$, take

$$t_0 = \sup\{t \ ; \ [s_0, t] \subset R\}. \tag{28.19}$$

Thus t_0 is the last point of $[0,1]$ before we leave R for the first time.

If $s_0 \notin R$, take $t_0 = s_0$.

Suppose now that we already defined $s_0 \leqslant t_0 \leqslant \cdots \leqslant s_n \leqslant t_n$, and that

$$\begin{cases} t_n = 1, \text{or else there is a sequence } \{\xi_k\} \text{ of points in} \\ ]t_n, 1] \setminus R \text{ that tends to } t_n. \end{cases} \tag{28.20}$$

Note that (28.20) holds for $n = 0$, because R is closed (which takes care of the case when $s_0 \notin R$), and by (29.19).

If $t_n = 1$, we stop the construction. Otherwise, (28.20) gives us a sequence $\{\xi_k\}$. By definition of R, each $\gamma\{\xi_k\}$ lie in some B_j, $j \in J_0$, and since J_0 is finite we can find $j_n \in J_0$ such that

$$K_1 = \{k \geqslant 0 \ ; \ \gamma\{\xi_k\} \in B_{j_n}\} \text{ is infinite.} \tag{28.21}$$

Recall from Lemma 28.15 that $\mathcal{A}(j_n)$ is finite. Since for each $k \in K_1$, $\gamma(\xi_k)$ lies in $\overline{a}$ for some $a \in \mathcal{A}(j_n)$ (by definition of $\mathcal{A}(j_n)$), we can find $a_n \in \mathcal{A}(j_n)$ such that

$$K_2 = \{k \in K_1 \ ; \ \gamma(\xi_k) \in \overline{a}_n\} \text{ is infinite.} \tag{28.22}$$

This implies in particular that

$$\gamma(t_n) \in \overline{a}_n \cap \overline{B}_{j_n}. \tag{28.23}$$

Take

$$s_{n+1} = \sup\left\{s \in [0,1] \ ; \ \gamma(s) \in \overline{a}_n \cap \overline{B}_{j_n}\right\}. \tag{28.24}$$

Note that $s_{n+1} > t_n$, by (28.21) and (28.22).

If $s_{n+1} \notin R$, we take $t_{n+1} = s_{n+1}$. Otherwise, we set

$$t_{n+1} = \sup\{t \in [s_{n+1}, 1] \ ; \ [s_{n+1}, t] \subset R\}. \tag{28.25}$$

We still have (28.20) for $n + 1$, because R is closed and by (28.25). It is also clear that $t_{n+1} \geqslant s_{n+1} > t_n$.

Thus we can construct our two sequences by induction. Let us now check that the construction stops after a finite number of steps, i.e., that $t_n = 1$ for some n. Obviously it is enough to check that

$$\text{the pairs } (j_n, a_n) \text{ are all different,} \tag{28.26}$$

because there are only finitely many possible pairs.

To prove (28.26) notice that when we choose (j_n, a_n), we make sure that there are points $\xi_k > t_n$ such that $\gamma(\xi_k) \in \overline{a}_n \cap B_{j_n}$. [See (28.21) and (28.22).] If there was an $m < n$ such that $j_m = j_n$ and $a_m = a_n$, we would have $s_{m+1} > t_n$, by the definition (28.24) of s_{m+1}. This would contradict the way we constructed our sequences (i.e., the fact that $s_0 \leqslant t_0 \leqslant s_1 \leqslant t_1 \cdots$). Thus (28.26) holds, and our construction stops after finitely many steps. Note that

$$\text{for every } j \in J_0, \text{there are at most } C \text{ integers } n \text{ such that } j_n = j, \tag{28.27}$$

where C is as in Lemma 28.15. This follows from our proof of (28.26).

Next we want to study the variations of $v \circ \gamma$. First note that

$$v \text{ has a continuous extension to } \Omega \cup \mathcal{S}. \tag{28.28}$$

Indeed, simply set

$$v(x) = \lim_{\substack{z \to x \\ z \in \Omega}} v(x) \text{ for } x \in \mathcal{S} \; ; \tag{28.29}$$

the limit exists by Proposition 25.1. To check (28.28) we only need to verify that v is continuous at all points of $\mathcal{S}$, and this can be done exactly like for the continuity of w at the end of Section 27. [See the few lines after (27.27).]

Note that (28.28) is a little stronger than the first statement of Proposition 28.2, because Γ_m does to meet $G \setminus \mathcal{S}$, by (28.12) [See also (28.7).] Since γ is continuous and $\gamma([0,1]) \subset \Gamma_m \subset \Omega \cup \mathcal{S}$, we get that

$$v \circ \gamma \text{ has a continuous extension to } [0,1]. \tag{28.30}$$

Denote by N the last value of n in the construction above, i.e., the integer such that $t_N = 1$. Then
(28.31)

$$v(\gamma(1)) - v(\gamma(0)) = \sum_{n=0}^{N} \{v(\gamma(t_n)) - v(\gamma(s_n))\} + \sum_{n=0}^{N-1} \{v(\gamma(s_{n+1})) - v(\gamma(t_n))\}.$$

Set

$$R_1 = \bigcup_{n=0}^{N}]s_n, t_n[. \tag{28.32}$$

Note that when $s_n \notin R$ we chose $t_n = s_n$, and so we are not adding any interval to R_1. When $s_n \in R$, the interval $[s_n, t_n]$ is contained in R by (28.25) (or (28.19)). Hence $R_1 \subset R$.

We have seen (just below (28.18)) that $v \circ \gamma$ has a derivative in L^1 in some neighborhood of R. Hence

$$v(\gamma(t_n)) - v(\gamma(s_n)) = \int_{s_n}^{t_n} \nabla v(\gamma(t)) \cdot \gamma'(t) dt \tag{28.33}$$

(the case when $s_n \notin R$ and $t_n = s_n$ is of course trivial). We can sum over n and get that

$$\sum_{n=0}^{N} \{v(\gamma(t_n)) - v(\gamma(s_n))\} = \int_{R_1} \nabla v(\gamma(t)) \cdot \gamma'(t) dt. \tag{28.34}$$

To control the remaining sum, let us show that for $0 \leqslant n \leqslant N-1$,

$$|v(\gamma(s_{n+1})) - v(\gamma(t_n))| \leqslant C r_{j_n}^{1/2}. \tag{28.35}$$

By definition (28.24), s_{n+1} is the limit of a sequence $\{\eta_k\}$ of points in $[0,1]$ such that $\gamma(\eta_k) \in \overline{a}_n \cap \overline{B}_{j_n}$. On the other hand, (28.21) and (28.22) say that t_n is the limit of a subsequence of $\{\xi_k\}$ for which $\gamma(\xi_k) \in \overline{a}_n \cap B_{j_n}$. Thus $\gamma(s_{n+1})$ and $\gamma(t_n)$ lie in $\overline{a}_n \cap \overline{B}_{j_n}$.

We checked in Section 25 that $f = v + iw$, and hence v itself, satisfies the requirement (21.1) [See near (25.22) and (25.23).]. Thus we can apply Lemma 21.3 with $B = B(x_{j_n}, 2r_{j_n})$ (note that $2r_{j_n} < 1$, by (28.10)). We can take for x and y (in (21.2) and (21.4)) any pair of points of a_n, but then also points of $\overline{a}_n \cap (\Omega \cup S)$, by (28.28). In particular we get that

$$|v(\gamma(s_{n+1})) - v(\gamma(t_n))| \leqslant C\, r_{j_n}^{1/2}. \tag{28.36}$$

Hence

$$\left|\sum_{n=0}^{N-1} \{v(\gamma(s_{n+1})) - v(\gamma(t_n))\}\right| \leqslant C \sum_{j \in J_0(\gamma)} r_j^{1/2}, \tag{28.37}$$

because (28.27) tells us that each r_j shows up at most C times in the sum. Then

$$\left|v(\gamma(1)) - v(\gamma(0)) - \int_{R_1} \nabla v(\gamma(t)) \cdot \gamma'(t)dt\right| \leqslant C \sum_{j \in J_0(\gamma)} r_j^{1/2}, \tag{28.38}$$

by (28.31), (28.34), and (28.37).

This is essentially the best we can do with ε and m fixed. Our next goal is to average this over m, and then let ε tend to 0 (to get rid of the right-hand side of (28.38).

For each $k \geqslant 0$, set $\varepsilon_k = 2^{-k}$ and choose a covering of $\mathcal{S}$ by disks $B_{j,k}$, $j \in J^k$, as in (28.8)-(28.11) and with $\varepsilon = \varepsilon_k$.

For all $m \in \mathbb{R}$ and all (large) $M > 0$, set

$$J^k(m, M) = \Big\{j \in J^k\,;\, B_j \subset B(0, M) \text{ and } B_j \text{ meets a connected component of } \Gamma_m \text{ with diameter } \geqslant 2^{-k}\Big\}. \tag{28.39}$$

This looks a little strange, but it will work. Also set, for each choice of $\tau > 0$ (small) and $M > 0$,

$$Z_{\tau,M} = \Big\{m \in \mathbb{R}\,; \Gamma_m \text{ does not meet } G \setminus \mathcal{S} \text{ and (28.41) holds for infinitely many values of } k\Big\}, \tag{28.40}$$

that is,

$$\sum_{j \in J^k(m,M)} r_{j,k}^{1/2} < \tau, \tag{28.41}$$

where we denote by $r_{j,k}$ the radius of $B_{j,k}$. Also set

$$Z = \bigcap_{\substack{\tau>0\\ M>0}} Z_{\tau,M}. \tag{28.42}$$

Lemma 28.43. — $H^1(\mathbb{R} \setminus Z) = 0$.

We shall prove this in the next section; in the mean time we want to finish the proof of Proposition 28.2.

Because of Lemma 28.43, we can restrict our attention to $m \in Z$. Note that v has a continuous extension to $\Omega \cup \Gamma_m$, by (28.28).

Let $\gamma : [0,1] \to \Gamma_m$ be a simple arc with a derivative in L^1, as in the statement of the proposition. Choose M so large that $\gamma([0,1]) \subset B(0, M-1)$. Also let $\tau > 0$ be small, and choose k such that (28.41) holds and

$$2^{-k} < \operatorname{Min}\left\{\frac{1}{2}, \operatorname{diam}(\gamma([0,1]))\right\}. \tag{28.44}$$

We can find k because $m \in Z_{\tau,M}$.

Let us do the construction of the beginning of this section, with $\varepsilon = \varepsilon_k$, our corresponding choice of disks $B_{j,k}$, and the curve γ. We get a closed set $R_1 \subset [0,1]$ (which still depends on τ), and for which (28.38) holds.

Let us check that

$$J_0(\gamma) \subset J^k(m, M), \tag{28.45}$$

where $J_0(\gamma)$ is the subset of J^k that shows up in (28.38) and is defined just after (28.13). Let $j \in J_0(\gamma)$ be given. By definition of $J_0(\gamma)$, $j \in J(\gamma)$ (the set in (28.13)). Then $B_j = B_{j,k}$ meets $\gamma([0,1])$, which is contained in a component of Γ_m with diameter $\geqslant 2^{-k}$ (by (28.44)). Also $B_j \subset B(0,M)$ because $\gamma([0,1]) \subset B(0, M-1)$, B_j meets $\gamma([0,1])$, and by (28.10) and (28.44). This proves (28.45) (compare with (28.39)).

Because of (28.41) and (28.45), the right-hand side of (28.38) is less than $C\tau$. Let us summarize the situation. Set

$$R_0 = \{t \in [0,1]\ ;\ \gamma(t) \in \Omega\}. \tag{28.46}$$

For each $\tau > 0$ we have found a closed subset R_1 of R_0 (see a little above (28.33) for the statement that $R_1 \subset R$, and use (28.18)) such that

$$\left| v(\gamma(1)) - v(\gamma(0)) - \int_{R_1} \nabla v(\gamma(t)) \cdot \gamma'(t) dt \right| \leqslant C\tau, \tag{28.47}$$

by (28.38). We also want to control

$$I = \int_{R_0} |\nabla v(\gamma(t))|\, |\gamma'(t)|\, dt. \tag{28.48}$$

Since γ is a simple curve and $\gamma' \in L^1$,

$$I \leqslant \int_{\gamma([0,1]) \setminus G} |\nabla v(y)|\, dH^1(y) \leqslant \int_{\Gamma_m \cap B(0,M) \setminus G} |\nabla v|\, dH^1, \tag{28.49}$$

by a change of variables, the injectivity of γ, and the definition of M.

This is the right time for imposing a second condition on m. Set

$$Z_1 = \left\{ m \in Z\ ;\ \int_{\Gamma_m \cap B(0,M) \setminus G} |\nabla v|\, dH^1 < +\infty \text{ for all } M > 0 \right\}. \tag{28.50}$$

Lemma 28.51. — $H^1(\mathbb{R} \setminus Z_1) = 0$.

To prove this, apply the co-area formula (see for instance [Fe], p.248) to the function v, on the domain $B(0,M) \setminus G$. We find that

$$\int_m dH^1_{|\Gamma_m \cap B(0,M) \cap \Omega} dm = \mathbb{1}_{B(0,M)\setminus G} |\nabla w| \, dx, \tag{28.52}$$

where dx denotes Lebesgue measure, and where (28.52) is an identity between finite measures. To be honest, we should only apply the co-area theorem to Lipschitz functions. Here we can do this on compact subsets of $B(0,M) \setminus G$, where v is even C^1, cover $B(0,M) \setminus G$ by an increasing sequence of such compact sets, and then take a limit.

Let us apply (28.52) to the function $|\nabla v|$. Since both $|\nabla v|$ and $|\nabla w|$ satisfy the estimate in (25.20), we get that

$$\int_m \left\{ \int_{\Gamma_m \cap B(0,M) \cap \Omega} |\nabla v| \, dH^1 \right\} dm = \int_{B(0,M)\setminus G} |\nabla v| \, |\nabla w| \, dx \leqslant C(M), \tag{28.53}$$

where we have to take $C(M) = CM^2$ when (13.1) holds, since (25.20) is only valid (with the constant 8π) for balls of radius $\leqslant 1$.

We deduce Lemma 28.51 from (28.53) by applying Fubini, and then taking a countable union in M. □

Let us return to our argument. Because of Lemma 28.51, we may restrict our attention to the case when $m \in Z_1$. Then the integral I above is finite.

Let us apply the argument that led to (28.47) to the restriction of γ to any interval $[s,t] \subset [0,1]$. The equivalent of (28.47) yields

$$|v(\gamma(t)) - v(\gamma(s))| \leqslant \int_{R_0 \cap [s,t]} |\nabla v(\gamma(t))| \, |\gamma'(t)| \, dt + C\tau \tag{28.54}$$

because the analogue of R_1 in this situation is contained in $R_0 \cap [s,t]$.

This estimate is true with all choices of τ (and with a constant C that does not depend on τ). Thus it is also true with $\tau = 0$. Thus $v \circ \gamma$ has bounded variation, and its total variation is less than I. Moreover, the derivative of $v \circ \gamma$ is absolutely continuous with respect to $\mathbb{1}_{R_0}(t) \, |\nabla v(\gamma(t))| \, |\gamma'(t)| \, dt$, with a density $\leqslant 1$. Hence $(v \circ \gamma)' \in L^1[0,1])$, in the distribution sense.

This gives the first part of (28.4). The second part comes from the fact that the contribution of $[0,1] \setminus R_0$ to $\int_0^1 (v \circ \gamma)'(t) dt$ is zero (by the absolute continuity mentioned above), and the fact that $(v \circ \gamma)'(t) = \nabla v(\gamma(t)) \cdot \gamma'(t)$ almost-everywhere on R_0 (by the chain rule).

This completes our proof of Proposition 28.2, modulo Lemma 28.43 that will be proved in the next section. □

To conclude this section, we give a corollary of the proof of Proposition 28.2.

Corollary 28.55. — *For almost-very $m \in \mathbb{R}$, we have that if $\gamma : [0,1] \to \Gamma_m$ is such that*

(28.56) $$\gamma' \in L^1([0,1])$$

and

(28.57) $$\nabla v(\gamma(t)) \cdot \gamma'(t) \geqslant 0 \text{ for almost-every } t \in R_0 = \{t \in [0,1] \ ; \ \gamma(t) \in \Omega\},$$

then $v(\gamma(1)) \geqslant v(\gamma(0))$.

The main point of this new statement is that here we do not require γ to be simple; the conclusion is weaker, too. We can keep the same argument as in the proof of Proposition 28.2, up until (28.47). [Indeed, the fact that γ is a simple arc is only used later, in the estimate of I.] From (28.47) and (28.57) we deduce that

(28.58) $$v(\gamma(1)) - v(\gamma(0)) \geqslant -C\tau$$

and, since τ was arbitrary, the conclusion of Corollary 28.55 follows.

This proves the corollary (modulo lemma 28.43). □

29. Proof of Lemma 28.43

Keep the notations of the previous section. Set

(29.1) $$Y = \{m \in \mathbb{R} \ ; \ \Gamma_m \text{ does not meet } G \setminus \mathcal{S}\}$$

and, for each $M > 0$ and $\tau > 0$,

(29.2) $$X_{\tau,M} = \{m \in Y \ ; \ (28.41) \text{ only holds for finitely many values of } k\}.$$

Then $Z_{\tau,M} = Y \setminus X_{\tau,M}$, and

(29.3) $$Z = Y \setminus \bigcup_{\tau,M} X_{\tau,M},$$

by the definitions (28.40) and (28.42). Since $X_{\tau,M}$ is an increasing function of M and a decreasing function of τ, we can make the union in (29.3) countable by restricting to integer values of M and rational values of τ. Since in addition $\mathbb{R} \setminus Y$ is at most countable (by (28.7)), Lemma 28.43 will follow as soon as we prove that

(29.4) $$H^1(X_{\tau,M}) = 0$$

for all choices of τ and M. Fix τ and M, and set

(29.5) $$A_k = \int_{m \in Y} \Big\{ \sum_{j \in J^k(m,M)} r_{j,k}^{1/2} \Big\} dm,$$

where $J^k(m,M)$ is as in (28.41) and (28.39).

Lemma 29.6. — *There is a constant $C(M)$ such that*

(29.7) $$A_k \leqslant C(M)\varepsilon_k^{1/2}.$$

Maybe we should remind the reader that we chose $\varepsilon_k = 2^{-k}$, and then we constructed our covering of $\mathcal{S}$ by disks $B_j = B_{j,k}$ so that (28.8)-(28.11) would hold with $\varepsilon = \varepsilon_k$.

We want to see first how to deduce (29.4) from Lemma 29.6, and then we shall prove the lemma.

So let us assume that (29.7) holds. Set

$$X(k) = \{m \in X_{\tau,M} \ ; \ (28.41) \text{ does not hold for } k\}. \tag{29.8}$$

Then $\tau\,|X(k)| \leqslant A_k$ by (29.5) and Chebychev, and hence

$$|X(k)| \leqslant \tau^{-1}A_k \leqslant \tau^{-1}C(M)\varepsilon_k^{1/2} = \tau^{-1}C(M)2^{-k/2} \tag{29.9}$$

by (29.7) and our earlier choice of ε_k. Set

$$E\,(k_0) = \bigcup_{k \geqslant k_0} X(k)\ ; \tag{29.10}$$

then $|E\,(k_0)| \leqslant 4\tau^{-1}C(M)2^{-k_0/2}$, by (29.9).

We claim that $X_{\tau,M} \subset E\,(k_0)$ for every k_0. Indeed, if $m \in X_{\tau,M}$, (28.41) only holds for finitely many values of k. In particular, for every k_0 we can find $k \geqslant k_0$ such that (28.41) does not hold for k. This exactly means that $m \in E\,(k_0)$, as we claimed.

The consequence of our claim is that $|X_{\tau,M}| \leqslant |E\,(k_0)|$, which is as small as we want. Thus (29.4) actually follows from (29.7), and Lemma 28.43 will fall as soon as we prove Lemma 29.6.

To prove this last lemma, first observe that when $j \in J^k(m, M)$, B_j meets a component of Γ_m of diameter $\geqslant 2^{-k}$ (see (28.39)). Since $r_{j,k} \leqslant \varepsilon_k = 2^{-k}$ by (28.10), we get that

$$H^1\,(\Gamma_m \cap 2B_{j,k}) \geqslant r_{j,k}. \tag{29.11}$$

Set

$$\Gamma_m^* = \Gamma_m \cap B(0, M+1) \cap \Omega. \tag{29.12}$$

Note that $2B_{j,k} \subset B(0, M+1)$ by (28.39) and because $r_{j,k} \leqslant 1$. Also, for $m \in Y$, H^1-almost all of Γ_m lies in Ω, because Γ_m does not meet $G \setminus \mathcal{S}$ (by (29.1)), and $H^1(\mathcal{S}) = 0$ (by (28.6)). Thus

$$H^1\,(\Gamma_m^* \cap 2B_{j,k}) \geqslant r_{j,k} \tag{29.13}$$

when $m \in Y$, by (29.11), and hence

$$A_k \leqslant \int_{m \in Y} \Big\{ \sum_{j \in J^k(m,M)} r_{j,k}^{-1/2} H^1\,(\Gamma_m^* \cap 2B_{j,k}) \Big\} dm, \tag{29.14}$$

by (29.5). Let us replace $H^1(\Gamma_m^* \cap 2B_{j,k})$ with $\int_{\Gamma_m^*} \mathbb{1}_{2B_{j,k}} dH^1$, and then apply Beppo-Levi to exchange the series and the integral. We get that

$$A_k \leqslant \int_{m\in Y} \left\{ \int_{\Gamma_m^*} \Big[\sum_j r_{j,k}^{-1/2} \mathbb{1}_{2B_{j,k}} \Big] dH^1 \right\} dm, \tag{29.15}$$

where we can now be generous and sum over all $j \in J^k$. Denote by

$$h_k(x) = \sum_{j\in J^k} r_{j,k}^{-1/2} \mathbb{1}_{2B_{j,k}}(x) \tag{29.16}$$

the function that we need to integrate. Let us use again the co-area formula (28.52), but with M replaced with $M+1$. We get that

$$\begin{aligned} A_k &\leqslant \int_m \left\{ \int_{\Gamma_m^*} h_k(x) dH^1(x) \right\} dm = \int_{B(0,M+1)\setminus G} h_k(x)\, |\nabla w(x)|\, dx \\ &\leqslant \|h_k\|_2 \left\{ \int_{B(0,M+1)\setminus G} |\nabla w|^2 \right\}^{1/2} = \|h_k\|_2 \left\{ \int_{B(0,M+1)\setminus G} |\nabla v|^2 \right\}^{1/2} \\ &\leqslant C(M)\, \|h_k\|_2 , \end{aligned} \tag{29.17}$$

by Cauchy-Schwarz, because $|\nabla w| = |\nabla v|$ everywhere on Ω, and by (25.20). The precise value of $C(M)$ will not matter.

We see that Lemma 29.6, and hence Lemma 28.43, will follow as soon as we prove that

$$\|h_k\|_2^2 \leqslant C\,\varepsilon_k. \tag{29.18}$$

Fix k, and decompose h_k into

$$h_k = \sum_\ell f_\ell, \tag{29.19}$$

where

$$f_\ell = \sum_{j\in J(\ell)} r_j^{-1/2} \mathbb{1}_{2B_j}, \tag{29.20}$$

where we set $r_j = r_{j,k}$ and $B_j = B_{j,k}$ for convenience, and

$$J(\ell) = \left\{ j \in J^k \,;\; 2^{-\ell} \leqslant r_j < 2^{-\ell+1} \right\}. \tag{29.21}$$

Recall from (28.11) that the disks $\frac{1}{5}B_j$, $j \in J^k$, are disjoint. Since for each ℓ the B_j, $j \in J(\ell)$, all have comparable radii, we get that

$$\sum_{j\in J(\ell)} \mathbb{1}_{2B_j}(x) \leqslant C \tag{29.22}$$

everywhere. Set

$$W_\ell = \bigcup_{j \in J(\ell)} 2B_j\ ; \tag{29.23}$$

then

$$f_\ell \leqslant C 2^{\ell/2} \mathbb{1}_{W_\ell} \tag{29.24}$$

by (29.20)-(29.23). Next

$$\begin{aligned} \|h_k\|_2^2 &= \sum_\ell \sum_m \langle f_\ell, f_m\rangle = \sum_\ell \|f_\ell\|_2^2 + 2\sum_\ell \sum_{m>\ell} \langle f_\ell, f_m\rangle \\ &\leqslant C \sum_\ell 2^\ell \left|W_\ell\right| + C \sum_\ell \sum_{m>\ell} 2^{(\ell+m)/2} \left|W_\ell \cap W_m\right|, \end{aligned} \tag{29.25}$$

by (29.24). Set

$$a_\ell = 2^{\ell/2} \left|W_\ell\right|^{1/2}. \tag{29.26}$$

Then

$$\begin{aligned} \sum_\ell a_\ell^2 &= \sum_\ell 2^\ell \left|W_\ell\right| \leqslant \sum_\ell 2^\ell \sum_{j \in J(\ell)} \left|2B_j\right| \\ &\leqslant C \sum_\ell \sum_{j \in J(\ell)} r_j^{-1} r_j^2 = C \sum_{j \in J^k} r_j \leqslant C\,\varepsilon_k, \end{aligned} \tag{29.27}$$

by (29.23), (29.21), and (28.10).

The first sum in the right-hand side of (29.25) is thus less than $C\varepsilon_k$, and we are left with the rectangular terms. Let us first sum over

$$\Delta_1 = \Big\{(\ell, m)\ ;\ \ell < m \text{ and } \left|W_\ell \cap W_m\right| \leqslant 2^{(\ell-m)/5} \left|W_\ell\right|\Big\}. \tag{29.28}$$

[This corresponds to a reasonable behavior: we can expect W_ℓ to be significantly larger than W_m when $\ell < m$.] Set

$$\Sigma_1 = \sum\sum_{(\ell,m)\,\in \Delta_1} 2^{(\ell+m)/2} \left|W_\ell \cap W_m\right|. \tag{29.29}$$

Then

$$\begin{aligned} \Sigma_1 &\leqslant \sum\sum_{(\ell,m)\,\in \Delta_1} 2^{(\ell+m)/2} \left|W_m\right|^{1/2} \left|W_\ell \cap W_m\right|^{1/2} \\ &\leqslant \sum\sum_{(\ell,m)\,\in \Delta_1} 2^{(\ell+m)/2} \left|W_m\right|^{1/2} 2^{(\ell-m)/10} \left|W_\ell\right|^{1/2} \\ &\leqslant \sum\sum_{(\ell,m)\,\in \Delta_1} 2^{(\ell-m)/10} a_\ell\, a_m. \end{aligned} \tag{29.30}$$

Let us cut this sum in slices where $m - \ell = n$. We find that

$$\Sigma_1 \leqslant \sum_{n>0} \sum_{\ell} 2^{-n/10} a_\ell \, a_{\ell+n} \leqslant \sum_{n>0} 2^{-n/10} \left\| \{a_\ell\} \right\|_2 \left\| \{a_{\ell+n}\} \right\|_2 \leqslant C \, \varepsilon_k, \tag{29.31}$$

by Cauchy-Schwarz and (29.27).

We still need to sum over

$$\Delta_2 = \Big\{ (\ell, m) \ ; \ \ell < m \text{ and } |W_\ell \cap W_m| > 2^{(\ell-m)/5} \, |W_\ell| \Big\}. \tag{29.32}$$

For $(\ell, m) \in \Delta_2$,

$$\begin{aligned} a_\ell = 2^{\ell/2} \, |W_\ell|^{1/2} &\leqslant 2^{\ell/2} 2^{(m-\ell)/10} \, |W_\ell \cap W_m|^{1/2} \\ &\leqslant 2^{m/2} 2^{(\ell-m)/2} 2^{(m-\ell)/10} \, |W_m|^{1/2} \leqslant 2^{-\frac{2}{5}(m-\ell)} a_m, \end{aligned} \tag{29.33}$$

and so

$$\begin{aligned} \Sigma_2 := \sum\sum_{(\ell,m) \in \Delta_2} 2^{(\ell+m)/2} \, |W_\ell \cap W_m| &\leqslant \sum\sum_{(\ell,m) \in \Delta_2} a_\ell \, a_m \\ &\leqslant \sum_m \Big\{ \sum_{\ell<m} 2^{-\frac{2}{5}(m-\ell)} a_m^2 \Big\} \leqslant C \sum_m a_m^2 \leqslant C \, \varepsilon_k, \end{aligned} \tag{29.34}$$

by (29.27) again. Altogether (29.18) follows from (29.25), (29.27), (29.31), and (29.34).

This completes our proof of Lemma 28.43. As was observed at the end of last section, Proposition 28.2 and Corollary 28.55 follow. □

30. There are no loops in Γ_m

Proposition 30.1. — *For all $m \in \mathbb{R}$, $\mathbb{R}^2 \setminus \Gamma_m$ has no bounded connected component.*

Let $m \in \mathbb{R}$ be given. We want to assume that $\mathbb{R}^2 \setminus \Gamma_m$ has a bounded component U and derive a contradiction. Let us first assume that U has a smooth exterior boundary and rapidly sketch an argument for this case; this will give an idea of our strategy for this section.

Since U does not meet Γ_m, $w(z) - m$ has a fixed sign on U, and hence the normal derivative $\partial w/\partial n$ also has a fixed sign (but may vanish) on the exterior boundary of U. The same thing holds for the tangential derivative $\partial v/\partial \tau = \partial w/\partial n$ (recall that v and w are conjugated). Since the exterior boundary of U is a loop, the only option is that v is constant on it. Note that w also is constant on that exterior boundary (which is contained in Γ_m), and so v is constant on the component(s) of Ω that meets the exterior boundary of U. We shall see later in this section that this is impossible.

Our intention is to give sense to the argument above. The main difficulty will be to find a loop γ in ∂U such that U always stays on the same side of γ (so that we can control the sign of $\partial v/\partial \tau$ along γ, and then use Corollary 28.55). Our first goal is to replace U with a simply connected domain $V \supset U$; this will simplify the manipulation of boundaries. Essentially, V will be the union of U and the closure of its interior components. Set

$$(30.2)\qquad \begin{cases} V = \Big\{ z \in \mathbb{R}^2 \ ; \text{ we can find a simple, polygonal Jordan curve } \Gamma \text{ in } U \\ \qquad\qquad \text{such that } z \text{ lies in the bounded component of } \mathbb{R}^2 \setminus \Gamma \Big\}. \end{cases}$$

It is clear that $U \subset V$, and also that $\partial V \subset \partial U$. Since U is bounded, V is also bounded. Let us check that

$$(30.3)\qquad V \text{ is simply connected.}$$

Let Γ be a loop in V; we want to show that Γ can be deformed into a trivial loop (reduced to a point) inside V. By approximation, we can assume that Γ is a polygonal loop. By simple manipulations (essentially, composing loops), we can reduce to the case when Γ is simple.

After these reductions, let us try to surround Γ by a curve in U. Each point of Γ can be surrounded by such a curve, by (30.2). By compactness of Γ, we can find a finite collection of simple, polygonal curves γ_n in U, such that Γ is contained in the union of the bounded components of $\mathbb{R}^2 \setminus \gamma_n$. Then we use the following lemma a few times.

Lemma 30.4. — *Let γ_1 and γ_2 be two simple polygonal loops in the plane. Denote by Ω_i the bounded component of $\mathbb{R}^2 \setminus \gamma_i$. If Ω_1 meets Ω_2, then we can find a simple polygonal loop γ_3 such that $\gamma_3 \subset \gamma_1 \cup \gamma_2$ and $\Omega_1 \cup \Omega_2$ is contained in the bounded component of $\mathbb{R}^2 \setminus \gamma_3$.*

Of course we are abusing notation here, we do not distinguish between the loops and their images. We shall leave the proof of this lemma to the reader. A probably not optimal proof would consist in observing first that we can also assume that γ_1 and γ_2 only meet transversally, and then take for γ_3 the boundary of the unbounded component of $\gamma_1 \cup \gamma_2$ and follow the arguments in Section 23. □

After a finite number of applications of Lemma 30.4, we obtain a polygonal Jordan curve $\gamma \subset U$ such that the whole Γ is contained in the bounded component Ω^* of $\mathbb{R}^2 \setminus \gamma$. By (30.2), $\Omega^* \subset V$ and, since Ω^* is simply connected, Γ can be contracted in Ω^*, hence in V. This proves (30.3).

Next we want to parameterize ∂V in a reasonable way. The most rapid option would probably be to use the boundary values of a conformal mapping $\psi : \mathbb{D} \to V$. Indeed we would get that $\psi' \in H^1$ (the Hardy space) because $H^1(\partial V) < +\infty$, if m is chosen correctly ; in particular we would get that for these values of m the boundary values of ψ on $\partial\mathbb{D}$ would have a derivative in L^1. Since we want to avoid giving too many references to [Po], we shall use a somewhat heavier construction, based on Section 23.

Fix an origin 0 in U, choose a sequence ε_n that tends to 0 and such that $\varepsilon_n < \frac{1}{10}\,\mathrm{dist}\left(0, \mathbb{R}^2 \setminus U\right)$ (to be safe), and do again the construction of the beginning of Section 23, with $G^0 = \partial V$ and $\varepsilon = \varepsilon_n$. Choose disks B_i, $i \in I(\varepsilon_n)$, and define $H(\varepsilon_n)$ as in (23.2)-(23.6), but this time let $U(\varepsilon_n)$ denote the connected component of $\mathbb{R}^2 \setminus \overline{H(\varepsilon_n)}$ that contains 0. Thus we are just exchanging the roles of 0 and ∞, but otherwise nothing changes seriously. Since $G^0 = \partial V$ is connected (by (30.3)), we get that $\partial U(\varepsilon_n)$ is composed of a single, piecewise C^1, Jordan curve that we shall call $\Gamma(n)$. See the description of $\partial U(\varepsilon)$ after (23.7), and Remark 23.17; the proofs are the same in the present situation.

Denote by ℓ_n the length of $\Gamma(n)$. It will be good to know that

$$C^{-1} \leqslant \ell_n \leqslant C \text{ for all } n, \tag{30.5}$$

with a constant C that is allowed to depend on m and U, but not on n.

To get this, let us put a first constraint on m. We shall later put another similar constraint, but at the end of the argument we shall see how to get rid of both constraints. In the mean time let us assume that

$$\Gamma_m \text{ does not meet } G \setminus \mathcal{S} \tag{30.6}$$

and

$$H^1\left(\Gamma_m \cap B(0,M)\right) < +\infty \text{ for all } M > 0. \tag{30.7}$$

We shall see soon why (30.7) implies (30.5), but let us first check that

$$\text{almost-every } m \in \mathbb{R} \text{ satisfies (30.6) and (30.7).} \tag{30.8}$$

We already know from (28.7) that (30.6) holds for almost every m, and so it is enough to take care of (30.7). Let us use the co-area formula (28.52) again, this time applied to the constant function 1. We get that

$$\begin{aligned} \int_m H^1\left(\Gamma_m \cap B(0,M) \cap \Omega\right) dm &= \int_{B(0,M)\setminus G} |\nabla w|\, dx \\ &\leqslant |B(0,M)|^{1/2} \left\{ \int_{B(0,M)\setminus G} |\nabla w|^2 \right\}^{1/2} < +\infty, \end{aligned} \tag{30.9}$$

and hence $H^1\left(\Gamma_m \cap B(0,M) \cap \Omega\right) < +\infty$ for almost-every m.

Note that when (30.6) holds, $H^1\left(\Gamma_m \cap B(0,M) \cap \Omega\right) = H^1\left(\Gamma_m \cap B(0,M)\right)$, because $\Gamma_m \cap G \subset \mathcal{S}$ and $H^1(\mathcal{S}) = 0$ (by (28.6)). Thus for every (fixed) M, $H^1\left(\Gamma_m \cap B(0,M)\right) < +\infty$ for almost-every m; (30.7) follows by taking a countable union.

Now let us check (30.5). The analogues of Lemma 23.18 and (23.22) in our situation are still valid, with $\Omega_0 = V$ (the component of $\mathbb{R}^2 \setminus \partial V$ that contains 0). In particular, every compact subset of V is contained in $U(\varepsilon_n)$ for n sufficiently large, and hence $\mathrm{diam}\left(U(\varepsilon_n)\right) \geqslant \frac{1}{2}\mathrm{diam}(U) > 0$ for n large enough. This forces $\ell_n \geqslant \frac{1}{2}\mathrm{diam}(U)$ for

n large, and the first inequality in (30.5) follows. [We may always remove the first values of n.]

For the second part of (30.5) we return to the construction of $H(\varepsilon_n)$ and $U(\varepsilon_n)$ from the disks B_i, $i \in I(\varepsilon_n)$. Recall in particular that we chose our centers x_i, $i \in I(\varepsilon_n)$, sufficiently far from each other, so that

$$\text{(30.10)} \qquad \text{the disks } B_i' = B\left(x_i, 10^{-3}\varepsilon_n\right), i \in I(\varepsilon_n), \text{ are disjoint.}$$

(See (23.3)). On the other hand

$$\text{(30.11)} \qquad H^1(\partial V \cap B_i') \geqslant 10^{-3}\varepsilon_n \text{ for all } i \in I(\varepsilon_n),$$

because B_i and B_i' are centered on $G^0 = \partial V$ and ∂V is connected (by (30.3)). Of course we can safely assume that $\varepsilon_n < \operatorname{diam}\partial V$.

From this and (30.10) we deduce that

$$\text{(30.12)} \qquad \#I(\varepsilon_n) \leqslant \sum_{i \in I(\varepsilon_n)} 10^3 \varepsilon_n^{-1} H^1(\partial V \cap B_i') \leqslant 10^3 \varepsilon_n^{-1} H^1(\partial V),$$

and then

$$\text{(30.13)} \qquad \ell_n = H^1(\partial U(\varepsilon_n)) \leqslant H^1(\partial H(\varepsilon_n)) \leqslant \sum_{i \in I(\varepsilon_n)} H^1(\partial B_i) \leqslant C\, H^1(\partial V)$$

because $\Gamma(n) = \partial U(\varepsilon_n)$ is a simple rectifiable curve, and by (23.6) and (30.12). This proves (30.5).

Let $z_n : \mathbb{S}^1 \to \Gamma(n)$ denote a parameterization of $\Gamma(n)$ with constant speed. Let us assume that

$$\text{(30.14)} \qquad z_n \text{ preserves the trigonometric sense,}$$

by which we mean that $U(\varepsilon_n)$ stays on our left when we follow $z_n(t)$ and t runs along $\mathbb{S}^1$ in the trigonometric sense. Of course this can be arranged.

We know from (30.5) that the z_n are Lipschitz, uniformly in n, and so we may always assume that $\{z_n\}$ converges uniformly on $\mathbb{S}^1$ to some Lipschitz function z. [Otherwise, replace $\{\varepsilon_n\}$ with a subsequence.] Note that $z(\mathbb{S}^1) \subset \partial V$, by (23.10). Our next goal will be to prove that sub-arcs of z satisfy the main hypothesis (28.57) of Corollary 28.55.

Since U is a component of $\mathbb{R}^2 \setminus \Gamma_m$, $w - m$ does not vanish on U and hence keeps a constant sign there. Without loss of generality, let us assume that

$$\text{(30.15)} \qquad w(x) - m > 0 \text{ on } U.$$

We want to check that

$$\text{(30.16)} \qquad (v \circ z)'(t) > 0 \text{ for almost-every } t \in R_0,$$

where $R_0 = \{t \in \mathbb{S}^1 \ ; \ z(t) \in \Omega\}$ is essentially as in Section 28, and where we abuse notations slightly and identify $\mathbb{S}^1$ with the torus to define derivatives. [We should really have considered $v \circ z\,(e^{it})$, $t \in \mathbb{R}$, instead of $v \circ z(t)$.]

Let $t \in \mathbb{S}^1$ be given, and set $z = z(t)$. Suppose that $z \in \Omega$. Then w is harmonic near z. It is not constant near z, because otherwise it would be equal to m in a neighborhood of z, and z would not lie in $\partial V \subset \partial U$. Let us first assume that $\nabla w(z) \neq 0$.

In this first case there is a small disk D centered at z such that $\Gamma_m \cap D$ is an analytic curve through z. Since $\partial V \subset \partial U \subset \Gamma_m$ and $z \in \partial V$, V cannot lie on both sides of Γ_m, and hence

$$V \cap D = U \cap D = \{x \in D \ ; \ w(x) > m\} \tag{30.17}$$

and $\partial V \cap D = \partial U \cap D = \Gamma_m \cap D$.

Then $\Gamma(n) \cap D$ is, for n large enough, a slightly corrugated curve that goes along Γ_m and lies in V. [The other side of $\partial H\,(\varepsilon_n)$ is not contained in $\Gamma(n)$, because $U\,(\varepsilon_n) \subset V$, by (23.21).] See Figure 30.1.

Let us assume also that $z'(t)$ exists. Then $z'(t) \neq 0$, because $\Gamma(n)$ was always parameterized with constant speed $\geqslant C^{-1}$ (by (30.5)), and because we know what $\Gamma(n)$ looks like in D. From (30.14) and our description of $\Gamma(n)$ and $U\,(\varepsilon_n)$ in D, we deduce that U and V lie on the left of z, when one looks in the direction of $z'(t)$. We also know that $\partial w / \partial n > 0$ at z (with a unit normal which points towards U and V, by (30.15) and because $\nabla w(z) \neq 0$ in the present case. Since w is conjugated to v we get that $\partial v / \partial \tau$, the tangential derivative of v in the direction of $z'(t)$, is positive. This proves (30.16) when $\nabla w(z) \neq 0$ and $z'(t)$ exists.

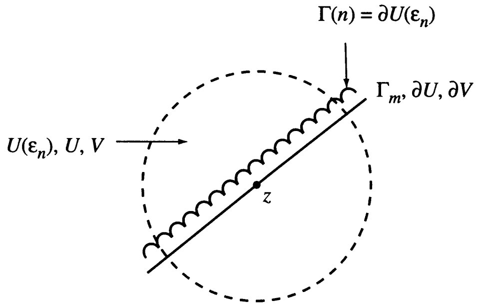


FIGURE 30.1

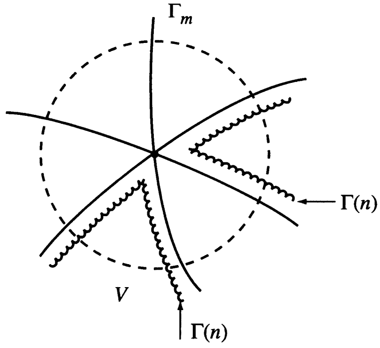


FIGURE 30.2

Next consider the case when $\nabla w(z) = 0$. By Puiseux's theorem, there is a small disk D centered at z such that $\Gamma_m \cap D$ is composed of a finite collection of analytic curves that converge to z in a starlike way. [See Figure 30.2.] Since $\partial V \subset \partial U \subset \partial \Gamma_m$, $V \cap D$ is composed of some of the sectors delimited by Γ_m. For n sufficiently large, $\Gamma(n) \cap D$ is composed of a finite number of curves that follow pieces of Γ_m. Figure 30.2 gives a reasonably realistic possibility, but we shall not need to know the combinatorics precisely. The main point is that since the curves $\Gamma(n)$ are parameterized at constant, never too small, speed, there are only finitely many points $t' \in \mathbb{S}^1$ such that $z(t') = z$. In other words, all points $t \in \mathbb{S}^1$ such that $z(t) \in \Omega$ and $\nabla w(z(t)) = 0$ are isolated in $\mathbb{S}^1$, and there is only countably many of them.

Since z is Lipschitz, $z'(t)$ exists for almost every t. Thus our first case occurs for almost every t such that $z(t) \in \Omega$. This proves (30.16).

Let us now put our second restriction on m: let us assume that m satisfies the condition of Corollary 28.55. As for our earlier conditions (30.6) and (30.7), this one is satisfied for almost-every m.

For each choice of $a > 0$ and $b \in \mathbb{R}$, the mapping $\gamma : [0,1] \to \Gamma_m$ defined by $\gamma(t) = z\left(e^{iat+ib}\right)$ satisfies the condition of Corollary 28.55: $\gamma' \in L^1([0,1])$ because z is Lipschitz, and (28.57) follows from (30.16). The conclusion of the corollary is that $v(z(t)) \geqslant v(z(s))$ for all choices of $s, t \in \mathbb{S}^1$. In other words, $v \circ z$ is constant on $\mathbb{S}^1$.

Since R_0 is open by definition this can only be compatible with (30.16) if R_0 is empty. On the other hand, we claim that

$$H^1\left(z\left(\mathbb{S}^1\right)\right) > 0. \tag{30.18}$$

Indeed z_n is the constant speed parameterization of $\Gamma(n) = \partial U(\varepsilon_n)$ which preserves the trigonometric sense (as in 30.14)), and hence it has winding number 1 around each point of $U(\varepsilon_n)$. Now we can apply Lemma 23.18 with $\Omega_0 = V$ (see half-way between (30.9) and (30.10)), and so V is the increasing union of the $U(\varepsilon_n)$. Therefore z has winding number 1 around every point of V, and of course this cannot happen if $\operatorname{diam}\left(z\left(\mathbb{S}^1\right)\right)$ is too small. This proves our claim (30.18).

Because of (30.6) and the fact that $z\left(\mathbb{S}^1\right) \subset \partial V \subset \partial U \subset \Gamma_m$, we now have that

$$z\left(\mathbb{S}^1\right) \subset G \cap \Gamma_m \subset \mathcal{S} \tag{30.19}$$

because R_0 is empty. Since $H^1(\mathcal{S}) = 0$ by (28.6), this contradicts (30.18).

So we finally reached the desired contradiction, but only under some additional conditions on m. Fortunately, these conditions are satisfied almost-everywhere, and so we proved that

$$\mathbb{R}^2 \setminus \Gamma_m \text{ has no bounded component for almost-every } m \in \mathbb{R}. \tag{30.20}$$

The other values of m are now easy to get. Let $m \in \mathbb{R}$ be an exception to (30.20), and let U be a bounded component of $\mathbb{R}^2 \setminus \Gamma_m$. As we said before , $w - m$ keeps the same sign on U, and we may assume without loss of generality that $w > m$ on U. Let z_0 be any point of U, and choose m_1 such that $m < m_1 < w(z_0)$ and for which

(30.20) holds. Denote by U_1 the connected component of z_0 in $\mathbb{R}^2 \setminus \Gamma_{m_1}$. As before $w - m_1$ keeps a constant sign on U_1, and so $w > m_1$ on U_1 (because of z_0). Thus U_1 does not meet Γ_m, and since it is connected and meets U, it must be contained in U. Then U_1 is bounded, a contradiction with the definition of m_1.

This completes our proof of Proposition 30.1. □

31. Almost-every Γ_m is composed of trees without endpoints

Let $m \in \mathbb{R}$ be given, and suppose that

(31.1) $$H^1(\Gamma_m \cap B(0,M)) < +\infty \text{ for all } M > 0.$$

We know from (30.8) that this is the case for almost-every m. Let us also suppose that $\Gamma_m \neq \varnothing$, and let Γ_m^0 be a connected component of Γ_m. Let us first show that

(31.2) $$\Gamma_m^0 \text{ is not bounded.}$$

Suppose to the contrary that Γ_m^0 is bounded. We want to find a bounded Jordan curve γ such that

(31.3) $$\gamma \text{ does not meet } \Gamma_m,$$

and

(31.4) $$\Gamma_m^0 \text{ is contained in the bounded component of } \mathbb{R}^2 \setminus \gamma.$$

First we want to apply Lemma 25.3 with G replaced by Γ_m. In the proof of that lemma, we only used the analogue of (31.1) for G and the fact that G is rectifiable (besides topological conditions). This is still the case here, because Γ_m is rectifiable. Indeed $\Gamma_m \cap G$ is rectifiable (since G is), and $\Gamma_m \cap \Omega$ is an at most countable union of analytic curves. [The case when w would be locally constant in a neighborhood of some point of Γ_m is excluded by (31.1).]

So we can apply Lemma 25.3 to Γ_m and its bounded component Γ_m^0, and we get a closed set $G^0 \subset \Gamma_m$ which is bounded, contains Γ_m^0, and for which $\operatorname{dist}\left(G^0, \Gamma_m \setminus G^0\right) > 0$.

Next apply apply the construction of Section 23 to surround G^0 by Jordan curves. Choose the parameter ε in the construction so small that the surrounding curves do not meet Γ_m. [See Remark 23.24 and in particular (23.26).] Since Γ_m^0 is connected, it is contained in the domain bounded by a single surrounding curve γ; this curve satisfies (31.3) and (31.4), as required.

Because of (31.3), $w(x)-m$ has a constant sign on γ; let us assume for instance that $w(x) > m$ on γ. Set $m_2 = \inf\{w(x)\ ;\ x \in \gamma\}$, and choose m_1 such that $m < m_1 < m_2$. Denote by U_1 the component of $\mathbb{R}^2 \setminus \Gamma_{m_1}$ that contains some given point of Γ_m^0. Since Γ_m^0 is connected and does not meet $\partial U_1 \subset \Gamma_{m_1}$, Γ_m^0 is contained in U_1. Then $w(x) < m_1$ on U_1, because $w(x) - m_1$ has a fixed sign on U_1, and U_1 contains Γ_m^0.

Then U_1 does not meet γ (by definition of m_2), and so U_1 is contained in the bounded component of $\mathbb{R}^2 \setminus \gamma$ (by (31.4)). This contradicts Proposition 30.1; (31.2) follows.

Now we want to show that "Γ_m^0 is a tree", by which we mean that

$$(31.5) \qquad \begin{cases} \text{for all choices of } x, y \in \Gamma_m^0, \text{ there is a unique} \\ \text{simple arc } \Gamma_m(x,y) \subset \Gamma_m^0 \text{ that connects them.} \end{cases}$$

For the existence, simply see Lemma 19.14, applied to Γ_m^0; the condition (19.1) is satisfied, by (31.1). To prove the uniqueness, let us assume that we have two simple arcs γ_1 and γ_2 supported on Γ_m^0, and which connect the same points x and y.

It will be more precise to consider parameterizations (rather than sets). So for $i = 1, 2$, $\gamma_i : I_i \to \Gamma_m^0$ is an injective continuous mapping, and we want to show that γ_1 and γ_2 are equivalent, i.e., that they can be obtained from each other by composing with a homeomorphism.

If $\gamma_1(I_1) \subset \gamma_2(I_2)$, it is fairly easy to show that γ_1 and γ_2 are equivalent. For instance, we may compose both arcs with the inverse of γ_2 (which is homeomorphism between $\gamma_2(I_2)$ and I_2), and we are reduced to the case when γ_2 is the identity on I_2 and $\gamma_1 : I_1 \to I_2$ is continuous, injective, and has the same endpoints. In this case γ_1 is a homeomorphism of I_1 to I_2, and our initial arcs γ_i were equivalent.

So we may assume that we can find $z_1 \in \gamma_1(I_1) \setminus \gamma_2(I_2)$. Let $J \subset I_1$ be a maximal interval such that $\gamma_1(J)$ contains z_1 and is disjoint from $\gamma_2(I_2)$. The two endpoints of $\gamma_1(J)$ lie in $\gamma_2(I_2)$, and there is a (simple) sub-arc γ_3 of γ_2 that connects them. We can glue γ_3 to the restriction of γ_1 to J, and we get a (closed) Jordan curve γ supported on Γ_m^0. This is impossible, because the bounded component of $\mathbb{R}^2 \setminus \gamma$ would necessarily contain bounded components of $\mathbb{R}^2 \setminus \Gamma_m$, in contradiction to Proposition 30.1. This proves (31.5).

Now we want to check that Γ_m^0 has no endpoints. Let us be more precise.

Lemma 31.6. — *Assume, in addition to (31.1), that*

$$(31.7) \qquad \Gamma_m \cap G \subset \mathcal{S}.$$

For every simple, rectifiable arc $\gamma : I \to \Gamma_m^0$, parameterized by arclength, we can find a simple rectifiable arc $\widetilde{\gamma} : \mathbb{R} \to \Gamma_m^0$, also parameterized by arclength, and which extends γ.

Note that (31.7) holds for almost every m, by (28.7). It is easy to show, just by chasing definitions, that every γ as above can be extended to a maximal interval $J \supset I$, with an extension which we shall still denote by γ, and which is still simple, supported on Γ_m^0, and parameterized by arclength. What we need to show is that $J = \mathbb{R}$.

So let us assume that $J \neq \mathbb{R}$, and try to derive a contradiction. Without loss of generality, we can assume that the final endpoint of J is 0.

Since γ is Lipschitz, it has a limit z_0 at 0. If $0 \notin J$, this means that the obvious extension of γ to $J \cup \{0\}$ is not simple. Then $z_0 \in \gamma(J)$, and Γ_m^0 contains a loop; we have seen just before the statement of Lemma 31.6 that this contradicts Proposition 30.1. So $0 \in J$. The same argument works for the initial endpoint of J, and so $J = [a, 0]$ for some $a \leqslant 0$, or $J = (-\infty, 0]$.

Let us call regular point of Γ_m^0 a point $z \in \Gamma_m^0$ such that for $r > 0$ small enough, $\Gamma_m^0 \cap B(z,r)$ is a simple C^1-curve through z and that crosses $B(z,r)$. Let us check that

$$H^1\text{-almost every point } z \in \Gamma_m^0 \text{ is a regular point of } \Gamma_m^0. \tag{31.8}$$

By (31.7) and (28.6), we need only consider points of $\Gamma_m^0 \cap \Omega$. But all points of $\Gamma_m^0 \cap \Omega$ such that $\nabla w(z) \neq 0$ are clearly regular points of Γ_m^0, and there are at most countably many points of $\Gamma_m^0 \cap \Omega$ where $\nabla w(z) = 0$. [Such points are isolated in Γ_m^0; note that (31.1) does not allow w to be locally constant near a point of Γ_m.] This proves (31.8).

Next we want to prove that

(31.9)

$$\Gamma_m^0 \setminus \{z\} \text{ has exactly 2 connected components when } z \text{ is a regular point of } \Gamma_m^0.$$

Let z be a regular point of Γ_m^0, and choose a small disk $B = B(z,r)$ such that $B \cap \Gamma_m^0$ is a simple C^1 curve that crosses B. Then $B \cap \Gamma_m^0 \setminus \{z\}$ has exactly two components, which we call γ_1 and γ_2. By Lemma 19.2 and (31.1), Γ_m^0 is arcwise connected, and so every point $y \in \Gamma_m^0 \setminus \{z\}$ can be connected to z by a rectifiable arc in Γ_m^0. We can stop this arc just before it hits z for the first time, and we get a rectifiable arc in $\Gamma_m^0 \setminus \{z\}$ that connects y to γ_1 or γ_2. Thus $\Gamma_m^0 \setminus \{z\}$ has at most two components.

To complete the proof of (31.9), we just need to check that γ_1 and γ_2 do not lie in the same component of $\Gamma_m^0 \setminus \{z\}$. If this were the case, Lemma 19.14 (applied to the component in question, maybe minus a small disk centered at z to make it compact) would give us a simple arc γ_3 supported on $\Gamma_m^0 \setminus \{z\}$ that connects γ_1 to γ_2. Then we could add to γ_3 a little arc of $\Gamma_m^0 \cap B$ and obtain a (closed) Jordan curve contained in Γ_m^0. As we have seen earlier (for instance before the statement of Lemma 31.6), this is impossible because of Proposition 30.1. This proves (31.9).

Return to Lemma 31.6 and our maximal arc $\gamma : J \to \Gamma_m^0$. Note that $J = \{0\}$ is impossible, because Γ_m^0 is not reduced to $\{z_0\}$ by (31.2), and hence it contains lots of nontrivial simple curves starting from $z_0 = \gamma(0)$ (by Lemma 19.14). Those curves would of course be extensions of the trivial arc γ. Set

$$J^* = \left\{ t \in \mathrm{int}(J) \ ; \ \gamma(t) \text{ is a regular point of } \Gamma_m^0 \right\}. \tag{31.10}$$

Let $t \in J^*$ be given, and set $z = \gamma(t)$. Since γ is simple, $\gamma(J) \setminus \{z\}$ has exactly two components. Call $\gamma_+ = \gamma_+(t)$ the component that contains z_0, and $\gamma_- = \gamma_-(t)$ the other one. Since $\gamma_\pm$ is connected and contained in $\Gamma_m^0 \setminus \{z\}$, it is contained in a component $\Gamma_{\pm} = \Gamma_{\pm}(t)$ of $\Gamma_m^0 \setminus \{z\}$. Note that $\Gamma_+ \neq \Gamma_-$, because the two small arcs γ_1 and γ_2 that compose $B \cap \Gamma_m^0$ (in the discussion above) are contained, one in γ_+ and the other one in γ_-.

Lemma 31.11. — *If t is close enough to 0, Γ_+ is bounded.*

First note that our condition on t may be needed. We expect Γ_m^0 to be an infinite tree with possibly many branches, and we don't want one of these branches to leave from γ_+ between z and z_0.

Let us assume that the lemma is false and try to get a contradiction. Choose a first point $t_1 \in J^*$ such that $\Gamma_+(t_1)$ is not bounded. Let M be so large that the whole arc of γ between $z_1 = \gamma(t_1)$ and z_0 is contained in $B(0, M-1)$. Choose $y_1 \in \Gamma_+(t_1) \setminus B(0,M)$. We can connect y_1 to z_0 by a simple arc $\xi_1 \subset \Gamma_+(t_1)$. To see this, first observe that if D is a small enough open disk centered at z, then $\Gamma_+(t_1) \setminus D$ is still connected and contains y_1 and z_0. The point of the manipulation is that now $\Gamma_+(t_1) \setminus D$ is closed and we can apply Lemma 19.14 to it to get ξ_1.

Denote by z_1' the first point of $\gamma(J)$ when we go from y_1 to z_0 along ξ_1. Since $\xi_1 \subset \Gamma_+(t_1)$ and $\gamma_-(t_1) \subset \Gamma_-(t_1) \neq \Gamma_+(t_1)$, z_1' must lie on $\gamma_+(t_1)$ and hence $z_1' = \gamma(t_1')$ for some $t_1' > t_1$. If $t_1' = 0$, we can extend γ by adding to it a parameterization by arclength of the arc of ξ_1 between z_1' and y_1; since by definition of z_1' this arc does not meet $\gamma(J)$, our extension of γ is still simple (and supported on Γ_m^0). This contradicts the maximality of J. Hence $t_1 < t_1' < 0$.

Since we assumed our lemma to fail, we can find $t_2 \in J^* \cap (t_1', 0)$ such that $\Gamma_+(t_2)$ is unbounded. As before, we choose $y_2 \in \Gamma_+(t_2) \setminus B(0,M)$, connect it to z_0 by a simple arc $\xi_2 \subset \Gamma_+(t_2)$, and let $z_2' = \gamma(t_2')$ denote the first point of $\gamma(J)$ that we hit when we go from y_2 to z_0 along ξ_2. For the same reason as before, $t_2 < t_2' < 0$.

Let us continue this construction; we get sequences $\{t_n\}$, $\{y_n\}$, $\{\xi_n\}$, and $\{t_n'\}$; the main point of the argument is that if ξ_n' denotes the arc of ξ_n between y_n and z_n',

(31.12) the arcs ξ_n' are disjoint.

To see this, let $k < n$ be given. Then $t_k' < t_n < t_n'$ by construction, and so $z_k' = \gamma(t_k') \in \gamma_-(t_n)$. The only point of $\xi_k' \cap \gamma(J)$ is z_k'. Since $z_k' \neq z_n = \gamma(t_n)$, ξ_k' is contained in $\Gamma_m^0 \setminus \{z_n\}$. Since ξ_k' is connected, it is contained in the component of $\Gamma_m^0 \setminus \{z_n\}$ that contains z_k'. This component is $\Gamma_-(t_n)$ because $z_k' \in \gamma_-(t_n)$ and $\gamma_-(t_n) \subset \Gamma_-(t_n)$.

Similarly, $z_n' \in \gamma_+(t_n)$ because $t_n' > t_n$. The only point of $\xi_n' \cap \gamma(J)$ is z_n' and, since $z_n' \neq z_n$ because $t_n' \neq t_n$, ξ_n' is contained in $\Gamma_m^0 \setminus \{z_n\}$. Since it is connected, it is contained in a single component of $\Gamma_m^0 \setminus \{z_n\}$, and since it contains $z_n' \in \gamma_+(t_n) \subset \Gamma_+(t_n)$, this component is $\Gamma_+(t_n)$. Altogether ξ_k' and ξ_n' lie in different components of $\Gamma_m^0 \setminus \{z_n\}$, and (31.12) follows.

For each n, $H^1(\xi'_n \cap B(0,M)) \geqslant 1$, because ξ'_n starts at $y_n \in \mathbb{R}^2 \setminus B(0,M)$ and ends at z'_n, which lies in $B(0,M-1)$ because all the curve γ between $z_1 = \gamma(t_1)$ and $z_0 = \gamma(0)$ is contained in $B(0,M-1)$. [Also recall that $t_1 < t'_n < 0$ by construction.]

From this and (31.12) we easily deduce that $H^1(\Gamma_m \cap B(0,M)) = +\infty$ because all the ξ'_n are supported on Γ_m. This contradicts (31.1). Lemma 31.11 follows from this contradiction. □

Choose $t \in J^*$ such that $\Gamma_+ = \Gamma_+(t)$ is bounded, and let B be a small open disk centered at $z = \gamma(t)$ and such that $\Gamma_m^0 \cap 2B$ is a simple, C^1-curve through $2B$. Then $G^0 = \Gamma_+ \setminus B$ is compact, connected, and not empty (because it contains z_0) if B is small enough.

Apply the construction of Section 23 (and in particular Remark 23.17) to G^0. We get a Jordan curve η such that

(31.13) $$G^0 \text{ is contained in the bounded component of } \mathbb{R}^2 \setminus \eta,$$

and

(31.14) $$\varepsilon \leqslant \operatorname{dist}(x, G^0) \leqslant 2\varepsilon \text{ for } x \in \eta.$$

where ε is the small parameter in the construction of Section 23, and (31.14) follows from (23.10). See Figure 31.1 already. Note that

(31.15) $$\operatorname{dist}(G^0, \Gamma_m \setminus \Gamma_+) > 0$$

because these two sets are disjoint, G^0 is compact, and $\Gamma_m \setminus \Gamma_+$ is closed. Hence

(31.16) $$\eta \text{ does not meet } G^0 \cup (\Gamma_m \setminus \Gamma_+)$$

if we choose ε small enough, by (31.14). Then the only place where η may meet Γ_m is in B, on the little arc $B \cap \overline{\Gamma}_+$. See Figure 31.1.

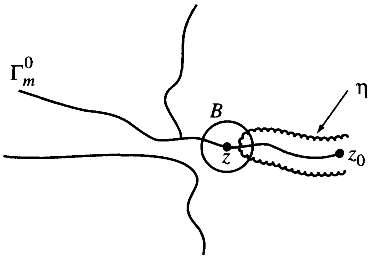


FIGURE 31.1

We are now ready to reach the desired contradiction. Since z is a regular point of Γ_0^m and $\Gamma_0^m \cap 2B$ is a smooth curve that crosses $2B$, we have a good control on what the curve η looks like inside B. By construction, if ε is small enough, η crosses Γ_m^0 exactly once, and transversally.

Recall from our proof of (31.8) that H^1-almost every point $z \in \Gamma_m^0$ is not only a regular point of Γ_m^0, but even a point of Ω such that $\nabla w(z) \neq 0$. Thus we can assume that our $t \in J^*$ was chosen so that $z = \gamma(t) \in \Omega$ and $\nabla w(z) \neq 0$ as well. In this case (and if B was chosen so small that $\nabla w \neq 0$ on B), the restriction of $w - m$ to η changes signs at the point where η crosses Γ_m^0. Since the rest of η does not meet Γ_m, $w - m$ does not change signs there. This is clearly impossible.

Recall that the present argument started when we assumed that we could find a maximal arc $\gamma : J \to \Gamma_m^0$ as in Lemma 31.6, but with $J \neq \mathbb{R}$. The contradiction that we just reached completes our proof of Lemma 31.6. □

32. Our final description of the levels sets Γ_m

So far we have only assumed that

$$\Omega = \mathbb{R}^2 \setminus G \text{ is connected} \tag{32.1}$$

(in addition to (13.1) or (13.2), of course), and we have proved that for almost-every $m \in \mathbb{R}$, Γ_m is either empty or a union of rectifiable (by (31.1)) trees (i.e., with no loops, as in Proposition 30.1), with infinite branches only (as in Lemma 31.6). Thus Γ_m may a priori look like the sets suggested by Figure 32.1.

Note that Figure 32.1 strongly suggest that almost-every Γ_m is actually composed of finitely many Jordan curves through ∞ (the true trees with many branches being exceptional). We shall only prove this under somewhat stronger assumptions on the behavior of (v, G) at ∞, and then there will be only one Jordan curve.

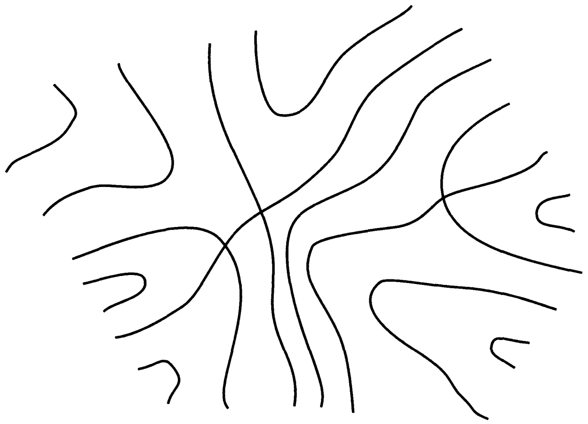


FIGURE 32.1. A few level sets Γ_m.

In this section we want to add a few hypotheses on the behavior of (v, G) at infinity, whose effect will be to bound by 2 the number of branches of Γ_m that escape to ∞.

Under these assumptions, we shall show that for almost-every $m \in \mathbb{R}$, Γ_m is either empty or a single Jordan curve through ∞. Then we shall check that (in the second case) v is strictly monotonous on the Jordan curve Γ_m.

Let us first describe our additional hypotheses on (v, G). We do not try to give optimal conditions here; the main point is to make sure that Γ_m has at most two infinite branches. First, we assume that

(32.2)
G has exactly one unbounded connected component, which we denote by G_{00}.

When (13.1) holds, this is always true, because G contains L and $G \setminus L$ is bounded.

Let us use (32.2) to normalize w; we require that

$$w(x) = 0 \text{ on } G_{00}. \tag{32.3}$$

[This is easy to get, because we can add a constant to w; also recall from Section 26 that w is defined and continuous on the whole plane and constant on each component of G.] We also assume that

(32.4) for all $m > 0$, we can find arbitrarily large radii R such that $\Gamma_m \cap \partial B_R = \varnothing$,

and

$$\begin{cases} \text{for all } m < 0, \text{ we can find arbitrarily large radii } R \\ \text{such that } \Gamma_m \cap \partial B_R \text{ has exactly two points.} \end{cases} \tag{32.5}$$

We shall see later that the hypotheses (32.1)-(32.5) are satisfied by (v, G) or by $(-v, G)$ when (v, G) looks sufficiently like a cracktip near ∞, and when (13.1) holds. Let us not worry about this now, and continue our description of the level sets Γ_m.

Proposition 32.6. — *Suppose that* (v, G) *satisfies* (13.1) *or* (13.2), *and the conditions* (32.1)-(32.5). *Then*

$$w(z) \leqslant 0 \text{ for all } z \in \mathbb{R}^2, \tag{32.7}$$

$$\Gamma_0 := \{z \in \mathbb{R}^2 \ ; \ w(z) = 0\} = G_{00} \ , \tag{32.8}$$

and, for almost-every $m < 0$,

$$\Gamma_m \text{ is a rectifiable Jordan curve through } \infty. \tag{32.9}$$

Let us first check (32.7). Because of (32.4), all the connected components of Γ_m are bounded when $m > 0$. Since for almost all $m > 0$, (31.2) tells us that no component of Γ_m is bounded, we see that $\Gamma_m = \varnothing$ for almost-all $m > 0$. In other words, for almost-all $m > 0$, w does not take the value m. Since w is continuous, it simply cannot take any positive value, and (32.7) holds.

Now we want to prove (32.9), and we give ourselves $m < 0$ such that all the conditions of Section 31 hold. Because of (32.5), Γ_m is not empty. Then we can apply Lemma 31.6 to a trivial arc, and we get a simple, rectifiable arc $\gamma : \mathbb{R} \to \Gamma_m$, parameterized by arclength. Because of (31.1), $\gamma(t)$ cannot stay too long in a given disk, and so

$$\lim_{t\to+\infty} |\gamma(t)| = \lim_{t\to-\infty} |\gamma(t)| = +\infty. \tag{32.10}$$

In other words, our simple curve $\Gamma = \gamma(\mathbb{R})$ goes through ∞.

To complete our proof of (32.9), we still need to check that $\Gamma_m = \Gamma$. Note that for R large, $\Gamma \cap \partial B_R$ contains at least two points, because Γ is simple and by (32.10). If in addition

$$\Gamma_m \cap \partial B_R \text{ has exactly two points,} \tag{32.11}$$

then

$$\Gamma \cap \partial B_R = \Gamma_m \cap \partial B_R. \tag{32.12}$$

Now suppose that $\Gamma_m \neq \Gamma$, and let z be a point of $\Gamma_m \setminus \Gamma$. Apply Lemma 31.6 to an initial arc reduced to $\{z\}$. This gives a rectifiable Jordan arc $\Gamma' \subset \Gamma_m$ that goes through ∞. For R large enough, $\Gamma' \cap \partial B_R$ contains at least 2 points; if we choose R so that in addition (32.11) holds (and we can do this with arbitrarily large values of R, by (32.5)), we get that

$$\Gamma' \cap \partial B_R = \Gamma_m \cap \partial B_R. \tag{32.13}$$

Hence $\Gamma' \cap \partial B_R = \Gamma \cap \partial B_R$ because (32.12) holds, and these two equal sets have exactly two points. Call these points x and y. If R was chosen large enough, the arc of Γ' between x and y contains z, which is not the case for the arc of Γ between x and y (by definition of z). This contradict (31.5). So we were wrong to suppose that $\Gamma_m \neq \Gamma$, and (32.9) holds.

We still need to prove (32.8), i.e., that

$$w(z) < 0 \text{ on } \mathbb{R}^2 \setminus G_{00}. \tag{32.14}$$

Let us first check that $w(z) < 0$ on Ω. Since Ω is connected, w is harmonic on Ω, and $w \leqslant 0$ everywhere, the only other option is that $w \equiv 0$ on Ω, hence also on $\mathbb{R}^2$. This is impossible here, for instance because it contradicts (32.5). Hence $w(z) < 0$ on Ω.

So we only need to exclude the case when $w(z) = 0$ on a component G_0 of G, $G_0 \neq G_{00}$. Because of (32.2), G_0 is bounded. By Lemma 25.3, we can find a compact set G^0 such that $G_0 \subset G^0 \subset G$ and

$$\operatorname{dist}\left(G^0, G \setminus G^0\right) > 0. \tag{32.15}$$

Then we can apply the construction of Section 23 to surround G^0 by a finite collection of (bounded) Jordan curves that satisfy (23.10). If the parameter ε is chosen small enough compared to the distance in (32.15), these Jordan curves do not meet G.

Finally, since G_0 is connected, we can find one curve γ among our surrounding Jordan curves such that

(32.16) $\qquad G_0$ is contained in the bounded component of $\mathbb{R}^2 \setminus \gamma$.

[The argument is the same as in Remark 23.17.] Set

$$m_0 = \sup\{w(z); z \in \gamma\}. \tag{32.17}$$

Then $m_0 < 0$ because γ is compact and contained in Ω, and we already know that $w(z) < 0$ on Ω. For $m_0 < m < 0$, call Ω_m the connected component of $\mathbb{R}^2 \setminus \Gamma_m$ that contains G_0. Then $w - m$ assumes a constant sign on Ω_m, and this sign is positive because $w = 0$ on G_0. Then Ω_m does not meet γ (by (32.17)), and Ω_m is contained in the bounded component of $\mathbb{R}^2 \setminus \gamma$, by (32.16). Thus we found a bounded component Ω_m of some $\mathbb{R}^2 \setminus \Gamma_m$. This contradiction with Proposition 30.1 completes our proof of (32.14), (32.8), and Proposition 32.6. □

Now we consider the variations of v along the Jordan curves of Proposition 32.6.

Lemma 32.18. — *For almost-every $m < 0$ (32.9) holds, v has a continuous extension to $\Omega \cup \Gamma_m$, and this extension is strictly monotone on Γ_m.*

Let $m < 0$ be such that (32.9) holds, such that $\Gamma_m \cap G \subset \mathcal{S}$ (as in (28.12)), and also such that we can apply Proposition 28.2. This is the case for almost all $m < 0$ by Propositions 32.6 and 28.2, and by (28.7).

Since $\Gamma_m \subset \Omega \cup \mathcal{S}$, v has a continuous extension to $\Omega \cup \Gamma_m$, by (28.28).

Since Γ_m is a Jordan curve, $\mathbb{R}^2 \setminus \Gamma_m$ has two components, which we call Ω_1 and Ω_2. By definition of Γ_m, $w(z) \neq m$ for all $z \in \Omega_1 \cup \Omega_2$. Since Ω_i is connected, $w(z) - m$ keeps a constant sign on Ω_i. Because of (32.5), we know that w takes values $> m$ as well as $< m$, and so the signs of $w(z) - m$ on Ω_1 and on Ω_2 are different. So we can assume that $w(z) < m$ on Ω_1 and $w(z) > m$ on Ω_2.

Let $\gamma : \mathbb{R} \to \Gamma_m$ be a parameterization of Γ_m by arclength. Note that $\gamma(t) \in \Omega$ for almost all $t \in \mathbb{R}$, because we have assumed that $\Gamma_m \cap G \subset \mathcal{S}$, and $H^1(\mathcal{S}) = 0$ (by (28.6)). In fact, for almost every t, $\nabla w(\gamma(t)) \neq 0$, because the set of points $z \in \Gamma_m$ such that $\nabla w(z) = 0$ is at most countable. [See for instance the proof of (31.8).]

Let us choose the orientation of Γ_m (or in other words, the direction of our parameterization by γ) so that Ω_2 lies on our left when we run along Γ_m. It is not too hard to check that the notion makes sense, i.e., that the condition that we get is the same when we look at any regular point of Γ_m. [One could for instance fix two such points, smooth Γ_m away from these points, and reduce to the simple case when Γ_m is smooth and we can use continuity.]

Now it is easy to check that if $t \in \mathbb{R}$ is such that $\gamma(t) \in \Omega$, $\nabla w(\gamma(t)) \neq 0$, and $\gamma'(t)$ exists, then $(v \circ \gamma)'(t) > 0$. This just uses our condition on orientations, the fact that $\gamma'(t) \neq 0$, and the fact that w is conjugated to v.

We can now apply Proposition 28.2 (because γ is simple), or even Corollary 28.55, to get that $v \circ \gamma$ is nondecreasing. To get that $v \circ \gamma$ is strictly increasing, we simply note that there is an open dense set on Γ_m (i.e., the set of points $z \in \Gamma_m \cap \Omega$ such that $\nabla w(z) \neq 0$) where v is strictly increasing.

Lemma 32.18 follows. □

CHAPTER H

THE MONOTONICITY FORMULA AND POINTS OF LOW ENERGY

33. Our tour of G_0

We shall need later to know how the boundary values of v vary when one turns around a bounded (nontrivial) component of G.

In this section G_0 is a fixed connected component of G, and we assume that G_0 is bounded and not reduced to a point. We want to describe a parameterization of G_0 by the unit circle that corresponds to the access from Ω. The most natural one would be the boundary values of a conformal mapping from $\mathbb{C} \setminus B(0,1)$ to $\mathbb{C} \setminus G_0 \simeq \mathbb{R}^2 \setminus G_0$, but we shall use constructions from the previous sections instead. The slight advantage will be that we won't have to use regularity properties of conformal mappings, and that we'll get a little more flexibility.

First choose a sequence $\{\delta_n\}$ which tends to 0, and for each n apply Lemma 25.3 to G_0, with $\delta = \delta_n$. We get a closed set $G^0 = G^0(n)$ such that

$$G_0 \subset G^0 \subset \{z \in G \; ; \; \operatorname{dist}(z, G) < \delta_n\} \tag{33.1}$$

and

$$d_n := \operatorname{dist}\left(G^0, G \setminus G^0\right) > 0. \tag{33.2}$$

We could have forgotten about this first stage and done most of the argument below, but it will be convenient later to have curves that surround G_0 and do not meet G, and this is the point to using G^0. Set

$$\varepsilon_n = \operatorname{Min}\left(\delta_n, 10^{-1} d_n\right), \tag{33.3}$$

and apply the construction of Section 23 to G^0 and with $\varepsilon = \varepsilon_n$. We get a collection of curves Γ_ε^j that surround G^0, but because G_0 is connected, there is a (unique) curve $\Gamma(n)$ among the Γ_ε^j such that

(33.4) $\qquad G_0$ is contained in the bounded component of $\mathbb{R}^2 \setminus \Gamma(n)$.

[See the proof of Remark 23.17.] Observe that

$$\operatorname{dist}(x, G_0) \leqslant \operatorname{dist}\left(x, G^0\right) + \delta_n \leqslant 2\varepsilon_n + \delta_n \leqslant 3\delta_n \tag{33.5}$$

for $x \in \Gamma(n)$, by (23.10), (33.1), and (33.3). Also

$$\operatorname{dist}(\Gamma(n), G) \geqslant \varepsilon_n \tag{33.6}$$

by (23.10), (33.2), and (33.3). In particular,

$$\Gamma(n) \text{ does not meet } G. \tag{33.7}$$

Let us check that

$$C^{-1} \leqslant H^1(\Gamma(n)) \leqslant C, \tag{33.8}$$

where C depends on G_0, but not on n. The first inequality is trivial, by (33.4) and the fact that $\operatorname{diam} G_0 > 0$. For the second one, let us even prove a bit more.

Lemma 33.9. — *For all disks $D = B(y, \rho)$ of radius $\rho \leqslant 1$,*

$$H^1(\Gamma(n) \cap D) \leqslant C\rho \tag{33.10}$$

as soon as $\varepsilon_n \leqslant \rho$.

The reader should not worry about the small restriction on ρ; we just decided to get a constant C that does not depend on ρ, and we want to use the local Ahlfors-regularity of G.

It is clear that (33.8) follows from Lemma 33.9, because all $\Gamma(n)$ lie in a fixed bounded set.

To prove the lemma we shall use the fact that $\Gamma(n) \subset \partial U(\varepsilon_n) \subset \partial H(\varepsilon_n)$ by (23.9), where $H(\varepsilon_n) = \bigcup_{i \in I(\varepsilon_n)} B_i$ as in (23.6). Set

$$J = \{i \in I(\varepsilon_n)\ ;\ \partial B_i \text{ meets } D\}. \tag{33.11}$$

We want to control the size of J. First recall from (23.3) that

$$\text{the disks } B_i' = B\left(x_i, 10^{-3}\varepsilon_n\right) \text{ are disjoint.} \tag{33.12}$$

Also, each disk B_i is centered on G^0, and so

$$H^1(G \cap B_i') \geqslant C^{-1}\varepsilon_n, \tag{33.13}$$

because G is locally Ahlfors-regular. [See (13.4).] Then

$$\begin{aligned} H^1(\Gamma(n) \cap D) &\leqslant H^1(\partial H(\varepsilon_n) \cap D) \leqslant \sum_{i \in J} H^1(\partial B_i) \leqslant 2\pi \sum_{i \in J} r_i \\ &\leqslant 4\pi\varepsilon_n(\#J) \leqslant C \sum_{i \in J} H^1(G \cap B_i') \leqslant CH^1(G \cap 2D) \leqslant C\rho \end{aligned} \tag{33.14}$$

because $r_i \leqslant 2\varepsilon_n$, by (33.13) and (33.12), because all B_i' are contained in $2D$ (by (33.11) and because $\varepsilon_n \leqslant \rho$), and by the upper bound in the local Ahlfors-regularity of G. [See (13.4).] The lemma follows. □

For each n denote by $z_n : \mathbb{S}^1 \to \Gamma(n)$ a parameterization of the Jordan curve $\Gamma(n)$ with constant speed. Chose z_n so that it "preserves the trigonometric sense", by which we mean that $z_n(t)$ turns around the bounded component of $\mathbb{R}^2 \setminus \Gamma(n)$ counterclockwise when $t \in \mathbb{S}^1$ turns around the unit disk counterclockwise, or that z_n has index 1 with respect to every point of that bounded component. By (33.8),

$$z_n \text{ is } C\text{-Lipschitz} \tag{33.15}$$

with a constant that does not depend on n. Modulo replacing $\{\delta_n\}$, $\{\varepsilon_n\}$, and $\{z_n\}$ with subsequences, we can assume that

$$\{z_n\} \text{ converges uniformly on } \mathbb{S}^1 \text{ to some Lipschitz function } z. \tag{33.16}$$

From (33.5) we easily deduce that $z\left(\mathbb{S}^1\right) \subset G_0$. Set

$$R = \left\{t \in \mathbb{S}^1 \ ; \ z(t) \text{ is a regular point of } G\right\}. \tag{33.17}$$

Lemma 33.18. — *R is a dense open subset of $\mathbb{S}^1$ and $H^1\left(\mathbb{S}^1 \setminus R\right) = 0$.*

It is clear that R is open, since

$$R' = \{x \in G_0 \ ; \ x \text{ is a regular point of } G\} \tag{33.19}$$

is itself open in G_0, and $R = z^{-1}(R')$. By Proposition 13.11,

$$H^1\left(G_0 \setminus R'\right) = 0. \tag{33.20}$$

We want to transfer this to the circle. Let $\tau > 0$ be small. By (33.20), we can cover $G_0 \setminus R'$ by disks $D_\ell = B\left(y_\ell, \rho_\ell\right)$, $\ell \in L$, with

$$\sum_{\ell \in L} \rho_\ell \leqslant \tau. \tag{33.21}$$

Since $G_0 \setminus R'$ is compact, we can assume that L is finite. Set $\rho = \operatorname{Min}\{\rho_\ell \ ; \ \ell \in L\}$. For all n such that $\varepsilon_n < \rho$, set

$$E_n' = \{y \in \Gamma(n) \ ; \ \operatorname{dist}\left(y, G_0 \setminus R'\right) \leqslant \rho\}. \tag{33.22}$$

Then

$$E_n' \subset \bigcup_{\ell \in L} B\left(y_\ell, \rho_\ell + \rho\right) \subset \bigcup_{\ell \in L} 2D_\ell,$$

and so

$$H^1\left(E_n'\right) \leqslant \sum_{\ell \in L} H^1\left(\Gamma(n) \cap 2D_\ell\right) \leqslant C \sum_{\ell \in L} \rho_\ell \leqslant C\tau \tag{33.23}$$

by Lemma 33.9 and (33.21). Next set

$$E_n = \left\{t \in \mathbb{S}^1 \ ; \ \operatorname{dist}\left(z_n(t), G_0 \setminus R'\right) \leqslant \rho\right\} = z_n^{-1}\left(E_n'\right) \tag{33.24}$$

Then $H^1\left(E_n\right) \leqslant C\tau$, because z_n is a parameterization of $\Gamma(n)$ at constant speed, and by (33.8). If n is also so large that $\left\|z_n - z\right\|_\infty < \rho$, then $\mathbb{S}^1 \setminus R = z^{-1}\left(G_0 \setminus R'\right)$ is contained in E_n. This proves that $H^1\left(\mathbb{S}^1 \setminus R\right) \leqslant C\tau$. Lemma 33.18 follows, since τ was arbitrary. □

We shall need to know more about z near regular points.

Lemma 33.25. — *For each $x \in R'$, there are exactly two points $t_1, t_2 \in \mathbb{S}^1$ such that $z\left(t_i\right) = x$.*

Recall that R' is the set of points $x \in G_0$ that are regular for G. The lemma should not be a surprise; we expect our tour of G_0 to visit x twice, one time for each access region to x.

For each $x \in R'$, choose a small radius $r(x)$ such that $B(x, 2r(x))$ is a disk of regularity, and denote by $\Omega_1(x)$ and $\Omega_2(x)$ the two components of $B(x, 2r(x)) \setminus G$.

Choose points $y_i(x)$, $i = 1, 2$, in the middle of $\Omega_i(x)$. Since Ω is assumed to be connected, $\Omega_i(x)$ is contained in the unbounded component of $\mathbb{R}^2 \setminus G^0$ (the set Ω_0 in Lemma 23.18 and (23.22)). Hence (23.22) says that $y_i(x) \in U\left(\varepsilon_n\right)$ for n large enough. This implies that for n large,

$$y_i(x) \text{ lies in the unbounded component of } \mathbb{R}^2 \setminus \Gamma(n). \tag{33.26}$$

See for instance (23.12).

Because $G \cap B(x, 2r(x))$ is a fairly flat C^1-curve that crosses $B(x, 2r(x))$, we have a very good idea of what the set $H\left(\varepsilon_n\right)$ in (23.6) looks like inside that disk. Set $B(x) = B(x, r(x))$. For n large enough, $\partial H\left(\varepsilon_n\right) \cap B(x)$ is the union of two slightly corrugated curves $\Gamma_1^n(x)$ and $\Gamma_2^n(x)$ that roughly follow $G^0 \cap B(x) = G \cap B(x)$, one on each side, as in Figure 33.1.

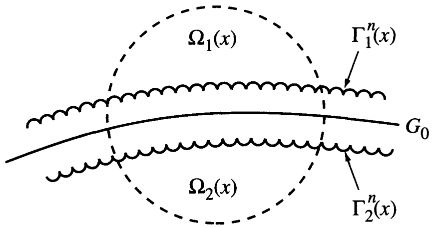


FIGURE 33.1

Since $\Gamma(n) \subset \partial H\left(\varepsilon_n\right)$ (by (23.9)), and on the other hand $\Gamma(n)$ separates G_0 from $y_1(x)$ and $y_2(x)$ for n large (by (33.26)), we get that for n large,

$$\Gamma(n) \cap B(x) = \partial H\left(\varepsilon_n\right) \cap B(x) = \Gamma_1^n(x) \cup \Gamma_2^n(x). \tag{33.27}$$

Because of this,

$$\left\{t \in \mathbb{S}^1 \; ; \; z_n(t) \in B(x)\right\} = I_1^n(x) \cup I_2^n(x), \tag{33.28}$$

where the sets

$$I_i^n(x) = z_n^{-1}\left(\Gamma_i^n(x)\right) \tag{33.29}$$

are disjoint intervals. From (33.8) and the fact that z_n runs along $\Gamma(n)$ with constant speed, we deduce that

$$C^{-1}r(x) \leqslant |I_i^n(x)| \leqslant Cr(x), \tag{33.30}$$

and also that

$$\operatorname{dist}\left(I_1^n(x), I_2^n(x)\right) \geqslant C^{-1}r(x), \tag{33.31}$$

because there are portions of $\Gamma(n) \cap B(x, 2r(x)) \setminus B(x)$ of lengths $\geqslant r(x)$ that z_n has to go through between $I_1^n(x)$ and $I_2^n(x)$.

For n large enough, $\Gamma_i^n(x)$ passes at distance $\leqslant 3\varepsilon_n$ from x. Hence we can find $t_i^n(x) \in I_i^n(x)$ such that

$$\left|z_n\left(t_i^n(x)\right) - x\right| \leqslant 3\varepsilon_n. \tag{33.32}$$

Extract a subsequence (that may depend on x) such that (after extraction) the $t_i^n(x)$ converge to limits $t_i = t_i(x)$. Then

$$z\left(t_i(x)\right) = x, \tag{33.33}$$

by (33.32), the uniform convergence (33.16), and the uniform Lipschitz estimate (33.15). Note that $t_1(x) \neq t_2(x)$, and even

$$|t_1(x) - t_2(x)| \geqslant C^{-1}r(x), \tag{33.34}$$

by (33.31).

To complete our proof of Lemma 33.25, we still need to check that $z^{-1}(x)$ has at most two points. Suppose that $s_1, s_2, s_3, \in \mathbb{S}^1$ are such that $z\left(s_i\right) = x$. Since $z_n\left(s_i\right)$ tends to x, we get that $s_i \in I_1^n(x) \cup I_2^n(x)$ for n large. [See (33.28).] In particular, two of the s_i lie in the same $I_i(x)$, and so they lie at distance $\leqslant |I_i^n(x)| \leqslant Cr(x)$ from each other (by (33.30)). Since we could have made this argument with any small value of $r(x)$, two of the s_i must be equal. Lemma 33.25 follows. □

We shall need even more notation. For all $x \in R'$, choose $r(x)$ as above, and let $t_1(x)$ and $t_2(x)$ be the two points given by Lemma 33.25. These are also the points that we constructed in the proof of that lemma. Denote by $I_i(x)$, $i = 1, 2$, the interval of $\mathbb{S}^1$ centered at $t_i(x)$ and with length $C_1^{-1}r(x)$, where we choose C_1 so large that

$$\operatorname{dist}\left(I_1(x), I_2(x)\right) \geqslant C_1^{-1}r(x) \tag{33.35}$$

(which is easy to get, by (33.34)) and

$$z\left(I_i(x)\right) \subset B\left(x, r(x)/2\right), \tag{33.36}$$

which we can arrange because $z\left(t_i(x)\right) = x$ and z is Lipschitz.

Let us check that for n large enough,

$$z_n(t) \in \Omega_i(x) \cap B(x) \text{ for } t \in I_i(x), \tag{33.37}$$

where $B(x) = B(x, r(x))$ as before, and maybe after exchanging the names of $\Omega_1(x)$ and $\Omega_2(x)$ (the two components of $2B(x) \setminus G$) or, equivalently, the names of $t_1(x)$ and $t_2(x)$.

The fact that $z_n(I_i(x)) \subset B(x)$ for n large is an easy consequence of (33.36) and the uniform convergence in (33.16).

To prove that $z_n(I_i(x))$ stays in the same component $\Omega_i(x)$ for n large, we use the uniform convergence (33.16) and the fact that all z_n were chosen to turn around G_0 in the same (counterclockwise) direction. For instance, in the situation of Figure 33.1, we must run along $\Gamma_1^n(x)$ from right to left and along $\Gamma_2^n(x)$ from left to right. If m and n are so large that $|z_m - z_n| < 10^{-2}r(x)$, say, $z_m(I_1(x))$ and $z_n(I_1(x))$ must lie in the same $\Omega_i(x)$. This proves our claim (33.37).

Next we want to transfer our notations to the circle. For each $t \in R$ (see the definition (33.17)), set $r_t = r(z(t))$, where $r(z(t))$ is as above. Then $B(z(t), 2r_t)$ is a disk of regularity for G. Define $\Omega_1(z(t))$ and $\Omega_2(z(t))$ as above, and choose the indices so that (33.37) holds with $t_1(x) = t$. Then set $t^* = t_2(x)$, $\Omega_t = \Omega_1(z(t))$, $\Omega_t^* = \Omega_2(z(t))$, $I_t = I_1(z(t))$, and $I_t^* = I_2(z(t))$.

Thus $t^* \neq t$, $z(t^*) = z(t)$, I_t and I_t^* are intervals of length $C_1^{-1}r_t$ centered at t and t^* respectively,

$$\operatorname{dist}(I_t, I_t^*) \geqslant C_1^{-1}r_t, \tag{33.38}$$

$$z(I_t) \text{ and } z(I_t^*) \text{ are contained in } G_0 \cap B(z(t), r_t/2) \tag{33.39}$$

and, for n large enough,

$$z_n(I_t) \subset \Omega_t \cap B(z(t), r_t) \tag{33.40}$$

and

$$z_n(I_t^*) \subset \Omega_t^* \cap B(z(t), r_t). \tag{33.41}$$

For $t \in R$, denote by Δ_t the line segment of length r_t that starts at $z(t)$, is perpendicular to G at that point, and lies in Ω_t except for its endpoint $z(t)$. See Figure 33.2.

Note that for n large,

$$\Gamma(n) \cap \Delta_t \text{ has exactly one point.} \tag{33.42}$$

This follows from our good description of $\Gamma(n)$ near $z(t)$; see near (33.27) and Figure 33.1. Thus we can define $s_n \in \mathbb{S}^1$ by

$$z_n(s_n) \in \Delta_t, \tag{33.43}$$

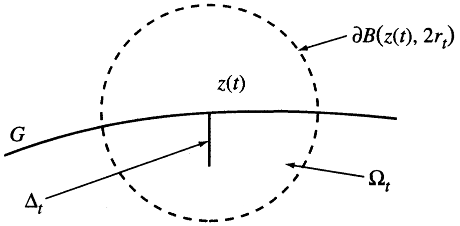


FIGURE 33.2

and then

$$\lim_{n\to\infty} z_n(s_n) = z(t), \tag{33.44}$$

by (33.5) (or our nice description of $\Gamma(n)$ near $z(t)$). Let us check that

$$\lim_{n\to\infty} s_n = t. \tag{33.45}$$

Suppose this fails. Then we can extract a subsequence for which $\lim_{n\to\infty} s_n = s \neq t$. Then $z(s) = z(t)$, by (33.44), the convergence of $z_n(s)$ to $z(s)$, and the fact that $z_n(s_n) - z_n(s)$ tends to 0 by the uniform Lipschitz bound in (33.15). Thus $s = t^*$ by Lemma 33.25, and hence $s_n \in I_t^*$ for n large. By (33.41), $z_n(s_n) \in \Omega_t^*$, which obviously contradicts (33.43). This proves our claim.

We now have enough general information on our tour of G_0 (i.e., the mapping z). In the next section we shall use z to study the variations of (the boundary values of) v along G_0.

34. Variations of v along our tour of G_0

We continue with the notations and assumptions of the previous section. In particular, G_0 is a bounded component of G, not reduced to a point, and $z : \mathbb{S}^1 \to G_0$ is the (essentially 2-to-1) parameterization of G_0 constructed in Section 33. Here we shall also use our assumptions of Section 32, because we want to apply Lemma 32.18.

Recall from (33.17) that

$$R = \left\{t \in \mathbb{S}^1 \ ; \ z(t) \text{ is a regular point of } G\right\}. \tag{34.1}$$

By lemma 14.1 and the definitions before (33.38),

$$v \text{ has a } C^1 \text{ extension to } \overline{\Omega}_t \cap B(z(t), r_t). \tag{34.2}$$

Thus we can set

$$u(t) = \lim_{\substack{z\to z(t)\\ z\in\Omega_t}} v(z) \quad \text{for } t \in R. \tag{34.3}$$

Lemma 34.4. — *The function u has a (unique) continuous extension to* $\mathbb{S}^1$. *It is even Hölder-continuous with exponent* 1/2.

Note that since R is dense in $\mathbb{S}^1$ (by Lemma 33.18), our extension is necessarily unique (if it exists). Thus it will be enough prove that

$$|u(t) - u(t')| \leqslant C\,|t - t'|^{1/2} \text{ for } t, t' \in R, \tag{34.5}$$

with a constant C that may depend on G_0.

Let $t, t' \in R$ be given, with $t \neq t$. For n large, let $s_n \in \mathbb{S}^1$ be defined by (33.43), and similarly define $s'_n \in \mathbb{S}^1$ by $z_n(s'_n) \in \Delta_{t'}$. Then

$$\lim_{n\to+\infty} s_n = t \text{ and } \lim_{n\to+\infty} s'_n = t', \tag{34.6}$$

by (33.45). Denote by γ_n the shortest arc of $\Gamma(n)$ between $z_n(s_n)$ and $z_n(s'_n)$. Then

$$\mathrm{diam}\gamma_n \leqslant C\,|t' - t| \text{ for } n \text{ large}, \tag{34.7}$$

by (34.6) and the uniform Lipschitz estimate (33.15). Also, γ_n does not meet G, by (33.7). Let us verify that we can apply Lemma 21.3 here. We have checked earlier that v (and even $f = v + iw$) satisfies the requirement (21.1); see (25.22), (25.23), and the sentence that follows. Also, $x = z_n(s_n)$ and $y = z_n(s'_n)$ lie in a same component of $B \setminus G$ for some disk B of radius $r_0 = C\,|t' - t|$, by (34.7) and because $\gamma_n \cap G = \varnothing$; this takes care of (21.2). The case when $r_0 > 1$ is not a serious problem because we can cut γ_n into boundedly many pieces with diameters < 1; anyway we shall not care about how the constant C in (34.5) depends on G_0. Altogether we can apply Lemma 21.3 and we get that

$$|v(z_n(s_n)) - v(z_n(s'_n))| \leqslant C\,|t' - t|^{1/2}, \tag{34.8}$$

with a constant C that does not depend on t, t' and n.

Note that $z_n(s_n)$ lies in Ω_t by definition, and tends to $z(t)$ by (33.44). Then $v(z_n(s_n))$ tends to $u(t)$, by (34.3). Similarly $v(z_n(s'_n))$ tends to $u(t')$, and (34.5) follows from (34.8). This proves Lemma 34.4. □

Since the extension is unique, we can safely denote it by u as well. Here is the main result of this section.

Proposition 34.9. — *There is a decomposition* $\mathbb{S}^1 = I^+ \cup I^-$ *into intervals with disjoint interiors, so that*

$$u \text{ is strictly increasing on } I^+, \tag{34.10}$$

and

$$u \text{ is strictly decreasing on } I^-. \tag{34.11}$$

Note that in (34.10) and (34.11), we use the order on $I^{\pm}$ that comes from the trigonometric sense on $\mathbb{S}^1$. Also, $I^{\pm}$ cannot be empty or reduced to one point, by (34.10), (34.11), and the continuity of u.

To prove the proposition, we shall try to focus on the dense set R as much as we can, to avoid complications with the parts of G that we do not control well. The general idea will be that we can approach G_0 with nearby level lines Γ_m, where we control the variations of v by Lemma 32.18. See Figure 34.1.

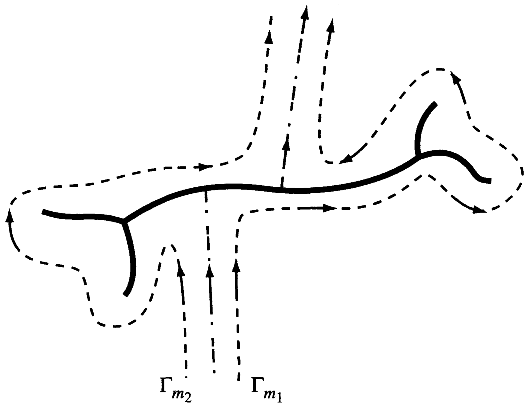


FIGURE 34.1. The set G_0 and three level sets Γ_m (including the central one which contains G_0). The arrows indicate the variations of v.

We start with a local description, essentially on R.

Lemma 34.12. — *For almost all $t \in \mathbb{S}^1$, u is differentiable at t and $u'(t) \neq 0$.*

Since $H^1\left(\mathbb{S}^1 \setminus R\right) = 0$ by Lemma 33.18, it will be enough to consider points of R. Let us first check that for each $t \in R$,

$$\nabla v(x) \neq 0 \text{ for } H^1\text{-almost every } x \in G \cap B\left(z(t), r_t\right), \tag{34.13}$$

where $\nabla v(x)$ denotes the continuous extension of ∇v to $\overline{\Omega}_t \cap B\left(z(t), r_t\right)$ given by (34.2). To prove this, we use the following classical result of F. and M. Riesz. [See for instance Corollary 4.2 p.65 in [Ga].]

Lemma 34.14. — *If the continuous function $f : \overline{B(0,1)} \to \mathbb{C}$ is holomorphic on $B(0,1)$, and if $f(x) = 0$ on a subset of positive H^1-measure of $\partial B(0,1)$, then $f \equiv 0$.*

The same result is easily seen to be valid on any simply connected bounded domain of class $C^{1+\varepsilon}$, $\varepsilon > 0$, because we can compose with a conformal mapping. This uses the fact that the conformal mappings from the unit disk to the given domain have C^1 extensions to the closed disk, with a derivative that does not vanish on the circle. See for instance Theorem 3.5 on page 48 of [Po].

In the present situation we apply (the analogue of) Lemma 34.14 to the derivative of $v+iw$ on a smooth subdomain of Ω_t whose boundary contains $G\cap B\,(z(t),r_t)$. We do not get into trouble because of regularity, because $G\cap B\,(z(t),2r_t)$ is in fact $C^{1+\varepsilon}$ for all $\varepsilon<1$. See for instance Sections 6 and 7 of [Bo] for a stronger result.

Thus if (34.13) fails, we get that $\nabla v\equiv 0$ on our smooth subdomain of Ω_t, hence on the whole Ω because we are currently assuming that Ω is connected. In general, this would be enough to give a complete description of (v,G), as in Section 18. Here we even assumed that (32.5) holds, and we get a contradiction. This proves (34.13).

Denote by $\partial v/\partial\tau$ the tangential derivative along G of our extension of v. For the moment, the precise sign of $\partial v/\partial\tau$ will not matter, so we don't need to orient G near $z(t)$. We have that

$$\frac{\partial v}{\partial\tau}(x)\neq 0 \text{ as soon as } \nabla v(x)\neq 0 \tag{34.15}$$

(hence, almost-everywhere on $G\cap B\,(z(t),r_t)$), because v satisfies the Neumann condition $\partial v/\partial n=0$ on $G\cap B\,(z(t),r_t)$.

We are now ready to prove Lemma 34.12. As was mentioned before, we may restrict our attention to R, because $H^1\left(\mathbb{S}^1\setminus R\right)=0$. For each $t\in R$, let I_t be the small interval centered at t_0 introduced before (33.38). Since R is a countable union of such intervals I_t, it is enough to show that for all $t\in R$,

$$u'(s) \text{ exists and } u'(s)\neq 0 \text{ for almost every } s\in I_t. \tag{34.16}$$

From (33.40) and our description of the curves $\Gamma(n)$ in $B\,(z(t),r_t)$ we easily deduce that the restriction of z to I_t is bilipschitz. In particular, it preserves sets of measure 0, and also

$$z'(s)\neq 0 \text{ for all } s\in I_t \text{ such that } z'(s) \text{ exists.} \tag{34.17}$$

For almost every $s\in I_t$, $z'(s)$ exists and so $u'(s)$ exists also. Moreover, $u'(t)\neq 0$ as soon as $\nabla v(z(t))\neq 0$ (by (34.15) and (34.17)). This happens almost everywhere on I_t, by (34.13), (33.39), and because the restriction of z to I_t preserves sets of measure 0. This completes our proof of (34.16), and Lemma 34.12 follows. □

Set

$$R^{\pm}=\{t\in R\;;\;\pm u'(t)>0\}, \tag{34.18}$$

where our condition $\pm u'(t)>0$ is meant to contain the differentiability of u at t, and we identify $\mathbb{S}^1$ with the torus to talk about the sign of $u'(t)$.

Let m_0 denote the constant value of w on G_0. We claim that for all $t\in R^{\pm}$ we can find a positive radius $r'_t\leqslant r_t$ such that

$$\pm\left(w(x)-m_0\right)<0 \text{ for } x\in\Omega_t\cap B\,(z(t),r'_t)\,. \tag{34.19}$$

Indeed $\partial v/\partial\tau(z(t))\neq 0$ because $u'(t)\neq 0$ (and z is Lipschitz), and then $\pm\partial w/\partial n(z(t))<0$, with a unit normal pointing towards Ω_t, because $\pm u'(t)>0$, w is conjugated to v, and because all $\Gamma(n)$ were set to turn around G_0 counterclockwise. Then (34.19) holds, with a suitably small choice of r_t', because ∇w is continuous on $\overline{\Omega}_t \cap B(z(t), r_t)$.

We shall later need to know that

$$\text{(34.20)} \qquad \begin{cases} \text{if } t \in R^{\pm}, \text{ there is a neighborhood } V \text{ of } t \text{ in } \mathbb{S}^1 \\ \text{such that } s \in R^{\pm} \text{ for almost all } s \in V, \end{cases}$$

where of course the point is that the set $R^{\pm}$ is the same for all s as the one for t. To see this, note that $\pm\partial v/\partial\tau(z(t)) > 0$, if we choose the orientation of G near $z(t)$ correctly. Then $\pm\partial v/\partial\tau(z(s)) > 0$ in a small neighborhood V of t, because $\partial v/\partial\tau$ is continuous near $z(t)$. Then $\pm u'(s) > 0$ for all $s \in V$ such that $z'(s)$ exists, because z preserves the orientation near t (with our choice of orientation on G). Our claim (34.20) follows.

Our sets $R^{\pm}$ are nice, because we have a good control of v on them, and

$$\text{(34.21)} \qquad H^1\left(\mathbb{S}^1 \setminus \left(R^+ \cup R^-\right)\right) = 0,$$

by Lemma 34.12. However we cannot prove easily that they are both nonempty, and so we introduce the slightly larger sets

$$\text{(34.22)} \quad S^{\pm} = \big\{t \in \mathbb{S}^1 \ ; \text{ there is a sequence } \{s_n\} \text{ in } \mathbb{S}^1 \text{ that converges to } t \text{ and such that } \pm\left(w\left(z_n\left(s_n\right)\right) - m_0\right) \leqslant 0 \text{ for infinitely many values of } n\big\}.$$

Let us check that

$$\text{(34.23)} \qquad R^{\pm} \subset S^{\pm}.$$

If $t \in R^+$ and n is large enough, we can find $s_n \in \mathbb{S}^1$ such that $z_n(s_n) \in \Delta_t$ (as in (33.43)). Then $z_n(s_n) \in \Omega_t \cap B(z(t), r_t')$ for n large (by (33.44)), and so $\pm\left(w\left(z_n\left(s_n\right)\right) - m_0\right) < 0$ (by (34.19)). Since s_n tends to t by (33.45), $t \in S^{\pm}$ as needed.

Lemma 34.24. — *$S^{\pm}$ is not empty.*

To prove this, it is enough to check that for all n

$$\text{(34.25)} \qquad \inf\{w(z) \ ; \ z \in \Gamma(n)\} \leqslant m_0 \leqslant \sup\{w(z) \ ; \ z \in \Gamma(n)\}.$$

Indeed (34.25) allows us to find $s_n^{\pm} \in \mathbb{S}^1$ such that $\pm\left(w\left(z_n\left(s_n\right)\right) - m_0\right) \leqslant 0$, and then we can find a subsequence $\{n_k\}$ for which $\left\{s_{n_k}^{\pm}\right\}$ converges to some limit $t^{\pm}$. Then $t^{\pm} \in S^{\pm}$, as we can see by keeping the values $s_{n_k}^{\pm}$ on our subsequence, and choosing $s_n = t^{\pm}$ for all other integers.

So it is enough to check (34.25). Suppose for instance that the first inequality fails, so that $w(z) > m_0$ on $\Gamma(n)$. Choose m such that $m_0 < m < \inf\{w(z) \ ; \ z \in \Gamma(n)\}$. Then the connected component of $\mathbb{R}^2 \setminus \Gamma_m$ that contains G_0 cannot meet $\Gamma(n)$ (because $w(z) < m$ on that component), hence is contained in the bounded component of $\mathbb{R}^2 \setminus \Gamma(n)$. This is impossible, by Proposition 30.1.

The proof of the second inequality in (34.25) is similar. Lemma 34.24 follows. □

Now we come to the place where we shall use level lines and Lemma 32.18.

Lemma 34.26. — *Suppose $t_0 \in S^-$ and $t_1, t_2, t_3 \in R^+$ are such that*

(34.27) $$t_0 < t_1 < t_2 < t_3 < t_0 \text{ in } \mathbb{S}^1.$$

Then

(34.28) $$u(t_1) \leqslant u(t_2) \leqslant u(t_3) \text{ or } u(t_3) \leqslant u(t_2) \leqslant u(t_1).$$

By (34.27), we mean that when we start from t_0 and run along $\mathbb{S}^1$ in the trigonometric direction, we hit t_1 first, then t_2, then t_3, before hitting t_0 again after one turn. In particular, the points t_i are all distinct.

To prove the lemma, we shall proceed by contradiction and assume that (34.28) does not hold. Thus

(34.29) $u(t_2)$ lies out of the closed interval with extremities $u(t_1)$ and $u(t_3)$.

Set $z_i = z(t_i)$ and $\Delta_i = \Delta_{t_i}$ for $0 \leqslant i \leqslant 3$. Also set

(34.30) $$\Delta_i' = \Delta_i \cap B(z_i, r),$$

where r is chosen so small that

(34.31) the disks $B(z_i, 2r)$, $0 \leqslant i \leqslant 3$, are disjoint,

(34.32) $v(z_2')$ lies out of the closed interval with endpoints $v(z_1')$ and $v(z_3')$ for all choices of points $z_i' \in \Delta_i'$, $1 \leqslant i \leqslant 3$,

and also that $r \leqslant r'_{t_i}$ for $1 \leqslant i \leqslant 3$, where r'_{t_i} is as in (34.19). The second property is easy to obtain, by its analogue (34.29) and by (34.3). The last condition $r_i < r'_{t_i}$ implies that

(34.33) $$w(x) < m_0 \text{ on } \Delta_1' \cup \Delta_2' \cup \Delta_3',$$

by (34.19) and because $t_i \in R^+$ for $1 \leqslant i \leqslant 3$.

Now we want to choose a level set Γ_m, $m < m_0$, that passes close to the z_i, $i \geqslant 1$. Since w is continuous and $w(x) < m_0$ on Δ_i', $i \geqslant 1$, w takes on Δ_i' all values of $m < m_0$ that are close enough to m_0. Hence

(34.34) $$\Gamma_m \text{ meets all } \Delta_i',\ 1 \leqslant i \leqslant 3,$$

as soon as $m < m_0$ is large enough. We choose m like this, but also such that

(34.35) Γ_m is a simple rectifiable curve through ∞,

and

(34.36) v has continuous extension to $\Omega \cup \Gamma_m$, which is strictly monotone on Γ_m.

Lemma 32.18 tells us that this is possible.

Denote by $z_i', i=1,2,3$, the point of $\Delta_i' \cap \Gamma_m$ that is closest to z_i. Thus

$$[z_i, z_i') \text{ does not meet } \Gamma_m. \tag{34.37}$$

Let us now construct a simple arc γ from z_1 to z_3. Let γ' denote the arc of Γ_m between z_1' and z_3'. This makes sense because of (34.35). Our arc γ starts from z_1, follows the segment $[z_1, z_1']$ up to z_1', then follows γ' up to z_3', and finally follows $[z_3', z_3]$ up to z_3. It is easy to see that γ is simple, but we shall not really need this fact. Also note that

$$w(x) < m_0 \text{ on } \gamma \setminus \{z_1, z_3\}, \tag{34.38}$$

by (34.33) and because $w(x) = m < m_0$ on $\gamma' \subset \Gamma_m$.

We also want to construct an arc γ_0, that almost connects z_0 to z_2, and that does not meet γ. The way we shall eventually get a contradiction is by showing that γ_0 must meet γ, for topological reasons connected to (34.27).

First we want to choose a curve $\Gamma(n)$ very close to G_0. Recall from (33.42) that for n large enough, $\Gamma(n) \cap \Delta_i$ has exactly one point y_i. By (33.44), y_i is as close to z_i as we want (provided that n is large). Thus, for n large,

$$y_i \in (z_i, z_i') \text{ for } 1 \leqslant i \leqslant 3. \tag{34.39}$$

Denote by $s_i = s_{n,i}$ the point of $\mathbb{S}^1$ such that $y_i = z_n(s_i)$. Then s_i tends to t_i (by (33.45)), and so (34.27) yields

$$t_0 < s_1 < s_2 < s_3 < t_0 \text{ in } \mathbb{S}^1 \tag{34.40}$$

for n large.

Note that G_0 is compact and does not meet Γ_m (because $w(x) = m_0$ on G_0, and $m < m_0$). Hence $\operatorname{dist}(G_0, \Gamma_m) > 0$, Γ_m does not meet $\Gamma(n)$ for n large (by (33.5)), and even

$$\Gamma_m \text{ lies in the unbounded component of } \mathbb{R}^2 \setminus \Gamma(n), \tag{34.41}$$

because Γ_m is connected and unbounded.

Now denote by ξ_n the point $\xi_n = z_n(s_n)$ associated to $t_0 \in S^-$ as in the definition (34.22) of S^-. Thus $N = \{n\ ;\ w(\xi_n) \geqslant m_0\}$ is infinite, and s_n tends to t_0. We choose $n \in N$ so large that (34.39)-(34.41) hold, and also

$$s_n < s_1 < s_2 < s_3 < s_n \text{ on } \mathbb{S}^1. \tag{34.42}$$

Our argument will be simpler if we modify slightly $\Gamma(n)$ near ξ_n. Recall from (33.7) that $\Gamma(n)$ does not meet G, and so $\xi_n \in \Omega$. Also, w cannot be constant near ξ_n; this is forbidden by our hypotheses (32.1) and (32.5), for instance. By the maximum principle, we can find points $\xi_n' \in \Omega$ such that $w(\xi_n') > w(\xi_n)$, and that are as close to ξ_n as we want. Set $m_1 = w(\xi_n')$. Since we can always replace ξ_n' with any point of

$[\xi_n, \xi_n']$, we have a continuum of values of m_1 to choose from. We decide to choose ξ_n' so that

(34.43) $$\Gamma_{m_1} \text{ is a simple rectifiable curve through } \infty.$$

This is possible because of Lemma 32.18, even with the constraint that ξ_n' be very close to ξ_n that will show up soon.

Note that

(34.44) $$m_1 > w(\xi_n) \geqslant m_0$$

because we took $n \in N$.

Let us deform our parameterization z_n of $\Gamma(n)$ in a small neighborhood of s_n, so as to get a mapping z_n' such that $z_n'(s_n) = \xi_n'$. Since we can choose ξ_n' very close to ξ_n, we can do this deformation without altering the main properties of z_n and $\Gamma(n)$. Let us require in particular that

(34.45) $$\Gamma = z_n'(\mathbb{S}^1) \text{ is a simple, piecewise } C^1 \text{ Jordan curve,}$$

(34.46) $$\Gamma_m \text{ lies in the unbounded component of } \mathbb{R}^2 \setminus \Gamma$$

(as in (34.41)),

(34.47) $$G_0 \text{ lies in the bounded component of } \mathbb{R}^2 \setminus \Gamma,$$

(34.48) $$\Gamma \cap \Delta_i = \{y_i\} \text{ for } i = 1, 2, 3,$$

and

(34.49) $$\xi_n' < y_1 < y_2 < y_3 < \xi_n' \text{ on } \Gamma,$$

which we can get because of (34.42). [Recall from just after (34.39) that $y_i = z_n(s_i)$ for $i \geqslant 1$.]

We are almost ready to construct our second curve γ_0. First choose a radius R so large that

(34.50) $$G_0 \cup \Gamma(n) \cup \Gamma \cup \gamma \subset B_R.$$

Denote by γ_0' the arc of Γ_m that goes from z_2' (our first point of $\Gamma_m \cap \Delta_2'$) to ∞, without meeting γ'. This arc γ_0' exists, because (34.32) and (34.36) tell us that z_2' lies outside of the closed arc of Γ_m between z_1' and z_3', and this arc is precisely γ'.

Our arc γ_0 is constructed as follows. [Also see Figure 34.2.] First we leave from $\xi_n' \in \Gamma_{m_1}$, and follow any branch of $\Gamma_{m_1} \setminus \{\xi_n'\}$ until we hit ∂B_R for the first time. Then we follow ∂B_R (in any direction), until we hit γ_0' for the first time. After this we follow γ_0' back to the point z_2', and finally we follow the segment Δ_2' up to the point y_2 where we stop. Note that γ_0' and both branches of $\Gamma_{m_1} \setminus \{\xi_n'\}$ meet ∂B_R, by (34.50) and because they are unbounded. Let us check that

(34.51) $$\gamma_0 \text{ does not meet } \gamma.$$

First, Γ_{m_1} does not meet γ, because $w(x) \leqslant m_0$ on γ (by (34.38)) and $w(x) > m_0$ on Γ_{m_1} (by (34.44)). Next ∂B_R does not meet γ, by (34.50). Our curve γ'_0 does not meet γ' by definition, and it does not meet the other pieces $[z_i, z'_i)$, $i = 1, 3$, of γ, by (34.37) and because $\gamma'_0 \subset \Gamma_m$. Hence γ'_0 does not meet γ. We are left with the segment $(z'_2, y_2]$, which does not meet γ' by (34.37) and does not meet the other $[z_i, z'_i)$ by (34.31) and (34.30). This proves (34.51).

Let us check also that

$$(34.52) \qquad \gamma_0 \text{ is simple.}$$

The arc of Γ_{m_1} that starts γ_0 is simple by (34.43), it does not meet ∂B_R (except at its end) by definition; it does not meet γ'_0 either, because $\gamma'_0 \subset \Gamma_m$ and $m < m_1$ (by (34.44)), and it does not meet $[y_2, z'_2)$ by (34.33) and (34.44). So our arc of Γ_{m_1} does not meet the rest of γ_0, and (34.52) follows from the construction of the rest of γ_0.

The situation at this point looks quite promising (in terms of getting the desired contradiction). The curves γ_0 and γ do not meet, but yet it seems from (34.49) that they should. [See Figure 34.2]. We want to get our contradiction from the following lemma.

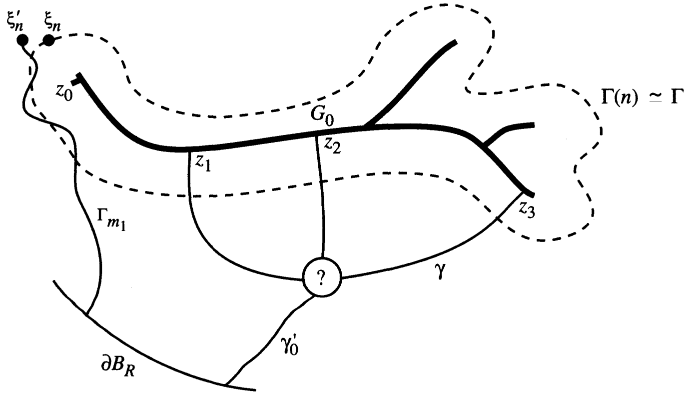


FIGURE 34.2. It looks like γ_0 should meet γ.

Lemma 34.53. — *If $y_0 < y_1 < y_2 < y_3 < y_0$ in Γ, γ_0 is a simple arc from y_0 to y_2, γ_1 is a simple arc from y_1 to y_3, and both arcs γ_0, γ_1 lie in the unbounded component of $\mathbb{R}^2 \setminus \Gamma$ except for their extremities y_i, then γ_0 meets γ_1.*

To prove the lemma, we can use a conformal mapping from the unit disk to the outside of Γ (including ∞) to reduce to the following.

Lemma 34.54. — *If $t_0 < t_1 < t_2 < t_3 < t_0$ in $\mathbb{S}^1$, η_0 is a simple arc from t_0 to t_2, η_1 is a simple arc from t_1 to t_3, and both arcs are contained in $B(0,1)$ except for their endpoints, then η_0 meets η_1.*

We leave the proof of Lemma 34.54 to the reader (see Figure 34.3). Note that the fact that our curves are simple is not needed, but allows a "simple" proof with the Jordan curve theorem. This completes our discussion of the proof of Lemma 34.53. □

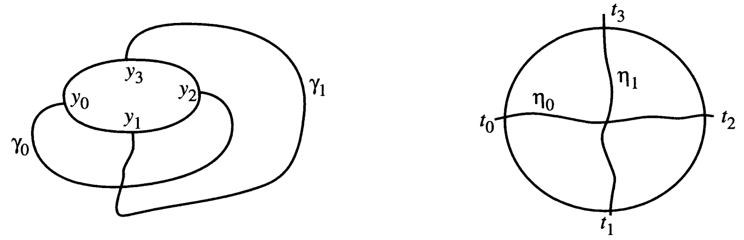


FIGURE 34.3. The situations of Lemmas 34.53 and 34.54.

Let us return to the proof of Lemma 34.26. As Figure 34.2 suggests, we are not yet in position to apply Lemma 34.53, because the curve Γ_{m_1} may cross Γ a few times. To fix this problem, set $X = \gamma_0 \cap \Gamma$ and

$$X^+ = \{x \in X \ ; \ x < y_1 < y_2 < y_3 < x \text{ on } \Gamma\}. \tag{34.55}$$

Note that $\xi_n' \in X^+$, by (34.49). The last point of X (when we go along γ_0, starting from ξ_n') is clearly y_2. Call x_0 the previous one. [It exists, because γ_0 does not meet Γ between z_2' and y_2 (by (34.48)), X is closed, and $\xi_n' \in X$]. Thus x_0 is the first point of $\Gamma \setminus \{y_2\}$ that we meet when we run along γ_0 backwards.

We have just seen that $x_0 \notin (y_2, z_2']$, (34.46) tells us that x_0 cannot lie on $\gamma_0' \subset \Gamma_m$, and (34.50) prevents x_0 from lying on ∂B_R. Thus x_0 lies on the arc of Γ_{m_1} that starts γ_0.

First suppose that $x_0 \in X^+$. Call γ_0'' the arc of γ_0 between x_0 and y_2. Then $\gamma_0'' \setminus \{x_0, y_2\}$ does not meet Γ (by definition of x_0), and it is even contained in the unbounded component of $\mathbb{R}^2 \setminus \Gamma$ (call it Ω_∞), because it contains a piece of ∂B_R. Also denote by γ'' the arc of γ between y_1 and y_3. Then $\gamma'' \setminus \{y_1, y_3\}$ lies in Ω_∞, by (34.48) and (34.46). Since $x_0 \in X^+$, we can apply Lemma 34.53 and we get that γ_0'' meets γ''. This obvious contradiction with (34.51) settles our first case.

So we may assume that $x_0 \in X^- = X \setminus X^+$. Denote by η_0 the arc of γ_0 between ξ_n' and x_0. We already know that η_0 is a sub-arc of Γ_{m_1}. Hence η_0 does not contain any y_i, $1 \leqslant i \leqslant 3$, by (34.44) and (34.33). Hence $\operatorname{dist}(X \cap \eta_0, \{y_1, y_2, y_3\}) > 0$, and the sets $\eta_0 \cap X^+$ and $\eta_0 \cap X^-$ are both closed (because they are the intersections of $\eta_0 \cap X$ with the two components of $\Gamma \setminus \{y_1, y_3\}$).

Since $\xi_n' \in X^+$ and $\eta_0 \cap X^+$ is closed, there is a last point $x_1 \in X^+$ when we run along η_0 from ξ_n' to x_0. Note that $x_1 \neq x_0$, because $x_0 \in X^-$. Since in addition

$\eta_0 \cap X^-$ is closed, there is a first point x_2 of $\eta_0 \cap X^-$ after x_1. Note that x_2 is also the first point of $\eta_0 \cap X$ after x_1, by definition of x_1.

Denote by η the arc of η_0 between x_1 and x_2. Then

$$\eta \text{ meets } \Gamma \text{ only at its endpoints,} \tag{34.56}$$

by what was just said. Also,

$$\eta \text{ is simple} \tag{34.57}$$

by (34.43) or (34.52), and

$$x_1 < y_1 < x_2 < y_3 < x_1 \text{ on } \Gamma, \tag{34.58}$$

because $x_1 \in X^+$ and $x_2 \in X^-$.

The easiest subcase now is when

$$\eta \setminus \{x_1, x_2\} \subset \Omega_\infty, \tag{34.59}$$

where Ω_∞ still denotes the unbounded component of $\mathbb{R}^2 \setminus \Gamma$. In this case we can apply Lemma 34.53 to the curves η (from x_1 to x_2) and γ'' (the sub-arc of γ from y_1 to y_3). As was observed before, $\gamma'' \setminus \{y_1, y_3\} \subset \Omega_\infty$, by (34.46), (34.48), the connectedness of γ'', and the fact that it contains some points of $\gamma' \subset \Gamma_m$. So (34.58) allows us to apply Lemma 34.53, and we get that γ'' meets η. This is impossible, because $\eta \subset \gamma_0$ and by (34.51).

In view of (34.56) the only case left when (34.59) fails is when

$$\eta \setminus \{x_1, x_2\} \subset \Omega_b, \tag{34.60}$$

where Ω_b denotes the bounded component of $\mathbb{R}^2 \setminus \Gamma$. In this case we want to use the same sort of argument, but with arcs in Ω_b. Thus we want to connect y_1 and y_3 by an arc in Ω_b [See Figure 34.4.]

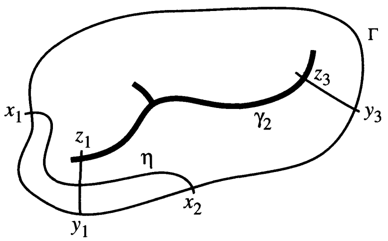


FIGURE 34.4

Let γ_2' be a simple arc in G_0 that connects z_1 to z_3. Such an arc exists, by Lemma 19.14. Let γ_2 be the simple curve that starts from y_1, follows $[y_1, z_1]$, runs along γ_2'

up to z_3, and then follows $[z_3, y_3]$ up to y_3. The two segments $[z_i, y_i)$ do not meet Γ (by (34.48)), and neither does G_0 (by (34.47)). Thus

$$\gamma_2 \setminus \{y_1, y_3\} \subset \Omega_b \tag{34.61}$$

(by (34.47) again). From (34.58) (34.60), (34.61), and the analogue for Ω_b of Lemma 34.54, we get that η meets γ_2. But this is impossible, because $w(x) = m_1$ on η (because $\eta \subset \eta_0 \subset \Gamma_{m_1}$), $w(x) \leqslant m_0$ on γ_2 (by (34.33) and because $w(x) \equiv m_0$ on G_0), and by (34.44). This contradiction in our last subcase completes the proof of Lemma 34.26. □

Lemma 34.62. — *If $t_0 \in S^-$ and $t_1, t_3 \in R^+$ are such that $t_0 < t_1 < t_3 < t_0$ in $\mathbb{S}^1$, then $u(t_1) < u(t_3)$.*

This looks like a substantial improvement over Lemma 34.26, but the hard work is done. Because of (34.20), there is a neighborhood V of t_1 in $\mathbb{S}^1$ such that $t \in R^+$ for almost all $t \in V$. We choose $t \in R^+$ just a little past t_1, so that

$$t_0 < t_1 < t < t_3 < t_0 \text{ in } \mathbb{S}^1, \tag{34.63}$$

and also so close to t_1 that

$$u(t) > u(t_1). \tag{34.64}$$

This is possible, because $u'(t_1) > 0$ by definition of R^+ (see (34.18)).

Let us apply Lemma 34.26 to our four points. We get that $u(t)$ lies between $u(t_1)$ and $u(t_3)$ (with possible equalities), and the only option compatible with (34.64) is that $u(t_1) < u(t) \leqslant u(t_3)$. This proves the lemma. □

Lemma 34.65. — *If $t_0 < t_1 < t_2 < t_3 < t_0$ in $\mathbb{S}^1$, we cannot have that $t_0, t_2 \in S^-$ and $t_1, t_3 \in R^+$.*

Indeed, if we apply Lemma 34.62 with $t_0 < t_1 < t_3 < t_0$, we get that $u(t_1) < u(t_3)$, while if we apply it with $t_2 < t_3 < t_1 < t_2$, we get that $u(t_3) < u(t_1)$. □

Denote by $I^\pm$ the closure of $R^\pm$ in $\mathbb{S}^1$. We know from (34.21) that $I^+ \cup I^- = \mathbb{S}^1$, because $R^+ \cup R^-$ is dense.

First we want to get rid of the case when $R^- = \varnothing$. Pick to $t_0 \in S^-$; this is possible, by Lemma 34.24. Set $J = \mathbb{S}^1 \setminus \{t_0\}$; J is an interval, and Lemma 34.62 tells us that the restriction of u to $R^+ \cap J$ is strictly increasing. Since R^+ is dense in J because $R^- = \varnothing$, we also get that the restriction of u to J is strictly increasing. [Recall from Lemma 34.4 that u is continuous.] This is impossible, because u is also continuous at t_0.

The case when $R^+ = \varnothing$ is also impossible, by the same argument (involving symmetric versions of Lemmas 34.26 and 34.62).

Fix points $t_- \in R^-$ and $t_+ \in R^+$. For each $t \in R^+$, $t \neq t_+$, denote by $J(t)$ the connected component of $\mathbb{S}^1 \setminus \{t, t_+\}$ that does not contain t_-. We claim that

$$J(t) \cap R^- = \varnothing \text{ for all } t \in R^+ \setminus \{t_+\}. \tag{34.66}$$

Indeed $R^- \subset S^-$ by (34.23), and then lemma 34.65 tells us that for all $s \in R^- \setminus \{t_-\}$, the two points s and t_- lie in the same component of $\mathbb{S}^1 \setminus \{t, t_+\}$. This proves (34.66).

From (34.66) we deduce that

$$J(t) \cap I^- = \varnothing \text{ for } t \in R^+ \{t_+\}, \tag{34.67}$$

because $J(t)$ is open and I^- is the closure of R^-. Then $J(t) \subset I^+$, since $\mathbb{S}^1 = I^+ \cup I^-$.

Thus every point of R^+ can be connected to t_+ by an arc in I^+, I^+ is connected, and hence I^+ is either a closed interval or the whole $\mathbb{S}^1$.

To exclude the last possibility, notice that since $t_- \in R^-$, (34.20) gives a small open interval V centered at t_- and which is H^1-almost entirely contained in R^-. Then $V \subset I^-$ and V does not meet any $J(t)$, $t \in \mathbb{R}^+ \setminus \{t_+\}$, by (34.67). Since V is open, we get that $V \cap I^+ = \varnothing$ and $I^+ \neq \mathbb{S}^1$.

Of course I^- is also an interval which does not get too close to t_+ (for the same reasons as for I^+), and I^- does not meet the interior of I^+, because it does not meet any of the $J(t), t \in R^+ \setminus \{t_+\}$ (by (34.67)).

To complete our proof of Proposition 34.9, we still need to check (34.10) and (34.11). However (34.10) is an easy consequence of Lemma 34.62 (applied with $t_0 = t_-$), the density of R^+ in I^+, and the continuity of u. The verification of (34.11) would be similar.

This completes our proof of Proposition 34.9. □

35. The monotonicity formula

The goal of this section is to prove a monotonicity formula similar to the one in [Bo], and which will be used later to find lots of low energy points in G.

We keep the same hypotheses as in Section 32, and even add a new one. Recall from (32.2) that we assume that G has exactly one unbounded component G_{00}. We want to suppose now that

$$G \setminus G_{00} \text{ is bounded,} \tag{35.1}$$

so that

$$m_0 = \inf \{w(z)\ ;\ z \in G\} \tag{35.2}$$

is finite. [Recall that w is continuous.] Set

$$\Omega_0 = \left\{z \in \mathbb{R}^2\ ; w(z) < m_0\right\}. \tag{35.3}$$

Then Ω_0 is open, and $\Omega_0 \subset \Omega$ by definition of m_0. For all $x \in \overline{\Omega}_0$ and $r > 0$, set

$$\Phi_x(r) = \frac{1}{r} \int_{\Omega_0 \cap B(x,r)} |\nabla v|^2 . \tag{35.4}$$

Proposition 35.5. — *For all $x \in \overline{\Omega}_0$, Φ_x is nondecreasing on $(0, +\infty)$.*

This is the main result of this section, but we shall also need to know about the cases when Φ_x is locally constant. Because of this, and even though the proof of Proposition 35.5 is very close to the argument in [Bo], we feel compelled to give a fairly detailed proof. This is also the reason why we don't want to derive the proposition from its analogue on simpler domains Ω_m, which would be easier to do.

Fix a point $x \in \overline{\Omega}_0$, and write $\Phi(r)$ instead of $\Phi_x(r)$ to simplify. By Fubini, $\varphi(r) = r\Phi(r)$ is the integral of its derivative $\varphi'(r) = \int_{\Omega_0 \cap \partial B(x,r)} |\nabla v|^2$; so Φ is differentiable almost-everywhere, its derivative is

$$\Phi'(r) = -\frac{1}{r}\Phi(r) + \frac{1}{r} \int_{\Omega_0 \cap \partial B(x,r)} |\nabla v|^2 \tag{35.6}$$

almost-everywhere, and Φ is the integral of its derivative. Thus it will be enough to prove that

$$\Phi(r) \leqslant \int_{\Omega_0 \cap \partial B(x,r)} |\nabla v|^2 \tag{35.7}$$

for almost every $r > 0$. Let us integrate by parts a first time to get that

$$\Phi(r) = \frac{1}{r} \int_{\Omega_0 \cap \partial B(x,r)} v \frac{\partial v}{\partial n} dH^1 , \tag{35.8}$$

where n is the unit normal to $\partial B(x,r)$ that points outside of $B(x,r)$. Thus $\partial v / \partial n$ will be the same thing as $\partial v / \partial r$. Let us (even) verify (35.8) by a limiting argument. Set

$$\Omega_m = \left\{ z \in \mathbb{R}^2 \ ; \ w(z) < m \right\} \tag{35.9}$$

for all $m < m_0$. Of course $\Omega_m \subset \Omega_0 \subset \Omega$, and Ω_0 is the increasing union of the Ω_m. Thus

$$\Phi(r) = \lim_{\substack{m \to m_0 \\ m < m_0}} \frac{1}{r} \int_{\Omega_m \cap B(x,r)} |\nabla v|^2 , \tag{35.10}$$

by monotone convergence. Let us also check that

$$\int_{\Omega_0 \cap \partial B(x,r)} v \frac{\partial v}{\partial n} \, dH^1 = \lim_{\substack{m \to m_0 \\ m < m_0}} \int_{\Omega_m \cap \partial B(x,r)} v \frac{\partial v}{\partial n} \, dH^1 \tag{35.11}$$

for almost every $r > 0$. Certainly (35.11) holds as soon as

$$\int_{\Omega_0 \cap \partial B(x,r)} |v| \, |\nabla v| \, dH^1 < +\infty, \tag{35.12}$$

and for (35.7) it is even enough to check that (35.12) holds for almost every r.

Let us use the fact that $H^1(G \cap B_R) < +\infty$ for all R; we get that $G \cap \partial B(x,r)$ is finite for H^1-almost all choices of r (by a Fubini-like argument that we used a few times before). For these values of r, $\partial B(x,r) \setminus G$ is composed of finitely many open intervals I, and on each of them the oscillation of v is at most $\int_I |\nabla v|$. Then (35.12) holds as soon $\displaystyle\int_{\partial B(x,r)\setminus G} |\nabla v| \, dH^1 < +\infty$. This is the case for almost all r, by Cauchy-Schwarz and because $\displaystyle\int_{B_R \setminus G} |\nabla v|^2 < +\infty$ for all R.

This proves that (35.12), and hence (35.11), hold for almost all $r > 0$. It was also possible to avoid the Fubini-like argument, and control the size of v on the potentially infinite number of components of $\partial B(x,r) \setminus G$ by connecting each of them to a reasonably large disk in Ω. This would use Sections 20 (to connect components to larger disks) and 21 (to control the values of v).

Because of (35.10) and (35.11), to prove that (35.8) holds for almost every $r > 0$ (which will be enough), it is enough to check that for almost-every $m < m_0$,

$$\int_{\Omega_m \cap B(x,r)} |\nabla v|^2 = \int_{\Omega_m \cap \partial B(x,r)} v \frac{\partial v}{\partial n} \, dH^1. \tag{35.13}$$

By proposition 32.6, we can restrict to the $m < m_0$ such that

$$\Gamma_m = w^{-1}(m) \text{ is a rectifiable Jordan curve through } \infty. \tag{35.14}$$

When (35.14) holds, Γ_m is even an analytic curve, because for $m < m_0$ the level set Γ_m is contained in Ω and w is harmonic on Ω. Thus there will be no regularity problem to integrate by parts. Also, $w - m$ changes signs when we cross Γ_m, and since the sign remains constant on each of the two components of $\mathbb{R}^2 \setminus \Gamma_m$, one of them must be Ω_m and the other one $\{z \,;\, w(z) > m\}$. In particular, $\partial \Omega_m = \Gamma_m$. Let us apply Green's theorem on $\Omega_m \cap B(x,r)$. We get that

$$\int_{\Omega_m \cap B(x,r)} |\nabla v|^2 = \int_{\partial(\Omega_m \cap B(x,r))} v \frac{\partial v}{\partial n} \, dH^1, \tag{35.15}$$

because v is harmonic. Notice that $\partial v / \partial n = \partial w / \partial \tau = 0$ on Γ_m, because w is conjugated to v and Γ_m is a level curve for w. Thus the only piece of boundary that gives a contribution to the right-hand side of (35.15) is $\Omega_m \cap \partial B(x,r)$; (35.13) follows.

This completes our verification of (35.8) for almost-every $r > 0$.

Because of (35.8), it will be enough to prove that

$$\frac{1}{r} \int_{\Omega_0 \cap \partial B(x,r)} v \frac{\partial v}{\partial n} \, dH^1 \leqslant \int_{\Omega_0 \cap \partial B(x,r)} |\nabla v|^2 \, dH^1 \tag{35.16}$$

for almost all $r > 0$. (Compare with (35.7)).

To prove (35.16), decompose $\Omega_0 \cap \partial B(x,r)$ into its connected components I_j, $j \in J$. Thus I_j is an open interval of $\partial B(x,r)$ (or maybe the whole $\partial B(x,r)$).

Denote by v_r and v_θ the radial and tangential derivatives of v on $\Omega_0 \cap \partial B(x,r)$. Then

$$(35.17)\quad \begin{aligned}\int_{I_j} |\nabla v|^2 \, dH^1 &= \int_{I_j} v_r^2 dH^1 + \int_{I_j} v_\theta^2 \, dH^1 \\ &:= A^2 + B^2 \geqslant 2AB = 2\Big\{\int_{I_j} v_r^2 \, dH^1\Big\}^{1/2} \Big\{\int_{I_j} v_\theta^2 \, dH^1\Big\}^{1/2}.\end{aligned}$$

Now we use a classical result of Wirtinger.

Lemma 35.18. — *Let $I = [a,b]$ be an interval, and f a function on I such that $f' \in L^2(I)$ and $\int_I f(x)dx = 0$. Then*

$$(35.19)\quad \int_I f(x)^2 dx \leqslant \Big(\frac{|I|}{\pi}\Big)^2 \int_I (f'(x))^2 \, dx.$$

Also, the only functions for which (35.19) *is an equality are the multiples of the function* $f_0(x) = \sin\Big[\frac{\pi(x-a)}{b-a} - \frac{\pi}{2}\Big]$.

See for instance Theorem 2.58 in [HaLiPo].

Denote by α_j the mean value of v on I_j. If I_j is an interval, Lemma 35.18 tells us that

$$(35.20)\quad \int_{I_j} (v-\alpha_j)^2 \, dH^1 \leqslant \frac{|I_j|^2}{\pi^2} \int_{I_j} v_\theta^2 \, dH^1.$$

In the remaining case when $I_j = \partial B(x,r)$, (35.20) is still true (and could even be improved), because we can always remove a point from the circle and apply the lemma to the remaining interval. [This amounts to forgetting a periodicity condition.] Finally,

$$(35.21)\quad \begin{aligned}\int_{I_j} v_r \,(v-\alpha_j) \, dH^1 &\leqslant \Big\{\int_{I_j} v_r^2 \, dH^1\Big\}^{1/2} \Big\{\int_{I_j} (v-\alpha_j)^2 \, dH^1\Big\}^{1/2} \\ &\leqslant \Big\{\int_{I_j} v_r^2 \, dH^1\Big\}^{1/2} \frac{|I_j|}{\pi} \Big\{\int_{I_j} v_\theta^2 \, dH^1\Big\}^{1/2} \\ &\leqslant \frac{|I_j|}{2\pi} \int_{I_j} |\nabla v|^2 \, dH^1\end{aligned}$$

by Cauchy-Schwarz, (35.20), and (35.17).

So far we did not use any specific property of $\partial\Omega_0$, but this will have to happen when we prove that

$$(35.22)\quad \int_{I_j} v_r dH^1 = 0.$$

Before we prove (35.22), let us say how to use it to complete our proof. By (35.22), we can remove α_j from the left-hand side of (35.21). We get that

$$(35.23)\quad \int_{\Omega_0\cap\partial B(x,r)} v\frac{\partial v}{\partial n}dH^1 = \sum_{j\in J}\int_{I_j} v_r v\, dH^1 \leqslant \sum_{j\in J}\frac{|I_j|}{2\pi}\int_{I_j}|\nabla v|^2\, dH^1$$
$$\leqslant \sum_{j\in J} r\int_{I_j}|\nabla v|^2\, dH^1 = r\int_{\Omega_0\cap\partial B(x,r)}|\nabla v|^2\, dH^1,$$

because $\partial v/\partial n = v_r$ by definitions, $\Omega_0\cap\partial B(x,r)$ is the disjoint union of the I_j, and by (35.22) and (35.21).

This is the same thing as (35.16). Thus Proposition 35.5 will follow from (35.22).

Now we want to prove (35.22). The argument will use the same sort of ingredients as for the verification of the coherence of our definition of w in Section 22, and it may even be that one can be deduced from the other. Let us first check that

$$(35.24)\qquad I_j = \bigcup_{m<m_0} I(j,m),$$

where $I(j,m)$ denotes the component of the center of I_j (or any given point of I_j if $I_j = \partial B(x,r)$) in $\Omega_m\cap\partial B(x,r)$. This is easy, because Ω_0 is the increasing union of the Ω_m, $m<m_0$, and then every compact subset of I_j is contained in Ω_m, $m<m_0$ sufficiently close to m_0.

Next we want to show that

$$(35.25)\qquad \int_{I(j,m)} v_r\, dH^1 = 0.$$

If we can do this, (35.22) will follow at once, provided that $\int_{\partial B(x,r)\setminus G}|\nabla v|\, dH^1 < +\infty$ (so that we can apply Lebesgue's dominated convergence theorem). Our convergence condition is satisfied for almost all $r>0$ (because $\int_{B_R\setminus G}|\nabla v|^2 < +\infty$ for all R), hence it will not disturb.

Also, it will be enough to prove (35.25) for almost-every $m<m_0$. So we may assume that (35.14) holds.

The easiest case is when $I(j,m) = \partial B(x,r)$. In this case Γ_m does not meet $\partial B(x,r)$; it does not meet $B(x,r)$ either (because it is connected and unbounded). Then $B(x,r)\subset\Omega_m$ and

$$(35.26)\qquad \int_{\partial B(x,r)} v_r\, dH^1 = \int_{\partial B(x,r)}\frac{\partial v}{\partial n}dH^1 = \int_{B(x,r)}\Delta v = 0.$$

If $I(j,m) = \partial B(x,r)\setminus\{y\}$ for some $y\in\partial B(x,r)$, then $\Gamma_m\cap\partial B(x,r) = \{y\}$ and Γ_m does not get inside $B(x,r)$ (by (35.14)). In this case also $B(x,r)\subset\Omega_m$ and (35.25) is proved as in (35.26).

We are left with the case when $I(j,m)$ is an interval of $\partial B(x,r)$ with distinct extremities a,b. Then a and b lie on $\partial\Omega_m = \Gamma_m$. Denote by $\gamma(a,b)$ the arc of Γ_m

between these two points, and by γ the Jordan arc that we get by completing $\gamma(a,b)$ by the arc $I(j,m)$. [This is a Jordan curve because $I(j,m) \subset \Omega_m$ does not meet Γ_m.] Let Ω^* denote the bounded component of $\mathbb{R}^2 \setminus \Gamma$. We claim that

$$\Omega^* \subset \Omega_m. \tag{35.27}$$

[See Figure 35.1.] Indeed, the two open arcs of Γ_m that compose $\Gamma_m \setminus \gamma(a,b)$ do not meet $\gamma = \partial\Omega^*$, and so they are both contained in the unbounded component of $\mathbb{R}^2 \setminus \gamma$ (because they are connected and unbounded). Thus Ω^* does not meet Γ_m. Since $w(z) < m$ near $I(j,m)$, we get (35.27).

Now v is harmonic on Ω^* and Green yields

$$0 = \int_{\Omega^*} \Delta v = \int_{\partial\Omega^*} \frac{\partial v}{\partial n} dH^1 = \int_{I(j,m)} \frac{\partial v}{\partial n} dH^1, \tag{35.28}$$

because $\partial v/\partial n = \partial w/\partial \tau = 0$ on Γ_m. This proves (35.25). Proposition 35.5 follows, as was said before. □

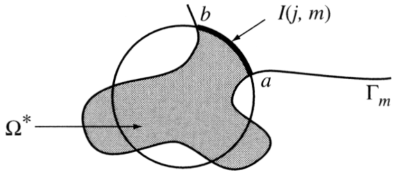


FIGURE 35.1

Now we want to discuss the case when Φ_x is constant on some interval $[r_1, r_2]$. The case when $\Phi_x(r) = 0$ for any $r > 0$ is excluded here, because we assume that Ω is connected and v is not constant. [See (32.1) and (32.5).]

Lemma 35.29. — *If Φ_x is constant on some interval $[r_1, r_2]$, $r_2 > r_1$, then we can find constants $\alpha, \beta, \theta_0 \in \mathbb{R}$ such that*

$$G = \left\{x - \rho e^{i\theta_0} \ ; \ \rho \geqslant 0\right\} \tag{35.30}$$

and

$$v\left(x + re^{i\theta}\right) = \alpha + \beta r^{1/2} \sin\Big(\frac{\theta - \theta_0}{2}\Big) \tag{35.31}$$

for $\theta_0 - \pi < \theta < \theta_0 + \pi$ and $0 < r < +\infty$. Moreover $\beta = \pm\left(2\Phi/\pi\right)^{1/2}$, where Φ denotes the constant value of Φ_x on $[r_1, r_2]$.

Suppose that Φ_x is constant on $[r_1, r_2]$. Then (35.7) is an equality for almost all $r \in (r_1, r_2)$, and so are all the inequalities that compose (35.23).

Suppose this is the case for r. Then $|I_j|/2\pi = r$ for all the sets I_j that give a nontrivial contribution to the sum in (35.23). Note that there is at least one such interval (call it I), because the right-hand side of (35.23) is positive (by (35.7) and because $\Phi_x(r) > 0$). Then $|I| = 2\pi r$, which means that

$$I = \left\{x + re^{i\theta} \ ; \ \theta_0 - \pi < \theta < \theta_0 + \pi\right\}, \tag{35.32}$$

where $\theta_0 = \theta_0(r)$ may depend on r, or else $I = \partial B(x,r)$. In both cases, I is the only component of $\Omega_0 \cap \partial B(x,r)$.

Since the first inequality in (35.23) is an equality, (35.21) (for $I_j = I$) is also an equality. Since both sides of (35.21) are positive, we can simplify and deduce from the second inequality that

$$\int_I (v - \alpha_j)^2 \, dH^1 = \frac{|I|^2}{\pi^2} \int_I v_\theta^2 \, dH^1, \tag{35.33}$$

and both sides of (35.33) are positive. Thus we are in a situation where (35.19) is an equality. If (35.32) holds, Lemma 35.18 yields that

$$v\left(x + re^{i\theta}\right) = \alpha(r) + \beta(r) \sin\frac{\theta - \theta_0(r)}{2} \quad \text{for} \ \ \theta_0(r) - \pi < \theta < \theta_0(r) + \pi. \tag{35.34}$$

Also, $\beta(r) \neq 0$ because both sides of (35.33) are positive.

In the remaining case when $I = \partial B(x,r)$, we can apply the same argument and get (35.34) with any value of θ_0. This is of course impossible, because $\beta(r) \neq 0$ and hence $v\left(x + re^{i\theta}\right)$ has a jump on $I \subset \Omega_0$.

Now we want to see how $\alpha(r)$, $\beta(r)$, and $\theta_0(r)$ depend on r. Rewrite (35.34) as

$$v\left(x + re^{i\theta}\right) = \alpha(r) + \beta(r)\cos(\theta_0(r)/2)\sin(\theta/2) - \beta(r)\sin(\theta_0(r)/2)\cos(\theta/2), \tag{35.35}$$

which holds for almost every $r \in (r_1, r_2)$ and then $\theta_0(r) - \pi < \theta < \theta_0(r) + \pi$.

Let $r_0 \in (r_1, r_2)$ be such that Ω_0 meets $\partial B(x, r_0)$. We know from (35.32) that this is the case for almost all $r_0 \in (r_1, r_2)$. Pick a little disk D_0 centered on $\partial B(x, r_0)$ and contained in Ω_0. Then replace our initial interval (r_1, r_2) with a smaller, but nontrivial interval such that $\partial B(x,r)$ meets $\frac{1}{2}D_0$ for all r in the new interval (r_1, r_2). For all $r \in (r_1, r_2)$ such that (35.35) holds, we can use linear algebra to recover the coefficients $\alpha(r)$, $\beta(r)\cos(\theta_0(r)/2)$, $\beta(r)\sin(\theta_0(r)/2)$ from the values of v on D_0. [In fact, a small number of values of θ would be enough.] This proves that these coefficients are (restrictions to a set of full measure of) smooth functions defined for $r \in (r_1, r_2)$. Then (35.35) holds for all $r \in (r_1, r_2)$ (by taking limits), and we can differentiate in (35.35).

Since (35.21) is an equality, we must also have $A^2 = B^2$ in (35.17), hence

$$\int_I v_r^2 \, dH^1 = \int_I v_\theta^2 \, dH^1 = \frac{1}{2} \int_I |\nabla v|^2 \, dH^1. \tag{35.36}$$

Call Φ the constant value of $\Phi_x(r)$ on (r_1, r_2). Since (35.7) is an equality (almost everywhere),

$$\int_I |\nabla v|^2 \, dH^1 = \Phi. \tag{35.37}$$

On the other hand, a direct computation with (35.35) or (35.34) gives that

$$\int_I v_\theta^2 dH^1 = \frac{\pi}{4r} \beta(r)^2, \tag{35.38}$$

and hence

$$\beta(r) = \pm \left(\frac{2r\Phi}{\pi}\right)^{1/2} = \beta\, r^{1/2}, \tag{35.39}$$

with $\beta = \pm\,(2\Phi/\pi)^{1/2}$, as in the statement of Lemma 35.29. Of course the sign is constant by continuity.

If we differentiate in (35.34) we get that

$$v_r\left(x + re^{i\theta}\right) = \beta'(r)\, \sin\Big(\frac{\theta - \theta_0(r)}{2}\Big) + h(\theta), \tag{35.40}$$

where $h(\theta)$ is some linear combination of $\cos\left((\theta - \theta_0(r))/2\right)$ and a constant, with coefficients that may depend on r. In particular, h is orthogonal to $\sin\left((\theta - \theta_0(r))/2\right)$ and

$$\int_I v_r^2 \, dH^1 = \pi r \beta'(r)^2 + \int_I h(\theta)^2 \, dH^1 = \frac{\Phi}{2} + \int_I h(\theta)^2 \, dH^1, \tag{35.41}$$

by (35.39). When we compare this with (35.36) and (35.37), we get that $h(\theta) \equiv 0$, and then that $\alpha(r)$ and $\theta_0(r)$ are constant.

This gives the same description of v as in (35.31), but only for $r_1 < r < r_2$. The general case follows easily, because Ω is connected and v is harmonic on Ω.

When (13.2) holds, we immediately deduce the description of G in (35.30) from (35.31), because (v, G) is a reduced minimizer.

When (13.1) holds, this brutal approach only gives that $G = G_0 \cup L$, where G_0 denotes the half-line in (35.30) and $L = (-\infty, -1]$ is as in the definition of our modified functional. [See Section 11.] However we must have $\partial v/\partial n = 0$ on L, and it is easy to see that this can only happen when $L \subset G_0$. For instance, we can say that L must be contained in a level set of w (since $\partial w/\partial \tau = 0$ on L), and we can check that G_0 is the only level set of w that contains a line segment.

This completes our proof of Lemma 35.29. □

Remark 35.42 (on cracktips). — In the situation of Lemma 35.29, we can say a little more, because many pairs (v, G) defined by (35.30) and (35.31) do not satisfy (13.1) or (13.2).

First, (35.30) and (35.31) can only define a global λ-minimizer (v, G) (as in (13.2)) when $\beta = \pm\sqrt{2\lambda/\pi}$, which corresponds to $\Phi_x(r) \equiv \lambda$. This was already observed in [MuSh]; the point is that when $\beta^2 < 2\lambda/\pi$, we can make G a little shorter near x and save more in length than we have to pay in energy, while when $\beta^2 > 2\lambda/\pi$ we can improve (v, G) by making the half-line G a little longer.

The situation for global λ-minimizers in $\mathbb{R}^2 \setminus (-\infty, 0]$ (see Definition 12.41) is a little different. If G in (35.30) contains $(-\infty, 0]$ strictly, then we still have the constraint $\beta^2 = 2\lambda/\pi$ as above, but if $G = (-\infty, 0]$, the argument suggested above only gives the constraint $\beta^2 \leqslant 2\lambda/\pi$ (because we are not allowed to make G shorter).

The situation for minimizers of the modified functional is similar. If G contains L strictly, we get the constraint $\beta^2 = 2\lambda/\pi$ (where now $\lambda = h'\left(H^1(G \setminus L)\right)$, and when $G = L$ we only know that $\beta^2 \leqslant 2\lambda/\pi$. This can be seen directly, or we could use the fact that blow-ups of (v, G) at the endpoint of L are λ-minimizers in $\mathbb{R}^2 \setminus (-\infty, 0)$ (by Proposition 12.42).

Of course the converse to these results, i.e., the fact that we listed all the constraints, is much less obvious, or else at least half of this paper is ridiculous.

36. $G \cap \partial\Omega_0$ contains only regular and spider points

In this section and the next one we want to use the monotonicity formula and blow up arguments to find lots of regular or spider points in G. We continue with the hypotheses of the previous section, and even add two. Let us assume that

$$\liminf_{r\to+\infty} \frac{1}{r} \int_{B(x,r)\setminus G} |\nabla v|^2 \leqslant \lambda, \tag{36.1}$$

where λ is as in the definition of a global λ-minimizer when (13.2) hold, and $\lambda = h'\left(H^1(G \setminus L)\right)$ (as usual) when (13.1) holds. Let us also assume that

$$(v, G) \text{ is not a “generalized cracktip''}, \tag{36.2}$$

by which we just mean that we cannot find $x \in \mathbb{R}^2$ and constants $\alpha, \beta, \theta_0 \in \mathbb{R}$ such that (35.30) and (35.31) hold. We use a slightly different name here not to conflict with the slightly more restrictive definition of cracktips given in Section 1.

Proposition 36.3. — *Under the hypotheses above, $m_0 < 0$ and every point of $G \cap \partial\Omega_0$ is a regular or a spider point of G.*

See (35.2) for the definition of m_0 and Section 13 for regular and spider points.

We start with a few easy observations. Let $x \in \overline{\Omega}_0$ be given, and let Φ_x be as in (35.4). Proposition 35.5 says that Φ_x is nondecreasing, and so it has limits at 0 and ∞. Then (36.1) tells us that

$$\lim_{r\to\infty} \Phi_x(r) \leqslant \lambda. \tag{36.4}$$

Set

$$\ell(x) = \lim_{r\to 0} \Phi_x(r) \tag{36.5}$$

Then (36.4) implies that $\ell(x) \leqslant \lambda$. Let us check that

$$\ell(x) < \lambda \text{ for all } x \in \overline{\Omega}_0. \tag{36.6}$$

Indeed, if $\ell(x) = \lambda$, then $\Phi_x(r)$ is constant and Lemma 35.29 says that (v, G) is a generalized cracktip. Since (36.2) excludes this case, (36.6) holds.

The main step in our proof will be to prove that $\ell(x) = 0$ (as in the next lemma). This will be done in the rest of this section. Then we shall need to worry about the difference between $\Omega_0 \cap B(x,r)$ and $B(x,r) \setminus G$ in the definition of Φ_x. This will be done in Section 37. In both parts of the argument, blow-ups will be useful.

Lemma 36.7. — *We have that $\ell(x) = 0$ for all $x \in \overline{\Omega}_0$, except perhaps for $x = -1$ when* (13.1) *holds.*

The case when (13.1) holds and $x = -1$ is a little special, and it will be easier to treat it separately later.

Note that $\ell(x) = 0$ on Ω trivially, because ∇v is continuous there. Thus we may restrict to $x \in G \cap \overline{\Omega}_0 = G \cap \partial\Omega_0$ (since $\Omega_0 \subset \Omega$, as we checked just below (35.3).

Fix a point $x \in G \cap \partial\Omega_0$. We shall proceed by contradiction and assume that $\ell(x) > 0$.

Let $\{t_n\}$ be any sequence of positive numbers with

$$\lim_{n\to+\infty} t_n = 0. \tag{36.8}$$

Later on, $\{t_n\}$ will be associated to a (converging) blow-up sequence $\{(v_n, G_n)\}$ as in Section 12, but for the moment we want to use the monotonicity formula and our assumption that $\ell(x) > 0$ to study the behavior of v on the circles $\partial B(x, t_n\rho)$ for (almost every) fixed $\rho > 0$. Some amount of notation will be useful.

For each $\rho > 0$ and $n \geqslant 0$, denote by $I(n)$ the longest component of $\Omega_0 \cap \partial B(x, t_n\rho)$. [In case of equality, choose a longest one at random.] Set

$$I'(n) = \{z \in \partial B(0,\rho)\ ;\ t_n z + x \in I(n)\} \tag{36.9}$$

and

$$f_n(z) = t_n^{-1/2}\left[v(t_n z + x) - \alpha(n)\right] \tag{36.10}$$

for $z \in I'(n)$, and where $\alpha(n)$ denotes the mean value of v on $I(n)$. The definition makes sense because $I(n) \subset \Omega_0 \subset \Omega$.

Lemma 36.11. — *For almost every $\rho > 0$, we can find a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and constants $\theta(\rho)$ and $\beta(\rho)$ such that:*

$$\text{the compacts sets } \partial B(0,\rho) \setminus I'(n_k) \text{ converge to } \left\{-e^{i\theta(\rho)}\right\}; \tag{36.12}$$

(36.13)
$$\{f_{n_k}\} \text{ converges to } f, \text{uniformly on compact subsets of } \partial B(0,\rho) \setminus \{-e^{i\theta(\rho)}\},$$

where f is defined by

$$f\left(\rho e^{i\theta}\right) = \beta(\rho)\sin\Big(\frac{\theta-\theta(\rho)}{2}\Big) \text{ for } \theta(\rho)-\pi < \theta < \theta(\rho)+\pi\ ; \tag{36.14}$$

$$\beta(\rho)^2 = \frac{2\rho\ell(x)}{\pi}. \tag{36.15}$$

The proof of this lemma will take some time. Clearly it will be enough to show that for all choices of $0 < a < b < +\infty$, almost all $\rho \in (a,b)$ have the property of the lemma. So we fix a and b as above and restrict our attention to $\rho \in (a,b)$.

Denote by E the set of $\rho \in (a,b)$ such that, for all $n \geqslant 0$, $\Phi_x(r)$ is differentiable at $r = t_n\rho$, and the various equalities of Section 35 (that were only proved for almost all r) hold with $r = t_n\rho$. Of course $H^1\left((a,b)\setminus E\right) = 0$ and we can restrict our attention to $\rho \in E$.

From (35.6) and the sentence that follows it, we get that

$$\begin{aligned}\Phi_x(bt_n) - \Phi_x(at_n) &= \int_{r=at_n}^{bt_n} \Phi_x'(r)dr \\ &= \int_{at_n}^{bt_n}\left\{\frac{1}{r}\int_{\Omega_0\cap\partial B(x,r)} |\nabla v|^2 - \frac{1}{r}\Phi_x(r)\right\}dr.\end{aligned} \tag{36.16}$$

For all $n \geqslant 0$ and $\rho \in (a,b)$, set

$$A_n(\rho) = \int_{\Omega_0\cap\partial B(x,t_n\rho)} |\nabla v|^2 \tag{36.17}$$

and

$$B_n(\rho) = \Phi_x(t_n\rho) = \frac{1}{t_n\rho}\int_{\Omega_0\cap B(x,t_n\rho)} |\nabla v|^2. \tag{36.18}$$

Note that

$$B_n(\rho) = \frac{1}{t_n\rho}\int_{\Omega_0\cap\partial B(x,t_n\rho)} v\,\frac{\partial v}{\partial n}\,dH^1 \tag{36.19}$$

for $\rho \in E$, by (35.8). Also set

$$\Delta_n(\rho) = A_n(\rho) - B_n(\rho). \tag{36.20}$$

Then (36.16) and the change of variables $\rho t_n = r$ yield

$$\Phi_x(bt_n) - \Phi_x(at_n) = \int_a^b \Delta_n(\rho)\frac{d\rho}{\rho}. \tag{36.21}$$

Note that

$$\Delta_n(\rho) \geqslant 0 \text{ for } \rho \in E, \tag{36.22}$$

by (35.7). Since $\lim_{n\to+\infty} \Phi(bt_n) = \lim_{n\to+\infty} \Phi(at_n) = \ell(x)$, (36.21) implies that $\int_a^b \Delta_n(\rho)d\rho/\rho$ tends to 0. Modulo extracting a subsequence from $\{t_n\}$ (which does not disturb for Lemma 36.11), we can assume that

$$(36.23) \qquad \int_a^b \Delta_n(\rho)\frac{d\rho}{\rho} \leqslant 4^{-n}.$$

Let us check that in this case

$$(36.24) \qquad \lim_{n\to+\infty} \Delta_n(\rho) = 0$$

for almost all $\rho \in (a,b)$. Set

$$(36.25) \qquad Z_n = \{\rho \in (a,b) \; ; \; \Delta_n(\rho) > 2^{-n}\}$$

and $Z'_n = \bigcup_{m\geqslant n} Z_m$. Then $|Z_n|_{d\rho/\rho} \leqslant 2^{-n}$ by Chebychev, and $|Z'_n|_{d\rho/\rho} \leqslant 2^{-n+1}$. If (36.24) fails, $\rho \in Z_m$ for infinitely many values of m, and hence $\rho \in Z'_n$ for all n. This proves our claim.

Now we fix $\rho \in E$ such that (36.24) holds and try to get the description of Lemma 36.11. Denote by I_j, $j \in J(n)$, the components of $\Omega_0 \cap \partial B(x, t_n\rho)$. Then

$$(36.26) \qquad B_n(\rho) = \frac{1}{t_n\rho}\sum_j \int_{I_j} v\frac{\partial v}{\partial n}\, dH^1 \leqslant \sum_j \frac{|I_j|}{2\pi\, t_n\,\rho}\int_{I_j} |\nabla v|^2\, dH^1,$$

by (36.19) and (35.23). Hence

$$\begin{aligned}(36.27) \quad \Delta_n(\rho) &= \int_{\Omega_0\cap\partial B(x,t_n\rho)} |\nabla v|^2 - B_n(\rho) \\ &\geqslant \sum_j \left\{1 - \frac{|I_j|}{2\pi\, t_n\,\rho}\right\}\int_{I_j} |\nabla v|^2 \geqslant \left[\operatorname{Min}_j \left\{1 - \frac{|I_j|}{2\pi\, t_n\,\rho}\right\}\right] A_n(\rho),\end{aligned}$$

by (36.17). Note that

$$(36.28) \qquad A_n(\rho) \geqslant B_n(\rho) = \Phi_x(t_n\rho) \geqslant \ell(x) > 0$$

by (36.22), (36.20), (36.18), (36.5), and our contradiction assumption. Hence the minimum in (36.27) tends to 0, because $\Delta_n(\rho)$ tends to 0 (by (36.24)).

Recall from just before Lemma 36.11 that $I(n)$ denotes the longest of the I_j, $j \in J(n)$. Then

$$(36.29) \qquad \lim_{n\to+\infty} \frac{|I(n)|}{2\pi\, t_n\,\rho} = 1.$$

Set $I^*(n) = \Omega_0 \cap \partial B(x, t_n\rho) \setminus I(n)$. Thus $I^*(n)$ is the union of all the intervals I_j, $j \in J(n)$, other that $I(n)$. For n large enough, $\{1 - |I_j|/2\pi\, t_n\,\rho\} > 1/2$ for all these I_j, and so

$$(36.30) \qquad \int_{I^*(n)} |\nabla v|^2 = \sum_{j; I_j \neq I(n)} \int_{I_j} |\nabla v|^2 \leqslant 2\Delta_n(\rho),$$

by the first inequality in (36.27). Hence

$$\lim_{n\to+\infty}\int_{I^*(n)}|\nabla v|^2 = 0, \tag{36.31}$$

by (36.24).

Next we want to analyze more precisely the contribution of $I(n)$ to (36.26) and (36.27). First recall from (35.21) and (35.22) that for all $j \in J(n)$,

$$\int_{I_j} v\,\frac{\partial v}{\partial n} = \int_{I_j}(v-\alpha_j)\,\frac{\partial v}{\partial n} \leqslant \frac{|I_j|}{2\pi}\int_{I_j}|\nabla v|^2 \leqslant t_n\rho\int_{I_j}|\nabla v|^2 \tag{36.32}$$

(where α_j denotes the mean value of v on I_j), and hence

$$\int_{I^*(n)} v\,\frac{\partial v}{\partial n} \leqslant t_n\rho\int_{I^*(n)}|\nabla v|^2, \tag{36.33}$$

$$B_n(\rho) = \frac{1}{t_n\rho}\int_{I(n)\cup I^*(n)} v\,\frac{\partial v}{\partial n} \leqslant \frac{1}{t_n\rho}\int_{I(n)} v\,\frac{\partial v}{\partial n} + \int_{I(n)^*}|\nabla v|^2 \tag{36.34}$$

(by (36.19) and (36.33)), and

$$\begin{aligned}\Delta_n(\rho) = A_n(\rho) - B_n(\rho) &\geqslant \int_{I(n)}|\nabla v|^2 - \frac{1}{t_n\rho}\int_{I(n)} v\,\frac{\partial v}{\partial n}\\ &\geqslant \frac{1}{t_n\rho}\left\{\frac{|I(n)|}{2\pi}\int_{I(n)}|\nabla v|^2 - \int_{I(n)} v\,\frac{\partial v}{\partial n}\right\}\end{aligned} \tag{36.35}$$

by (36.20), (36.17), (36.34), because $I^*(n) = \Omega_0\cap\partial B(x, t_n\rho)\setminus I(n)$, and since $|I(n)| \leqslant 2\pi\, t_n\rho$.

Next return to (35.21). Set $I = I(n)$ to save some space and denote by $\alpha(n)$ the mean value of v on I (as before the statement of Lemma 36.11). Observe that

$$\int_I v\,\frac{\partial v}{\partial n} = \int_I (v-\alpha(n))\,v_r \tag{36.36}$$

(with the notations of (35.21)), by (35.22). From (35.21) we deduce that

$$\frac{|I|}{2\pi}\int_I|\nabla v|^2 - \int_I v\frac{\partial v}{\partial n} \geqslant \left\{\int_I v_r^2\right\}^{1/2}\left\{\frac{|I|}{\pi}\left[\int_I v_\theta^2\right]^{1/2} - \left[\int_I (v-\alpha(n))^2\right]^{1/2}\right\}, \tag{36.37}$$

because the right-hand side of (36.37) is the difference between two consecutive lines of (35.21). Set

$$\delta_n = \left[\int_{I(n)} v_\theta^2\right]^{1/2} - \frac{\pi}{|I(n)|}\left[\int_{I(n)}(v-\alpha(n))^2\right]^{1/2}. \tag{36.38}$$

Then

$$\Delta_n(\rho) \geqslant \frac{|I(n)|}{\pi\, t_n\,\rho}\left\{\int_{I(n)} v_r^2\right\}^{1/2}\delta_n, \tag{36.39}$$

by (36.35) and (36.37). We want to deduce from this that δ_n tends to 0. First note that

(36.40)
$$\lim_{n\to+\infty} \int_{I(n)} |\nabla v|^2 = \lim_{n\to+\infty} A_n(\rho) = \lim_{n\to+\infty} B_n(\rho) = \lim_{n\to+\infty} \Phi_x(t_n\rho) = \ell(x) > 0,$$

by (36.17) and (36.31), (36.24) and (36.20), (36.18), (36.5), and our contradiction assumption. Since

(36.41)
$$\int_{I(n)} |\nabla v|^2 - \Delta_n(\rho) \leqslant \frac{1}{t_n\rho} \int_{I(n)} v\frac{\partial v}{\partial n} \leqslant \int_{I(n)} |\nabla v|^2$$

by the first inequality in (36.35) and (36.32), (36.40) and (36.24) imply that

(36.42)
$$\lim_{n\to+\infty} \left\{ \frac{1}{t_n\rho} \int_{I(n)} v\frac{\partial v}{\partial n} \right\} = \ell(x).$$

Set

(36.43)
$$a_n = \left\{ \int_{I(n)} v_r^2 \right\}^{1/2} \text{ and } b_n = \left\{ \int_{I(n)} v_\theta^2 \right\}^{1/2}.$$

When we multiply (35.21) by $1/t_n\rho$, both sides tend to $\ell(x)$ (by (36.42), (35.22), (36.29), and (36.40)); hence

(36.44)
$$\lim_{n\to+\infty} a_n b_n = \frac{\ell(x)}{2}$$

(look at the second line of (35.21)). On the other hand

(36.45)
$$a_n^2 + b_n^2 = \int_{I(n)} |\nabla v|^2,$$

which tends to $\ell(x)$ by (36.40). Hence $(a_n - b_n)^2$ tends to 0, and

(36.46)
$$\lim_{n\to+\infty} a_n^2 = \lim_{n\to+\infty} b_n^2 = \frac{\ell(x)}{2}.$$

We immediately deduce from this, (36.39), (36.24), and (36.29) that

(36.47)
$$\lim_{n\to+\infty} \delta_n = 0.$$

[Recall from the definition (36.38) and (35.21) or Lemma 35.18 that $\delta_n \geqslant 0$.]

Let us renormalize all this, and then we shall take limits. Let $I'(n)$ and f_n be as in (36.9) and (36.10). Thus f_n has mean value 0 on $I'(n)$, and it is almost optimal for Lemma 35.18, in the sense that if we denote by f'_n its (tangential) derivative,

(36.48)
$$\left\{ \int_{I'(n)} (f'_n)^2 \right\}^{1/2} - \frac{\pi}{|I'(n)|} \left\{ \int_{I'(n)} f_n^2 \right\}^{1/2} = \delta_n$$

(by (36.38) and a change of variables), while

$$\lim_{n\to+\infty} \int_{I'(n)} (f_n')^2 = \frac{\ell(x)}{2} \tag{36.49}$$

(by (36.46) and (36.43)).

Let us check that for n large,

$$\partial B(0,\rho) \setminus I'(n) \text{ is not empty.} \tag{36.50}$$

Otherwise, $g(z) = f_n(\rho z)$ would be smooth on the circle, we could write the Fourier expansion

$$g(z) = \sum_{k\in\mathbb{Z}} c_k\, z^k, \tag{36.51}$$

and then

$$\|g'\|_2^2 = 2\pi \sum_k k^2 c_k^2, \tag{36.52}$$

$$\|g\|_2^2 = 2\pi \sum_k c_k^2 \leqslant \|g'\|_2^2 \tag{36.53}$$

because $c_0 = 0$, so that finally

$$\|f_n'\|_2^2 = \rho^{-1}\,\|g'\|_2^2 \geqslant \rho^{-1}\,\|g\|_2^2 = \rho^{-2}\,\|f_n\|_2^2 = \Big(\frac{2\pi}{|I'(n)|}\Big)^2\,\|f_n\|_2^2 . \tag{36.54}$$

This is not compatible with (36.48) and (36.49) when n is large enough, because δ_n tends to 0. Our claim (36.50) follows.

For n large, $\partial B(0,\rho) \setminus I'(n)$ is a nonempty closed interval of $\partial B(0,\rho)$, and (36.29) tells us that its diameter tends to 0. Thus, modulo extracting a new subsequence, we may assume that it converges to a point $\xi(\rho) = -e^{i\theta(\rho)}$ in $\partial B(0,\rho)$. This gives (36.12).

Modulo a third sequence extraction, we can also assume that $\{f_n\}$ converges to a limit f, uniformly on every compact subset of $I' = \partial B(0,\rho) \setminus \{\xi(\rho)\}$. This comes from our uniform estimate (36.49) on $\|f_n'\|_2$, and the fact that all f_n have mean value 0.

To complete the proof of Lemma 36.11, we still have to check that f satisfies (36.14) and (36.15).

First note that $\|f'\|_2 \leqslant (\ell(x)/2)^{1/2}$, by (36.49) and Fatou. On the other hand

$$\|f\|_2 = \lim_{n\to+\infty} \|f\|_2 = \frac{|I'|}{\pi}\Big(\frac{\ell(x)}{2}\Big)^{1/2}, \tag{36.55}$$

because $\{f_n\}$ converges to f uniformly on compact subsets of I', f_n is also uniformly bounded by (36.49) (and the fact that it has mean value 0), and then by (36.48) and (36.49).

Thus f is one of the extremal functions in Lemma 35.18, and so f has the form (36.14) for some $\beta(\rho)$. Finally (36.15) follows from (36.14) and (36.55) by a computation.

This completes our proof of Lemma 36.11. □

We may now return to the proof of Lemma 36.7. We want to use Lemma 36.11 to show that limits of (v, G) under blow-ups at the point x are "generalized cracktips".

Let us keep our initial sequence $\{t_n\}$ that tends to 0, set

$$G_n = t_n^{-1}(G - x), \tag{36.56}$$

and define v_n on $\mathbb{R}^2 \setminus G_n$ by

$$v_n(z) = t_n^{-1/2} v\left(t_n z + x\right). \tag{36.57}$$

Thus $\{(v_n, G_n)\}$ is the blow-up sequence associated to (t_n) and the fixed point $x \in G$ in (12.7) and (12.8). Let us assume that

$$\{(v_n, G_n)\} \text{ converges to a limit } (u, K), \tag{36.58}$$

with the notion of convergence that was described a little before Lemma 12.4.

Note that we can always find sequences $\{t_n\}$ like this. In fact, Lemma 12.5 tells us that for each sequence $\{t_n\}$ that tends to 0, we can extract a subsequence for which (36.58) holds.

We want to use Lemma 36.11 to prove that (u, K) is almost a "generalized cracktip", and then get a contradiction because the cracktip is at best a generalized λ'-minimizer for the wrong value of λ'.

Let D be any compact disk contained in $\mathbb{R}^2 \setminus K$. Since $\operatorname{dist}(D, K) > 0$ and $\{G_n\}$ converges to K (by (36.58)), G_n does not meet D for n large. Moreover,

$$\nabla v_n \text{ converges to } \nabla u \text{ uniformly on } D, \tag{36.59}$$

again by (36.58). See the definition of convergence before Lemma 12.4.

Select an origin x_D inside D, and set

$$a_D(n) = v_n\left(x_D\right) - u\left(x_D\right). \tag{36.60}$$

Then (36.59) yields that

$$v_n(z) - a_D(n) \text{ converges to } u \text{ uniformly in } D. \tag{36.61}$$

Denote by F the set of radii $\rho > 0$ that have the property of Lemma 36.11. Also set

$$F(D) = \{\rho \in F \,;\, \partial B(0, \rho) \text{ meets the interior of } D\}. \tag{36.62}$$

Let $\rho \in F(D)$ be given. Define f_n by (36.10), but on the possibly larger (and more natural) domain of definition $\partial B(0, \rho) \setminus G_n$. Note that

$$f_n(z) = v_n(z) - t_n^{-1/2}\alpha(n) \text{ on } \partial B(0, \rho) \setminus G_n, \tag{36.63}$$

by (36.57). Denote by $\{n_k\}$ the subsequence given by Lemma 36.11 (because $\rho \in F$), and let f be as in (36.14). Thus

$$\{f_{n_k}\} \text{ converges to } f \text{ on } \partial B(0,\rho) \setminus \{-e^{i\theta(\rho)}\}. \tag{36.64}$$

Set $g_k = v_{n_k} - a_D(n_k) - f_{n_k}$ on $D \cap \partial B(0,\rho)$ (which does not meet G_{n_k} for k large enough). Then $g_k = t_{n_k}^{-1/2}\alpha(n_k) - a_D(n_k)$ is constant (by (36.63)), and $\{g_k\}$ converges to $u - f$ on $D \cap \partial B(0,\rho) \setminus \{e^{-i\theta(\rho)}\}$ by (36.61) and (36.64). Denote by $\gamma_D(\rho)$ the (constant) limit. Thus

$$\gamma_D(\rho) = \lim_{k\to+\infty} \left\{t_{n_k}^{-1/2}\alpha(n_k) - a_D(n_k)\right\}, \tag{36.65}$$

and

$$u(z) = f(z) + \gamma_D(\rho) \text{ on } D \cap \partial B(0,\rho) \setminus \{-e^{i\theta(\rho)}\}. \tag{36.66}$$

Since u is continuous on D and f has a jump of size $2\beta(\rho) \neq 0$ at $-e^{i\theta(\rho)}$ (by (36.14) and (36.15)),

$$-e^{i\theta(\rho)} \notin \operatorname{int}(D). \tag{36.67}$$

Thus (36.66) and (36.14) yield

$$u\left(\rho e^{i\theta}\right) = \beta(\rho)\sin\left(\frac{\theta - \theta(\rho)}{2}\right) + \gamma_D(\rho) \tag{36.68}$$

for all $\theta \in (\theta(\rho) - \pi, \theta(\rho) + \pi)$ such that $\rho e^{i\theta} \in D$.

Now we want to study the dependence of our coefficients on ρ and D. As in Section 35 (see between (35.35) and (35.36)), the coefficients $\beta(\rho)e^{i\theta(\rho)/2}$ and $\gamma_D(\rho)$ can be computed (by linear algebra) from the values of u on any small interval of $D \cap \partial B(x,\rho)$. Then they depend smoothly on ρ (with D fixed). That is, we can find smooth functions on $\{\rho > 0;\ \partial B(0,\rho) \text{ meets } \operatorname{int}(D)\}$ that coincide with $\beta(\rho)e^{i\theta(\rho)/2}$ and $\gamma_D(\rho)$ on $F(D)$.

Note that $\beta(\rho)e^{i\theta(\rho)/2}$ does not depend on our choice of D, and even though $\gamma_D(\rho)$ may depend on D, $\gamma'_D(\rho) = \partial\gamma_D(\rho)/\partial\rho$ depends only on ρ, not on D. This is because, in the definition (36.65), the only dependence of $\gamma_D(\rho)$ on ρ comes from the $t_{n_k}^{-1/2}\alpha(n_k)$, which do not depend on D.

Because of all this, the coefficients $\beta(\rho)e^{i\theta(\rho)/2}$ and $\gamma'_D(\rho)$ are defined and smooth on the complement of

$$Z = \{\rho > 0\ ;\ \partial B(0,\rho) \subset K\}. \tag{36.69}$$

This is because if $\rho \in (0,+\infty) \setminus Z$, we can apply the preceding arguments to a disk $\subset \mathbb{R}^2 \setminus K$ centered on $\partial B(0,\rho)$. The reader should not be too impatient here; we agree that Z must be empty, but we shall only prove it later.

Let $\rho \in F \setminus Z$ be given. To avoid convergence issues, assume that $\partial B(0,\rho) \cap K$ is finite. We know that this is the case for almost all ρ, because $H^1(K)$ is locally finite and by a standard Fubini-like argument. Note that $\beta(\rho)$ never vanishes, so we can find smooth determinations of $\beta(\rho)$ and $\theta(\rho)$ locally. Then we can differentiate (36.68) with respect to ρ. We get that

$$\frac{\partial u}{\partial \rho}\left(\rho e^{i\theta}\right) = \beta'(\rho)\sin\Big(\frac{\theta-\theta(\rho)}{2}\Big) + h_\rho(\theta), \tag{36.70}$$

with

$$h_\rho(\theta) = -\frac{1}{2}\theta'(\rho)\,\beta(\rho)\cos\Big(\frac{\theta-\theta(\rho)}{2}\Big) + \gamma_D'(\rho). \tag{36.71}$$

Then

$$\begin{aligned}\int_{\partial B(0,\rho)\setminus K}\Big(\frac{\partial u}{\partial \rho}\Big)^2 dH^1 &= \rho\int_{\theta(\rho)-\pi}^{\theta(\rho)+\pi}\ \Big|\frac{\partial u}{\partial \rho}\left(\rho e^{i\theta}\right)\Big|^2 d\theta\\ &= \rho\beta'(\rho)^2\int_\theta \sin^2\Big(\frac{\theta-\theta(\rho)}{2}\Big)d\theta + \rho\int_\theta h_\rho(\theta)^2 d\theta,\end{aligned} \tag{36.72}$$

because $h_\rho(\cdot)$ is orthogonal to $\sin\left(\cdot - \theta(\rho)/2\right)$. Since $\beta(\rho) = \pm\left(2\rho\,\ell(x)/\pi\right)^{1/2}$, $\beta'(\rho)^2 = \ell(x)/2\pi\rho$ and hence

$$\int_{\partial B(0,\rho)\setminus K}\left(\frac{\partial u}{\partial \rho}\right)^2 dH^1 = \frac{\ell(x)}{2} + \rho\int_\theta h_\rho(\theta)^2 d\theta. \tag{36.73}$$

We want to show that

$$\theta'(\rho) = \gamma_D'(\rho) = 0, \tag{36.74}$$

and so it will be enough to show that $\int_\theta h_\rho(\theta)^2 d\theta = 0$, or that

$$\int_{\partial B(0,\rho)\setminus K}\left(\frac{\partial u}{\partial \rho}\right)^2 dH^1 \leqslant \frac{\ell(x)}{2}. \tag{36.75}$$

To get this we shall use the uniform convergence of ∇v_n to ∇u on the compact subsets of $\mathbb{R}^2 \setminus K$, which comes from (36.58). Note that $-e^{i\theta(\rho)} \in K$, because otherwise we could choose a compact disk $D \subset \mathbb{R}^2 \setminus K$ centered at $-e^{i\theta(\rho)}$ and contradict (36.67). Then (36.12) says that every compact subset T of $\partial B(0,\rho) \setminus K$ is contained in $I'\,(n_k)$ for n large enough. Then

$$\begin{aligned}\int_T\left(\frac{\partial u}{\partial \rho}\right)^2 &= \lim_{n\to+\infty}\int_T\left(\frac{\partial v_n}{\partial \rho}\right)^2 \leqslant \liminf_{k\to+\infty}\int_{I'(n_k)}\left(\frac{\partial v_{n_k}}{\partial \rho}\right)^2\\ &= \liminf_{k\to+\infty}\int_{I(n_k)}\left(\frac{\partial v}{\partial \rho}\right)^2 = \frac{\ell(x)}{2}\end{aligned} \tag{36.76}$$

because ∇v_n converges to ∇u uniformly on T, by (36.9), (36.57) and a change of variables (to get the second equality), and by (36.46) and (36.43) (applied to n_k). Since this holds for all choices of T, we get (36.75), and then (36.74).

Let Z be as in (36.39), and let I be a component of $(0,+\infty) \setminus Z$. We know from (36.15), (36.74), and the remark before (36.69) that $\beta(\rho) = \pm\left(2\rho\,\ell(x)/\pi\right)^{1/2}$ (with a constant sign) on I, and that $e^{i\theta(\rho)}$ is constant on I. As for $\gamma_D(\rho)$, we only know that

$\gamma'_D(\rho) = 0$ on I, so that $\gamma_D(\rho)$ is only "locally constant". Let us be more precise. For each connected component $\mathcal{O}$ of

$$V_I = \{z \in \mathbb{R}^2 \setminus K; |z| \in I\}, \tag{36.77}$$

we can define $\gamma_D(\rho)$ for all compact disks $D \subset \mathcal{O}$ and all $\rho \in F(D)$. Here again the reader is probably shocked that we even consider the possibility that V_I may not be connected, but we just don't want to prove immediately that this is impossible. Let us check that $\gamma_D(\rho)$ does not depend on D or ρ. First, (36.74) and the regularity of $\gamma_D(\cdot)$ on $F(D)$ imply that $\gamma_D(\rho) = \gamma_D(\rho')$ for all choices of $\rho, \rho' \in F(D)$. Also, it is clear from (36.66) that $\gamma_D(\rho) = \gamma_{D'}(\rho)$ whenever $\rho \in F$ and $\mathrm{int}(D) \cap \mathrm{int}(D') \cap \partial B(0,\rho) \neq \varnothing$. From these two observations it is easy to deduce that $\gamma_D(\rho)$ does not depend on $D \subset \mathcal{O}$ or $\rho \in F(D)$. Altogether we found constants $\beta_I = \pm\,(2\ell(x)/\pi)^{1/2}$ and θ_I that depend only on I, and a constant $\gamma_{\mathcal{O}}$ that depends on $\mathcal{O}$ as well, such that

$$u\left(\rho e^{i\theta}\right) = \beta_I\, \rho^{1/2} \sin\left(\frac{\theta - \theta_I}{2}\right) + \gamma_{\mathcal{O}} \tag{36.78}$$

for all $\rho \in I$ and $t \in (\theta_I - \pi, \theta_I + \pi)$ such that $\rho e^{i\theta} \in \mathcal{O}$.

We are now fairly close to the desired contradiction. By (36.58), (u, K) is the limit of some blow-up sequence of (v, G) at the point x. If (13.2) holds, Proposition 12.44 tells us that (u, K) is a global λ-minimizer. If (13.1) holds and $x \notin L$, then we can use Proposition 12.12 to get the same conclusion, with $\lambda = h'\left(H^1(G \setminus L)\right)$ as usual. In the remaining case when (13.1) holds and $x \in L$, note that $x \neq -1$ because we excluded this case in the statement of Lemma 36.7; hence Proposition 12.42 says that (u, K) is a global λ-minimizer in $\mathbb{R}^2 \setminus \mathbb{R}$.

Let us first use this to show that Z is empty. The simplest proof at this point is to say that $\mathbb{R}^2 \setminus K$ has no bounded connected component, by Lemma 15.1 and Remark 15.8, but we can also proceed directly, as follows. Let z be any regular point of K, and let B be a disk of regularity centered at z. [See Definition 13.6.] Call Ω_1 and Ω_2 the connected components of $B \setminus K$. On each Ω_i, u is given by (36.78), for some values of the constants β_I, θ_I, and $\gamma_{\mathcal{O}}$ that may depend on i. Because u minimizes $\int_{\mathbb{R}^2 \setminus K} |\nabla u|^2$ locally, its boundary values on $K \cap B$ (with access from Ω_i) satisfy the Neumann condition $\partial u/\partial n = 0$. Then $K \cap B$ is contained in a level set of the conjugated function w defined by

$$w\left(\rho e^{i\theta}\right) = -\beta_I\, \rho^{1/2} \cos\left(\frac{\theta - \theta_I}{2}\right) \tag{36.79}$$

for $\rho \geqslant 0$ and $|\theta - \theta_I| \leqslant \pi$. These level sets are the half-line

$$K_I = \left\{-\rho e^{i\theta_I}\ ;\ \rho \geqslant 0\right\}, \tag{36.80}$$

and a collection of parabolas.

If Z was not empty, K would contain a circle $\partial B(0,R)$, Proposition 13.11 would say that $\partial B(0,R)$ contains regular points for K, and then some nontrivial arc of circle would be contained in one of the level sets above. This is impossible, and so Z is empty and there is only one interval $I = (0, +\infty)$.

Now we can easily exclude the case when (u, K) is a global minimizer on $\mathbb{R}^2 \setminus \mathbb{R}$. In this case also, almost every point of $\mathbb{R}$ is a regular point for K (or equivalently, is the center of some small disk B such that $K \cap B = \mathbb{R} \cap B$). We did not state this fact yet, but the proof is the same as for (13.12) above. It only uses measure theory and the local Ahlfors-regularity of $K \setminus \mathbb{R}$. By the argument above, for almost every point $z \in \mathbb{R}$ there is a small interval of $\mathbb{R}$ centered at z which is contained in a level line of w above, and hence in K_I. This is clearly false. So we can assume that

$$(u, K) \text{ is a global } \lambda\text{-minimizer } \big(\text{in } \mathbb{R}^2\big)\,. \tag{36.81}$$

Next we want to check that $K = K_I$, where K_I is still as in (36.80). From (36.78), which now holds with fixed values of β_I and θ_I, we easily deduce that $K_I \subset K$.

Suppose that K is not contained in K_I. Since K is Ahlfors-regular, $H^1(K \setminus K_I) > 0$, and by Proposition 13.11 we can find a point $z \in K \setminus K_I$ which is regular for K. Let B be a small disk of regularity centered at z. By the argument above, $K \cap B$ is contained in one of the level sets of w above, and so it is an arc of parabola.

We can find a competitor $(\widetilde{u}, \widetilde{K})$ which is strictly better that (u, K), as follows. First, $\widetilde{K}$ is obtained from K by replacing $K \cap B$ with the line segment with the same endpoints. Call Ω_1, Ω_2 the two connected components of $B \setminus K$, and $\widetilde{\Omega}_1, \widetilde{\Omega}_2$ the corresponding components of $B \setminus \widetilde{K}$. On each Ω_i, u is defined by (36.78), with some constant $\gamma_{\mathcal{O}}$. We define $\widetilde{u}$ on $\widetilde{\Omega}_i$ by the same formula, with the same constant; outside of B, we keep $\widetilde{u} = u$. It is easy to see that $(\widetilde{u}, \widetilde{K})$ is an acceptable competitor for (u, K). Note that we did not change the values of ∇u; we simply added or subtracted constants locally. Since $\widetilde{K}$ is strictly shorter that K, we get the desired contradiction with (36.81).

So $K = K_I$, there is only one component $\mathcal{O}$, and (u, K) is a generalized cracktip. [Compare (36.80) and (36.78) with (35.30) and (35.31).] Now Remark 35.42 and (36.81) imply that $\beta_I = \pm\,(2\lambda/\pi)^{1/2}$. But we already knew that $\beta_I = \pm\,(2\ell(x)/\pi)^{1/2}$ (see just above (36.78)). Since we know from (36.6) that $\ell(x) < \lambda$, this gives the desired contradiction.

Our proof of Lemma 36.7 is now complete. □

Note that our argument fails when (13.1) holds and $x = -1$, because (u, K) is a global λ-minimizer in $\mathbb{R}^2 \setminus (-\infty, 0]$, and Remark 35.42 does not exclude the case when $K = (-\infty, 0]$ and $\beta_I^2 < 2\lambda/\pi$.

37. Points of low energy (continued)

In this section we want to complete our proof of Proposition 36.3. The assumptions are the same as in Section 36.

Lemma 37.1. — *The case when $m_0 = 0$ does not occur.*

Assume instead that $m_0 = 0$. Then $\Omega_0 = \{z \in \mathbb{R}^2 \, ; \, w(z) < 0\} = \mathbb{R}^2 \setminus G_{00}$, by (35.3) and Proposition 32.6, and hence $G = G_{00}$ (by (35.2)) and in particular G is connected. Thus we could use [Bo] and conclude immediately when (13.2) holds, but it will be useful to review the argument anyway.

Sublemma 37.2. — *If* (13.2) *holds, every point G is a regular or a spider point of G. If* (13.1) *holds, every point of $G \setminus L$ is a regular or a spider point of G.*

Let $x \in G$ be given, and suppose that $x \in G \setminus L$ if (13.1) holds. Choose a sequence $\{t_n\}$ of positive numbers that tends to 0, and such that if $\{(v_n, G_n)\}$ denotes the blow-up sequence defined by (36.56) and (36.57), then

$$\{(v_n, G_n)\} \text{ converges to a limit } (u, K). \tag{37.3}$$

See before Lemma 12.4 for the precise definition of convergence, and Lemma 12.5 for the existence of a sequence $\{t_n\}$ like this. Also note that

$$(u, K) \text{ is a global } \lambda\text{-minimizer.} \tag{37.4}$$

by Proposition 12.12 or Proposition 12.44.

Let D be any compact disk in $\mathbb{R}^2 \setminus K$. Then $D \subset \mathbb{R}^2 \setminus G_n$ for n large, and $\{\nabla v_n\}$ converges to ∇u uniformly on D, by (37.3). Hence

$$\int_D |\nabla u|^2 = \lim_{n\to+\infty} \int_D |\nabla v_n|^2 = \lim_{n\to+\infty} t_n^{-1} \int_{x+t_nD} |\nabla v|^2 , \tag{37.5}$$

by (36.57) and a change of variables. Since $m_0 = 0$, $\Omega_0 = \mathbb{R}^2 \setminus G$ and hence

$$x + t_n D \subset \Omega_0 \text{ for } n \text{ large enough.} \tag{37.6}$$

Choose R so large that $D \subset B(0, R)$. Then $x + t_n D \subset \Omega_0 \cap B\,(x, t_n R)$ and

$$t_n^{-1} R^{-1} \int_{x+t_nD} |\nabla v|^2 \leqslant \Phi_x\,(t_n R)\,, \tag{37.7}$$

by the definition (35.4) of Φ_x. By (36.5) and Lemma 36.7, $\Phi_x\,(t_n R)$ tends to $\ell(x) = 0$ when n tends to $+\infty$. Hence $\int_D |\nabla u|^2 = 0$, by (37.5), and u is constant on the connected component of D in $\mathbb{R}^2 \setminus K$ (because it is harmonic).

Sublemma 37.8. — *If u is constant on some connected component of $\mathbb{R}^2 \setminus K$, then x is a regular or a spider point of G.*

Indeed (37.4) allows us to apply Lemma 18.1. We get that K is a line or a propeller; it cannot be empty because it contains 0 (since $x \in G$). Now Corollary 13.32 says that x is a regular or a spider point of G, as needed. □

We recorded this as a sublemma because it will be used a few times later. Note that we only used (37.6), and the fact that $x \notin L$ if (13.1) holds (through (37.4)), not our hypothesis that $m_0 = 0$.

Sublemma 37.2 is an immediate consequence of Sublemma 37.8 and the preceding discussion. □

Remark 37.9. — In our proof of Sublemma 37.2, we only used our hypothesis that $m_0 = 0$ to get (37.6). When $m_0 \neq 0$, we can still say the following. Let $x \in G$ be given, and suppose that $x \in G \setminus L$ if (13.1) holds. Let $\{t_n\}$ be a sequence of positive numbers that tends to 0, and suppose that (37.3) holds. If there exists a compact disk D (with positive diameter) such that (37.6) holds, then x is a regular or a spider point of G. The proof is the same.

Let us return to the proof of Lemma 37.1. First assume that (13.2) holds. Let $\gamma : I \to G$ be a simple piecewise C^1 arc such that $|\gamma'(t)| = 1$ almost everywhere and for which the interval of definition I is maximal. Such a curve exists, essentially by abstract nonsense. If I has an endpoint $a \in \mathbb{R}$, then $\gamma(a)$ is easily defined by continuity (even if $a \notin I$), because γ is Lipschitz. Sublemma 37.2 even allows us to extend γ beyond a, but this extension cannot be simple (because I is maximal). This implies that G contains a loop, and then $\mathbb{R}^2 \setminus G$ has a bounded component. This is not possible, by Section 15. Hence a does not exist, and $I = \mathbb{R}$.

Because $H^1(G \cap B(0,R)) < +\infty$ for all R, $\gamma(t)$ tends to infinity in both directions. Thus $\Gamma = \gamma(\mathbb{R})$ is a Jordan curve through ∞, and $\mathbb{R}^2 \setminus G$ has at least two components (because $\Gamma \subset G$). This contradiction with our assumption (32.1) settles the case when (13.2) holds.

We are left with the case when ($m_0 = 0$ and) (13.1) holds. Let us first check that $G = L$. Suppose not, pick a point $x \in G \setminus L$ and let $\gamma : I \to G \setminus L$ be a simple, piecewise C^1 arc through x, such that $|\gamma'(t)| = 1$ almost everywhere, and for which I is maximal.

If $I = \mathbb{R}$, then $\gamma(I)$ is a Jordan curve through infinity (because $H^1(G \cap B(0,R)) < +\infty$ for all R, as above), hence $\mathbb{R}^2 \setminus G$ has a least two components, in contradiction with (32.1).

So I has at least one endpoint $a \in \mathbb{R}$. As before, $\gamma(a)$ is defined because γ is Lipschitz. We claim that $\gamma(a) \in L$. Otherwise, Lemma 37.2 allows us to extend γ a little across a. Since I is maximal, the extension cannot be simple (because it takes values in $G \setminus L$). Then $G \setminus L$ contains a loop and we get a contradiction, as before. So $\gamma(a) \in L$.

If I has two endpoints $a, b \in \mathbb{R}$, then $\gamma(a)$ and $\gamma(b)$ both lie in L, and we get a loop in G by completing the arc $\gamma(t)$ with the line segment $[\gamma(a), \gamma(b)] \subset L$.

Thus we are left with the case when I has exactly one endpoint $a \in \mathbb{R}$. We construct a Jordan curve Γ through infinity by gluing $\gamma(I)$ with the half-line $(-\infty, \gamma(a)] \subset L$. Of course $\Gamma \subset G$, and so $\mathbb{R}^2 \setminus G$ has at least two components, in contradiction with (32.1).

This series of contradictions proves our claim that $G = L$ (when $m_0 = 0$ and (13.1) holds). Let us check now that (v, G) is a generalized cracktip.

Set $P = \{z \in \mathbb{C} \ ; \ \mathrm{Re}(z) > 0\}$ and $\widetilde{v}(z) = v\left(z^2 - 1\right)$ for $z \in P$. Then $\widetilde{v}$ is harmonic on P and satisfies the usual Neumann condition $\partial \widetilde{v}/\partial n = 0$ on ∂P, because it is locally energy-minimizing (by the invariance of our energy integrals under conformal mappings). Also,

$$\int_{B(0,R)\cap P} |\nabla \widetilde{v}|^2 = \int_{B(-1,R^2)\setminus L} |\nabla \widetilde{v}|^2 \leqslant CR^2, \tag{37.10}$$

because otherwise we could add $\partial B\left(-1, R^2\right)$ to G and replace v with a constant in $B\left(-1, R^2\right)$. We can extend $\widetilde{v}$ by continuity on $\overline{P}$, and then by symmetry on the whole plane. We get a harmonic function $\widetilde{v}$ on the whole plane; see the arguments in Section 14 for more details in slightly more complicated situations. Let $\widetilde{w}$ be a harmonic function such that $\widetilde{v} + i\widetilde{w}$ is holomorphic. Because of (37.10), $\widetilde{v} + i\widetilde{w}$ is a polynomial of degree at most 1, and a simple computation shows that (v, G) is a "generalized cracktip" (with the definition just after (36.2)). This contradicts our assumption (36.2).

This completes our proof of Lemma 37.1. □

We may now return to the main statement in Proposition 36.3.

Since $m_0 < 0$ and $w(z) = 0$ on G_{00} (by (35.2)),

$$\partial\Omega_0 \text{ does not meet } G_{00}. \tag{37.11}$$

For the rest of the proof, we let $x \in G \cap \partial\Omega_0$ be given, and we want to prove that x is a regular or a spider point of G.

Pick a blow-up sequence $\{t_n\}$ such that $\lim_{n\to+\infty} t_n = 0$ and for which (37.3) holds. Note that $x \notin L$ when (13.1) holds, by (37.11). Thus Proposition 12.12 or Proposition 12.44 tells us that (u, K) is a global λ-minimizer (as in (37.4)). We want to show that u is constant on some component of $\mathbb{R}^2 \setminus K$, and then conclude with Sublemma 37.8. Note that we can extract as many subsequences as we like from $\{t_n\}$, because this will not affect (37.3) or (37.4).

If we can find a nontrivial compact disk $D \subset \mathbb{R}^2 \setminus K$ such that (37.6) holds, then we can use Remark 37.9 to conclude. So we may assume that this is not the case. Let us see what this means. Set

$$\Omega_0^n = \{z \ ; \ x + t_n z \in \Omega_0\}. \tag{37.12}$$

Note that Ω_0^n is open. We may as well assume that the sets $\mathbb{R}^2 \setminus \Omega_0^n$ converge to some closed set $\mathbb{R}^2 \setminus \Omega_0^\infty$, because otherwise we can extract a subsequence from $\{t_n\}$ and make it true. If Ω_0^∞ is not empty, then it contains a nontrivial disk $2D$, and it is easy to check that D satisfies (37.6). Thus we may assume that

$$\text{the sets } \mathbb{R}^2 \setminus \Omega_0^n \text{ converge to } \mathbb{R}^2. \tag{37.13}$$

For each n, choose a point $z_n \in \Omega_0$ such that $|z_n - x| \leqslant 2^{-n} t_n$. Set

$$m(n) = w(z_n) < m_0. \tag{37.14}$$

Note that w takes all the values of $m \in (m(n), m_0)$ on the interval $[x, z_n]$. Because of this, we may as well assume that $m(n)$ has the properties of Proposition 32.6 and Lemma 32.18, since otherwise we could replace z_n with some other point of $\Omega_0 \cap [x, z_n]$. In particular,

$$\Gamma_{m(n)} \text{ is a rectifiable Jordan curve through } \infty. \tag{37.15}$$

Choose a point $z_n' \in \Gamma_{m(n)} \cap \partial B(x, t_n)$ and denote by ξ_n the subarc of $\Gamma_{m(n)}$ from z_n to z_n'. We may assume that

$$\xi_n \subset \Gamma_{m(n)} \cap \overline{B}(x, t_n) \subset \Omega_0, \tag{37.16}$$

because otherwise we can replace z_n' with an earlier point of $\xi_n \cap \partial B(x, t_n)$. Set

$$\varphi(z) = t_n^{-1}(z - x) \text{ for } z \in \mathbb{R}^2, \tag{37.17}$$

and then

$$\widetilde{\xi}_n = \varphi(\xi_n). \tag{37.18}$$

Thus $\widetilde{\xi}_n$ is a simple arc from $\varphi(z_n)$ to $\varphi(z_n')$, and

$$\widetilde{\xi}_n \subset \overline{B}(0,1) \setminus G_n \tag{37.19}$$

by (37.16) and (36.56).

We shall need to distinguish between two cases.

Case 1. — We first assume that we can find $\eta > 0$ such that, for infinitely many values of n, there is a point $y_n \in \xi_n$ such that

$$\operatorname{dist}(y_n, G) \geqslant \eta t_n. \tag{37.20}$$

Replace $\{t_n\}$ with a subsequence, so that now y_n exists for all n, and even $\{\varphi(y_n)\}$ converges to some limit $y^* \in \overline{B}(0,1)$. [Note that $|\varphi(y_n)| \leqslant 1$, by (37.19).] Then

$$\operatorname{dist}(y^*, K) \geqslant \eta, \tag{37.21}$$

by (37.20) and (37.3). [See also the definitions (37.17) and (36.56).] Set

$$w_n(z) = t_n^{-1/2} w(t_n z + x) - t_n^{-1/2} w(t_n y^* + x) \tag{37.22}$$

on $\mathbb{R}^2$. Thus $v_n + i w_n$ is holomorphic on $\mathbb{R}^2 \setminus G_n$, because $v + iw$ is holomorphic on $\mathbb{R}^2 \setminus G$.

Denote by V the connected component of y^* in $\mathbb{R}^2 \setminus K$. From (37.3) we know that

$$\{\nabla w_n\} \text{ converges uniformly on compact subsets of } \mathbb{R}^2 \setminus K. \tag{37.23}$$

From this and the fact that $w_n(y^*) = 0$ for all n we deduce that $\{w_n\}$ converges uniformly on compact subsets of V to some limit w_∞; then $u + iw_\infty$ is holomorphic on V (because it is the locally uniform limit of functions $v_n + iw_n + C_n$ that are holomorphic on the domains $\mathbb{R}^2 \setminus G_n \simeq \mathbb{C} \setminus G_n$) We want to show that

$$w_\infty(z) \geqslant 0 \text{ on } V. \tag{37.24}$$

Let $z \in V$ be given. Because of (37.13), we can find points $\alpha_n^* \in \mathbb{R}^2 \setminus \Omega_0^n$ such that

$$\lim_{n\to+\infty} \alpha_n^* = z. \tag{37.25}$$

Set $\alpha_n = x + t_n\alpha_n^*$. Then $w(\alpha_n) \geqslant m_0 > w(y_n)$, because $\alpha_n^* \in \mathbb{R}^2 \setminus \Omega_0^n$, by definition (37.12) of Ω_0^n, because $\Omega_0 = \{z \; ; \; w(z) < m_0\}$, because $y_n \in \xi_n$, and by (37.16). Hence

$$w_n(\alpha_n^*) > w_n(\varphi(y_n)), \tag{37.26}$$

by (37.22). Next let us check that

$$w_\infty(z) = \lim_{n\to+\infty} w_n(z) = \lim_{n\to+\infty} w_n(\alpha_n^*). \tag{37.27}$$

The first equality is just the definition of w_∞. For the second one, note that if D is a small compact disk centered at z and contained in V, then (37.23) implies that the functions $|\nabla w_n|$ are uniformly bounded on D. From (37.25) we deduce that $\alpha_n^* \in D$ for n large, then that $|w_n(z) - w_n(\alpha_n^*)| \leqslant C|z - \alpha_n^*|$, which tends to 0. This proves (37.27). The same proof also gives that

$$\lim_{n\to+\infty} w_n(\varphi(y_n)) = \lim_{n\to+\infty} w_n(y^*) = 0, \tag{37.28}$$

because $y^* = \lim_{n\to+\infty} \varphi(y_n)$ and by (37.21).

Now (37.24) easily follows from (37.26), (37.27), and (37.28).

From (37.24) and (37.28) we see that the harmonic function w_∞ reaches its minimum on V at the point y^*. Since y^* is interior (by (37.21)), w_∞ is constant on V, and so is u. We can conclude using Sublemma 37.8.

Case 2. — We may now assume that we cannot find $\eta > 0$ as in Case 1.

This means that

$$\lim_{n\to+\infty} \eta_n = 0, \tag{37.29}$$

where

$$\eta_n = t_n^{-1}\sup\{\text{dist}(y, G) \; ; \; y \in \xi_n\} = \sup\left\{\text{dist}(\widetilde{y}, G_n) \; ; \; \widetilde{y} \in \widetilde{\xi}_n\right\} \tag{37.30}$$

(by (37.17) and (37.18)).

Recall from (37.19) that $\widetilde{\xi}_n$ is a compact subset of $\overline{B}(0,1)$. Thus we can extract a new subsequence from $\{t_n\}$, so that $\{\widetilde{\xi}_n\}$ converges to a limit $\widetilde{\xi}$. Then

$$\widetilde{\xi} \text{ is a compact, connected subset of } K \cap \overline{B}(0,1). \tag{37.31}$$

Indeed, $\widetilde{\xi}$ is connected because each $\widetilde{\xi}_n$ is connected, and $\widetilde{\xi} \subset K$ because of (37.29) and the convergence of $\{G_n\}$ to K. The other properties listed in (37.31) are obvious.

Note that $0 \in \widetilde{\xi}$ because the initial point z_n of $\xi_n = \varphi^{-1}(\widetilde{\xi}_n)$ lies at distance $\leqslant 2^{-n}t_n$ from x. (See after (37.13)). Similarly, $\widetilde{\xi}$ contains a point of $\partial B(0,1)$, because $z_n' \in \partial B(x, t_n)$ and $z_n' \in \xi_n$. (See after (37.15)).

Thus $H^1(\widetilde{\xi}) \geqslant 1$. On the other hand, (37.4) and Proposition 13.11 tell us that almost-every point of $\widetilde{\xi} \subset K$ is a regular point of K. So we can choose a point $y \in \widetilde{\xi}$ such that $0 < |y| < 1$ and y is a point of regularity of K.

Set $B = B(y, 2r)$, where r is chosen so small that

$$B \text{ is a disk of regularity for } K \text{ and } B \subset B(0,1) \setminus \{0\}. \tag{37.32}$$

Other similar constraints on r will show up in the next few lines.

For all n, denote by y_n a point of G_n that minimizes $|y_n - y|$, and set $B_n = B(y_n, r)$. We claim that if r is chosen small enough, then

$$B_n \text{ is a disk of regularity for } G_n \text{ for } n \text{ large enough.} \tag{37.33}$$

To see this we want to apply Lemma 13.17. Denote by D the tangent line to K at y. Since K is a C^1 curve near y, D satisfies (13.19) and (13.20) with G, r, ε replaced with $K, 3r, \varepsilon/3$ (say), at least if r is small enough. Then for n large enough, D still satisfies (13.19) and (13.20) with G, r replaced with $G_n, 2r$; this follows from the convergence of $\{G_n\}$ to K, as in (37.3).

To verify (13.18), consider the set $A(\delta) = \{z \in B(y, 3r)\ ;\ \mathrm{dist}(z, K) \geqslant \delta\}$, where $\delta > 0$ will be chosen soon. Since $A(\delta)$ is a compact subset of $\mathbb{R}^2 \setminus K$, (37.3) says that for n large,

$$\int_{A(\delta)} |\nabla v_n| \leqslant \int_{A(\delta)} |\nabla u| + \frac{\varepsilon}{2} r^{3/2} \leqslant \varepsilon\, r^{3/2}, \tag{37.34}$$

where the second inequality holds if r is small enough, because ∇u has continuous extensions on both sides of $\mathbb{R}^2 \setminus K$ near y. [See Section 14]. Besides

$$\begin{aligned}\int_{B(y,2r)\setminus(G_n \cup A(\delta))} |\nabla v_n| &\leqslant \left\{\int_{B(y,2r)\setminus G_n} |\nabla v_n|^2\right\}^{1/2} |B(y,2r) \setminus A(\delta)|^{1/2} \\ &\leqslant C r^{1/2}\, |B(y,2r) \setminus A(\delta)|^{1/2},\end{aligned} \tag{37.35}$$

by Cauchy-Schwarz, (13.5), and (36.57). The right-hand side of (37.55) can be made as small as we want compared with $r^{3/2}$, by choosing δ very small. Hence the analogue of (13.18) for G_n and $B(y_n, 2r)$ holds (by (37.34) and (37.35)), we can apply lemma 13.17, and our claim (37.33) follows.

Denote by Ω_1 and Ω_2 the two components of $B \setminus K$. Fix an origin α somewhere in the middle of $\Omega_1 \cap \frac{1}{3}B$, and denote by $\Omega_{1,n}$ (respectively, $\Omega_{2,n}$) the component of $B_n \setminus G_n$ that contains (respectively, does not contain) α.

For n large enough, $\widetilde{\xi}_n$ passes at distance $\leqslant 10^{-2}r$ from y, because $y \in \widetilde{\xi}$ and $\widetilde{\xi}$ is the limit of the curves $\widetilde{\xi}_n$. Hence for n large enough there is a sub-arc $\widetilde{\xi}'_n$ of $\widetilde{\xi}_n$ such that

$$\widetilde{\xi}'_n \subset \widetilde{\xi}_n \cap B_n \tag{37.36}$$

and

$$\widetilde{\xi}'_n \text{ meets } B(y_n, r/50) \text{ and } \partial B(y_n, 9r/10). \tag{37.37}$$

[See Figure 37.1].

Note that $\widetilde{\xi}'_n$ does not meet G_n, by (37.19). Hence it is contained in $\Omega_{1,n}$ or in $\Omega_{2,n}$. Let us extract a new subsequence from $\{t_n\}$ so that $\widetilde{\xi}'_n$ is contained in the same $\Omega_{i,n}$ for all n. Without loss of generality, we can assume that $i = 1$, so that

$$\widetilde{\xi}'_n \subset \Omega_{1,n} \text{ for all } n. \tag{37.38}$$

Note that

$$\operatorname{dist}(z, G_n) \leqslant \eta_n \text{ for } z \in \widetilde{\xi}'_n, \tag{37.39}$$

by (37.30). [See Figure 37.1 for a vague description of the geometric situation.]

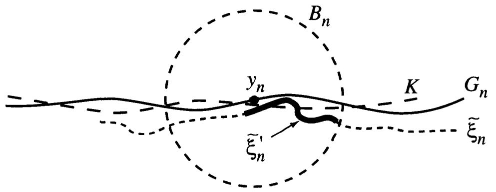


FIGURE 37.1

Define w_n on $\mathbb{R}^2$ by

$$w_n(z) = t_n^{-1/2} w(t_n z + x) - t_n^{-1/2} w(t_n \alpha + x), \tag{37.40}$$

where α still denotes our origin in Ω_1. Then

$$v_n + i w_n \text{ is holomorphic on } \mathbb{R}^2 \setminus G_n, \tag{37.41}$$

because $v + iw$ is holomorphic on $\mathbb{R}^2 \setminus G$.

Note that $\{\nabla w_n\}$ converges uniformly on compact subsets of Ω_1, by (37.3). Since in addition $w_n(\alpha) = 0$ for all n, $\{w_n\}$ converges uniformly on compact subsets of Ω_1 to a limit w_∞, and $u + i w_\infty$ is holomorphic on Ω_1 (by (37.41)).

By construction, w_n is constant on $\widetilde{\xi}'_n$, because $\widetilde{\xi}'_n \subset \widetilde{\xi}_n$, and by (37.16), (37.18), and (37.40). Denote by $m'(n)$ the constant value of w_n on $\widetilde{\xi}'_n$. Let us first check that

$$w_\infty(z) \geqslant \limsup_{n\to+\infty} m'(n) \text{ for all } z \in \Omega_1 \cap B(y, 2r/3). \tag{37.42}$$

Let $z \in \Omega_1 \cap B(y, 2r/3)$ be given. By (37.13) we can find points $z(n) \in \mathbb{R}^2 \setminus \Omega_0^n$ such that $\{z(n)\}$ converges to z. It is easy to see that $z(n) \in \Omega_{1,n}$ for n large enough (essentially, by (37.3) and definitions). By the same argument as for (37.27), the $|\nabla w_n|$ are uniformly bounded on some small disk around z, and hence

$$w_\infty(z) = \lim_{n\to+\infty} w_n(z) = \lim_{n\to+\infty} w_n(z(n)). \tag{37.43}$$

On the other hand $z(n) \in \mathbb{R}^2 \setminus \Omega_0^n$, hence $x + t_n z(n) \in \mathbb{R}^2 \setminus \Omega_0$ (by (37.12)), $w(x + t_n z(n)) \geqslant m_0 > m(n)$ (by definition of Ω_0 and (37.14)), and then $w_n(z(n)) > m'(n)$ (by (37.40) and because $m(n)$ is the constant value of w on ξ_n). Now (37.42) follows from this and (34.43).

To simplify our discussion, let us extract a new subsequence so that $m'(n)$ has a limit $m'(\infty)$. [We do not exclude the unlikely case where $m'(\infty) = -\infty$ a priori.] Then (37.42) says that $w_\infty(z) \geqslant m'(\infty)$ on $\Omega_1 \cap B(y, 2r/3)$. If we do not have that

$$w_\infty(z) > m'(\infty) \text{ on } \Omega_1 \cap B(y, 2r/3), \tag{37.44}$$

then we can conclude, because $w_\infty(z) \equiv m'(\infty)$ on $\Omega_1 \cap B(y, 2r/3)$ by the maximum principle, and then u is also constant on the component of $\mathbb{R}^2 \setminus K$ that contains Ω_1, so that we can apply Sublemma 37.8 as usual. So we may assume that (37.44) holds.

Our plan is to deduce from (37.44) that for n large enough

$$w_n(z) > m'(n) \text{ on } \Omega_{1,n} \cap B(y, r/10), \tag{37.45}$$

say. Since $w_n(z) = m'(n)$ on $\widetilde{\xi}'_n$ (by definition of $m'(n)$), this will contradict (37.37) or (37.38), and we will be rid of our last case. Set

$$\Omega'_n = \Omega_{1,n} \cap B(y, r/2). \tag{37.46}$$

Then $\partial\Omega'_n = \partial_1 \cup \partial_2 \cup \partial_3$, where

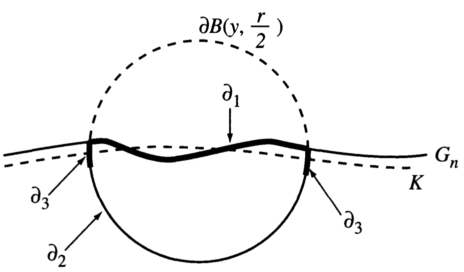


FIGURE 37.2

(37.47) $$\partial_1 = \partial\Omega_{1,n} \cap B(y, r/2) \subset G_n,$$

(37.48) $$\partial_2 = \{z \in \Omega_{1,n} \cap \partial B(y, r/2)\ ;\ \mathrm{dist}(z, K) \geqslant \varepsilon r\},$$

(37.49) $$\partial_3 = \{z \in \Omega_{1,n} \cap \partial B(y, r/2)\ ;\ \mathrm{dist}(z, K) < \varepsilon r\},$$

and the small constant $\varepsilon > 0$ will be chosen soon. [See Figure 37.2.]

Let us check that

(37.50) $$w_n(z) > m'(n) \text{ on } \partial_1.$$

Indeed $w(z) = m(n) < m_0$ on ξ_n (by (37.16) and (37.14)), and $w(z) \geqslant m_0$ on G (by definition of m_0). Then $w_n(z)$ is strictly larger on $\partial_1 \subset G_n$ than its constant value $m'(n)$ on $\widetilde{\xi}'_n$; this proves (37.50).

Note that ∂_2 does not depend on n (for n large enough), and that it is a compact subset of $\Omega_1 \cap B(y, 2r/3)$. Then $\{w_n\}$ converges to w_∞ uniformly on (the constant set) ∂_2. Since $\inf\{w_\infty(z)\ ;\ z \in \partial_2\} > m'(\infty)$ (by (37.44)) and $m'(\infty)$ is the limit of the $m'(n)$, we get that for n large enough,

(37.51) $$w_n(z) \geqslant m'(n) \text{ on } \partial_2.$$

Near the middle of ∂_2, we can do a little better. Choose a compact arc of circle $I \subset \partial B(y, r/2) \cap \Omega_1$, which will not depend on ε. Note that $I \subset \partial_2$ for n large (and if ε is not too large). By (37.44), we can find $\delta > 0$ such that $w_\infty(z) \geqslant m'(\infty) + 2\delta$ on I, and so

(37.52) $$w_n(z) \geqslant m'(n) + \delta \text{ on } I$$

for n large enough.

Let ∂ denote any of the two small arcs of circle that compose ∂_3. We claim that

(37.53) $$|w_n(z) - w_n(z')| \leqslant C\varepsilon^{1/2} \text{ for } z, z' \in \partial,$$

with a constant C that does not depend on n or ε. Because of the regularity of G_n near ∂ (see (37.33)), (37.53) follows from the fact that

(37.54) $$|w_n(z) - w_n(z')| \leqslant C\rho^{1/2} \text{ for } z, z' \in D$$

whenever D is a disk of radius ρ such that $2D \subset \Omega_{1,n}$.

To go from (37.54) to (37.53), we can decompose ∂ into a geometric series of Whitney arcs of circles, to which we can apply (37.54). To prove (37.54) we simply note that w_n is harmonic in $2D$ and $\int_{2D} |\nabla w_n|^2 \leqslant C\rho$ by (13.5). This proves (37.53).

Note that each arc ∂ has an extremity in ∂_2. Therefore

(37.55) $$w_n(z) \geqslant m'(n) - C\varepsilon^{1/2} \text{ on } \partial_3$$

for n large enough, by (37.51) and (37.53).

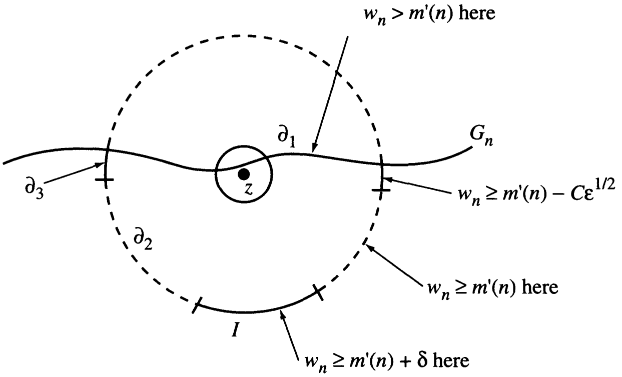


FIGURE 37.3

For $z \in \Omega_{1,n} \cap B(y, r/10)$ and $E \subset \partial\Omega'_n$, denote by $\omega_z(E)$ the harmonic measure of E in the domain Ω'_n and centered at z. Then for n large

$$w_n(z) \geqslant m'(n)\,\omega_z(\partial_1) + m'(n)\,\omega_z(\partial_2 \setminus I) + [m'(n) + \delta]\,\omega_z(I) + \left[m'(n) - C\varepsilon^{1/2}\right]\omega_z(\partial_3), \tag{37.56}$$

by (37.50), (37.51), (37.52), and (37.55). Then

$$w_n(z) - m'(n) \geqslant \delta\omega_z(I) - C\varepsilon^{1/2}\omega_z(\partial_3) \tag{37.57}$$

(because $\omega_z(\partial\Omega'_n) = 1$). It is clear that if ε is small enough, $\omega_z(\partial_3) \leqslant \omega_z(I)$ for all $z \in \Omega_{1,n} \cap B(y, r/10)$ (and n large enough). [See Figure 37.3.] Now we can choose ε so small (depending on δ in particular) that (37.45) follows from (37.57).

As was announced after (34.45), this allows us to get a contradiction in our only remaining case. Proposition 36.3 follows. □

CHAPTER I

CONCLUSIONS

38. The final contradiction

In this section we continue with the same notations and assumptions as above, and reach a contradiction. A precise recapitulation of what this means will be done in the next section.

Let $z_0 \in G$ be such that

$$w(z_0) = m_0 := \inf\{w(z)\ ;\ z \in G\}. \tag{38.1}$$

Note that $m_0 < 0$ by Proposition 36.3, and that z_0 exists because w is continuous, $G \setminus G_{00}$ is bounded (by (35.1)), and $w(z) = 0$ on G_{00} (by the normalization (32.3)). Denote by G_0 the connected component of z_0 in G. Obviously, $G_0 \neq G_{00}$ (by (32.3)).

Our general strategy will be to check first that

$$G_0 \neq \{z_0\}, \tag{38.2}$$

and then use our description of the variations of v around G_0 (in Section 34) and the regularity of a good piece of G_0 (as in Section 36) to find a regular or a spider point of G_0 where v does not jump. This will contradict the results of Section 16.

Let us first check (38.2). Suppose that $G_0 = \{z_0\}$. Then $z_0 \notin \partial\Omega_0$, because otherwise Proposition 36.3 would say that z_0 is a regular or a spider point of G. Then there is an open neighborhood V of z_0 such that $w(z) \geqslant m_0$ on V. By Lemma 25.3, there is a compact set G^0 such that $G_0 = \{z_0\} \subset G^0 \subset G \cap V$ and $\operatorname{dist}(G^0, G \setminus G^0) > 0$. We can use the construction of Section 23 to surround G^0 by a Jordan curve $\Gamma \subset \Omega \cap V$. [See in particular (23.10) (to show that $\Gamma \subset \Omega \cap V$ is ε is small enough) and Remark 23.17.]

Note that w is harmonic and $\geqslant m_0$ on a neighborhood of Γ; hence the maximum principle says that $w(z) > m_0$ on Γ. [The other option is that w and v are constant on a component of Ω, but this is ruled out by (32.1) and (32.5), say.]

Now choose m such that $m_0 < m < \inf\{w(z)\ ;\ z \in \Gamma\}$, and let W denote the component of z_0 in $\mathbb{R}^2 \setminus \Gamma_m$. Then $w(z) < m$ on W (by connectedness and because $w(z_0) = m_0$), hence W does not meet Γ and is contained in the bounded component of $\mathbb{R}^2 \setminus \Gamma$. This contradiction with Proposition 30.1 proves (38.2).

Now we want to use our tour $z : \mathbb{S}^1 \to G_0$ of G_0 and Proposition 34.9. Let us first check that

$$z(t) \in \partial\Omega_0 \cap G \text{ for all } t \in I^+, \tag{38.3}$$

where I^+ is the interval of $\mathbb{S}^1$ that shows up in Proposition 34.9.

Recall from a little bit after Lemma 34.65 that I^+ is the closure of R^+ in $\mathbb{S}^1$, where R^+ is the set defined in (34.18) and (34.1). Since z is Lipschitz, it is enough to check that $z(t) \in \partial\Omega_0$ for $t \in R^+$. This last follows at once from (34.19). The reader should not be shocked by this short proof: the reduction to the dense set R where $z(t)$ is a regular point is natural, and for these points we are essentially saying that if the boundary values of v near $z(t)$ (and with access from Ω_t) are strictly increasing, then $\partial w/\partial n < 0$ near $z(t)$ and $z(t)$ is accessible from Ω_0.

From (38.3) and Proposition 36.3 we deduce that

$$z(t) \text{ is a regular or a spider point of } G \text{ for all } t \in I^+. \tag{38.4}$$

Note that

$$I_0^+ = \{t \in I^+\ ;\ z(t) \text{ is a spider point of } G\} \tag{38.5}$$

does not have any accumulation point (by (38.4) and because I^+ is closed); hence I_0^+ is finite.

For $t \in I^+ \setminus I_0^+$, $z(t)$ is a regular point of G and so $t \in R$. We know from Section 33 (and in particular the discussion a little before (33.38)) that there is a unique point $t^* \in \mathbb{S}^1$, $t^* \neq t$, such that $z(t^*) = z(t)$. [This point t^* corresponds to the access to $z(t)$ from the other region Ω_t^*.]

Lemma 38.6. — *Set $I^\# = I^+ \setminus I_0^+$ and $\varphi(t) = t^*$ for $t \in I^\#$. Then*

$$\varphi(t) \in \mathbb{S}^1 \setminus I^+ \subset I^- \text{ for } t \in I^\#, \tag{38.7}$$

and $\varphi : I^\# \to I^-$ is continuous and strictly decreasing.

Of course the continuity of φ on $I^\#$ does not prevent the existence of jumps at the points of I_0^+. When we say that $\varphi : I^\# \to I^-$ is decreasing, we use the orders on the intervals I^+ and I^- that come from the trigonometric orientation of $\mathbb{S}^1$.

Let us first check (38.7). We claim that

$$\text{the restriction of } z \text{ to } I^+ \text{ is injective.} \tag{38.8}$$

Note that (38.7) will follow from this, because $\varphi(t) \neq t$ and $z(\varphi(t)) = z(t)$, so that t and $\varphi(t)$ cannot both lie on I^+. The fact that $\mathbb{S}^1 \setminus I^+ \subset I^-$ comes from Proposition 34.9.

To prove (38.8), suppose that we can find $a, b \in I^+$ such that $a < b$ (in I^+) and $z(a) = z(b)$. Set

$$b' = \sup \{t \in [a,b] \ ; \ \text{the restriction of } z \text{ to } [a,t) \text{ is injective}\}, \tag{38.9}$$

where $[a,b]$ and $[a,t)$ denote the subintervals of I^+ that the reader imagines. Then $b' > a$ (in I^+), because z is obviously injective in a small neighborhood of a (by (38.4) and the construction of z). Also, $z(b') \in z([a,b'))$ by definition of b' (and because z is injective in a small neighborhood of b'). Next set

$$a' = \sup \{t \in [a,b') \ ; \ z(t) = z(b')\}. \tag{38.10}$$

Then $a' < b'$ (in I^+), because $z(b')$ is a regular or a spider point (and by construction of z). The restriction of z to (a',b') is injective (by (38.9)), and $z(a') = z(b')$. Hence G_0 contains a loop and $\mathbb{R}^2 \setminus G$ has a bounded component. This contradiction with Lemma 15.1 proves (38.8) and (38.7).

To prove that φ is decreasing, let $t_1, t_2 \in I^\#$ be given, with $t_1 < t_2$ in I^+. Set $\Delta_i = \Delta_{t_i}$, where Δ_{t_i} is the line segment of length r_{t_i} that starts at $z(t_i)$, is perpendicular to G at that point, and lies in Ω_{t_i} (except for its endpoint $z(t_i)$). See the definition after (33.41), or just Figure 33.2. Similarly set $\Delta_i^* = \Delta_{\varphi(t_i)}$.

Recall that z was constructed as the limit on $\mathbb{S}^1$ of parameterizations z_n of curves $\Gamma(n)$. We know from (33.42) that for n large, $\Gamma(n) \cap \Delta_i$ has exactly one point. Call this point y_i, and define $s_{i,n} \in \mathbb{S}^1$ by $z_n(s_{i,n}) = y_i$. Similarly, for n large enough and $i = 1, 2$, $\Gamma(n) \cap \Delta_i^*$ has exactly one point, which we call $y_i^* = z_n(s_{i,n}^*)$.

Since $t_1 \neq t_2$, (38.8) tells us that $z(t_1) \neq z(t_2)$, and so the four points $t_1, t_2, \varphi(t_1), \varphi(t_2)$ are all distinct (because each $z(t_i)$ has exactly two inverse images under z). We want to show that

$$t_1 < t_2 < \varphi(t_2) < \varphi(t_1) < t_1 \text{ in } \mathbb{S}^1, \tag{38.11}$$

and as usual it will be slightly easier to proceed by contradiction. So let us assume that (38.11) does not hold. Then

$$t_1 < t_2 < \varphi(t_1) < \varphi(t_2) < t_1 \text{ in } \mathbb{S}^1, \tag{38.12}$$

because we know that $t_1, t_2 \in I^+$ and $\varphi(t_1), \varphi(t_2) \in \mathbb{S}^1 \setminus I^+$. Note that

$$\lim_{n \to +\infty} s_{n,i} = t_i \text{ and } \lim_{n \to +\infty} s_{n,i}^* = \varphi(t_i) \tag{38.13}$$

for $i = 1, 2$, by (33.45). Hence

$$s_{n,1} < s_{n,2} < s_{n,1}^* < s_{n,2}^* < s_{n,1} \quad \text{on } \mathbb{S}^1 \tag{38.14}$$

for n large enough, and so

$$y_1 < y_2 < y_1^* < y_2^* < y_1 \quad \text{on } \Gamma(n) \tag{38.15}$$

as well. Set

$$\gamma_i = [y_i, z(t_i)] \cup [z(t_i), y_i^*] \tag{38.16}$$

for $i = 1, 2$. For n large enough, the simple curve γ_i is contained in the bounded component of $\mathbb{R}^2 \setminus \Gamma(n)$, except for its endpoints y_i and y_i^*. Also $\gamma_1 \cap \gamma_2 = \emptyset$, by (33.44) and because $z(t_1) \neq z(t_2)$. We get the desired contradiction from (38.15) and Lemma 34.54. [See Figure 38.1.]

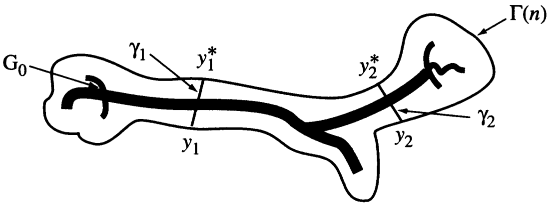


FIGURE 38.1

Thus $\varphi : I^{\#} \to I^-$ is strictly decreasing. We still need to check that φ is continuous. Let $t_1 \in I^{\#}$ be given, and let us just verify that φ is continuous from the right at t_1; the continuity from the left would be the same. Observe that in the argument above $|z(t_2) - z(t_1)| \leqslant C\,|t_2 - t_1|$ (because z is Lipschitz), and then $|y_2^* - y_1^*| \leqslant C'\,|t_2 - t_1|$ for n large enough (and t_2 close enough to t_1) because G is a nice curve near $z(t_1)$. The constant C' may depend on t_1, but not on n or t_2. Then $\left|s_{2,n}^* - s_{1,n}^*\right| \leqslant C''\,|t_2 - t_1|$ (because $\Gamma(n)$ is parameterized by z_n with constant speed, and we have uniform bounds on its length), and hence $|\varphi(t_2) - \varphi(t_1)| \leqslant C''\,|t_2 - t_1|$ (by (38.13)).

This completes our proof of Lemma 38.6. □

Remark 38.17. — The only property of I^+ that was used here is (38.4) (and the fact that I^+ is a closed interval). Thus Lemma 38.6 still holds with I^+ replaced with a slightly larger interval $\widehat{I}^+$ (that contains I^+ in its interior).

Next we want to say a little more about the behavior of φ near points of I_0^+. For each $t \in I_0^+$, set

$$\varphi(t^{\pm}) = \lim_{s \to t^{\pm}} \varphi(s). \tag{38.18}$$

The limits exist by monotonicity. If t is one of the endpoints of I^+, (38.18) still makes sense, because we can define φ on $\widehat{I}^+ \setminus I_0^+$ for some slightly larger interval $\widehat{I}^+ \supset I^+$, as in Remark 38.17. It will be good to know that

$$\varphi(t^+) \neq \varphi(t^-), \tag{38.19}$$

i.e., that φ has a nonzero jump at t.

Let $t \in I_0^+$ be given, and let $t_1, t_2 \in I^{\#}$ be very close to t, and such that $t_1 < t < t_2$ in I^+. [If t is an endpoint of I^+, replace I^+ with a slightly larger $\widehat{I}^+$, as in Remark 38.17.] Keep the same notations as in the proof of Lemma 38.6, in particular concerning the intervals Δ_i, Δ_i^* and the points y_i, y_i^*, $i = 1, 2$. [Also see Figure 38.2.]

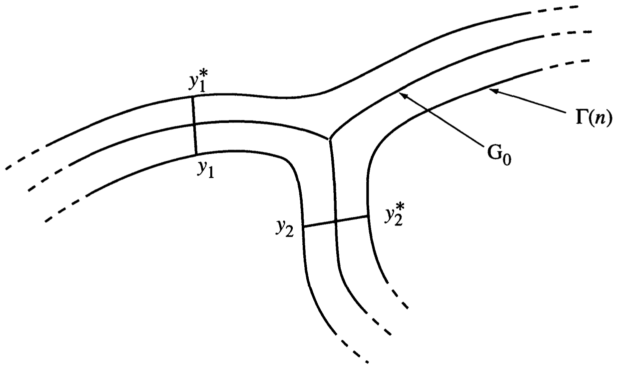


FIGURE 38.2

For n large enough, the length of the shortest arc of $\Gamma(n)$ between y_2^* and y_1^* is at least C, where C does not depend on n or the choice of t_1, t_2 (provided that they stay close enough to t). Then $\left|s_{1,n}^* - s_{2,n}^*\right| \geqslant C'$ for n large enough (because z_n is Lipschitz with uniform bounds, see (33.15)), and hence $|\varphi(t_1) - \varphi(t_2)| \geqslant C'$, by (38.13). This proves (38.19).

Recall from Proposition 34.9 and Lemma 34.4 that we have a continuous function u on $\mathbb{S}^1$, which is strictly increasing on I^+ and strictly decreasing on I^-. Set

$$u^*(t) = u(\varphi(t)) \text{ for } t \in I^\# = I^+ \setminus I_0^+. \tag{38.20}$$

Lemma 38.6 tells us that u^* is continuous on $I^\#$, and strictly increasing (as a composition of two strictly decreasing functions). Also, the jumps of u^* at points of I_0^+ are strictly positive, because u is strictly decreasing on I^- (by (34.11)), because $\varphi(t^+)$ and $\varphi(t^-)$ lie in I^- (as limits of points of I^-), and $\varphi(t^+) < \varphi(t^-)$ in I^- (by Lemma 38.6 and (38.19)).

Extend u^* to I^+ by taking

$$u^*(t) = u\left(\varphi\left(t^+\right)\right) = \lim_{s \to t^+} u^*(s) \text{ for } t \in I_0^+. \tag{38.21}$$

Write $I^+ = [a, b]$. Then $u(a) < u(t)$ for all $t \in \mathbb{S}^1 \setminus \{a\}$, by Proposition 34.9. Thus

$$u(a) < u^*(a), \tag{38.22}$$

because $\varphi(a) \neq a$ if $a \in I^\#$, and $\varphi(a^+) \neq a$ (because $\varphi(a^+) < \varphi(a^-)$ in I^-) if $a \in I_0^+$. Similarly, $u(b) > u(t)$ for all $t \in \mathbb{S}^1 \setminus \{b\}$,

$$u(b) > u^*(b) \text{ if } b \in I^\#, \tag{38.23}$$

$$u(b) > u\left(\varphi\left(b^-\right)\right) = \lim_{s \to b^-} u^*(s) \text{ if } b \in I_0^+ \tag{38.24}$$

(again because $\varphi(b^-) \neq b$). Now set

$$t_0 = \inf\{t \in (a, b) \ ; \ u(t) \geqslant u^*(t)\} \tag{38.25}$$

the existence of t_0, and also the fact that $t_0 \in (a,b)$, are easily deduced from the continuity of u and (38.22)-(38.24). If $t_0 \in I^{\#}$, we get that

$$u^*(t_0) = u(t_0) \tag{38.26}$$

because u and u^* are continuous near t. If $t_0 \in I_0^+$, we get that $u(t_0) \geqslant u^*(t_0)$, by (38.21) and because u is continuous. Since u^* has a positive jump at t_0, we get that

$$u(t_0) > \lim_{t \to t_0^-} u^*(t), \tag{38.27}$$

and then $u(t) > u^*(t)$ for lots of points $t \in (a, t_0)$. This contradicts the definition of t_0 in (38.25); hence the only possibility is that $t_0 \in I^{\#}$ and (38.26) holds.

Since $t_0 \in I^{\#}$, $z(t_0)$ is a regular point of G, $t_0 \in R$ (see (34.1)), and

$$u(t_0) = \lim_{\substack{z \to z(t_0) \\ z \in \Omega_{t_0}}} v(z), \tag{38.28}$$

as in (34.3). Similarly, $\varphi(t_0) \in R$, and

$$u^*(t_0) = u(\varphi(t_0)) = \lim_{\substack{z \to z(t_0) \\ z \in \Omega^*_{t_0}}} v(z). \tag{38.29}$$

Thus (38.26) says that v has no jump at the regular point $z(t_0)$.

On the other hand, we are in position to apply Lemma 16.1, because $\Omega = \mathbb{R}^2 \setminus G$ is connected (by (32.1)), and because G_0 does not meet L when (13.1) holds. [Recall from the first lines of this section that G_0 is a component of G, and $G_0 \neq G_{00}$.]

So Lemma 16.1 contradicts (38.26). We are now reasonably happy. We made various assumptions on a minimizer (v,G) (including that (v,G) is not a "generalized cracktip"), and we finally got a contradiction. In the next section we shall sort out what this means.

39. The main technical statements

Let us summarize in this section what we proved since Section (13). We started with a pair (v,G), and we assumed that

(39.1) (v,G) is a reduced minimizer of the modified functional

(as in (13.1); see Definition 11.3), or

(39.2) (v,G) is a reduced global λ-minimizer

(as in (13.2); see Definition 12.1). We also assumed that $\mathbb{R}^2 \setminus G$ is connected (see (26.1), (32.1)), but in view of Lemma 15.1, it is equivalent to assume that

(39.3) $\mathbb{R}^2 \setminus G$ has only one unbounded connected component.

We further assumed that

(39.4) G has exactly one unbounded connected component G_{00}, and that

$$G \setminus G_{00} \text{ is bounded} \tag{39.5}$$

(see (32.2) and (35.1)). The next conditions concern the level sets

$$\Gamma_m = \{z \in \mathbb{R}^2 \; ; \; w(z) = m\} \tag{39.6}$$

of the function w constructed in Section 22. Recall that w is continuous on $\mathbb{R}^2$ and $v + iw$ is holomorphic on $\mathbb{R}^2 \setminus G \approx \mathbb{C} \setminus G$. We assumed that w satisfies (32.4) and (32.5), under the normalization (32.3) that $w = 0$ on G_{00}.

Our final real assumption was that

$$\liminf_{r \to +\infty} \frac{1}{r} \int_{B(x,r) \setminus G} |\nabla v|^2 \leqslant \lambda \tag{39.7}$$

for $x \in \mathbb{R}^2$, where $\lambda = h'\left(H^1(G \setminus L)\right)$ when (39.1) holds. (See (36.1)).

We finally assumed in (36.2) that (v, G) is not a "generalized cracktip", and then we got a contradiction. Recall from just after (36.2) that we call generalized cracktip a pair (v, G) such that

$$G = \{x - \rho e^{i\theta_0} \; ; \; \rho \geqslant 0\} \tag{39.8}$$

and

$$\begin{cases} v\left(x + r^{i\theta}\right) = \alpha + \beta \, r^{1/2} \sin\left(\dfrac{\theta - \theta_0}{2}\right) \\ \text{for } \theta_0 - \pi < \theta < \theta_0 + \pi \text{ and } 0 < r < +\infty, \end{cases} \tag{39.9}$$

for some choice of $x \in \mathbb{R}^2$ and $\alpha, \beta, \theta_0 \in \mathbb{R}$.

Let us summarize this in two statements. These are not definitive; in particular Theorem 39.10 is not a good as Theorem 1.16.

Theorem 39.10. — *Let (v, G) be a reduced global λ-minimizer (as in Definition 12.1). Suppose that* (39.3), (39.4), (39.5), (32.3), (32.4), (32.5), *and* (39.7) *hold. Then we can find $x \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$, and $\theta_0 \in \mathbb{R}$ such that* (39.8) *and* (39.9) *hold, with $\beta = \sqrt{2\lambda/\pi}$.*

The only apparently new thing in this statement is the precise value of β, but Remark 35.42 tells us that $\beta = \pm\sqrt{2\lambda/\pi}$, and the negative sign is excluded by (32.4) and (32.5).

Theorem 39.11. — *Let (v, G) be a reduced minimizer of the modified functional (as in Definition 11.3). Suppose that* (32.3), (32.4), (32.5) *and* (39.7) *hold, with $\lambda = h'\left(H^1(G \setminus L)\right)$ (and h as in* (11.4)*). Then we can find $x \in \mathbb{R}$, $x \geqslant -1$, $\alpha \in \mathbb{R}$, and $0 \leqslant \beta \leqslant \sqrt{2\lambda/\pi}$ such that* (39.8) *and* (39.9) *hold with $\theta_0 = 0$.*

Here again the precision on β comes from Remark 35.42, the fact that we can take $\theta_0 = 0$ and $x \geqslant -1$ comes from the constraint that $L \subset G$, and we did not need to mention (39.3), (39.4), and (39.5) because they are automatically satisfied, since $G \setminus L$ is bounded by definition of a minimizer.

40. Theorem 1.16 and a variant with blow-ins

In this section (v, G) is a reduced global λ-minimizer (see definition 12.1); we want to prove Theorem 1.16 and a variant with "blow-ins".

Definition 40.1. — A blow-in of (v, G) is a pair (u, K) for which we can find a point $y \in G$ and a sequence $\{t_n\}$ of positive numbers such that

$$\lim_{n \to +\infty} t_n = +\infty, \tag{40.2}$$

and

$$\{(v_n, G_n)\} \text{ converges to } (u, K), \tag{40.3}$$

where v_n and G_n are defined by (12.7) *and* (12.8) with $y_n = y$, and the notion of convergence is the one described before Lemma 12.4.

Note that blow-ins exist. Lemma 12.5 tells us that from each sequence $\{t_n\}$ we can extract a subsequence for which (40.3) holds. Of course (v, G) could have lots of different blow-ins, coming from different sequences. The notion will be useful because Proposition 12.44 tells us that blow-ins of (v, G) are also global λ-minimizers, and in some situations they may be simpler. Let us now state our companion to Theorem 1.16. We continue to call "generalized cracktip" any pair (v, G) such that (39.8) and (39.9) hold for some choice of $x \in \mathbb{R}^2$ and $\alpha, \beta, \theta_0 \in \mathbb{R}$.

Theorem 40.4. — *Let (v, G) be a (reduced) global λ-minimizer. Suppose that we can find a blow-in (u, K) of (v, G) which is a generalized cracktip. Then (v, G) is a generalized cracktip.*

Note that in this case we also get that $\beta = \pm\sqrt{2\lambda/\pi}$, by Remark 35.42.

Also, it should be observed that if (v, G) is a (reduced) global λ-minimizer and we can find a sequence $\{t_n\}$ such that $\lim_{n \to +\infty} t_n = +\infty$ and $\{t_n^{-1} G\}$ converges to a half-line, then (v, G) is a generalized cracktip. This is because we can always extract a subsequence of $\{t_n\}$ so that (40.3) holds for some (u, K) (by Lemma 12.5), and then (u, K) is a global minimizer (by Proposition 12.44). Since K is a half-line, (u, K) is a generalized cracktip (for instance by [Bo], but we could also use the Léger formula in [Lé2] or the end of our proof of Lemma 37.1 (see four lines above (37.10))). Now we can apply Theorem 40.4 and (v, G) is a generalized cracktip.

The rest of this section is devoted to the proof of Theorems 1.16 and 40.4. We start with a simpler version of Theorem 40.4.

Proposition 40.5. — *If (v, G) is a reduced global λ-minimizer and all the blow-ins of (v, G) are generalized cracktips, then (v, G) is a generalized cracktip.*

Let us assume for convenience that $0 \in G$ (otherwise, we could translate v and G). Let us first define a function $\gamma(t)$ to measure the closeness of (v, G) to cracktips at the various scales t. Denote by H_λ the set of generalized cracktips (i.e. pairs (v^*, G^*) as in (39.8) and (39.9)), with the natural constraint $\beta^2 = 2\lambda/\pi$. For each $t > 0$, set

$$G_t = t^{-1}G, \tag{40.6}$$

$$v_t(x) = t^{-1/2}v(tx) \text{ for } x \in \mathbb{R}^2 \setminus G_t, \tag{40.7}$$

and then

$$\gamma(t) = \inf_{(v^*, G^*) \in H_\lambda} \left\{ d\,(G_t, G^*) + \int_{B(0,1)\setminus(G_t \cup G^*)} |\nabla v_t - \nabla v^*| \right\}, \tag{40.8}$$

where

$$\begin{aligned} d\,(G_t, G^*) = \sup\{\mathrm{dist}\,(z, G^*) \ ; \ z \in G_t \cap B(0,1)\} \\ + \sup\{\mathrm{dist}\,(z, G_t) \ ; \ z \in G^* \cap B(0,1)\}. \end{aligned} \tag{40.9}$$

Lemma 40.10. — $\lim_{t\to+\infty} \gamma(t) = 0$.

It will be enough to check that

$$\lim_{n\to+\infty} \gamma\,(t_n) = 0 \tag{40.11}$$

for every sequence $\{t_n\}$ such that (40.2) and (40.3) hold for some generalized cracktip (u, K). Indeed, if $\gamma(t)$ does not tend to 0, then we can find $\{t_n\}$ such that (40.2) holds and $\gamma\,(t_n)$ stays away from 0. Then we can extract a subsequence for which (40.3) holds for some (u, K) (by Lemma 12.5), and the assumption in Proposition 40.5 says that (u, K) is a generalized cracktip. Our claim follows.

So let $\{t_n\}$ satisfy (40.2) and (40.3) for some generalized cracktip (u, K), and let us prove (40.11). Of course we want to try $(v^*, G^*) = (u, K)$ in the definition of $\gamma\,(t_n)$. The fact that

$$\lim_{n\to+\infty} d\,(G_{t_n}, K) = 0 \tag{40.12}$$

comes straight from the definition of convergence in (40.3). To estimate

$$I_n = \int_{B(0,1)\setminus(G_{t_n} \cup K)} |\nabla v_{t_n} - \nabla u|, \tag{40.13}$$

we cut the domain of integration into two pieces A_1 and A_2. Set

$$A_1 = \{z \in B(0,1) \ ; \ \mathrm{dist}(z, K) \geqslant \delta\}. \tag{40.14}$$

Note that for each choice of $\delta > 0$, $\overline{A}_1$ is a compact subset of $\mathbb{R}^2 \setminus K$ that does not depend on n, so $\overline{A}_1$ does not meet G_{t_n} for n large enough, and

$$\lim_{n\to+\infty} \int_{A_1} |\nabla v_{t_n} - \nabla u| = 0 \tag{40.15}$$

because $|\nabla v_{t_n}|$ converges to $|\nabla u|$ uniformly on A_1, by (40.3). On the other hand, if $A_2 = B(0,1) \setminus (G_{t_n} \cup K \cup A_1)$,

$$(40.16) \quad \int_{A_2} |\nabla v_{t_n} - \nabla u| \leqslant |A_2|^{1/2} \left\{ \int_{A_2} (|\nabla v_{t_n}| + |\nabla u|)^2 \right\}^{1/2} \leqslant C\,|A_2|^{1/2} \leqslant C\delta^{1/2},$$

by Cauchy-Schwarz and (13.5). Thus we can make I_n as small as we wish (for n large) by first choosing δ very small, and then using (40.15). In other words, I_n tends to 0. This proves (40.11), and Lemma 40.10 follows. □

For each $t > 0$, choose a pair (v_t^*, G_t^*) in H_λ such that

$$(40.17) \qquad d(G_t, G_t^*) + \int_{B(0,1)\setminus(G_t \cup G_t^*)} |\nabla v_t - \nabla v_t^*| \leqslant 2\gamma(t).$$

Lemma 40.18. — *Denote by x_t the endpoint of G_t^*. Then $\lim_{t\to+\infty} x_t = 0$.*

Let us first check that there are arbitrarily large values of t for which $x_t \in B(0,1/2)$. Choose any sequence $\{t_n\}$ such that (40.2) and (40.3) hold for some (u,K). Then K is a half-line, by assumption on (v,G). Call x its endpoint, and set $a = (1+2\,|x|)^{-1}$, say. Then $\{G_{t_n}\}$ converges to K (by (40.3)), and hence $\{G_{at_n}\}$ converges to aK. Since aK is a half-line with an endpoint in $B(0,1/2)$, and $\gamma(at_n)$ tends to 0, $x_{at_n} \in B(0,1/2)$ for n large.

Now observe that if $x_t \in (0,1/2)$ and $2t \leqslant t' \leqslant 4t$, say, then

$$(40.19) \qquad |x_{t'}| \leqslant \frac{1}{2}\,|x_t| + 10\gamma(t) + 10\gamma(t').$$

This is because G_t^* is very close to G_t, $G_{t'}^*$ is very close to $G_{t'}$, and $G_{t'} = \frac{t}{t'} G_t$. This estimate is quite rough, but because of Lemma 40.10 it is still enough to imply that $\lim_{t\to+\infty} x_t = 0$, as needed. [Start with $x_t \in (0,1/2)$ and iterate (40.19).] □

Lemma 40.20. — *For each (small) $\delta > 0$, there is a $t_0 > 0$ such that for each $t \geqslant t_0$,*

$$(40.21) \qquad G_t \cap B(0,1) \setminus B(0,\delta) \text{ is a } C^1 \text{ curve.}$$

Let $\delta > 0$ be given, and let t be so large that $x_t \in B(0,\delta/3)$. Let $r = r(\delta) < 10^{-2}$ be a small constant, to be chosen soon, and let $B = B(x,r)$ be any disk centered on G_t, with radius r, and such that

$$(40.22) \qquad B \subset B(0,1) \setminus B(0, 2\delta/3).$$

We want to apply Lemma 13.17 to B and the pair (v_t, G_t). Let D be the line that contains G_t^*. Then (13.19) and (13.20) hold if $\gamma(t)$ is small enough (compared with the constant ε of Lemma 13.17), by (40.17). As for (13.18), observe that since $x_t \in B(0,\delta/3)$ lies at distance $\geqslant \delta/3$ from B, a direct computation with (39.9) yields

$$(40.23) \qquad \int_{B\setminus G_t^*} |\nabla v_t^*| \leqslant C(\delta) r^2 \leqslant \frac{\varepsilon}{2} r^{3/2}$$

if r is small enough. Since

$$\int_{B\setminus(G_t^*\cup G_t)} |\nabla v_t - \nabla v_t^*| \leqslant 2\gamma(t) \tag{40.24}$$

by (40.17), (13.18) holds if $\gamma(t)$ is small enough.

Thus we can apply Lemma 13.17, and we get that $B(x, r/2)$ is a disk of regularity for G_t. Note that this holds for all choices of $x \in G_t$ such that (40.22) holds. Let us also require that $\gamma(t) < 10^{-2}r$, say. Then $G_t \cap B(0,1)$ is so close to the line segment $G_t^* \cap B(0,1)$ (by (40.17)) that we can easily deduce that

$$G_t \cap B(0,1/2) \setminus B(0,\delta) \text{ is a } C^1\text{-curve} \tag{40.25}$$

from our local description of G_t.

Of course (40.25) is not exactly the same as (40.21), but we can easily get (4.21) by applying (4.25) to $2t$ and with $\delta/2$. This proves the lemma. □

Remark 40.26. — As we already mentioned earlier, Lemma 13.17 actually gives more regularity than what we stated: Theorem 4.8 in [Da], for instance, gives uniform $C^{1+\varepsilon}$ bounds for some $\varepsilon > 0$, and much more is true. Because of this, we even have that for $t \geqslant t_0$, $G_t \cap B(0,1) \setminus B(0,\delta)$ is a $C^{1+\varepsilon}$ curve, with uniform estimates (that depend on δ).

The proof of Lemma 40.20 also gives that for t large enough and all radii $\rho \in (1/2, 1)$, $G_t \cap \partial B(0,\rho)$ has exactly one point. Hence

$$G \cap \partial B(0,\rho) \text{ has exactly one point} \tag{40.27}$$

for ρ large enough, and consequently

$$\mathbb{R}^2 \setminus G \text{ has only one unbounded component,} \tag{40.28}$$

as required in (39.3).

We also deduce from Lemma 40.20 (applied with $\delta = 1/3$) and some gluing that

$$G \setminus B(0, t_0/3) \text{ is a } C^1 \text{ curve that escapes to } \infty. \tag{40.29}$$

Hence (39.4) and (39.5) are satisfied.

Next we want to check the conditions (32.3)-(32.5) on the level sets of w.

Let $r > 0$ be small (essentially as small as in the proof of Lemma 40.20 with $\delta = 1/4$, say). For t large, denote by z_t the only point of $G_t \cap \partial B(0,1/2)$, and set $B_t = B(z_t, r)$.

We know from (the proof of) Lemma 40.20 that if r is small enough, then for t large B_t is a disk of regularity for G_t, and also $G_t \cap B_t$ is a $C^{1+\varepsilon}$ curve, with estimates that do not depend on t large enough. [See Remark 40.26.] Denote by $\Omega_t^\pm$ the two connected components of $B_t \setminus G_t$, with Ω_t^+ on the right of $G_t \cap B_t$ when one looks at B_t from the origin. We know from Section 14 that v_t has a C^1 extension to $\overline{\Omega_t^\pm}$ (we may have to replace B_t with $\frac{1}{2}B_t$ for this, but this does not matter), and the proof also gives a control on the continuity of ∇v_t on $\overline{\Omega_t^\pm}$ which does not depend on t (large enough). Thus there is a function $h(\rho)$ that tends to 0 (when ρ tends to 0^+) such that for t large enough, and each choice of $\pm$,

$$|\nabla v_t(x) - \nabla v_t(y)| \leqslant h(|x-y|) \text{ when } x, y \in \Omega_t^\pm. \tag{40.30}$$

We claim that for t large enough,

$$\left|\frac{\partial v_t}{\partial r}(z) - \left(\pm\frac{\beta}{\sqrt{2}}\right)\right| \leqslant \frac{|\beta|}{10} \text{ for } z \in \partial B(0, 1/2) \cap \Omega_t^\pm, \tag{40.31}$$

where $\partial v_t/\partial r$ denotes the radial derivative of v_t and β is the constant associated to (v_t^*, G_t^*) as in (39.9). Note that $\beta = \pm\sqrt{2\lambda/\pi}$, and the sign cannot depend on t (large enough), because (40.17) would not allow such a brutal discontinuity of v_t^*.

Because of (40.30), it is enough to prove (40.31) with the smaller constant $|\beta|/20$, but only on the smaller set

$$E_t = \left\{z \in \partial B(0, 1/2) \cap \Omega_t^\pm\ ;\ \operatorname{dist}(z, G_t) \geqslant \rho\right\}, \tag{40.32}$$

where the constant ρ is chosen so small that $h(\rho') \leqslant |\beta|/20$ for $\rho' \leqslant \rho$. Next

$$\left|\frac{\partial v_t}{\partial r}(z) - \frac{\partial v_t^*}{\partial r}(z)\right| \leqslant C\rho^{-2}\gamma(t) \text{ for } z \in E_t \tag{40.33}$$

(and t large enough), by (40.17) and the fact that $\nabla v_t - \nabla v_t^*$ is harmonic away from $G_t \cup G_t^*$.

Now (40.31) follows from a stupid computation on $\partial v_t^*/\partial r$ near G_t^*, and the fact that the radius r of B_t can be made as small as we want.

Suppose for definiteness that $\beta > 0$. If $\beta < 0$, we may as well apply our argument to the pair $(-v, G)$. Set

$$w_t(z) = t^{-1/2}w(tz) \text{ on } \mathbb{R}^2 \setminus G_t; \tag{40.34}$$

thus $v_t + iw_t$ is holomorphic on $\mathbb{R}^2 \setminus G_t$.

From (40.31) we deduce that

$$\pm\frac{\partial w_t}{\partial \theta} \geqslant \frac{\beta}{4} \text{ on } \partial B(0, 1/2) \cap \Omega_t^\pm, \tag{40.35}$$

where $\partial w_t/\partial\theta$ denotes the angular derivative of w_t (i.e., half of the tangential derivative).

Because of (40.35), and the fact that $w_t(z) = 0$ on G_t, $w_t(z) \leqslant -\beta r/4$ at both points of $\partial B(0, 1/2) \cap \partial B_t$. Since $\partial w_t/\partial\theta$ is as close as we want to $\partial w_t^*/\partial\theta$ on $\partial B(0, 1/2) \setminus B_t$ (by (40.17) and the harmonicity of $\nabla v_t - \nabla v_t^*$, as for (40.33)), we easily deduce from this that

$$w_t(z) \leqslant -\frac{\beta r}{4} \text{ on } \partial B(0, 1/2) \setminus B_t. \tag{40.36}$$

Set $\Gamma_{m,t} = \{z \in \mathbb{R}^2 \ ; \ w_t(z) = m\}$. From (40.35) and (40.36) we deduce that

$$\Gamma_{m,t} \cap \partial B(0,1/2) = \varnothing \text{ for } m > 0, \tag{40.37}$$

$$\Gamma_{0,t} \cap \partial B(0,1/2) = \{z_t\}, \tag{40.38}$$

and

$$\Gamma_{m,t} \cap \partial B(0,1/2) \text{ has exactly two points when } -\frac{\beta r}{4} < m < 0. \tag{40.39}$$

[We do not care about large negative values of m.] The desired estimates (32.4) and (32.5) follow from this, because

$$\Gamma_m \cap \partial B(0,t/2) = t\left\{\Gamma_{m/\sqrt{t},t} \cap \partial B(0,1/2)\right\}. \tag{40.40}$$

We still need to verify (39.7). Probably the most natural way to do this would be to observe that if $\{t_n\}$ satisfies (40.2) and (40.3) then

$$\limsup_{n\to+\infty} \int_{B(0,1)\smallsetminus G_n} |\nabla v_n|^2 \leqslant \int_{B(0,1)\smallsetminus K} |\nabla u|^2. \tag{40.41}$$

Note that this is not the inequality that one gets by Fatou. We can get it by looking closely at the proof of Proposition 12.44 (i.e., of the fact that (u,K) is a global λ-minimizer). The idea is that if (40.41) failed, we would be able to construct better competitors for (v,G). Such competitors would be obtained by replacing (v,G) in big disks $B(0,t_n)$ with dilations of (u,K), plus a small term to correct the slightly different boundary values on $\partial B(0,t_n)$. We would win a significant amount of energy because (40.41) fails, and we would almost not lose on lengths or because of the correction, by the argument in Section 12. In the present situation, it is just as easy to prove a stronger variant of (40.41), as follows.

Lemma 40.42. — *Set*

$$\alpha(t) = \int_{B(0,1/2)\smallsetminus(G_t\cup G_t^*)} |\nabla v_t - \nabla v_t^*|^2 \tag{40.43}$$

for $t > 0$. *Then*

$$\lim_{t\to+\infty} \alpha(t) = 0. \tag{40.44}$$

The proof is a lot like our estimate in I_n in (40.13). Set

$$A_1 = \{z \in B(0,1/2) \ ; \ \operatorname{dist}(z,G_t^*) \geqslant \delta\}. \tag{40.45}$$

Note that $\operatorname{dist}(A_1, G_t \cup G_t^*) \geqslant \delta/2$ for t large, and then

$$|\nabla v_t - \nabla v_t^*| \leqslant C\delta^{-1}\gamma(t) \text{ on } A_1, \tag{40.46}$$

by (40.17) and because $\nabla v_t - \nabla v_t^*$ is harmonic on $\mathbb{R}^2 \smallsetminus (G_t \cup G_t^*)$. Then

$$\int_{A_1} |\nabla v_t - \nabla v_t^*|^2 \leqslant C\delta^{-2}\gamma(t)^2 \tag{40.47}$$

for t large, and the left-hand side of (40.47) will tend to 0, no matter which choice of δ we make. Next consider $A_2 = B(0,\rho) \setminus (G_t \cup G_t^*)$ for $0 < \rho < 1/2$. Then

$$\int_{A_2} |\nabla v_t - \nabla v_t^*|^2 \leqslant C\rho, \tag{40.48}$$

by (13.5). Finally set

$$A_3 = B(0,1/2) \setminus (G_t \cup G_t^* \cup A_1 \cup A_2). \tag{40.49}$$

For each choice of ρ, there is a constant $C(\rho)$ such that for t large enough,

$$|\nabla v_t| + |\nabla v_t^*| \leqslant C(\rho) \text{ on } A_3. \tag{40.50}$$

The estimate for $|\nabla v_t^*|$ is very easy, because we have a formula for v_t^*, and Lemma 40.18 tells us that $x_t \in B(0,2/3)$ for t large.

The estimate for $|\nabla v_t|$ will be a consequence of Remark 40.26. First, we can find a small radius $r = r(\rho)$ such that for all $x \in G_t \cap B(0,2/3) \setminus B(0,\rho)$, $B(x,r)$ is a disk of regularity for G_t and

$$|\nabla v_t(z) - \nabla v_t(z')| \leqslant C_1(\rho) \text{ for } z, z' \in \Omega, \tag{40.51}$$

where Ω is any of the two components of $B(x,r) \setminus G_t$. The proof is the same as for (40.30).

For each choice of $B(x,r)$ and Ω as above, choose $z \in \Omega$ such that $\operatorname{dist}(z, G_t) \geqslant r/10$. Then

$$|\nabla v_t(z) - \nabla v_t^*(z)| \leqslant C r^{-1} \gamma(t) \leqslant 1 \tag{40.52}$$

for t large enough, by (40.17) and harmonicity. Hence $|\nabla v_t(z)| \leqslant C_2(\rho)$, and (4.51) yields

$$|\nabla v_t(z')| \leqslant C_3(\rho) \text{ for } z' \in \Omega. \tag{40.53}$$

This proves (40.50) for all points of A_3 that lie at distance $\leqslant r/2$ from G_t. For the rest of A_3, we can use (40.17) (and anyway we won't need them because δ will be chosen very small).

From (40.50) we deduce that

$$\int_{A_3} |\nabla v_t(z) - \nabla v_t^*(z)|^2 \leqslant C(\rho)^2 |A_3| \leqslant 2C(\rho)^2 \delta \tag{40.54}$$

(by the definitions (40.45) and (40.49) of A_1 and A_3).

For each small $\varepsilon > 0$, we can choose ρ so small that $C\rho < \varepsilon/3$ in (40.48), and then δ such that $2C(\rho)^2\delta < \varepsilon/3$ in (40.54), and since for t large enough the right-hand side of (40.47) is also smaller than $\varepsilon/3$, we get that $\alpha(t) < \varepsilon$ for t large enough. This proves (40.44) and Lemma 40.42. □

Note that

$$\lim_{t \to +\infty} \int_{B(0,1/2) \setminus G_t^*} |\nabla v_t^*|^2 = \frac{\lambda}{2} \tag{40.55}$$

by (39.9), the fact that $\beta^2 = 2\lambda/\pi$, and Lemma 40.18. We omit the easy computation. Then

$$\lim_{t\to+\infty} \frac{2}{t}\int_{B(0,t/2)\smallsetminus G} |\nabla v|^2 = 2\lim_{t\to+\infty}\int_{B(0,1/2)\smallsetminus G_t} |\nabla v_t|^2 = \lambda, \tag{40.56}$$

by (40.7) and Lemma 40.42. In particular, (39.7) holds.

We completed all the necessary verifications. Theorem 39.10 applies, and (v, G) is a generalized cracktip.

Proposition 40.5 follows. □

Proof of Theorem 1.16. — Let (v, G) be as in Theorem 1.16. Thus (v, G) is a global λ-minimizer (with $\lambda = 1$, but this does not really matter) and there is a connected component G_{00} of G such that $G \smallsetminus G_{00}$ is bounded. We want to show that (v, G) is one of the easy solutions (where G is a line or a propeller and v is locally constant) or a generalized cracktip.

Lemma 40.57. — *If (u, K) is a blow-in of (v, G) (as in* (40.2) *and* (40.3)*), then K is connected.*

Let $y \in G$ and $\{t_n\}$ be such that (40.2) and (40.3) hold. We may assume that $y = 0$, because otherwise we could translate G and v. Then

$$K = \lim_{n\to+\infty} G_n = \lim_{n\to+\infty} \frac{1}{t_n} G. \tag{40.58}$$

We want to show that for each $x \in K \smallsetminus \{0\}$, there is a curve in K that goes from 0 to x. Fix $x \in K \smallsetminus \{0\}$ and, for each n, denote by x_n the point of G_n that is closest to x. Thus

$$x = \lim_{n\to+\infty} x_n \tag{40.59}$$

by (40.58), $|t_n x_n|$ tends to $+\infty$ by (40.2), and hence $t_n x_n \in G_{00}$ for n large enough (because $G \smallsetminus G_{00}$ is bounded).

Fix an origin $\widetilde{x}_0 \in G_{00}$, and apply Lemma 19.14 to G_{00}, $\widetilde{x}_0$, and $t_n x_n$. We get a simple rectifiable curve $\widetilde{\Gamma}_n \subset G_{00}$ that goes from $\widetilde{x}_0$ to $t_n x_n$. Set $\Gamma_n = t_n^{-1}\widetilde{\Gamma}_n$ and $\ell_n = H^1(\Gamma_n)$. We claim that

$$\{\ell_n\} \text{ is bounded.} \tag{40.60}$$

Suppose not. Then we can extract a subsequence from $\{t_n\}$ so that ℓ_n tends to $+\infty$. Set

$$\delta_n = \sup\{|z| \ ; \ z \in \Gamma_n\}. \tag{40.61}$$

Then δ_n tends to $+\infty$ as well, because $\operatorname{diam}\gamma_n \geqslant C^{-1}\ell_n$, because $\widetilde{\Gamma}_n \subset G_{00}$ and by (13.4). Set

$$\Gamma_n^* = \Gamma_n \cup \left[t_n^{-1}\widetilde{x}_0, x_n\right], \tag{40.62}$$

and then choose a point z_n such that

$$z_n \text{ lies in a bounded component of } \mathbb{R}^2 \setminus \Gamma_n^*, \tag{40.63}$$

and

$$|z_n| \geqslant \delta_n - 1. \tag{40.64}$$

Such a point is easy to find. For instance we can choose a regular point ξ_n of Γ_n very far from the origin, and try two points $z_n^{\pm}$ in $\mathbb{R}^2 \setminus \Gamma_n$, very close to ξ_n, and on different sides of Γ_n (locally). The points $z_n^{\pm}$ cannot both lie in the unbounded component of $\mathbb{R}^2 \setminus \Gamma_n^*$, for instance because the winding numbers of the closed curve Γ_n^* around these points are different. We can even choose z_n out of G_n.

Note that (v_n, G_n) is also a global λ-minimizer, and so we can apply Lemma 20.1 to it. We get an escape path $\gamma_n = \gamma_{z_n}$, which is even defined on $[0, +\infty)$ because of Remark 20.5.

From (20.4) it is clear that $\gamma_n(s)$ eventually leaves the (bounded) component of z_n in $\mathbb{R}^2 \setminus \Gamma_n^*$. Set

$$s_n = \inf \{s \geqslant 0 \; ; \; \gamma_n(s) \in \Gamma_n^*\} \tag{40.65}$$

and $y_n = \gamma_n(s_n)$. Then

$$y_n \in \Gamma_n^* \setminus \Gamma_n \subset \left[t_n^{-1}\widetilde{x}_0, x_n\right], \tag{40.66}$$

because the image of γ_n does not meet $\Gamma_n \subset G_n$ (by (20.4)). Then $|y_n - z_n| \geqslant \delta_n/2$ for n large, by (40.64) and because $\{x_n\}$ is bounded (by (40.59)). Hence $s_n \geqslant C^{-1}\delta_n$ (by (20.3)), and $\operatorname{dist}(y_n, G_n) \geqslant C^{-1}\delta_n$ (by (20.4)). When δ_n is too large, this is incompatible with (40.66) and the fact that $x_n \in G_n$. Our claim (40.60) follows from this contradiction.

Because of (40.60), we can find parameterizations f_n of the curves Γ_n that are defined on $[0, 1]$ and Lipschitz with uniform bounds. Then we can extract a subsequence of $\{f_n\}$ that converges uniformly on $[0, 1]$ to some limit f. It is clear that $f([0, 1])$ is an arc in K that contains 0 and x. Lemma 40.57 follows. □

Let us now apply the main result in [Bo]. We get that all blow-ins of (v, G) are generalized cracktips or trivial minimizers associated to lines or propellers as in (1.10) and (1.11). [The case of the empty set is not possible by our definition of blow-ins and because we implicitly assumed that G is not empty.]

If all the blow-ins of (v, G) are generalized cracktips, we can apply Proposition 40.5 and conclude. So we are left with the easier case when for at least one blow-in (u, K) of (v, G), K is a line or a propeller. In this case Lemma 18.26 tells us that G is a line or a propeller, and v is locally constant on each component of $\mathbb{R}^2 \setminus G$.

This completes our proof of Theorem 1.16. □

Proof of Theorem 40.4. — Let (v, G) be as in the statement, and let $y \in G$ and $\{t_n\}$ be such that (40.2) and (40.3) hold and (u, K) is a generalized cracktip. We may assume that $y = 0$, since otherwise we can translate v and G.

For $t > 0$, define G_t, v_t, and $\gamma(t)$ as in (40.6)-(40.9). Thus $(v_n, G_n) = (v_{t_n}, G_{t_n})$ with these new notations. Also, (40.11) holds with the same proof as above.

If $\lim_{t\to+\infty} \gamma(t) = 0$, then all blow-ins of (v, G) are generalized cracktips or minimizers associated to lines, and we can conclude as in the proof of Theorem 1.16. We can also follow quietly the proof of Proposition 40.5 and see that it works.

So we may assume that $\gamma(t)$ does not tend to 0. Let $\tau > 0$ be very small (to be chosen later), and set

$$t_n^* = \sup\{t \in [0, t_n] \ ; \ \gamma(t) \geqslant \tau\}. \tag{40.67}$$

Then t_n is well defined for n large (because $\gamma(t)$ does not tend to 0, and if τ is small enough), and even

$$\lim_{n\to+\infty} t_n^* = +\infty \tag{40.68}$$

(because t_n tends to $+\infty$). We want to use $\{t_n^*\}$ to construct an unlikely blow-in of (v, G). Modulo extracting a subsequence, we can assume that

$$\left\{\left(v_{t_n^*}, G_{t_n^*}\right)\right\} \text{ converges to some limit } (u^*, K^*). \tag{40.69}$$

Denote by γ^* the analogue of the function γ, but for the pair (u^*, K^*). That is, set

$$\gamma^*(s) = \inf_{(v^*, G^*)\in H_\lambda} \left\{ d\left(K_s^*, G^*\right) + \int_{B(0,1)\setminus(K_s^*\cup G^*)} |\nabla u_s^* - \nabla v^*| \right\}, \tag{40.70}$$

with the same conventions as in (40.6)-(40.9).

Lemma 40.71. — *We have that*

$$\gamma^*(s) \leqslant \tau \text{ for } s \geqslant 1. \tag{40.72}$$

Let us first note that

$$\gamma\left(t^*\right) \leqslant \left(\frac{t}{t^*}\right)^{3/2} \gamma(t) \text{ for } 0 < t^* \leqslant t < +\infty. \tag{40.73}$$

This is easy to check. If $(v^*, G^*) \in H_\lambda$ is a competitor in the definition (40.8) of $\gamma(t)$, then $\left(v_{t^*/t}^*, G_{t^*/t}^*\right)$ (with the same notations as in (40.6) and (40.7)) is a competitor in the definition of $\gamma(t^*)$, the distance in (40.9) is at most multiplied by t/t^* (because of (40.6)), and similarly the integral is multiplied by less than $(t/t^*)^{3/2}$. The precise power will not matter anyway.

By (40.73) (with t and t^* exchanged) and (40.67), $\gamma\left(t_n^*\right) \geqslant \tau$. Since $\gamma\left(t_n\right)$ tends to 0 and $\gamma\left(t_n^*\right) \geqslant \tau$ does not tend to 0, (40.73) also implies that

$$\lim_{n\to+\infty} \frac{t_n}{t_n^*} = +\infty. \tag{40.74}$$

Next

$$\gamma(st_n^*) < \tau \text{ for } 1 < s \leqslant \frac{t_n}{t_n^*}, \tag{40.75}$$

by (40.67). For each $s > 1$,

$$\{(v_{st_n^*}, G_{st_n^*})\} \text{ converges to } (u_s^*, K_s^*), \tag{40.76}$$

by (40.69). Then

$$\gamma^*(s) \leqslant \liminf_{n\to+\infty} \gamma(st_n^*) \leqslant \tau, \tag{40.77}$$

by the compactness of our set H_λ of generalized cracktips, the obvious continuity of $d(G_t, G^*)$ with respect to G_t, Fatou, (40.74), and (40.75). This proves the lemma. □

For each $s > 1$, choose $(\widetilde{u}_s, \widetilde{K}_s) \in H_\lambda$ such that

$$d\left(K_s^*, \widetilde{K}_s\right) + \int_{B(0,1)\setminus\left(K_s^*\cup\widetilde{K}_s\right)} |\nabla u_s^* - \nabla\widetilde{u}_s| \leqslant 2\gamma^*(s). \tag{40.78}$$

Let us first assume that for s large enough,

$$\text{the endpoint } \widetilde{x}_s \text{ of } \widetilde{K}_s \text{ lies in } B(0, 1/10). \tag{40.79}$$

If our constant τ is chosen small enough, then for s large

$$K_s^* \cap B(0,1) \setminus B(0,1/4) \text{ is a } C^1 \text{ curve.} \tag{40.80}$$

The proof is the same as for (40.21) with $\delta = 1/4$.

From all this we easily deduce that

$$K^* \setminus B(0, s_0) \text{ is connected for some } s_0 > 0. \tag{40.81}$$

Then we can apply Theorem 1.16, and we get that

$$(u^*, K^*) \text{ is a generalized cracktip or } K^* \text{ is a line or a propeller.} \tag{40.82}$$

Let us try to prove (40.82) also when (40.79) does not hold for s large. First note that if $\widetilde{x}_s \in B(0, 9/10)$ for some $s > 1$, then $\widetilde{x}_{s'} \in B(0, 1/10)$ for $10s \leqslant s' \leqslant 100s$. This is because (40.76) and (40.77) say that in the unit disk, $\widetilde{K}_s$ is very close to K_s^* and $\widetilde{K}_{s'}$ is very close to $K_{s'}^* = \frac{s}{s'} K_s^*$. Of course we need τ to be small enough here.

Thus if (40.79) does not hold for all $s \geqslant 10$,

$$\widetilde{x}_s \in \mathbb{R}^2 \setminus B(0, 9/10) \text{ for } s > 1. \tag{40.83}$$

In this case also we can apply the argument of Lemma 40.20 (but this time in $B(0, 2/3)$, say, to stay sufficiently far from the points $\widetilde{x}_s$), and we get that for all $s > 1$,

$$K_s^* \cap B(0, 1/2) \text{ is a } C^1 \text{ curve.} \tag{40.84}$$

Then K^* is connected, and we can deduce (40.82) from [Bo].

Thus we proved that (40.82) holds in both cases. Recall that (u^*, K^*) is a blow-in of (v, G). So if K^* is a line or a propeller, G also is a line or a propeller. See the end of the proof of Theorem 1.16 (starting a little below (40.66)).

We are left with the case when (u^*, K^*) is a generalized cracktip. Then

$$\lim_{n\to+\infty} \gamma(t_n^*) = 0, \tag{40.85}$$

by (40.69) and (40.11). On the other hand, $\gamma(t_n^*) \geqslant \tau$ (see a little below (40.73)). This contradiction completes our proof of Theorem 40.4. □

41. Cracktips are global minimizers

In this section we complete our proof of (1.15).

In the first sections we have assumed that (1.15) fails, and then constructed a modified functional (see Section 11) and a minimizer (v, G) of that functional. The next stage is the following.

Lemma 41.1. — *(v, G) is a generalized cracktip.*

We already know from Proposition 11.5 that (v, G) is a minimizer of the modified functional. It is reduced by construction, but otherwise we could always replace it with a reduced minimizer. We want to apply Theorem 39.11, so let us check the hypotheses.

We start with (32.3)-(32.5). Recall from (10.24) that v is the limit of some sequence v_{R_m}. [The pairs (v_R, G_R) were themselves obtained as minimizers of some other functionals J_R.] The convergence of $\{v_{R_m}\}$ is uniform on every compact subset of $\mathbb{R}^2 \setminus G$ and, since all our functions are harmonic, ∇v_{R_m} converges to ∇v at least pointwise. Then Lemma 10.14 implies that

$$|\nabla v(x) - \nabla u_0(x)| \leqslant C\,|x|^{-1} \tag{41.2}$$

for all $x \in \mathbb{R}^2 \setminus L$ such that $|x| \geqslant 2R_0$. Here u_0 is the reference cracktip function defined by (2.2), and R_0 is some constant. [It shows up in Proposition 5.1 for the first time, but this does not matter here.]

We want to use (41.2) to study the variations of w (the conjugate function) on ∂B_R for R large. Denote by w_0 the function conjugated to u_0 and normalized by $w_0(x) = 0$ on $(-\infty, 0]$. Then

$$w_0(r\cos\theta, r\sin\theta) = -\sqrt{2/\pi}\, r^{1/2} \cos(\theta/2) \tag{41.3}$$

for $r > 0$ and $-\pi \leqslant \theta \leqslant \pi$.

From (41.2) we deduce that

$$|w(x) - w_0(x)| \leqslant C \text{ for } |x| \geqslant 2R_0 \tag{41.4}$$

(because w and w_0 coincide on L). In particular,

$$w(R\cos\theta, R\sin\theta) \leqslant -\frac{R^{-1/2}}{10} \text{ for } |\theta| \leqslant \frac{3\pi}{4} \tag{41.5}$$

if R is large enough.

Another consequence of (41.2) is that w is strictly monotone on each of the two arcs of $\partial B(0,R)$ where $3\pi/4 \leqslant \theta \leqslant \pi$ and $-\pi \leqslant \theta \leqslant -3\pi/4$. Therefore

$$\Gamma_m = \{z\ ;\ w(z) = m\} \text{ does not meet } \partial B(0,R) \text{ when } m > 0, \tag{41.6}$$

$$\Gamma_0 \cap \partial B(0,R) = \{-R\}, \tag{41.7}$$

and

$$\Gamma_m \cap \partial B(0,R) \text{ has exactly two points when } -\frac{R^{1/2}}{10} < m < 0. \tag{41.8}$$

In particular, (32.3)-(32.5) hold.

The last condition (39.9) holds, by (10.28) and because $\lambda = h'\left(H^1(G \setminus L)\right) \geqslant 1$ by (3.3) and (3.5). So we can apply Theorem 39.11, and (v,G) is a generalized cracktip. This proves the lemma. □

Note that the constant β in the representation of v by (39.9) must be $\sqrt{2/\pi}$, for instance because of (41.2). Thus (v,G) is a translation of our reference cracktip (u_0,K_0).

Suppose that $G \neq L$, i.e., that $G = (-\infty, x_0]$ for some $x_0 > -1$. Note that (u_0,K_0) is a blow-up of (v,G) at x_0: it is the limit of the sequence (v_n,G_n) defined by (12.7) and (12.8) with $y_n = x_0$ and $t_n = 2^{-n}$, say. Then Proposition 12.12 says that (u_0,K_0) is a global λ-minimizer, and hence $\lambda = 1$ (by Remark 35.42). Here we are working under the assumption that (u_0,K_0) is not a global minimizer; thus the current case when $G \neq L$ is impossible.

We are left with the case when $G = L$, and unfortunately some energy estimates will be needed to show that this case is impossible as well. The general principle is easy: if we finally arrived to the minimizer $G = L$, then L itself should have been a significantly better competitor than $K_0 = (-\infty, 0]$ for our local functionals J_R, which is not the case.

Recall that our pair (v,G) was obtained as the limit of minimizers (v_{R_m}, G_{R_m}) of the functionals J_{R_m}, for some sequence $\{R_m\}$ that tends to $+\infty$. [See a little under (10.21) and (10.24).] Set

$$E(m) = \int_{B(0,R_m)\setminus G_{R_m}} |\nabla v_{R_m}|^2, \tag{41.9}$$

and also call $D_m = B(0,R_m)$ and $\partial_m = \partial B(0,R_m) \setminus L$. We want to replace v_{R_m} with the simpler function u_m which is defined and continuous on $\overline{D}_m \setminus L$, coincides with u_0 and v_{R_m} on ∂_m, is harmonic on $D_m \setminus L$, and for which

$$E'(m) = \int_{D_m \setminus L} |\nabla u_m|^2 \tag{41.10}$$

is minimal. Note that $E(m) \leqslant E'(m)$, because v_{R_m} minimizes $E(m)$ among continuous functions on $\overline{D}_m \setminus G_{R_m}$ that coincide with u_0 on ∂_m, and u_m is such a function.

Lemma 41.11. — $\lim_{m\to+\infty} (E'(m) - E(m)) = 0$.

We want to use v_{R_m} to construct a fairly good competitor for $E'(m)$. This will be easier after a conformal mapping. Set $\varphi(z) = (z+1)^{1/2}$ for $z \in \mathbb{C} \setminus L$, where we choose the obvious determination of the square root, with values in the half-plane $P^+ = \{z : \Re z > 0\}$. Set $B = B\left(0, (R_0+1)^{1/2}\right)$, where R_0 is the constant of Proposition 5.1. Then

$$\varphi(G_{R_m} \setminus L) \subset B, \tag{41.12}$$

by (5.2). Also,

$$\varepsilon_m = \sup\{\Re z \ ; \ z \in \varphi(G_{R_m} \setminus L)\} \tag{41.13}$$

tends to 0, because G_{R_m} tends to $G = L$.

Denote by H_m the isosceles trapezoid with vertices $\pm 2(R_0+1)^{1/2} i$ and $\varepsilon_m \pm (R_0+1)^{1/2} i$. [See Figure 41.1.] Choose a diffeomorphism $\psi : 3B \cap P^+ \to 3B \cap P^+ \setminus \overline{H}_m$ such that $\psi(z) = z$ near $P^+ \cap \partial(3B)$ and

$$|D\psi(z) - \mathrm{Id}| \leqslant C\, \varepsilon_m \text{ on } 3B \cap P^+. \tag{41.14}$$

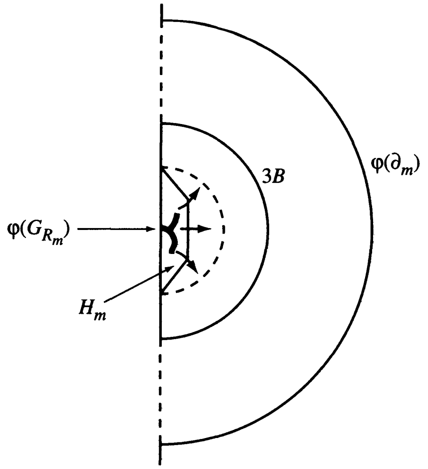


FIGURE 41.1. The arrows indicate the action of ψ.

Set $\widetilde{v} = v_{R_m} \circ \varphi^{-1}$ on $\varphi\left(\overline{D}_m \setminus G_{R_m}\right)$, and then $\widetilde{u} = \widetilde{v} \circ \psi$ on $\varphi\left(\overline{D}_m \setminus L\right)$. Finally set $u = \widetilde{u} \circ \varphi$ on $\overline{D}_m \setminus L$. Then

$$\int_{\varphi(D_m \setminus G_{R_m})} |\nabla \widetilde{v}|^2 = E(m), \tag{41.15}$$

by (41.9) and the conformal invariance of energy integrals. The composition with ψ does not change anything out of $3B$, and multiplies the energy in $3B$ by at most $1 + C\varepsilon_m$, by (41.14). Thus

$$\int_{\varphi(D_m \setminus L)} |\nabla \widetilde{u}|^2 \leqslant E(m) + C\,\varepsilon_m \int_{3B \cap P^+ \setminus H_m} |\nabla \widetilde{v}|^2 . \tag{41.16}$$

Also

$$\int_{3B \cap P^+ \setminus H_m} |\nabla \widetilde{v}|^2 = \int_{\varphi^{-1}(3B \cap P^+ \setminus H_m)} |\nabla v_{R_m}|^2 \leqslant C, \tag{41.17}$$

by conformal invariance of energies, because $\varphi^{-1}\left(3B \cap P^+\right) = B\left(-1, 9\left(R_0 + 1\right)\right)$, and by Lemma 4.12. We do not care if C is very large, as long as it does not depend on m. Finally,

$$\int_{D_m \setminus L} |\nabla u|^2 = \int_{\varphi(D_m \setminus L)} |\nabla \widetilde{u}|^2 \leqslant E(m) + C\,\varepsilon_m. \tag{41.18}$$

Now u is an allowed competitor in the definition (41.10) of $E'(m)$, and so

$$E'(m) \leqslant \int_{D_m \setminus L} |\nabla u|^2 \leqslant E(m) + C\,\varepsilon_m. \tag{41.19}$$

Lemma 41.11 follows, because ε_m tends to 0. □

Next we want to compare $E'(m)$ with

$$E_0(m) = \int_{D_m \setminus (-\infty, 0]} |\nabla u_0|^2 , \tag{41.20}$$

where u_0 still denotes our reference cracktip function. Let us use Green's theorem on $\Omega = D_m \setminus (-\infty, 0]$ (as we did for (4.32)). We get that

$$\begin{aligned} E'(m) - E_0(m) &= \int_\Omega \left\{|\nabla u_m|^2 - |\nabla u_0|^2\right\} = \int_\Omega \nabla\left(u_m - u_0\right) \cdot \nabla\left(u_m + u_0\right) \\ &= -\int_\Omega \left(u_m - u_0\right) \Delta\left(u_m + u_0\right) + \int_{\partial\Omega} \left(u_m - u_0\right) \frac{\partial\left(u_m + u_0\right)}{\partial n}. \end{aligned} \tag{41.21}$$

Since $\Delta\left(u_m + u_0\right) = 0$ on Ω, $u_m - u_0 = 0$ on ∂_m, and $\partial\left(u_m + u_0\right)/\partial n = 0$ on $[-R_m, -1]$, we are only left with the double contribution of $[-1, 0]$ in the boundary integral. On $[-1, 0]$, $\partial u_0/\partial n = 0$, and $\partial u_m/\partial n$ shows up twice, with opposite values. Thus

$$\begin{aligned} E'(m) - E_0(m) &= \int_{-1}^0 \mathrm{Jump}\left(u_m - u_0\right) \frac{\partial u_m}{\partial y} = \int_{-1}^0 \mathrm{Jump}\left(u_0\right) \frac{\partial u_m}{\partial y} \\ &= 2\sqrt{2/\pi} \int_{-1}^0 |x|^{1/2} \frac{\partial u_m(x)}{\partial y} dx, \end{aligned} \tag{41.22}$$

by (2.2). Set $\widetilde{u}_m(z) = u_0(z+1)$. We shall see soon that $\widetilde{u}_m$ is a good approximation to u_m. At any rate,

$$(41.23)\qquad \int_{-1}^{0} |x|^{1/2} \frac{\partial \widetilde{u}_m}{\partial y}(x)dx = \frac{1}{2}\sqrt{2/\pi}\int_{-1}^{0} |x|^{1/2}(x+1)^{-1/2}\,dx.$$

Set $x = \sin^2 u - 1$, $0 < u < \pi/2$, so that $dx = 2\sin u\cos u$, $|x|^{1/2} = \cos u$, $(x+1)^{-1/2} = (\sin u)^{-1}$, and

$$(41.24)\qquad \int_{-1}^{0} |x|^{1/2}(x+1)^{-1/2}\,dx = \int_{0}^{\pi/2} 2\cos^2 u\; du = \frac{\pi}{2}.$$

Thus the main term in our computation of the right-hand side of (41.22) is 1.

The function $u_m - \widetilde{u}_m$ is continuous on $\overline{D}_m \setminus L$, harmonic on $D_m \setminus L$, and satisfies the Neumann condition $\partial(u_m - \widetilde{u}_m)/\partial n = 0$ on L. Also,

$$(41.25)\qquad |(u_m - \widetilde{u}_m)(z)| = |u_0(z) - u_0(z+1)| \leqslant C R_m^{-1/2}$$

on ∂_m, by definitions and a crude estimate. By the maximum principle for solutions of the Dirichlet-Neumann problem (i.e., because $u_m - \widetilde{u}_m$ minimizes $\int_{D_m \setminus L} |\nabla(u_m - \widetilde{u}_m)|^2$ for the given boundary data on ∂_m), (41.25) holds in $\overline{D}_m \setminus L$, and in particular on the circle $\partial B(-1, R_m - 1)$.

Define the auxiliary function f on $D = \{z \in \overline{B}(0, (R_m-1)^{1/2})\ ;\ \Re(z) > 0\}$ by

$$(41.26)\qquad f(z) = (u_m - \widetilde{u}_m)(z^2 - 1).$$

Then f is continuous on D, harmonic inside, and satisfies the usual Neumann condition on the imaginary axis. We can extend f to $\overline{B} = \overline{B}(0, (R_m-1)^{1/2})$ by symmetry, and we get a continuous function on $\overline{B}$ which is harmonic in B. [See the argument after (4.53) for more details.] Thus

$$(41.27)\qquad |\nabla f(z)| \leqslant C R_m^{-1} \text{ on } B(0,1),$$

by (41.25), and

$$(41.28)\qquad |\nabla(u_m - \widetilde{u}_m)(\xi)| \leqslant C\, R_m^{-1}(\xi+1)^{-1/2} \text{ on } B(-1,1) \setminus L,$$

by composing with $\xi \to (\xi+1)^{1/2}$. We can plug this into (41.22), and we get that

$$(41.29)\qquad \begin{aligned} |E'(m) - E_0(m) - 1| &= \left| 2\sqrt{2/\pi}\int_{-1}^{0} |x|^{1/2} \frac{\partial(u_m - \widetilde{u}_m)}{\partial y}(x)dx \right| \\ &\leqslant C\, R_m^{-1}\int_{-1}^{0} |x|^{1/2}(x+1)^{-1/2}\,dx \leqslant C\, R_m^{-1}. \end{aligned}$$

Altogether,

$$(41.30)\qquad \lim_{m\to+\infty} |E(m) - E_0(m) - 1| = 0,$$

because of Lemma 41.11. Since $J_{R_m}(u_0, K_0) = 1 + E_0(m)$ by (3.2), (2.1), (3.3), and (41.20), we get that

$$\Delta(R_m) := J_{R_m}(u_0, K_0) - J_{R_m}(v_{R_m}, G_{R_m}) \leqslant 1 + E_0(m) - E(m) \tag{41.31}$$

(by (3.2) and (41.9)), and hence

$$\limsup_{m \to +\infty} \Delta(R_m) \leqslant 0, \tag{41.32}$$

by (41.30).

On the other hand, $\Delta(R)$ is a nondecreasing function of R, by (3.2) and the fact that U_R in (3.1) has more and more elements (so that $(v_R, G_R) \in U_{R'}$ for $R' > R$, for instance). Our assumption that (u_0, K_0) is not a minimizer also implies that $\Delta(R) > 0$ for $R > 1$, as in (3.9) and (3.10). Thus $\limsup_{m \to +\infty} \Delta(R_m) \geqslant \Delta(1) > 0$, in contradiction with (41.32).

This final contradiction completes our proof of (1.15). □

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