AST

Appendix B. Trurston's hyperbolic Dehn filling theorem

Astérisque, tome 272 (2001), p. 179-196

<http://www.numdam.org/item?id=AST_2001__272__179_0>

APPENDIX B

THURSTON'S HYPERBOLIC DEHN FILLING THEOREM

We give a proof of Thurston's hyperbolic Dehn filling theorem for completeness. In the manifold case, the proof is given in Thurston's notes [**Thu1**], and it has been generalized to orbifolds by Dunbar and Meyerhoff [**DuM**].

We follow Thurston's proof [**Thu1**], taking care of the smoothness of the variety of representations. For the smoothness, we use an argument from [**Zh1, Zh2**]. There is another approach in [**PP**] without using these results in the manifold case.

We prove the theorem for manifolds in Section B.1, and for orbifolds in Section B.2. In Section B.3 we prove it for a special case of manifolds with totally geodesic boundary.

B.1. The manifold case

Let M be a compact 3-manifold with boundary $\partial M = T^2_1 \cup \cdots \cup T^2_k$ a non-empty union of tori, whose interior is complete hyperbolic with finite volume. Thurston's hyperbolic Dehn filling theorem provides a parametrization of a space of hyperbolic deformations of this structure on $\operatorname{int}(M)$. The parameters for these deformations are the generalized Dehn filling coefficients, which describe the metric completion of the ends of $\operatorname{int}(M)$.

For each boundary component T^2_j we fix two oriented simple closed curves μ_j and λ_j that generate $\pi_1(T^2_j)$. The completion of the structure on the j-th end of $\operatorname{int}(M)$ is described by the generalized Dehn filling coefficients $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\} = S^2$, so that the structure at the j-th end is complete iff $(p_j, q_j) = \infty$. The interpretation of the coefficients $(p_j, q_j) \in \mathbb{R}^2$ is the following:

- If $p_j, q_j \in \mathbb{Z}$ are coprime, then the completion at the j-th torus is a non-singular hyperbolic 3-manifold, which topologically is the Dehn filling with surgery meridian $p_j\mu_j + q_j\lambda_j$.
- When $p_j/q_j \in \mathbb{Q} \cup \{\infty\}$, let $m_j, n_j \in \mathbb{Z}$ be coprime integers such that $p_j/q_j = m_j/n_j$. The completion is a cone 3-manifold obtained by gluing a torus with singular core. The surgery meridian is $m_j\mu_j + n_j\lambda_j$ and the cone angle of the singular component is $2\pi|m_j/p_j|$.

- When $p_j/q_j \in \mathbb{R} - \mathbb{Q}$, then the completion (by equivalence classes of Cauchy sequences) is not topologically a manifold. These singularities are called of Dehn type, cf. [**Ho2**].

***Theorem B.1.1* (Thurston's hyperbolic Dehn filling [Thu1]).** — *There exists a neighborhood of $\{\infty, \ldots, \infty\}$ in $S^2 \times \cdots \times S^2$ such that the complete hyperbolic structure on* $\text{int}(M)$ *has a space of hyperbolic deformations parametrized by the generalized Dehn filling coefficients in this neighborhood.*

Proof. — The proof has three main steps. The first one is the construction of the algebraic deformation of the holonomy of the complete structure on $\text{int}(M)$. The second step is to associate generalized Dehn filling coefficients to this deformation and the third one is the construction of the developing maps with the given holonomies. These steps are treated in paragraphs B.1.1, B.1.2 and B.1.3 respectively.

B.1.1. Algebraic deformation of the holonomies. — We recall some notation. Let $R(M) = \text{Hom}(\pi_1(M), SL_2(\mathbb{C}))$ be the variety of representations of $\pi_1(M)$ into $SL_2(\mathbb{C})$, and $X(M) = R(M)//SL_2(\mathbb{C})$ its variety of characters. Both are affine algebraic complex varieties (not necessarily irreducible). For a representation $\rho \in R(M)$, its character χ_ρ is its projection to $X(M)$ and can be viewed as the map $\chi_\rho : \pi_1(M) \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \text{trace}(\rho(\gamma))$, for every $\gamma \in \Gamma$. Given an element $\gamma \in \pi_1(M)$ we will also consider the rational function

$$\begin{aligned} I_\gamma : X(M) &\longrightarrow \mathbb{C} \\ \chi_\rho &\longmapsto \chi_\rho(\gamma) = \text{trace}(\rho(\gamma)). \end{aligned}$$

Recall that $\mu_1, \ldots, \mu_k$ is a family of simple closed curves, one for each boundary component of M. We will consider the map

$$I_\mu = (I_{\mu_1}, \ldots, I_{\mu_k}) : X(M) \to \mathbb{C}^k.$$

Let $\rho_0 \in R(M)$ be a lift of the holonomy representation of $\text{int}(M)$ and let $\chi_0 \in X(M)$ denote its character. The main result we need about deformations is the following:

Theorem B.1.2. — *The map $I_\mu = (I_{\mu_1}, \ldots, I_{\mu_k}) : X(M) \to \mathbb{C}^k$ is locally bianalytic at χ_0.*

Proof. — We follow the proofs of [**Thu1**] and [**Zh1, Zh2**]. We prove first that I_μ is open at χ_0. Let $X_0(M)$ be any irreducible component of $X(M)$ that contains χ_0. In order to prove that I_μ is open we use the following two facts:

- By an estimate of Thurston [**Thu1**, Thm. 5.6], see also [**CS**, Thm. 3.2.1], $\dim X_0(M) \geq k$.
- The character χ_0 is an isolated point of $I_\mu^{-1}(I_\mu(\chi_0))$, by Mostow rigidity theorem.

By the openness principle [**Mum2**], it follows that I_μ is open at χ_0.

Moreover, I_μ is either locally bianalytic or a branched cover. Let $V_0 \subset X(M)$ and $V_1 \subset \mathbb{C}^k$ be respective neighborhoods of χ_0 and $I_\mu(\chi_0)$ such that the restriction $I_\mu|_{V_0} : V_0 \to V_1$ is either bianalytic or a branched cover. If $I_\mu|_{V_0}$ was a branched cover, then the ramification set would be a proper subvariety $W \subset V_0$ such that $\overline{I_\mu(W)}$ would be also a proper subvariety of V_1. In addition, if $I_\mu|_{V_0}$ was a branched cover, then the restriction of $I_\mu|_{V_0 - W}$ would be a cover of $V_1 - \overline{I_\mu(W)}$ of degree $d > 1$. Hence it suffices to show that $I_\mu|_{V_0}$ has only one preimage in a Zariski dense subset of $V_1 \subset \mathbb{C}^k$. This set is

$$S = \left\{ \left(\epsilon_1 2\cos\frac{\pi}{n_1}, \ldots, \epsilon_q 2\cos\frac{\pi}{n_q} \right) \mid n_i > N_0 \right\}$$

for some N_0 sufficiently large, where the coefficients $\epsilon_i = \pm 1$ are chosen so that $I_\mu(\chi_0) = (\epsilon_1 2, \ldots, \epsilon_q 2)$. We have that for $\chi \in X(M)$ in a neighborhood of χ_0, if $I_\mu(\chi) \in S$ then χ is the character of the holonomy of a hyperbolic orbifold, and therefore it is unique by Mostow rigidity.

Along this proof we have used twice that deformations of the holonomy imply deformations of the structure (every time we used Mostow rigidity). The techniques in Paragraph B.1.3 below apply to construct such deformations of structures. □

Remark B.1.3. — A stronger version of this theorem can be found in Kapovich's book, [**Kap**, Thm. 9.34 and Remark 9.41], where the dimension of certain cohomology groups with twisted coefficients are computed. These computations are an infinitesimal rigidity result, similar to rigidity results of Calabi-Weil [**Wei**], Raghunathan [**Rag**] and Garland [**Garl**], and they imply Theorem B.1.2 above.

B.1.2. Dehn filling coefficients. — In order to define the Dehn filling coefficients (p_j, q_j), we must introduce first the holomorphic parameters u_j and v_j. Following [**Thu1**], if we view the holonomy of μ_j and λ_j as affine transformations of $\mathbb{C} = \partial\mathbb{H}^3 - \{\infty\}$ (∞ being a point fixed by μ_j and λ_j), u_j and v_j are branches of the logarithm of the linear part of the holonomy of μ_j and λ_j respectively. For the definition we use branched coverings.

Definition B.1.4. — Let $U \subset \mathbb{C}^k$ be a neighborhood of the origin and $W \subset X(M)$ a neighborhood of χ_0. We define $\pi : U \to W$ to be the branched covering such that

$$I_{\mu_j}\pi(u) = \epsilon_j 2\cosh(u_j/2) \qquad \text{for every } u = (u_1, \ldots, u_k) \in U,$$

and $\epsilon_j \in \{\pm 1\}$ is chosen so that $I_{\mu_j}(\pi(0)) = \chi_0(\mu_j) = \operatorname{trace}(\rho_0(\mu_j)) = \epsilon_j 2$.

Remark B.1.5. — From this definition, $u = (u_1, \ldots, u_k)$ is just the parameter of a neighborhood of the origin $U \subset \mathbb{C}^k$ and its geometric interpretation comes from the branched covering $\pi : U \to V \subset X(M)$. We also remark that $\pi(u) = \chi_0$ iff $u = 0$.

Now we shall associate a representation to each $u \in U$ by considering an analytic section

$$s: V \subset X(M) \longrightarrow R(M)$$

such that $s(\chi_0) = \rho_0$. This section may be constructed easily by using [**CS**, Prop. 3.1.2] or [**Po1**, Prop. 3.2]. We use the notation:

$$\rho_u = s(\pi(u)) \in R(M) \qquad \text{for every } u \in U.$$

Lemma B.1.6. — *For $j = 1, \dots, k$, there is an analytic map $A_j : U \to SL_2(\mathbb{C})$ such that for every $u \in U$:*

$$\rho_u(\mu_j) = \epsilon_j A_j(u) \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix} A_j(u)^{-1} \qquad \textit{with } \epsilon_j = \pm 1.$$

Proof. — Let $\epsilon_j \in \{\pm 1\}$ be such that $I_{\mu_j}(\chi_0) = \chi_0(\mu_j) = \epsilon_j 2$. We fix a vector $w_2 = (w_2^1, w_2^2) \in \mathbb{C}^2$ that is not an eigenvector for ρ_0 and we set

$$w_1(u) = \big(w_1^1(u), w_1^2(u)\big) = \left(\epsilon_j \rho_u(\mu_j) - e^{-u_j/2}\right) w_2.$$

Since $\epsilon_j e^{\pm u_j/2}$ are the eigenvalues for $\rho_u(\mu_j)$, the following is the matrix of a change of basis that has the required properties for the lemma:

$$A_j(u) = \frac{1}{\sqrt{w_1^1(u) w_2^2 - w_1^2(u) w_2^1}} \begin{pmatrix} w_1^1(u) & w_2^1 \\ w_1^2(u) & w_2^2 \end{pmatrix}.$$

□

Lemma B.1.7. — *There exist unique analytic functions $v_j, \tau_j : U \to \mathbb{C}$ such that $v_j(0) = 0$ and for every $u \in U$:*

$$\rho_u(\lambda_j) = \pm A_j(u) \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix} A_j(u)^{-1}.$$

In addition:

i) $\tau_j(0) \in \mathbb{C} - \mathbb{R}$;
ii) $\sinh(v_j/2) = \tau_j \sinh(u_j/2)$;
iii) v_j *is odd in* u_j *and even in* u_l, *for* $l \neq j$;
iv) $v_j(u) = u_j(\tau_j(u) + O(|u|^2))$.

Proof. — The existence and uniqueness of v_j and τ_j, as well as point ii), follow straightforward from the commutativity between λ_j and μ_j. We remark that the uniqueness of v_j uses the hypothesis $v_j(0) = 0$, because this fixes the branch of the logarithm. To prove i) we recall that $\rho_0(\mu_j)$ and $\rho_0(\lambda_j)$ generate a rank two parabolic group, because ρ_0 is the holonomy of a complete structure. In particular 1 and $\tau_j(0)$ generate a lattice in $\mathbb{C}$ and therefore $\tau_j(0) \notin \mathbb{R}$.

To prove iii) we remark that the points $(\pm u_1, \pm u_2, \dots, \pm u_k)$ project to the same character in $X(M)$ independently of the signs $\pm$, hence:

$$v_j(\pm u_1, \pm u_2, \dots, \pm u_k) = \pm v_j(u_1, u_2, \dots, u_k).$$

This equality, combined with points i) and ii) imply that v_j is odd in u_j and even in u_l for $l \neq j$. Finally, iv) follows easily from the previous points. □

Definition B.1.8. — **[Thu1]** For $u \in U$ we define the *generalized Dehn filling coefficients* of the j-th cusp $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\} \cong S^2$ by the formula:

$$\begin{cases} (p_j, q_j) = \infty & \text{if } u_j = 0 \\ p_j u_j + q_j v_j = 2\pi\sqrt{-1} & \text{if } u_j \neq 0 \end{cases}$$

The equality $v_j = u_j(\tau_j(u) + O(|u|^2))$, with $\tau_j(0) \in \mathbb{C} - \mathbb{R}$, implies:

Proposition B.1.9. — *The generalized Dehn filling coefficients are well defined and*

$$\begin{aligned} U &\longrightarrow S^2 \times \cdots \times S^2 \\ u &\longmapsto (p_1, q_1), \ldots, (p_k, q_k) \end{aligned}$$

defines a homeomorphism between U and a neighborhood of $\{\infty, \ldots, \infty\}$. □

B.1.3. Deforming developing maps. — Let $D_0 : \widetilde{\operatorname{int} M} \to \mathbb{H}^3$ be the developing map for the complete structure on $\operatorname{int}(M)$, with holonomy ρ_0. The following proposition completes the proof of Theorem B.1.1.

Proposition B.1.10. — *For each $u \in U$ there is a developing map $D_u : \widetilde{\operatorname{int} M} \to \mathbb{H}^3$ with holonomy ρ_u, such that the completion of $\operatorname{int} M$ is given by the generalized Dehn filling coefficients of u.*

Proof. — We write $\operatorname{int}(M) = N \cup C_1 \cup \cdots \cup C_k$, where $N \cong M$ is a compact core of $\operatorname{int}(M)$, $C_j \cong T^2 \times [0, +\infty)$, $C_j \cap N \cong T^2 \times [0,1]$ and $C_j \cap C_l = \varnothing$ for $j \neq l$. We construct D_u separately for $\widetilde{N}$ and for $\widetilde{C}_j$, and then glue the pieces. We construct a family of maps $\{D_u\}_{u \in U}$ that will be continuous on u for the compact C^1-topology. This means that if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence in U converging to $u_\infty \in U$, then D_{u_n} converges to D_{u_∞} uniformly on compact subsets, and the tangent map of D_{u_n} also converges to D_{u_∞} uniformly on compact subsets.

Lemma B.1.11. — *There exists a family of local diffeomorphism $D_u^0 : \widetilde{N} \to \mathbb{H}^3$, which depends on $u \in U$ continuously for the compact C^1-topology, such that D_u^0 is ρ_u-equivariant and $D_0^0 = D_{0|\widetilde{N}}$.*

Proof. — This is a particular case of **[CEG**, Lemma 1.7.2], but we repeat their proof here because we will use the gluing technique. We fix $u \in U$ and we construct D_u^0 a family continuous on u for the compact C^1-topology.

We start with a finite covering $\{U_1, \ldots, U_n\}$ of a neighborhood of N. Let $p : \widetilde{\operatorname{int}(M)} \to M$ denote the universal covering projection and let V_1 be a connected component of $p^{-1}U_1 = \bigsqcup_{\gamma \in \pi_1} \gamma V_1$. We define $\Delta_1 : V_1 \to \mathbb{H}^3$ to be the restriction of D_0^0 and we extend it ρ_t-equivariantly to $p^{-1}U_1 = \bigsqcup_{\gamma \in \pi_1} \gamma V_1$.

We would like to define $\Delta_i : p^{-1}U_i \to \mathbb{H}^3$ in the same way and to construct D^0_u by gluing $\Delta_1, \dots, \Delta_n$, but we must be careful with the equivariance and the continuity on u for the compact $\mathcal{C}^1$-topology. The next step will be to try to extend Δ_1 to a map on $p^{-1}U_1 \cap p^{-1}U_2$. To be precise, we take $\{U'_1, \dots, U'_n\}$ a shrinking of $\{U_1, \dots, U_n\}$ that covers N and we will define $\Delta_2 : p^{-1}U'_1 \cap p^{-1}U'_2 \to \mathbb{H}^3$ as an extension of $\Delta_1|_{p^{-1}U'_1}$. Thus we define $\Delta_2|_{p^{-1}U'_1} = \Delta_1|_{p^{-1}U'_1}$ and we extend it to $p^{-1}U'_2$ as follows.

We choose V_2 a connected component of $p^{-1}U_2$ and V'_2 a connected component of the smaller neighborhood $p^{-1}U'_2$ which is contained in V_2. In particular

$$\overline{V'_2} \subset \operatorname{int}(V_2).$$

Let $\phi : \overline{V_2} \to [0,1]$ a $\mathcal{C}^\infty$-bump function such that:

- ϕ restricted to $V'_2 \cap p^{-1}U'_1$ is constant equal to 1,
- the closure of the support of ϕ is contained in $V_2 \cap p^{-1}U_1$.

By using ϕ, we define $f : V_2 \to \mathbb{H}^3$ as:

$$f = \phi\Delta_1 + (1-\phi)D^0_0.$$

This is, f equals Δ_1 on the intersection $V'_2 \cap p^{-1}U'_1$ and equals D^0_0 on $V_2 - p^{-1}U_1$. In addition f depends continuously on u for the compact $\mathcal{C}^1$-topology (we remark that ϕ is independent of u). We define $\Delta_2|_{V'_2} = f|_{V'_2}$ and we extend it ρ_u-equivariantly to $p^{-1}U'_2 = \bigsqcup_{\gamma\in\pi_1} \gamma V'_2$. In this way Δ_2 is ρ_u-equivariant and the construction depends continuously on u for the compact $\mathcal{C}^1$-topology.

Now we can continue by an inductive process and make successive shrinking to get D^0_u defined on a neighborhood of $\widetilde{N}$. Finally, since D^0_0 is a local diffeomorphism and N is compact, the compact $\mathcal{C}^1$-topology implies that D^0_u is a local diffeomorphism for u close to 0. □

Lemma B.1.12. — *There exists a family of local embeddings $D^j_u : \widetilde{C}_j \to \mathbb{H}^3$ which is continuous on $u \in U$ for the compact $\mathcal{C}^1$-topology, such that D^j_u is ρ_u-equivariant, $D^j_0 = D_{0|\widetilde{C}_j}$ and the structure on C_j can be completed as described by the generalized Dehn filling parameters.*

Before proving this lemma, we prove the following one, that concludes the proof of Proposition B.1.10.

Lemma B.1.13. — *There exists a family of local embeddings $D_u : \widetilde{\operatorname{int}(M)} \to \mathbb{H}^3$ which depends continuously on $u \in U$ for the compact $\mathcal{C}^1$-topology, such that D_u is ρ_u-equivariant and D_0 is the developing map of the complete structure on* int(M). *In addition, away from a compact set it coincides with the maps $D^1_u, \dots D^k_u$ of Lemma B.1.12.*

Proof of Lemma B.1.13. — We already have D^0_u and D^j_u, defined on the respective universal coverings of N and C_j. We want to glue these maps by using bump functions again. Recall that $C_j \cong T^2 \times [0,+\infty)$ and $N \cap C_j \cong T^2 \times [0,1]$. Thus it suffices to work with a partition of the unit, subordinate to the covering $\{[0,3/4),(1/4,1]\}$ of the interval $[0,1]$, to glue the maps on the universal coverings. □

Proof of Lemma B.1.12. — The universal covering $\widetilde{C}_j$ is homeomorphic to $\mathbb{R}^2 \times [0,+\infty)$. We suppose that the action of the fundamental group is given by:

$$\begin{array}{rcl}\mu_j : \mathbb{R}^2 \times [0,+\infty) & \longrightarrow & \mathbb{R}^2 \times [0,+\infty) \\ (x,y,t) & \longmapsto & (x+1,y,t)\end{array} \quad \text{and} \quad \begin{array}{rcl}\lambda_j : \mathbb{R}^2 \times [0,+\infty) & \longrightarrow & \mathbb{R}^2 \times [0,+\infty) \\ (x,y,t) & \longmapsto & (x,y+1,t)\end{array}.$$

By Lemmas B.1.6 and B.1.7, we may assume that

$$\rho_u(\mu_j) = \pm \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix} \qquad \rho_u(\lambda_j) = \pm \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix}.$$

Since the cusp is complete for the initial hyperbolic structure when $u = 0$, we also assume that the restriction of the developing map $D_0^j = D_{0|\widetilde{C}_j}$ is:

$$\begin{array}{rcl} D_0^j : \mathbb{R}^2 \times [0,+\infty) & \longrightarrow & \mathbb{H}^3 \cong \mathbb{C} \times (0,+\infty) \\ (x,y,t) & \longmapsto & (x + \tau_j(0)y, e^t) \end{array}$$

Here we use the half space model $\mathbb{H}^3 \cong \mathbb{C} \times (0,+\infty)$ for the hyperbolic space.

We consider the family of maps $D_u^j : \mathbb{R}^2 \times [0,+\infty) \to \mathbb{H}^3$ defined by

$$D_u^j(x,y,t) = \begin{cases} \left(\dfrac{e^{u_j x + v_j(u) y} - 1}{e^{u_j/2} - e^{-u_j/2}}, e^{t + \mathrm{Re}(u_j x + v_j(u) y)} \right) & \text{if } u_j \neq 0; \\ (x + \tau_j(u) y, e^t) & \text{if } u_j = 0. \end{cases}$$

The map D_u^j is ρ_u-equivariant and it is also a local diffeomorphism. Since $v_j(u) = u_j(\tau_j(u) + O(|u|^2))$, D_u^j varies continuously on $u \in U$ for the compact C^1-topology.

The following claim finishes the proof of the lemma.

Claim B.1.14. — *The hyperbolic structure on C_j induced by D_u^j is complete iff $u_j = 0$. If $u_j \neq 0$ then the metric completion of C_j is the completion described by the Dehn filling parameters.*

Proof. — When $u_j = 0$, the structure is complete since it is the quotient of a horoball by a rank two parabolic group (cf. **[BP]** or **[Rat]**).

We assume that $u_j \neq 0$. For every $t \in [0,+\infty)$, $D_u^j(\mathbb{R}^2 \times \{t\})$ is the set of points that are at distance $d(t)$ from the geodesic γ having end-point

$$\overline{\gamma} \cap \partial \mathbb{H}^3 = \left\{ \frac{-2}{\sinh(u_j/2)}, +\infty \right\}.$$

The distance $d(t)$ satisfies

$$\sinh(d(t))\, 2\, |\sinh(u_j/2)| e^t = 1.$$

In particular, for a fixed $u \in U$, $d(t) \to 0$ when $t \to +\infty$, and

$$D_u^j\left(\mathbb{R}^2 \times [t,+\infty)\right) = \mathcal{N}_{d(t)}(\gamma) - \gamma,$$

where $\mathcal{N}_{d(t)}(\gamma)$ is the tubular neighborhood of γ of radius $d(t)$.

We deal first with the case where (p_j, q_j) is a pair of coprime integers. Let $r_j, s_j \in \mathbb{Z}$ be such that $p_j s_j - q_j r_j = 1$. Consider the linear isomorphism

$$\begin{aligned}\Phi : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2\\ (a,b) &\longmapsto (p_j a + r_j b, q_j a + s_j b)\end{aligned}$$

The equality $p_j u_j + v_j q_j = 2\pi\sqrt{-1}$ implies that

$$D_u^j(\Phi(a,b),t) = \left(\frac{e^{a2\pi\sqrt{-1}+bl_j} - 1}{e^{u_j/2} - e^{-u_j/2}}, e^{t+b\,\mathrm{Re}(l_j)}\right)$$

where $l_j = r_j u_j + s_j v_j$. An easy computation shows that

$$\mathrm{Re}(l_j) = \mathrm{Im}(u_j \overline{v}_j)/(2\pi),$$

which is non-zero, because $v_j(u) = u_j(\tau_j(u) + O(|u|^2))$ and $\mathrm{Im}(\tau_j(0)) \neq 0$.

It follows that for every $t > 0$, $D_u^j : \mathbb{R}^2 \times [t, +\infty) \to \mathbb{H}^3$ factorizes to a homeomorphism

$$(\mathbb{R}^2 \times [t,+\infty))\big/\langle p_j\mu_j + q_j\lambda_j\rangle \cong \mathcal{N}_{d(t)}(\gamma) - \gamma,$$

where $\langle p_j\mu_j + q_j\lambda_j\rangle$ denotes the cyclic group generated by $p_j\mu_j + q_j\lambda_j$. In addition, the holonomy of $r_j\mu_j + s_j\lambda_j$ preserves γ and acts on γ as a translation of length $\mathrm{Re}(l_j) \neq 0$. It follows that the completion of C_j is obtained by adding the quotient of γ by this translation, and topologically this is the Dehn filling with meridian $p_j\mu_j + q_j\lambda_j$.

Next we study the case where $p_j/q_j \in \mathbb{Q} \cup \{\infty\}$. Let (m_j, n_j) be a pair of coprime integers such that $m_j/n_j = p_j/q_j$ and set $\alpha_j = 2\pi m_j/p_j$. Consider the singular space $\mathbb{H}^3_{\alpha_j}$ defined in the first chapter. The developing map $D_u^j : \widetilde{C}_j \to \mathbb{H}^3 - \gamma$ induces a developing map $D'_u : \widetilde{C}_j \to \mathbb{H}^3_{\alpha_j} - \Sigma$, because the universal coverings of $\mathbb{H}^3 - \gamma$ and of $\mathbb{H}^3_{\alpha_j} - \Sigma$ are isometric. Now the discussion in the precedent case applies, and we conclude that the completion of C_j consist of adding a singular geodesic, with cone angle α_j, and the topological filling meridian is $m_j\mu_j + n_j\lambda_j$.

In the last case, where $p_j/q_j \in \mathbb{R} - \mathbb{Q}$, the holonomy of C_j acts faithfully on the geodesic γ. Since $\pi_1(C_j) \cong \mathbb{Z} \oplus \mathbb{Z}$, this action is non discrete. It follows easily that the completion cannot be Hausdorff.

This finishes the proof of the claim, of Proposition B.1.10 and of Theorem B.1.1. □

The proof yields not only the existence of a one parameter family of cone 3-manifold structures but also gives a path of corresponding holonomies in the variety $R(M)$ of representations of $\pi_1(M)$ into $SL_2(\mathbb{C})$. A corollary of the proof of Thurston's hyperbolic Dehn filling theorem is the following:

Corollary B.1.15. — *For any real numbers $\alpha_1, \dots, \alpha_k \geq 0$ there exist $\varepsilon > 0$ and a path $\gamma : [0, \varepsilon) \to R(M)$, such that, for every $t \in [0, \varepsilon)$, $\gamma(t)$ is a lift of the holonomy of a hyperbolic structure on M corresponding to the generalized Dehn filling coefficients*

$$((p_1, q_1), \dots, (p_k, q_k)) = ((2\pi/(\alpha_1 t), 0), \dots, (2\pi/(\alpha_k t), 0)).$$

When $\alpha_j t = 0$, the structure at the j-th cusp is complete; otherwise its completion is a cone 3-manifold obtained by adding to T_j^2 a solid torus with meridian curve μ_j and singular core with cone angle $\alpha_j t$.

B.2. The orbifold case

B.2.1. Dehn filling on orbifolds. — Let $\mathcal{O}$ be a compact 3-orbifold whose boundary components are Euclidean 2-orbifolds. Each component of $\partial\mathcal{O}$ is one of the following:

- a 2-torus $T^2 \cong S^1 \times S^1$;
- a pillow $P = S^2(2,2,2,2) \cong T^2/(\mathbb{Z}/2\mathbb{Z})$;
- a turnover $S^2(n_1, n_2, n_3)$ with $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$.

A turnover cannot bound the quotient of a solid torus, hence we cannot do any Dehn filling on it. This is coherent with the fact that turnovers are rigid, and do not allow to define Dehn filling parameters. For a 2-torus T^2, we define the Dehn filling coefficients exactly in the same way as for manifolds. Next we give the details of the definition for a pillow.

Definition B.2.1. — A *solid pillow* is a 3-ball with two unknotted singular arcs with ramification indices 2. In a solid pillow, a *meridian disc* is a proper non-singular disc of the solid pillow that splits it into two balls with a singular arc each one (see Figure 1).

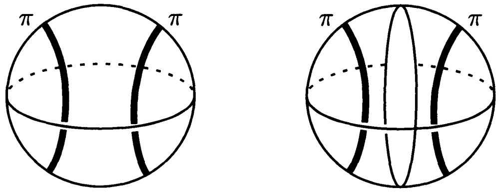

FIGURE 1. The solid pillow. Its boundary is the pillow. The figure on the right represents a meridian disc in the solid pillow.

A meridian disc in a solid pillow is unique up to orbifold isotopy. The solid pillow is the quotient of the solid torus by $\mathbb{Z}/2\mathbb{Z}$, and the meridian disc of the solid pillow lifts to two parallel meridian discs of the solid torus.

The boundary of the solid pillow is the pillow $S^2(2,2,2,2)$, hence we have the following definition.

Definition B.2.2. — Let $\mathcal{O}$ be a 3-orbifold, let $P \subset \partial\mathcal{O}$ be a boundary component with $P \cong S^2(2,2,2,2)$, and let $\mu \subset P$ be a simple closed curve that splits P into two discs with two cone points each one. The *Dehn filling* of $\mathcal{O}$ with *surgery meridian*

μ is the orbifold $\mathcal{O} \cup_\phi S$, where S is a solid pillow and $\phi : P \to \partial S$ is an orbifold homeomorphism that identifies μ with the boundary of a meridian disc.

As for manifolds, the Dehn filling only depends on the orbifold isotopy class of the surgery meridian. To describe the orbifold isotopy classes of these curves, we need to recall some elementary facts about the fundamental group of the pillow. Since $S^2(2,2,2,2) \cong T^2/(\mathbb{Z}/2\mathbb{Z})$, we have an exact sequence:

$$1 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \pi_1(S^2(2,2,2,2)) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

The sequence splits, and the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}\oplus\mathbb{Z}$ by mapping each element to its inverse.

We list some elementary properties of the fundamental group of an orbifold in the following lemma, whose proof is an easy exercise.

Lemma B.2.3. — *Given $\gamma \in \pi_1(S^2(2,2,2,2))$ with $\gamma \neq 1$, then:*

i) Either γ is torsion free or has order two.

ii) The element γ is torsion free iff $\gamma \in \ker\left(\pi_1(S^2(2,2,2,2)) \to \mathbb{Z}/2\mathbb{Z}\right) \cong \mathbb{Z} \oplus \mathbb{Z}$.

iii) If γ is torsion free, then γ is represented by n times a simple closed loop that splits $S^2(2,2,2,2)$ into two discs with two singular points each one. □

Definition B.2.4. — We call $\ker\left(\pi_1(S^2(2,2,2,2)) \to \mathbb{Z}/2\mathbb{Z}\right) \cong \mathbb{Z} \oplus \mathbb{Z}$ the *torsion free* subgroup of $\pi_1(S^2(2,2,2,2))$.

Remark B.2.5. — For a Dehn filling on a pillow, the surgery meridian gives, up to sign, a primitive element of the torsion free subgroup of $\pi_1(S^2(2,2,2,2))$. Thus, to describe a Dehn filling it suffices to give a primitive element of the torsion free subgroup, up to sign.

B.2.2. The hyperbolic Dehn filling theorem. — Let $\mathcal{O}$ be a compact 3-orbifold with boundary such that $\mathrm{int}(\mathcal{O})$ is hyperbolic with finite volume. Each boundary component of $\mathcal{O}$ is a Euclidean 2-orbifold. As for manifolds, the completion of the deformed hyperbolic structures on $\mathrm{int}(\mathcal{O})$ is described by generalized Dehn filling parameters. Assume that $\partial\mathcal{O}$ has

- k non-singular tori,
- l pillows, and
- m turnovers,

with $k + l > 0$. For each torus T^2_j in $\partial\mathcal{O}$, we fix μ_j and λ_j two generators of $\pi_1(T^2_j)$, that are represented by two simple loops in T^2_j. For each pillow P^2_j in $\partial\mathcal{O}$, we also fix μ_j and λ_j two generators of the torsion free subgroup of $\pi_1(P^2_j)$, that represent two simple closed curves in P^2_j (each curve bounds a disc with two cone points).

For a torus T^2_j ($j \leq k$), the interpretation of the the generalized Dehn filling coefficients is the same as in the manifold case. For a pillow P^2_j ($k+1 \leq j \leq k+l$), we also associate generalized Dehn filling coefficients $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\}$ such that

$(p_j, q_j) = \infty$ iff the structure at the j-th cusp is complete. When $(p_j, q_j) \in \mathbb{R}^2$ the interpretation is very similar to the manifold case:

- If $p_j, q_j \in \mathbb{Z}$ are coprime, then the completion at the j-th end is a non-singular hyperbolic 3-orbifold, which topologically is the Dehn filling with surgery meridian $p_j\mu_j + q_j\lambda_j$.
- When $p_j/q_j \in \mathbb{Q} \cup \{\infty\}$, let $m_j, n_j \in \mathbb{Z}$ be coprime integers such that $p_j/q_j = m_j/n_j$. The completion is a cone 3-manifold obtained by gluing a solid pillow with singular core. This core is a segment with silvered boundary (see Figure 2), and therefore there are singularities which are not of cyclic type. The surgery meridian is $m_j\mu_j + n_j\lambda_j$ and the cone angle of the singular component is $2\pi|m_j/p_j|$.
- When $p_j/q_j \in \mathbb{R} - \mathbb{Q}$, the completion (by equivalence classes of Cauchy sequences) is not topologically a manifold. These singularities are called of Dehn type, cf. **[Ho2]**.

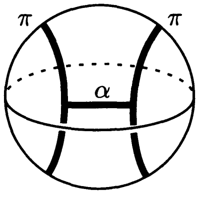

FIGURE 2. The solid pillow with a singular soul with cone angle α.

***Theorem B.2.6* (Thurston's hyperbolic Dehn filling for orbifolds [DuM])**

There exists a neighborhood of $\{\infty, \ldots, \infty\}$ in $S^2 \times \cdots \times S^2 \cong (S^2)^{k+\ell}$ such that the complete hyperbolic structure on int$(\mathcal{O})$ *has a space of hyperbolic deformations parametrized by the generalized Dehn filling coefficients in this neighborhood.*

Proof. — The proof has the same steps as in the manifold case, but it is more involved. We give the three main steps in the next three paragraphs.

B.2.3. Algebraic deformation of holonomies. — The holonomy representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$ may not lift to a representation into $SL_2(\mathbb{C})$, because $\pi_1(\mathcal{O})$ has elements of finite order which are rotations. One could work with the variety of representations into $PSL_2(\mathbb{C})$, but in order to use some results of Section B.1, we will work with representations of $\mathcal{O} - \Sigma$ into $SL_2(\mathbb{C})$, where Σ is the ramification set of $\mathcal{O}$.

Assume that Σ consists of n_0 circles and n_1 arcs (thus it has $n_0 + n_1$ components). Let $\gamma_1, \ldots, \gamma_{n_0+n_1} \in \pi_1(\mathcal{O} - \Sigma)$ represent meridians of the components of Σ. Let $\rho_0 : \pi_1(\mathcal{O} - \Sigma) \to SL_2(\mathbb{C})$ denote a lift of the restriction of the holonomy of int$(\mathcal{O})$.

Its character is denoted by χ_0. We define:

$$\mathcal{R}(\mathcal{O}) = \{\rho \in R(\mathcal{O} - \Sigma) \mid \operatorname{trace}(\rho(\gamma_j)) = \operatorname{trace}(\rho_0(\gamma_j)), \text{ for } j = 1, \ldots, n_0 + n_1\},$$
$$\mathcal{X}(\mathcal{O}) = \{\chi \in X(\mathcal{O} - \Sigma) \mid \chi(\gamma_j) = \chi_0(\gamma_j), \text{ for } j = 1, \ldots, n_0 + n_1\}.$$

A representation in $\mathcal{R}(\mathcal{O})$ composed with the natural projection $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ factors to a representation $\pi_1(\mathcal{O}) \to PSL_2(\mathbb{C})$ because the restriction $\operatorname{trace}(\rho(\gamma_j)) = \operatorname{trace}(\rho_0(\gamma_j))$ implies that $\rho(\gamma_j)$ is a rotation of the same angle as $\rho_0(\gamma_j)$. For the same reason, if $\chi_\rho \in \mathcal{X}(\mathcal{O})$, then ρ factors to a representation $\pi_1(\mathcal{O}) \to PSL_2(\mathbb{C})$.

The elements $\mu_1, \ldots, \mu_{k+l} \in \pi_1(\mathcal{O})$ represent a family of simple closed curves, one for each boundary component of $\mathcal{O}$ different from a turnover. As in the manifold case, we consider the map

$$I_\mu = (I_{\mu_1}, \ldots, I_{\mu_{k+l}}) : \mathcal{X}(\mathcal{O}) \longrightarrow \mathbb{C}^{k+l}.$$

Let χ_0 be the character of the holonomy ρ_0 of the complete structure on $\operatorname{int} \mathcal{O}$.

Theorem B.2.7. — *The map $I_\mu = (I_{\mu_1}, \ldots, I_{\mu_{k+l}}) : \mathcal{X}(\mathcal{O}) \to \mathbb{C}^{k+l}$ is locally bianalytic at χ_0.*

Proof. — The proof or the theorem follows the same scheme as the proof of Theorem B.1.2 in the manifold case. The only difference is the lower bound of the dimension of $\mathcal{X}_0(\mathcal{O})$, where $\mathcal{X}_0(\mathcal{O})$ is a component of $\mathcal{X}(\mathcal{O})$ that contains χ_0. This is done in the following lemma.

Lemma B.2.8. — $\dim(\mathcal{X}_0(\mathcal{O})) \geq k + l$.

Proof of Lemma B.2.8. — Let $X_0(\mathcal{O} - \Sigma)$ be a component of $X(\mathcal{O} - \Sigma)$ that contains $\mathcal{X}_0(\mathcal{O})$. Let $\mathcal{N}(\Sigma)$ denote a tubular neighborhood of Σ. By Thurston's estimate [**Thu1**, Thm 5.6], see also [**CS**, Thm. 3.2.1], we have that

$$\dim(X_0(\mathcal{O} - \Sigma)) \geq k + n_0 - \frac{3}{2}\chi(\partial(\mathcal{O} - \mathcal{N}(\Sigma))),$$

because $k + n_0$ is the number of torus components of $\partial(\mathcal{O} - \mathcal{N}(\Sigma))$. Since each pillow meets 4 singular arcs, each turnover meets 3 singular arcs, and each singular arc meets the boundary twice, we have that

$$4l + 3m = 2n_1.$$

In addition the Euler characteristic of the boundary is

$$\chi(\partial(\mathcal{O} - \mathcal{N}(\Sigma))) = -m - 2l.$$

Combining these equalities, Thurston's estimate can be reformulated as:

$$\dim(X_0(\mathcal{O} - \Sigma)) \geq k + n_0 + n_1 + l.$$

Since $\mathcal{X}_0(\mathcal{O})$ is a subset of $X_0(\mathcal{O} - \Sigma)$ defined by $n_0 + n_1$ equations, lemma B.2.8 follows. This also finishes the proof of Theorem B.2.7 □

B.2.4. Generalized Dehn filling coefficients. — As in the manifold case, by using Theorem B.2.7, we choose a neighborhood $V \subset \mathcal{X}(\mathcal{O})$ of χ_0, a neighborhood $U \subset \mathbb{C}^{k+l}$ of the origin, and a branched covering $\pi : U \to V \subset \mathcal{X}(\mathcal{O})$ of order 2^{k+l} defined by

$$I_{\mu_j}(\pi(u)) = I_{\mu_j}(\pi(u_1, \ldots, u_{k+l})) = \epsilon_j 2\cosh(u_j/2) \qquad \text{for } j = 1, \ldots, k+l,$$

where the coefficients $\epsilon_j \in \{\pm 1\}$ are chosen so that $I_{\mu_j}(\chi_0) = \mathrm{trace}(\rho_0(\mu_j)) = \epsilon_j 2$.

Following the manifold case, we choose an analytic section $s : V \to \mathcal{R}(\mathcal{O})$ and we use the notation $\rho_u = s(\pi(u)) \in \mathcal{R}(\mathcal{O})$.

Recall that for $j = 1, \ldots, k$, the j-th boundary component of $\mathcal{O}$ is a torus T_j and μ_j and λ_j generate $\pi_1(T_j^2)$. For $j = k+1, \ldots, k+l$, the j-th boundary component of $\mathcal{O}$ is a pillow P_j and μ_j and λ_j generate the torsion free subgroup of $\pi_1(P_j)$. We choose $\theta_j \in \pi_1(P_j)$ any element of order two, so that the following is a presentation of the fundamental group

$$\pi_1(P_j) = \langle \mu_j, \lambda_j, \theta_j \mid \mu_j\lambda_j = \lambda_j\mu_j,\ \theta_j^2 = 1,\ \theta_j\mu_j\theta_j = \mu_j^{-1},\ \theta_j\lambda_j\theta_j = \lambda_j^{-1} \rangle.$$

We recall that $\pi_1(P_j)$ is a quotient of $\pi_1(P_j - \Sigma)$.

Lemma B.2.9. — *Let $\widetilde{\mu}_j, \widetilde{\theta}_j \in \pi_1(P_j - \Sigma)$ be two elements that project to $\mu_j, \theta_j \in \pi_1(P_j)$.*

i) For $j = 1, \ldots, k+l$, there is an analytic map $A_j : U \to PSL_2(\mathbb{C})$ such that for every $u \in U$:

$$\rho_u(\widetilde{\mu}_j) = \varepsilon_j A_j(u) \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix} A_j(u)^{-1}.$$

ii) In addition, for $j = k+1, \ldots, k+l$, $A_j : U \to PSL_2(\mathbb{C})$ satisfies:

$$\rho_u(\widetilde{\theta}_j) = \pm A_j(u) \begin{pmatrix} i & 0 \\ -i(e^{u_j/2} - e^{-u_j/2}) & -i \end{pmatrix} A_j(u)^{-1}$$

where $i = \sqrt{-1}$.

Proof. — We give only the proof for pillows, the proof for tori being the proof of Lemma B.1.6. We fix $w_3 \in \mathbb{C}^2$ such that w_3 is not an eigenvector for $\rho_u(\widetilde{\theta}_j)$ and set $w_2(u) = (\rho_u(\widetilde{\theta}_j) - i)w_3 \neq 0$, so that $(\rho_u(\widetilde{\theta}_j) + i)w_2(u) = 0$, because $\pm i$ are the eigenvalues for $\rho_u(\widetilde{\theta}_j)$.

The matrix $\rho_0(\widetilde{\mu}_j)$ is parabolic, hence it does not diagonalize. This means that $\rho_0(\widetilde{\mu}_j)$ has only a one dimensional eigenspace with eigenvalue ϵ_j. Therefore, up to replacing i by $-i$, we have:

$$\ker(\rho_0(\widetilde{\mu}_j) - \epsilon_j \,\mathrm{Id}) \cap \ker(\rho_0(\widetilde{\theta}_j) + i\,\mathrm{Id}) = \{0\}.$$

In particular $w_2(u) = (w_2^1(u), w_2^2(u))$ is not an eigenvector for $\rho_0(\widetilde{\mu}_j)$. As in Lemma B.1.6, we take $w_1(u) = (w_1^1(u), w_1^2(u)) = (\epsilon_j\rho_u(\mu_j) - e^{-u_j/2})w_2$, where $\epsilon_j = \pm 1$ is the eigenvalue for $\rho_0(\widetilde{\mu}_j)$. We define:

$$A_j(u) = \frac{1}{\sqrt{w_1^1(u)w_2^2(u) - w_1^2(u)w_2^1(u)}} \begin{pmatrix} w_1^1(u) & w_2^1(u) \\ w_1^2(u) & w_2^2(u) \end{pmatrix},$$

and it is clear from this construction that i) holds.

To prove ii), since $(\rho_u(\widetilde{\theta}_j) + i)w_2(u) = 0$, we have that $\rho_u(\widetilde{\theta}_j)$ is of the form:

$$\rho_u(\widetilde{\theta}_j) = \pm A_j(u) \begin{pmatrix} * & 0 \\ * & -i \end{pmatrix} A_j(u)^{-1}.$$

Therefore point ii) follows from the fact that $\rho_u(\widetilde{\theta}_j) \in SL_2(\mathbb{C})$ and from the relation $\rho_u(\theta_j)\rho_u(\mu_j)\rho_u(\theta_j^{-1}) = \pm\rho_u(\mu_j^{-1})$, because ρ_u factors to a representation of $\pi_1(P_j)$ into $PSL_2(\mathbb{C})$. □

The following lemma has exactly the same proof as Lemma B.1.7, again because ρ_u factors to a representation of $\pi_1(P_j)$ into $PSL_2(\mathbb{C})$.

Lemma B.2.10. — *Let $\widetilde{\lambda}_j \in \pi_1(P_j - \Sigma)$ be an element that projects to $\lambda_j \in \pi_1(P_j)$. For $j = 1, \dots, k + l$, there exist unique analytic functions $v_j, \tau_j : U \to \mathbb{C}$ such that $v_j(0) = 0$ and for every $u \in U$:*

$$\rho_u(\widetilde{\lambda}_j) = \pm A_j(u) \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix} A_j(u)^{-1}.$$

In addition:

i) $\tau_j(0) \in \mathbb{C} - \mathbb{R}$;
ii) $\sinh(v_j/2) = \tau_j \sinh(u_j/2)$;
iii) v_j *is odd in* u_j *and even in* u_r, *for* $r \neq j$;
iv) $v_j = u_j(\tau_j(u) + O(|u|^2))$. □

Following the manifold case, we define:

Definition B.2.11 (**[Thu1]**). — For $u \in U$ and $j = 1, \dots, k+l$, we define the *generalized Dehn filling coefficients* of the j-th cusp $(p_j, q_j) \in \mathbb{R}^2 \cup \{\infty\} \cong S^2$ by the formula:

$$\begin{cases} (p_j, q_j) = \infty & \text{if } u_j = 0; \\ p_j u_j + q_j v_j = 2\pi\sqrt{-1} & \text{if } u_j \neq 0. \end{cases}$$

The following proposition follows also from Lemma B.2.10 i) and iv).

Proposition B.2.12. — *The generalized Dehn filling coefficients are well defined and*

$$U \longrightarrow S^2 \times \overset{(k+l)}{\cdots} \times S^2$$
$$u \longmapsto (p_1, q_1), \dots, (p_{k+l}, q_{k+l})$$

defines a homeomorphism between U and a neighborhood of $\{\infty, \dots, \infty\}$. □

B.2.5. Deformation of developing maps. — Let $D_0 : \widetilde{\operatorname{int}\mathcal{O}} \to \mathbb{H}^3$ be the developing map for the complete structure on $\operatorname{int}\mathcal{O}$, with holonomy ρ_0. The following is the orbifold version of Proposition B.1.10 and completes the proof of Theorem B.2.6.

Proposition B.2.13. — *For each $u \in U$ there is a developing map $D_u : \widetilde{\operatorname{int}\mathcal{O}} \to \mathbb{H}^3$ with holonomy ρ_u, such that the completion of* $\operatorname{int}\mathcal{O}$ *is given by the generalized Dehn filling coefficients of u.*

Proof. — The proof is analogous to the proof of Proposition B.1.10, but it needs to be adapted to orbifolds.

First we need an orbifold version of Lemma B.1.11. In the proof of Lemma B.1.11 we use a finite covering $\{U_1, \dots, U_n\}$ of a neighborhood of a compact core of the manifold $N \subset \operatorname{int}(M)$, such that each U_i is simply connected. In the orbifold case, we have to use simply connected subsets U_i such that if $U_i \cap \Sigma \neq \varnothing$ then U_i is the quotient of a ball by an orthogonal rotation. With this choice of U_i, one can generalize the argument in Lemma B.1.11 by using the fact that, for every torsion element $\gamma \in \pi_1(\mathcal{O})$, the fixed point set of $\rho_u(\gamma)$ depends analytically on $u \in U$. By using these remarks, Lemma B.1.11 can easily be generalized to orbifolds, as well as Lemma B.1.13.

It only remains to prove a version of Lemma B.1.12 for orbifolds. This lemma gives the precise developing maps for the ends. In the orbifold case, we have to distinguish the kind of end of $\operatorname{int}(\mathcal{O})$, according to the associated component of $\partial\mathcal{O}$. For tori, Lemma B.1.12 applies. We do not have to worry about turnovers because they are rigid. Thus we only need an orbifold version of Lemma B.1.12 for pillows. It is Lemma B.2.14 below, that concludes the proof of Proposition B.2.13. Let C_j denote the j-th end of $\mathcal{O}$. If $k+1 \leq j \leq k+l$ then $C_j \cong P_j \times [0, +\infty)$, where P_j is a pillow.

Lemma B.2.14. — *For $k+1 \leq j \leq k+l$, there exists a family of local embeddings $D_u^j : \widetilde{C}_j \to \mathbb{H}^3$ which is continuous on $u \in U$ for the compact C^1-topology, such that:*

i) D_u^j is ρ_u-equivariant,
ii) $D_0^j = D_{0|\widetilde{C}_j}$ and
iii) the structure on C_j can be completed as described by the generalized Dehn filling parameters.

Proof. — The universal covering $\widetilde{C}_j$ is homeomorphic to $\mathbb{R}^2 \times [0, +\infty)$. With the notation above, the group $\pi_1(C_j) \cong \pi_1(P_j)$ is generated by μ_j, λ_j and θ_j. We may assume that their action on $\widetilde{C}_j$ by deck transformations is the following:

$$\begin{cases} \mu_j(x,y,t) = (x+1, y, t) \\ \lambda_j(x,y,t) = (x, y+1, t) \\ \theta_j(x,y,t) = (-x, -y, t) \end{cases} \qquad \text{for every } (x,y,t) \in \mathbb{R}^2 \times [0, +\infty).$$

By Lemmas B.2.9 and B.2.10, we may assume:

$$\rho_u(\mu_j) = \pm \begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2} \end{pmatrix}, \qquad \rho_u(\lambda_j) = \pm \begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2} \end{pmatrix}$$

$$\text{and} \quad \rho_u(\theta_j) = \pm \begin{pmatrix} i & 0 \\ -i(e^{u_j/2} - e^{-u_j/2}) & -i \end{pmatrix}.$$

When $u = 0$, the cusp is complete and therefore the developing map $D_0^j = D_{0|\widetilde{C}^j}$ is:

$$\begin{aligned} D_0^j : \mathbb{R}^2 \times [0, +\infty) &\longrightarrow \mathbb{H}^3 \cong \mathbb{C} \times (0, +\infty) \\ (x, y, t) &\longmapsto (x + \tau_j(0)y, e^t) \end{aligned}$$

The family of maps $D_u^j : \mathbb{R}^2 \times [0, +\infty) \to \mathbb{H}^3$ that proves the lemma is the following:

$$D_u^j(x, y, t) = \begin{cases} \left(\dfrac{a(u,t)\, e^{u_j x + v_j(u) y} - 1}{e^{u_j/2} - e^{-u_j/2}}, a(u,t)\, e^{t + \mathrm{Re}(u_j x + v_j(u) y)} \right) & \text{if } u_j \neq 0; \\ (x + \tau_j(u) y, e^t) & \text{if } u_j = 0. \end{cases}$$

where $a(u, t) = \left(1 + e^t |e^{u_j/2} - e^{-u_j/2}|\right)^{-1/2}$. We remark that in Lemma B.1.13 we used the same family but with $a(u,t) \equiv 1$, since we did not require the equivariance by θ_j.

The family D_u^j is a family of ρ_u-equivariant local diffeomorphisms that depends continuously on $u \in U$ for the compact $\mathcal{C}^1$-topology. The completion of C_j for the structure induced by D_u^j is the one described by the Dehn coefficients and it can be proved in the same way as Claim B.1.14.

This concludes the proof of Lemma B.2.14 and of Theorem B.2.6. □

B.3. Dehn filling with totally geodesic turnovers on the boundary

The aim of this last section is to prove Proposition B.3.1, which is a version with boundary of the hyperbolic Dehn filling theorem, used in Chapter 7.

Let N^3 be a three manifold with boundary and let $\Sigma \subset N^3$ be a 1-dimensional properly embedded submanifold. This is the case for instance when N^3 is the underlying space of an orbifold and Σ its branching locus.

We will assume that every component of ∂N^3 is a 2-sphere that intersects Σ in three points. We define the non-compact 3-manifold with boundary

$$M^3 = N^3 - \Sigma.$$

Each component of ∂M^3 is a disjoint union of 3 times punctured spheres. Each end of M^3 is the product of $[0, +\infty)$ with a torus or an annulus, according to whether the corresponding component of Σ is a circle or an arc.

We also assume that M^3 admits a hyperbolic structure with totally geodesic boundary, whose ends are cusps (of rang one or two, according to whether the corresponding component of Σ is an arc or a circle). As a metric space M^3 is complete of finite volume, and the boundary components are three times punctured spheres. The double of M^3 along the boundary is a complete hyperbolic manifold with finite volume and without boundary. Let k denote the number of connected components of Σ.

Proposition B.3.1. — *For any real numbers $\alpha_1, \ldots, \alpha_k \geq 0$ there exist $\varepsilon > 0$ and a path $\gamma : [0, \varepsilon) \to R(M^3)$, such that, for every $t \in [0, \varepsilon)$, $\gamma(t)$ is a lift of the holonomy of a hyperbolic structure on M^3 whose metric completion is a cone manifold structure with totally geodesic boundary, topological type (N^3, Σ), and cone angles $(\alpha_1 t, \ldots, \alpha_k t)$.*

Proof. — We follow the same argument as in the proof of Theorem B.1.1. For the algebraic part, we choose $\{\mu_1, \ldots, \mu_k\} \subset \pi_1(M^3)$ a system of meridians for Σ. As in Theorem B.1.2 we have:

Proposition B.3.2. — *The map $I_\mu = (I_{\mu_1}, \ldots, I_{\mu_k}) : X(M^3) \to \mathbb{C}^k$ is locally bianalytic at χ_0, where χ_0 is the character of the lift of holonomy of the complete structure on M^3.* □

The proof of Proposition B.3.2 follows precisely the same argument as Theorem B.1.2: Thurston's estimate gives $\dim(X_0(M^3)) \geq k$, and one can also apply the argument about Mostow rigidity to the double of M^3. Moreover we use the following lemma:

Lemma B.3.3. — *Let $\rho_0 : \pi_1(S^2 - \{*, *, *\}) \to SL_2(\mathbb{R})$ be the holonomy of a hyperbolic turnover or of a hyperbolic 3 times punctured sphere. Let $\{\rho_t\}_{t\in[0,\varepsilon)}$ be a deformation of ρ_0 in $R(S^2 - \{*, *, *\}, SL_2(\mathbb{C}))$ such that, for each meridian $\mu \in \pi_1(S^2 - \{*, *, *\})$ and for each $t \in (0, \varepsilon)$, $\rho_t(\mu)$ is a rotation. Then ρ_t is conjugate to the holonomy of a hyperbolic turnover (i.e. it is Fuchsian).*

Proof. — If ρ_0 is a holonomy representation, then ρ_0 is irreducible. Since irreducibility is an open property, we may assume that ρ_t is irreducible. The group $\pi_1(S^2 - \{*, *, *\})$ is free of rank 2, generated by two meridians a and b such that the product ab is also a meridian. To prove the claim we use the fact that the conjugacy class of an irreducible representation is determined by the traces of a, b and ab. Thus if $\rho_t(a)$, $\rho_t(b)$ and $\rho_t(ab)$ are rotations, then ρ_t is conjugate to the holonomy of the hyperbolic turnover that has cone angles given by $\rho_t(a)$, $\rho_t(b)$ and $\rho_t(ab)$. This finishes the proof. □

By using the fact that deformations of holonomy imply deformations of the structure and Lemma B.3.3 we obtain the following remark:

Remark B.3.4. — When we deform the holonomy of a hyperbolic cone structure on M^3 with totally geodesic boundary so that the meridians are mapped to rotations, then the deformed representations are still the holonomy of a hyperbolic structure with totally geodesic boundary.

All the explicit deformations constructed in Section B.1 can be used here, combined with Lemma B.3.3, to prove Proposition B.3.1. □