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Quasi-affinoid varieties

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QUASI-AFFINOID VARIETIES

1. Introduction

In [**6**], we developed the commutative algebra of rings of separated power series and the local theory of quasi-affinoid varieties. The goal of this paper is to define the category of quasi-affinoid varieties and to treat the basic sheaf theory. The Quasi-Affinoid Acyclicity Theorem, the main result of this paper, is proved in Theorem 3.2.4. This paper uses the Nullstellensatz (Theorem 4.1.1) and results from Subsections 5.3 and 5.4 of [**6**], and the Quantifier Elimination Theorem of [**7**].

Let $X := \operatorname{Max} A$, where A is a K-quasi-affinoid algebra. Other than the canonical topology on X induced by the complete, nontrivial, ultrametric absolute value $|\cdot| : K \to \mathbb{R}_+$, there are two G-topologies we consider in this paper, the "wobbly" G-topology on X and the "rigid" G-topology on X. Both of these G-topologies are based on the same collection of admissible open sets, namely the system of R-subdomains U of X, defined in [**6**, Definition 5.3.3]. By [**6**, Theorem 5.3.5], an R-subdomain U has a canonical A-algebra of quasi-affinoid functions. In this manner X is endowed with a quasi-affinoid structure presheaf $\mathcal{O}_X$, which to each R-subdomain U of X, assigns the A-algebra $\mathcal{O}_X(U)$ of quasi-affinoid functions on U. The fact that $\mathcal{O}_X$ is a presheaf is one of the principal results of [**6**]. (See [**6**, Theorem 5.3.5 ff].)

The wobbly and rigid G-topologies on X differ, however, in the systems of admissible open coverings that they assign to X. In Subsection 2.2 we define the wobbly sheaf $\mathcal{W}_X$ to be the sheafification of $\mathcal{O}_X$ with respect to the wobbly G-topology. We show that wobbly coverings of X (finite coverings by R-domains) are $\mathcal{W}_X$-acyclic, and give a basic finiteness theorem for the wobbly sheaves based on [**6**, Theorem 6.1.2]. This finiteness theorem in various guises is a key feature that appears in many of the applications of the theory, for example the results of [**7**]. When X carries the wobbly G-topology, however, morphisms of affinoid varieties Y (carrying the usual strong affinoid G-topology) into X are not continuous (unless Y is finite), and the quasi-affinoid structure presheaf is not a sheaf.

The rigid G-topology on X, defined in subsection 2.3 assigns to X the largest collection of coverings, the quasi-affinoid coverings, such that morphisms of affinoid varieties into X are continuous. We conclude this paper, in Section 3, by proving that any quasi-affinoid covering of X is $\mathcal{O}_X$-acyclic. In particular, $\mathcal{O}_X$ is a sheaf for the rigid G-topology. Thus, the category of quasi-affinoid varieties (in the rigid G-topology) is an "extension" of the category of affinoid varieties and enjoys many similar properties from the point of view of analytic geometry and commutative algebra. It should be remarked that if X is affinoid (and infinite) then there is an R-subdomain U of X such that $\mathcal{O}_X^{\text{affinoid}}(U) \neq \mathcal{O}_X^{\text{quasi-affinoid}}(U)$ as $\mathcal{O}_X(X)$-algebras, because $\mathcal{O}_X^{\text{quasi-affinoid}}(U)$ is always a Noetherian ring.

In Subsection 2.1 we define the system of admissible open sets on a quasi-affinoid variety X, and we define the quasi-affinoid structure presheaf $\mathcal{O}_X$. In Subsection 2.2 we define the system of wobbly admissible open coverings of X to be finite coverings of X by R-domains and prove various properties of the sheafification $\mathcal{W}_X$ of the presheaf $\mathcal{O}_X$ with respect to the wobbly G-topology. In Subsection 2.3 we define the system of rigid admissible coverings of X. This is the G-topology we adopt for the category of quasi-affinoid varieties. We also give a simple characterization of rigid ("quasi-affinoid") coverings in terms of "quasi-affinoid generating systems".

In Subsection 3.1 we give an intrinsic characterization of quasi-affinoid coverings in terms of refinements by "closed" R-subdomains and we also prove some lemmas about refinements of quasi-affinoid coverings by certain closed R-subdomains that will be used in the Quasi-affinoid Acyclicity Theorem. Subsection 3.2 is devoted to the proof of this theorem.

The remainder of Section 1 is devoted to a summary, drawn from **[2]** of the definition of Čech Cohomology with coefficients in a presheaf and to statements of the basic comparison theorems.

1.1. Čech Cohomology with Coefficients in a Presheaf. — Let X be a set and let $\mathfrak{T}$ be a collection of "open" subsets of X, closed under finite intersections. A *presheaf* $\mathcal{F}$ on X is a map from $\mathfrak{T}$ to the class of abelian groups such that for all $U \subset V \subset W \in \mathfrak{T}$, there is a "restriction" homomorphism $\mathcal{F}(V) \to \mathcal{F}(U) : f \mapsto f|_U$ such that $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity and

$$\begin{array}{ccc} & \mathcal{F}(V) & \\ \nearrow & & \searrow \\ \mathcal{F}(W) & \longrightarrow & \mathcal{F}(U) \end{array}$$

commutes.

Let $\mathfrak{A} = \{U_i\}_{i\in I}$ be a covering of X by elements $U_i \in \mathfrak{T}$. For $(i_0, \dots, i_q) \in I^{q+1}$, put

$$U_{i_0 \dots i_q} := \bigcap_{j=0}^{q} U_{i_j}.$$

The $\mathbb{Z}$-module of *q-cochains* on $\mathfrak{A}$ with values in $\mathcal{F}$ is

$$C^q(\mathfrak{A}, \mathcal{F}) := \prod_{(i_0,\dots,i_q)\in I^{q+1}} \mathcal{F}(U_{i_0\dots i_q}), \qquad q \geq 0,$$
$$C^q(\mathfrak{A}, \mathcal{F}) := (0), \qquad q < 0.$$

The $(i_0, \dots, i_q)$-component of a q-cochain f is denoted $f_{i_0\dots i_q} \in \mathcal{F}(U_{i_0\dots i_q})$. We define the *coboundary* homomorphisms $d^q : C^q(\mathfrak{A}, \mathcal{F}) \to C^{q+1}(\mathfrak{A}, \mathcal{F})$ by $d^q := 0$ if $q < 0$, and for $q \geq 0$,

$$d^q(f)_{i_0\dots i_{q+1}} := \sum_{j=0}^{q+1} (-1)^j f_{i_0\dots \widehat{\imath}_j \dots i_{q+1}}\Big|_{U_{i_0\cdots i_{q+1}}},$$

where the notation $\widehat{\imath}_j$ means omit i_j. Note that $d^{q+1} \circ d^q = 0$, so $C^\bullet(\mathfrak{A}, \mathcal{F})$ is a chain complex, called the *Čech complex of cochains on* $\mathfrak{A}$ *with values in* $\mathcal{F}$. We denote the corresponding *cohomology complex* $H^\bullet(\mathfrak{A}, \mathcal{F})$, where

$$H^q(\mathfrak{A}, \mathcal{F}) := \operatorname{Ker} d^q / \operatorname{Im} d^{q-1}.$$

If $X \in \mathfrak{T}$, we define the *augmentation homomorphism*

$$\varepsilon : \mathcal{F}(X) \longrightarrow C^0(\mathfrak{A}, \mathcal{F}) : f \longmapsto (f|_{U_{i_0}})_{i_0 \in I},$$

with image contained in $\operatorname{Ker} d^0$. The covering $\mathfrak{A}$ is *$\mathcal{F}$-acyclic* iff the sequence

$$0 \to \mathcal{F}(X) \xrightarrow{\varepsilon} C^0(\mathfrak{A}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathfrak{A}, \mathcal{F}) \xrightarrow{d^1} \dots$$

is exact; i.e., iff ε induces an isomorphism of $\mathcal{F}(X)$ with $C^0(\mathfrak{A}, \mathcal{F})$ and $H^q(\mathfrak{A}, \mathcal{F}) = (0)$ for $q \neq 0$. A q-cochain f is an *alternating cochain* iff for all permutations π of $\{0, \dots, q\}$,

$$f_{i_{\pi(0)}\dots i_{\pi(q)}} = (\operatorname{sgn} \pi) f_{i_0\dots i_q}.$$

The alternating q-cochains form a submodule $C^q_{\mathrm{a}}(\mathfrak{A}, \mathcal{F})$ of $C^q(\mathfrak{A}, \mathcal{F})$. As d^q maps alternating cochains into alternating cochains, the modules $C^q_{\mathrm{a}}(\mathfrak{A}, \mathcal{F})$ constitute a subcomplex $C^\bullet_{\mathrm{a}}(\mathfrak{A}, \mathcal{F})$ of $C^\bullet(\mathfrak{A}, \mathcal{F})$ called the *Čech complex of alternating cochains on* $\mathfrak{A}$ *with values in* $\mathcal{F}$. The corresponding cohomology modules are denoted by

$$H^q_{\mathrm{a}}(\mathfrak{A}, \mathcal{F}) := H^q(C^\bullet_{\mathrm{a}}(\mathfrak{A}, \mathcal{F})).$$

There is no essential difference between the complexes $C^\bullet_{\mathrm{a}}(\mathfrak{A}, \mathcal{F})$ and $C^\bullet(\mathfrak{A}, \mathcal{F})$, since both yield the same cohomology.

Proposition 1.1.1. — *The injection* $\iota : C^\bullet(\mathfrak{A}, \mathcal{F}) \to C^\bullet_{\mathrm{a}}(\mathfrak{A}, \mathcal{F})$ *induces bijections* $H^q(\iota) : H^q_{\mathrm{a}}(\mathfrak{A}, \mathcal{F}) \xrightarrow{\sim} H^q(\mathfrak{A}, \mathcal{F})$, *for all* q.

Let $\mathfrak{A} = \{U_i\}_{i\in I}$ and $\mathfrak{B} = \{V_j\}_{j\in J}$ be $\mathfrak{T}$-coverings of X. Then $\mathfrak{B}$ is a *refinement of* $\mathfrak{A}$ iff for each $j \in J$ there is some $i \in I$ such that $V_j \subset U_i$.

Proposition 1.1.2 ([**2**, Proposition 8.1.3.4]). — *Let $\mathfrak{A}$ and $\mathfrak{B}$ be open coverings which are refinements of each other. Assume $X \in \mathfrak{T}$. Then the covering $\mathfrak{A}$ is $\mathcal{F}$-acyclic if, and only if, $\mathfrak{B}$ is $\mathcal{F}$-acyclic.*

For the next propositions, it is convenient to define some notation. Let $\mathfrak{A} = \{U_i\}_{i\in I}$ be a $\mathfrak{T}$-covering of X and let $V \in \mathfrak{T}$; then

$$\mathfrak{A}|_V := \{V \cap U_i\}_{i\in I}$$

is a $\mathfrak{T}$-covering of V which is called the *restriction of* $\mathfrak{A}$ to V. We define the presheaf $\mathcal{F}|_V$ on $(V, \mathfrak{T}|_V)$ by restricting the domain of $\mathcal{F}$ to $\mathfrak{T}|_V$.

Proposition 1.1.3 ([**2**, Theorem 8.1.4.2]). — *Assume that all coverings $\mathfrak{A}|_{V_{j_0\cdots j_q}}$ and $\mathfrak{B}|_{U_{i_0\cdots i_p}}$ are $\mathcal{F}$-acyclic. Then,*

$$H^r(\mathfrak{A}, \mathcal{F}) \cong H^r(\mathfrak{B}, \mathcal{F})$$

for all r. In particular, if $X \in \mathfrak{T}$, the covering $\mathfrak{A}$ is $\mathcal{F}$-acylic if, and only if, $\mathfrak{B}$ is $\mathcal{F}$-acyclic.

Proposition 1.1.4 ([**2**, Corollary 8.1.4.3]). — *Assume that $\mathfrak{B}$ is a refinement of $\mathfrak{A}$ and that $\mathfrak{B}|_{U_{i_0\cdots i_p}}$ is $\mathcal{F}$-acyclic for all indices $i_0, \ldots, i_p \in I$ and for all p. Then, if $X \in \mathfrak{T}$, the covering $\mathfrak{A}$ is $\mathcal{F}$-acyclic if, and only if, $\mathfrak{B}$ is $\mathcal{F}$-acyclic.*

Proposition 1.1.5 ([**2**, Corollary 8.1.4.4]). — *Assume that the covering $\mathfrak{B}|_{U_{i_0\cdots i_p}}$ is $\mathcal{F}$-acyclic for all indices $i_0, \ldots, i_p \in I$ and for all p. Then, if $X \in \mathfrak{T}$, the covering $\mathfrak{A} \times \mathfrak{B} := \{U_i \cap V_j\}_{\substack{i\in I\\ j\in J}}$ of X is $\mathcal{F}$-acyclic if, and only if, $\mathfrak{A}$ is $\mathcal{F}$-acyclic.*

We assume that the reader is familiar with the following concepts, which can be found in [**2**, Chapter 9]: G-topology ([**2**, Definition 9.1.1.1]); sheaf and stalks ([**2**, Definition 9.2.1.2 ff]); sheafification ([**2**, Definition 9.2.2.1]); and locally G-ringed space ([**2**, Section 9.3.1]).

2. G-Topologies and the Structure Presheaf

Recall that a G-topology on a set X is determined by a system $\mathfrak{T}$ of admissible open sets, and for each admissible open U, a system $\operatorname{Cov} U$ of admissible coverings of U by admissible open sets (see [**2**, Definition 9.1.1.1]). Let A be a quasi-affinoid algebra (i.e., $A = S_{m,n}/I$, see [**6**]) and put $X := \operatorname{Max} A$. In this section, we will consider two G-topologies on X, the wobbly G-topology and the rigid G-topology. The admissible open sets in both of these topologies will be the same, namely the collection of R-subdomains of X. The systems of admissible open coverings, however, will be different.

For each R-subdomain $U \subset X$, we have shown ([**6**, Subsection 5.3]) that there is a uniquely determined A-algebra $\mathcal{O}_X(U)$ that satisfies the Universal Mapping Property of [**6**, Definition 5.3.4] and such that $\operatorname{Max} \mathcal{O}_X(U) = U$. Note that $\mathcal{O}_X(X) = A$. In fact, the Universal Mapping Property for R-subdomains ([**6**, Theorem 5.3.5]) shows that $\mathcal{O}_X$, so defined, is presheaf. This is summarized in Subsection 2.1.

In Subsection 2.2, we show that $\mathcal{O}_X$ is not a sheaf with respect to the wobbly G-topology on X, and we discuss a few properties of its sheafification $\mathcal{W}_X$ with respect to the wobbly G-topology.

In Subsection 2.3, we define the class of quasi-affinoid coverings, and the rigid G-topology of a quasi-affinoid variety X. In particular, it is with respect to this G-topology that we show in Subsection 3.2 that $\mathcal{O}_X$ is indeed a sheaf. We also define the category of quasi-affinoid varieties and prove that fiber products and disjoint unions exist in this category (but the disjoint union of two quasi-affinoid subdomains is not necessarily a quasi-affinoid subdomain).

2.1. Open Sets and the Structure Presheaf. — The notion of quasi-affinoid subdomain of a quasi-affinoid variety X was defined in [**6**, Section 5.3] by means of the following universal property.

Definition 2.1.1. — Let $X = \operatorname{Max} A$ be a quasi-affinoid variety and let $U \subset X$. Then U is a *quasi-affinoid subdomain* of X iff there is a quasi-affinoid variety Y and a quasi-affinoid map $\varphi : Y \to X$ with $\varphi(Y) \subset U$ such that φ represents all quasi-affinoid maps into U in the sense of [**6**, Definition 5.3.4].

A certain class of quasi-affinoid subdomains plays a key role in the local theory, that is the class of quasi-rational subdomains and, by iteration, R-subdomains (see [**6**, Definition 5.3.3 and Theorem 5.3.5]). Recall that if $f_1, \ldots, f_r$, $g_1, \ldots, g_s$, $h \in A$ generate the unit ideal of the quasi-affinoid algebra A, then

$$U := \{x \in \operatorname{Max} A : |f_i(x)| \le |h(x)| \text{ and } |g_j(x)| < |h(x)|, 1 \le i \le r, 1 \le j \le s\}$$

is a quasi-rational subdomain of $X = \operatorname{Max} A$; indeed the quasi-affinoid map induced by the natural K-algebra homomorphism

$$A \to A\left\langle \frac{f}{h} \right\rangle \left[\left[\frac{g}{h}\right]\right]_s \tag{2.1.1}$$

represents all quasi-affinoid maps into U (the latter ring is defined in [**6**, Definition 5.3.1]). When $s = 0$ (i.e., when there are no g's), we will find it convenient to denote U by

$$X\left(\frac{f}{h}\right).$$

Other special types of quasi-rational subdomains are those of the form

$$X(f) := \{x \in X : |f_i(x)| \le 1,\ 1 \le i \le r\},$$
$$X(f, g^{-1}) := \{x \in X : |f_i(x)| \le 1,\ |g_j(x)| \ge 1,\ 1 \le i \le r,\ 1 \le j \le s\}.$$

Unlike the affinoid case, a quasi-rational subdomain of a quasi-rational subdomain of X, although it is by definition an R-subdomain of X, need not itself be a quasi-rational subdomain of X (see [**6**, Example 5.3.7]). In order to keep track of the complexity of R-subdomains, we define the notion of level.

Definition 2.1.2. — Let $X = \operatorname{Max} A$ be a quasi-affinoid variety. We define the class of R-subdomains of X of level $\le \ell$ inductively as follows. Any quasi-rational subdomain of X is an *R-subdomain of X of level* ≤ 1. If U is an R-subdomain of X of level $\le \ell$, then any quasi-rational subdomain V of U is an *R-subdomain of X of level* $\le \ell + 1$.

The class of R-subdomains of X is closed under finite intersections.

Definition 2.1.3. — Let $X = \operatorname{Max} A$ be quasi-affinoid. By $\mathfrak{T}$, denote the system of R-subdomains of X; note that $\varnothing, X \in \mathfrak{T}$ and that $\mathfrak{T}$ is closed under finite intersection. The elements of $\mathfrak{T}$ are the *admissible open* sets. Using (2.1.1) and Definition 2.1.2, we inductively assign to each $U \in \mathfrak{T}$ a generalized ring of fractions over A, which we denote $\mathcal{O}_X(U)$. The map $U \mapsto \mathcal{O}_X(U)$ is called the *quasi-affinoid structure presheaf* on $(X, \mathfrak{T})$.

By [**6**, Theorem 5.3.5], the natural K-algebra homomorphism $A \to \mathcal{O}_X(U)$ represents all quasi-affinoid maps into U. This has the following consequence.

Theorem 2.1.4. — *$\mathcal{O}_X$ is a presheaf on $(X, \mathfrak{T})$.*

When $U \subset X$ is an affinoid R-subdomain of X (see [**6**, Proposition 5.3.8]), it follows from [**6**, Theorem 5.3.5] that $\mathcal{O}_X^{\text{affinoid}}(U) = \mathcal{O}_X^{\text{quasi-affinoid}}(U)$. But, taking $X := \operatorname{Max} K\langle\xi_1\rangle$, for example, it can easily be seen that $\mathcal{O}_X^{\text{affinoid}} \neq \mathcal{O}_X^{\text{quasi-affinoid}}$ as presheaves. Indeed, put

$$U := \{x \in X : |x| < 1\}.$$

Then $\mathcal{O}_X^{\text{quasi-affinoid}}(U) = K[\![\xi_1]\!]_s$ is a ring of separated power series, hence is Noetherian. On the other hand,

$$\mathcal{O}_X^{\text{affinoid}}(U) = \varprojlim_{\varepsilon \in \sqrt{|K^{\circ\circ} \setminus \{0\}|}} K\langle\xi_1\rangle\left\langle\frac{\xi_1}{\varepsilon}\right\rangle$$

is not Noetherian.

In [**6**, Theorem 6.2.2], we showed that a quasi-affinoid subdomain V of X is a finite union of R-subdomains $U_0, \ldots, U_p$ of X. The covering $\{U_i\}$ of V so obtained is admissible in the sense of Subsection 2.2, but it is not, in general, a"quasi-affinoid" covering in the sense of Subsection 2.3.

2.2. The Wobbly G-Topology. — Recall that the intersection of finitely many R-domains is an R-domain [**6**, Section 5.3]. This allows us to make the following definition.

Definition 2.2.1. — Let A be a quasi-affinoid algebra, $X := \operatorname{Max} A$. The *wobbly G-topology* on X is defined by taking the admissible open sets of X to be the system of R-subdomains of X. For each admissible open U, we take the admissible coverings of U to be the system of all finite coverings of U by admissible open sets.

This definition admits finite coverings of $X = \operatorname{Max} A$ by disjoint admissible open sets, for example, when $A = T_1$,

$$U_0 := \{x \in X : |\xi(x)| < 1\}, \quad U_1 := \{x \in X : |\xi(x)| = 1\}$$

is such a covering. Moreover, the complement of any R-subdomain of X is a finite disjoint union of R-subdomains of X by an easy extension of [**6**, Section 5.3]. It follows that any wobbly admissible cover of X has a wobbly admissible refinement by finitely many pairwise disjoint R-subdomains.

Definition 2.2.2. — Let A be a quasi-affinoid algebra, $X := \operatorname{Max} A$. Define $\mathcal{W}_X$, the *wobbly sheaf* on X, to be the sheafification (see [**2**, Section 9.2.2)], with respect to the wobbly G-topology on X, of the presheaf $\mathcal{O}_X$. For each admissible open U, we have

$$\mathcal{W}_X(U) = \varinjlim \mathcal{O}(U_0) \oplus \cdots \oplus \mathcal{O}(U_p),$$

where the direct limit runs over the directed system of all (wobbly) admissible open coverings of $\{U_0, \ldots, U_p\} \subset \operatorname{Cov} U$.

By the preceding remark, observe that the characteristic function of any R-subdomain of X belongs to the ring $\mathcal{W}_X(X)$; hence $\mathcal{W}_X(X) \neq \mathcal{O}_X(X)$ when X is infinite. In particular this shows that $\mathcal{O}_X$ is not in general a sheaf with respect to the wobbly G-topology.

Proposition 2.2.3. — *Let $X = \operatorname{Max} A$, where A is a quasi-affinoid algebra, and let $\mathfrak{A}$ be a wobbly admissible covering of X, i.e., a finite covering of X by R-subdomains. Then $\mathfrak{A}$ is $\mathcal{W}_X$-acyclic.*

Proof. — Since the intersection of two R-subdomains is an R-subdomain, and since the complement of any R-subdomain is a finite disjoint union of R-subdomains, there is an admissible refinement $\mathfrak{B} = \{V_j\}_{j \in J}$ of $\mathfrak{A}$ by finitely many pairwise disjoint R-subdomains. By Proposition 1.1.4, it suffices to prove that disjoint wobbly coverings are universally $\mathcal{W}_X$-acyclic; i.e., for each R-subdomain X' of X, a disjoint wobbly covering of X' is $\mathcal{W}_X$-acyclic. To see this, observe that $C^q(\mathfrak{B}|_{X'}, \mathcal{W}_X) = (0)$ for $q \neq 0$ because the elements of $\mathfrak{B}|_{X'}$ are pairwise disjoint, and the map

$$\varepsilon : \mathcal{W}_X(X') \longrightarrow C^0(\mathfrak{B}|_{X'}, \mathcal{W}_X) : f \longmapsto (f|_{V_j})_{j \in J}$$

is a bijection, by definition of $\mathcal{W}_X$. □

Remark 2.2.4

(i) The stalks of the wobbly sheaf on X agree with those of the rigid structure presheaf: for each $x \in X$,

$$\mathcal{W}_{X,x} = \mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}_X(U).$$

This follows from the representation in Definition 2.2.2 of $\mathcal{W}_X(U)$ as a direct limit.

(ii) There is a natural map of Max $\mathcal{W}_X(X)$ onto the space Cont $\mathcal{O}_X(X)$ of continuous valuations (for the definition of Cont $\mathcal{O}_X(X)$ see [**4**]). This is because a point of Cont $\mathcal{O}_X(X)$ is uniquely determined by the collection of quasi–rational subdomains to which it belongs. The mapping is in general not injective.

(iii) Note that if $U \subset V$ are two R-subdomains of X, then the canonical restriction $\mathcal{W}_X(V) \to \mathcal{W}_X(U)$ is surjective; i.e., $\mathcal{W}_X$ is a flasque sheaf in the sense of [**3**, Exercise II.1.16].

(iv) We may reformulate [**6**, Theorem 6.1.2] in terms of the wobbly sheaf, as follows.

Theorem. — *Let $\pi : Y \to X$ be a quasi-affinoid map with finite fibers. Then the induced morphism of sheaves on X*

$$\mathcal{W}_X \longrightarrow \pi_* \mathcal{W}_Y$$

(where $\pi_* \mathcal{W}_Y$ is the direct image sheaf) is finite.

This theorem is false upon replacing $\mathcal{W}$ by the rigid structure presheaf $\mathcal{O}$ (see [**6**, Example 6.1.3]). This finiteness theorem in various guises is a key feature of the proofs of the quantifier elimination theorems of [**5**] and [**7**].

(v) Let A be an affinoid algebra of positive Krull dimension. Then the identity map

$$id : \operatorname{Sp} A \to \operatorname{Max} A$$

is not continuous if Sp A carries the strong affinoid G-topology of [**2**, Section 9.1.4] and Max A carries the wobbly G-topology induced by regarding A as a quasi-affinoid algebra (though the inverse image of an admissible open is admissible open).

2.3. Quasi-Affinoid Coverings and the Rigid G-Topology. — In this section, we define the weakest G-topology on $X = \operatorname{Max} A$, A quasi-affinoid, such that each R-subdomain of X is admissible open and such that each quasi-affinoid morphism $\varphi : Y \to X$, with Y an affinoid variety carrying the strong affinoid G-topology ([**2**, Section 9.1.4]) is continuous. Let U be an R-subdomain of X. Since $\varphi^{-1}(U)$ is admissible open in Y, specifying such a topology is equivalent to specifying an appropriate system of admissible open coverings of X. We call such coverings quasi-affinoid coverings, and we prove a simple sufficient condition for a finite covering of X by R-subdomains to be a quasi-affinoid covering.

Definition 2.3.1. — Let $X = \operatorname{Max} A$, where A is a quasi-affinoid algebra. A covering $\mathfrak{A}$ of X is said to be a *quasi-affinoid covering* iff $\mathfrak{A}$ is a finite covering by R-subdomains $U_0, \ldots, U_p$ such that for every quasi-affinoid morphism $\varphi : Y \to X$, where Y is an affinoid variety, the covering $\{\varphi^{-1}(U_0), \ldots, \varphi^{-1}(U_p)\}$ of Y has a finite refinement by rational subdomains of Y.

In other words, $\{U_0, \ldots, U_p\}$ is a quasi-affinoid covering of X iff for all $\varphi : Y \to X$ with Y affinoid, $\{\varphi^{-1}(U_0), \ldots, \varphi^{-1}(U_p)\}$ is an admissible open covering of Y, where Y is given the strong G-topology (in the sense of [**2**, Section 9.1.4]). Theorem 3.1.5 gives a more intrinsic characterization of the class of quasi-affinoid coverings.

Definition 2.3.2. — Let $X = \operatorname{Max} A$, where A is quasi-affinoid. The *rigid G-topology* on X is defined by taking the admissible open sets to be the system of R-subdomains of X. For each admissible open set U, we take the admissible coverings of U to be the system of all quasi-affinoid coverings of U.

In the rest of this section, we give a simple characterization of the rigid G-topology on a quasi-affinoid X that will be useful in Subsection 3.1, where we give a more intrinsic characterization of the rigid G-topology.

Definition 2.3.3. — Let X be quasi-affinoid. A system $\{X_i\}_{i \in I}$ of affinoid R-subdomains of X (i.e., R-subdomains of X that are, in fact, affinoid, see [**6**, Proposition 5.3.8]) is a *system of definition for the rigid G-topology of X* iff for any quasi-affinoid map $\varphi : Y \to X$, where Y is an affinoid variety, $\varphi(Y) \subset X_i$ for some i.

The different representations of a quasi-affinoid algebra A as a quotient of a ring of separated power series give (possibly different) systems of definition, as we see below.

Definition 2.3.4. — Let $A = S_{m,n}/I$ be a representation of the quasi-affinoid algebra A as a quotient of a ring of separated power series. Put $X := \operatorname{Max} A$, and for each $\varepsilon \in \sqrt{|K \setminus \{0\}|}$, $\varepsilon < 1$, put

$$X_\varepsilon := \operatorname{Max}(S_{m,n}/I)\left\langle \frac{\rho_1^\ell}{e}, \ldots, \frac{\rho_n^\ell}{e} \right\rangle = \operatorname{Max} T_{m,n}(\varepsilon)/\iota_\varepsilon(I) \cdot T_{m,n}(\varepsilon),$$

where $e \in K^\circ$ is chosen so that $\varepsilon^\ell = |e|$ for some $\ell \in \mathbb{N}$. (See [**6**, Section 3.2].) This is the intersection of X with a closed polydisc; it is an R-subdomain of X which is, in fact, affinoid. Note that X_ε depends on the representation $A = S_{m,n}/I$. (For definitions of $T_{m,n}(\varepsilon)$ and $\iota_\varepsilon(I)$ see [**6**, Section 3.2].)

We now show that $\{X_\varepsilon\}_{\varepsilon<1}$ is a system of definition for X.

Lemma 2.3.5. — *Let X and $\{X_\varepsilon\}_{\varepsilon<1}$ be as above. Then $\{X_\varepsilon\}_{\varepsilon<1}$ is a system of definition.*

Proof. — This follows from the affinoid Maximum Modulus Principle ([**2**, Proposition 6.2.1.4]) and from the fact that the X_ε are R-subdomains of X, that are affinoid.

Let $\psi^* : A \to C$ be a K-algebra homomorphism, where C is an affinoid algebra. Put $Y := \operatorname{Max} C$. By the Nullstellensatz, [**6**, Theorem 4.1.1], $|\psi^*(\overline{\rho}_i)(y)| < 1$ for all $y \in Y$, where $\overline{\rho}_i$ is the image of ρ_i in $S_{m,n}/I$, $1 \le i \le n$. By the Maximum Modulus Principle,

$$\max_{1\le i\le n} \|\psi^*(\overline{\rho}_i)\|_{\sup} =: \varepsilon < 1.$$

Hence $\psi(Y) \subset X_\varepsilon$. □

The next proposition shows that any system of definition characterizes the quasi-affinoid coverings, hence the rigid G-topology.

Proposition 2.3.6. — *Let X be quasi-affinoid and let $\{X_i\}_{i\in I}$ be a system of definition for X. A covering $\mathfrak{A} = \{U_0, \ldots, U_p\}$ of X by R-subdomains U_i is a quasi-affinoid covering if, and only if, for each $i \in I$, the covering $\{X_i \cap U_0, \ldots, X_i \cap U_p\}$ of the affinoid variety X_i has a finite refinement by rational domains.*

Proof

($\Rightarrow$) This is immediate.

($\Leftarrow$) Let $\psi : Z \to X$ be a quasi-affinoid map, with Z affinoid. We must show that $\{\psi^{-1}(U_0), \ldots, \psi^{-1}(U_p)\}$ has a finite refinement by rational domains. For some $i \in I$, $\psi(Z) \subset X_i$, and $\{X_i \cap U_0, \ldots, X_i \cap U_p\}$ has a finite refinement by rational domains, which we pull back to Z via ψ. □

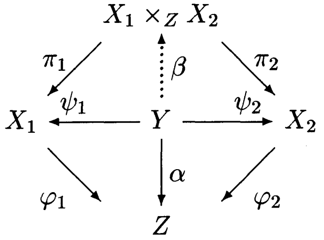
Remark 2.3.7. — Let $\{Y_i\}_{i\in I}$ be a system of definition for the rigid G-topology on X. Then by Lemma 2.3.5, $\{Y_i\}_{i\in I}$ must be a covering of X by affinoid subdomains because each $X_\varepsilon \subset Y_i$ for some i and $\{X_\varepsilon\}_{\varepsilon<1}$ coverings X. Unless $\{Y_i\}_{i\in I}$ is finite, however, it is *not* an admissible (quasi-affinoid) covering of X. And if it is finite, then X itself must be affinoid by [**6**, Proposition 5.3.8].

Using the rigid G-topology of the last subsection, we now define the category of quasi-affinoid varieties. Let $\varphi : X \to Y$ be a quasi-affinoid morphism (see Definition 2.3.8, below). It follows from the definition of R-subdomain that $\varphi^{-1}(U)$ is an R-subdomain of X for any R-subdomain U of Y. To check that φ is continuous, it remains to show that if $\{U_0, \ldots, U_p\}$ is a quasi-affinoid covering of Y then $\{\varphi^{-1}(U_0), \ldots, \varphi^{-1}(U_p)\}$ is a quasi-affinoid covering of X. Let Z be an affinoid variety and let $\psi : Z \to X$ be a quasi-affinoid map. The fact that the covering $\{\psi^{-1}(\varphi^{-1}(U_i))\}$ of Z has a finite refinement by rational domains then follows from the facts that $\psi^{-1}(\varphi^{-1}(U_i)) = (\varphi \circ \psi)^{-1}(U_i)$, and $\{U_i\}$ is a quasi-affinoid covering of Y. Note, moreover, that the induced maps $\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ of stalks are local homomorphisms for each $x \in X$.

Definition 2.3.8. — Let A be a quasi-affinoid algebra and let $X := \operatorname{Max} A$. The *quasi-affinoid variety* $\operatorname{Sp} A$ is the locally G-ringed space $(X, \mathcal{O}_X)$, where X carries the rigid G-topology. (The Acyclicity Theorem, Theorem 3.2.4, guarantees that $\mathcal{O}_X$ is a sheaf on X for its rigid G-topology.) A *morphism* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (φ, φ^*) such that $\varphi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is a K-algebra homomorphism and φ is the map from $X = \operatorname{Max} \mathcal{O}_X(X)$ to $Y = \operatorname{Max} \mathcal{O}_Y(Y)$ induced by the Nullstellensatz ([**6**, Theorem 4.1.1]).

Fiber products and direct sums exist in this category.

Proposition 2.3.9. — *The category of quasi-affinoid varieties admits fiber products; i.e., if $\varphi_1 : X_1 \to Z$ and $\varphi_2 : X_2 \to Z$ are quasi-affinoid morphisms, then there is a quasi-affinoid variety $X_1 \times_Z X_2$ and quasi-affinoid morphisms $\pi_i : X_1 \times_Z X_2 \to X_i$ such that, given any quasi-affinoid variety Y and morphisms ψ_i and α as shown, there is a unique morphism β that makes*



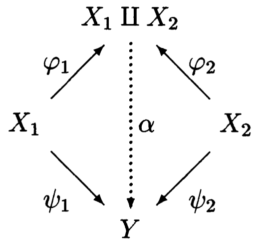
commute.

Proof. — This is just the dual diagram obtained from the diagram of [**6**, Proposition 5.4.3]. Thus,

$$X_1 \times_Z X_2 = \operatorname{Sp}(\mathcal{O}_{X_1}(X_1) \otimes^s_{\mathcal{O}_Z(Z)} \mathcal{O}_{X_2}(X_2)),$$

and the morphisms π_i are dual to the corresponding K-algebra homomorphisms of [**6**, Proposition 5.4.3]. □

Proposition 2.3.10. — *The category of quasi-affinoid varieties admits disjoint unions; i.e., if X_1 and X_2 are quasi-affinoid varieties, then there is a quasi-affinoid variety $X_1 \amalg X_2$ and morphisms $\varphi_1 : X_i \to X_1 \amalg X_2$ such that for any quasi-affinoid variety Y and morphisms $\psi_i : X_i \to Y$, there exists a unique morphism α that makes*



commute.

Proof. — This is the dual of the diagram one obtains for direct sums of quasi-affinoid algebras (see [**6**, Lemma 5.4.1]). Thus

$$X_1 \amalg X_2 = \mathrm{Sp}(\mathcal{O}_{X_1}(X_1) \oplus \mathcal{O}_{X_2}(X_2)).$$

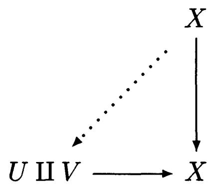
□

For completeness, we include the following.

Corollary 2.3.11. — *Let $\varphi : X \to Y$ be a quasi-affinoid morphism, and let U, V be quasi-affinoid subdomains of Y. Then $U \cap V$ is a quasi-affinoid subdomain of Y and $\varphi^{-1}(U)$ is a quasi-affinoid subdomain of X.*

Proof. — It suffices to note that $U \cap V = U \times_Y V$ and $\varphi^{-1}(U) = U \times_Y X$. That the Universal Mapping Property for quasi-affinoid domains (see [**6**, Section 5.3]) is satisfied is a consequence of Proposition 2.3.9. □

Unlike the situation for affinoid subdomains (see [**2**, Proposition 7.2.2.9]), the disjoint union of two quasi-affinoid subdomains may fail to be a quasi-affinoid subdomain. For example, take $X := \mathrm{Sp}\, S_{1,0}$, $U := \mathrm{Sp}\, S_{1,0}[\![\xi]\!]_s$, $V = \mathrm{Sp}\, S_{1,0}\langle \xi^{-1} \rangle$. Then the diagram



cannot be completed as required; i.e., the closed unit disc is the *set-theoretic* disjoint union of the open unit disc and an annulus, but not as quasi-affinoid subdomains.

3. Coverings and Acyclicity

In Subsection 3.2 we prove our main theorem, that quasi-affinoid coverings are $\mathcal{O}_X$-acyclic (which has the consequence that $\mathcal{O}_X$ is a sheaf for the rigid G-topology on X). The proof follows the general outline given in [**2**, Chapter 8] for the affinoid case. To make it work in our context requires the characterization of quasi-affinoid covers given in Subsection 3.1. This relies on the quantifier elimination of [**7**].

3.1. Refinements by closed R-subdomains. — We define a special class of quasi-affinoid subdomains, the closed R-subdomains, that facilitate our computations and are general enough for our purposes. In Theorems 3.1.4 and 3.1.5, we give the more intrinsic characterization of the quasi-affinoid coverings as those that have a finite refinement by closed R-subdomains.

Definition 3.1.1. — Let $X = \operatorname{Sp} A$ be a quasi-affinoid variety. The class of *closed R-subdomains of X of level $\leq \ell$* is defined inductively as follows. If $f_1, \ldots, f_n, g \in A$ generate the unit ideal of A, then

$$X\left(\frac{f}{g}\right) := \operatorname{Max} A\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\rangle$$

is a closed R-subdomain of X of level ≤ 1. If $U \subset X$ is a closed R-subdomain of level $\leq \ell$, and V is a closed R-subdomain of U of level ≤ 1, then V is a closed R-subdomain of X of level $\leq \ell + 1$. (Unlike the affinoid case, there may exist closed R-subdomains of X of level > 1; see [**6**, Example 5.3.7].)

Remark 3.1.2. — Note that a closed R-subdomain U of X is relatively affinoid in the sense that $X_\varepsilon \cap U$ is an affinoid rational subdomain of the affinoid variety X_ε (defined in Definition 2.3.4). Thus by Lemma 2.3.5, any finite covering of X by closed R-subdomains is a quasi-affinoid (admissible) covering of X.

Our next goal is to show that a quasi-affinoid covering has a refinement by finitely many closed R-subdomains. The first step is to prove a shrinking lemma for R-subdomains that contain an affinoid. We recall here the definition made in [**6**, Section 5.3]. Write the quasi-affinoid algebra $A = S_{m,n}/I$, and suppose $f_1, \ldots, f_r$, $g_1, \ldots, g_s$, $h \in A$ generate the unit ideal. Put $X := \operatorname{Max} A$. Then

$$U := \{x \in X : |f_i(x)| \leq |h(x)| \text{ and } |g_j(x)| < |h(x)|,\ 1 \leq i \leq r,\ 1 \leq j \leq s\}$$

is an R-subdomain of X of level ≤ 1, and

$$\mathcal{O}_X(U) = A\left\langle \frac{f_1}{h}, \ldots, \frac{f_r}{h}\right\rangle \left[\left[\frac{g_1}{h}, \ldots, \frac{g_s}{h}\right]\right]_s = S_{m+r,n+s}/J,$$

where

$$J := I + \sum_{i=1}^{r}(h\xi_{m+i} - f_i) + \sum_{j=1}^{s}(h\rho_{n+j} - g_j).$$

Let $\delta \in \sqrt{|K \setminus \{0\}|}$, say $\delta^\ell = |e|$ for some $e \in K^{\circ\circ}$. We can "shrink" the R-subdomain U to a smaller closed R-subdomain $U(\delta)$ by replacing the strict inequalities $|g_j(x)| < |h(x)|$ with the more restrictive weak inequalities $|g_j(x)| \leq \delta|h(x)|$; i.e., $|g_j^\ell(x)| \leq |eh^\ell(x)|$. We have

$$\mathcal{O}_X(U(\delta)) = (S_{m+r,n+s}/J)\left\langle \frac{\rho_{n+1}^\ell}{e}, \ldots, \frac{\rho_{n+s}^\ell}{e} \right\rangle.$$

The point here is to emphasize that $U(\delta)$ is, in fact, a closed R-subdomain with $U(\delta) \subset U$.

This construction can be carried out for an R-subdomain U of any level. Write

$$\mathcal{O}_X(U) = S_{m+r,n+s}/J,$$

where $J \supset I$ is given exactly as in [**6**, Definition 5.3.3]. Then

$$U(\delta) := \mathrm{Max}(S_{m+r,n+s}/J)\left\langle \frac{\rho_{n+1}^\ell}{e}, \ldots, \frac{\rho_{n+s}^\ell}{e} \right\rangle$$

is a closed R-subdomain with $U(\delta) \subset U$. Note that the closed R-subdomain $U(\delta)$ may depend on the presentation of U.

Lemma 3.1.3. — *(In the above notation.) Let U be an R-subdomain of*

$$X = \mathrm{Max}\, S_{m,n}/I.$$

Suppose $\varphi : Y \to X$ is a quasi-affinoid morphism with Y affinoid and $\varphi(Y) \subset U$. Then for some $\delta \in \sqrt{|K \setminus \{0\}|}$, $\delta < 1$, $\varphi(Y) \subset U(\delta)$.

Proof. — Write

$$\mathcal{O}_X(U) = S_{m+r,n+s}/J,$$

as above, let $\overline{\rho}_j$ be the image of ρ_j in $\mathcal{O}_X(U)$, $1 \leq j \leq n+s$, and let $\varphi^* : \mathcal{O}_X(U) \to \mathcal{O}_Y(Y)$ be the K-algebra homomorphism corresponding to φ. Put

$$\delta := \max_{1 \leq j \leq s} \|\varphi^*(\overline{\rho}_{n+j})\|_{\sup}.$$

By the Nullstellensatz and the Maximum Modulus Principle, $\delta \in \sqrt{|K \setminus \{0\}|}$ and $\delta < 1$. Then $\varphi(Y) \subset U(\delta)$. □

Theorem 3.1.5 characterizes quasi-affinoid coverings in terms of finite coverings by closed R-subdomains. For the proof of the Acyclicity Theorem of Subsection 3.2, however, we require some precise information about the complexity of the resulting refinements by closed R-subdomains. This is contained in Theorem 3.1.4.

Theorem 3.1.4. — *A covering $\mathfrak{A} = \{U_0, \ldots, U_p\}$ of X by finitely many R-subdomains of level ≤ 1 is a quasi-affinoid covering if, and only if, it has a refinement by finitely many closed R-subdomains of X of level ≤ 1.*

Proof

($\Leftarrow$) Immediate by Remark 3.1.2 and Lemma 2.3.5.

($\Rightarrow$) Assume each U_i is of level ≤ 1. Let $X := \operatorname{Sp} A$ and let $A = S_{m,n}/I$ be a representation of A as quotient of a ring of separated power series.

Let $\varepsilon \in \sqrt{|K \setminus \{0\}|}$, $\varepsilon < 1$, and consider the covering $\{X_\varepsilon \cap U_i\}_{0 \leq i \leq p}$ of the affinoid variety X_ε. By assumption, this covering has a refinement by finitely many rational domains, hence by Lemma 3.1.3, for some $\delta \in \sqrt{|K \setminus \{0\}|}$ with $\delta < 1$, $\{X_\varepsilon \cap U_i(\delta)\}_{0 \leq i \leq p}$ is a covering of X_ε. We may therefore define the function $\delta(\varepsilon)$ by

$$\delta(\varepsilon) := \inf\{\delta \in \sqrt{|K \setminus \{0\}|} : X_\varepsilon \cap U_0(\delta), \ldots, X_\varepsilon \cap U_p(\delta) \text{ covers } X_\varepsilon\}.$$

The function $\delta(\varepsilon)$ is definable in the sense of [**7**, Definition 2.7]. Therefore, by the Quantifier Elimination Theorem [**7**, Theorem 4.2], there are $c, \varepsilon_0 \in \sqrt{|K \setminus \{0\}|}$, $\varepsilon_0 < 1$, and $\alpha \in \mathbb{Q}$ such that for $1 > \varepsilon \geq \varepsilon_0$,

$$\delta(\varepsilon) = c\varepsilon^\alpha.$$

Let $e \in K^\circ$ satisfy $|e| = \varepsilon_0^\ell$. Since $\delta(\varepsilon) < 1$, we have two possibilities.

***Case* (A).** — $\lim_{\varepsilon \to 1} \delta(\varepsilon) < 1$.

Choose $\delta \in \sqrt{|K \setminus \{0\}|}$, $\delta < 1$, with $\lim_{\varepsilon \to 1} \delta(\varepsilon) < \delta$. Then $\{U_0(\delta), \ldots, U_p(\delta)\}$ is the desired refinement of $\mathfrak{A}$ by closed R-subdomains of level ≤ 1.

***Case* (B).** — $\lim_{\varepsilon \to 1} \delta(\varepsilon) = 1$.

In this case, $c = 1$ and $\alpha > 0$. Write $\alpha = a/b$, $a, b \in \mathbb{N}$. Since each U_i is of level ≤ 1, we may write

$$\mathcal{O}_X(U_i) = A\left\langle \frac{f_{i1}}{h_i}, \ldots, \frac{f_{ir_i}}{h_i} \right\rangle \left[\left[\frac{g_{i1}}{h_i}, \ldots, \frac{g_{is_i}}{h_i} \right]\right]_s,$$

where $A = S_{m,n}/I$, as above, and

$$J_i := (f_{i1}, \ldots, f_{ir_i}, g_{i1}, \ldots, g_{is_i}, h_i)$$

is the unit ideal for $0 \leq i \leq p$.

Let $\overline{\rho}_i$ be the image of ρ_i in $\mathcal{O}_X(X)$. Define

$$X_j := \{x \in X : |\overline{\rho}_j(x)| = \max_{1 \leq i \leq n} |\overline{\rho}_i(x)| \text{ and } |\overline{\rho}_j(x)| \geq \varepsilon_0\}.$$

Note that X is covered by X_{ε_0} and the X_j. For $x \in X_j$, we have

$$\delta(|\overline{\rho}_j(x)|) = |\overline{\rho}_j(x)|^{a/b} < |\overline{\rho}_j(x)|^{a/2b} < 1. \tag{3.1.1}$$

Put

$$U'_{ij} := \operatorname{Max} A \left\langle \frac{f_{i1}}{h_i}, \ldots, \frac{f_{ir_i}}{h_i}, \frac{g_{i1}^{2b}}{\rho_j^a h_i^{2b}}, \ldots, \frac{g_{is_i}^{2b}}{\rho_j^a h_i^{2b}}, \frac{e}{\rho_j^\ell}, \frac{\rho_1}{\rho_j}, \ldots, \frac{\rho_n}{\rho_j} \right\rangle.$$

By (3.1.1),

$$\{U'_{ij}\}_{\substack{0\le i\le p\\ 1\le j\le n}} \cup \{X_{\varepsilon_0}\cap U_i\}_{0\le i\le p}$$

is a refinement of $\mathfrak{A}$ that covers X because $\{U'_{ij}\}_{0\le i\le p}$ covers X_j. Since X_{ε_0} is affinoid and $\{U_i\}_{0\le i\le p}$ is a quasi-affinoid covering, there are finitely many rational subdomains V_j of X_{ε_0} such that $\{V_j\}_{0\le j\le q}$ is a covering of X_{ε_0} that refines $\{X_{\varepsilon_0}\cap U_i\}_{0\le i\le p}$. By [**2**, Theorem 7.2.4.2], each V_j is of level ≤ 1 (in fact defined by polynomial inequalities). Moreover, each U'_{ij} is of level ≤ 1. To see this, observe first that in the definition of U'_{ij} we may assume that $\ell = a$ and hence that U'_{ij} is defined by the inequalities

$$\begin{aligned} |f_{ik}| &\le |h_i| && k=1,\ldots,r_i,\\ |g_{ik}^{2b}| &\le |\rho_j^a h_i^{2b}| && k=1,\ldots,s_i,\\ |e| &\le |\rho_j^a| &&\\ |\rho_k| &\le |\rho_j| && k=1,\ldots,n. \end{aligned}$$

These inequalities are equivalent to

$$\begin{aligned} |\rho_j^a f_{ik} h_i^{2b-1}| &\le |\rho_j^a h_i^{2b}| && k=1,\ldots,r_i,\\ |e f_{ik}^{2b}| &\le |\rho_j^a h_i^{2b}| && k=1,\ldots,r_i,\\ |g_{ik}^{2b}| &\le |\rho_j^a h_i^{2b}| && k=1,\ldots,s_i,\\ |e h_i^{2b}| &\le |\rho_j^a h_i^{2b}| &&\\ |\rho_k \rho_j^{a-1} h_i^{2b}| &\le |\rho_j^a h_i^{2b}| && k=1,\ldots,n. \end{aligned}$$

This is immediate from the fact that J_i is the unit ideal and the Nullstellensatz ([**6**, Theorem 4.1.1]). The functions occurring in the second set of inequalities generate the unit ideal and thus these inequalities define U'_{ij} as a closed R-subdomain of X of level ≤ 1. Therefore

$$\{U'_{ij}\}_{0\le i\le p} \cup \{V_j\}_{0\le j\le q}$$

is the desired refinement of $\mathfrak{A}$. □

In fact, the generalization of Theorem 3.1.4 to level $\le \ell$, $\ell > 1$, is true, as can be seen by a careful examination of the proof of Theorem 3.1.5, but since we do not need this extra information, we do not keep track of it in the proof. Though we don't use it, we include the following theorem which gives a complete characterization of quasi-affinoid coverings.

Theorem 3.1.5. — *A covering is quasi-affinoid if, and only if, it has a refinement by finitely many closed R-subdomains.*

Proof

($\Leftarrow$) Immediate, by Remark 3.1.2 and Lemma 2.3.5.

($\Rightarrow$) Let $X = \operatorname{Max} S_{m,n}/I$ and suppose $U_0,\ldots,U_p$ is a quasi-affinoid covering of X. Suppose $X_0,\ldots,X_n$ is a covering of X by closed R-subdomains. It suffices to show that each quasi-affinoid covering $\{X_j \cap U_i\}_{0\le i\le p}$ of X_j, $0 \le j \le n$, has a refinement by finitely many closed R-subdomains.

Fix $e \in K^{\circ\circ}\setminus\{0\}$, and consider the following covering of X by closed R-subdomains $X_0, \ldots, X_n$:

$$X_0 := X\left(\frac{\rho_1}{e}, \ldots, \frac{\rho_n}{e}\right),$$
$$X_j := X\left(\frac{e}{\rho_j}, \frac{\rho_i}{\rho_j}\right)_{1\le i\le n}, \qquad 1 \le j \le n.$$

Since X_0 is affinoid and $\{U_i\}_{0\le i\le p}$ is a quasi-affinoid covering of X, $\{X_0 \cap U_i\}_{0\le i\le p}$ has a refinement by finitely many rational domains.

Observe that

$$\mathcal{O}_X(X_j) = S_{m+n,n} \Big/ (I + (\xi_{m+j}\rho_j - e) + \sum_{i\ne j}(\xi_{m+i}\rho_j - \rho_i)),$$

$1 \le j \le n$. Making the substitutions $\rho_i = \xi_{m+i}\rho_j$, $i \ne j$, we may write

$$\mathcal{O}_X(X_j) = S_{m+n,1}/I_j,$$

for the corresponding ideal I_j. Thus, we have reduced the theorem to the case $n = 1$; i.e.,

$$X = \operatorname{Max} S_{m,1}/I.$$

Let $\varepsilon \in \sqrt{|K\setminus\{0\}|}$, and consider the covering $\{X_\varepsilon \cap U_i\}_{0\le i\le p}$ of the affinoid variety X_ε. By assumption, this covering has a finite refinement by rational domains, hence by Lemma 3.1.3, for some $\delta \in \sqrt{|K\setminus\{0\}|}$, $\delta < 1$, $\{X_\varepsilon \cap U_i(\delta)\}$ is a covering of X_ε. We may therefore define the function $\delta(\varepsilon)$ by

$$\delta(\varepsilon) := \inf\{\delta \in \sqrt{|K\setminus\{0\}|} : X_\varepsilon \cap U_0(\delta), \ldots, X_\varepsilon \cap U_p(\delta) \text{ covers } X_\varepsilon\}.$$

The function $\delta(\varepsilon)$ is definable in the sense of [**7**, Definition 2.7]. Therefore, by the Quantifier Elimination Theorem [**7**, Theorem 4.2], there are c, $\varepsilon_0 \in \sqrt{|K\setminus\{0\}|}$, $\varepsilon_0 < 1$, and $\alpha \in \mathbb{Q}$ such that for $\varepsilon \ge \varepsilon_0$,

$$\delta(\varepsilon) = c\varepsilon^\alpha.$$

Since $\delta(\varepsilon) < 1$, we have two possibilities.

***Case* (A).** — $\lim_{\varepsilon\to 1}\delta(\varepsilon) < 1$.

Choose $\delta \in \sqrt{K\setminus\{0\}|}$, $\delta < 1$, with $\lim_{\varepsilon\to 1}\delta(\varepsilon) < \delta$. Then $\{U_0(\delta), \ldots, U_p(\delta)\}$ is the desired refinement of $\{U_i\}$.

***Case* (B).** — $\lim_{\varepsilon\to 1}\delta(\varepsilon) = 1$.

In this case $c = 1$ and $\alpha > 0$. Write $\alpha = a/b$, $a, b \in \mathbb{N}$. Let $\overline{\rho}_1$ be the image of ρ_1 in $\mathcal{O}_X(X)$. When $|\overline{\rho}_1(x)| \geq \varepsilon_0$, we have

$$\delta(|\overline{\rho}_1(x)|) = |\overline{\rho}_1(x)|^{a/b} < |\overline{\rho}_1(x)|^{a/2b} < 1.$$

Write

$$\mathcal{O}_X(U_i) = S_{m+r_i,1+s_i}/J_i,$$

where J_i is determined according to [**6**, Definition 5.3.3]. Put

$$U_i' := \operatorname{Max} S_{m+r_i+1+s_i,1+s_i}/J_i',$$

where

$$J_i' = J + (\rho_1 \xi_{m+r_i+1} - \varepsilon_0) + \sum_{j=1}^{s_i} (\rho_1^a \xi_{m+r_i+1+j} - \rho_{1+j}^{2b}).$$

Put

$$J_i'' := S_{m+r_i+1+s_i,1} \cap J_i'.$$

By inspection,

$$U_i' = \operatorname{Max} S_{m+r_i+1+s_i,1}/J_i''$$

is exactly the closed R-subdomain obtained from U_i by replacing each strict inequality $|f| < |g|$ that occurred in its definition by the weak inequality $|f| \leq |\rho_1|^{a/2b}|g|$. Now, $\{U_i'\}_{0 \leq i \leq p} \cup \{X_{\varepsilon_0} \cap U_i\}_{0 \leq i \leq p}$ is a refinement of $\{U_i\}_{0 \leq i \leq p}$. As above, we find a refinement of the covering $\{X_{\varepsilon_0} \cap U_i\}_{0 \leq i \leq p}$ of the affinoid variety X_{ε_0} by rational domains $\{V_j\}_{0 \leq j \leq q}$. Finally $\{U_i'\}_{0 \leq i \leq p} \cup \{V_j\}_{0 \leq j \leq q}$ is the desired refinement of $\{U_i\}_{0 \leq i \leq p}$ by closed R-subdomains of X. □

Theorem 3.1.4, together with the following lemmas, provide the successively simpler refinements of a quasi-affinoid covering that are required to prove the Acyclicity Theorem of the next section.

Definition 3.1.6. — Let A be quasi-affinoid, $X := \operatorname{Max} A$. A *rational covering* of X is a covering of the form

$$\left\{ X\left(\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}\right) \right\}_{1 \leq i \leq n},$$

where $f_1, \ldots, f_n \in A$ generate the unit ideal. Clearly, any rational covering is quasi-affinoid.

Lemma 3.1.7. — *Any finite covering of X by closed R-domains of level ≤ 1 has a refinement which is a rational covering.*

Proof. — Exactly as in [**2**, Lemma 8.2.2.2]. □

Definition 3.1.8. — Let A be quasi-affinoid, $X := \operatorname{Max} A$. Let $f_1, \ldots, f_n \in A$. A *Laurent covering* of X is a covering of the form

$$\{X(f_1^{\alpha_1}, \ldots, f_n^{\alpha_n})\}_{(\alpha_1,\ldots,\alpha_n)\in\{1,-1\}^n}.$$

Any Laurent covering is quasi-affinoid.

Lemma 3.1.9. — *Let $\mathfrak{A}$ be a rational covering of X. Then there is a Laurent covering $\mathfrak{B}$ of X such that for each $V \in \mathfrak{B}$, the covering $\mathfrak{A}|_V$ is a rational covering of V generated by units $f_1, \ldots, f_n$ of $\mathcal{O}(V)$ such that there are $F_1, \ldots, F_n \in \mathcal{O}_X(X)$ with $f_i = F_i|_V$, $1 \le i \le n$.*

Proof. — As in [**2**, Lemma 8.2.2.3]. □

Lemma 3.1.10. — *Let $\mathfrak{A}$ be a rational covering of X generated by units of $\mathcal{O}_X(X)$. Then there is a Laurent covering $\mathfrak{B}$ which is a refinement of $\mathfrak{A}$.*

Proof. — As in [**2**, Lemma 8.2.2.4]. □

3.2. The Quasi-Affinoid Acyclicity Theorem. — The Quasi-affinoid Acyclicity Theorem, Theorem 3.2.4, is the main result of this paper. It follows immediately that the quasi-affinoid structure presheaf $\mathcal{O}_X$ is a sheaf for the rigid G-topology of the quasi-affinoid variety X.

Lemma 3.2.1 (cf. [9]). — *Let $X = \operatorname{Max} A$ be quasi-affinoid and let $f \in A$. Then the covering $\mathfrak{A} := \{X(f), X(f^{-1})\}$ of X is $\mathcal{O}_X$-acyclic.*

Proof. — We follow [**2**, Section 8.2.3], which treats the affinoid case. Since there are only two open sets in $\mathfrak{A}$, the alternating Čech cohomology modules $C_a^q(\mathfrak{A}, \mathcal{O}_X) = (0)$ if $q \neq 0, 1$. Thus, by Proposition 1.1.1, it suffices to prove that the sequence

$$0 \longrightarrow \mathcal{O}_X(X) \xrightarrow{\varepsilon} C_a^0(\mathfrak{A}, \mathcal{O}_X) \xrightarrow{d^0} C_a^1(\mathfrak{A}, \mathcal{O}_X) \longrightarrow 0$$

is exact, where the augmentation homomorphism ε is defined by

$$\varepsilon(g) := (g|_{X(f)}, g|_{X(f^{-1})}).$$

Since $A = \mathcal{O}_X(X)$, the above sequence may be written

$$0 \longrightarrow A \xrightarrow{\varepsilon} A\langle f\rangle \times A\langle f^{-1}\rangle \xrightarrow{d^0} A\langle f, f^{-1}\rangle \longrightarrow 0,$$

where ε is induced by the canonical inclusions of A in $A\langle f\rangle$ and $A\langle f^{-1}\rangle$, and

$$d^0(g_0, g_1) := g_1 - g_0.$$

Let η and ζ be indeterminates. It is sufficient to establish the exactness of the following commutative diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\zeta - f)A\langle\zeta\rangle \times (1 - f\eta)A\langle\eta\rangle & \xrightarrow{\lambda'} & (\zeta - f)A\langle\zeta, \zeta^{-1}\rangle & \longrightarrow & 0 \\
 & & & & \cap\downarrow & & \cap\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\iota} & A\langle\zeta\rangle \times A\langle\eta\rangle & \xrightarrow{\lambda} & A\langle\zeta, \zeta^{-1}\rangle & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varepsilon} & A\langle f\rangle \times A\langle f^{-1}\rangle & \xrightarrow{d^0} & A\langle f, f^{-1}\rangle & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The map $\iota : A \to A\langle\zeta\rangle \times A\langle\eta\rangle$ is the canonical injection, λ is determined by

$$\lambda : A\langle\zeta\rangle \times A\langle\eta\rangle \longrightarrow A\langle\zeta, \zeta^{-1}\rangle : (h_0(\zeta), h_1(\eta)) \longmapsto h_1(\zeta^{-1}) - h_0(\zeta),$$

and λ' is induced by λ.

The columns are exact because

$$A\langle f\rangle = A\langle\zeta\rangle/(\zeta - f), \qquad A\langle f^{-1}\rangle = A\langle\eta\rangle/(1 - f\eta),$$
$$A\langle f, f^{-1}\rangle = A\langle\zeta, \zeta^{-1}\rangle/(\zeta - f).$$

To check the exactness of the first two rows, we require the direct sum decomposition

$$A\langle\zeta\rangle \oplus \zeta^{-1}A\langle\zeta^{-1}\rangle = A\langle\zeta, \zeta^{-1}\rangle = A\langle\zeta, \eta\rangle/(\zeta\eta - 1). \tag{3.2.1}$$

This follows from the fact that for any complete quasi-Noetherian B-ring $B \subset K^\circ$, we have the direct sum decomposition of $B\langle\xi_1, \ldots, \xi_m\rangle[\![\rho_1, \ldots, \rho_n]\!]$-modules

$$B\langle\xi_1, \ldots, \xi_{m+2}\rangle[\![\rho_1, \ldots, \rho_n]\!] = M \oplus N,$$

where

$$M := \left\{ \sum_\nu \rho^\nu \left(\sum_{\mu_{m+2} \geq \mu_{m+1}} a_{\mu\nu}\xi^\mu \right) \right\}$$
$$N := \left\{ \sum_\nu \rho^\nu \left(\sum_{\mu_{m+2} < \mu_{m+1}} a_{\mu\nu}\xi^\mu \right) \right\}.$$

This decomposition induces the corresponding decomposition on $S_{m+2,n}$, which, in turn, induces the decomposition (3.2.1). From (3.2.1), we obtain

$$(\zeta - f)A\langle\zeta,\zeta^{-1}\rangle = (\zeta - f)A\langle\zeta\rangle \oplus (1 - f\zeta^{-1})A\langle\zeta^{-1}\rangle.$$

This yields the surjectivity of λ' and (3.2.1) yields the surjectivity of λ. In particular, the first row is exact. To check the exactness of the second row, note that

$$\lambda\left(\sum_{i\geq 0} a_i\zeta^i, \sum_{i\geq 0} b_i\eta^i\right) = \sum_{i\geq 0} b_i\zeta^{-i} - \sum_{i\geq 0} a_i\zeta^i = 0$$

if, and only if, $a_i = b_i = 0$ for $i > 0$ and $a_0 = b_0$ (see the discussion following [**6**, Definition 5.2.7]).

To see that ε is injective, let $g \in A$, $g \neq 0$. Then, since being 0 is a local property, there is some maximal ideal $\mathfrak{m}$ of A such that the image of g in the localization $A_{\mathfrak{m}}$ is not zero. Thus, by the Krull Intersection Theorem [**8**, Theorem 8.10], the image of g in the completion $\widehat{A}_{\mathfrak{m}}$ is not zero. Since $\{X(f), X(f^{-1})\}$ covers X, the conclusion follows from [**6**, Proposition 5.3.6 (ii)]. Now, by some diagram-chasing, the third row is exact. □

Corollary 3.2.2. — *Let $X = \operatorname{Max} A$ be quasi-affinoid, then any Laurent covering (see Definition 3.1.8) of X is $\mathcal{O}_X$-acyclic.*

Proof. — Use Lemma 3.2.1 and apply Proposition 1.1.5 inductively. □

In fact, the rest of the proof of the $\mathcal{O}_X$-acyclicity of quasi-affinoid coverings holds in greater generality.

Proposition 3.2.3. — *Let $\mathcal{F}$ be a presheaf on the quasi-affinoid variety X. Assume that Laurent coverings are universally $\mathcal{F}$-acyclic on X; i.e., that for each R-subdomain $X' \subset X$, all Laurent coverings of X' are $\mathcal{F}$-acyclic. Then all quasi-affinoid coverings of X are $\mathcal{F}$-acyclic.*

Proof. — The proposition is proved by induction on the complexity of the quasi-affinoid covering, after successive simplifications.

***Claim* (A).** — *Rational coverings (see Definition 3.1.6) generated by invertible functions are universally $\mathcal{F}$-acyclic.*

By Lemma 3.1.10, such a covering is refined by a Laurent covering. Apply Proposition 1.1.4.

***Claim* (B).** — *Rational coverings are universally $\mathcal{F}$-acyclic.*

Let $\mathfrak{A}$ be a rational covering. By Lemma 3.1.9, there is a Laurent covering $\mathfrak{B}$ such that for each $V \in \mathfrak{B}$, the covering $\mathfrak{A}|_V$ is a rational covering of V generated by units of $\mathcal{O}_X(V)$, hence is $\mathcal{F}$-acyclic by Claim A. For $U \in \mathfrak{A}$, $\mathfrak{B}|_U$ is a Laurent covering, hence by assumption is $\mathcal{F}$-acyclic. Since $\mathfrak{B}$ is $\mathcal{F}$-acyclic by assumption, the claim follows from Proposition 1.1.3.

***Claim* (C).** — *Coverings by closed R-domains of level ≤ 1 are universally $\mathcal{F}$-acyclic.*

Let $\mathfrak{A}$ be such a covering. By Lemma 3.1.7, $\mathfrak{A}$ has a rational refinement, $\mathfrak{B}$. Now Claim C follows from Claim B and Proposition 1.1.4.

***Claim* (D).** — *Quasi-affinoid coverings by R-domains of level ≤ 1 are universally $\mathcal{F}$-acyclic.*

Let $\mathfrak{A} = \{U_0, \ldots, U_p\}$ be such a covering. By Theorem 3.1.4, $\mathfrak{A}$ has a refinement $\mathfrak{B}$ by finitely many closed R-subdomains of level ≤ 1. For each $U_{i_0\ldots i_r}$, and each r, $\mathfrak{B}|_{U_{i_0\ldots i_r}}$ is a covering of $U_{i_0\ldots i_r}$ by finitely many closed R-subdomains, which, as subdomains of $U_{i_0\ldots i_r}$ have level ≤ 1. Therefore Claim D follows from Claim C and Proposition 1.1.4.

We now conclude the proof of the theorem.

Let $\mathfrak{A} = \{U_0, \ldots, U_p\}$ be a quasi-affinoid covering of X. We say that $\mathfrak{A}$ is of *type* $\leq (\ell, j)$ iff $U_0, \ldots, U_j$ are of level $\leq \ell + 1$ and $U_{j+1}, \ldots, U_p$ are of level $\leq \ell$.

Order the types lexicographically. We prove the claim by induction on (ℓ, j). When $\ell = 1$, $j = -1$, this is Claim D. Suppose the claim holds for quasi-affinoid coverings of type $\leq (\ell, j)$, and let

$$\mathfrak{B} = \{U_0, \ldots, U_j, U'_{j+1}, U_{j+2}, \ldots, U_p\}$$

be a quasi-affinoid covering of type $\leq (\ell, j+1)$. Now, since U'_{j+1} is of level $\leq \ell + 1$, there is an R-subdomain U_{j+1} of level $\leq \ell$ such that $U'_{j+1} \subset U_{j+1}$ and U'_{j+1} is of level ≤ 1 in U_{j+1}. Consider the covering

$$\mathfrak{A} := \{U_0, \ldots, U_{j+1}, \ldots, U_p\},$$

which is a quasi-affinoid covering of type $\leq (\ell, j)$, hence by the inductive hypothesis is $\mathcal{F}$-acyclic.

To apply Proposition 1.1.4, we consider the coverings $\mathfrak{B}|_{U_{i_0\ldots i_r}}$. If some index $i_s \neq j+1$, then $U_{i_0\ldots i_r} \subset U_{i_s}$, and $\mathfrak{B}|_{U_{i_0\cdots i_r}}$ is refined by the trivial covering $\{U_{i_0\ldots i_r}\} \cap U_{i_s}$. In this case, $\mathfrak{B}|_{U_{i_0\cdots i_r}}$ is $\mathcal{F}$-acyclic by Proposition 1.1.2 since the trivial covering is $\mathcal{F}$-acyclic. It remains to consider the covering

$$\mathfrak{B}|_{U_{j+1}} = \{U_{j+1} \cap U_0, \ldots, U_{j+1} \cap U_j, U'_{j+1}, U_{j+1} \cap U_{j+2}, \ldots, U_{j+1} \cap U_p\}.$$

This is a covering of U_{j+1} of type $\leq (\ell, j)$, which is $\mathcal{F}$-acyclic by the inductive hypothesis. Now, since $\mathfrak{A}$ is $\mathcal{F}$-acyclic, $\mathfrak{B}$ must also be $\mathcal{F}$-acyclic by Proposition 1.1.2. To finish the proof, note that any quasi-affinoid covering of type $\leq (\ell + 1, -1)$ is of type $\leq (\ell, p)$ for some p. □

***Theorem 3.2.4* (Quasi-Affinoid Acyclicity Theorem).** — *Let X be a quasi-affinoid variety. Any quasi-affinoid covering of X is $\mathcal{O}_X$-acyclic.*

Proof. — This is an immediate consequence of Corollary 3.2.2 and Proposition 3.2.3. □

Corollary 3.2.5. — *Let X be a quasi-affinoid variety. Then $\mathcal{O}_X$ is a sheaf with respect to the rigid G-topology on X.*

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