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LEONARD LIPSHITZ

Model completeness and subanalytic sets

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MODEL COMPLETENESS AND SUBANALYTIC SETS

1. Introduction

The class of real subanalytic sets was defined by Gabrielov [**2**], where he proved that the class is closed under complementation. Real subanalytic sets have attracted extensive study; in particular, Hironaka [**7**] proved uniformization and rectilinearization theorems for real subanalytic sets. In [**1**], Denef and van den Dries introduced the class of p-adic subanalytic sets and showed how to develop both the real and p-adic theories from a suitable analytic quantifier elimination theorem. In [**9**] an analogous quantifier elimination theorem was proved for K an algebraically closed field, complete with respect to a non-Archimedean absolute value, using the functions of $S = \cup_{m,n} S_{m,n}$. (See below.) That paper developed a theory of subanalytic sets (termed rigid subanalytic sets). This theory was developed further in [**10**], [**11**] and [**12**]. In [**17**]–[**21**], Schoutens developed a theory of subanalytic sets (which he termed strongly subanalytic), over such fields. This theory used a class of functions somewhat smaller than $T = \cup T_m$. (The T_m are the Tate rings of strictly convergent power series over K.)

In this paper we prove a quantifier elimination theorem (Theorem 4.2) for algebraically closed extension fields of K in language $L_{\mathcal{E}}$, the language of valued rings augmented with function symbols for the members of $\mathcal{E}$, where $\mathcal{E} = \mathcal{E}(\mathcal{H})$ is a class of analytic (partial) functions obtained from $\mathcal{H} \subset S$ by closing up with respect to "differentiation" and existential definition (see below for precise definitions). For suitable choice of $\mathcal{E} = \mathcal{E}(T)$ this gives a quantifier elimination theorem (Corollary 4.4) in $L_{\mathcal{E}(T)}$ (or a quantifier simplification theorem, Corollary 4.5, in L_T, the language of valued rings augmented with function symbols for the members of T) suitable for developing the theory of subanalytic sets based on T, which we term K-affinoid (Corollaries 5.4 and 5.5). These results have been used by Gardener and Schoutens in their proof, [**3**], [**4**], and [**22**], of a quantifier elimination theorem in the language L_T^D ($= L_T$ enriched by "restricted division" (see below)). Section 2 contains precise definitions of what we mean by "closed under differentiation and existential definitions", in all characteristics. Section 3 gives the Weierstrass Preparation and Division Theorems for these classes of functions that we need for all the Elimination Theorems in Section 4. Section 5 contains the application of the Elimination Theorems to the theory of Subanalytic Sets.

We recall some of the basic definitions. K is a field complete with respect to a non-Archimedean absolute value $|\cdot| : K \to \mathbb{R}_+$. We do not assume that K is algebraically closed. $K^\circ = \{x \in K : |x| \leq 1\}$ is the valuation ring of K, and $K^{\circ\circ} = \{x \in K : |x| < 1\}$ is the maximal ideal of K°. $T_m = T_m(K)$ is the (Tate) ring of strictly convergent power series over K and $S_{m,n} = S_{m,n}(E, K)$ is a ring of separated power series over K (see [**13**, Definition 2.1.1]). Recall that $T_{m+n} \subset S_{m,n}$ and that elements of $S_{m,n}$ represent analytic functions $(K^\circ)^m \times (K^{\circ\circ})^n \to K$.

The language of multiplicatively valued rings is

$$L = (0, 1, +, \cdot, |\cdot|, \overline{0}, \overline{1}, \bar{\cdot}, \overline{<}).$$

The symbols 0, 1, +, $\cdot$ denote the obvious elements and operations on the field; $\overline{0}$, $\overline{1}$, $\bar{\cdot}$ denote the obvious elements and multiplication on the value group $\cup\{\overline{0}\}$; $|\cdot|$ denotes the valuation and $\overline{<}$ the order relation on the value group $\cup\{\overline{0}\}$. Section 0 of [**1**] provides all the background about first order languages that we will need.

A structure F (for a language L') has elimination of quantifiers if every subset of F^m defined by an L'-formula is in fact defined by a quantifier free L'-formula. We say that F has quantifier simplification (or is model complete) if every subset of F^m defined by an L' formula is in fact defined by an existential L'-formula.

In [**13**] we defined certain open domains in K^m which we termed R-domains ([**13**, Definition 5.3.3]) and showed that each R-domain U carries a canonical ring of functions denoted $\mathcal{O}(U)$; R-domains generalize the Rational Domains of Affinoid geometry.

2. Existentially Defined Analytic Functions

As usual K is a complete non-Archimedean valued field. Let F be a complete field extending K and let F_{alg} be its algebraic closure. In general F_{alg} will not be complete. However if $F' \subset F_{\text{alg}}$ is a finitely generated extension of F, then F' is complete and hence the power series $f \in S_{m,n}$ actually define analytic functions $(F_{\text{alg}}^\circ)^m \times (F_{\text{alg}}^{\circ\circ})^n \to F_{\text{alg}}$. By the Nullstellensatz ([**13**, Theorem 4.1.1]) there is a map

$$\tau_m : (F_{\text{alg}}^\circ)^m \to \operatorname{Max} T_m(F).$$

Since $T_m(K) \subset T_m(F)$ we may therefore regard any R-domain $U \subset \operatorname{Max} T_m(K)$ as a subset of $(F_{\text{alg}}^\circ)^m$. In this section we set up the formalism for the quantifier elimination theorem.

The (not necessarily algebraically closed) field K will be the field over which the functions in our language are defined in the sense that these functions will all be elements of generalized rings of fractions (see below) defined over K. Formulas in the language define subsets of $(F_{\text{alg}}^\circ)^m$. The Quantifier Elimination Theorem (Theorem 4.2) is uniform in the sense that if φ is defined over K then there is a quantifier-free formula φ^*, also defined over K, such that for each complete F with $K \subset F$, φ and φ^* define the same subset of $(F_{\text{alg}}^\circ)^m$.

In [**1**] and [**9**] the quantifier elimination takes place in a language L_{an}^D which has symbols for all functions built up from a suitable class of analytic functions and "restricted division" D, where $D(x,y) = x/y$ if $|x| \le |y| \neq 0$ and $D(x,y) = 0$ otherwise. In this paper the use of "restricted division" is replaced by that of generalized rings of fractions (see definition below). This is necessary for us because Theorems 3.1 and 3.3 give definitions of the Weierstrass data in terms of functions, but do not in general produce representations of the Weierstrass data by (definable) D-terms. (In the special case that $\mathcal{H} = S$, definability issues drop away and the treatment in this paper is easily seen to be equivalent to the treatment of [**9**] using restricted division. See Corollary 4.3).

Definition 2.1 (*cf.* [**13**, Definition 5.3.1]). — We define the *generalized rings of fractions over T_m* inductively as follows: T_m is a generalized ring of fractions, and if A is generalized ring of fractions and $f, g \in A$ then both $A\langle f/g\rangle$ and $A[\![f/g]\!]_s$ are generalized rings of fractions.

$S_{m,n} = T_{m,n}[\![\xi_{m+1}, \ldots, \xi_{m+n}]\!]_s$ is a generalized ring of fractions over T_{m+n}.

Definition 2.2. — Let $\varphi : T_m \to A$ be a generalized ring of fractions and let Φ : $\operatorname{Max} A \to \operatorname{Max} T_m$ be the induced map. We define the *domain of A*, $\operatorname{Dom} A \subset \operatorname{Max} T_m$, by saying that $x \in \operatorname{Dom} A$ iff there is a quasi-rational subdomain U (see [**13**, Definition 5.3.3]) of $\operatorname{Max} T_m$ with $x \in U$, such that

$$\Phi^{-1}(U) \to U$$

is bijective.

Remark 2.3

(i) The set $\operatorname{Dom} A$ does not depend on the representation of the generalized ring of fractions A as a quasi-affinoid T_m-algebra. Suppose that $\varphi : T_m \to A$ and $\psi : T_m \to B$ are isomorphic quasi-affinoid T_m-algebras, i.e. there is a K-algebra isomorphism σ such that

$$\begin{array}{ccc} A & \xrightarrow{\quad\sigma\quad} & B \\ & {\scriptstyle\varphi}\nwarrow \quad \nearrow{\scriptstyle\psi} & \\ & T_m & \end{array}$$

commutes. By the Nullstellensatz [**13**, Theorem 4.1.1]

$$X := \operatorname{Max} T_m \cap \operatorname{Max} A = \operatorname{Max} T_m \cap \operatorname{Max} B.$$

Let $x \in X$ and suppose there is a quasi-rational subdomain $x \in U \subset X$ such that $\Phi^{-1}(U) \to U$ is bijective, where $\Phi : \operatorname{Max} A \to \operatorname{Max} T_m$ corresponds to φ (as in Definition 2.2). Let Ψ correspond to ψ. Since σ is an isomorphism, $\Psi^{-1}(U) \to U$ is bijective. Since the argument is symmetric in A and B, this shows that $\operatorname{Dom} A$ is independent of the presentation of A as a T_m-algebra. (Note however that $\operatorname{Dom} A$ is not in general a quasi-affinoid subdomain in the sense of [**13**, Definition 5.3.4].)

(ii) Let $\varphi : T_m \to A$ be a generalized ring of fractions. It follows from the Nullstellensatz, ([**13**, Theorem 4.1.1]), that

$$\operatorname{Dom} A\langle f/g\rangle = \{x \in \operatorname{Dom} A : |f(x)| \leq |g(x)| \neq 0\}, \text{ and}$$
$$\operatorname{Dom} A[\![f/g]\!]_s = \{x \in \operatorname{Dom} A : |f(x)| < |g(s)|\}.$$

(iii) Let $\varphi : T_m \to A$ be a generalized ring of fractions. By ground field extension ([**13**, Definition 5.4.9 and Proposition 5.4.10]) $A \subset A' = S_{0,0}(E,F) \otimes^s_{S_{0,0}(E,K)} A$ and we may regard $\operatorname{Dom} A$ as a subset of $(F^\circ_{\mathrm{alg}})^m$ and each $f \in A$ as determining an analytic function $\operatorname{Dom} A \to F_{\mathrm{alg}}$. In fact, given $x \in \operatorname{Dom} A$, there is a unique power series $\overline{f} \in K[\![\xi]\!]$ and a rational polydisc $x \in U \subset \operatorname{Dom} A$ such that $\overline{f}(y - x)$ converges on U and $f(y) = \overline{f}(y - x)$ for all $y \in U$.

(iv) As we noted in the discussion before Definition 2.1, in this paper we work with generalized rings of fractions instead of with D-functions. Any element f of a generalized ring of fractions A over T_m defines a partial function on $\operatorname{Dom} A \subset \operatorname{Max} T_m$. We may regard f as a total function by assigning $f(x) = 0$ for $x \in \operatorname{Max} T_m \setminus \operatorname{Dom} A$. It is a consequence of (ii) above that such functions are represented by D-terms in the sense of [**9**, Section 3.2], and conversely.

We will see below that the Weierstrass data of a power series are existentially definable from f and its partial derivatives. In characteristic $p \neq 0$, "partial derivatives" must be interpreted as Hasse derivatives which we define next.

Definition 2.4. —Let $f \in R[\![\xi_1, \dots, \xi_m]\!]$, R a commutative ring, and let $t = (t_1, \dots, t_m)$. The *Hasse Derivatives* of f, denoted $D_\nu f \in R[\![\xi]\!]$, $\nu = (\nu_1, \dots, \nu_m) \in \mathbb{N}^m$, are defined by the equation

$$f(\xi + t) = \sum_{\nu \in \mathbb{N}^m} (D_\nu f)(\xi) t^\nu.$$

(See [**5**] or [**6**, Section 3].)

Remark 2.5

(i) In characteristic zero the Hasse derivatives are constant multiples of the usual partial derivatives. In fact

$$\frac{\partial^{|\nu|}}{\partial \xi_1^{\nu_1} \dots \partial \xi_m^{\nu_m}} f = \nu_1! \dots \nu_m! D_\nu f.$$

Hence the partial derivatives of f and the Hasse derivatives of f are quantifier free definable from each other (*cf.* Definition 2.7). The following facts are not hard to prove. Proofs can be found in [**5**] or [**6**, Section 3].

(ii) In characteristic $p \neq 0$ the situation is more complicated. If

$$\nu = (0, \dots, 0, p^n, 0 \dots, 0)$$

with p^n in the i^{th} position denote D_ν by D_i^n. Then the whole family of Hasse derivatives is generated by the D_i^n under composition. In particular $D_i^m D_j^n = D_j^n D_i^m$ and $D_\nu = D_1^{\nu_1} D_2^{\nu_2} \dots D_m^{\nu_m}$.

(iii) Suppose the characteristic is $p \neq 0$ and let $f \in R[\![\xi_1, \dots, \xi_m]\!]$. Fix $i, 1 \leq i \leq m$, and write

$$f = \sum_{j=0}^{p-1} f_j(\xi_1, \dots, \xi_{i-1}, \xi_i^p, \xi_{i+1}, \dots, \xi_m)\xi_i^j.$$

The power series f_j are uniquely determined by this equation, so we may define

$$\delta_{\xi_i, j}(f) := f_j.$$

If f converges on a rational polydisc $0 \in U \subset (F_{\text{alg}}^\circ)^m$, so do the $\delta_{\xi_i,j}(f)$. We call the $\delta_{\xi_i,j}(f)$ the *p-components* of f. By induction, we define the *$p^{\ell+1}$-components* of f to be the $\delta_{\xi_i,j}(g)$, where g is a p^ℓ-component of f. Thus,

$$f = \sum_{j=0}^{p^\ell - 1} f_{\ell j}(\xi_1, \dots, \xi_{i-1}, \xi_i^{p^\ell}, \xi_{i+1}, \dots, \xi_m)\xi_i^j,$$

where the $f_{\ell j}$ are p^ℓ-components of f with respect to ξ_i.

It is not hard to show that the D_ν are existentially definable from the $\delta_{\xi_i,j}$ and conversely. Indeed the D_ν are linear combinations of compositions of the $\delta_{\xi_i,j}$ with polynomial coefficients, and conversely.

(iv) The following properties of the D_ν follow easily from the definition

(a) $D_{\underline{0}} = id$
(b) $D_\nu c = 0$ for $c \in R$, $\nu \neq \underline{0}$
(c) $D_\nu(f + g) = D_\nu f + D_\nu g$
(d) $D_\mu \circ D_\nu = \binom{\mu+\nu}{\mu} D_{\mu+\nu}$, where $\binom{\mu+\nu}{\mu} = \prod_i \binom{\mu_i+\nu_i}{\nu_i}$
(e) $D_\nu(f \cdot g) = \sum_{\mu+\mu'=\nu} (D_\mu f)(D_{\mu'} g)$.
(f) a chain rule (see [**5**]).

Definition 2.6. — Let $\varphi : T_m \to A$ be a generalized ring of fractions and let $f \in A$. Using Remark 2.3 we define $\Delta(f)$ to be the collection of functions $\operatorname{Dom} A \to F_{\text{alg}}$ determined by the $D_\nu f$, $\nu \in \mathbb{N}^m$. In other words $\Delta(f)$ is the smallest collection of functions $\operatorname{Dom} A \to F_{\text{alg}}$ containing f and closed under the Hasse derivatives.

Definition 2.7. — Let $\mathcal{H} \subset \cup_{m,n} S_{m,n}$ be any collection such that $\Delta(\mathcal{H}) \subset \mathcal{H}$. (In the most important application $\mathcal{H} = T = \cup_m T_m$; another possibility is $\mathcal{H} = \Delta(f_1, \dots, f_n)$.) Let

$$L_{\mathcal{H}} := L(0, 1, +, \cdot, \{f\}_{f \in \mathcal{H}}; |\cdot|, \overline{0}, \overline{1}, \bar{\cdot}, \overline{<})$$

be the first-order language of multiplicatively valued rings, augmented by symbols for the functions of $\mathcal{H}$. A subset $X \subset (F_{\text{alg}}^\circ)^m$ is said to be *definable* (respectively, *existentially definable, quantifier-free definable*) in $L_{\mathcal{H}}$ iff there is an $L_{\mathcal{H}}$-formula (respectively, an existential $L_{\mathcal{H}}$-formula, quantifier-free $L_{\mathcal{H}}$-formula) $\varphi(\xi_1,\ldots,\xi_m)$ such that

$$(a_1,\ldots,a_m)\in X \Longleftrightarrow \varphi(a_1,\ldots,a_m) \text{ is true.}$$

A partial function $f : X \to F_{\text{alg}}$ is said to be definable (respectively, existentially definable, quantifier-free definable) in $L_{\mathcal{H}}$ iff its graph (and domain) are. The $\mathcal{H}$-subanalytic sets discussed in Section 5 are exactly the sets existentially definable in $L_{\mathcal{H}}$. A function f is quantifier free (respectively, existentially) definable from functions $g_1,\ldots,g_\ell$ if there is a quantifier-free (respectively, existential) formula φ in the language L of multiplicatively valued rings, such that

$$y = f(x) \Leftrightarrow \varphi(x,y,g_1(x),\ldots,g_\ell(x)).$$

We next define the class of functions $\mathcal{E}(\mathcal{H})$ all of whose "derivatives" are existentially definable from $\mathcal{H}$. The Quantifier Elimination Theorem (Theorem 4.2) applies to the language $L_{\mathcal{E}(\mathcal{H})}$ where $\mathcal{H} = \Delta(\mathcal{H})$. Since all functions of $\mathcal{E}(\mathcal{H})$ are existentially definable in $L_{\mathcal{H}}$ a corresponding quantifier simplification theorem for the language $L_{\mathcal{H}}$ follows.

Definition 2.8. — The collection $\mathcal{E}(\mathcal{H})$ consists of all functions $f : X \to F_{\text{alg}}$ such that $f \in A$ and $X = \operatorname{Dom} A$ for some generalized ring of fractions $\varphi : T_m \to A$, and such that the members of $\Delta(f)$ are all existentially definable in $L_{\mathcal{H}}$. We define the language $L_{\mathcal{E}}$ in analogy to Definition 2.7, i.e., $L_{\mathcal{E}}$ is the language of multiplicatively valued rings augmented by symbols for the functions of $\mathcal{E}(\mathcal{H})$.

The languages $L_{\mathcal{H}}$ (or $L_{\mathcal{E}(\mathcal{H})}$) are three-sorted languages. The three sorts are F°, $F^{\circ\circ}$ and $|F^\circ|$. (See [**9**, Sections 3.1–3.7].)

We shall use the following in Section 3.

Remark 2.9

(i) Let $\operatorname{Char} K = p \neq 0$, let $f(y)$ be a convergent power series in y, let $\overline{y} \in K$ sufficiently near 0, and let $\ell \in \mathbb{N}$. There is a polynomial $\overline{f}(y)$ such that

$$\overline{f}(y) \equiv f(y) \mod (y-\overline{y})^{p^\ell}$$

and $\overline{f}$ is existentially definable from the p^ℓ-components of f with respect to y. To see this write

$$f = \sum_{j=0}^{p^\ell-1} f_{\ell j}(y^{p^\ell})y^j$$

and let $\overline{f} = \sum_{j=0}^{p^\ell-1} f_{\ell j}(\overline{y}^{p^\ell})y^j$. By Remark 2.5(iii), $\overline{f}$ is existentially definable from Δf.

(ii) If $f(x,y) \in \mathcal{E}(\mathcal{H})$ and $f = \sum f_i(x)y^i$ then each $f_i \in \mathcal{E}(\mathcal{H})$.

3. Existential Definability of Weierstrass Data

Let A be a generalized ring of fractions over T and let $f, g \in A\langle\xi\rangle[\![\rho]\!]_s$ with f regular of degree s in y (where y is either ξ_m or ρ_n). By the Weierstrass Division and Preparation Theorems ([**13**, Theorem 2.3.8 and Corollary 2.3.9]) we can write

$$f = uP \quad \text{and} \quad g = qf + r$$

where u, P, q and r are as described in those theorems.

In this section we show that all the members of $\Delta(u)$ and $\Delta(P)$ are existentially definable from $\Delta(f)$ and all the members of $\Delta(q)$ and $\Delta(r)$ are existentially definable from $\Delta(f)$ and $\Delta(g)$. These results are needed for the Elimination Theorem (Theorem 4.2).

Analogous questions in the real case are considered in [**23**]. For completeness, we include proofs below not only in characteristic p but also in characteristic zero.

Theorem 3.1 **(Weierstrass Preparation for $\mathcal{E}$).** — *Let $\varphi : T_m \to A$ be a generalized ring of fractions and let $f \in A\langle\xi\rangle[\![\rho]\!]_s$. Suppose f is regular of degree s in ξ_M (respectively, in ρ_N) in the sense of* [**13**, Definition 2.3.7]. *By* [**13**, Corollary 2.3.9], *there exist a uniquely determined polynomial $P \in A\langle\xi'\rangle[\![\rho]\!]_s[\xi_M]$ (respectively, $P \in A\langle\xi\rangle[\![\rho']\!]_s[\rho_N]$) monic and regular of degree s and a unit $u \in A\langle\xi\rangle[\![\rho]\!]_s$ such that*

$$f = u \cdot P.$$

(Here $\xi' := (\xi_1, \ldots, \xi_{M-1})$ and $\rho' := (\rho_1, \ldots, \rho_{N-1})$.) Each member of $\Delta(u)$ and $\Delta(P)$ is existentially definable in $L_{\Delta(f)}$. Hence if $f \in \mathcal{E}(\mathcal{H})$, then $u, P \in \mathcal{E}(\mathcal{H})$.

Proof. — Let y denote the variable (either ξ_M or ρ_N) in which f is regular and let x denote the other variables. With this notation the above equation becomes

$$f(x,y) = u(x,y)[y^s + a_{s-1}(x)y^{s-1} + \cdots + a_0(x)].$$

We must show that each member of $\Delta(u)$ and $\Delta(a_j)$ $j = 0, \ldots, s-1$ is existentially definable in $L_{\Delta(f)}$, i.e. from $\Delta(f)$.

For each $x \in \operatorname{Dom} A\langle\xi'\rangle[\![\rho]\!]_s$ (respectively $\operatorname{Dom} A\langle\xi\rangle[\![\rho']\!]_s$), let $\overline{y}_1(x), \ldots, \overline{y}_s(x)$ be the s roots of the equation $f(x,y) = 0$ with $|y| \leq 1$ (respectively < 1). Then the $a_j(x)$ are symmetric functions of the $\overline{y}_i(x)$, say $a_j(x) = \sigma_j(\overline{y}_1(x), \ldots, \overline{y}_s(x))$.

We consider the cases $\operatorname{Char} K = 0$ and $\operatorname{Char} K = p \neq 0$ separately.

Case **(A).** — Characteristic $K = 0$.

By Remark 2.5(i) we may work with the usual partial derivatives instead of the Hasse derivatives.

For each partition $\mathcal{P}: s = s_1 + s_2 + \cdots + s_m$, with the $s_i \geq 1$, let $\varphi_{\mathcal{P}}$ be the formula

$$\left(\bigwedge_{i=1}^{s} |y_i| \Box \overline{1}\right) \wedge (y_1 = y_2 = \cdots = y_{s_1}) \wedge (y_{s_1+1} = \cdots = y_{s_1+s_2}) \wedge \ldots$$

$$\cdots \wedge (y_{s_1+\cdots+s_{m-1}+1} = \cdots = y_s) \wedge \bigwedge_{j=0}^{s_1-1} \frac{\partial^j f}{\partial y^j}(y_1) = 0 \wedge \ldots$$

$$\cdots \wedge \bigwedge_{j=0}^{s_m-1} \frac{\partial^j f}{\partial y^j}(y_{s_1+\cdots+s_{m-1}+1}) = 0 \wedge \bigwedge_{i \neq j} (y_{s_1+\cdots+s_i} \neq y_{s_1+\cdots+s_j}),$$

where $\Box$ is $<$ or $\leq$ depending on whether y is a ξ_M or ρ_N. Hence $\varphi_{\mathcal{P}}$ expresses the fact that y_1 is a root of $f = 0$ of multiplicity s_1, y_{s_1+1} is a root of multiplicity s_2, etc. For each $j = 0, \ldots, s-1$, let $\varphi_j(x, w_j)$ be the formula

$$\exists y_1 \ldots \exists y_s \big[\bigvee_{\mathcal{P}} \varphi_P \wedge w_j = \sigma_j(y_1, \ldots, y_s)\big].$$

Then φ_j is an existential definition of $a_j(x)$. We must further show that u and the derivatives of the $a_j(x)$ are existentially definable. Notice that the $\overline{y}_i(x)$ may not be differentiable even at points where the $a_j(x)$ are analytic. Let $P(x, y) = y^s + a_{s-1}(x)y^{s-1} + \cdots + a_0(x)$. Then

$$f(x, y) = u(x, y)P(x, y). \tag{3.1}$$

Next we show that $u(x, y)$ is existentially definable. This is obvious from (3.1) except perhaps when $y = \overline{y}_i(x)$ for some i (i.e. when $P(x, y) = 0$). Note that

$$P, \frac{\partial P}{\partial y}, \frac{\partial^2 P}{\partial y^2}, \ldots, \frac{\partial^s P}{\partial y^s} = s! \neq 0$$

are all existentially definable. It is now easy to see that if $\overline{y}$ is an s_i-fold root of $f(x, y) = 0$ then $u(x, \overline{y})$ is defined by

$$\frac{\partial^{s_i} f}{\partial y^{s_i}}(x, \overline{y}) = u(x, \overline{y}) \frac{\partial^{s_i} P}{\partial y^{s_i}}(x, \overline{y}).$$

Iterating, we see that $\dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial y^2}, \ldots$ are all existentially definable from $\Delta(f)$.

Differentiating (3.1) with respect to x_1 we get

$$\frac{\partial f}{\partial x_1} = \frac{\partial u}{\partial x_1} P + u \frac{\partial P}{\partial x_1} = \frac{\partial u}{\partial x_1} P + u \left[\frac{\partial a_{s-1}}{\partial x_1} y^{s-1} + \cdots + \frac{\partial a_0}{\partial x_1}\right].$$

So, if $\overline{y}_1, \ldots, \overline{y}_s$ satisfy $P(x, y) = 0$, then

$$u^{-1}(x, \overline{y}_i) \frac{\partial f}{\partial x_1}(x, \overline{y}_i) = a'_{s-1} \overline{y}_i^{s-1} + \cdots + a'_0, \tag{3.2}$$

where we write a'_j for $\dfrac{\partial a_j}{\partial x_1}$. If the roots $\overline{y}_1, \ldots, \overline{y}_s$ of $P = 0$ are distinct then the equations (3.2) uniquely determine the a'_j. (The coefficient matrix of the system of linear equations (3.2) is the Vandermonde matrix with determinant $\prod_{i<j}(\overline{y}_i - \overline{y}_j) \neq 0$).

If $\overline{y}_i$ is a root of $P = 0$ of multiplicity s_i we replace the s_i identical equations in (3.2) by the subsystem

$$\begin{aligned}
u^{-1}(x,\overline{y}_i)\frac{\partial f}{\partial x_1}(x,\overline{y}_i) &= \frac{\partial P}{\partial x_1}(x,\overline{y}_i),\\
u^{-1}(x,\overline{y}_i)\frac{\partial^2 f}{\partial y \partial x_1}(x,\overline{y}_i) &= \frac{\partial^2 P}{\partial y \partial x_1}(x,\overline{y}_i) + u^{-1}(x,\overline{y}_i)\frac{\partial u}{\partial y}(x,\overline{y}_i)\frac{\partial P}{\partial x_1}(x,\overline{y}_i),\\
&\cdots\\
u^{-1}(x,\overline{y}_i)\frac{\partial^{s_i} f}{\partial y^{s_i-1}\partial x_1}(x,\overline{y}_i) &= \frac{\partial^{s_i} P}{\partial y^{s_i-1}\partial x_1}(x,\overline{y}_i) + \dots.
\end{aligned}$$

to obtain a system of equations that we denote (3.2)$'$. The coefficient matrix of the resulting system of equations is nonsingular (see Remark 3.2 below) and hence the new system of equations defines the $\frac{\partial a_j}{\partial x_1}$. Existential definitions of $\frac{\partial u}{\partial x_1}$ and the higher derivatives of the a_j and u are obtained by iterating.

***Case* (B).** — Characteristic $K = p \neq 0$.

We follow the same general outline as in Case A and indicate the necessary changes. In characteristic zero we used the derivatives $\frac{\partial^k f}{\partial y^k}(\overline{y})$ to detect the multiplicity of a root $\overline{y}$ of $f = 0$. In Characteristic p we use the device of Remark 2.9(i). If we choose $p^\ell > s$ then the multiplicity of $\overline{y}$ as a root of $f = 0$ is the same as the multiplicity of $\overline{y}$ as a root of $\overline{f}(y) = 0$, and since $\overline{f}$ is a polynomial in y, the multiplicity of y as a zero of $\overline{f}$ is existentially definable from the coefficients of $\overline{f}$, which are by Remark 2.9(i) existentially definable from the p^ℓ components of f. Hence P is existentially definable from the p^ℓ-components of f with respect to y and hence from the Hasse derivatives $D_\nu f$ for $\nu = (0,\dots,0,i)$, $i = 0,\dots,p^\ell - 1$.

Next we must show that u and all its Hasse derivatives with respect to y are existentially definable. From the equation

$$f = uP,$$

u is existentially definable, except when $P = 0$, i.e. except when $y = \overline{y}_i$ for some i. If $\overline{y}$ is a zero of P of order $\alpha \leq s$ then $u(\overline{y})$ is (existentially) defined using

$$\overline{f}(y) \equiv u(\overline{y})P \mod (y - \overline{y})^{\alpha+1}$$

where $\overline{f}$ is the polynomial as in Remark 2.9(i) and $p^\ell > s$. In fact, for any $\beta \in \mathbb{N}$ we can existentially define a polynomial $\overline{u}$ such that $\overline{u} \equiv u \mod (y - \overline{y})^\beta$ by considering the congruence $\overline{f} \equiv uP \mod (y - \overline{y})^{\beta+\alpha}$.

Let D^i_y denote $D_{(0,\ldots,0,i)}$. Then

$$D^i_y f = \sum_{j+k=i} D^j_y u D^k_y P$$

(see Remark 2.5(iv)(e)). Since P is a polynomial in y the $D^i_y P$ are all quantifier free definable. We proceed inductively

$$D^1_y f = (D^1_y u)P + uD^1_y P.$$

This defines $(D^1_y u)$ except when $y = \overline{y}$ is a zero of P. But for such $\overline{y}$ we consider a congruence of the form

$$D^1_y f \equiv (D^1_y u)P + uD^1_y P \mod (y - \overline{y})^\beta.$$

By Remark 2.9(i), for any $\beta \in \mathbb{N}$ we can existentially define a polynomial congruent to $D^1_y f \mod (y - \overline{y})^\beta$. We saw above that we can existentially define a polynomial $\overline{u}(y) \equiv u(y) \mod (y - \overline{y})^\beta$. Hence we can existentially define $D^1_y u$ modulo $(y - \overline{y})^\beta$ for any β. From this, for β large enough, an existential definition of $(D^1_y u)(\overline{y})$ follows. Next we use

$$D^2_y f = (D^2_y u)P + (D^1_y u)(D^1_y P) + u(D^2_y P)$$

and the same argument to see that we can existentially define $D^2_y u \mod (y - \overline{y})^\beta$ for any β. The same devices allow us to obtain existential definition of the other Hasse derivative of u and P. We do an example that will convince the reader, and show that $D^2_{x_1} D^1_y u$ and $D^2_{x_1} D^1_y P$ are existentially definable. (Here $D^i_{x_1} = D_{(i,0,\ldots,0)}$. Observe also that $D^i_{x_1} D^j_y = D_{(i,0,\ldots,0,j)}$.) We again start with the equation

$$f = uP.$$

Thus

$$D^1_{x_1} f = (D^1_{x_1} u)P + u(D^1_{x_1} P). \tag{3.3}$$

Let the distinct zeros of P be $\overline{y}_1, \ldots, \overline{y}_d$ and let $\overline{y}_i$ have multiplicity α_i. Then $D^1_{x_1} P$, which is a polynomial in y of degree $\leq s-1$ is determined by the congruences

$$D^1_{x_1} f \equiv u(\overline{y}_i)(D^1_{x_1} P) \mod (y - \overline{y}_i)^{\alpha_i}, \quad i = 1, \ldots, d.$$

$D^1_{x_1} u$ is determined by equation (3.3), except where $y = \overline{y}_i$ for some i. But as above $D^1_{x_1} u \mod (y - \overline{y}_i)^\beta$ can be existentially defined by looking at (3.3) $\mod (y - \overline{y}_i)^{p^\ell}$ for large enough ℓ and using the fact that $D^1_{x_1} f \mod (y - \overline{y}_i)^{p^\ell}$ and $u \mod (y - \overline{y})^{p^\ell}$ are existentially definable from the $D^j_y D^1_{x_1} f$ and $D^j_y f$. To obtain the "second derivative" with respect to x_1 we apply $D^2_{x_1}$ to the equation $f = uP$:

$$D^2_{x_1} f = (D^2_{x_1} u)P + (D^1_{x_1} u)(D^1_{x_1} P) + u(D^2_{x_1} P). \tag{3.4}$$

Looking at this equation modulo the $(y - \overline{y}_i)^{\alpha_i}$ and using the facts that $P \equiv 0 \mod (y - \overline{y}_i)^{\alpha_i}$ and that we have existentially defined polynomials congruent to $(D^1_{x_1} u)$ and u modulo $(y - \overline{y}_i)^{\alpha_i}$, gives an existential definition of (the polynomial in y)

$D^2_{x_1}P$. Then $D^2_{x_1}u$ is determined when y is different from all the $\overline{y}_i$ by (3.4) and $D^2_{x_1}u$ mod $(y-\overline{y}_i)^\beta$ (for any β) is determined by looking at (3.4) modulo a high enough power of $y-\overline{y}_i$ and using the facts that we have existentially defined polynomials congruent to $D^1_{x_1}u$ and u modulo any specified power of $y-\overline{y}_i$. Next apply D^1_y to (3.3):

$$(3.5)\qquad D^1_yD^1_{x_1}f = (D^1_yD^1_{x_1}u)P + (D^1_{x_1}u)(D^1_yP) + (D^1_yu)(D^1_{x_1}P) + u(D^1_yD^1_{x_1}P).$$

As above, first determine $D^1_yD^1_{x_1}P$ by looking at this equation $\mod (y-\overline{y}_i)^{\alpha_i}$ and then determine $D^1_yD^1_{x_1}u$ for $y \neq \overline{y}_i$, $i=1,\dots,d$ and $D^1_yD^1_{x_1}u \mod (y-\overline{y}_i)^\beta$ for any β. Finally apply D^1_y to (3.4) to obtain

$$\begin{aligned} D^1_yD^2_{x_1}f = (D^1_yD^2_{x_1}u)P + (D^2_{x_1}u)(D^1_yP) + (D^1_yD^1_{x_1}u)(D^1_{x_1}P) \\ + (D^1_{x_1}u)(D^1_yD^1_{x_1}P) + (D^1_yu)(D^2_{x_1}P) + u(D^1_yD^2_{x_1}P). \end{aligned}$$

Exactly as above, first determine $D^1_yD^2_{x_1}P$ and then $D^1_yD^2_{x_1}u$ for $y\neq\overline{y}_i$, $i=1,\dots,d$, and finally $D^1_yD^2_{x_1}u \mod (y-\overline{y}_i)^\beta$ for any β. □

Remark 3.2. — Assume the characteristic of K is zero. Let $s_1+s_2+\cdots+s_m=s$ and let the Y_{ij} be variables $i=1,\dots,m$; $j=1,\dots,s_i$.

$$\det\begin{bmatrix} Y_{11}^{s-1} & Y_{11}^{s-2} & \dots & Y_{11} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{1s_1}^{s-1} & Y_{1s_1}^{s-2} & \dots & Y_{1s_1} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{ms_m}^{s-1} & Y_{ms_m}^{s-2} & \dots & Y_{ms_m} & 1 \end{bmatrix} = \prod_{(i,j)<(s,t)} (Y_{ij}-Y_{st})$$

where $<$ is the lexicographic ordering.

For each i and j, differentiate $j-1$ times with respect to Y_{ij}. Then set all the $Y_{ij}=Y_i$ (a new variable) for each $i=1,\dots,m$. The resulting determinant is a nonzero constant times a product of powers of (Y_i-Y_j), $i\neq j$. Call this function $\overline{V}(Y_1,\dots,Y_m)$.

Then the determinant of the coefficient matrix of the system of equations (3.2)′ occurring in the proof of Theorem 3.1 is $\overline{V}(\overline{y}_1,\dots,\overline{y}_m)\neq 0$ where $\overline{y}_1,\dots,\overline{y}_m$ are the distinct roots of $f(x,y)=0$, and $\overline{y}_i$ is a root of multiplicity s_i.

***Theorem 3.3* (Weierstrass Division for $\mathcal{E}$)**. — *Let $\varphi: T_m \to A$ be a generalized ring of fractions and let $f,g\in A\langle\xi\rangle[\![\rho]\!]_s$. Suppose f is regular of degree s in ξ_M (respectively, ρ_N) in the sense of* [**13**, Definition 2.3.7]. *Then by* [**13**, Theorem 2.3.8] *there exist unique elements*

$$r\in A\langle\xi'\rangle[\![\rho]\!]_s[\xi_M]$$

(respectively, $r \in A\langle\xi\rangle[\![\rho']\!]_s[\rho_N]$) of degree $s-1$ and $q \in A\langle\xi\rangle[\![\rho]\!]_s$ such that

$$g = qf + r.$$

(Here $\xi' := (\xi_1, \dots, \xi_{M-1})$ and $\rho' := (\rho_1, \dots, \rho_{N-1})$.) Furthermore, each member of $\Delta(q)$ and $\Delta(r)$ is existentially definable in $L_{\Delta(f)\cup\Delta(g)}$. Hence if $f, g \in \mathcal{E}(\mathcal{H})$ then $q, r \in \mathcal{E}(\mathcal{H})$.

Proof. — We follow the same notational convention as in the proof of Theorem 3.1 — i.e. we let y denote ξ_M (respectively ρ_N) and let x denote the other variables. Let $r = \sum_{i=1}^{s-1} r_i(x)y^i$, and let $\overline{y}_1(x), \dots, \overline{y}_s(x)$ be the roots of $f(x,y) = 0$. Then

$$g(x, \overline{y}_i) = \sum_{j=0}^{s-1} r_j(x)\overline{y}_i^j. \tag{3.6}$$

***Case* (A).** — Characteristic $K = 0$.

Again in this case we may consider the usual derivatives. If the $\overline{y}_i$ are all distinct then (3.6) has coefficient matrix the Vandermonde matrix and (3.6) determines the $r_j(x)$. If $\overline{y}_i$ is a root of $f = 0$ of multiplicity s_i, replace the corresponding s_i identical equations in (3.6) by the equations

$$\frac{\partial^\ell g}{\partial y^\ell}(x, \overline{y}_i) = \frac{\partial^\ell r}{\partial y^\ell}(x, \overline{y}_i), \qquad \ell = 0, \dots, s_i - 1.$$

The resulting system again has nonsingular coefficient matrix (see Remark 3.2) and hence determines the $r_j(x)$.

Existential definitions of the derivatives of the r_j are obtained in a way similar to that employed in the proof of Theorem 3.1 to obtain those for the derivative of the a_j. The same arguments also give existential definitions of q and its derivatives from $\Delta(f) \cup \Delta(g)$.

***Case* (B).** — Characteristic $K = p \neq 0$.

We proceed in a way entirely analogous to the characteristic p case of the proof of Theorem 3.1. □

4. The Elimination Theorem

We prove an elimination theorem that both generalizes that of **[9]** and provides a basis for the theory of affinoid subanalytic sets (i.e., the images of affinoid maps) as the elimination theorem of **[9]** provided a basis for the theory of quasi-affinoid subanalytic sets. We follow the strategy of **[1]**, first using parameterized Weierstrass Preparation (and Division) to reduce to the case that some variable occurs polynomially and then using an algebraic elimination theorem. Where **[1]** used Macintyre's elimination theorem **[16]** we use the elimination theorem of **[24]**.

To obtain parametrized Weierstrass division from the usual one, [**1**] used restricted division by coefficients (with parameters). The fact that functions are not canonically represented by terms in a first order language leads to difficulties in our situation, since we have extra definability conditions to satisfy. It turns out that the generalized rings of fractions (see Definition 2.1) allow us to carry out the necessary divisions while retaining definability properties in a natural way. Furthermore, [**1**] works over discretely valued fields K, where multiplication by a uniformizing parameter for the maximal ideal of K° can be used to witness strict inequalities. As in [**9**], we use variables ranging over $F_{\mathrm{alg}}^{\circ\circ}$ to witness strict inequalities: our fields F_{alg} are never discretely valued, being algebraically closed. As we remarked in [**13**, Example 2.3.5], the class of Weierstrass automorphism for the resulting rings of analytic functions is not large enough to transform every nonzero function to one that is regular. Thus we employ Weierstrass Preparation and Division and the double induction of [**9**] to reduce to an application of the algebraic elimination theorem for algebraically closed valued fields of [**24**].

Let A be a quasi-affinoid algebra. Recall that we showed in [**13**, Section 5.2] that $A\langle\xi\rangle[\![\rho]\!]_s \subset A[\![\xi,\rho]\!]$, so we may write

$$f = \sum f_{\mu\nu}\xi^\mu\rho^\nu, \quad f_{\mu\nu} \in A,$$

for any $f \in A\langle\xi\rangle[\![\rho]\!]_s$.

Lemma 4.1. — *Let A be a generalized ring of fractions over T, and let*

$$f = \sum f_{\mu\nu}\xi^\mu\rho^\nu \in A\langle\xi\rangle[\![\rho]\!]_s.$$

Then there are: $c \in \mathbb{N}$, A-algebras $A_{\mu\nu}$, $|\mu| + |\nu| \leq c$, each a generalized ring of fractions, and elements $g_{\mu\nu} \in A_{\mu\nu}\langle\xi\rangle[\![\rho]\!]_s$ such that

(i) *$f_{\mu\nu}(x)g_{\mu\nu}(x,\xi,\rho) = f(x,\xi,\rho)$ for every $x \in \mathrm{Dom}\, A_{\mu\nu}$,*

(ii) *each $g_{\mu\nu}$ is preregular of degree (μ,ν) in the sense of* [**13**, Definition 2.3.7], *and*

(iii) $\mathrm{Dom}\, A = Z(f) \cup \bigcup_{|\mu|+|\nu|\leq c} A_{\mu\nu}$,

where $Z(f) := \{x \in \mathrm{Dom}\, A : f(x,\xi,\rho) \equiv 0\}$. If $f \in \mathcal{E}$, then $Z(f)$ is quantifier-free definable in $L_{\mathcal{E}}$ and each $g_{\mu\nu} \in \mathcal{E}$.

Proof. — Writing A as a quotient of a ring of separated power series and applying [**13**, Lemma 3.1.6] to a preimage of f, we obtain a $c \in \mathbb{N}$ and elements $h_{\mu\nu} \in A\langle\xi\rangle[\![\rho]\!]_s$ such that

$$f = \sum_{|\mu|+|\nu|\leq c} f_{\mu\nu}\xi^\mu\rho^\nu(1+h_{\mu\nu}) \text{ and}$$
$$|h_{\mu\nu}(y)| < 1 \text{ for all } y \in \mathrm{Max}\, A\langle\xi\rangle[\![\rho]\!]_s.$$

(Hence each $1 + h_{\mu\nu}$ is a unit of $A\langle\xi\rangle[\![\rho]\!]_s$.)

For each $(\mu_0, \nu_0) \in \mathbb{N}^m \times \mathbb{N}^n$ with $|\mu_0| + |\nu_0| \leq c$, we define the generalized ring of fractions $A_{\mu_0\nu_0}$ from A in the obvious way so that the inequalities

$$\begin{aligned} |f_{\mu_0\nu_0}(x)| &\geq |f_{\mu\nu}(x)| && \text{for all } |\mu| + |\nu| \leq c, \\ |f_{\mu_0\nu_0}(x)| &> |f_{\mu\nu}(x)| && \text{for all } \nu < \nu_0 \text{ and } |\mu| + |\nu| \leq c, \\ |f_{\mu_0\nu_0}(x) &> |f_{\mu\nu_0}(x)| && \text{for all } \mu > \mu_0 \text{ and } |\mu| + |\nu_0| \leq c \end{aligned}$$

hold for all $x \in \operatorname{Dom} A_{\mu_0\nu_0}$. Indeed, $A_{\mu_0\nu_0}$, so defined, has the property that $x \in \operatorname{Dom} A_{\mu_0\nu_0}$ if, and only if, $f_{\mu_0\nu_0}(x) \neq 0$ and the above inequalities hold.

Now, for $|\mu| + |\nu| \leq c$, $f_{\mu\nu}/f_{\mu_0\nu_0} \in A_{\mu_0\nu_0}$, so we may put

$$g_{\mu_0\nu_0} := \xi^{\mu_0}\rho^{\nu_0}(1 + h_{\mu_0\nu_0}) + \sum_{\substack{|\mu|+|\nu|\leq c, \\ (\mu,\nu)\neq(\mu_0,\nu_0)}} \frac{f_{\mu\nu}}{f_{\mu_0\nu_0}} \xi^{\mu}\rho^{\nu}(1 + h_{\mu\nu}), \quad |\mu_0| + |\nu_0| \leq c.$$

Finally, suppose $f \in \mathcal{E}$. Since $f_{\mu\nu}(x) \neq 0$ for $x \in \operatorname{Dom} A_{\mu\nu}$, and $f_{\mu\nu}(x) \in \mathcal{E}$ by Remark 2.9(ii), condition (i) implies that $g_{\mu\nu} \in \mathcal{E}$. To see this inductively, apply $D_{\nu'}$ to (i), use the product formula of Remark 2.5(iv)(e) and solve for $D_{\nu'} g_{\mu\nu}$. Furthermore,

$$Z(f) = \{x \in \operatorname{Dom} A : f_{\mu\nu}(x) = 0, |\mu| + |\nu| \leq c\},$$

which is a quantifier-free $L_{\mathcal{E}}$-definition. □

***Theorem 4.2* (Quantifier Elimination Theorem).** — *Let $\mathcal{H} \subset S$ with $\mathcal{H} = \Delta(\mathcal{H})$, let $\mathcal{E} := \mathcal{E}(\mathcal{H})$, and let Φ be an $L_{\mathcal{E}}$-formula. Then there is a quantifier-free $L_{\mathcal{E}}$-formula Ψ such that for every complete field F extending K, $F^{\circ}_{\mathrm{alg}} \models \Phi \leftrightarrow \Psi$; i.e., Φ and Ψ define the same subset of $(F^{\circ}_{\mathrm{alg}})^m$.*

Proof. — Recall that $L_{\mathcal{E}(\mathcal{H})}$ is a three-sorted language. We shall use the following convention which will greatly simplify notation. The ξ_i will denote variables of the first sort (that range over F°) and the ρ_j will denote variables of the second sort (that range over $F^{\circ\circ}$); x will denote a string of variables of sorts one and two. Observe that a quantified variable of the third sort (that ranges over $|F^{\circ}|$) can always be replaced by a quantified variable of the first sort — if v is a variable of the third sort replace it by $|\xi|$ where ξ is a variable of the first sort. Hence we need only eliminate quantified variables of sorts one and two. (Alternatively, a quantified variable of the third sort can be eliminated by a direct application of the quantifier elimination theorem of [**24**]). After routine manipulations we may assume that Φ is of the form $\exists \xi\rho\varphi(v, x, \xi, \rho)$, where φ is a conjunction of atomic formulas; i.e., formulas of the form

$$t_1(v) \cdot |f(x, \xi, \rho)| \;\square\; t_2(v) \cdot |g(x, \xi, \rho)|,$$

where $\square$ is either $<$ or $=$; $f, g \in A\langle\xi\rangle[\![\rho]\!]_s \cap \mathcal{E}$ for some fixed generalized ring of fractions over T; v denotes a string of variables of the third sort and the t_i are terms of the third sort containing no variables of sorts one or two. (Observe that the negation of such a formula is a disjunction of such formulas.)

For such formulas φ, we may define $\ell(\varphi)$ to be the number of functions in the formula that actually depend on (ξ, ρ). Writing

$$\xi = (\xi_1, \dots, \xi_m) \quad \text{and} \quad \rho = (\rho_1, \dots, \rho_n),$$

we induct on the triples (m, n, ℓ), ordered lexicographically.

Let $f_1, \dots, f_\ell$ be the functions that occur in φ and depend on (ξ, ρ). Write

$$f_i = \sum f_{i\mu\nu} \xi^\mu \rho^\nu = \sum f_{i\nu} \rho^\nu \in A\langle\xi\rangle[\![\rho]\!]_s \cap \mathcal{E},$$

where $f_{i\mu\nu} \in A \cap \mathcal{E}$ and $f_{i\nu} \in A\langle\xi\rangle \cap \mathcal{E}$. Applying Lemma 4.1 to $f = f_1$ yields rings $A_{\mu\nu}$ and elements $g_{\mu\nu} \in A_{\mu\nu}\langle\xi\rangle[\![\rho]\!]_s$ preregular of degree (μ, ν).

Consider the formulas

$$\varphi_0 := x \in Z(f) \wedge \varphi \quad \text{and} \quad \varphi_{\mu\nu} := x \in \operatorname{Dom} A_{\mu\nu} \wedge \varphi.$$

By Lemma 4.1(iii), Φ is equivalent to the disjunction

$$\exists\, \xi \rho \varphi_0(\xi, \rho) \vee \bigvee \exists\, \xi \rho \varphi_{\mu\nu}(\xi, \rho).$$

Let φ_0' result from φ_0 by replacing f by 0 and let $\varphi_{\mu\nu}'$ result from $\varphi_{\mu\nu}$ by replacing f by $f_{\mu\nu} \cdot g_{\mu\nu}$. Note that $\ell(\varphi_0') < \ell(\varphi)$ and $\ell(\varphi_{\mu\nu}') = \ell(\varphi)$. By induction, we may assume that Φ is of the form $\exists\, \xi \rho \varphi_{\mu\nu}'$. Iterating this procedure reduces us to the case that Φ is of the form $\exists\, \xi \rho \varphi$, where the functions occurring in φ are $a_i(x) \cdot f_i(x, \xi, \rho)$, and each $f_i(x, \xi, \rho)$ is preregular of degree (μ_i, ν_i) with $f_{i\mu_i\nu_i} = 1$, $1 \le i \le \ell$.

Consider the $L_{\mathcal{E}}$-formulas

$$\varphi_0 := \varphi \wedge \bigwedge_{i=1}^{\ell} |f_{i\nu_i}(x, \xi)| = 1 \quad \text{and} \quad \varphi_i := \varphi \wedge |f_{i\nu_i}(x, \xi)| < 1.$$

Clearly, Φ is equivalent to the disjunction

$$\exists\, \xi \rho \varphi_0(\xi, \rho) \vee \bigvee \exists\, \xi \rho \varphi_i(\xi, \rho),$$

and we may consider the disjuncts separately.

***Case* (A).** — $\Phi = \exists\, \xi \rho \varphi_i(\xi, \rho)$.

We have that Φ is equivalent to

$$\exists\, \xi \rho \rho_{n+1} \varphi \wedge |f_{i\nu_i} - \rho_{n+1}| = 0.$$

Observe that $f_{i\nu_i} - \rho_{n+1}$ is preregular of degree $(\mu_i, 0)$. Hence, after a Weierstrass automorphism involving only the ξ's, we may assume that $f_{i\nu_i} - \rho_{n+1}$ is regular in ξ_m. (Recall that Weierstrass automorphisms preserve membership in $\mathcal{E}$.) After applying Weierstrass Preparation (Theorem 3.1) to $f_{i\nu_i} - \rho_{n+1}$ and Weierstrass Division (Theorem 3.3) with divisor $f_{i\nu_i} - \rho_{n+1}$ to the other functions in Φ, we may assume that all the functions occurring in Φ are polynomials in ξ_m. We may now apply the algebraic elimination theorem of **[24]** to find a formula

$$\Psi = \exists\, \xi_1 \dots \xi_{m-1} \rho\, \rho_{n+1} \psi$$

equivalent to Φ. Since $(m-1, n+1, \ell(\psi)) < (m, n, \ell(\varphi))$, we are done by induction.

***Case* (B).** — $\Phi = \exists\, \xi\rho\varphi_0(\xi, \rho)$.

We have that Φ is equivalent to

$$\Psi := \exists\, \xi\xi_{m+1}\rho\varphi \wedge \left|\left(\prod_{i=1}^{\ell} f_{i\nu_i}\right)\xi_{m+1} - 1\right| = 0.$$

Observe that $h = (\prod_{i=1}^{\ell} f_{i\nu_i})\xi_{m+1} - 1$ is preregular of degree $(\sum \mu_i, 1, 0)$. Hence after a Weierstrass automorphism involving only $\xi_1, \ldots, \xi_{m+1}$ we may assume that h is regular in ξ_{m+1}. Let f_i' result from f_i by multiplying by $(\prod_{j\neq i} f_{j\nu_j})\xi_{m+1}$ and replacing the coefficient $(\prod_{j=1}^{\ell} f_{j\nu_i})\xi_{m+1}$ (of ρ^{ν_i}) by 1. Then each f_i' is preregular of degree $(0, \nu_i)$. Let Ψ' result from ψ by replacing each f_i by f_i'. Then Ψ is equivalent to Ψ'. After a Weierstrass automorphism among the ρ's we may assume that each f_i' in Ψ' is regular in ρ_n. Applying Weierstrass Preparation (Theorem 3.1) to each f_i' with respect to ρ_n and to h with respect to ξ_{m+1}, and then Weierstrass Division (Theorem 3.3) with divisor h, we may assume that each function occurring in Ψ' is a polynomial in both ρ_n and ξ_{m+1}. We may now apply the algebraic elimination theorem of **[24]** to find a formula

$$\Psi'' = \exists\, \xi_1, \ldots, \xi_m, \rho_1, \ldots, \rho_{n-1}\psi$$

equivalent to Φ. Since $(m, n-1, \ell(\psi)) < (m, n, \ell(\varphi))$, we are done by induction. □

Taking $\mathcal{H} = S(E,K) = \cup S_{m,n}(E,K)$ we obtain the following strengthened version of the elimination theorem of **[9]**. Observe that in this case every (partial) function of $\mathcal{E}(S(E,K))$ is represented by a D-term (i.e., a function in the language L_{an}^{D} of **[9]**), and conversely, as in Remark 2.3(iv).

Corollary 4.3. — *F_{alg}° admits elimination of quantifiers in the language $L_{S(E,K)}$. The elimination is uniform in F and depends only on $S(E,K)$.*

Taking $\mathcal{H} = T(K) = \cup T_m(K)$ we obtain the following quantifier elimination theorem.

***Corollary 4.4* (Quantifier Elimination over $\mathcal{E}(T)$).** — *F_{alg}° admits elimination of quantifiers in the language $L_{\mathcal{E}(T(K))}$. The elimination is uniform in F and depends only on K.*

Observing that every member of $\mathcal{E}(T)$ is existentially definable over T gives us the following quantifier simplification (model completeness) theorem, which provides the basis of the theory of affinoid subanalytic sets discussed in Section 5.

***Corollary 4.5* (Quantifier Simplification over T)**

(i) *F_{alg}° is model complete in the language $L_{T(K)}$.*

(ii) *Every subset of* $(F^{\circ}_{\mathrm{alg}})^m$ *definable by an* $L_{T(K)}$*-formula is definable by an existential* $L_{T(K)}$*-formula.*

5. Subanalytic Sets

In this section we explain how the basic properties of subanalytic sets based on the functions in $T = \cup T_m$ (or on any set of functions $\mathcal{H} \subset S$, with $\mathcal{H} = \Delta(\mathcal{H})$) follow from Corollary 4.5.

Definition 5.1. — Let K be a complete, non-Archimedean valued field and let $\mathcal{H} \subset S = \cup_{m,n} S_{m,n}(E, K)$. Let F be a complete field extending K and let F_{alg} be its algebraic closure. A subset $X \subset (F^{\circ}_{\mathrm{alg}})^m$ is called *globally* $\mathcal{H}$*-semianalytic* iff X is defined by a quantifier-free $L_{\mathcal{H}}$-formula. A subset $X \subset (F^{\circ\circ}_{\mathrm{alg}})^m$ is called $\mathcal{H}$*-subanalytic* iff it is the projection of a globally $\mathcal{H}$-semianalytic set (or equivalently is defined by an existential $L_{\mathcal{H}}$-formula). When $\mathcal{H} = T(K)$ we use the terms K*-affinoid semianalytic* and K*-affinoid subanalytic* and when $\mathcal{H} = S(E, K)$ we use the terms (E, K)*-quasi-affinoid-semianalytic* and (E, K)*-quasi-affinoid-subanalytic.*

The following is a restatement of Theorem 4.2 (the Elimination Theorem).

Theorem 5.2. — *Let* $\mathcal{H} \subset S(E, K)$ *with* $\mathcal{H} = \Delta(\mathcal{H})$*. The* $\mathcal{H}$*-subanalytic sets are exactly the* $L_{\mathcal{H}}$*-definable sets. In particular, the class of* $\mathcal{H}$*-subanalytic sets is closed under complementation and (metric) closure.*

The following can be proved by a small modification of the arguments of [**9**, Section 5] in characteristic zero. The characteristic $p \neq 0$ case requires a larger modification. Details are given in [**14**].

Corollary 5.3. — *Every* $\mathcal{H}$*-subanalytic set is a finite disjoint union of* F_{alg}*-analytic,* $\mathcal{H}$*-subanalytic submanifolds.*

We restate the above results in the special case that $\mathcal{H} = T(K)$.

Corollary 5.4. — *The class of* K*-affinoid-subanalytic sets is closed under complementation and closure.*

Corollary 5.5. — *Each* K*-affinoid-subanalytic set is a finite disjoint union of* K*-affinoid-subanalytic sets which are also* F_{alg}*-analytic submanifolds. If* X *is such a set, this allows us to define the dimension of* X*,* $\dim X$*, to be the maximum dimension of an* F_{alg}*-analytic submanifold that occurs in a smooth subanalytic stratification, or equivalently, the maximum dimension of an* F_{alg}*-analytic submanifold of* X*.*

Remark 5.6

(i) The theory of subanalytic sets developed in [**9**] (and there termed rigid) is the special case of Theorem 5.2 with $\mathcal{H} = S$.

(ii) The Łojasiewicz inequalities proved in **[9]** for S-subanalytic sets also hold for $\mathcal{H}$-subanalytic sets. This is immediate since $\mathcal{H} \subset S$.

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