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COMPLEX FOLIATIONS WITH ALGEBRAIC LIMIT SETS

by

César Camacho & Bruno Azevedo Scárdua

Dedicated to Adrien Douady on the occasion of his $60^{\underline{th}}$ birthday.

Abstract. — We regard the problem of classification for complex projective foliations with algebraic limit sets and prove the following:

Let $\mathcal{F}$ be a holomorphic foliation by curves in the complex projective plane $\mathbb{C}P(2)$ having as limit set some singularities and an algebraic curve $\Lambda \subset \mathbb{C}P(2)$. If the singularities $\operatorname{sing}\mathcal{F} \cap \Lambda$ are generic then either $\mathcal{F}$ is given by a closed rational 1-form or it is a rational pull-back of a Riccati foliation $\mathcal{R} : p(x)dy - (a(x)y^2 + b(x)y)dx = 0$, where Λ corresponds to $\overline{(y = 0)} \cup \overline{(p(x) = 0)}$, on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.

The proof is based on the solvability of the generalized holonomy groups associated to a reduction process of the singularities $\operatorname{sing}\mathcal{F} \cap \Lambda$ and the construction of an affine transverse structure for $\mathcal{F}$ outside an algebraic curve containing Λ.

1. Introduction

Let $\mathcal{F}$ be a holomorphic codimension one foliation on the complex projective 2-space $\mathbb{C}P(2)$. Given any leaf L of $\mathcal{F}$ the *limit set* of L is defined as $\lim(L) = \bigcap_\nu \overline{L \backslash K_\nu}$ where $K_\nu \subset K_{\nu+1}$ is an exhaustion of L by compact subsets $K_\nu \subset L$. The *limit set* of the foliation $\mathcal{F}$ is defined as $\lim \mathcal{F} = \overline{\bigcup_L \lim(L)}$. We are interested in classifying those foliations whose limit set is a union of singularities of $\mathcal{F}$ and an algebraic curve $\Lambda \subset \mathbb{C}P(2)$. There are two reasons for this, first because these foliations exhibit the simplest dynamic behavior we can imagine and also because they must support an important class of first integrals. The parallel with the actions of Kleinian groups on the Riemann sphere comes naturally to mind. These foliations will correspond to actions with a finite set of limit points (one or two) while the first integrals of these foliations will correspond to the automorphic functions of such Kleinian group

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actions. Here we will show that this similarity is not only apparent. Indeed the Kleinian groups will appear naturally as the holonomy groups of the Riccati foliation that, it will be shown here, is the ultimate model for these foliations.

The problem of classifying such foliations $\mathcal{F}$ was considered in **[1]** and **[17]**. In both cases it is proved that, under generic assumptions, there are a rational map $F\colon \mathbb{C}P(2) \to \mathbb{C}P(2)$ and a linear foliation $\mathcal{L} : \lambda_1 x dy - \lambda_2 y dx = 0$ on $\mathbb{C}P(2)$ such that $\mathcal{F} = F^*(\mathcal{L})$. In particular, it follows that no saddle-nodes appear in the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$, and in fact all the singularities as well as all the holonomy groups appearing in this resolution are abelian and linearizable. Using **[9]** we can construct examples where $\mathcal{F}$ is a *Riccati foliation* with algebraic limit set on $\mathbb{C}P(2)$, containing the invariant line $\overline{(y = 0)}$:

$$\mathcal{F} : p(x)dy - (y^2 a(x) + yb(x))dx = 0$$

where $a(x), b(x), p(x)$ are polynomials, and $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$ is an affine chart (see Example 1.3 below) and $\Lambda \cap \mathbb{C}^2 = (p(x) = 0) \cup (y = 0)$.

In the Riccati case, the holonomy groups are solvable and we have an additional compatibility condition as in **[2]**. However we may have saddle-nodes in the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$. The aim of this paper is to solve the problem above in the case the foliation may have certain saddle-node singularities in its resolution along Λ.

Let therefore $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ and let $\Lambda \subset \mathbb{C}P(2)$ be an algebraic invariant curve (perhaps reducible). We will say that $\operatorname{sing}\mathcal{F} \cap \Lambda$ *has* the pseudoconvexity property (*psdc*) if the *invariant (by $\mathcal{F}$) part* Γ of the resolution divisor D of $\operatorname{sing}\mathcal{F} \cap \Lambda$ is connected and its complement is a *Stein manifold* (alternatively, Γ is a very ample divisor on the ambient (algebraic) manifold of the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$ denoted by $\widetilde{\mathbb{C}P(2)}$), so that we can apply Levi's extension theorem **[21]** which allows us to extend analytically to all $\mathbb{C}P(2)$, any analytic object defined on a neighborhood of Γ. This property is verified if $\mathcal{F}$ has no dicritical singularities over Λ **[1]**. There is another remarkable case where property (*psdc*) is verified, as we can find in **[17]**. A singularity $q_o \in \operatorname{sing}\widetilde{\mathcal{F}} \cap D$ is a *corner* if $q_o = D_i \cap D_j$, where $D_i \neq D_j$ are invariant components of D.

Also, we say that a saddle-node singularity $q_o \in D$ is in *good position* relatively to D, if its *strong separatrix* is contained in some component of Γ. A saddle-node $x^{k+1}dy - y(1 + \lambda x^k)dx + \mathrm{h.o.t.} = 0$ is *analytically normalizable* if we may choose local coordinates (x, y) as above for which we have $\mathrm{h.o.t.} = 0$. In this case it will be called *normally hyperbolic* if we have $\lambda \notin \mathbb{Q}$. In this case we call $(x = 0)$ the *strong separatrix* and $(y = 0)$ the *central manifold* of the saddle-node. We recall that according to **[12]** a saddle-node singularity is analytically classified by the local holonomy of this strong separatrix. In particular, the saddle-node is analytically normalizable if, and only if, its strong separatrix holonomy is an analytically normalizable flat diffeomorphism.

Finally, we introduce the following technical condition (see Example 1.4):

$(\mathcal{C}_1)$ *The saddle-nodes in the resolution of* $\operatorname{sing}\mathcal{F}\cap\Lambda$ *are analytically normalizable, and the ones in the corners are normally hyperbolic.*

Our main result is the following:

Theorem 1.1. — *Let* $\mathcal{F}$ *be a codimension one holomorphic foliation on* $\mathbb{C}P(2)$ *having as limit set some singularities and an algebraic curve* $\Lambda\subset\mathbb{C}P(2)$. *Assume that* $\operatorname{sing}\mathcal{F}\cap\Lambda$ *satisfies property (psdc) and condition* $\mathcal{C}_1$. *Then, either* $\mathcal{F}$ *is given by a closed rational* 1*-form or it is a rational pull-back of a Riccati foliation* $\mathcal{R}: p(x)dy-(a(x)y^2+b(x)y)dx=0$, *where* Λ *corresponds to* $\overline{(y=0)}\cup\overline{(p(x)=0)}$, *on* $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$.

The proof of this theorem relies on the study of the *singular* and *virtual* holonomy groups **[2]**, **[5]**, **[19]** and **[1]** respectively, of the irreducible components of the divisor given by the resolution of $\operatorname{sing}\mathcal{F}\cap\Lambda$. *The limit set of the leaves* $\widetilde{L}$ *of* $\widetilde{\mathcal{F}}$ *induces discrete pseudo-orbits in each of these groups, so that they are solvable* **[14]**. The solvability of these groups, allows (under our restrictions on $\operatorname{sing}\mathcal{F}\cap\Lambda$) the construction of a "transversely formal" meromorphic 1-form $\widetilde{\eta}$, defined over the invariant part Γ of the resolution divisor of $\operatorname{sing}\mathcal{F}\cap\Lambda$. This 1-form is closed and satisfies the relation $d\widetilde{\omega}=\widetilde{\eta}\wedge\widetilde{\omega}$, where $\widetilde{\omega}$ is a meromorphic 1-form with isolated singularities which defines the foliation $\widetilde{\mathcal{F}}$, obtained from the resolution of $\operatorname{sing}\mathcal{F}\cap\Lambda$. Moreover, $\widetilde{\eta}$ has (simple) poles over Γ which coincides with the limit set of $\widetilde{\mathcal{F}}$. Using a result of Hironaka-Matsumara (see **[5]**, **[8]**), we conclude that (since $\widetilde{\mathbb{C}P(2)}\backslash\Gamma$ is a Stein manifold) the 1-form $\widetilde{\eta}$ is in fact rational on $\widetilde{\mathbb{C}P(2)}$. This corresponds to the existence of a Liouvillian first integral for $\mathcal{F}$ on $\mathbb{C}P(2)$, and also to the existence of an affine transverse structure for $\widetilde{\mathcal{F}}$ in $\widetilde{\mathbb{C}P(2)}\backslash C$, where $C\subset\widetilde{\mathbb{C}P(2)}$ is an algebraic invariant curve containing $\widetilde{\Lambda}$, where $\widetilde{\Lambda}$ is the strict transform of Λ, **[18]**. This affine transverse structure can be extended as a projective transverse structure to $(\widetilde{\mathbb{C}P(2)}\backslash C)\cup\widetilde{\Lambda}$. In particular, all the singular holonomy groups associated to the components of Γ are solvable analytically normalizable. This implies by (a careful reading of the last part of) **[2]** that either $\mathcal{F}$ is given by a closed rational 1-form or by a rational pull-back of a Riccati foliation.

Example 1.2. — Let $\mathcal{F}$ be a rational pull-back of a hyperbolic linear foliation $\mathcal{L}$: $xdy-\lambda ydx=0, \lambda\in\mathbb{C}\backslash\mathbb{R}$, on $\mathbb{C}P(2)$. Clearly $\mathcal{F}$ has an algebraic limit set consisting of some singularities and an algebraic curve Λ as in Theorem 1.1.

Example 1.3. — Let us take any finitely generated group of Moebius transformations $G\subset\mathrm{SL}(2,\mathbb{C})$. Assume that the limit set of G is a single point, which can be assumed to be the origin $0\in\overline{\mathbb{C}}$. The limit point 0 is a fixed point of G. According to **[9]** we can find a Riccati foliation $\mathcal{F}: p(x)dy-(a(x)y^2+b(x)y+c(x))dx=0$ on $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$, whose holonomy group of the line $\overline{(y=0)}$ is conjugated to the group G. Moreover we can assume that the singularities of $\mathcal{F}$ over this horizontal line are reduced and non degenerate. The line $\overline{(y=0)}$ is invariant by $\mathcal{F}$ so that $c(x)=0$, and also it is contained in the limit set of $\mathcal{F}$ and satisfies condition $\mathcal{C}_1$ in the statement above. This example can also be seen in $\mathbb{C}P(2)$ using a birational transformation. This will create a dicritical singularity. This example will satisfy the (*psdc*) property for a proper choice of Λ.

Example 1.4. — This is a counterexample to a more general statement. Let $\mathcal{F}$ be given by $\omega = dy - (a(x)y + b(x))dx = 0$ over $\mathbb{C}^2 \subset \mathbb{C}P(2)$. If we consider the vector field $X(x,y) = (1, a(x)y + b(x))$, then X is complete and tangent to $\mathcal{F}$ over $\mathbb{C}^2$. Moreover the orbits of X are diffeomorphic to $\mathbb{C}$. It is not difficult to see, using the flow of X, that the leaves of $\mathcal{F}$ accumulate the line at infinity $L_\infty = \mathbb{C}P(2) \backslash \mathbb{C}^2$, so that $\lim \mathcal{F} = L_\infty$. However, generically, the resolution of $\operatorname{sing} \mathcal{F} \cap L_\infty$ exhibits some non analytically normalizable saddle-node. Indeed, this resolution is quite simple and shows that there are saddle-nodes with non convergent central manifolds **[5]**. On the other hand, in general, $\mathcal{F}$ is not a rational pull-back of a Riccati foliation of the form stated in Theorem 1.1.

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2. Formal normal forms and resolution of singularities

Let $\mathcal{F}$ and $\Lambda \subset \lim \mathcal{F}$ be as in Theorem 1.1. Let $\pi \colon (\widetilde{\mathbb{C}P(2)}, \widetilde{\mathcal{F}}, D) \to (\mathbb{C}P(2), \mathcal{F}, \Lambda)$ be the resolution morphism of Seidenberg, for $\operatorname{sing} \mathcal{F} \cap \Lambda$ **[20]**. Thus $\widetilde{\mathbb{C}P(2)}$ is a compact complex surface which is obtained from $\mathbb{C}P(2)$ by a finite sequence of blowing-up's, denoted π. The proper morphism π induces therefore a foliation by curves $\widetilde{\mathcal{F}} = \pi^* \mathcal{F}$ on $\widetilde{\mathbb{C}P(2)}$. The divisor $D = \pi^{-1}(\Lambda)$ of the resolution is a finite union $D = \bigcup_{j=0}^m D_j$, of projective lines $D_j \cong \mathbb{C}P(1)$, $j \neq 0$, and of the strict transform of Λ, $D_0 = \overline{\pi^{-1}(\Lambda \backslash \operatorname{sing} \mathcal{F})}$. The foliation $\widetilde{\mathcal{F}}$ has singularities of the following two types (called *irreducible* singularities):

(i) $xdy - \lambda y dx + \text{h.o.t.} = 0$ (non degenerate)
(ii) $y^{p+1}dx - [x(1 + \lambda y^p) + \text{h.o.t.}]dy = 0$ (saddle-node).

We consider the foliation $\widetilde{\mathcal{F}} = \pi^* \mathcal{F}$ and denote by Γ the invariant (by $\widetilde{\mathcal{F}}$) part of D, which consists of the invariant projective lines and of the strict transform of Λ. Let ω be a rational 1-form which defines $\mathcal{F}$ on $\mathbb{C}P(2)$ and denote by $\widetilde{\omega}$ the strict transform of $\pi^* \omega$. Therefore the 1-form $\widetilde{\omega}$ has isolated singularities and we can assume that its polar set intersects the divisor D transversely and at regular points of $\widetilde{\mathcal{F}}$. Clearly we have $\lim(\widetilde{\mathcal{F}}) \subset \Gamma$.

Lemma 2.1. — *We have* $\lim(\widetilde{\mathcal{F}}) = \Gamma$. *In particular all the saddle-nodes in* $\operatorname{sing} \widetilde{\mathcal{F}} \cap \Gamma$ *are in good position with respect to* Γ.

Proof. — Recall that by hypothesis Γ is connected. Let us fix a saddle-node $q_o \in D_j$, which is not a corner. Assume by contradiction that the strong separatrix S of q_o is not contained in D_j. We consider the local of S around q_o at a small transverse disk $\Sigma \cong \mathbb{D}$, with $\Sigma \cap S = q_1 \notin \operatorname{sing} \widetilde{\mathcal{F}}$. This holonomy map $h_o\colon (\Sigma, q_1) \to (\Sigma, q_1)$ is a flat local diffeomorphism, that is, a local diffeomorphism tangent to the identity. Thus, the orbits of h_o accumulate the origin q_1, so that the local leaves of $\widetilde{\mathcal{F}}$ around q_o and crossing Σ, must accumulate the strong separatrix S, and therefore we obtain $S \subset \lim \mathcal{F}$ (notice that S is transverse to D, so that it corresponds to a separatrix of $\mathcal{F}$ not contained in Λ), which contradicts the hypothesis that $\lim \mathcal{F} = \Lambda$. Now, we fix a saddle-node on a corner $q_o = D_i \cap D_j$. First we prove that if $\lim \widetilde{\mathcal{F}}$ contains the central manifold of q_o, say D_i then it contains the strong manifold, in this case D_j. In fact, by the hypothesis we may write $\widetilde{\mathcal{F}}$ as

$$y(1 + \lambda x^k)dx - x^{k+1}dy = 0, \quad D_i = (y = 0), \quad D_j = (x = 0).$$

Now, in a sector $(x, y) \in U \times \mathbb{C}$ near $0 \in \mathbb{C}^2$, where $U = \{x \in \mathbb{C}^*;\ \operatorname{Re}(x^p) > 0\}$ the leaves of $\widetilde{\mathcal{F}}$ have a saddle-like behavior in the sense that there are sections $\Sigma_i = (x = 1)$ and $\Sigma_j = (y = 1)$, such that any leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}\big|_{U\times\mathbb{C}}$, not contained in $(y = 0)$, is at a positive distance from $0 \in \mathbb{C}^2$ and, if we denote $r_\nu = \Sigma_\nu \cap \widetilde{L}$, we have: if $r_i \to (0, 1)$ then $r_j \to (1, 0)$.

Now we prove the converse: If $\lim \widetilde{\mathcal{F}}$ contains the central manifold of q_o, D_i, then it contains the strong manifold, D_j. In fact, using the normal form above we may conclude that the local holonomy of the central manifold around q_o, is linearizable of the form $h\colon (\Sigma_i, (1, 0)) \to (\Sigma_i, (1, 0)), h(y) = \exp(2\pi i\lambda) \cdot y$. If $\lambda \in \mathbb{R}\backslash\mathbb{Q}$, then it is a non rational rotation so that the accumulations of the leaves in the section Σ_i do not correspond to algebraic limit sets. Thus $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and therefore, either h or h^{-1} is an attractor, so that any leaf which intersects Σ_i accumulates the origin $(0, 1) \in \Sigma_i$.

We also remark that $\lim \widetilde{\mathcal{F}}$ contains all the strong separatrices of the saddle-nodes in Γ. In fact, from the analytic normal form above we have a multivalued first integral $f(x, y) = (y/x^\lambda) \exp(1/kx^k)$. This first integral shows that the leaves accumulate on the strong manifold $(x = 0)$. Also from the same arguments of **[1]**, **[17]** we have (in the non degenerated corners) the passage of the limit set $\lim(\widetilde{L})$ from one to the other adjacent component of D: It is in fact, only necessary to use the fact that if q_o is a non degenerate corner say of the form, $xdy - \lambda ydx + \text{h.o.t.} = 0$ such that $\lambda \in \mathbb{R}_+ \backslash \mathbb{Q}_+$, then by Poincaré Linearization Theorem this singularity is linearizable, and therefore it is not difficult to see that the local leaves around q_o are not proper. On the other hand, in the case $\lambda \in \mathbb{R}_-$, any leaf which accumulates q_o and which is not a separatrix, accumulates both separatrices. Finally, we remark that since by the hypothesis the limit set of $\mathcal{F}$ is algebraic, it follows that all the strong manifolds in $\operatorname{sing} \widetilde{\mathcal{F}} \cap \Gamma$ are contained in Γ, that is, the saddle-nodes are in good position relatively to Γ. $\square$

Now we fix local transverse sections $\mathcal{S}_j$, $\mathcal{S}_j \cap D_j = p_j \notin \operatorname{sing} \widetilde{\mathcal{F}}$, $(\mathcal{S}_j, p_j) \cong (\mathbb{C}, 0)$. Let us write G_j for the holonomy group $\operatorname{Hol}(\widetilde{\mathcal{F}}, D_j, \mathcal{S}_j)$ of $D_j \subset \Gamma$ (see **[2]**). The following definition is found in **[1]**:

Definition 2.2. — The *virtual holonomy group* of $\widetilde{\mathcal{F}}$ relative to the component D_j at the section $\mathcal{S}_j$ is defined as

$$\operatorname{Hol}^v(\widetilde{\mathcal{F}}, D_j, \mathcal{S}_j) = \{f \in \operatorname{Diff}(\mathcal{S}_j, p_j) \mid \widetilde{L}_z = \widetilde{L}_{f(z)},\ \forall\, z \in (\mathcal{S}_j, p_j)\}$$

Clearly this group contains the *holonomy group* of $\widetilde{\mathcal{F}}$ relative to D_j at the section $\mathcal{S}_j$, denoted by $\operatorname{Hol}(\widetilde{\mathcal{F}}, D_j, \mathcal{S}_j)$ (see **[2]** for the definition of the holonomy group). Let us write G_j^v for the virtual holonomy group $\operatorname{Hol}^v(\widetilde{\mathcal{F}}, D_j, \mathcal{S}_j)$.

We will write *projective* holonomy group to denote the holonomy group of any component D_j of D.

We denote by $\operatorname{Diff}(\mathbb{C}, 0)$ respectively $\widehat{\operatorname{Diff}}(\mathbb{C}, 0)$ the group of germs of biholomorphisms respectively the group of formal biholomorphisms of $(\mathbb{C}, 0)$. We also denote by $\mathcal{X}(\mathbb{C}, 0)$, respectively $\widehat{\mathcal{X}}(\mathbb{C}, 0)$ the Lie algebra of the germs of singular holomorphic vector fields at $0 \in \mathbb{C}$, respectively the Lie algebra of singular formal vector fields in one complex variable.

According to Lemma 2.1 the limit set of any non algebraic leaf $\widetilde{L}$ induces discrete pseudo-orbits in each projective virtual holonomy group. These groups are solvable as a consequence of the following result due to I. Nakai:

Proposition 2.3 (**[14]**). — *Let $G \subset \operatorname{Diff}(\mathbb{C}, 0)$ be a subgroup which has some discrete pseudo-orbit. Then G is solvable.*

Corollary 2.4. — *Let $\mathcal{F}$ be as in Theorem 1.1. Then each projective or virtual holonomy group of* $\operatorname{sing} \mathcal{F} \cap \Lambda$ *is solvable.*

We also have the following result concerning subgroups with discrete pseudo-orbits:

Theorem 2.5 (**[10]**). — *Let $G \subset \operatorname{Diff}(\mathbb{C}, 0)$ be a nonabelian subgroup with discrete pseudo-orbits outside the origin. Then G is either formally conjugate to some group*

$$G_\nu^2 := \langle z \mapsto az, z \mapsto z/(1+z^\nu)^{1/\nu} \rangle$$

where a^ν has order 2; or it is analytically conjugate to some group

$$G_{\nu,\tau}^2 := \langle z \mapsto az, z \mapsto z/(1+z^\nu)^{1/\nu}, z \mapsto z/(1+\tau z^\nu)^{1/\nu} \rangle$$

where a^ν has order 2 and $\tau \in \mathbb{C} \setminus \mathbb{R}$; or finally it is analytically conjugate to some group

$$G_\nu^n := \langle z \mapsto az, z \mapsto z/(1+z^\nu)^{1/\nu} \rangle$$

where a^ν has order $n \in \{3, 4, 6\}$.

We shall consider the subgroups

$$\mathbb{H}_k = \left\{\varphi \in \operatorname{Diff}(\mathbb{C},0) \mid \varphi(z)^k = \frac{\mu_\varphi z^k}{1+a_\varphi z^k}, \quad \mu_\varphi \in \mathbb{C}^*,\ a_\varphi \in \mathbb{C}\right\}$$

where $k \in \mathbb{N}^*$. According to [4] any solvable non abelian subgroup of $\operatorname{Diff}(\mathbb{C},0)$ is formally conjugated to a subgroup of some $\mathbb{H}_k$, this conjugacy is analytic except for some special case. We also use the following result:

Lemma 2.6 **([15])**. — *Let $G \subset \widehat{\operatorname{Diff}}(\mathbb{C},0)$ be a subgroup. Then:*

(i) G is abelian if, and only if, there exists a formal vector field $\xi \in \widehat{\mathcal{X}}(\mathbb{C},0)$ such that $g * \xi = \xi$, $\forall\, g \in G$. In the case G is not linearizable the vector field ξ is unique.

(ii) G is solvable non abelian if, and only if, there exists a formal vector field $\xi \in \widehat{\mathcal{X}}(\mathbb{C},0)$ such that $g * \xi = c_g \cdot \xi$, $c_g \in \mathbb{C}^*$, $\forall\, g \in G$, where $c_g \neq 1$ for some $g \in G$. The vector field ξ is unique up to multiplicative constants.

As it is well-known [11], given a formal vector field $\xi = a(z)d/dz$, with $\xi(0) = 0$, there exists a formal diffeomorphism $\Theta \in \widehat{\operatorname{Diff}}(\mathbb{C},0)$, such that

$$\Theta_*\xi(z) = \xi_{\lambda,k}(z) := \frac{z^{k+1}}{1+\lambda z^k},$$

where $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ are formal invariants associated to ξ. It is clear that if $\varphi \in \widehat{\operatorname{Diff}}(\mathbb{C},0)$ satisfies $\varphi_*\xi_{0,k}(z) = \xi_{0,k}(z)$, then $\varphi(z) \in \mathbb{H}_k$. On the other hand it is not difficult to see that if $G \subset \widehat{\operatorname{Diff}}(\mathbb{C},0)$ is solvable and non abelian, then the vector field ξ given by Lemma 2.6 above must exhibit $\lambda = 0$ [15]. Therefore we have the following definition:

Definition 2.7. — Let $G \subset \widehat{\operatorname{Diff}}(\mathbb{C},0)$ be a solvable subgroup. A formal *normalizing* coordinate $w \in (\mathbb{C},0)$ for G is anyone for which the vector field ξ of Lemma 2.6 above writes as $\xi(w) = \xi_{\lambda,k}(w)$.

Clearly, if G is solvable and non abelian, then the formal normalizing coordinate is unique up to composition with elements $\varphi \in \mathbb{H}_k$, where k is given by G as above.

If G is abelian then given two normalizing coordinates u and w with $u'(0) = w'(0)$, we have $u = \varphi(w)$, where $\varphi(z) = \exp t\xi_{\lambda,k}(z)$ for some $t \in \mathbb{C}$.

The group G is called *analytically normalizable* if the associated vector field ξ is convergent, otherwise we will say that G is *non analytically normalizable.* In other words, a solvable (perhaps abelian) subgroup $G \subset \operatorname{Diff}(\mathbb{C},0)$ is analytically normalizable if it is analytically conjugated to its formal model.

Proposition 2.8. — *Let $\mathcal{F}, \Lambda$ be as in Theorem 1.1. Denote by D the resolution divisor of* $\operatorname{sing}\mathcal{F} \cap \Lambda$. *Let D_j be a component of D. Assume that the holonomy of the component D_j is solvable non abelian $G_j \subset \mathbb{H}_k$ by a formal conjugation. Then, given a singularity $q_o \in \operatorname{sing}\widetilde{\mathcal{F}} \cap D_j$ there exists a formal diffeomorphism $\Phi \in \widehat{\operatorname{Diff}}(\widetilde{\mathbb{C}P(2)}, q_o)$, such that $\Phi^*(\mathcal{F})$ has one of the following normal forms where $\Phi^*(D_j) = (y = 0)$:*

(a) $\omega_\lambda = xdy - \lambda ydx$, $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ *(q_o is formally linearizable non resonant);*
(b) $\omega_{n/m} = nxdy + mydx$, $n, m \in \mathbb{N}$, $\langle n, m\rangle = 1$ *(q_o has a formal first integral);*
(c) $\omega_{k,\ell} = kxdy + \ell y(1 + \frac{\sqrt{-1}}{2\pi}x^\ell y^k)dx$ *(q_o is resonant non formally linearizable);*
(d) $\omega_k = y^{k+1}dx - xdy$ *(q_o is a saddle-node with strong manifold tangent to D_j);*
(e) $\omega^{p,\lambda} = x^{p+1}dy - y(1+\lambda x^p)dx$ *(q_o is a saddle-node with strong manifold transverse to D_j).*

Moreover, we may assume that Φ converges except for case (c), and that G_j is analytically normalizable except for cases (b) and (c).

Proof. — First we assume that q_o is non degenerate, say

$$\widetilde{\omega}(u,v) = udv - \lambda vdu + \text{h. o. t.}, \qquad \lambda \in \mathbb{C}^*$$

for some local *holomorphic* coordinates (u, v) centered at q_o. If $\lambda \notin \mathbb{Q}$ then it follows that $\widetilde{\omega}$ is formally linearizable at q_o, that is, we have (a). Assume now that $\lambda = -n/m \in \mathbb{Q}_-$ with $n, m \in \mathbb{N}$, $\langle n, m\rangle = 1$. Then we consider the local holonomy $\varphi(v)$ of D_j at q_o. According to the hypothesis on the holonomy group of D_j, there exists a formal change of coordinates $\widehat{\psi} \in \widehat{\mathrm{Diff}}(\mathbb{C}, 0)$ such that

$$\widehat{\psi} \circ \varphi \circ \widehat{\psi}^{-1}(w) = \frac{cw}{(1 + aw^k)^{1/k}}$$

where the linear part is $c = \exp(\frac{2k\pi\sqrt{-1}n}{m})$. On the other hand it is well-known that an homography which is not tangent to the identity is linearizable by another homography. If $kn/m \notin \mathbb{N}$, then $c \neq 1$ and therefore the singularity is therefore formally linearizable as in (b). Assume that q_o is not formally linearizable and (therefore) that $k/n = \ell/m$ for some $\ell \in \mathbb{N}$. Then according to **[11]** there exists a formal conjugacy at q_o which takes $\widetilde{\omega}$ into the form (c).

Assume now that q_o is a saddle-node singularity. If the strong manifold of q_o is tangent to D_j then $c = 1$ in the expression of $\widehat{\psi} \circ \varphi \circ \widehat{\psi}^{-1}(w)$ above and therefore this local holonomy is formally conjugated to the local holonomy of the strong manifold $(v = 0)$ of the saddle-node $v^{k+1}du - udv = 0$. Therefore $\widetilde{\omega}$ must be of the form (d) above. If the strong manifold of q_o is transverse to D_j then its has a formal normal form as in (e) as a consequence of **[12]**.

Now we remark that in case (a) G_j must be analytically normalizable because it contains an element with nonperiodic linear part. In case (d) G_j is analytically normalizable because it contains the holonomy of the strong manifold of ω_k, which is assumed to be analytically normalizable. We remark that in case (e) G_j is again analytically normalizable because, as it follows from Lemma 2.1, q_o is a corner. Indeed, in this case G_j contains the holonomy of the central manifold of $\omega^{p,\lambda}$, which is a nonrational linearizable rotation, and therefore has nonperiodic linear part. Finally we remark that according to **[13]** a singularity q_o has a formal first integral if, and only if, q_o has a holomorphic first integral, so that we can assume that Φ is convergent in case (a). This finishes the proof of the proposition. $\square$

Remark 2.9. — According to **[11]**, **[12]**, **[13]** any reduced singularity admits a formal integrating factor.

3. Virtual holonomy and Singular holonomy

In this section we follow [5], [19] and [2]. Let us consider the following situation: $\mathcal{F}$ is a foliation on a compact complex surface M, $D \subset M$ is a compact (codimension one) invariant divisor with normal crossings, $D = \bigcup_j D_j$ where the D_j are irreducible smooth components. As in §2 we fix local transverse sections $\mathcal{S}_j$, $\mathcal{S}_j \cap D_j = p_j \notin \operatorname{sing} \mathcal{F}$, $(\mathcal{S}_j, p_j) \cong (\mathbb{C}, 0)$, and write G_j for the holonomy group $\operatorname{Hol}(\mathcal{F}, D_j, \mathcal{S}_j)$ of D_j (see [2]). Given any other transverse section Σ_j to $\mathcal{F}$ such that $\Sigma_j \cap D_j = q_j$, there is a conjugacy between $\operatorname{Hol}(\mathcal{F}, D_j, \Sigma_j)$ and $\operatorname{Hol}(\mathcal{F}, D_j, \mathcal{S}_j)$ induced by lifting to the leaves of $\mathcal{F}$ a simple path joining p_j to q_j, in the leaf $D_j \backslash \operatorname{sing} \mathcal{F}$. Thus, up to conjugacy, we can identify these groups and in particular $\operatorname{Hol}(\mathcal{F}, D_j, \Sigma_j)$ is solvable if, and only if, G_j is.

Now we fix a *corner* $q_o = D_i \cap D_j$. We assume that all the virtual the holonomy groups G_ν^v are solvable (perhaps abelian) for $\nu \in \{i, j\}$. In the non abelian case we denote by k_ν the *ramification order* of G_ν^v, so that $G_\nu \subset G_\nu^v \subset \mathbb{H}_{k_\nu}$ by a formal conjugacy.

The following lemma holds in general, *i.e.*, also for non normally hyperbolic saddle-nodes:

Lemma 3.1. — *Assume that the holonomy group G_j is analytically normalizable, and that if q_o is a saddle-node then its strong manifold is contained in D_j. Then $\mathcal{F}$ is analytically normalizable at the singular point q_o.*

Proof. — First we assume that G_j is non abelian, so that there exists a local holomorphic coordinate $z \in \Sigma_j$, $z(q_j) = 0$, where Σ_j is a local transverse section with $\Sigma_j \cap D_j = q_j$ close to q_o, such that the local holonomy of $\mathcal{F}$ due to q_o and relative to D_j writes

$$\varphi(z) = \frac{\lambda z}{(1 + az^k)^{1/k}}.$$

Now, if $\lambda^k \neq 1$ then we can linearize this local holonomy and therefore the singularity q_o, which is not a saddle-node (recall that the holonomy of the strong manifold of a saddle-node is never linearizable). Assume now that we have $\lambda^k = 1$. In this case we have $\varphi(z)^k = z/(1 + az^k)$. If q_o is not a saddle-node then as in the proof of Proposition 2.8 it follows from **[11]** that q_o must be analytically conjugated to a singularity of the form $\omega_{k,\ell}$ as in Proposition 2.8, because the holonomies are analytically conjugated. If q_o is a saddle-node then by the hypothesis the strong manifold is contained in D_j. Therefore, by **[12]**, the analytic normalization of the holonomy of the strong manifold implies the analytic normalization of the singularity q_o. Now we consider the case G_j is abelian and analytically normalizable. According to the techniques of construction of integrating factors (see Lemma 5.4 below or **[16]**), we can construct a closed meromorphic 1-form Ω_j in a neighborhood of q_o, and which satisfy $\widetilde{\omega} \wedge \Omega_j \equiv 0$. This implies that q_o is analytically normalizable **[3]**, **[11]**, **[13]**. $\square$

According to Lemma 3.1 above, except for the non analytically normalizable cases on the holonomy groups, there exists a neighborhood U of q_o where $\mathcal{F}$ can be written in an analytic normal form.

We shall describe the notion of singular holonomy introduced in **[5]** and used in **[2]**, **[19]**:

3.1. The case of a formal holomorphic first integral. — Let us assume that q_o is a non degenerate resonant corner say, $\mathcal{F} : nxdy + mydx + \mathrm{h.o.t.} = 0$ in a neighborhood of $q_o : x = y = 0$, where $n/m \in \mathbb{Q}_+$. We will assume that $\mathcal{F}$ has a formal first integral at q_o and therefore **[13]**, a local holomorphic first integral on a neighborhood of q_o. Thus this singularity is linearizable **[13]** and we can define the *Dulac correspondence* in a neighborhood of the singularity q_o. This correspondence is defined as follows: By the hypothesis $q_o \in D_j$ is a linearizable singularity corresponding to a local holomorphic first integral of $\mathcal{F}$, therefore we can choose local coordinates $(x, y) \in U$, a neighborhood of q_o, centered at this point, such that $D_i \cap U = (x = 0)$, $D_j \cap U = (y = 0)$, and such that $\mathcal{F}\big|_U$ is given by $nxdy + mydx = 0$. We fix the local transverse sections as $\Sigma_j = (x = 1)$ and $\Sigma_i = (y = 1)$, such that $\Sigma_i \cap D_i = q_i \neq q_o$ and $\Sigma_j \cap D_j = q_j \neq q_o$. Let us denote by $G_j = \mathrm{Hol}(\mathcal{F}, D_j, \Sigma_j)$ and by $G_i = \mathrm{Hol}(\mathcal{F}, G_i, \Sigma_i)$. Also denote by $h_o \in \mathrm{Hol}(\mathcal{F}, D_i, \Sigma_i)$ the element corresponding to the corner q_o. Then we have $h_o(x) = \exp(-2\pi \frac{n}{m}\sqrt{-1}) \cdot x$. The Dulac correspondence is therefore given by $\mathcal{D}$: $(\Sigma_i, q_i) \to (\Sigma_j, q_j)$, $\mathcal{D}(x_o) = x_o^{m/n}$. We use this correspondence in order to associate to G_i a subgroup $G_j * (\mathcal{D}_* G_i) \subset \mathrm{Diff}(\Sigma_j, q_j)$. Given an element $h \in G_i$ we look for elements $h^{\mathcal{D}} \in \mathrm{Diff}(\Sigma_j, q_j)$, which are solutions of the *adjunction equation* $h^{\mathcal{D}} \circ \mathcal{D} = \mathcal{D} \circ h$.

Case 1. — G_i is abelian: Take any element $h \in G_i$. Since G_i is abelian we have $h(x) = \mu x \widetilde{h}(x^m)$ for some $\widetilde{h} \in \mathcal{O}_1, \widetilde{h}(0) = 1$. We take $\mu_1 = \mu^{m/n}$ and $h_1 = \widetilde{h}^{m/n}$ be one of the n-roots of μ^m and $\widetilde{h}^m$ respectively. Then we define $h^{\mathcal{D}} : (\Sigma_j, q_j) \to (\Sigma_j, q_j)$, by

$$h^{\mathcal{D}}(y) = \mu_1 y h_1(y^n)$$

Clearly we have $h^{\mathcal{D}} \in G_j^v$.

Case 2. — $G_i \subset \mathbb{H}_{k_i}$ is non abelian, analytically normalizable and $nk_i/m = k_j \in \mathbb{N}$: In this case we have an analytic embedding $G_i \subset \mathbb{H}_{k_i}$. Take an element

$$h(x) = \frac{\lambda x}{(1 + ax^{k_i})^{1/k_i}} \in G_i.$$

We consider determinations of

$$y_o \longmapsto \frac{\lambda^{m/n} y_o}{\left(1 + a y_o^{nk_i/m}\right)^{m/nk_i}}$$

By definition the maps $h^{\mathcal{D}}$ are all these determinations. Clearly the maps $h^{\mathcal{D}}$ belong to the virtual holonomy group G_j^v.

Case 3. — G_i is solvable non abelian and not analytically normalizable: In this case it follows from **[4]** that the group of the commutators $[G_i, G_i]$ is cyclic, say, $[G_i, G_i] = \langle h_1 \rangle$ for some $h_1 \in G_i$ and G_i is generated by some power or root of h_o and some power h_1^ℓ, $\ell \in \mathbb{Z}$. Notice that if $n = m = 1$ then we have $\mathcal{D}(x) = x$, and given any $h \in G_i$, we may define $h^{\mathcal{D}} \in G_j^v$ as $h^{\mathcal{D}}(y) = h(y)$, in the coordinates above. Thus we may assume that $n \neq m$. We regard this case: First we consider the case the virtual holonomy group G_j^v is abelian. Then all its elements commute with the local holonomy g_o around q_o, associated to the separatrix contained in D_j. Therefore, using the same construction of Case 1 above we may consider the adjunction of G_j^v to the holonomy group G_i, as a subgroup of the virtual holonomy group G_i^v. If G_j^v contains some element g of infinite order then we have two possibilities to consider:

(a) g has non periodic linear part: In this case, g induces an element h in the adjunction holonomy and therefore in the virtual holonomy $h \in G_i^v$, which also has non periodic linear part. This implies that G_i^v (which is solvable by hypothesis), is analytically normalizable **[4]**. Therefore we may exclude this case.

(b) Every element g in G_j^v has periodic linear part: In this case we may find some non trivial element $g \in G_j^v$, which is tangent to the identity $g(y) = y + ay^{\ell+1} + \text{h.o.t.}$, $a \neq 0$. Then, g induces an element h in the virtual holonomy G_i^v, which has infinity order and some power tangent to the identity. Moreover, since G_j^v is analytically normalizable, it follows that g and h are analytically normalizable. This implies that the powers of h are analytically normalizable and therefore since the group of flat elements in G_i^v is cyclic, G_i^v is analytically normalizable. We exclude therefore this case, and conclude that all the elements in G_j^v have finite order. It follows that G_j^v is a group whose elements are rational rotations, and that each finitely generated subgroup is in fact a finite linearizable group. In particular G_j is a finite linearizable group.

Now we consider the case G_j^v is solvable non abelian, and analytically normalizable. In this case, once again we may use the same procedure of Case 2 above in order to induce non trivial analytically normalizable flat elements in the virtual holonomy G_i^v, and conclude that this is in fact analytically normalizable. Thus we exclude this case.

Summarizing, we conclude that *if G_i is exceptional* (that is, solvable non abelian and not analytically normalizable) *then G_j^v is either a group of rational rotations and therefore with finite finitely generated subgroups, or an exceptional group.* In this last case, we use **[4]** to conclude that G_j^v is generated by some root of the local holonomy

g_o associated to q_o (we may have $g_o = \mathrm{Id}$), and some flat element g_1. Moreover each flat element in G_j^v is some power of g_1. We may then use the ideas of **[14]** in order to conclude that some power of h_1 corresponds to some power of g_1 by means of the Dulac correspondence adjunction. In fact, we can give the ideas: It is possible to use the Dulac corresponce given by $\mathcal{D}(x) = x^{m/n}$, in order to consider the sets of "pseudo-orbits"

$$\{g_1^{k_r} \circ \mathcal{D} \circ h_1^{\ell_r} \circ \mathcal{D}^{-1} \circ \cdots \circ g_1^{k_2} \circ \mathcal{D} \circ h_1^{\ell_2} \circ \mathcal{D}^{-1} \circ g_1^{k_1} \circ \mathcal{D} \circ h_1^{\ell_1}(x)\} \subset \Sigma_i,$$

where $x \in \Sigma_i$, ℓ_t, k_t are integral numbers, and $\mathcal{D}^{-1}$ is the correspondence $\Sigma_j \to \Sigma_i$, $y \mapsto y^{n/m}$. These sets are contained in a same leaf of $\mathcal{F}$ for each fixed x, and as in **[14]**, if the powers g_1^ℓ, and h_1^k are never related by the congugation equation, then we will have accumulations for the leaves of $\mathcal{F}$, outside the origin in Σ_i. On the other hand, in the case we are interested in, we have discrete intersections of the leaves with the transverse sections, outside the origin, so that we will conclude that some power h_1^ℓ, passes to the virtual holonomy G_j^v, as some power g_1^k.

3.2. The case of a non degenerate non resonant corner. — Let us assume that q_o is a non degenerate non resonant corner say, $\mathcal{F} : xdy - \lambda ydx + \text{h.o.t.} = 0$ in a neighborhood of $q_o : x = y = 0$, where $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. We only need to consider the following case (see Definition 4.2): *G_i is abelian and G_j is solvable non abelian.*

In this case the singularity q_o is analytically linearizable. In fact, since the group G_j is solvable non abelian, and since the local holonomy φ associated to q_o has non periodic linear part it follows that the group G_j is analytically normalizable [**4**], and therefore there exists an analytic coordinate $w \in \Sigma_j, w(q_j) = 0$, such that

$$\varphi(w) = \frac{aw}{(1 + bw^{k_j})^{1/k_j}},$$

where $a^n \neq 1, \forall\, n \in \mathbb{N}^*$. Thus, we can change coordinates analytically in order to have $\varphi(u) = au$ for some coordinate $u = \phi(w)$, $\phi \in \mathbb{H}_{k_j}$. This implies that the singularity q_o is analytically linearizable **[13]**.

The adjunction holonomy group $G_j * (\mathcal{D}_* G_i) \subset \mathrm{Diff}(\Sigma_j, q_j)$ is defined as follows: There exists an analytic embedding $G_j \subset \mathbb{H}_{k_j}$. We may also assume that we have $\Sigma_i, \Sigma_j, D_i \cap U, D_j \cap U, G_i, G_j, q_i, q_j$ given in terms of (x, y) as in 3.1 above.

Since G_i is abelian and contains a non resonant linearizable diffeomorphism, G_i is linearizable. In fact, we can assume that the holonomy group $\mathrm{Hol}(\mathcal{F}, D_i, \Sigma_i)$ is linear in the local coordinate $x\big|_{\Sigma_i}$. Therefore any element $h \in G_i$ corresponds to an element $h(x) = \mu_h \cdot x$ in $\mathrm{Diff}(\Sigma_i, q_i)$. In the present case the Dulac correspondence is given by $\mathcal{D}(x_o) = x_o^{-\lambda}$. We are looking for elements $h^{\mathcal{D}} \in \mathrm{Diff}(\Sigma_j, q_j)$, which satisfy $h^{\mathcal{D}} \circ \mathcal{D} = \mathcal{D} \circ h$. Therefore we choose $h^{\mathcal{D}}$ as

$$h^{\mathcal{D}}(y) = \mu_h^{-\lambda} \cdot y$$

where $\mu_h^{-\lambda}$ runs over the solutions of $z^{-1/\lambda} = \mu_h$. Clearly the diffeomorphims $h^{\mathcal{D}}$ belong to the virtual holonomy G_j^v.

3.3. The case of a saddle-node corner. — Assume now that q_o is a saddle-node singularity which will be assumed to be analytically normalizable. Thus, there exists a system of local coordinates $(x, y) \in U$ centered at q_o such that, $\mathcal{F}\big|_U$ is given by $x^{k+1}dy - y(1 + \lambda x^k)dx = 0$. We assume that $D_i = (x = 0)$ and $D_j = (y = 0)$. We also introduce the transverse sections $\Sigma_i = (y = e^{1/k})$ and $\Sigma_j = (x = 1)$. The leaves of $\mathcal{F}\big|_U$ are the level curves of the multiform first integral $f(x, y) = yx^{-\lambda}\exp\big(1/kx^k\big)$. Therefore, the *Dulac correspondence* is defined by

$$D\colon \Sigma_i \longrightarrow \Sigma_j, \quad x_o \longmapsto y_o = \mathcal{D}(x_o) = x_o^{-\lambda}\exp\big(1/kx_o^k\big).$$

Now we show how the Dulac correspondence can still be used to define an adjunction for the holonomy. This adjunction will be from the strong manifold to the central manifold, that is, from G_i to G_j above. Given an element $h \in G_i$ we look for elements $h^{\mathcal{D}} \in \mathrm{Diff}(\Sigma_j, q_j)$, which are solutions of the *adjunction equation* $h^{\mathcal{D}} \circ \mathcal{D} = \mathcal{D} \circ h$.

Case 1. — G_i is solvable non abelian. In this case we have $\lambda = 0$ and

$$h_o(x) = \frac{x}{(1 + ax^k)^{1/k}}$$

and therefore $k_i = k$. We claim that G is analytically normalizable. In fact, since it is solvable it follows from **[4]** that it is analytically normalizable if, and only if, the (abelian) subgroup of flat elements in G is analytically normalizable. But this is clear because this group contains h_o. This also implies that we can assume that x is a normalizing coordinate for G_i. The Dulac correspondence $\mathcal{D}$ is given by $\mathcal{D}(x_o) = \exp\big(1/kx_o^k\big)$. Now we take any element $h \in G_i$ and write

$$h(x) = \frac{\mu x}{(1 + ax^k)^{1/k}}.$$

Since h is of the form given above we have the adjunction equation as

$$h^{\mathcal{D}}(\exp\big(1/kx^k\big)) = \exp\big(a/k\mu^k\big) \cdot \big(\exp\big(1/kx^k\big)\big)^{\mu^{-k}}.$$

Thus we have linear solutions of the form $h^{\mathcal{D}}(y) = \exp\big(a/k\mu^k\big) \cdot y^{\mu^{-k}}$. However these are not uniform analytic functions. *Assume that h^k is tangent to* 1, that is, $\mu^k = 1$. In this case we can take solutions of the form $g(y) = \exp\big(a/k\big) \cdot y$. These are linear diffeomorphisms and we will denote any of them by $h^{\mathcal{D}}$ as before. Notice that *if q_o is a corner then by the normal hyperbolicity hypothesis G_i must be abelian.*

Case 2. — G_i is abelian. Let us consider once again the local holonomy $h_o \in G_i$ associated to the strong manifold of the saddle-node q_o. First notice that we have

$$h_o(x) = \exp\Big(\frac{x^{k+1}}{1 + \lambda x^k}\frac{d}{dx}\Big).$$

Let us denote by $\xi(x) = \dfrac{x^{k+1}}{1+\lambda x^k}\dfrac{d}{dx}$. Once again we claim that G is analytically normalizable, and it follows from the fact that h_o is analytically normalizable and that G is abelian. It also is clear that h_o is a normal form diffeomorphism in the sense of [**11**], and therefore if $z \in (\Sigma_i, q_i)$ is a normalizing coordinate for G_i, then we must have $h_o(z) = \exp\left(\mu \dfrac{z^{k+1}}{1+\lambda z^k}\dfrac{d}{dz}\right)$, for some $\mu \in \mathbb{C}^*$. Thus it follows that

$$\xi(x) = \frac{x^{k+1}}{1+\lambda x^k}\frac{d}{dx} = \mu \frac{z^{k+1}}{1+\lambda z^k}\frac{d}{dz}.$$

On the other hand the local holonomy $g_o \in \operatorname{Diff}(\Sigma_j, q_j)$ associated to the local separatrix $(y=0)$; i.e, the holonomy of the *central manifold* of the saddle-node, is linear in $y\big|_{\Sigma_j}$ given by

$$g_o(y) = \exp\left(2\pi\lambda\sqrt{-1}y\frac{d}{dy}\right) = \exp(2\pi\lambda\sqrt{-1})y.$$

Thus it is natural to pass the elements $h \in G_i$, tangent to the identity, to G_j as linear maps in the coordinate $y\big|_{\Sigma_j}$. Let $h \in G_i$ with $h'(0) = 1$. We have $h = \exp(t_h\xi)$ for some $t_h \in \mathbb{C}$. Now we remark that if we consider the multiform function $f(x) = x^{-\lambda}\exp\left(1/kx^k\right)$ defined on the transversal $\Sigma_i = (y = e^{1/k})$, it extends to a local multiform first integral for $\mathcal{F}$. Therefore given any element $\gamma \in \pi_1(D_i \backslash \operatorname{sing}\mathcal{F})$, if we denote by $h_\gamma \in G_i$, the corresponding holonomy diffeomorphism associated to this homotopy class, then we obtain $f_o(x) \circ h_\gamma = \exp(2\pi c_\gamma\sqrt{-1}) \cdot f_o(x)$, for any fixed determination $f_o(x)$ of $f(x)$. Therefore we obtain

$$x^{-\lambda}\exp\left(1/kx^k\right) \circ \exp\left(t_{h_\gamma}\frac{x^{k+1}}{1+\lambda x^k}\frac{d}{dx}\right) = \exp(2\pi c_\gamma\sqrt{-1}) \cdot x^{-\lambda}\exp\left(1/kx^k\right).$$

Take now an element $h \in G_i$ tangent to the identity say

$$h(x) = \exp\left(t_h\frac{x^{k+1}}{1+\lambda x^k}\frac{d}{dx}\right).$$

Since we have $f_o(x) \circ h(x) = \exp(2\pi c_h\sqrt{-1}) \cdot f_o(x)$ for some $c_h \in \mathbb{C}$, we consider $h^{\mathcal{D}}(y) := \exp(2\pi c_h\sqrt{-1})y$. It is now clear that $h^{\mathcal{D}} \in \operatorname{Diff}(\mathbb{C}, 0)$ satisfies the adjunction equation. Finally we observe that by our construction we have $h^{\mathcal{D}} \in G_j^v$.

Remark 3.2. — As it is well-known a resonant nonlinearizable corner $q_o = D_i \cap D_j$, has a formal normal form,

$$\omega = [n(1 + (\lambda - 1)(x^m y^n)^k)xdy + m(1 + \lambda(x^m y^n)^k)y\,dx] = 0,$$

where $\lambda = n/m$ [**11**]. As we have seen in Proposition 2.8, when one of the virtual holonomy groups is nonabelian, then we have a formal normal form

$$\omega = kxdy + \ell y\Big(1 + \frac{\sqrt{-1}}{2\pi}x^\ell y^k\Big)dx = 0.$$

These formal normal forms admit *integrating factors* (that is, formal functions h such that $\Omega = \omega/h$ is closed), and give us closed meromorphic 1-forms Ω which have non simple poles along $(x \cdot y = 0)$. Moreover, these closed 1-forms are unique up to multiplication by constants (this is a consequence of the fact that the singularity admits no meromorphic first integral, except the constants). Later on, we will see that this last remark shows that it is not necessary for our purposes, to define any adjunction associated to such a corner (see Case C in the proof of Proposition 5.6).

Definition 3.3 (**[5]**, **[19]**). — Under the hypothesis of 3.1, 3.2 or 3.3 above for q_o, G_i^v and G_j^v, we define the group $G_j * (\mathcal{D}_* G_i)$ as the subgroup of $\mathrm{Diff}(\Sigma_i, q_i)$ generated by G_j and by the elements $h^{\mathcal{D}}$ where $h \in G_i$. This group will be called the *Dulac adjunction of G_i to G_j*(1).

Given a singularity $q_o = D_i \cap D_j$ as in cases 3.1, 3.2 or 3.3 above we assume that G_i is analytically normalizable, and choose analytic normalizing coordinates $(x, y) \in U$ for the singularity q_o (see Lemma 3.1 and Proposition 2.8), as in these cases, *i.e.*, coordinates that give the foliation $\mathcal{F}|_U$ in its local normal form. We choose $D_i \cap U = (x = 0)$ and $D_j \cap U = (y = 0)$. We may also choose x in such a way that the restriction $x|_{\Sigma_i}$ is an analytic normalizing coordinate for the holonomy group G_i (Lemma 3.1). However, it is not always true that $y|_{\Sigma_j}$ also normalizes G_j. This is the subject of the following lemma.

Lemma 3.4. — *The adjunction holonomy group $G_j * (\mathcal{D}_* G_i)$ is a solvable group if, and only if, $y|_{\Sigma_j}$ normalizes G_j.*

Lemma 3.4 is proved in **[2]**(2). It also follows, under our hypothesis of analytic normalization and normal hyperbolicity on the saddle-nodes, from an equivalent result of **[16]**, which is stated in terms of the solvability and convergence of some formal Lie algebras of vector fields.

In other words, Lemma 3.4 says that if $q_o = D_i \cap D_j$ with G_i, G_j solvable as above, then we can normalize simultaneously G_i, G_j and the singularity q_o if, and only if, the adjunction holonomy group $G_j * (\mathcal{D}_* G_i)$ is a solvable group. This same lemma holds in the formal case, where G_i is not assumed to be analytically normalizable. We will use this lemma in order to "glue" certain integrating factors associated to adjacent components D_i and D_j (see Proposition 5.2 and Proposition 5.6).

In order to proceed to the definition of the singular holonomy groups of the components D_j we introduce an order in the resolution divisor (here we follow as in **[2]**). This order is defined as follows: Let $\mathcal{Z}$ be the (finite) family of connected components

(1)This adjunction process, defined preliminary for holonomy groups, must be iterated whenever it is possible.

(2)We will give a more general version of this lemma in §5 Lemma 5.7.

of

$$D^* = D \backslash \{\text{saddle-nodes}\} \cup \{\text{nonlinearizable resonant singularities}\}$$

Given two elements $A, B \in \mathcal{Z}$ we will say that $A > B$ if and only if there exist components D_i, D_j of D such that $D_i \cap D_j = \{q\}$ is either a nonlinearizable resonant singularity, or a saddle-node, $D_i^* = D_i - \{q\} \subset A$, $D_j^* = D_j - \{q\} \subset B$, and the strong manifold of q lies over D_i. Clearly $>$ defines a partial order on $\mathcal{Z}$. Each totally ordered subfamily $\mathcal{Z}_1 \subset \mathcal{Z}$ has a maximal element. Take a supremum say $A_o \in \mathcal{Z}$. It is clear that given a component D_j such that

$$D_j \backslash \{\text{saddle-nodes and resonant nonlinearizable singularities}\}$$

belongs to A_o then no adjunction can be defined from any other component B of $\mathcal{Z}$ to A_o. Thus we consider the *singular holonomy* of any fixed component D_j in A_o as the maximal subgroup $\mathrm{Hol}^{\mathrm{sing}}(\mathcal{F}, D_j, \mathcal{S}_j) = G_j^{\mathrm{sing}} \subset \mathrm{Diff}(\mathcal{S}_j, p_j)$ obtained by iterating the adjunction process at all the other corners of D_j, $D_j \cap D_k$, $k \neq j$ where D_k is in A_o. Now we consider the new family $\mathcal{Z}_1 = \mathcal{Z} - \{A_o\}$. Take a maximal element $A_1 \in \mathcal{Z}_1 = \mathcal{Z} - \{A_o\}$ with respect to the (induced partial order) $>$, and consider the same iteration of the adjunction process for the corners in A_1 adding to this the adjuncted singular holonomy of D_j. This defines the singular holonomy group of any component $D_k \in A_1$. Thus we may exhaust $\mathcal{Z}$ and define the adjunction of the holonomy for all the components of D.

Definition 3.5 (**[5]**, **[19]**). — The subgroup

$$\mathrm{Hol}^{\mathrm{sing}}(\mathcal{F}, D_j, \mathcal{S}_j) = G_j^{\mathrm{sing}} \subset \mathrm{Diff}(\mathcal{S}_j, p_j)$$

obtained by the algorithmic process above will be called the *singular holonomy group* associated to the component D_j of D. The singular holonomy group is defined up to conjugacies, depending on the choice of the transverse sections $\mathcal{S}_j$.

Using the fact that the singular holonomy is always contained in the virtual holonomy, we conclude from Corollary 2.4 that:

Corollary 3.6. — *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ having as limit set some singularities and an algebraic curve Λ as in Theorem 1.1. Then each projective singular holonomy group of* $\mathrm{sing}\,\mathcal{F} \cap \Lambda$ *is solvable with discrete pseudo-orbits outside the origin.*

4. Logarithmic derivatives of an integrable differential 1-form

In what follows $\mathcal{F}$ is a foliation on a complex surface M defined by a meromorphic integrable 1-form $\omega = 0$ outside its polar divisor $(\omega)_\infty$. We consider a bimeromorphic map $\pi \colon \widetilde{M} \to M$ (*e.g.* a resolution morphism as in §2) and denote by $\widetilde{\omega}$ the pull-back 1-form $\pi^*\omega$. The 1-form $\widetilde{\omega}$ is meromorphic in $\widetilde{M}$, but may have non-isolated singularities (in the case of the resolution morphism these singularities appear over the projective lines D_j introduced by π). On the other hand, for the cases we are interested in, we may assume that ($(\omega)_\infty$ and therefore) $(\widetilde{\omega})_\infty$, has non invariant codimension one irreducible components, and meets the divisor transversely at regular points of $\widetilde{\mathcal{F}}$. In this case, the singular set $\operatorname{sing}\widetilde{\mathcal{F}}$ is contained in the zero divisor $(\widetilde{\omega})_0$.

Definition 4.1 **([18])**. — Let $D \subset \widetilde{M}$ be an invariant divisor (not necessarily irreducible). A meromorphic 1-form $\widetilde{\eta}$ defined on $\widetilde{M}$ is called a *logarithmic derivative of $\widetilde{\omega}$ adapted to D* if

(1) $d\widetilde{\omega} = \widetilde{\eta} \wedge \widetilde{\omega}$, $d\widetilde{\eta} = 0$,

(2) the polar divisor $(\widetilde{\eta})_\infty$ of $\widetilde{\eta}$ has order one along (any component of) D, and consists of the union of $(\widetilde{\omega})_\infty \cup (\widetilde{\omega})_0$ and an invariant divisor of $\mathcal{F}$,

(3) the residue of $\widetilde{\eta}$ along any *noninvariant* irreducible component L of $(\widetilde{\omega})_\infty \cup (\widetilde{\omega})_0$ is equal to either $-$(the order of the poles of $\widetilde{\omega}$ along L), or (the order of $(\widetilde{\omega})_0$ along L), respectively. We remark that, *a priori*, $\widetilde{\eta}$ may have nonsimple poles on $\widetilde{M}$.

The existence of an adapted logarithmic derivative is related to some conditions in the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$ **[2]**, **[19]** (see also Theorem 5.1 below).

Definition 4.2 **([2])**. — The foliation $\mathcal{F}$ has a *Liouvillian resolution* (*relative to* Λ) if the divisor $D = \bigcup_{j=0}^{m} D_j$ of the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$ satisfies:

(i) The singular holonomy group of every D_i is solvable.

(ii) If $q = D_i \cap D_j$ is a non degenerate corner such that $G_i \subset \mathbb{H}_{k_i}$ and $G_j \subset \mathbb{H}_{k_j}$ are nonabelian, then q is a resonant singularity. Moreover, if q is linearizable (has a holomorphic first integral), then we can find local coordinates $(x, y) \in U$ centered at q such that $D_i \cap U = (x = 0)$, $D_j \cap U = (y = 0)$ and $\mathcal{F}\big|_U : k_j x dy + k_i y dx = 0$.

(iii) If $q = D_i \cap D_j$ is a saddle-node corner whose strong manifold is contained in D_i, then G_i is solvable, and G_j is abelian analytically linearizable.

The motivation for the definition above is given by **[2]**, **[18]** and the following result:

Proposition 4.3 **([2])**. — *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{C}P(2)$ given by a rational 1-form ω, and suppose that there exists an algebraic $\mathcal{F}$-invariant curve Λ having only non dicritical singularities. Assume that each component of the resolution divisor of* $\operatorname{sing}\mathcal{F} \cap \Lambda$ *contains a linearizable non resonant singularity and that all the saddle-nodes in the resolution divisor are in good position relatively to this divisor. Then the following two assertions are equivalent:*

(i) The 1-form ω admits a rational logarithmic derivative η, adapted to the invariant curve $\Lambda \subset \mathbb{C}P(2)$.

(ii) $\mathcal{F}$ exhibits a Liouvillian resolution relative to Λ, and all the singular holonomy groups are analytically normalizable.

Theorem 4.4. — *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ having as limit set some singularities and an algebraic curve Λ. Assume that* $\operatorname{sing}\mathcal{F} \cap \Lambda$ *satisfies (C_1) and that the invariant part of the resolution divisor is connected. Then $\mathcal{F}$ has a Liouvillian resolution relative to Λ.*

Proof. — According to Corollary 3.6, it remains to prove that

(1) if $q = D_i \cap D_j \in \operatorname{sing}\widetilde{\mathcal{F}}$ is a non degenerate singularity such that the holonomy groups of D_i and of D_j are non abelian. Then q is resonant. Moreover, if q has a holomorphic first integral, then we can write it as $k_j x dy + k_i y dx = 0$ for some local coordinates with $D_j : \{y = 0\}$ and $D_i : \{x = 0\}$.

(2) If $q = D_i \cap D_j \in \operatorname{sing}\widetilde{\mathcal{F}}$ is a saddle-node whose strong manifold is tangent to D_i, then the holonomy group of D_j is abelian analytically linearizable.

Proof of (1). — In fact, if the holonomy group of D_j is not abelian then, since the virtual holonomy G_j^v exhibits discrete pseudo-orbits, it follows that G_j^v contains only elements with periodic linear part **[10]**. In particular the singularity q must be a resonant singularity. Assume now that q is linearizable say, as $mxdy + nydx = 0$ with $m, n \in \mathbb{N}$, $\langle m, n\rangle = 1$, and $(y = 0) \subset D_j$ and $(x = 0) \subset D_i$ as usual. We must prove that $m/n = k_j/k_i$. We consider first the case $m = n = 1$, and G_i is analytically normalizable. In this case as in item 3.1 of §3. we have that the Dulac correspondence is given by

$$\mathcal{D} \colon \Sigma_i \to \Sigma_j \quad \mathcal{D}(\{(x,1)\}) = \{(1,x)\} \in \Sigma_j.$$

We take any element $h \in G_i$, write

$$h(x) = \frac{\lambda x}{(1 + a x^{k_i})^{1/k_i}}$$

and define

$$G_j^v \ni h^{\mathcal{D}}(y) = y \mapsto \frac{\lambda y}{\left(1 + a y^{k_i}\right)^{1/k_i}}$$

using the fact that $m = n = 1$ (see 3.1 §3). Applying now Theorem 2.5 for G_i and G_j^v we conclude that $k_i \le k_j$. The same way, we may conclude that $k_j \le k_i$ and therefore $k_i = k_j$. Now we treat the case $m \ne n$ as in 3.1 §3. We repeat the main argument: Take elements $g_1 \in G_j^v$ and $h_1 \in G_i^v$ that are tangent to the identity of orders k_j and k_i respectively. Using the Dulac correspondence $\mathcal{D}(x) = x^{m/n}$, we introduce the sets of "pseudo-orbits"

$$\left\{g_1^{k_r} \circ \mathcal{D} \circ h_1^{\ell_r} \circ \mathcal{D}^{-1} \circ \cdots \circ g_1^{k_2} \circ \mathcal{D} \circ h_1^{\ell_2} \circ \mathcal{D}^{-1} \circ g_1^{k_1} \circ \mathcal{D} \circ h_1^{\ell_1}(x)\right\} \subset \Sigma_i,$$

where $x \in \Sigma_i$, ℓ_t, k_t are integral numbers, and $\mathcal{D}^{-1}$ is the inverse correspondence $\Sigma_j \to \Sigma_i$, $y \mapsto y^{n/m}$. These sets are contained in a same leaf of $\mathcal{F}$ for each fixed x, and as in Theorem 2.5 and in Nakai's Theorem **[14]**, if the powers g_1^ℓ, and h_1^k are never related by the conjugation equation, then we will have accumulation points for the leaves of $\mathcal{F}$, outside the origin in Σ_i. Thus we conclude that some power h_1^ℓ, passes to the virtual holonomy G_j^v, as some power g_1^k. This implies that $m/n = k_j/k_i$.

Proof of (2). — In fact, choose analytic coordinates (x, y) at q, such that $\widetilde{\mathcal{F}}$ is given by $x^{p+1}dy - y(1 + \lambda x^p)dx = 0$. By hypothesis the saddle-node is normally hyperbolic so that the holonomy group G_j contains some element with non periodic linear part.

According to **[10]** since G has some discrete pseudo-orbit this implies that G_j cannot be solvable non abelian. Thus it follows that the holonomy group G_j is abelian and, since it contains some analytically linearizable non rational rotation, G_j is analytically linearizable. □

Remark 4.5. — Since $G_j^{\text{sing}} \subset G_j^v$ also has discrete orbits, it follows (with the same proof) that in (2) above we have G_j^{sing} abelian analytically linearizable.

5. Formal logarithmic derivatives near the limit set

In this section we prove a generalization of Theorem 4.4 above:

Theorem 5.1. — *Let $\mathcal{F}$ be a foliation on M a complex projective surface, and let $\Lambda \subset M$ be an invariant irreducible analytic curve. Assume that* $\operatorname{sing}\mathcal{F} \cap \Lambda$ *satisfies property $(\mathcal{C}_1)$ and that all the saddle-nodes in the resolution divisor are in good position. If $\mathcal{F}$ is given by a meromorphic 1-form ω with isolated singularities, then the following assertions are equivalent:*

(i) The strict transform $\widetilde{\omega}$ of the 1-form ω, admits a transversely formal logarithmic derivative $\widetilde{\eta}$ over Γ, adapted to the invariant curve Γ.

(ii) $\mathcal{F}$ has a Liouvillian resolution relative to Λ.

We shall describe briefly the notion of transversely formal object. Suppose that $\Gamma \subset M$ is an algebraic codimension one divisor. We denote by $\mathcal{I}_\Gamma$ the sheaf of ideals defining $\Gamma \subset M$, and by $\mathcal{O}_M$ the sheaf of regular functions on M. For any $m \in \mathbb{N}$ we have $(\mathcal{I}_\Gamma)^{m+1} \subset (\mathcal{I}_\Gamma)^m \subset \mathcal{O}_M$.

The *infinitesimal tubular neighborhood of order* m of Γ in M is the locally ringed space

$$\left(M, \frac{\mathcal{O}_M}{(\mathcal{I}_\Gamma)^m}\right)$$

The *formal completion* of M *along* Γ is the locally ringed space $\widehat{\Gamma} = (M, \mathcal{O}_{\widehat{\Gamma}})$, whose structural sheaf is defined by the projective limit

$$\mathcal{O}_{\widehat{\Gamma}} = \varprojlim \frac{\mathcal{O}_M}{(\mathcal{I}_\Gamma)^m}$$

The *ring of (transversely) formal rational functions on* M *along* Γ, $K(\widehat{\Gamma})$, is defined as the ring of rational functions on the formal completion $\widehat{\Gamma}$ of M along Γ.

It is proved in **[7]** that:

(i) $K(\widehat{\Gamma}) = H^o(\widehat{\Gamma}, \mathcal{K}_{\widehat{\Gamma}})$ where $\mathcal{K}_{\widehat{\Gamma}}$ is the quotient field of the sheaf of total quotient rings of $\mathcal{O}_{\widehat{\Gamma}}$,

(ii) there exists a natural inclusion of sheaves $\mathcal{K}_\Gamma \to \mathcal{K}_{\widehat{\Gamma}}$, where $\mathcal{K}_\Gamma$ is the sheaf of germs of meromorphic functions defined on neighborhoods of points in Γ.

We refer to **[5]** for more details.

With the notions above we extend the notion of logarithmic derivative to the notion of *transversely formal logarithmic derivative* over a resolution divisor Γ.

The first step in the proof of Theorem 5.1 is the following (see also **[16]** for similar constructions):

Proposition 5.2. — *Let $\mathcal{F}$ be a foliation in the complex surface M, and let Λ be an invariant compact curve such that* $\operatorname{sing}\mathcal{F}\cap\Lambda$ *satisfies* $(\mathcal{C}_1)$ *and all the saddle-nodes in the resolution divisor are in good position. Assume that $\mathcal{F}$ has a Liouvillian resolution relative to Λ. Let $D_j \subset \Gamma$ be an irreducible component of the resolution divisor of* $\operatorname{sing}\mathcal{F}\cap\Lambda$. *Then there exists a closed transversely formal meromorphic* 1*-form $\widetilde{\eta}_j$ defined over D_j, such that $\widetilde{\eta}_j$ is a transversely formal logarithmic derivative of $\widetilde{\omega}$, adapted to D_j.*

Proof. — It follows from the definition of Liouvillian resolution that the groups G_j^{sing} are solvable. We distinguish the non abelian case and the abelian case considered in the following two lemmas. Clearly they complete the proof of Proposition 5.2. □

Lemma 5.3. — *Under the hypothesis of Proposition 5.2, assume that G_j^{sing} is non abelian and formally embedded in $\mathbb{H}_{k_j}$. Let $\ell_j = \operatorname{ord}((\widetilde{\omega})_0, D_j)$, then there exists a transversely formal closed logarithmic derivative $\widetilde{\eta}_j$ of $\widetilde{\omega}$, adapted to D_j, such that* $\operatorname{Res}_{D_j}\widetilde{\eta}_j = k_j + 1 + \ell_j$.

Proof. — In order to simplify the notation we write $k = k_j$. We follow **[16]**: Given any regular point $q \in D_j$ and a local transverse section Σ_q with $\Sigma_q \cap D_j = q$, we can choose a formal coordinate $y \in \Sigma_q$ centered at q, such that y defines the formal embedding $G_j \cong \operatorname{Hol}(\widetilde{\mathcal{F}}, D_j, \Sigma_q) \subset \mathbb{H}_{k_j}$. Given such a coordinate we consider the formal vector field $\widehat{\xi}_q = y^{k+1}d/dy$. We have $g * \widehat{\xi}_q = c_g \cdot \widehat{\xi}_q$, $\forall\, g \in \operatorname{Hol}(\widetilde{\mathcal{F}}, D_j, \Sigma_q)$. Thus $\widehat{\xi}_q$ can be extended into a global section $\widehat{\sigma}$ of the quotient sheaf $\widehat{\mathcal{S}}/\mathbb{C}^*$, where $\widehat{\mathcal{S}}$ is the sheaf of the transversely formal symmetries of $\widetilde{\mathcal{F}}$ over $D_j \backslash \operatorname{sing}\widetilde{\mathcal{F}}$ **[16]**. We define the 1-form $\widetilde{\eta}_j = d(\widetilde{\omega}(\widehat{\sigma}))/\widetilde{\omega}(\widehat{\sigma})$, which is a well defined transversely formal closed 1-form over $D_j \backslash \operatorname{sing}\widetilde{\mathcal{F}}$. The 1-form $\widetilde{\eta}_j$ is a logarithmic derivative of $\widetilde{\omega}$ adapted to $D_j \backslash \operatorname{sing}\widetilde{\mathcal{F}}$.

An alternative way of constructing $\widetilde{\eta}_j$ is the following (**[2]**, **[5]**, **[19]**): For each point $q \in D_j \backslash \operatorname{sing}\widetilde{\mathcal{F}}$ as above, we extend the coordinate y into a transversely formal coordinate defined over a certain neighborhood of q in $D_j \backslash \operatorname{sing}\widetilde{\mathcal{F}}$. This local extension is a consequence of the local trivialization for $\widetilde{\mathcal{F}}$. Thus we obtain a collection $\{(x_\alpha, y_\alpha), U_\alpha\}_{\alpha\in A}$ where the U_α are open sets which cover a neighborhood of $D_j \backslash \operatorname{sing}\widetilde{\mathcal{F}}$ in $\widetilde{\mathbb{C}P(2)}$, and also $(y_\alpha = 0) = U_\alpha \cap D_j$, $\widetilde{\omega}\big|_{U_\alpha} = g_\alpha dy_\alpha$ for some transversely formal function g_α over $U_\alpha \cap D_j$. The coordinates x_α are analytic in U_α and the coordinates y_α are transversely formal over $U_\alpha \cap D_j$, obtained as the extensions of the coordinates y above. Finally if $U_\alpha \cap U_\beta \neq \phi$ then we have

$$y_\alpha^k = \frac{a\overline{y}_\beta{}^k}{1 + b\overline{y}_\beta{}^k}$$

for some $a, b \in \mathbb{C}$. Define the 1-form

$$\widetilde{\eta}_j\big|_{U_\alpha \cap D_j} := (k+1)\frac{dy_\alpha}{y_\alpha} + \frac{dg_\alpha}{g_\alpha}.$$

It is not difficult to see that this is a well-defined closed transversely formal meromorphic 1-form over $D_j \setminus \operatorname{sing} \widetilde{\mathcal{F}}$, and satisfies the relation $d\widetilde{\omega} = \widetilde{\eta}_j \wedge \widetilde{\omega}$. Moreover $(\widetilde{\eta}_j)_\infty$ is invariant and contains $(y_\alpha = 0)$ as a simple pole of residue $k+1$.

The extension of the 1-form $\widetilde{\eta}_j$ to a singularity $q \in D_j$ is done as follows. According to Proposition 2.8 there exists a formal logarithmic derivative η_q defined at the singular point q. If q is a saddle-node then by the hypothesis we can take η_q meromorphic in a neighborhood of q. This same holds in the case q admits a formal first integral or is a nonresonant nondegenerate singularity (Proposition 2.8). In general, in the nondegenerate case the formal integrating factors η_q can be extended in a transversely formal way along the separatrices through q, as a consequence of the resommation properties of the integrating factors along the separatrices through q **[19]**(3). Thus, over a neighborhood of q in D_j, the difference $\widetilde{\eta}_j - \eta_q$ writes $\widetilde{\eta}_j - \eta_q = h_q \cdot \widetilde{\omega}$ for some transversely formal function h_q, defined over a punctured neighborhood of q in D_j, which is an integrating factor for $\widetilde{\omega}$, that is $d(h_q \cdot \widetilde{\omega}) = 0$. We will show that h_q extends in a transversely formal way to a neighborhood of q in D_j (see Remark 2.9 and the first part of the proof of Lemma 5.3 above). We consider the following cases:

(a) q is formally linearizable non resonant, of the form ω_λ in Proposition 2.8. In this case we know that there exist formal coordinates (x, y) such that $D_j = (y = 0)$, $D_i = (x = 0)$ and $\widetilde{\omega}(x, y) = g(xdy - \lambda ydx)$, for some transversely formal function g over a neighborhood of q in D_j. We also have

$$\eta_q = \frac{dg}{g} + a\frac{dx}{x} + b\frac{dy}{y}$$

where $a, b \in \mathbb{C}^*$ satisfy

$$(*) \qquad 1 + \lambda = a + b\lambda.$$

(3)We say that a 1-form $\widetilde{\omega}$ admits a *formal integrating factor* $\widehat{h}$ defined at q; if we have

$$(*) \qquad d\Big(\frac{\widetilde{\omega}}{\widehat{h}}\Big) = 0$$

where $\widehat{h}$ is a formal series at q. Equation $(*)$ exhibits *resommation properties for* $\widehat{h}$ along a certain separatrix S of $\widetilde{\omega} = 0$ at q, if $\widehat{h}$ can be written $\widehat{h}(x, y) = \sum_{j=0}^{+\infty} a_j(x) y^j$, where $(x, y) \in U$ is a local holomorphic coordinate centered at q, such that $S \cap U = \{y = 0\}$, $a_j(x)$ is a holomorphic function converging in a small disk $\mathbb{D}_q \subset S$ centered at q, not depending on $j \in \mathbb{N}$. This occurs, by a Briot-Bouquet type argument **[11]**, **[12]**, **[16]**, **[19]**, for any separatrix of a non-degenerate singularity, and for the strong separatrix of a saddle-node.

We have

$$\widetilde{\eta}_j - \eta_q = h_q.\widetilde{\omega}$$

as transversely formal objects over $(y = 0), (x \neq 0)$. The function $h_q(x, y)$ is a transversely formal function defined over $(y = 0), (x \neq 0)$, and since the 1-form $\frac{1}{gxy}\widetilde{\omega}(x, y)$ is closed it follows that we have $d(h_q xyg) \wedge \left(\frac{dy}{y} - \lambda\frac{dx}{x}\right) = 0$. Now, since $\lambda \in \mathbb{C}\backslash\mathbb{Q}$, it follows from **[18]** that $h_q xyg = \mu \in \mathbb{C}$ is constant (use Laurent series). Therefore, the formal expression $h_q(x, y)$ extends as a formal meromorphic extension to $(x = 0)$.

(b) q has a formal first integral. In this case, we can choose (x, y) as above such that $\widetilde{\omega}(x, y) = g(nxdy + mydx)$ as in Proposition 2.8 (b). We have

$$\eta_q = \frac{dg}{g} + a\frac{dx}{x} + b\frac{dy}{y}$$

where $a, b \in \mathbb{C}^*$ satisfy $n - m = na - mb$. The formal expression

$$d(h_q xyg) \wedge \left(\frac{dy}{y} + \frac{m}{n}\frac{dx}{x}\right) = 0$$

implies that $(h_q xyg)(x, y) = \varphi(x^m y^n)$ for some formal function $\varphi(z)$ in one variable. Therefore, the formal expression for $h_q(x, y)$ extends as $\varphi(x^m y^n)/xyg$ to $(x = 0)$.

(c) q has a non linear formal normalization as in Proposition 2.8 (c). In this case we may choose (x, y) such that $\widetilde{\omega} = g\big(kxdy + \ell y(1 + \frac{\sqrt{-1}}{2\pi}x^\ell y^k)dx\big)$. We choose

$$\eta_q = (k + 1)\frac{dy}{y} + (\ell + 1)\frac{dx}{x} + \frac{dg}{g}.$$

Then we have

$$d(h_q g x^{\ell+1} y^{k+1}) \wedge d\Big(\frac{1}{x^\ell y^k} - \frac{\sqrt{-1}\ell}{2\pi}\log x\Big) = 0.$$

On the other hand, since the singularity q has a non linear formal normalization, it follows that its local holonomy is not formally linearizable. This implies that $h_q g x^{\ell+1} y^{k+1} = \mu \in \mathbb{C}$ is a constant. Therefore $h_q(x, y)$ extends to $(x = 0)$.

(d) q is a saddle-node whose strong manifold is tangent to D_j. We may choose analytic coordinates (x, y) such that $(y = 0) \subset D_j$ and $\widetilde{\omega} = g(y^{k+1}dx - x(1+\lambda y^k)dy)$. We define

$$\eta_q = \frac{dx}{x} + (k + 1)\frac{dy}{y} + \frac{dg}{g}.$$

Then the difference $\widetilde{\eta}_j - \eta_q$ has simple poles over $D_j\backslash\{q\} : (x \neq 0), (y = 0)$, therefore it follows that $\eta_q = \widetilde{\eta}_j$ in $(x \neq 0)$: In fact, since a saddle-node admits no formal first integral, it follows that up to multiplicative constants the 1-form

$$\frac{dx}{x} - \frac{dy}{y^{k+1}} - \lambda\frac{dy}{y}$$

is the only closed formal 1-form which defines $\mathcal{F}$ at q, and it has non simple poles over $(y = 0)$. Hence $\widetilde{\eta}_j$ extends as η_q to $(x = 0)$.

(e) q is a saddle-node whose strong manifold is transverse to D_j. In particular G_j is abelian analytically linearizable (see Proposition 4.3). This implies that $\operatorname{sing}\widetilde{\mathcal{F}}\cap D_j$ *contains no saddle-nodes in good position with respect to D_j and that the nondegenerate singularities are analytically linearizable.* Using the techniques of **[2]**, **[19]**, **[16]** one constructs a meromorphic closed 1-form $\widetilde{\Omega}_j$ in a neighborhood V_j of D_j minus the saddle-nodes in $\operatorname{sing}\widetilde{\mathcal{F}}\cap D_j$. This 1-form satisfies $\widetilde{\Omega}_j\wedge\widetilde{\omega}=0$, and $\widetilde{\Omega}_j$ has simple poles, all of them contained in the divisor D. We have $\widetilde{\omega}=h_j\cdot\widetilde{\Omega}_j$ for some meromorphic h_j in V_j. Redefine now the 1-form $\widetilde{\eta}_j=dh_j/h_j$. Clearly $\widetilde{\eta}_j$ is a candidate for a logarithmic derivative of $\widetilde{\omega}$ adapted to D_j. Notice that $\widetilde{\eta}_j$ *has residue* 1 *over* D_j: In fact, the poles of $\widetilde{\Omega}_j$ are simple, and $\widetilde{\omega}$ has isolated singularities. We must show that $\widetilde{\eta}_j$ extends meromorphically to the saddle-node singularities in D_j. Fix such a singularity $q_o\in D_j$. We must have $q_o=D_l\cap D_j$ for some D_l (the saddle-nodes are in good position). We will show the extension of $\widetilde{\eta}_j$ to q_o using the fact that the adjunction holonomy group $G_j*\mathcal{D}_*(G_l)$ is solvable. Choose analytic coordinates $(x,y)\in U$ centered at q, such that $D_l\cap U=(x=0)$, and $D_j=(y=0)$, and such that $\widetilde{\omega}\big|_U(x,y)=g\cdot(x^{p+1}dy-ydx)$ for some meromorphic function g and some $p\in\mathbb{N}$. As we know the holonomy group G_l of the component D_l, must be solvable nonabelian $G_l\subset\mathbb{H}_p$. Moreover we can assume that the analytic coordinate $x\big|_{\Sigma_l}$, where $\Sigma_l=(y=e^{1/p})$, normalizes the group $\operatorname{Hol}(\widetilde{\mathcal{F}},D_l,\Sigma_l)\cong G_l$. Take an element tangent to the identity $h\in G_l$, say

$$h(x)=\frac{x}{(1+ax^p)^{1/p}}.$$

Then the corresponding element $h^{\mathcal{D}}\in G_j*\mathcal{D}_*(G_l)$ is given by $h^{\mathcal{D}}(y)=e^{a/p}\cdot y$ Let now $z\in(\Sigma_j,q_j)$ be any analytic coordinate, where $\Sigma_j=(x=1)$ and $q_j=\Sigma_j\cap D_j$, which linearizes the holonomy group $G_j\cong\operatorname{Hol}(\widetilde{\mathcal{F}},D_j,\Sigma_j)$. Then the fact that the singular holonomy group G_j^{sing} is solvable implies that either $y=\varphi(z)$ for some diffeomorphism $\varphi\in\mathbb{H}_k$ when $G_j^{\text{sing}}\subset\mathbb{H}_k$ by analytic conjugacy, or $y=\mu\cdot z$ for some $\mu\in\mathbb{C}^*$ when G_j^{sing} is abelian analytically linearizable. According to this we can assume that the analytic coordinate $y\big|_{\Sigma_j}$ linearizes G_j. This implies that the 1-forms $\widetilde{\eta}_j$ and

$$\eta_{q_o}:=\frac{dy}{y}+(p+1)\frac{dx}{x}+\frac{dg}{g}$$

coincide over the transverse section Σ_j. This same argument shows that they coincide over an open neighborhood of q_j in D_j. Therefore $\widetilde{\eta}_j$ extends as η_{q_o} to the singularity q_o. □

Lemma 5.4. — *Under the hypothesis of Proposition 5.2, assume that G_j^{sing} is abelian. There exists a transversely formal logarithmic derivative $\widetilde{\eta}_j$ defined over D_j, this 1-form is obtained as follows: There exists a closed transversely formal 1-form $\widetilde{\omega}_j$ defined over $D_j\setminus\operatorname{sing}\widetilde{\mathcal{F}}$, which defines $\widetilde{\mathcal{F}}$ around $D_j\setminus\operatorname{sing}\widetilde{\mathcal{F}}$ in the sense that $\widetilde{\omega}_j\wedge\widetilde{\omega}=0$.*

The 1-form $\widetilde{\eta}_j$ is obtained by extending $d\widetilde{h}_j/\widetilde{h}_j$, where $\widetilde{\omega}=\widetilde{h}_j\cdot\widetilde{\omega}_j$, to the singularities $q\in\operatorname{sing}\widetilde{\mathcal{F}}\cap D_j$.

Proof. — In this case the group G_j^{sing} may be non linearizable. Anyway, according to Lemma 2.6, there exists a formal singular holomorphic vector field in one complex variable $\xi\in\widehat{\mathcal{X}}(\mathbb{C},0)$ which is invariant by the natural action induced by G_j^{sing}, *i.e.*, $g*\xi=\xi$, $\forall\, g\in G_j^{\text{sing}}$. Now, using the techniques of **[16]** we can extend the vector $\widehat{\xi}$ into a *transversely formal symmetry* $\widehat{\sigma}$ for $\widetilde{\mathcal{F}}$, through the holonomy, (here one may use the invariance above), that is, a global section of the sheaf $\widehat{\mathcal{S}}$ of transversely formal symmetries of $\widetilde{\mathcal{F}}$ over $D_j\setminus\operatorname{sing}\widetilde{\mathcal{F}}$. This section induces a transversely formal integrating factor $\widehat{h}_j=\widetilde{\omega}(\widehat{\sigma})$ for $\widetilde{\omega}$ over $D_j\setminus\operatorname{sing}\widetilde{\mathcal{F}}$. We define the closed transversely formal 1-form $\widetilde{\eta}_j=d\widehat{h}_j/\widehat{h}_j$ over D_j. Since $\widetilde{\omega}/\widehat{h}_j$ is closed we have $d\widetilde{\omega}=\widetilde{\eta}_j\wedge\widetilde{\omega}$. Clearly $\widetilde{\eta}_j$ is a logarithmic derivative of $\widetilde{\omega}$ adapted to D_j and defined over $D_j\setminus\operatorname{sing}\widetilde{\mathcal{F}}$.

Now we show that this logarithmic derivative extends to the singularities in D_j. Fix a singularity $q\in\operatorname{sing}\widetilde{\mathcal{F}}\cap D_j$. As in the proof of Lemma 5.3, there exists a transversely formal logarithmic derivative η_q defined over a neighborhood of q in D_j, and we have $\widetilde{\eta}_j-\eta_q=h_q\cdot\widetilde{\omega}$ for some transversely formal function h_q which is an integrating factor for $\widetilde{\omega}$, defined over a punctured neighborhood of q in D_j. We consider the following cases:

(a) q is formally linearizable non resonant, of the form ω_λ in Proposition 2.8. In this case the same arguments of the proof of Lemma 5.3 above, apply to show that we have $hxyg=\text{cte}$ and therefore we have a natural extension of $\widetilde{\eta}_j$ to q.

(b) q has a formal first integral as $\omega_{k,\ell}$ in Proposition 2.8 (b). In this case, as above we can choose (x,y) above such that $\widetilde{\omega}(x,y)=g(nxdy+mydx)$, and

$$\eta_q=\frac{dg}{g}+a\frac{dx}{x}+b\frac{dy}{y}$$

where $a,b\in\mathbb{C}^*$ with $n-m=na-mb$. The formal expression $h_q(x,y)$ extends therefore as $\varphi(x^my^n)/xyg$ to $(x=0)$.

(c) q has a non linear formal normalization as in Proposition 2.8 (c). As above we conclude that $h_q(x,y)$ extends to $(x=0)$.

(d) q is a saddle-node with the strong manifold tangent to D_j. As above the difference $\widetilde{\eta}_j-\eta_q$ has simple poles over $D_j\setminus\{q\}:(x\neq 0),(y=0)$, and therefore it follows that $\eta_q=\widetilde{\eta}_j$ in $(x\neq 0)$. This implies that $\widetilde{\eta}_j$ extends as η_q to $(x=0)$.

(e) q is a saddle-node whose strong manifold is transverse to D_j. In particular G_j^v is abelian analytically linearizable. This case is done as above, by using the fact that the adjunction holonomy groups associated to D_j are all abelian. This ends the proof of Lemma 5.4. □

Remark 5.5. — The following proposition is one of the main tools in this work, and we refer to **[19]**, **[2]** for a more general version and some details. This also may follow from the results in [**16**], our hypothesis on the saddle-node singularities and the fact that the *virtual holonomy* groups are solvable (Corollary 2.4) and "contain" all the information concerning the foliation near the resolution divisor so that, in particular they must contain the "generalized holonomy" of [**16**].

Proposition 5.6. — *Let $\mathcal{F}$ be a foliation on a complex surface M, and let Λ be an invariant compact curve such that the saddle-nodes in the resolution divisor of* $\operatorname{sing}\mathcal{F}\cap\Lambda$ *are in good position. Assume that $\mathcal{F}$ has a Liouvillian resolution relative to Λ, and that* $\operatorname{sing}\mathcal{F}\cap\Lambda$ *satisfies property C_1. There exists a closed transversely formal 1-form $\widetilde{\eta}$ defined over Γ, such that $d\widetilde{\omega}=\widetilde{\eta}\wedge\widetilde{\omega}$. This form $\widetilde{\eta}$ is a logarithmic derivative for $\widetilde{\omega}$ adapted to Γ.*

Proof. — We give the proof of the most relevant points in the construction. Let D_j be any component of Γ which we will write as $\Gamma=\bigcup_{i=0}^{m}D_i$. We consider a 1-form $\widetilde{\eta}_j$ over D_j given by Proposition 5.2 above. Fix a corner $q_o=D_i\cap D_j$ where D_i and D_j are components of Γ. We will prove that

$D_i\cap D_j\neq\varnothing\Rightarrow\widetilde{\eta}_i=\widetilde{\eta}_j$, *(for suitable and fixed choices of the 1-form $\widetilde{\eta}_j$), as formal expressions at q_o. In the case the 1-forms are meromorphic we also have the equality holding in a neighborhood of q_o.*

There are some cases to consider:

Case A. — q_o is a saddle-node singularity. In this case we have analytic coordinates (x,y) centered at q_o such that $\widetilde{\omega}=g\cdot\big(x(1+\lambda y^k)dy-y^{k+1}dx\big)$ and $D_i=(x=0)$, $D_j=(y=0)$. We have $\widetilde{\eta}_i-\widetilde{\eta}_j=f_{ij}\cdot\widetilde{\omega}$ for some formal meromorphic function f_{ij} such that $d(f_{ij}\cdot\widetilde{\omega})=0$ in a neighborhood of q_o. This implies that $f_{ij}gxy^{k+1}=a\in\mathbb{C}$. Therefore

$$\widetilde{\eta}_i-\widetilde{\eta}_j=a\cdot\Big(\lambda\frac{dy}{y}-\frac{dx}{x}+\frac{dy}{y^{k+1}}\Big),$$

and hence since the poles of $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ are simple we have $a=0$ and therefore $\widetilde{\eta}_i=\widetilde{\eta}_j$ in a neighborhood of q_o.

Now we assume that q_o is not a saddle-node. We have that q_o is a non degenerate singularity of the form $xdy-\lambda ydx+\text{h.o.t.}=0$ where (x,y) are local holomorphic coordinates centered at q_o and such that $D_i:(x=0)$ and $D_j:(y=0)$. As above $\widetilde{\eta}_i-\widetilde{\eta}_j=f_{ij}\cdot\widetilde{\omega}$, but now f_{ij} is a formal meromorphic function at q_o, which satisfies $d(f_{ij}\cdot\widetilde{\omega})=0$. We introduce the following types of singularity:

- ***type (1)*** **:** q_o admits a holomorphic first integral, G_i is abelian or nonabelian analytically normalizable, $G_i\subset\mathbb{H}_{k_i}$, with $nk_i/m\in\mathbb{N}$ as in Definition 4.2 and Theorem 4.4.
- ***type (2)*** **:** q_o is a non resonant analytically linearizable singularity and G_i is abelian.

We will use the following lemma whose proof we have extracted from [**19**] for the reader's convenience:

Lemma 5.7. — *Let $q_o = D_i \cap D_j$ be a corner of type* (1) *or* (2) *as above. Assume that the adjunction holonomy group $G_j * (\mathcal{D}_* G_i)$ is solvable. Then we may construct $\widetilde{\eta}_j$, $\widetilde{\eta}_i$ in such a way that we have $\widetilde{\eta}_j = \widetilde{\eta}_i$ as formal expressions at q_o. Moreover, for the cases G_i, G_j are analytically normalizable, let $(x, y) \in U$ be* analytic *coordinates such that $(x = 0) = D_i \cap U$, $(y = 0) = D_j \cap U$, $\widetilde{\mathcal{F}}\big|_U$ is in the normal form $xdy - \lambda ydx = 0$, $\lambda \in (\mathbb{C} \backslash \mathbb{Q}) \cup \mathbb{Q}_-$, and such that $x\big|_{\Sigma_i}$ is an analytic normalizing coordinate for* $\mathrm{Hol}(\widetilde{\mathcal{F}}, D_i, \Sigma_i)$, $\Sigma_i = (x = 1)$. *Then the adjunction $G_j * (\mathcal{D}_* G_i)$ is solvable if, and only if, $y\big|_{\Sigma_j}$ $(\Sigma_j = (x = 1))$ is an analytic normalizing coordinate for* $\mathrm{Hol}(\widetilde{\mathcal{F}}, D_j, \Sigma_j)$.

Proof. — We denote by $\ell_j = \mathrm{ord}((\widetilde{\omega})_o, D_j)$. First we assume that q_o is of type (2) We have G_i linearizable so that each $h \in \mathrm{Hol}(\widetilde{\mathcal{F}}, D_i, \Sigma_i)$ writes $h(x) = \mu_h \cdot x$ and induces an element $h^{\mathcal{D}}(y) = \nu_h \cdot y$ in $\mathrm{Diff}(\Sigma_j, q_j)$ where $\nu_h = \mu_h^{-\lambda}$. In particular the local holonomy h_0 around q_o induces $h_0^{\mathcal{D}}(y) = \left(e^{-2\pi i/\lambda}\right)^{-\lambda} y = y$. Thus we have two possibilities:

Case 1. — G_i is cyclic generated by h_0. In this case any element $h \in G_i$ writes $h = h_0^{m_h}$ for some $m_h \in \mathbb{Z}$. Now, if G_j is also abelian then we may assume that $\widetilde{\eta}_j = \widetilde{\eta}_i$ (are meromorphic and coincide) in a neighborhood of q_o. If G_j is nonabelian say, $G_j \subset \mathbb{H}_{k_j}$ by an analytic embedding, then we have necessarily $\mathrm{Res}_{D_j} \widetilde{\eta}_j = k_j + 1 + \ell_j$ (Lemma 5.3). This fixes $\mathrm{Res}_{D_i} \widetilde{\eta}_j$ as $\mathrm{Res}_{D_i} \widetilde{\eta}_j = 1 - k_j\lambda + \ell_i$ as it follows from equation $(*)$ in the proof of Lemma 5.3. On the other hand we may (since G_i is linearizable), choose $\widetilde{\eta}_i$ such that $\mathrm{Res}_{D_i} \widetilde{\eta}_i = 1 - k_j\lambda + \ell_i$ and therefore $\mathrm{Res}_{D_j} \widetilde{\eta}_i = k_j + \ell_i + 1$. Thus $\widetilde{\eta}_j - \widetilde{\eta}_i$ is holomorphic in a neighborhod of q_o and since

$$(\widetilde{\eta}_j - \widetilde{\eta}_i) \wedge \left(\frac{dy}{y} - \lambda \frac{dx}{x}\right) = 0$$

and $\lambda \notin \mathbb{Q}$, it follows that $\widetilde{\eta}_j = \widetilde{\eta}_i$ around q_o. Alternatively, we may use the fact that G_i is generated by h_0 in order to extend $\widetilde{\eta}_j\big|_{\Sigma_i}$ to a neighborhod of D_i and therefore obtain $\widetilde{\eta}_j = \widetilde{\eta}_i$ around q_o for $\widetilde{\eta}_i =$ extension of $\widetilde{\eta}_j\big|_{\Sigma_i}$ above obtained.

Case 2. — G_i is not cyclic. In this case there exists $h \in G_i$ such that $h^m \neq h_0^n$, $\forall\, (n, m) \ni \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\}$. Thus $\mu_h^m \neq e^{-2\pi i n/\lambda}$ and therefore $\mu_h^{-\lambda} \neq e^{2\pi i n/m}$ $\forall\, (n, m) \in \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\}$.

This shows that $h^{\mathcal{D}} \in G_j * (\mathcal{D}_* G_i) \subset \mathrm{Diff}(\Sigma_j, q_j)$ is of the form $h^{\mathcal{D}}(y) = \mu_h^{-\lambda} \cdot y$ and is not a rational rotation. Now, fixed a normalizing coordinate $z \in (\Sigma_j, q_j)$, $z(q_j) = 0$, for $G_j * (\mathcal{D}_* G_i)$ as a solvable group then (in the nonabelian case) we may write each element $g \in G_j * (\mathcal{D}_* G_i)$ as $g(z) = az / \sqrt[k_j]{1 + bz^{k_j}}$. In particular we may write $h^{\mathcal{D}}(z) = a \cdot z / \sqrt[k_j]{1 + bz^{k_j}}$ where $a = \mu_h^{-\lambda}$ is not a root of 1. This implies that $h^{\mathcal{D}}$ is linearizable by some coordinate $Z = T(z)$, where T is an homography. Therefore we may assume that $h^{\mathcal{D}}(z) = \mu_h^{-\lambda} \cdot z$. Since $\mu_h^{-\lambda}$ is not a root of 1 it follows that $z = \alpha \cdot y$ for some $\alpha \in \mathbb{C}^*$. Thus $y\big|_{\Sigma_j}$ also normalizes the group $G_j * (\mathcal{D}_* G_i) \supset G_j$. In particular we obtain $\widetilde{\eta}_j = \widetilde{\eta}_i$ around q_o.

– Now we consider the case q_o is of type (1) with G_i solvable nonabelian:

We write $\widetilde{\mathcal{F}}\big|_U : nxdy + mydx = 0$. Any element $h \in G_i$ writes

$$h(x) = \frac{ax}{\sqrt[k_i]{1 + bx^{k_i}}}$$

and induces an element $h^{\mathcal{D}} \in G_j * (\mathcal{D}_* G_i)$ of the form

$$h^{\mathcal{D}}(y) = \frac{a^{m/n}y}{\sqrt[k_j]{1 + by^{k_j}}} \in \mathrm{Diff}(\Sigma_j, q_j).$$

This implies that $G_j * (\mathcal{D}_* G_i)$ is nonabelian and we can choose a normalizing coordinate $z \in \Sigma_j$, $z(q_j) = 0$, such that $h^{\mathcal{D}}(z) = a^{m/n}z/\sqrt[k_j]{1 + cz^{k_j}}$. Now, if we choose $h \in G_i$ such that $b \cdot c \neq 0$ then necessarily we have $z^{k_j} = T(y^{k_j})$ for some homography $T(Z) \in \mathrm{SL}(2, \mathbb{C})$ and therefore $y\big|_{\Sigma_j}$ also normalizes the group $G_j * (\mathcal{D}_* G_i) \subset \mathrm{Diff}(\Sigma_j, q_j)$. In particular $\widetilde{\eta}_j = \widetilde{\eta}_i$ in a neighborhood of q_o.

– Finally, we consider the case where q_o is of type (1) with G_i abelian:

In this case again we write $\widetilde{\mathcal{F}}\big|_U : nxdy + mydx = 0$, and now any element $h \in G_i$, writes $h(x) = \mu x h_1(x^m)$ and induces $h^{\mathcal{D}}(y) = \mu^{m/n}\, y \cdot h_1^{m/n}(y^n)$ in the group $G_j * (\mathcal{D}_* G_i)$.

If G_i is not cyclic then there exists h such that $h^k \neq h_0^\ell$, $\forall\,(k, \ell) \in \mathbb{Z} \times \mathbb{Z}\backslash\{(0,0)\}$ and therefore we may proceed as above and conclude that $y\big|_{\Sigma_j}$ normalizes $G_j * (\mathcal{D}_* G_i)$ (notice that G_i not cyclic $\Rightarrow$ G_i analytically normalizable $\Rightarrow$ G_j is analytically normalizable). Thus we one reduced to the case G_i is cyclic. Thus G_i is in fact finite (because it is abelian and contains a rational rotation, h_0). We may consider two distinguished situations:

• If G_i is generated by h_0 then since $h_0^{\mathcal{D}} = \mathrm{Id}$ it follows that $G_j * (\mathcal{D}_* G_i) = G_j$ and we may extend $\widetilde{\eta}_j\big|_{\Sigma_i}$ to a neighborhood of D_i and therefore assume that $\widetilde{\eta}_j = \widetilde{\eta}_i$ at q_o.

• If G_i is not generated by h_0 then we have $G_i = \langle h_0^{1/\ell} \rangle$ for some $\ell \in \mathbb{N}$ and $h_0^{1/\ell} \in G_i$ induces an element $\big(h_0^{1/\ell}\big)^{\mathcal{D}} \in G_j * (\mathcal{D}_* G_i)$ that writes $\big(h_0^{1/\ell}\big)^{\mathcal{D}}(y) = e^{2\pi i/\ell}\, y$ which is not the identity. Since $G_j * (\mathcal{D}_* G_i) \supset G_j$ and is also solvable, we can construct $\widetilde{\eta}_j$ in such a way that $\big(h_0^{1/\ell}\big)^{\mathcal{D}}_* \widetilde{\eta}_j = \widetilde{\eta}_j$ in Σ_j, and this assures that $\big(h_0^{1/\ell}\big)_* (\widetilde{\eta}_j\big|_{\Sigma_i}) = (\widetilde{\eta}_j\big|_{\Sigma_i})$ so that $\widetilde{\eta}_j\big|_{\Sigma_i}$ can be extended to a neighborhood of D_i and we may assume that $\widetilde{\eta}_j = \widetilde{\eta}_i$ at q_o. This ends the proof of the lemma. □

Case B. — $\lambda \notin \mathbb{Q}$. In this case, q_o is formally linearizable **[3]**, moreover, according to Definition 4.2 (ii) and Theorem 4.4, at least one of the holonomy groups of D_i and D_j is abelian, say G_i is abelian formally linearizable. However it is not clear that q_o is analytically linearizable, so that it is not immediate that we can introduce the adjunction holonomy group $G_j * \mathcal{D}_*(G_i)$. If both virtual holonomy groups are abelian then we can proceed as in the proof of Lemma 5.4 and construct the 1-forms $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ from integrating factors, say $\widetilde{\eta}_\nu = d\log h_\nu$, in a neighborhood of q_o, where $\widetilde{\omega}/h_\nu$

is closed, $\nu = i, j$. Clearly the quotient h_i/h_j is a first integral for $\widetilde{\omega}$ at q_o. Since the singularity is not formally linearizable it follows that $h_i = \text{cte} \cdot h_j$ and therefore we have $\widetilde{\eta}_i = \widetilde{\eta}_j$. Thus we may assume that G_j is non-abelian. But since it contains a diffeomorphism with non-periodic linear part (the local holonomy around q_o), it follows that G_j^v is analytically normalizable, and so it is the singularity q_o. This says that q_o is of type (2). The glueing of the forms $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ is therefore given by the fact that the adjunction holonomy group $G_j * \mathcal{D}_*(G_i)$ is solvable ([**2**], [**19**]) (see Lemma 5.7).

Case C. — q_o is not linearizable and $\lambda = -n/m \in \mathbb{Q}_-$ in the usual notation. If both singular holonomy groups are abelian, then according to the proof of Lemma 5.4 we construct the 1-forms $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ from integrating factors, say $\widetilde{\eta}_\nu = d \log h_\nu$, in a neighborhood of q_o, where $\widetilde{\omega}/h_\nu$ is closed, $\nu = i, j$. Clearly the quotient h_i/h_j is a first integral for $\widetilde{\omega}$ at q_o. Since the singularity is not formally linearizable it follows that $h_i = \text{cte} \cdot h_j$ and therefore we have $\widetilde{\eta}_i = \widetilde{\eta}_j$. Now we assume at least that only one of the holonomy groups say, G_j is nonabelian. In this case we have formal coordinates (x, y) at q_o such that

$$\widetilde{\omega} = g\Big(kxdy + \ell y\Big(1 + \frac{\sqrt{-1}}{2\pi}x^\ell y^k\Big)dx\Big) \qquad \text{(Proposition 2.8)}.$$

Thus we have as usual $\widetilde{\eta}_i - \widetilde{\eta}_j = h_q \cdot \widetilde{\omega}$ where

$$d(h_q g x^{\ell+1} y^{k+1}) \wedge d\Big(\frac{1}{x^\ell y^k} - \frac{\sqrt{-1}\ell}{2\pi} \log x\Big) = 0.$$

On the other hand, since the singularity q_o has a non linear formal normalization, it follows that $h_q g x^{\ell+1} y^{k+1} = \mu \in \mathbb{C}$ is a constant. Finally we have $\mu = 0$ because both $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ have simple poles over Γ.

Case D. — q_o has a holomorphic first integral, $\lambda = -n/m$ as usual. If both virtual holonomy groups G_j^v and G_i^v are abelian, then we can consider the adjunction process from D_i to D_j and conversely. Thus it will follow that the 1-forms $\widetilde{\eta}_j$ and $\widetilde{\eta}_i$ may be constructed in compatible way, so that we have $\widetilde{\eta}_i = \widetilde{\eta}_j$ at q_o (indeed, q_o is of type (1)). Thus we may assume that some of the virtual holonomy groups say, G_j^v is non-abelian. If it is analytically normalizable then we have q_o of type (1) and we may apply Lemma 5.7. Let us however give a sole argument. Assume that G_i^v is also non-abelian. Then the 1-forms $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ have residues given by $\operatorname{Res}_{D_i} \widetilde{\eta}_i = \ell_i + k_i + 1$ and $\operatorname{Res}_{D_j} \widetilde{\eta}_j = \ell_j + k_j + 1$, in the usual notation. On the other hand we have that $m/n = k_j/k_i$ (Theorem 4.4). Using this and equation $(*)$ in the proof of Lemma 5.3, it follows that $\operatorname{Res}_{D_i} \widetilde{\eta}_j = \operatorname{Res}_{D_i} \widetilde{\eta}_i$ and also $\operatorname{Res}_{D_j} \widetilde{\eta}_j = \operatorname{Res}_{D_i} \widetilde{\eta}_j$. Thus, the difference $\widetilde{\eta}_j - \widetilde{\eta}_i$ is a closed (formal) holomorphic 1-form at q_o, so that we may write it as $\widetilde{\eta}_j - \widetilde{\eta}_i = d\varphi_{ij}(x^m y^n)$ for some (holomorphic) formal function $\varphi(z)$. Now, as in 3.1 of §3 *we can consider the adjunction holonomy groups* obtained from G_j and G_i. These groups are solvable and we have $G_i * \mathcal{D}^*(G_j) \subset G_i^v$ and $G_j * \mathcal{D}^*(G_i) \subset G_j^v$.

The solvability of these groups allows us to extend the 1-form $\widetilde{\eta}_i$ as a transversely formal 1-form over $D_j \setminus \operatorname{sing} \widetilde{\mathcal{F}}$, and therefore the integral $\varphi(x^m y^n)$ also extends as a transversely formal first integral over $D_j \setminus \operatorname{sing} \widetilde{\mathcal{F}}$. This implies, in the case $\varphi(z)$ is non-constant, that the virtual holonomy group G_j^v is finite **[13]**, **[16]**, and we would have a contradiction. Thus we may assume that G_j^v is non-abelian, but G_i^v is abelian. In particular *we may perform the adjunction from D_i to D_j and obtain a solvable subgroup $G_j * \mathcal{D}^*(G_i) \subset G_j^v$* (see 3.1 §3). We write the difference $\widetilde{\eta}_i - \widetilde{\eta}_j = \Omega_{ij}$, for some closed (formal) meromorphic 1-form at q_o. This 1-form can be extended by holonomy to a transversely formal closed meromorphic 1-form over $D_j \setminus \operatorname{sing} \widetilde{\mathcal{F}}$. This implies, in the $\Omega_{ij} \neq 0$, that the virtual holonomy G_j^v is abelian **[16]**, **[19]**, and we would have a contradiction. Thus $\Omega_{ij} = 0$ and therefore $\widetilde{\eta}_i = \widetilde{\eta}_j$ at q_o. Thus in any case either the glueing of $\widetilde{\eta}_i$ and $\widetilde{\eta}_j$ is immediate, or it is given by the fact that the adjunction holonomy is well-defined and solvable as it follows from Lemma 5.7.

Remark 5.8. — The solvability of the *virtual* holonomy groups is enough, under our hypothesis of normal hyperbolicity on the saddle-nodes, to conclude that we may choose all the 1-forms $\widetilde{\eta}_j$ in a simultaneously compatible way (see Remark 5.5). For instance we discuss the case the holonomy of D_j is finite, and there are D_i, D_k with non abelian holonomies such that $D_i \cap D_j \neq \varnothing \neq D_k \cap D_j$. In this case the 1-form $\widetilde{\eta}_j$ is not unique *over a neighborhood of $D_j \setminus \operatorname{sing} \widetilde{\mathcal{F}}$*. However, if $q_i = D_i \cap D_j$ and $q_k = D_k \cap D_j$ are not saddle-nodes then (they have finite local holonomies and therefore, by **[13]**, these singularities admit local holomorphic first integrals so that) we may perform the adjunction from the holonomy of D_j to D_k and D_i and conversely. Thus we consider the case where q_i and q_j are saddle-nodes. Since the holonomy of D_j is linearizable, it follows that these saddle-nodes are not in good position with respect to D_j, that is, their strong manifolds lie over D_i and D_k respectively. But in this case we can perform the adjunction of the holonomy of D_i to D_j and of the holonomy of D_k to D_j (see the construction of the singular holonomy which is below Lemma 3.4). Therefore, the fact that the singular holonomy G_j^{sing} is abelian linearizable (*cf.* Remark 4.5), is enough to assure the compatibility of the forms $\widetilde{\eta}_j$, $\widetilde{\eta}_i$ and $\widetilde{\eta}_k$ **[19]**. The collection of 1-forms $(\widetilde{\eta}_i)_{i=0}^m$ defines therefore a transversely formal 1-form $\widetilde{\eta}$ over Γ, which is a logarithmic derivative of $\widetilde{\omega}$, adapted to Γ.

Proof of Theorem 5.1. — According to Proposition 5.6, it remains to prove (i)⇒(ii) in Theorem 5.1. This is proved with a geometric interpretation of **[16]**, or following the steps of an analogous result stated in **[2]**, one may also find a proof in **[19]**, using our hypothesis of normal hyperbolicity on the saddle-nodes. □

6. Rationality of formal logarithmic derivatives

As in §2 we denote by Γ the invariant part of the divisor D, obtained in the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$. The following proposition is a consequence of **[5]** which is based on a theorem of Hironaka-Matsumara **[8]**:

Proposition 6.1. — *Let $\widetilde{\alpha}$ be a transversely formal differential form defined over Γ in $\widetilde{\mathbb{C}P(2)}$, where $\Gamma \subset \widetilde{\mathbb{C}P(2)}$ is a normal crossing divisor, and $\widetilde{\mathbb{C}P(2)}$ is is obtained from $\mathbb{C}P(2)$ by a finite sequence of blowing-ups. Assume that $\Gamma \subset \widetilde{\mathbb{C}P(2)}$ satisfies property (psdc). Then $\widetilde{\alpha}$ extends rationally to $\widetilde{\mathbb{C}P(2)}$.*

Proposition 6.2. — *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$, having as limit set some singularities and an algebraic curve Λ. Assume that* $\operatorname{sing}\mathcal{F} \cap \Lambda$ *satisfies property* $(\mathcal{C}_1)$ *and* (*psdc*). *Then there exists a closed rational* 1*-form $\widetilde{\eta}$ which is a rational logarithmic derivative for $\widetilde{\omega}$, adapted to the invariant part of the resolution divisor of* $\operatorname{sing}\mathcal{F} \cap \Lambda$. *In particular all the projective singular holonomy groups of* $\operatorname{sing}\mathcal{F} \cap \Lambda$ *are solvable analytically normalizable.*

Proof. — According to Theorem 4.4, $\mathcal{F}$ has a Liouvillian resolution relative to Λ. Using Propositions 5.6 and 6.1 we conclude that $\widetilde{\omega}$ admits a rational logarithmic derivative $\widetilde{\eta}$ adapted to Γ. We apply Proposition 6.1 to conclude the rationality of $\widetilde{\eta}$. The last part of the statement is a consequence of **[16]** or, also, of an improvement of **[2]** found in **[19]**. □

7. Proof of Theorem 1.1

According to Proposition 6.2 we know that $\widetilde{\mathcal{F}}$ (that is $\widetilde{\omega}$) admits a rational logarithmic derivative $\widetilde{\eta}$ on $\widetilde{\mathbb{C}P(2)}$, which is adapted to Γ. According to Proposition 6.2 above, all the singular holonomy groups appearing in the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$ are analytically normalizable. Thus we may apply the last part of **[2]** which assures that either $\mathcal{F}$ is given by a closed rational 1-form or $\mathcal{F}$ is a rational pull-back of a Riccati foliation as in Theorem 1.1. □

We can relax the hypothesis on the limit set of $\mathcal{F}$ if we assume that $\mathcal{F}$ has a transcendent leaf whose limit set is an algebraic curve Λ plus some singularities but, in order to prove that $\Gamma = \lim(\widetilde{L})$ for the corresponding transcendent leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$, we have to make an additional hypothesis on $\operatorname{sing}\mathcal{F} \cap \Lambda$.

$(\mathcal{C}_2)$ Λ *contains all its local separatrices, and all the saddle-nodes appearing in the resolution of* $\operatorname{sing}\mathcal{F} \cap \Lambda$ *have their strong manifolds contained in the limit set* $\lim \widetilde{L}$.

Using the same techniques as in the proof of Theorem 1.1 we can prove:

Theorem 7.1. — *Let $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$, having a transcendent leaf whose limit set is an algebraic curve Λ. Assume that* $\operatorname{sing}\mathcal{F} \cap \Lambda$ *satisfies property* (*psdc*), *conditions* $(C_1),(C_2)$. *Then, either* $\mathcal{F}$ *is given by a closed rational* 1*-form or it is a rational pull-back of a Riccati foliation* $\mathcal{R}: p(x)dy-(a(x)y^2+b(x)y)dx=0$, *where* Λ *corresponds to* $\overline{(y=0)}\cup\overline{(p(x)=0)}$, *on* $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$.

A few words should be said, concerning the relations between our result and groups of linear rational transformations. Let G be a finitely generate *Fuchsian group*, i.e, a properly discontinuous group of diffeomorphisms of $\overline{\mathbb{C}}$, carrying a certain circle $C(G)$, (the *principal circle*), into itself. It is known that the limit set of G lies on $C(G)$. Moreover it is well-known that *if* $\lim(G)$ *has more than two points, either* $\lim(G)=C(G)$, *or* $\lim(G)\subset C(G)$ *is a nowhere dense perfect subset* **[6]**.

Let us assume that $\lim(G)=C(G)$. Using **[9]** we can realize G as the "suspension holonomy" of the line $\overline{(y=0)}\subset\overline{\mathbb{C}}\times\overline{\mathbb{C}}$, of a Riccati foliation

$$\mathcal{R}(G): p(x)dy-(a(x)y^2+b(x)y+c(x))dx=0$$

on $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$. The foliation $\mathcal{R}(G)$ satisfies $\lim(\mathcal{R}(G))=M^3(G)$, for a real 3-dimensional singular subvariety $M^3(G)$, which is singular along the invariant vertical fibers $\overline{\mathbb{C}}_x$: $x\times\overline{\mathbb{C}}, p(x)=0$, and such that the intersection $M^3(G)\cap\overline{\mathbb{C}}_x$ is a principal circle of a Fuchsian group conjugate to G, provided that the fiber $\overline{\mathbb{C}}_x$ is non invariant. Conversely, one can ask whether a foliation whose limit set is an invariant real singular hypersurface as $M^3(G)$ above, is in fact the pull-back of a Riccati foliation. This problem remains open.

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