JACOB PALIS

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A GLOBAL VIEW OF DYNAMICS AND A CONJECTURE ON THE DENSENESS OF FINITUDE OF ATTRACTORS

by

Jacob Palis

To Adrien Douady for his lasting contribution to mathematics (July 1995)

Abstract. — A view on dissipative dynamics, i.e. flows, diffeomorphisms, and transformations in general of a compact boundaryless manifold or the interval is presented here, including several recent results, open problems and conjectures. It culminates with a conjecture on the denseness of systems having only finitely many attractors, the attractors being sensitive to initial conditions (chaotic) or just periodic sinks and the union of their basins of attraction having total probability. Moreover, the attractors should be stochastically stable in their basins of attraction. This formulation, dating from early 1995, sets the scenario for the understanding of most nearby systems in parametrized form. It can be considered as a probabilistic version of the once considered possible existence of an open and dense subset of systems with dynamically stable structures, a dream of the sixties that evaporated by the end of that decade. The collapse of such a previous conjecture excluded the case of one dimensional dynamics: it is true at least for real quadratic maps of the interval as shown independently by Swiatek, with the help of Graczyk [GS], and Lyubich [Ly1] a few years ago. Recently, Kozlovski [Ko] announced the same result for C^3 unimodal mappings, in a meeting at IMPA. Actually, for one-dimensional real or complex dynamics, our main conjecture goes even further: for most values of parameters, the corresponding dynamical system displays finitely many attractors which are periodic sinks or carry an absolutely continuous invariant probability measure. Remarkably, Lyubich [Ly2] has just proved this for the family of real quadratic maps of the interval, with the help of Martens and Nowicki [MN].

1. Introduction and Main Conjecture

In the sixties two main theories in dynamics were developed, one of which was designed for conservative systems and called KAM for Kolmogorov-Arnold-Moser. A later important development in this area, in the eighties, was the Aubry-Mather theory

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for periodic (and Cantori) motions, which has been more recently further improved by Mather. Other outstanding results have been obtained even more recently by Eliasson, Herman, Mañé and others.

The focus of this paper, however, will be the surprising unfolding of the other theory that has been constructed for general systems (nonconservative, dissipative) and called hyperbolic: it deals with systems with hyperbolic limit sets. This means for a diffeomorphism f on a manifold M, that the tangent bundle to M at L, the limit set of f, splits up into df-invariant continuous subbundles $T_L M = E^s \oplus E^u$ such that $df \mid E^s$, $df^{-1} \mid E^u$ are contractions, with respect to some Riemmanian metric. As for most of the concepts in the sequel, a similar definition holds for a non-invertible map g, requiring $dg \mid E^u$ to be invertible. And for a flow X_t, $t \in \mathbb{R}$, we add to the splitting a subbundle in the direction of the vector field that generates the flow

$$T_L M = E^s \oplus E^u \oplus E^0$$

and require for some Riemannian metric and some constants C, $0 < \lambda < 1$, that

$$||dX_t \mid E^s||, \quad ||dX_{-t} \mid E^u|| \leq C\, e^{\lambda t}, \quad t \in \mathbb{R}.$$

See **[PT]**, specially chapter seven, for details and many of the notions presented here.

The concept of hyperbolicity was introduced by Smale, with important contributions to its development as a theory being also given by some of his students at the time, as well as Anosov, Arnold, Sinai and others. Initially, it was created to help pursue the "lost dream" referred to in the abstract: to find an open and dense subset of dynamically (structurally) stable systems; i.e., systems that when slightly perturbed in the C^r-topology, $r \geq 1$, remain with the same dynamics, modulo homeomorphisms of the ambiente space that preserve orbit structures, in the case of flows, or are conjugacies, in the case of transformations. It has actually transcended this objective, loosing through a series of counter-examples its projected character of much universality, i.e. its validity for an open dense subset of systems. But it became the ground basis for a notable evolution that dynamics experienced in the last twenty five years or so. Still, based on previous results, specially by Anosov and by ourselves, Smale and I were able to formulate in 1967 what was called the Stability Conjecture, that would fundamentally tie together hyperbolicity and dynamical stability: a system is C^r-stable if its limit set is hyperbolic and, moreover, stable and unstable manifolds meet transversally at all points. For stability restricted to the limit set, the transversality condition is substituted by the nonexistence of cycles among the transitive (dense orbit), hyperbolic subsets of the limit set.

The theory of hyperbolic systems, i.e. systems with hyperbolic limit sets, was quite developed especially for flows and diffeomorphisms, and it was perhaps even near completion (an exageration!), by the end of the sixties and beginning of the seventies. That included some partial classifications, and an increasing knowledge of their ergodic properties, due to Sinai, Bowen, Ruelle, Anosov, Katok, Pesin, Franks, Williams, Shub, Manning, among several others.

More or less at the same time, the proof of one side of the Stability Conjecture was completed through the work of Robbin, Robinson and de Melo. However, the outstanding part of this basic question from the 60's was proved to be true only in the middle 80's, in a remarkable work of Mañé **[M2]** for diffeomorphisms, followed ten years later by an again remarkable paper of Hayashi **[Ha]** for flows: C^1 dynamically stable systems must be hyperbolic. Before, by 1980, Mañé had proved the two-dimensional version of the result, but independent and simultaneous proofs were also provided by Liao and Sannami. Other partial contributions should be credited to Pliss, Doering, Hu and Wen. A high point in Hayashi's work is his connecting lemma creating homoclinic orbits by C^1 small perturbations of a flow or diffeomorphism: an unstable manifold accumulating on some stable one can be C^1 perturbed to make it intersect one another (the creation of homoclinic or heteroclinic orbits). This fact has been at this very moment sparkling some advance of dynamics in the lines proposed here, as it will be pointed out later.

While the ergodic theory of dynamical systems, as suggested by Kolmogorov and more concretely by Sinai, was being successfully developed, the hope of proposing some global structure for dynamics in general grew dimmer and dimmer in the seventies. This was due to new intricate dynamical phenomena that were presented or suggested all along the decade. First, Newhouse **[N]** extended considerably his previous results, showing that infinitely many sinks occur for a residual subset of an open set of C^2 surface diffeomorphisms near one exhibiting a homoclinic tangency. Perhaps equally or even more striking at the time, was the appearance of attractors having sensitivity with respect to initial conditions in their basin.

Although there are several possible definitions of an attractor, here we will just require it to be invariant, transitive (dense orbit) and attracting all nearby future orbits or at least a Lebesgue positive measure set in the ambient manifold. If A is an attractor for f with basin $B(A)$, we say that it is sensitive to initial conditions, or chaotic if there is $\varepsilon > 0$ such that with total probability on $B(A) \times B(A)$, for each pair of points (x, y) there is an integer $n > 0$ so that $f^n(x)$ and $f^n(y)$ are more than ε apart, where the distances are considered with respect to some Riemannian metric. The definition for flows is entirely similar. Chaotic attractors became also known as strange. The first one, beyond the hyperbolic attractors which are not just sinks, is due to Lorenz **[L]**. Proposed numerically by Lorenz in 1963, it's a rather striking fact that only in the middle seventies most of us became acquainted with Lorenz-like attractors through the examples of Guckenheimer and Williams, which we now call geometric ones. It is still an open and interesting question if the original Lorenz's equations

$$\dot{x} = -10x + 10y$$
$$\dot{y} = 28x - y - xz$$
$$\dot{z} = -(8/3)z + xy$$

in fact correspond to a flow displaying a strange attractor.

Subsequently, again based on numerical experiments, Hénon **[He]** asked about the possible existence of a strange attractor, but now in two dimensions, for certain quadratic diffeomorphisms of the plane

$$f_{a,b}(x, y) = (1 - ax^2 + y, bx)$$

for $a \approx 1.4$ and $b \approx 0.3$. Finally, by the end of the decade, Feigenbaum **[F]** and independently Coullet-Tresser **[CT]** suggested another kind of attractor, now for quadratic maps of the interval and related to a limiting map of a sequence of transformations exhibiting period-doubling bifurcations of periodic orbits. Almost immediately after that, Jacobson **[J]** exhibited strange attractors in the same setting. All this, together with the unsuccessful atempts of the sixties, led to a common belief that perhaps no such a global scenario for dynamics was possible.

However, a series of important results on strange attractors for maps, concerning their persistence, i.e. their existence for a positive Lebesgue measure set in parameter space, and the fact that they carry physical or SRB (Sinai-Ruelle-Bowen) invariant measures, provided by Jacobson **[J]** for the interval, Benedicks-Carleson **[BC]**, Mora-Viana **[MV]**, Benedicks-Young **[BY1]**, **[BY2]** and Diaz-Rocha-Viana **[DRV]** for Hénon-like maps (small C^r perturbations of Hénon maps, $r \geq 1$), were about to take place in the next fifteeen years or so. Perhaps even more striking is the recent proof that they are stochastically stable, a recent remarkable result of Benedicks-Viana **[BV1]**, **[BV2]**. In proving this fact, Benedicks and Viana first showed for Hénon-like attractors, that there are "no holes" in the basin of attraction with respect to the SRB measure, a question I heard from Ruelle and Sinai more than a decade ago: a.e. in the attractor with respect to the SRB measure, there are stable manifolds and their union covers Lebesgue a.a. points in the basin of attraction (and, thus, the union of the stable manifolds is dense in the basin of attraction) **[BV1]**. The concepts of SRB measure and stochastic stability will be presented below. Almost simultaneously, after previous pioneering work of Arnold and Herman (see **[Ar]**, **[H]**), the theory of one-dimensional dynamics experienced a great advance, due to Yoccoz, Sullivan, McMullen, Lyubich, Douady, Hubbard, Swiatek and an impressive number of other mathematicians (see **[dMS]** and **[dM]**).

Such developments, as well as my own work with Takens and Yoccoz, **[PT1]**, **[PT2]**, **[PY1]**, and many inspiring conversations with colleagues, former and present students, among them de Melo, Pujals, Takens, Yoccoz and above all Viana, made me progressively acquire a new global view of dynamics, emphasizing a much more probabilistic approach to the question. I was then able to formulate, just prior to the meeting at the Université Paris-Sud, Orsay in honour of A. Douady, in 1995, the following conjecture:

Global conjecture on the finitude of attractors and their metric stability

(I) *Denseness of finitude of attractors – there is a C^r $(r \geq 1)$ dense set D of dynamics such that any element of D has finitely many attractors whose union of basins of attraction has total probability;*

(II) *Existence of physical (SRB) measures – the attractors of the elements in D support a physical measure;*

(III) *Metric stability of basins of attraction – for any element in D and any of its attractors, for almost all small C^r perturbations in generic k-parameter families of dynamics, $k \in \mathbb{N}$, there are finitely many attractors whose union of basins is nearly (Lebesgue) equal to the basin of the initial attractor; such perturbed attractors support a physical measure;*

(IV) *Stochastic stability of attractors – the attractors of elements in D are stochastically stable in their basins of attraction;*

(V) *For generic families of one-dimensional dynamics, with total probability in parameter space, the attractors are either periodic sinks or carry an absolutely continuous invariant measure.*

As mentioned in the abstract, Lyubich **[Ly2]** solved the last item of the conjecture for families of quadratic maps of the interval, setting the stage for its full solution for one-dimensional real dynamics.

We close this section by briefly recalling the notions of physical or SRB measures and sthocastic stability. Let A be an attractor for a C^r, $r \geq 1$, map and let $B(A)$ be its basin of attraction. We call an invariant probability measure μ with support in A, a physical or SRB measure if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x)) = \int g\, d\mu$$

for every continuous function g and for a positive Lebesgue measure set of x in $B(A)$. Notice that if the attractor is just a periodic sink, then we take the Dirac measure equally distributed at its orbit as SRB measure. To define stochastic stability, we first assume that μ is an SRB measure as above, with the property holding for a.e. $x \in B(A)$ (like is the case of Lorenz and Hénon-like attractors as commented above). Let now $\{f_n\}$ be independent, identically distributed random variables in the space of C^r maps, $r \geq 1$, and $f_n \in B_\varepsilon(f)$ with (a convenient) probability distribution θ_ε, where $B_\varepsilon(f)$ denotes the ε-neighbourhood of f. For a point $z_0 \in B(A)$, let $z_j = f_j \circ \cdots \circ f_1(z_0)$, $j \geq 1$. We say that (f, A, μ) is stochastically stable on the basin of attraction $B(A)$ if for every neighbourhood V of μ in the weak* topology, the weak* limit of $\frac{1}{n}\sum_{j=0}^{n-1}\delta_{z_j}$ is in V, for every small enough $\varepsilon > 0$ and $(m \times \theta_\varepsilon^{\mathbb{N}})$-almost all $(z_0, f_1, f_2, \dots)$. Here, δ_{z_j} stands for the Dirac measure at z_j and m for Lebesgue measure. We observe that when we consider parametrized families of maps with finitely many parameters, the distribution θ_ε is to be interpreted as random (Lebesgue) choice of parameters. And this is precisely the meaning we want to give to stochastic stability in the conjecture above.

There is another, formally not equivalent, definition of stochastic stability: instead of different maps $f_1, \dots, f_j$ above, one considers only the initial one f but lets the images of an initial point have small random fluctuations. That is, trajectories $(x_0, x_1, x_2, \dots)$ of the perturbed system are obtained by letting each x_{i+1} be chosen at random in a small neighbourhood of $f(x_i)$. However, in all known stable cases, like the hyperbolic systems treated by Kifer and Young (see **[K]**), as well as the Hénon-like diffeomorphisms, dealt with by Benedicks-Viana, both definitions can be applied.

In the case of flows, the definition is basically the same, the random fluctuations taking place at "infinitesimal" intervals of time dt. More formally, one considers a stochastic differential equation like (see **[K]**)

$$dx = X_0(x)\,dt + dY(x),$$

where X_0 is the initial (unperturbed) vector field and dY is a stochastic differential, for instance $dY = \varepsilon dW$ where dW corresponds to the standard Brownian motion. Then, for ε small, the weak* limit of $\frac{1}{T}\int_0^T \delta_{\xi_t}$ should be close to the SRB measure of the vector field X_0 for almost all trajectories of the stochastic flow ξ_t associated to this equation.

At this point, the following remark is in order concerning the definition of attractor. Essentially in all known cases, the basin of attraction is a neighborhood of the set. Still we don't know that such a property is common to a dense subset of all dynamics. For possible relevant situations where the basin contains a set of positive probability but not a full neighborhood of the attractor, random perturbations might cause orbits to escape from the basin of the attractor. In fact this is the case for the random perturbations of flows through Brownian motion, even when the basin is a neighborhood. A related possible situation corresponds to having more than one SRB measure with supports contained in a same (transitive) attractor, so that random orbits cross several of the basins of these measures. In these cases, stochastic stability should mean that the weak* limit of $\frac{1}{n}\sum_{j=0}^{n-1}\delta_{z_j}$ is close to the convex hull of those SRB measures if ε is small (see above and **[V2]**).

2. Other conjectures and some recent results

In his famous essay on the stability of the solar system, written around 1890, Poincaré introduced the notion of homoclinic orbits: in the past and future they converge to the same periodic orbit or are in the intersection of the stable and unstable manifolds (sets) of such a periodic orbit. Subsequently, not only he expressed amazement with the inherent dynamical complexity, but stressed its importance:

"Rien n'est plus propre à nous donner une idée de la complication du problème de trois corps et en général de tous les problèmes de dynamique..."

Indeed, on one hand we have that a transversal homoclinic orbit for a diffeomorphism implies the presence of the dynamics of the horseshoe, as proved by Smale in the early sixties (see [**PT**]). On the other, a generic unfolding of a homoclinic tangency of a dissipative (determinant of the Jacobian smaller than one) surface diffeomorphism yields a rather striking number of rich, intricate dynamical phenomena:

(a) residual subsets of intervals in the parameter line whose corresponding diffeomorphisms display infinitely many coexisting sinks
(b) cascade of period-doubling bifurcations of periodic points (sinks)
(c) positive Lebesgue measure sets in the parameter line whose corresponding diffeomorphism display Hénon-like attractors.

These facts were proved in the late seventies and the eighties by Newhouse, Yorke-Alligood and Mora-Viana extending the fundamental work of Benedicks-Carleson (see [**PT**]). They are also valid in higher dimensions, when the dimension of the unstable manifold of the periodic point is one (codimension-one case) and the product of any two eigenvalues of the derivative of the map at this point has norm less than one (sectionally dissipative). See [**PV**] for the existence of infinitely many coexisting sinks in such a case. It is to be noticed that there are also plenty of values of the parameter such that the corresponding diffeomorphisms have infinitely many coexisting Hénon-like attractors [**C**].

The results above show how Poincaré once and again had a great mathematical insight. In accordance with his view, I have proposed some time ago the following conjectures:

Conjecture I. — *C^r near any surface diffeomorphism exhibiting one of the above bifurcating phenomena (a), (b), (c), there exists a diffeomorphism displaying a homoclinic tangency, $r \geq 1$. Thus, C^r near any of such bifurcating phenomena we may find all the others. The same in higher dimensions, the homoclinic tangency being now of codimension-one and sectionally dissipative.*

Conjecture II. — *In any dimension, the diffeomorphisms exhibiting either a homoclinic tangency or a (finite) cycle of hyperbolic periodic orbits with different stable dimensions (heterodimensional cycles) are C^r dense in the complement of the closure of the hyperbolic ones, $r \geq 1$.*

Notice that for surface diffeomorphisms, it is not possible to have an heterodimensional cycle. So, we have conjectured that diffeomorphisms exhibiting Hénon-like attractors or repellers (or infinitely many sinks or sources) are dense in the complement of the hyperbolic ones. The same question may be posed for non-invertible maps. Also for flows, but in this case one has to add, to homoclinic tangencies and heterodimensional cycles, flows exhibiting singular attractors, like the Lorenz-like ones, and singular cycles, i.e. cycles involving periodic orbits and singularities, studied in **[LP]**. This conjecture for flows will be mentioned again at the end of the paper, when discussing new Lorenz-like attractors. In general, we have presented in **[PT]**, page 134, a notion of a homoclinic bifurcating dynamical system and formulate a less explicit but similar conjecture in terms of a such a notion.

Progress has been made on conjecture I, by Ures **[U]**, who has showed that for all known cases of diffeomorphisms exhibiting Hénon-like attractors, they may be C^r approximated by one with homoclinic tangencies. Similarly for the limiting map of cascades of period-doubling bifurcations in many interesting cases **[CE1, CE2]**. On the other hand, Pujals and Sambarino seem to be making good progress in showing in any dimension, that a diffeomorphism with infinitely many sinks can be C^1 approximated by one displaying a homoclinic tangency, which is codimension one and sectionally dissipative.

More concretely, Pujals and Sambarino **[PS]** have provided a positive solution to conjecture II for surface diffeomorphisms and $r = 1$. From the comments above, this implies that conjecture I would also be true in the same setting. Remarkably, they also obtain from their method a similar result for surface diffeomorphisms whose topological entropy changes under small C^1 perturbations.

Related to conjecture II, we wish to pose the following weaker version of it:

Conjecture III. — *The subset of dynamical systems that either have their limit set consisting of finitely many hyperbolic periodic orbits or else they have transversal homoclinic orbits, are C^r dense in the set of all dynamical systems, $r \geq 1$.*

Again, Pujals and Sambarino, made some progress towards a positive answer to this question in dimensions higher than two and $r = 1$.

Somewhat related to our main conjecture, stated in the previous section, and initially motivated by **[TY]**, we formulate yet another conjecture.

Conjecture IV. — *When unfolding a homoclinic tangency for a codimension-one sectionally dissipative diffeomorphism, the set of parameter values corresponding to diffeomorphisms with infinitely many sinks or infinitely many Hénon-like attractors has (Lebesgue) measure zero. The same for generic k-parameter families of dynamical systems, $k \in \mathbb{N}$.*

In all the previous assertions concerning homoclinic tangencies for surface diffeomorphisms, the corresponding periodic point may be part of a larger hyperbolic (basic) set. The problem is then translated to the understanding of the arithmetic difference of Cantor sets in the line, obtained from the intersections of the stable and unstable foliations of the hyperbolic set with a line transverse to the leaves at the homoclinic tangency. Such Cantor sets are regular (bounded geometry) if the diffeomorphism is of class C^2 **[PT]**. One can also consider a topology for such Cantor sets. Trying to go beyond the notion of thickness of Cantor sets used by Newhouse (see **[PT]**), I asked if for a residual subset of pairs C_1, C_2 of regular Cantor sets, either $C_1 - C_2$ has measure zero or else it contains an interval. The first case would correspond to $HD(C_1 \times C_2) < 1$ and the second to $HD(C_1 \times C_2) > 1$, where HD stands for Hausdorff dimension. Recently, Moreira and Yoccoz **[Mor]**, **[MorY1]**, **[MorY2]** have proved this fact, even more strongly, for an open and dense subset of pairs of Cantor sets, extending partial previous results in **[PT1]**, **[PT2]** and **[PY1]**. The question for affine Cantor sets is still open:

Conjecture V. — *If C_1, C_2 are affine Cantor sets and $HD(C_1 \times C_2) > 1$, then $C_1 - C_2$ contains an interval. In particular, the question can be posed for most or an open and dense subset of pairs of affine Cantor sets.*

I believe that the result of Moreira-Yoccoz is a major contribution to a more profound understanding of the unfolding of homoclinic bifurcations, which we consider to be central to dynamics. In the previous works, it has been shown that if $HD(C_1 \times C_2) < 1$ then the set of parameter values corresponding to hyperbolicity has full (Lebesgue) density at the value corresponding to the homoclinic tangency **[PT1]**, **[PT2]**. Conversely, this is not so if $HD(C_1 \times C_2) > 1$ as proved in **[PY1]**. The results of Moreira-Yoccoz enriches the picture: the set of parameter values corresponding to hyperbolicity or generalized homoclinic tangencies (tangencies between stable and unstable leaves of the foliations) has full density at that same parameter value. In another development, it is shown in **[PY2]** that at the homoclinic bifurcation parameter value, attractors have Lebesgue density equal to zero. This is another indication that attractors perhaps are not so abundant, as Newhouse's result may at first glance suggest.

Concerning attractors in particular carrying an SRB measure, a focusing point of the main conjecture in this article, there has been an explosion of new relevant results, adding to the important development on strange attractors mentioned before. This raises the hope of more progress in the questions posed here as well as in some partial classification of attractors with the properties mentioned in that conjecture.

First of all, Alves **[A]** has shown the existence of SRB measures for Viana's attractors with multidimensional positive Lyapunov exponents **[V1]**. Together with Bonatti, they are announcing a series of results aiming at the following one: a positive Lyapunov exponent on (Lebesgue) many points suffices for the existence of SRB measures **[ABV]**.

These results have to do with an important class of diffeomorphisms called partially hyperbolic. This means that there is a continuous invariant decomposition

$$TM = E^1 \oplus E^2$$

where E^1 is uniformly expanding or uniformly contracting and it dominates E^2 [**PT**, page 161]. Shub [**S**], Mañé [**M1**] and Bonatti-Diaz [**BD**] have constructed partially hyperbolic but not hyperbolic diffeomorphisms which are robustly transitive: every C^1 small perturbation is also transitive. Very recently, Diaz-Pujals-Ures [**DPU**] have shown that partial hyperbolicity is in fact a necessary condition for C^1 robust transitiveness in three dimensions. Some time earlier, Pugh and Shub had shown that for volume preserving diffeomorphisms, partial hyperbolicity is a main ingredient for robust ergodicity; see [**PSh**] and references therein.

Concerning flows, Rovella [**R**] has constructed a few years ago an important variation of Lorenz geometric attractor, which is not robust but only persists for perturbations of the flow corresponding to a positive Lebesgue measure set of parameter values. The global spiralling attractors constructed in [**PRV1**], [**PRV2**], whose reduced one-dimensional map exhibits infinitely many critical points, and the critical geometric Lorenz attractors in [**LV1**], [**LV2**] also display measure theoretical persistence. In this setting, de Melo and Martens [**dMM**] have built up families of Lorenz-like maps that are full in the sense that exhibit all combinatorial types. An oustanding question that remained open since the seventies is whether there exist robust transitive attractors for flows containing a singularity with more than one expanding eigenvalue. This has been positively answered very recently by Bonatti-Pumariño-Viana [**BPV**].

The theory of singular or Lorenz-like attractors, specially in dimension three, attained sharp progress in recent years through a series of papers by Morales, Pacifico and Pujals. They have constructed new relevant types of singular attractors in [**MPa**], [**MPu1**], [**MPu2**], [**MPP1**], [**MPP2**] and [**MPP3**]. In some cases, these attractors are across the boundary of hyperbolic flows. They also proved the following partial characterization [**MPP4**]: robustly transitive singular sets, i.e. invariant sets containing singularities, of three dimensional flows are necessarily attractors or repellers and they are singularly hyperbolic, i.e. there exists a continuous invariant decomposition $TM = E^s \oplus E^{cu}$ such that E^s is uniformly contracting, E^{cu} is volume expanding and E^s dominates E^{cu}. Moreover, the derivatives of the flow at the singularities contained in the attractor have eigenvalues with the same relative distribution as in the geometric Lorenz attractors: the unstable (positive) eigenvalue has norm bigger than that of the weak stable (least negative) one. Here again, as well as in [**DPU**] above, Hayashi's connecting lemma is used. Such developments, I believe, should be very inspiring in tackling Conjecture II for flows, at least in three dimensions.

Let me finish by briefly remarking that the scenario we have proposed for most dynamical systems in finite dimension, may apply to some infinite dimensional systems whose solutions are often only defined for positive time (semiflow), like dissipative evolution equations (Euler, Navier-Stokes). Putting the view presented here together with several conjectures/results in this setting, we would have the following conjecture: For most such systems in parametrized form (finitely many parameters) and most initial conditions, the solutions are global and converge to finitely many finite-dimensional attractors with nice ergodic properties, as above.

As a sign of much activity in the area, many of the references below are to appear. They are, however, available at the Internet, for instance at the authors web page at IMPA.

References

[ABV] J.F. Alves, C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems: the expanding case*, to appear in Invent. Math.

[A] J.F. Alves, *SRB measures for nonhyperbolic systems with multidimensional expansion*, thesis IMPA, 1997, to appear in Ann. Sci. ENS.

[Ar] V.I. Arnold, *On the mapping of the circle into itself*, Izvestia Akad. Nauk., USSR Math. Series bf 25 (1961), 21–86.

[BC] M. Benedicks and L. Carleson, *The dynamics of the Hénon maps*, Ann. of Math. 133 (1991), 73–169.

[BV1] M. Benedicks and M. Viana, *Solution of the basin problem for Hénon-like attractors*, preprint 1999.

[BV2] M. Benedicks and M. Viana, *Stochastic stability and other statistical properties for Hénon-like maps*, in preparation.

[BY1] M. Benedicks and L.-S. Young, *Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps*, Ergod. Th. & Dynam. Sys. 12 (1992), 13–37.

[BY2] M. Benedicks and L.-S. Young, *SBR-measures for certain Hénon maps*, Invent. Math. 112 (1993), 541–576.

[BD] C. Bonatti and L.J. Díaz, *Nonhyperbolic transitive diffeomorphisms*, Ann. of Math. 143 (1996), 357–396.

[BPV] C. Bonatti, A. Pumariño, and M. Viana, *Lorenz attractors with arbitrary expanding dimension*, C.R. Acad. Sci. Paris 325 (1997), 883-888.

[C] E. Colli, *Infinitely many coexisting strange attractors*, to appear, Annales de l'Inst. Henri Poincaré, Analyse Non linéaire 15 (1998), 539-579.

[CE1] E. Catsigeras and H. Enrich, *Homoclinic tangencies near cascades of period doubling bifurcations*, to appear, Ann. de l'Inst. Henri Poincaré Analyse Nonlinéaire.

[CE2] E. Catsigeras and H. Enrich, *Persistence of the Feigenbaum attractor in one-parameter families*, preprint 1997.

[CT] P. Coullet and C. Tresser, *Iterations d'endomorphismes et group de renormalisation*, CRAS Paris 287 (1978), 577–580.

[dMM] W. de Melo and M. Martens, *Universal models for Lorenz maps*, preprint 1997, to appear, Ergodic Th. and Din. Syst.

[dM] W. de Melo, *Rigidity and renormalization in one-dimensional dynamics*, Procs. International Congress of Mathematicians, ICM98-Berlin.

[dMS] W. de Melo and S. van Strien, *One-dimensional dynamics*, Springer-Verlag, 1993.

[DRV] L.J. Díaz, J. Rocha, and M. Viana, *Strange attractors in saddle-node cycles: prevalence and globality*, Invent. Math. 125 (1996), 37– 74.

[DPU] L.J. Díaz, E. Pujals, and R. Ures, *Normal hyperbolicity and robust transitivity*, preprint 1997, to appear in Acta Math.

[F] M. Feigenbaum, *Qualitative universality for a class of nonlinear transformations*, J. Stat. Phys. 19 (1978), 25–52.

[GS] J. Graczyk and G. Swiatek, *Generic hyperbolicity in the logistic family*, Annals of Math. 146 (1997), 1–52.

[Ha] S. Hayashi, *Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows*, Annals of Math. 145 (1997), 81–137.

[He] M. Hénon, *A two dimensional mapping with a strange attractor*, Comm. Math. Phys. 50 (1976), 69–77.

[H] M. Herman. *Sur la conjugaison différentiable des difféomorphims du cercle à des rotations*, Publ. Math. 49 (1979), 5–233.

[J] M. Jacobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. 81 (1981), 39–88.

[K] Y. Kifer, *Random perturbations of dynamical systems*, Birkhäuser, 1988.

[Ko] O. Kozlovski, *Structural stability in one-dimensional dynamics*, PhD thesis, Univ. Amsterdam, 1997.

[L] E.N. Lorenz, *Deterministic nonperiodic flow*, J. Atmosph. Sci. 20 (1963), 130–141.

[LP] R. Labarca and M.J. Pacifico, *Stability of singular horseshoes*, Topology 25 (1986), 337–352.

[LV1] S. Luzzatto and M. Viana, *Positive Lyapunov exponents for Lorenz-like maps with criticalities*, these proceedings.

[LV2] S. Luzzatto and M. Viana, *Lorenz-like attractors without invariant foliations*, in preparation.

[Ly1] M. Lyubich, *Dynamics of quadratic polynomials I-II*, Acta Math., 178 (1997), 185–297.

[Ly2] M. Lyubich, *Almost every real quadratic map is either regular or stochastic*, preprint 1997.

[M1] R. Mañé, *Contribution to the stability conjecture*, Topology 17 (1978), 383–396.

[M2] R. Mañé, *A proof of the C^1 stability conjecture*, Publ. Math. I.H.E.S. 66 (1988), 161–210.

[MN] M. Martens and T. Nowicki, *Invariant measures for Lebesgue typical quadratic maps*, these proceedings.

[Mor] C. Moreira, *Stable intersections of Cantor sets and homoclinic bifurcations*, Annales de l'Institut Henri Poincaré, Analyse Non Linéaire, vol. 13, nº 6, (1996), 741–781.

[MorY1] C. Moreira and J.-C. Yoccoz, *Stable intersections of regular Cantor sets with large Hausdorff dimensions*, preprint 1998.

[MorY2] C. Moreira and J.-C. Yoccoz, *Stable homoclinic tangencies for hyperbolic sets of large Hausdorff dimension*, in preparation.

[MV] L. Mora and M. Viana, *Abundance of strange attractors*, Acta Math. 171 (1993), 1–71.

[MPa] C. Morales and M.J. Pacifico, *New singular strange attractors arising from hyperbolic systems*, preprint 1997.

[MPu1] C. Morales and E. Pujals, *Singular strange attractors on the boundary of Morse-Smale systems*, Ann. Sci. E.N.S. 30 (1997), 693–718.

[MPu2] C. Morales and E. Pujals, *Strange attractors containing a singularity with two positive multipliers*, Comm. Math. Phys. 196 (1998), 671-679.

[MPP1] C. Morales, M.J. Pacifico, and E. Pujals, *New strange attractors on the boundary of hyperbolic systems*, preprint 1997.

[MPP2] C. Morales, M.J. Pacifico, and E. Pujals, *Global attractors from the explosion of singular cycles*, C.R. Acad. Sci. Paris 325 (1997), 1317-1322.

[MPP3] C. Morales, M.J. Pacifico, and E. Pujals, *Singular hyperbolic sets*, Proc. Amer. Math. Soc. 127 (1999), 3343-3400.

[MPP4] C. Morales, M.J. Pacifico, and E. Pujals, *On C^1 robust singular transitive sets for three-dimensional flows*, C.R. Acad. Sci. Paris 326 (1998), 81-86.

[N] S. Newhouse, *The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms*, Publ. Math. I.H.E.S. 50 (1979), 101–151.

[PRV1] M.J. Pacifico, A. Rovella, and M. Viana, *Infinite-modal maps with global chaotic behaviour*, Annals of Math. 148 (1998), 1-44.

[PRV2] M.J. Pacifico, A. Rovella, and M. Viana, *Persistence of global spiraling attractors*, in preparation.

[PS] E. Pujals and M. Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms: A conjecture of Palis*, preprint 1998, to appear in Annals of Math.

[PSh] C. Pugh and M. Shub, *Stable ergodicity and julienne quasi-conformality*, preprint 1997.

[PT] J. Palis and F. Takens, *Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations*, Cambridge University Press, 1993.

[PT1] J. Palis and F. Takens, *Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms*, Inventiones Math. 82 (1985), 397–422.

[PT2] J. Palis and F. Takens, *Hyperbolicity and the creation of homoclinic orbits*, Annals of Math. 125 (1987), 337–374.

[PV] J. Palis and M. Viana, *High dimension diffeomorphisms displaying infinitely many periodic attractors*, Annals of Math. 140 (1994), 207–250.

[PY1] J. Palis and J.-C. Yoccoz, *Homoclinic tangencies for hyperbolic sets of large Hausdorff dimension*, Acta Mathematica, vol. 172 (1994), 91–136.

[PY2] J. Palis and J.-C. Yoccoz, *Non-uniform hyperbolic horseshoes arising from homoclinic bifurcations*, to appear in C.R. Acad. Sci. Paris.

[R] A. Rovella, *The dynamics of perturbations of the contracting Lorenz attractor*, Bull. Braz. Math. Soc. vol. 24, nº 2 (1993), 233–259.

[S] M. Shub, *Topological transitive diffeomorphisms on T^4*, Lecture Notes in Math. 206 (1971), 39, Springer-Verlag.

[TY] L. Tedeschini-Lalli and J.A. Yorke, *How often do simple dynamical processes have infinitely many coexisting sinks?* Comm. Math. Phys. 106 (1986), 635–657.

[U] R. Ures, *On the approximation of Hénon-like attractors by homoclinic tangencies*, Ergodic Theory and Dynamical Systems, vol. 15 (1995), 1223–1230.

[V1] M. Viana, *Multidimensional nonhyperbolic attractors*, Publ. Math. IHES (1998).

[V2] M. Viana, *Dynamics: a probabilistic and geometric perspective*, Procs. International Congress of Mathematicians, ICM98-Berlin.

J. Palis, Instituto de Matemática Pura e Aplicada - IMPA, Est. D. Castorina 110, 22460-320 Rio de Janeiro, Brazil • *E-mail :* jpalis@impa.br