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by

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Abstract. — We investigate the problem of optimizing, for a fixed integer k and real u and on all sets $K = \{a_1 < a_2 < \cdots < a_k\} \subset \mathbb{Z}$, the measure of the set of $\alpha \in [0, 1]$ where the absolute value of the trigonometric sum $S_K(\alpha) = \sum_{j=1}^{k} e^{2\pi i \alpha a_j}$ is greater than $k - u$. When u is sufficiently small with respect to k we are able to construct a set K_{ex} which is very close to optimal. This set is a union of a finite number of arithmetic progressions. We are able to show that any more optimal set, if one exists, has a similar structure to that of K_{ex}. We also get tight upper and lower bounds on the maximal measure.

1. Introduction

Let k be a positive integer and $u < k$ a positive real. For a set

$$K = \{a_1 < a_2 < \cdots < a_k\}, \quad a_j \in \mathbb{Z}, \quad 1 \leq j \leq k,$$

let

$$S_K(\alpha) = \sum_{j=1}^{k} e^{2\pi i \alpha a_j}, \quad s_K(\alpha) = |S_K(\alpha)|,$$

$$E_{K,u} = \{\alpha \in [0, 1) : \quad s_K(\alpha) \geq k - u\}$$

and

$$\mu_K(u) = \mu(E_{K,u}),$$

where μ is the Lebesgue measure on $[0, 1]$ normalized so $\mu([0, 1]) = 1$.

This work deals with the following problem, first raised at the talk of Freiman and Yudin at the Number Theory Conference (Vladimir, 1968):

Problem 1. — *Find the set K which maximizes $\mu_K(u)$ and find the maximal value.*

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We denote by $\mu_{\max}(k,u)$ the supremum of $\mu_K(u)$ on all sets K of size k.
The first results on this problem were obtained by Freiman and Yudin:

***Theorem 1* (Freiman, [2, page 144]).** — *For $u = 1$, $a_1 = 0$ and $a_k < 0.05k^{3/2}$, the maximal measure is*

$$\mu_{\max}(k,u) = \frac{2\sqrt{6}}{\pi}k^{-3/2} + O(k^{-2})$$

and it is attained by K if and only if K is an arithmetic progression.

***Theorem 2* (Yudin, [4]).** — *For $u = o(k)$*

$$\mu_{\max}(k,u) = \frac{2\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}(1+o(1))$$

as $k \longrightarrow \infty$.

In [**1**] Freiman treated the problem assuming the ratio u/k is small enough. He sketched an approach for attacking the problem and conjectured it would prove that the best set is an arithmetic progression.

The purpose of this work is to carry out and extend Freiman's approach (and also that of [**4**]). It will turn out that once u is sufficiently large it is no longer true that an arithmetic progression attains $\mu_{\max}(k,u)$. We are unable to find a set which does. Nevertheless, we do describe to some extent the "structure" of the maximal set. To make precise what this means, we will introduce and use the following terminology

Definition 1.1. — Let k and u be as above.

1. For any $\psi \in [0,1]$ we let $\mathbb{K}_\psi$ be the collection of all sets $K \subset \mathbb{Z}$ of size k that satisfy $\mu_K(u) \geq \psi$.
2. A collection $\mathbb{K}$ of sets is said to be "good for ψ", or to be a G_ψ collection, if it satisfies the following two properties:
   (a) We have $\mathbb{K} \subset \mathbb{K}_\psi$,
   (b) For any set $K \subset \mathbb{Z}$ of size k there exists a set $K' \in \mathbb{K}$ such that $\mu_K(u) \leq \mu_{K'}(u)$.

Our main results are of three types. We are able to describe the "structure" of sets in $\mathbb{K}_\psi$ for a ψ which is very close to maximal. In addition we construct a certain subcollection of this which has property G_ψ. The subclass we describe is not a singleton but it does have a rather simple structure: it is essentially the union of arithmetic progressions, and we have a fairly accurate information about the location and length of all these sequences. Lastly, we get a good bound on $\mu_{\max}(k,u)$.

The type of results we get is dictated by our method of proof, which could be describe as an iteration of four steps.

1. We first guess a set K expected to have a large $\mu_K(u)$. We will take $\psi = \mu_K(u)$.
2. We get information about sets K_1 that have an even higher $\mu_{K_1}(u)$. Typically this information consists of knowledge that most elements are contained inside arithmetic progressions of relatively short length. This is what we mean by describing the "structure" of $\mathbb{K}_\psi$.

3. Given a set $K_1 \in \mathbb{K}_\psi$, we give a procedure for obtaining out of K_1 a set K_2 such that $\mu_{K_2}(u) \geq \mu_{K_1}(u)$. The procedure usually involves compressing the elements contained in the short progressions described above to form short progressions, possibly with a single gap. Sets obtained in this way will form a subclass $\mathbb{K}$ that has property G_ψ by construction.
4. We use the knowledge of $\mathbb{K}_\psi$ to get an improved bound on $\mu_{\max}(k, u)$.

Our results apply under the assumptions $k/30000 \geq u > 1$ and $k \geq$ Const, with Const an unspecified constant. We note that this second assumption is only forced on us because we are using lemma 5.2 which is ineffective. If an effective bound is supplied for that lemma, it will be very easy to deduce an effective lower bound on k as well.

We state a simplified version of the results here under the additional assumption $u > k^{2/3}$. With these assumptions, it follows (proposition 3.1) that for an arithmetic progression K of difference 1 and length k there exists some $\beta_{k,u}$(definition 3.2) such that

$$E_{K,u} = [-\beta_{k,u}, \beta_{k,u}] \pmod 1.$$

We will see in proposition 3.4 that

$$\beta_{k,u} \approx \frac{\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}.$$

We describe a certain basic set K_{ex} (a more precise description will be given in construction 6.1). Set $m_0 = k - 5u/12$ and $\beta = \beta_{m_0,u}$. To first order, $\beta_{m_0,u} \approx \beta_{k,u}(1 + 5u/8k)$. The set K_{ex} is the union of an arithmetic progression of length m_0, symmetric around 0, and for any non zero integer n an arithmetic progression of length

$$\frac{1}{2}m_n = \frac{u}{(\pi n)^2}\left(1 - \frac{(-1)^n}{2}\right),$$

centered around $\frac{n}{\beta}$. All the arithmetic progressions here have difference 1. The structure of K_{ex} and the particular values of the m_n are chosen in such a way that the contributions of the shorter progressions to $S_{K_{ex}}(\alpha)$ exactly compensates for the decline of the contribution of the large progression when $|\alpha| > \beta/2$. We show in proposition 6.12 that $\mu_{K_{ex}}(u) \approx 2\beta$. The results are now as follows.

1. $$\mu_{\max}(k, u) = 2\beta_{m_0,u}\left(1 + O\left(\left(\frac{u}{k}\right)^2\right)\right).$$
2. A set $K \in \mathbb{K}_{\mu_{K_{ex}}(u)}$ has the following structure (similar to that of K_{ex}).
   (a) All but
   $$\frac{5}{12}u + O\left(\frac{u^2}{k}\right)$$
   elements of K are contained in a short arithmetic progression of length $1/4\beta$. To state the other results we will assume that this progression is

symmetric around 0 and its difference is 1. The general case is essentially the same by translation and dilation which change nothing.

(b) Most other elements are contained in a union of short arithmetic progressions with centres near $\frac{n}{\beta}$ and $-\frac{n}{\beta}$ for $n \in \mathbb{N}$. each such short progression has length $2k$ at the most. The number of elements contained in progressions near $\pm\frac{n}{\beta}$ is

$$m_n + O\left(\frac{u^2}{k}\right).$$

(c) The number of elements not contained in any of the progressions above is $O\left((u/k)^{1/2}u\right)$.

3. The following subclass of $\mathbb{K}_{\mu_{K_{ex}}(u)}$ is of type $G_{\mu_{K_{ex}}(u)}$. It consists of the sets where all the elements *contained* in the progression described in (a) above in fact *form* an arithmetic progression except that one gap may persist.

All O terms can and will be made explicit although no claim for best bounds is made.

Here is a brief summary of the contents of this paper. In section 3 we estimate $\mu(E_{K,u})$ in the case where K is an arithmetic progression, and we prove the lower bound:

$$\mu(E_{K,u}) \leq \frac{2\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}$$

for such a progression. In section 4 we prove the upper bound

$$\mu(E_{K,u}) \leq \frac{d}{k}\left(\frac{u}{k}\right)^{1/2}$$

holds for an explicitly given $d \approx 4$ and all sets K under some mild restrictions on k and u. In section 5 we consider a set

$$K \in \mathbb{K}_{\psi}, \quad \psi = \frac{2\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}.$$

We show that this implies that $E_{K,u}$ is contained in a union of small intervals and that K has most of its elements contained in an arithmetic progression of short length. We then perform the first of our "compression arguments" mentioned above and construct a G_{ψ} subclass consisting of the sets where these elements form an arithmetic progression with at most one gap. The construction of K_{ex} is described in section 6. In sections 7 to 9 we describe the structure of $\mathbb{K}_{\mu_{K_{ex}}(u)}$ and also describe a $G_{\mu_{K_{ex}}(u)}$ subclass.

Some of the results of this paper appeared in [**1**]. We follow [**1**] very closely in sections 3 to 5. We note that the argument in [**1**, p. 368] may be completed to give the result that, the part of K not in an arithmetic progression is bounded in size by cu with $c \longrightarrow 1/(2-4/\pi)$ as $u/k \longrightarrow 0$. This result is improved here to $c \approx 5/12$.

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2. Notation and terminology

An arithmetic progression $\{a, a+q, a+2q, \ldots, a+(n-1)q\}$ is said to have length n and difference q. We will often make the distinction between a set being *contained* in an arithmetic progression, which means it is a subset of the above, and set *forming* and arithmetic progression. We will also sometime talk about an interval of integers. This will mean a set of the form $\{x \in \mathbb{Z} : s \leq x \leq t\}$. This interval has length $t-s$. Notice that a subset of $\{a, a+q, a+2q, \ldots, a+(n-1)q\}$ is contained in an arithmetic progression of length n but in an interval of length $(n-1)q$.

In this paper, excluding the introduction, we will make a non-standard use of the notations $O(1)$ and $o(\epsilon)$. The notation $O(1)$ will mean having absolute value ≤ 1. In particular, we will write $a = b + O(1)c$ to mean $|a-b| \leq c$. We will use $o(\epsilon)$ to refer to a quantity which is very small and will be discarded in the computation by swallowing it in a larger quantity. A typical use of this will be for example $8(1+o(\epsilon)) \leq 9$. The reader will have to check for himself or herself that such an argument is justified, which should not be too hard. This notation is used because we have been asked to give explicit, while not best possible, upper bounds for everything.

3. The case of arithmetic progressions

As remarked in the introduction, we always assume that $u > 1$. In this section we want to determine $s_K(\alpha), E_{K,u}$ and $\mu_K(u)$ when K is an arithmetic progression. It will be occasionally convenient to write, when K is of length k and difference 1, $s_k(\alpha)$ for $s_K(\alpha)$, $E_{k,u}$ for $E_{K,u}$ and $\mu_k(u)$ for $\mu_K(u)$.

We first note that, for any set K and integers d and m,

$$s_{dK+m}(\alpha) = s_K(d\alpha). \tag{1}$$

Therefore

$$E_{dK+m,u} = \langle d \rangle^{-1}(E_{K,u}). \tag{2}$$

Here, the map $\langle d \rangle : [0,1) \to [0,1)$ is defined by $\langle d \rangle(\alpha) =$ fractional part of $d\alpha$. For $F \subset [0,1)$, $\langle d \rangle^{-1}(F)$ denotes the inverse image of F under $\langle d \rangle$. It is easy to deduce from this that

$$\mu_{dK+m}(u) = \mu_K(u).$$

These observations allow us to reduce to the case of difference 1.

Proposition 3.1. — *Let K be an arithmetic progression of length k and difference* 1. *Assume $k \geq 2u$.*

1. *We have*

$$s_K(\alpha) = \left| \frac{\sin(\pi\alpha k)}{\sin(\pi\alpha)} \right| \quad \textit{when } \alpha \neq 0.$$

2. *The set $E_{k,u}$ is a single interval modulo* 1, *i.e.,*

$$E_{K,u} = [-\beta, \beta] \pmod 1,$$

For some $\beta \in \mathbb{R}$.

It will be worth while to give the number β appearing in the proposition a special notation.

Definition 3.2. — Under the assumptions of proposition 3.1, define $\beta_{k,u} > 0$ by the equality

$$E_{K,u} = [-\beta_{k,u}, \beta_{k,u}] \pmod 1,$$

for K an arithmetic progression of length k and difference 1. Note that

$$s_K(\beta_{k,u}) = k - u.$$

Corollary 3.3. — *If K is an arithmetic progression of length k and difference d, then*

$$E_{K,u} = \bigcup_{q=0}^{d-1} \left[\frac{q}{d} - \frac{\beta_{k,u}}{d}, \frac{q}{d} + \frac{\beta_{k,u}}{d}\right] \pmod 1.$$

Proof. — From proposition 3.1 and (2) it follows that

$$E_{K,u} = \langle d\rangle^{-1}([-\beta_{k,u}, \beta_{k,u}]) = \bigcup_{q=0}^{d-1} \left[\frac{q}{d} - \frac{\beta_{k,u}}{d}, \frac{q}{d} + \frac{\beta_{k,u}}{d}\right] \pmod 1.$$

□

Proof of proposition 3.1. — By (1) it is enough to consider any arithmetic progression of difference 1. In particular, one can take

$$K = \{-\frac{k-1}{2}, \dots, \frac{k-1}{2}\}.$$

Note that this set might be composed of half integers but that makes no difference here. We get for $\alpha \neq 0$,

$$(3) \qquad S_K(\alpha) = \sum_{n=-\frac{k-1}{2}}^{\frac{k-1}{2}} e^{2\pi i\alpha n} = e^{(1-k)\pi i\alpha} \sum_{n=0}^{k-1} e^{2\pi i\alpha n}$$
$$= e^{(1-k)\pi i\alpha} \frac{e^{2\pi i\alpha k} - 1}{e^{2\pi i\alpha} - 1} = \frac{(e^{\pi i\alpha k} - e^{-\pi i\alpha k})/2i}{(e^{\pi i\alpha} - e^{-\pi i\alpha})/2i} = \frac{\sin(\pi\alpha k)}{\sin(\pi\alpha)}.$$

Taking absolute values gives the first assertion. We now have

$$E_{K,u} = \left\{\alpha \in (0,1) : \quad \left|\frac{\sin(\pi\alpha k)}{\sin(\pi\alpha)}\right| \geq k - u\right\} \cup \{0\}.$$

This set is symmetric around 1/2. It is thus sufficient to consider its intersection with the interval $(0, \frac{1}{2})$. On this set S_K is in fact positive. Indeed, since by assumption $2u \leq k$, we find

$$\frac{1}{2}k \leq k - u \leq \frac{|\sin(\pi\alpha k)|}{\sin(\pi\alpha)} \leq \frac{1}{\sin(\pi\alpha)} \leq \frac{\pi/2}{\pi\alpha},$$

and therefore

$$(4) \qquad \pi\alpha k \leq \pi,$$

which shows that $\sin(\pi\alpha k) \geq 0$. We can also write

$$S_K(\alpha) = \sum_{n=-\frac{k-1}{2}}^{\frac{k-1}{2}} e^{2\pi i\alpha n} = \sum_{n=-\frac{k-1}{2}}^{\frac{k-1}{2}} \cos(2\pi\alpha n).$$

By (4), each term in this sum, and therefore $S_K(\alpha) = |S_K(\alpha)|$, is decreasing in α. Thus, $E_{k,u} \cap [0, \frac{1}{2}]$ is an interval. □

Proposition 3.4. — *When $k > 3u$ we have*

$$\beta_{k,u} = \frac{\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}\left(1 + \frac{3u}{20k} + O(1)\left(\frac{u}{k}\right)^2\right).$$

Corollary 3.5. — *When $k > 3u$ we have*

$$\mu_{\max}(k,u) \geq \mu_K(u) \geq \frac{2\sqrt{6}}{\pi}\left(\frac{u}{k}\right)^{\frac{1}{2}}\frac{1}{k}.$$

Proof of proposition 3.4. — Set $\beta = \pi\beta_{k,u}$. Then

$$u = k - \frac{\sin(k\beta)}{\sin(\beta)}.$$

We notice first that

$$\begin{aligned} 0 &\leq \frac{\sin(k\beta)}{\sin(\beta)} - \frac{\sin(k\beta)}{\beta} = \frac{\sin(k\beta)}{\sin(\beta)}\left(\frac{\beta - \sin(\beta)}{\beta}\right) \\ &\leq k\left(\frac{\beta^3/6}{\beta}\right) = \frac{k^3\beta^2}{6}\frac{1}{k^2}. \end{aligned}$$

Now expand

$$\begin{aligned} k - \frac{\sin(k\beta)}{\beta} &= \beta^{-1}(k\beta - \sin(k\beta)) \\ &= \beta^{-1}\left(\frac{(k\beta)^3}{6} - \frac{(k\beta)^5}{120} + \frac{(k\beta)^7}{7!} - \dots\right) \qquad (5) \\ &= \frac{k^3\beta^2}{6}\left(1 - \frac{6(k\beta)^2}{120} + \frac{6(k\beta)^4}{7!} - \dots\right). \end{aligned}$$

It follows that

$$\frac{k^3\beta^2}{6}\left(1 - \frac{(k\beta)^2}{20} + \frac{(k\beta)^4}{840}\right) \geq u \geq \frac{k^3\beta^2}{6}\left(1 - \frac{(k\beta)^2}{20} - \frac{1}{k^2}\right).$$

Thus,

$$\frac{6u}{k^3}\left(1 - \frac{(k\beta)^2}{20} + \frac{(k\beta)^4}{840}\right)^{-1} \leq \beta^2 \leq \frac{6u}{k^3}\left(1 - \frac{(k\beta)^2}{20} - \frac{1}{k^2}\right)^{-1}.$$

We now plug here our first estimate $k\beta \leq \pi$ from (4) to iterate estimates on $(k\beta)^2$. First, since we assume $k > 3u > 3$,

$$1 - \frac{(k\beta)^2}{20} - \frac{1}{k^2} > \frac{1}{3},$$

so

$$(k\beta)^2 < 18\frac{u}{k}.$$

Applying this and using $1/k^2 < u/(3k)$ we now get

$$6\frac{u}{k} \le (k\beta)^2 < 6\frac{u}{k}\left(1+\frac{2u}{k}\right).$$

Thus

$$\frac{6u}{k^3}\left(1-\frac{6u}{20k}+\frac{6^2}{840}\left(\frac{u}{k}\right)^2\right)^{-1} \le \beta^2$$

and

$$\frac{6u}{k^3}\left(1-\frac{6u}{20k}-\frac{12}{20}\left(\frac{u}{k}\right)^2-\left(\frac{u}{k}\right)^2\right)^{-1} \ge \beta^2.$$

The proposition follows easily from this. □

4. An upper bound for $\mu_{\mathbf{max}}(k,u)$

We will prove the upper bound

$$\mu_{\max}(k,u) \le \frac{d}{k}\left(\frac{u}{k}\right)^{\frac{1}{2}}$$

for a constant $d \approx 4.04$ that will be defined later. Note that this is of the same type as the lower bound we got in corollary 3.5. We will need a few lemmas.

Lemma 4.1. — *For any u and k we have*

$$\mu_{\max}(k,u) \le \frac{k}{(k-u)^2}.$$

Proof. — Since $s_K(\alpha) \ge k-u$ on a set of measure $\mu_K(u)$, we have

$$(k-u)^2\mu_K(u) \le \int_0^1 s_K(\alpha)^2 d\alpha.$$

The right hand side can be explicitly computed.

$$\begin{aligned}\int_0^1 s_K(\alpha)^2 d\alpha &= \int_0^1 S_K(\alpha)\overline{S_K(\alpha)}d\alpha \\ &= \int_0^1 \left(\sum_{n=1}^k e^{2\pi i\alpha a_n}\right)\left(\sum_{m=1}^k e^{-2\pi i\alpha a_m}\right)d\alpha \\ &= \sum_{n,m=1}^k \int_0^1 e^{2\pi i\alpha(a_n-a_m)}d\alpha \\ &= \sum_{n=m=1}^k \int_0^1 1 d\alpha = k.\end{aligned}$$

This immediately implies the result. □

Lemma 4.2. — *Let $p_1, p_2, \dots, p_k$ be real positive numbers such that $\sum_{i=1}^{k} p_i = 1$. Let a_i, $i = 1, \dots, k$, be integers. Set $\phi(\alpha) = \sum_{i=1}^{k} p_i e^{2\pi i \alpha a_i}$. Then,*

$$|\phi(\alpha_1 + \alpha_2)| \geq |\phi(\alpha_1)||\phi(\alpha_2)| - \sqrt{1 - |\phi(\alpha_1)|^2}\sqrt{1 - |\phi(\alpha_2)|^2}.$$

Proof. — We reproduce the proof given in [**5**, Lemma 1]. Let

$$\begin{aligned} v_0 &= (\sqrt{p_1}, \dots, \sqrt{p_k}), \\ v_1 &= (\sqrt{p_1}e^{2\pi i \alpha_1 a_1}, \dots, \sqrt{p_k}e^{2\pi i \alpha_1 a_k}), \\ v_2 &= (\sqrt{p_1}e^{2\pi i \alpha_2 a_1}, \dots, \sqrt{p_k}e^{2\pi i \alpha_2 a_k}), \end{aligned}$$

be three unit vectors in $\mathbb{C}^k$. Then $|\phi(\alpha_1 + \alpha_2)| = \cos\theta(v_1, v_2)$ and $|\phi(\alpha_i)| = \cos\theta(v_i, v_0)$ for $i = 1, 2$, where $\theta(v, w)$ is the angle between the vectors v and w. Since $\theta(v_1, v_2) \leq \theta(v_1, v_0) + \theta(v_2, v_0)$ we have

$$\begin{aligned} |\phi(\alpha_1 + \alpha_2)| &\geq \cos\theta(v_1, v_0)\cos\theta(v_2, v_0) - \sin\theta(v_1, v_0)\sin\theta(v_2, v_0) \\ &= \cos\theta(v_1, v_0)\cos\theta(v_2, v_0) \\ &\quad - \sqrt{1 - \cos^2\theta(v_1, v_0)}\sqrt{1 - \cos^2\theta(v_2, v_0)} \\ &= |\phi(\alpha_1)||\phi(\alpha_2)| - \sqrt{1 - |\phi(\alpha_1)|^2}\sqrt{1 - |\phi(\alpha_2)|^2}. \end{aligned}$$

□

Corollary 4.3. — *For any set K and real numbers u_1, u_2 we have $E_{K,u_1} + E_{K,u_2} \subseteq E_{K,2(u_1+u_2)}$.*

Proof. — When $u_1 = u_2$ this was obtained in [**5**]. In the general case, putting in lemma 4.2 $p_i = \frac{1}{k}$ and multiplying by k^2 we get

$$ks_K(\alpha_1 + \alpha_2) \geq s_K(\alpha_1)s_K(\alpha_2) - \sqrt{k^2 - s_K(\alpha_1)^2}\sqrt{k^2 - s_K(\alpha_2)^2}.$$

If we assume $s_K(\alpha_i) \geq k - u_i$ for $i = 1, 2$, then we get

$$\begin{aligned} ks_K(\alpha_1 + \alpha_2) &\geq (k - u_1)(k - u_2) - \sqrt{k^2 - (k - u_1)^2}\sqrt{k^2 - (k - u_2)^2} \\ &= k^2 - k(u_1 + u_2) + u_1u_2 - \sqrt{u_1(2k - u_1)}\sqrt{u_2(2k - u_2)}. \end{aligned}$$

By dropping the term u_1u_2 and replacing $2k - u_i$ by k we see that

$$ks_K(\alpha_1 + \alpha_2) \geq k^2 - k(u_1 + u_2) - k(2\sqrt{u_1}\sqrt{u_2}).$$

Since $2\sqrt{u_1}\sqrt{u_2} \leq u_1 + u_2$ we get

$$ks_K(\alpha_1 + \alpha_2) \geq k^2 - 2k(u_1 + u_2),$$

which implies the result. □

Lemma 4.4. — *If $E \subset [0, 1)$ is closed and $\mu(E) \leq \frac{1}{35}$, then*

$$\mu(E + E \pmod 1) \geq 2\mu(E).$$

Proof. — This is a result of Macbeath and Kneser (see [**6**] for reference). Also, this follows easily from the Theorem in [**3**, p. 46]. The referee informs me that this is also due to Raikov [**7**] with the relaxed condition $\mu(E) \leq \frac{1}{2}$. □

Proposition 4.5. — *Put*

$$c = \frac{\sqrt{2}-1}{4\sqrt{2}-1} \approx 0.09 \ , \ d = \frac{1}{\sqrt{c}(1-c)^2} \approx 4.04 \ .$$

For $k \geq 50$ *and* $15u < k$ *we have*

$$\mu_{\max}(k,u) \leq \frac{d}{k}\left(\frac{u}{k}\right)^{\frac{1}{2}}.$$

Proof. — More precise restrictions on k and u are in fact $k \geq 35/(1-c)^2$ and $u < ck$. From corollary 4.3 and lemma 4.4 it follows that

$$\mu_K(4^s u) \geq 2^s \mu_K(u)$$

for every positive integer s for which $\mu_K(4^{s-1}u) \leq \frac{1}{35}$. On the other hand, by lemma 4.1 we have, for any s with $4^s u \leq k$,

$$\mu_K(4^s u) \leq \frac{k}{(k-4^s u)^2}.$$

It follows that

$$\mu_{\max}(k,u) \leq \min\left(\frac{k}{2^s(k-4^s u)^2}\right),$$

where the minimum is taken over all the integers s such that

$$4^s u \leq k \quad \text{and} \quad \frac{k}{(k-4^{s-1}u)^2} \leq \frac{1}{35}. \tag{6}$$

To get a good upper bound we choose

$$s = [\log_4(ck/u)] + 1,$$

where [ ] is the integral part and $\log_4$ is log in base 4. The conditions of the proposition guarantee that s is in the range (6). Also set $t = \log_4(ck/u) - (s-1)$. Note that $t \in [0,1)$. We have

$$4^s = 4 \cdot 4^{s-1} = 4^{1-t}\left(\frac{ck}{u}\right).$$

Therefore we obtain the bound

$$\mu_{\max}(k,u) \leq \frac{1}{k}\left(\frac{u}{k}\right)^{\frac{1}{2}} \frac{1}{\sqrt{4^{1-t}c}\,(1-4^{1-t}c)^2}.$$

Consider now $\sqrt{4^{1-t}c}\,(1-4^{1-t}c)^2$ as a function of t. Its maximal value over $[0,1]$ is easily found. It is obtained at $t=1$ and equals $1/d$. This finishes the proof. □

Remark 4.6. — Here is the reasoning behind the choice of s. We are trying to minimize a function of the integer s. The replacement for a differential when computing a "critical value" in this situation is the difference of two successive values, but we may also consider the quotient of two such values, which is more natural here. Therefore, we look for an integer s for which the ratio of the expressions at s and $s-1$ is closest to 1, i.e.,

$$2\left(\frac{k-4\left(4^{s-1}u\right)}{k-\left(4^{s-1}u\right)}\right)^2 \approx 1.$$

Solving this gives

$$4^{s-1} \approx \frac{ck}{u}.$$

Remembering that $s \in \mathbb{Z}$ makes our choice clear. As mentioned above, we have arranged things so that this s will be in the range we are considering.

5. Structure of K with large $E_{K,u}$

In this section we will describe the structure of sets in the class $\mathbb{K}_\psi$, where ψ is roughly the measure attained by an arithmetic progression. More precisely, set $d_1 = \frac{2\sqrt{6}}{\pi}$. We let

$$\psi = \frac{d_1}{k}\left(\frac{u}{k}\right)^{\frac{1}{2}}$$

and consider from now on a set $K \in \mathbb{K}_\psi$. Towards the end of this section we will also describe a subclass satisfying property G_ψ.

Our initial restrictions on k and u in this section are that the restrictions of proposition 4.5 are satisfied for k and 4^3u. It is enough to require $k \geq 50$ and $1000u < k$. These assumptions will be strengthen later. Let d be defined as in proposition 4.5.

Lemma 5.1. — *For a constant $c_1 \approx 0.75$ there exists $i \in \{0,1,2\}$ such that*

$$\mu(E_{K,4^iu} + E_{K,4^iu}) \leq (2+c_1)\mu(E_{K,4^iu}).$$

Proof. — Assume by contradiction that for all $0 \leq i \leq 2$ we have

$$\mu(E_{K,4^iu} + E_{K,4^iu}) > (2+c_1)\mu(E_{K,4^iu}).$$

Applying corollary 4.3 repeatedly we get

$$\mu(E_{K,4^3u}) > (2+c_1)^3\mu(E_{K,u}).$$

substituting the lower bound we imposed on $\mu(E_{K,u})$ and the upper bound of proposition 4.5 on $\mu(E_{K,4^3u})$ we get the inequality

$$2^3 d > d_1(2+c_1)^3.$$

We fix c_1 so that this last inequality fails, i.e., $c_1 = 2((d/d_1)^{1/3} - 1)$. The approximation to c_1 is recovered from the estimate $d/d_1 \approx 2.59$. □

For a positive integer q we set

$$E_{q,\delta} = \bigcup_{r=0}^{q-1}[\frac{r}{q} - \frac{\delta}{2}, \frac{r}{q} + \frac{\delta}{2}] \pmod 1.$$

The following lemma is proved in **[5]** on p.154-159.

Lemma 5.2. — *Let $F \subset [0,1)$ be a closed set such that $\mu(F) \leq Const$ for some unspecified constant $Const$. Suppose that there exists $0 < c < 1$ such that*

$$\mu(F+F) \leq (2+c)\mu(F).$$

Then there exist $\beta \in [0,1)$ and a positive integer q such that

$$F \subseteq \beta + E_{q,\delta},$$

where $\delta = \frac{(1+c)}{q}\mu(F)$.

Lemma 5.3. — *Suppose $F \subseteq \beta + E_{q,\delta}$, where $\delta = \frac{(1+c)}{q}\mu(F), 0 < c < 1$, Suppose in addition that $\mu(F) > 0$, that $0 \in F$ and that $-F = F \pmod 1$. Then $F \subseteq E_{q,\delta}$ and*

$$E_{q,2\delta} \subseteq F+F+F \pmod 1.$$

Proof. — To see that $F \subseteq E_{q,\delta}$ note first that $E_{q,\delta}$ is stable under translation by $1/q$. Therefore, we may assume that $|\beta| \leq 1/2q$. We know that $0 \in F \subseteq \beta + E_{q,\delta}$. This implies that $|\beta| \leq \delta/2q$. Finally, as F is stable under negation,

$$F \subseteq (\beta + E_{q,\delta}) \cap (-\beta + E_{q,\delta}) = E_{q,\delta-|\beta|} \subseteq E_{q,\delta}.$$

When $\beta = 0$ the second part of the lemma is proved at the same place the previous lemma was. □

From now, until the end of the paper, excluding section 6, we will be working under the following additional assumption

Assumption 5.4. — Our u and k are such that

$$\frac{4.04}{k}\left(\frac{16u}{k}\right)^{\frac{1}{2}} < Const,$$

where $Const$ is the unknown constant of lemma 5.2.

Making this assumption allows one to use lemma 5.2 for our purposes. It is enough to require that k is big enough, of course. This assumption makes the results of this paper ineffective. It is our hope, however, that one can give an effective bound in lemma 5.2, and thereby for the entire paper.

Proposition 5.5. — *If $K \in \mathbb{K}_\psi$, then there exist integers q and $i \in \{0,1,2\}$ and a positive real number δ such that*

1. *We have the inclusions*

$$E_{K,4^i u} \subseteq E_{q,\delta} \text{ and } E_{K,10\cdot 4^i u} \supseteq E_{q,2\delta}.$$

2. *We have the inequality*

$$\delta \geq q^{-1}2^i\frac{d_1}{k}\left(\frac{u}{k}\right)^{\frac{1}{2}}.$$

Proof. — Let $i \in \{0,1,2\}$ be the smallest integer for which the assertion of Lemma 5.1 holds. This lemma precisely says that Lemmas 5.2 and 5.3 can be applied in succession to $F = E_{K,4^i u}$. The implication is that there exist some positive integer q and some positive real number δ such that

$$E_{K,4^i u} \subseteq E_{q,\delta} \tag{7}$$

and

$$E_{q,2\delta} \subseteq E_{K,4^i u} + E_{K,4^i u} + E_{K,4^i u}.$$

Corollary 4.3 implies that

$$E_{K,10\cdot 4^i u} \supseteq E_{K,4^i u} + E_{K,4^i u} + E_{K,4^i u}.$$

This gives the first assertion. The inclusion (7) implies

$$q\delta = \mu(E_{q,\delta}) \geq \mu(E_{K,4^i u}) \geq 2^i \mu(E_{K,u}) \geq 2^i \frac{d_1}{k}\left(\frac{u}{k}\right)^{\frac{1}{2}}.$$

This gives the second assertion. □

Proposition 5.6. — *Let q be a positive integer. If*

$$E_{K,u} \supset \{0, q^{-1}, \ldots, \frac{q-1}{q}\},$$

then there exists an integer r such that the set

$$K_r = \{a \in K : \quad a \equiv r \pmod q\}$$

satisfies

$$|K_r| \geq k - 2u.$$

Proof. — We have

$$\begin{aligned} q(k-u)^2 &\leq \sum_{r=0}^{q-1} |S_K(r/q)|^2 = \sum_{r=0}^{q-1} \sum_{m,n=1}^{k} e^{2\pi i r(a_n - a_m)/q} \\ &= \sum_{m,n=1}^{k} \sum_{r=0}^{q-1} \left(e^{2\pi i (a_n - a_m)/q}\right)^r = \sum_{a_m \equiv a_n \ (q)} q \\ &= q \sum_{r=0}^{q-1} |K_r|^2 \leq (q \max |K_r|) \sum_{r=0}^{q-1} |K_r| = kq \max |K_r|, \end{aligned}$$

and therefore

$$\max |K_r| \geq k\left(1 - \frac{u}{k}\right)^2 \geq k - 2u.$$

□

Let, for $\theta > 0$,

$$b_\theta = \theta - \int_0^\theta |\cos(\pi\alpha)|d\alpha.$$

Clearly, $b_\theta > 0$. It is also easy to see that b_θ is increasing in θ because its derivative with respect to θ is $1 - |\cos(\pi\theta)| \geq 0$. Finally, one checks that $2b_{1/2} = b_1$.

Proposition 5.7. — *Assume $E_{K,u} \supset [0,\delta]$ and set*

$$\ell_i = a_{k+1-i} - a_i, \quad i = 1, \ldots, k.$$

Then, for every $0 < \theta < 1/2$,

$$|\{i : \quad |\ell_i|\delta \geq \theta\}| \leq \frac{u}{b_\theta}.$$

Proof. — We have

$$\begin{aligned}\delta(k-u) &\leq \int_0^\delta s_K(\alpha)d\alpha \leq \frac{1}{2}\sum_{n=1}^k \int_0^\delta \left|e^{2\pi i\alpha a_{k+1-n}} + e^{2\pi i\alpha a_n}\right| d\alpha \\ &= \frac{1}{2}\sum_{n=1}^k \int_0^\delta \left|e^{2\pi i\alpha\ell_n} + 1\right| d\alpha = \sum_{n=1}^k \int_0^\delta |\cos(\pi\alpha\ell_n)|d\alpha.\end{aligned}$$

We bound each term from above. If $|\ell_n|\delta \leq \theta$, we use the trivial estimate

$$\int_0^\delta |\cos(\pi\alpha\ell_n)|d\alpha \leq \delta.$$

Otherwise, we make the change of variables

$$\int_0^\delta |\cos(\pi\alpha\ell_n)|d\alpha = \frac{1}{|\ell_n|}\int_0^{|\ell_n|\delta} |\cos(\pi\alpha)|d\alpha.$$

When $\theta \leq |\ell_n|\delta \leq 1$ we use the estimate

$$\begin{aligned}\frac{1}{|\ell_n|}\int_0^{|\ell_n|\delta} |\cos(\pi\alpha)|d\alpha &= \frac{1}{|\ell_n|}\left(\int_0^\theta |\cos(\pi\alpha)|d\alpha + \int_\theta^{|\ell_n|\delta} |\cos(\pi\alpha)|d\alpha\right) \\ &\leq \frac{1}{|\ell_n|}\left(\theta - b_\theta + |\ell_n|\delta - \theta\right) = \delta - b_\theta\frac{1}{|\ell_n|} \\ &\leq \delta(1 - b_\theta).\end{aligned}$$

When $|\ell_n|\delta > 1$ one finds similarly

$$\begin{aligned}\frac{1}{|\ell_n|}\int_0^{|\ell_n|\delta} |\cos(\pi\alpha)|d\alpha &\leq \delta - b_1\frac{[|\ell_n|\delta]}{|\ell_n|} \\ &\leq \delta - \frac{1}{2}b_1\delta = \delta(1 - b_{\frac{1}{2}}) \leq \delta(1 - b_\theta).\end{aligned}$$

Therefore,

$$\delta(k-u) \leq \delta\left(k - b_\theta \cdot |\{i : \quad |\ell_i|\delta \geq \theta\}|\right),$$

which proves what we wanted. $\square$

Lemma 5.8. — *Suppose $k > 30000u$ and $K \in \mathbb{K}_\psi$. Then there exists a unit vector v such that for any $\alpha \in E_{K,u}$ we have* $\mathrm{Angle}(S_K(\alpha), v) < \pi/2$. *In addition there is a subset K_0 of K with at least $k - 2000u$ elements such that the following is satisfied: For any $a \in K_0$ and any $\alpha \in E_{K,u}$ one has* $\mathrm{Angle}(v, e^{2\pi i\alpha a}) < \pi/4$.

Proof. — By proposition 5.5 there exist $i \in \{0, 1, 2\}$, a positive integer q and a real number δ such that

$$E_{K,u} \subseteq E_{K,4^i u} \subseteq E_{q,\delta}, \quad E_{K,10\cdot 4^i u} \supseteq E_{q,2\delta}.$$

Consider a parameter $\theta < 1/4$ to be set later. According to proposition 5.7 all but $10 \cdot 4^i u / b_{2\theta}$ elements of K are in an interval of length ℓ such that $\ell\delta < 2\theta$, or equivalently $2\pi\ell(\delta/2) < 2\pi\theta$. Further, all but $2 \cdot 4^i u$ elements of these are in the same congruence class modulo q. We may translate K by an integer to make this interval symmetric around 0 and the residue class be that of 0. Now denote by K_0 the intersection of K with the interval and the residue class of 0, and let $\bar{K} = K - K_0$. We will show the lemma with $v = 1$. One easily sees that the condition of the lemma is now equivalent to $\mathrm{Re}\, S_K(\alpha)/|S_K(\alpha)| > \sqrt{2}/2$. We will in fact show this for all α in the bigger set $E_{q,\delta}$. Consider such an α and $a \in K_0$. Suppose first that α is in the interval of $E_{q,\delta}$ around 0. Since $|a| \leq \ell/2$ we have

$$|\arg(e^{2\pi i a\alpha})| = |2\pi a\alpha| \leq 2\pi(\ell/2)(\delta/2) < \pi\theta.$$

Therefore, $\mathrm{Re}\, e^{2\pi i a\alpha} > \cos(\theta\pi)$. Now, since elements of K_0 are divisible by q, it is easily seen that they behave the same on all intervals of $E_{q,\delta}$. Thus, the same estimate is true for any $\alpha \in E_{q,\delta}$. Since K_0 contains at least $k - (10b_{2\theta}^{-1} + 2)4^i u$ elements, this implies that for $\alpha \in E_{q,\delta}$ we have

$$\mathrm{Re}\, S_{K_0} > \cos(\theta\pi)\left(k - (10b_{2\theta}^{-1} + 2)4^i u\right).$$

Therefore

$$\begin{aligned}\frac{\mathrm{Re}\, S_K(\alpha)}{|S_K(\alpha)|} &\geq \frac{\mathrm{Re}\, S_{K_0}(\alpha) - |S_{\bar{K}}(\alpha)|}{k} \\ &> \frac{\cos(\theta\pi)(k - (10b_{2\theta}^{-1} + 2)4^i u) - (10b_{2\theta}^{-1} + 2)4^i u}{k}.\end{aligned}$$

Thus, the lemma will be true if we can find a $\theta < 1/4$ for which the right hand side is larger than $\sqrt{2}/2$. Clearly the worst possible case is when $i = 2$, in which we need to solve

$$\frac{\cos(\theta\pi)(k - (160b_{2\theta}^{-1} + 32)u) - (160b_{2\theta}^{-1} + 32)u}{k} > \frac{\sqrt{2}}{2}.$$

This inequality is equivalent to

$$\frac{k}{u} > (160b_{2\theta}^{-1} + 32)(1 + \cos(\pi\theta))\left(\cos(\pi\theta) - \frac{\sqrt{2}}{2}\right)^{-1}.$$

It remains to numerically find the minimum of the expression on the right over $\theta \in [0, 1/4]$. This is found to be about 29439, located around $\theta = 0.19272$. The result follows with the bound on the number of elements outside K_0 being $160b_{2\theta}^{-1}+32 \approx 1860$ times u. □

Proposition 5.9. — *Suppose $k > 30000u$. Then, the following subclass of $\mathbb{K}_\psi$ has property G_ψ. A set in the class can be written as a disjoint union*

$$M = M_0 \cup \bar{K} \text{ with } |\bar{K}| < 2000u,$$

where M_0 is an arithmetic progression with at most one gap.

Proof. — Suppose $K \in \mathbb{K}_\psi$. We will show how to find a set M in the subclass described above such that $\mu_M(u) \geq \mu_K(u)$. By the previous lemma we know that we have a decomposition $K = K_0 \cup \bar{K}$ and that there is a unit vector v such that for all $\alpha \in E_{K,u}$ both the sum $S_K(\alpha)$ and any individual term $e^{2\pi i\alpha a}$, with $a \in K_0$, form an angle of $< \pi/4$ with v. In the situation just described it is easily seen that by replacing $a > b \in K_0$ by $c = a - qt, d = b + qt$, such that $c > d$ and $c, d \notin K_0$, we enlarge the value of $S_K(\alpha)$ for $\alpha \in E_{q,\delta}$. This is because the contribution of the pair (c,d) is larger than that of (a, b) and has the same direction which forms an acute angle with the rest of the sum. Therefore, such a change can only increase the value of $\mu_{K,u}$. All that remain to do then is to show that by repeated application of this we transform K_0 into a set M_0 which is an arithmetic progression of difference q with possibly one gap. To see this we may again assume that elements of K_0 are divisible by q. Suppose that $e = \max(K_0)$, $f = \min(K_0)$ and consider the set

$$K_{comp} = \{x \in q\mathbb{Z} : \quad f \leq x \leq e \quad \text{and} \quad x \notin K_0\}.$$

Let $c = \max(K_{comp})$ and $d = \min(K_{comp})$. Suppose that $K_{comp} \neq \phi$ and that $c \neq d$. Since $a = c + q \in K_0$ and $b = d - q \in K_0$, we may perform a transformation as above. It is clear that each step decreases the sum of the absolute values of all the differences between the elements of K_0. Therefore the process has to stop. The computation above shows that it stops only when K_{comp} has at most one element. This means that the resulting set, has at most one gap. □

6. A close to maximal set

In this section we describe a set K_{ex} which we suspect to be very close to maximal. Just how close will become evident later on. We will begin with parameters m_0 and w and construct a set $M(m_0, w)$. This will roughly be our set K_{ex} except that we can not guarantee in general that it will have exactly k elements. We will choose the parameters so that it has about k elements and then take out as many elements as we need to get it to be of the right size.

We assume we are given an odd positive integer m_0 and a real number w which satisfy the assumption $m_0 > 30000w$. Let M_0 be the set $\{-(m_0-1)/2, \ldots, (m_0-1)/2\}$ of size m_0. Clearly $|S_{M_0}| = S_{M_0}$. We write β for $\beta_{m_0,w}$. According to definition 3.2, β satisfies $S_{M_0}(\beta) = m_0 - w$ and furthermore $E_{M_0}(w) = [-\beta, \beta]$.

Construction 6.1. — Given m_0 and w we construct the set $M(m_0, w)$ as follows:

$$M = M(m_0, w) = M_0 \cup \overline{M} = \bigcup_{n \in \mathbb{Z}} M_n, \tag{8}$$

where each M_n is an arithmetic progression of difference 1 centered (as best possible) around $\frac{n}{\beta}$. For $n > 0$, the length of the two progressions $M_{\pm n}$ is the same and is denoted by $\frac{m_n}{2}$. To fully determine $M(m_0, w)$ one only needs to give the number m_n. It will be defined to be the largest even integer smaller than a constant c_n, whose description is given in definition 6.3 below, and which has the approximation, given in proposition 6.8,

$$c_n \approx \frac{2w}{(\pi n)^2}\left(1 - \frac{(-1)^n}{2}\right).$$

We note that $c_n < 2$ for $n >> 0$, hence $m_n = 0$ for all but finitely many values of n. Also m_n is always non-negative (see remark 6.9).

To define the constants c_n we need an auxiliary function f.

Definition 6.2. — We define a function $f = f_{m_0,w}$ on $[0, 1]$ as

$$f_{m_0,w}(r) = \begin{cases} s_{m_0}(\beta r) - m_0 + w & \text{if } \ r \in [\frac{1}{2}, 1] \\ f_{m_0,w}(1 - r) & \text{if } r \in [0, \frac{1}{2}]. \end{cases}$$

Note that $f(r)$ is a continuous function such that

$$f(0) = f(1) = s_{m_0}(\beta) - (m_0 - w) = 0$$

by the definition of $\beta = \beta_{m_0,w}$. Also, by construction, f is symmetric around $1/2$, which implies that in its real Fourier expansion all the sin functions do not appear. Finally, for all $r \in [0, 1]$, $f(r) \geq 0$. It is enough to check this by symmetry for $r \geq 1/2$, in which case $\beta r \in E_{M_0}(w)$ hence $s_{m_0}(\beta r) \geq m_0 - w$.

Definition 6.3. — Define real numbers $c_n = c_n(m_0, w)$ for $n \geq 0$ in such a way that the real Fourier expansion of f is

$$f(r) = c_0 - \sum_{n=1}^{\infty} c_n \cos(2\pi n r). \tag{9}$$

Define m_n as the largest even number smaller than c_n.

We have the usual integral expansions of c_n,

$$c_0 = \int_0^1 f(r)\ dr = 2\int_{\frac{1}{2}}^1 f(r)\ dr \tag{10}$$

and

$$c_n = -4\int_{\frac{1}{2}}^1 f(r) \cos(2\pi n r)\ dr. \tag{11}$$

The Fourier expansion (9) clearly converges pointwise on $[0,1]$ because f is continuous and piecewise differentiable. Substituting $r=0$ we have

$$c_0 = \sum_{n=1}^{\infty} c_n. \tag{12}$$

Remark 6.4. — The heuristic reasoning behind construction 6.1 is as follows: We are looking for a set M of the form (8). The number β is defined so that $E_{M,w} \subseteq [-\beta,\beta]$ whatever $\overline{M}$ is, so best we can hope for is near equality. We would also like to make $\bar{m} = \sum_{n>0} m_n$ (and therefore $m = |M|$) as large as possible. Since S_{M_0} is real and large, it is easily seen that the best way to enlarge S_M is to contribute to its real part. Thus we assume from the start that $\overline{M}$ is symmetric around 0. Then S_M is real valued. Therefore, the condition for α to be in $E_{M,w}$ becomes $S_M(\alpha) \geq k-w$, which is equivalent to

$$\bar{m} - S_{\overline{M}}(\alpha) \leq S_{M_0}(\alpha) - m_0 + w \tag{13}$$

Suppose it was possible to have $\beta \in E_{M,w}$. Then we get $\bar{m} - S_{\overline{M}}(\beta) = 0$ and thus $e^{2\pi a\beta i} = 1$ for $a \in \overline{M}$. This implies that each $a \in \overline{M}$ is of the form $a = \frac{n}{\beta}$ for some $n \neq 0$. Set

$$e_n = |\{a = \pm\frac{n}{\beta} \in \overline{M}\}|.$$

We can therefore write

$$f_{\overline{M}}(r) := \bar{m} - S_{\overline{M}}(\beta r) = \bar{m} - \sum_{n=1}^{\infty} e_n \cos(2\pi n r).$$

The function $f_{\overline{M}}$ satisfies $f_{\overline{M}} \leq f$ since this is true on $[1/2,1]$ by (13) and since both sides are symmetric for replacing r by $1-r$. Conversely, for any symmetric $f_{\overline{M}} \leq f$ we can, replacing f by $f_{\overline{M}}$ in definition 6.3 and what follows, find constants e_n and create an $\overline{M}$ that will satisfy (13). But since we want the largest $\bar{m} = e_0$ it is clear we should take $f_{\overline{M}} = f$. Then we make the necessary adjustments to get from the c_n to a true candidate for $\overline{M}$ by taking m_n to be the largest even number smaller than c_n and $\overline{M}$ as a union of arithmetic progressions centered on $\pm n/\beta$ and of length $m_n/2$ each, which is just the construction 6.1.

We now derive estimates on the parameters of M and the size of $E_{M,w}$.

Lemma 6.5. — *Let n be an integer. Then,*

$$\sup_{\substack{z\in\mathbb{C} \\ |z|\leq \frac{1}{n}}} \frac{\sin(nz)}{\sin(z)} \leq e^{\frac{1}{n}} \cdot 1.2n.$$

Proof. — We have

$$\frac{\sin(nz)}{\sin(z)} = \frac{e^{inz} - e^{-inz}}{e^{iz} - e^{-iz}} = e^{i(1-n)z}\frac{e^{2inz}-1}{e^{2iz}-1} = e^{i(1-n)z}\sum_{k=0}^{n-1} e^{2ikz}.$$

If $z = x - iy$, then, since $|e^{iz}| = e^y$, we get the upper bound

$$\left|\frac{\sin(nz)}{\sin(z)}\right| \leq e^{-(n-1)y} \sum_{k=0}^{n-1} e^{2ky} = \sum_{\substack{1-n \leq l \leq n-1 \\ 2|n-1-l}} e^{ly}.$$

It is enough to find the maximal value of the last expression for $-1/n \leq y \leq 1/n$. It is clear that the expression is symmetric in y. The derivative is given by

$$\sum_{\substack{1 \leq l \leq n-1 \\ 2|n-1-l}} l(e^{ly} - e^{-ly}),$$

which is clearly positive for positive y. Therefore, the maximal value is obtained at $y = 1/n$ and equals

$$e^{-\frac{n-1}{n}} \sum_{k=0}^{n-1} e^{\frac{2k}{n}} = e^{-\frac{n-1}{n}} \frac{e^2 - 1}{e^{\frac{2}{n}} - 1}.$$

Using the inequality $e^x - 1 \geq x$ we obtain

$$\left|\frac{\sin(nz)}{\sin(z)}\right| \leq e^{-\frac{n-1}{n}} (e^2 - 1)\frac{n}{2} = e^{\frac{1}{n}} \frac{e^2 - 1}{2e} n \leq e^{\frac{1}{n}} \cdot 1.2n.$$

□

Let

$$f(r) = w + \sum_{j=1}^{\infty} a_j r^{2j}$$

be the expansion of $f(r)$ on the interval $[1/2, 1]$. In other words, it is the Taylor expansion of $s_{m_0}(\beta r) - m_0 + w$ around 0. Note that the odd coefficients vanish because s_{m_0} is an even function, and that since its value at 0 is m_0 the constant coefficient is indeed w. Let

$$R_2(r) := f(r) - w - a_1 r^2$$

be the error term in the quadratic approximating of $f(r)$.

Lemma 6.6. — *Let $c = 6(w/m_0)(1 + 2w/m_0)$ and define*

$$h(r) := 2m_0 \frac{(cr^2)^2}{1 - cr^2}.$$

Then, for $r \in [1/2, 1]$ and any $n \geq 1$,

$$\left|\frac{d^n}{dr^n} R_2(r)\right| \leq \frac{d^n}{dr^n} h(r).$$

Proof. — We give an upper bound on the coefficients a_j. From the explicit description of s_{m_0} given in part 1 of proposition 3.1, we see the the complex function $\sin(m_0 z)/\sin(z)$ extends the real function $s_{m_0}(\alpha/\pi)$. Consider the Taylor expansion

$$\sin(m_0 z)/\sin(z) = \sum_{j=0}^{\infty} b_j z^j.$$

By lemma 6.5 we see that since $m_0 \geq 3$, $s_{m_0}(z/\pi) = \sin(m_0 z)/\sin(z)$ is bounded by $2m_0$ when $|z| \leq 1/m_0$. We use the Cauchy integral formula on a circle C_{m_0} of radius $1/m_0$ around 0 to obtain the estimate

$$|b_j| = \left| \frac{1}{2\pi i} \oint_{C_{m_0}} \frac{\sin(m_0 z)/\sin(z)}{z^{j+1}} dz \right| < 2{m_0}^{j+1}.$$

From the definition of f we see that for $j > 0$ we have $a_j = b_{2j}(\pi\beta)^{2j}$. From the bound on β in proposition 3.4 we get for $j > 0$

$$|a_j| < 2{m_0}^{2j+1} \left(\left(\frac{6w(1 + 2u/m_0)}{m_0} \right)^{\frac{1}{2}} \frac{1}{m_0} \right)^{2j} = 2m_0 c^j.$$

One easily checks that

$$h(r) = 2m_0 \sum_{j=2}^{\infty} c^j r^{2j}.$$

The bound is now clear. $\square$

Corollary 6.7. — *We have the following estimates for $r \in [1/2, 1]$ and $\delta \in [0, 1/2]$.*

(6.7.1) $$R_2(r) \leq 75 \frac{w^2}{m_0} r^4.$$

(6.7.2) $$R_2'(r) \leq 300 \frac{w^2}{m_0} r.$$

(6.7.3) $$|w + a_1| \leq 75 \frac{w^2}{m_0}.$$

(6.7.4) $$|f'(r) + 2wr| \leq 450 \frac{w^2}{m_0} r.$$

(6.7.5) $$f(1 - \delta) \leq 2\delta w \left(1 + 225 \frac{w}{m_0}\right).$$

Proof. — To prove (6.7.1) we note that

$$R_2(r) \leq h(r) \leq 2m_0 \frac{c^2}{1 - c} r^4 = 2 \cdot 6^2 \frac{w^2}{m_0} r^4 (1 + o(\epsilon)) \leq 75 \frac{w^2}{m_0} r^4.$$

Similarly, we get (6.7.2) because

$$\begin{aligned}(14)\qquad h'(r) &= 4m_0c^2r^3\frac{2-r^2c}{(1-cr^2)^2} \le 8m_0\frac{c^2}{(1-c)^2}r\\ &\le 8\cdot 36\frac{w^2}{m_0}r(1+o(\epsilon)) \le 300\frac{w^2}{m_0}r.\end{aligned}$$

Since $f(1)=0$ we find

$$|w+a_1| = |f(1)-w-a_1 1^2| = |R_2(1)| \le h(1),$$

which is $\le 75w^2/m_0$ by (6.7.1). This gives (6.7.3). We get (6.7.4) by

$$|f'(r)+2wr| = |R_2'(r)+2(w+a_1)r| \le h'(r)+2\cdot 75\frac{w^2}{m_0}r \le 450\frac{w^2}{m_0}r,$$

where the last two inequalities follow from (6.7.3) and (6.7.2) respectively. Finally, (6.7.5) is derived from (6.7.4) by the mean value theorem: Since $f(1)=0$ we can find $1-\delta<\rho<1$ such that $|f(1-\delta)| \le \delta|f'(\rho)|$. By (6.7.4) we find

$$|f'(\rho)| \le 2u\rho + 450\frac{w^2}{m_0}\rho \le 2u + 450\frac{w^2}{m_0} = 2u\left(1+225\frac{w}{m_0}\right),$$

which finishes the proof. □

Proposition 6.8. — *We have the following estimates on the coefficients c_n.*

$$(6.8.1)\qquad c_0 = \frac{5}{12}w + 75\frac{w^2}{m_0}O(1).$$

$$(6.8.2)\qquad c_n = \frac{2w}{(\pi n)^2}\left(1-\frac{(-1)^n}{2}\right) + \frac{1500}{(\pi n)^2}\frac{w^2}{m_0}O(1).$$

Remark 6.9. — In particular, c_n is positive for every n and hence we can indeed define m_n as we did in construction 6.1.

Lemma 6.10. — *For any C^∞ real valued function g and any nonzero integer n we have*

$$\int_{\frac12}^1 \cos(2\pi nr)g(r)dr = \frac{1}{(2\pi n)^2}\left(g'(1)-(-1)^n g'\left(\frac12\right) - \int_{\frac12}^1 \cos(2\pi nr)g''(r)dr\right).$$

Proof. — We use integration by parts twice to get

$$\begin{aligned}&\operatorname{Re}\int_{\frac12}^1 e^{2\pi inr}g(r)dr\\ &= \operatorname{Re}\left(\frac{1}{2\pi ni}\left(g(1)-(-1)^n g\left(\frac12\right) - \int_{\frac12}^1 e^{2\pi inr}g'(r)dr\right)\right)\\ &= \operatorname{Re}\left(\frac{-1}{(2\pi ni)^2}\left(g'(1)-(-1)^n g'\left(\frac12\right) - \int_{\frac12}^1 e^{2\pi inr}g''(r)dr\right)\right).\end{aligned}$$

Writing out the real part in terms of cos functions gives the result. □

Proof of proposition 6.8. — Taking $g(r) = 1$ and $g(r) = r^2$ respectively in lemma 6.10 we get

$$\int_{\frac{1}{2}}^{1} \cos(2\pi nr)dr = 0, \tag{15}$$

$$\int_{\frac{1}{2}}^{1} \cos(2\pi nr)r^2 dr = \frac{2}{(2\pi n)^2}\left(1 - \frac{(-1)^n}{2}\right) \leq \frac{3}{(2\pi n)^2}. \tag{16}$$

Now we use $g(r) = R_2(r)$ as defined in lemma 6.6. In the inequality of lemma 6.10 we can replace R_2 and its derivatives by the function h and its derivative, as follows from lemma 6.6. This gives

$$\left|\int_{\frac{1}{2}}^{1} \cos(2\pi nr)R_2(r)dr\right| \leq \frac{1}{(2\pi n)^2}\left(h'(1) + h'(\frac{1}{2}) + \int_{\frac{1}{2}}^{1} h''(r)dr\right).$$

Evaluating the right hand side we find

$$\left|\int_{\frac{1}{2}}^{1} \cos(2\pi nr)R_2(r)dr\right| \leq \frac{2h'(1)}{(2\pi n)^2} \leq 150\frac{w^2}{m_0}\frac{1}{(\pi n)^2}, \tag{17}$$

where the inequality follows from the estimate (14). By definition,

$$f(r) = R_2(r) + w + a_1 r^2 = R_2(r) + w(1 - r^2) + (w + a_1)r^2.$$

Here, the main term is $w(1 - r^2)$. Therefore, The main terms in the estimates on c_n and c_0 are

$$\begin{aligned} c_n &\approx -4w\int_{\frac{1}{2}}^{1} (1 - r^2)\cos(2\pi nr)\, dr = \frac{2w}{(\pi n)^2}\left(1 - \frac{(-1)^n}{2}\right), \\ c_0 &\approx 2w\int_{\frac{1}{2}}^{1} (1 - r^2)dr \qquad\qquad = \frac{5}{12}w. \end{aligned}$$

The error term for c_n is now obtained by integrating $R_2(r)+(w+a_1)r^2$ multiplied by $\cos(2\pi nr)$ and the appropriate constant, and the integral is estimated using (15),(16) and (17). For $n > 0$ we get

$$\begin{aligned} \left|c_n - \frac{2w}{(\pi n)^2}\left(1 - \frac{(-1)^n}{2}\right)\right| &= 4\left|\int_{\frac{1}{2}}^{1} \cos(2\pi nr)\left(R_2(r) + (w + a_1)r^2\right) dr\right| \\ &\leq 4\left(150\frac{w^2}{m_0}\frac{1}{(\pi n)^2} + 75\frac{w^2}{m_0}\frac{3}{(\pi n)^2}\right) \\ &= 1500\frac{w^2}{m_0}\frac{1}{(\pi n)^2}. \end{aligned}$$

Similarly for c_0 we find

$$\begin{aligned}\left|c_0 - \frac{5}{12}w\right| &= 2\left|\int_{\frac{1}{2}}^{1}(f(r) - w(1-r^2))dr\right| \\ &\leq 2\left|\int_{\frac{1}{2}}^{1} R_2(r)dr\right| + 2\left|\int_{\frac{1}{2}}^{1}(a_1+w)r^2dr\right| \\ &\leq 2\left|\int_{\frac{1}{2}}^{1} 75\frac{w^2}{m_0}r^4dr\right| + 2\cdot\frac{1}{3}\cdot\frac{7}{8}(a_1+w) \\ &\leq \frac{2}{5}\cdot 75\frac{w^2}{m_0} + \frac{2}{3}\cdot\frac{7}{8}\cdot 75\frac{w^2}{m_0} \leq 75\frac{w^2}{m_0}.\end{aligned}$$

□

We now wish to modify the set M to make it our candidate set K_{ex}. We need a few more computations.

Lemma 6.11. — *Let ω be a real constant. Then we have*

$$\sum_{n=1}^{\infty}\min\left(\frac{\omega}{n^2}, 2\right) \leq 4.2\sqrt{\omega}.$$

Proof. — Recall that $\sum 1/n^2 = \pi^2/6$. We begin by considering small values of ω. If $\omega \leq 2$, then the sum is clearly $\omega\pi^2/6 \leq \sqrt{\omega}\sqrt{2}\pi^2/6$. Similar computations show that when $2 < \omega \leq 4$ we get a bound of $\sqrt{\omega}\pi^2/3$ and when $4 < \omega \leq 9$ we get a bound of

$$3\left(\frac{\pi^2}{6} - \frac{1}{4}\right)\sqrt{\omega} \leq 4.2\sqrt{\omega},$$

which is the largest so far. Now suppose that $9 \leq \omega$. Let $x = \sqrt{\omega/2}$. One checks that $2x^2/(x-1) \leq 3x+2$, It is easily seen that the largest n for which $\omega/n^2 \geq 2$ is $[x]$. Thus we have

$$\begin{aligned}\sum_{n=1}^{\infty}\min\left(\frac{\omega}{n^2}, 2\right) &\leq 2[x] + \omega\sum_{n=[x]+1}^{\infty}\frac{1}{n^2} \leq 2[x] + \omega\int_{[x]}^{\infty}\frac{dt}{t^2} = 2[x] + \frac{\omega}{[x]} \\ &\leq 2(x-1) + \frac{\omega}{x-1} = 2x + \frac{2x^2}{x-1} - 2 \\ &\leq 5x = \frac{5}{\sqrt{2}}\sqrt{\omega} \leq 4.2\sqrt{\omega}.\end{aligned}$$

□

Proposition 6.12. — *Suppose k and u satisfy the condition $1 < u \leq k/30000$. Then there exist $m_0, \bar{m} \in \mathbb{Z}$, $w \in \mathbb{R}$ and a set $K_{ex} = K_{ex}(k,u)$, such that the following are* *satisfied.*

(6.12.1) $$k = m_0 + \tilde{m}.$$

(6.12.2) $$M_0 \subseteq K_{ex} \subseteq M(m_0, w).$$

(6.12.3) $$E_{K_{ex},u} \supseteq [-\beta, \beta], \quad \textit{with } \beta = \beta_{m_0,w}.$$

(6.12.4) $$c_0(m_0, w) \geq \tilde{m} \geq c_0(m_0, w) - 5.2\sqrt{w}.$$

(6.12.5) $$u = w + \left(\frac{w}{k}\right)^3 w + 2\left(\frac{w}{k}\right)^{3/2}.$$

Remark 6.13. — Note that $c_0(m_0, w)$ depends on w and m_0 but to first order is just $\frac{5}{12}w \approx \frac{5}{12}u$.

Proof of proposition 6.12. — We start by choosing w such that (6.12.5) is satisfied. For a given m_0 set $\tilde{m} = \sum_{n=1}^{\infty} m_n$ and $m = m_0 + \tilde{m} = |M(m_0, w)|$. We take the smallest m_0 for which $m \geq k$. We now pull elements out of the sets M_n until we obtain our set K_{ex} (the choice of which elements to take is arbitrary, hence K_{ex} is not uniquely defined). By (12) and the construction of the m_n in 6.1 we find the left inequality in (6.12.4). To get the inequality on the right we need to compute the sum of the differences between c_n and m_n and count how many elements we take out of the M_n in the final step. Recall that m_n was taken to be the largest even integer smaller than c_n. Therefore,

$$c_0 - \tilde{m} = \sum_{n=1}^{\infty} c_n - m_n \leq \sum_{n=1}^{\infty} \min(c_n, 2).$$

Let $\omega = (3w/\pi^2)(1 + o(\epsilon))$. It follows from (6.8.2) that

$$c_0 - \tilde{m} \leq \sum_{n=1}^{\infty} \min\left(\frac{\omega}{n^2}, 2\right),$$

which by lemma 6.11 is

$$\leq 4.2\sqrt{\frac{3}{\pi^2}}(1 + o(\epsilon))\sqrt{u} \leq 2.4\sqrt{u}.$$

The number of elements we may have to take out of $M(m_0, w)$ to get K_{ex} is bounded from above by $|M(m_0, w)| - |M(m_0 - 2, w)|$. The difference in size between the two sets is caused by the fact that we have $m_n(m_0, w) - m_n(m_0 - 2, w) = 2$ every time there is a multiple of 2 between $c_n(m_0, w)$ and $c_n(m_0 - 2, w)$. With the constant ω as before, this does not occur once $n > \sqrt{\omega/2}$. Thus,

$$m - k \leq 2 + 2\sqrt{\omega/2} \leq 2 + 0.78\sqrt{u} \leq 2.78\sqrt{u}.$$

This gives (6.12.4) as $2.4 + 2.78 \leq 5.2$. It remains to show (6.12.3). From the definition of f we see that for $r \geq 1/2$,

$$S_{M_0}(\beta r) - m_0 + w = f(r) = c_0 - \sum_{n=1}^{\infty} c_n \cos(2\pi n r) = \sum_{n=1}^{\infty} c_n(1 - \cos(2\pi n r)),$$

where the last equality follows from (12). Since $m_n \leq c_n$ we find

$$S_{M_0}(\beta r) - m_0 + w \geq \sum_{n=1}^{\infty} m_n(1-\cos(2\pi nr)) = \tilde{m} - \sum_{n=1}^{\infty} m_n \cos(2\pi nr)$$

and, rearranging terms

$$S_{M_0}(\beta r) + \sum_{n=1}^{\infty} m_n \cos(2\pi nr) \geq m_0 + \tilde{m} - w = m - w. \tag{18}$$

Comparing the situation for $r < \frac{1}{2}$, we see that S_{M_0} is larger while the rest is symmetric around $\frac{1}{2}$ and therefore the last inequality holds for $r \in [0,1]$. Each term $m_n \cos(2\pi nr)$ is very close to $S_{M_n}(r\beta) + S_{M_{-n}}(r\beta)$. We measure the difference in the following lemma.

Lemma 6.14. — *Let $A = \{s, s+1, \ldots, t-1, t\}$ be an arithmetic progression of length $l = t - s + 1$. Suppose $\alpha \in [0, 1/l]$ and suppose $x \in \mathbb{R}$ satisfies $|x - x'| \leq 1/2$, with $x' = (s+t)/2$. Then,*

$$|\operatorname{Re}(le^{2\pi i x\alpha} - S_A(\alpha))| \leq \pi l\alpha + \frac{\pi^2}{6} l^3 \alpha^2.$$

Proof. — Suppose first that $x = x'$. Then,

$$|\operatorname{Re}(le^{2\pi i x\alpha} - S_A(\alpha))| = |l - e^{-2\pi i x\alpha} S_A(\alpha)|.$$

It is easy to see that the expression inside the absolute value is real, positive and equal to

$$l - s_l(\alpha) = l - \frac{\sin(\pi l\alpha)}{\sin(\pi\alpha)} \leq \frac{\sin(\pi l\alpha)}{\pi\alpha} \leq \frac{\pi^2}{6} l^3 \alpha^2,$$

where the last inequality follows from (5). To complete the proof all we have to do is to compute

$$|\operatorname{Re}(le^{2\pi i x\alpha} - le^{2\pi i x'\alpha})| = |l(\cos(2\pi x\alpha) - \cos(2\pi x'\alpha))| \leq l|2\pi(x - x')\alpha| \leq \pi l\alpha.$$

$\square$

Applying the last lemma in our situation we see, using the fact that $\sum {m_n}^3 \leq \tilde{m}^3$, that

$$\sum_{n=1}^{\infty} m_n \cos(2\pi nr) - 2\operatorname{Re} S_{M_n}(\beta r) \leq \pi\tilde{m}\beta + \frac{\pi^2}{6}\tilde{m}^3\beta^2$$

$$\leq \left(\pi \frac{w}{2}\frac{\sqrt{6}}{\pi}\left(\frac{w}{m_0}\right)^{\frac{1}{2}}\frac{1}{m_0} + \frac{\pi^2}{6}\frac{w^3}{8}\frac{6}{\pi^2}\left(\frac{w}{m_0}\right)\frac{1}{{m_0}^2}\right)(1 + o(\epsilon))$$

by (6.8.1) and proposition 3.4

$$\leq \left(\frac{w}{k}\right)^3 w + 2\left(\frac{w}{k}\right)^{3/2}.$$

Thus, our choice of w in (6.12.5) guarantees by (18) that

$$S_M(\beta r) \geq m - w - \left(\frac{w}{k}\right)^3 w - 2\left(\frac{w}{k}\right)^{3/2} = m - u$$

for all $r \in [0,1]$. Therefore, $E_{M,u} \supseteq [-\beta, \beta]$. To finish we just need to note that taking elements out of M does not change the situation as is easily seen. ☐

7. Structure of the maximal set

In the final three sections we try to determine the structure of a set K in the class $\mathbb{K}_{\mu_{K_{ex}}(u)}$, i.e., a set which is "better" than the example we produced in the last section.

Our assumptions are as usual: $1 < u \leq k/30000$ and u and k satisfy assumption 5.4. Since $\mu_{K_{ex}}(u)$ is greater than the one for arithmetic progressions, we certainly know that our results from section 5 Apply here. Therefore, we can write K in the form $K = K_0 \cup \overline{K}$, where K_0 is the set whose existence is guaranteed by lemma 5.8, $|K_0| = k_0$, $|\overline{K}| = \bar{k}$, and $k_0 + \bar{k} = k$. What we will try to do is determine the structure of $\overline{K}$. Since, as we saw in proposition 5.9, There is, for any set in $\mathbb{K}_{\mu_{K_{ex}}(u)}$, a better one with the corresponding K_0 forming an arithmetic progression with at most a single gap, it is no harm to assume that our sets are already of this type. We will assume in fact that K_0 is an arithmetic progression (without a gap), that it has an odd number of elements and that its difference is 1. The modifications required to cover the general case will be explained in the end. Since by (2) we are always allowed to translate our set, we can assume

$$K_0 = \left\{-\frac{k_0 - 1}{2}, \ldots, \frac{k_0 - 1}{2}\right\}.$$

We also make the following *shortcut*:

$$\bar{k} < \frac{1}{2}u. \tag{19}$$

The justification for this is as follows: we have already seen this estimate with $\frac{1}{2}$ replaced by 2000 in proposition 5.9. Shortly (in (7.4.1)) we will see that $\bar{k}$ is to first order $\frac{5}{12}u$, where the second order terms depend on the above mentioned constant. Iterating this we can assume in advance that the constant is say $\frac{1}{2}$.

Let $\beta' = \beta_{k_0,u}$. Set

$$K_n = K \cap [(n - \frac{1}{8})/\beta', (n + \frac{1}{8})/\beta'], \quad k_n = |K_n| + |K_{-n}|. \tag{20}$$

We also write $g = f_{k_0,u}$. By definition 6.2,

$$k_0 - u = S_{K_0}(\beta' r) - g(r). \tag{21}$$

This function has a Fourier expansion similar to (9) and the coefficients $c_n(k_0, u)$ will be denoted d_n. Obviously

$$E_{K,u} \subseteq [-\beta', \beta']. \tag{22}$$

Definition 7.1. — A constant $\delta \geq 0$ (depending on K and u) is defined by the equation

$$\mu_K(u) = 2(1-\delta)\beta'.$$

In this section, δ appears in the error terms for the estimates of the k_n. To get absolute bounds we will bound δ in section 9.

The basis for the estimates is a bound for $\operatorname{Re} S_{\overline{K}}$.

Proposition 7.2. — *If $\alpha = \beta' r \in E_{K,u}$, then*

$$\bar{k} - \operatorname{Re} S_{\overline{K}}(\alpha) \leq g(r)\left(1 + \frac{2\bar{k}}{k}\right) \leq g(r)\left(1 + \frac{u}{k}\right).$$

Corollary 7.3. — *For all $r \in [0,1]$,*

$$\bar{k} - \operatorname{Re} S_{\overline{K}}(\beta' r) \leq \begin{cases} g(r) + \frac{u^2}{k} & \text{if } \beta' r \in E_{K,u} \\ u & \text{otherwise.} \end{cases} \tag{23}$$

Proof. — When $\beta' r \in E_{K,u}$ this is clear from the proposition since $g(r) \leq u$. Otherwise we just use the trivial upper bound $2\bar{k}$, which is $< u$ by (19). □

Proof of proposition 7.2. — Assume $\alpha \in E_{K,u}$ but drop α from the notation. Since S_{K_0} is real valued we find

$$(S_{K_0} + \operatorname{Re} S_{\overline{K}})^2 + \operatorname{Im} S_{\overline{K}}^2 = |S_K|^2 \geq (k-u)^2 = (S_{K_0} + \bar{k} - g(r))^2,$$

where the last equality follows from (21). Expanding this, cancelling $\operatorname{Re} S_{\overline{K}}^2 + \operatorname{Im} S_{\overline{K}}^2$ on the left with $\bar{k}^2$ on the right and cancelling out $S_{K_0}^2$ one gets

$$2S_{K_0} \operatorname{Re} S_{\overline{K}} \geq g(r)^2 + 2S_{K_0}\bar{k} - 2\bar{k}g(r) - 2S_{K_0}g(r).$$

Rearranging terms we find

$$2S_{K_0}(\bar{k} - \operatorname{Re} S_{\overline{K}}) \leq g(r)(2\bar{k} + 2S_{k_0} - g(r))$$

and hence

$$\bar{k} - \operatorname{Re} S_{\overline{K}} \leq g(r)\left(1 + \frac{\bar{k} - g(r)/2}{S_{K_0}}\right) \leq g(r)\left(1 + \frac{\bar{k}}{S_{K_0}}\right),$$

because $g \geq 0$. Now we use (19) to get

$$S_{K_0} \geq k_0 - u = k - \bar{k} - u \geq k - \frac{3u}{2} \geq \frac{1}{2}k.$$

Substituting this in the previous inequality we get

$$\bar{k} - \operatorname{Re} S_{\overline{K}} \leq g(r)\left(1 + \frac{2\bar{k}}{k}\right).$$

Using (19) again we get the second inequality. □

Recall that the numbers d_n are the Fourier coefficients of g and that their integral representations are given in (10) and (11), with f replaced by g.

Proposition 7.4. — *We have the following inequalities.*

$$\bar{k} \le d_0 + \frac{u^2}{k} + 6u\sqrt{\delta}. \tag{7.4.1}$$

$$2\bar{k} \mp k_n \le 2d_0 \mp d_n + 2\frac{u^2}{k} + 24u\sqrt{\delta}. \tag{7.4.2}$$

Proof. — We multiply (23) by the positive functions $1 \pm \cos(2\pi nr)$ and integrate from $\frac{1}{2}$ to 1. By definition 7.1 the second case in (23) occurs on a set of measure at most δ of r's. On this small set we bound $1 \pm \cos(2\pi nr)$ by 2. We easily obtain

$$\int_{\frac{1}{2}}^{1} (\bar{k} - \operatorname{Re} S_{\overline{K}}(\beta' r))(1 \pm \cos(2\pi nr))dr \le \frac{1}{2}d_0 \mp \frac{1}{4}d_n + \frac{u^2}{2k} + 2\delta u. \tag{24}$$

Similarly, multiplying by 1 and integrating, we get

$$\int_{\frac{1}{2}}^{1} (\bar{k} - \operatorname{Re} S_{\overline{K}}(\beta' r))dr \le \frac{1}{2}d_0 + \frac{u^2}{2k} + \delta u.$$

We will derive (7.4.1), the derivation of (7.4.2) being similar and simpler. We expand the left hand side of (24).

$$\begin{aligned} &\int_{\frac{1}{2}}^{1} (\bar{k} - \operatorname{Re} S_{\overline{K}}(\beta' r))(1 \pm \cos(2\pi nr))dr \\ &= \frac{1}{2}\bar{k} - \int_{\frac{1}{2}}^{1} \operatorname{Re} S_{\overline{K}}(\beta' r)dr \mp \int_{\frac{1}{2}}^{1} \operatorname{Re} S_{\overline{K}}(\beta' r)\cos(2\pi nr)dr = \\ &= \frac{1}{2}\bar{k} - \sum_{a\in\overline{K}} \int_{\frac{1}{2}}^{1} \cos(2\pi|a|\beta' r)dr \\ &\mp \sum_{a\in\overline{K}} \int_{\frac{1}{2}}^{1} \cos(2\pi|a|\beta' r)\cos(2\pi nr)dr. \end{aligned} \tag{25}$$

We will split the summands in the last sum as

$$\begin{aligned} &\int_{\frac{1}{2}}^{1} \cos(2\pi|a|\beta' r)\cos(2\pi nr)dr \\ &= \frac{1}{2}\int_{\frac{1}{2}}^{1} \cos(2\pi(|a|\beta' + n)r)dr + \frac{1}{2}\int_{\frac{1}{2}}^{1} \cos(2\pi(|a|\beta' - n)r)dr. \end{aligned}$$

In case $a \in K_n \cup K_{-n}$ we will further split

$$\frac{1}{2}\int_{\frac{1}{2}}^{1} \cos(2\pi(|a|\beta' - n)r)dr = \frac{1}{4} + \frac{1}{2}\left(\int_{\frac{1}{2}}^{1} \cos(2\pi(|a|\beta' - n)r)dr - \frac{1}{2}\right).$$

We wish to obtain a lower bound for the left hand side of (24). We take all non-constant terms in the last 2 identities, replace them by the negatives of their absolute values and plug into (25). This gives

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1} (\bar{k} - \operatorname{Re} S_{\overline{K}}(\beta' r))(1 \pm \cos(2\pi n r))dr \\
&\geq \frac{1}{2}\bar{k} \mp \frac{1}{4}k_n \\
&- \frac{1}{2} \sum_{a \in K_n \cup K_{-n}} \left| \int_{\frac{1}{2}}^{1} \cos(2\pi(|a|\beta' - n)r)dr - \frac{1}{2} \right| \\
&- \sum_{a \in \overline{K}} \left| \int_{\frac{1}{2}}^{1} \cos(2\pi |a| \beta' r)dr \right| \\
&- \frac{1}{2} \sum_{a \in \overline{K} - (K_n \cup K_{-n})} \left| \int_{\frac{1}{2}}^{1} \cos(2\pi(|a|\beta' + n)r)dr \right| \\
&- \frac{1}{2} \sum_{a \in \overline{K} - (K_n \cup K_{-n})} \left| \int_{\frac{1}{2}}^{1} \cos(2\pi(|a|\beta' - n)r)dr \right|.
\end{aligned}
\tag{26}
$$

The last four terms are error terms. The last three have a common form, namely, they are sums of terms of the form $\int_{1/2}^{1} \cos(2\pi b r)\, dr$, with b sufficiently large. In fact, b will be $|a|\beta'$ or $|a|\beta' + n$ for $a \in \overline{K}$ in the second and third terms respectively, or $|a|\beta' - n$ for $a \in \overline{K} - K_{\pm n}$ in the fourth term. By (20) we have in all cases that $|b| > \frac{1}{8}$.

Lemma 7.5. — *Suppose $|b| > \frac{1}{8}$ and that $\delta' < 1$. Then*

$$\left| \int_{\frac{1}{2}}^{1} \cos(2\pi b r)dr \right| \leq \frac{12}{\pi} |\sin(\pi b(1 - \delta'))| + \frac{3}{2}\delta'.$$

Proof. — By using the trivial bound $|\cos(x)| \leq 1$ on $[1 - \delta', 1] \cup \frac{1}{2}[1 - \delta', 1]$ we get

$$\left| \int_{\frac{1}{2}}^{1} \cos(2\pi b r)dr \right| \leq \left| \int_{\frac{1-\delta'}{2}}^{1-\delta'} \cos(2\pi b r)dr \right| + \frac{3}{2}\delta'.$$

Now set $b' = b(1 - \delta')$. Then

$$
\begin{aligned}
&\left| \int_{\frac{1-\delta'}{2}}^{1-\delta'} \cos(2\pi b r)dr \right| = \left| \frac{1}{2\pi b}(\sin(2\pi b') - \sin(\pi b')) \right| = \\
&= \left| \frac{1}{2\pi b}(2\sin(\pi b')\cos(\pi b') - \sin(\pi b')) \right| \leq \frac{2+1}{2\pi(1/8)} |\sin(\pi b')| = \frac{12}{\pi} |\sin(\pi b')|.
\end{aligned}
$$

□

Applying the lemma just proved to the second error term we get

$$\sum_{a\in\overline{K}}\left|\int_{\frac{1}{2}}^{1}\cos(2\pi|a|\beta' r)\,dr\right| \le \frac{12}{\pi}\sum_{a\in\overline{K}}|\sin(\pi a\beta'(1-\delta'))| + \frac{3}{2}\bar{k}\delta'. \tag{27}$$

By the definition 7.1 of δ and by proposition 7.2 we know that there exists some $\delta' < \delta$ such that

$$\bar{k} - \operatorname{Re} S_{\overline{K}}(\beta'(1-\delta')) \le g(1-\delta')\left(1+\frac{u}{k}\right). \tag{28}$$

The left hand side can be expanded as

$$\sum_{a\in\overline{K}}(1-\cos(2\pi a\beta'(1-\delta'))) = 2\sum_{a\in\overline{K}}\sin^2(\pi a\beta'(1-\delta')).$$

Using (6.7.5) to bound $g(1-\delta)$ we get the estimate

$$\sum_{a\in\overline{K}}\sin^2(\pi a\beta'(1-\delta')) \le \delta' u\left(1+225\frac{u}{k_0}\right)\left(1+\frac{u}{k}\right) \le \delta u\left(1+300\frac{u}{k}\right).$$

Lemma 7.6. — *For any positive real numbers $x_1,\dots,x_r$ we have*

$$\sum_{i=1}^{r} x_i \le \sqrt{r\sum_{i=1}^{r} x_i^2}.$$

Proof. — This is just the Cauchy-Schwartz formula for the vectors $(1,\dots,1)$ and $(x_1,x_2,\dots,x_r)$. □

Applying this lemma in our situation to the numbers $|\sin(\pi a\beta'(1-\delta'))|$ for $a\in\overline{K}$ we find

$$\sum_{a\in\overline{K}}|\sin(\pi a\beta'(1-\delta'))| \le \sqrt{\bar{k}\delta u(1+300u/k)}.$$

By (27),

$$\sum_{a\in\overline{K}}\left|\int_{\frac{1}{2}}^{1}\cos(2\pi a\beta' r)\,dr\right| \le \frac{12}{\pi}\sqrt{\bar{k}u\,(1+300u/k)\,\delta} + \frac{3}{2}\bar{k}\delta.$$

Using (19) we see that the right hand side is smaller than

$$\frac{12}{\pi}\sqrt{\frac{1}{2}u\delta(1+o(\epsilon))} + \frac{3}{4}u\delta \le 2.9u\delta^{\frac{1}{2}} + \frac{3}{4}u\delta.$$

Clearly the same bound holds for the

$$\sum\left|\int_{\frac{1}{2}}^{1}\cos(2\pi(a\beta'\pm n)r)\,dr\right|$$

and therefore the last three terms in (26) can be bounded by $5.8u\sqrt{\delta}+(3/2)u\delta$.

We now handle the first error term.

Lemma 7.7. — *If $|b| < \frac{1}{8}$ and $\delta' < 1/2$, then*

$$\left|\int_{\frac{1}{2}}^{1} \cos(2\pi br)\, dr - \frac{1}{2}\right| \le \frac{1}{2}(1 - \cos(2\pi b(1-\delta'))) + \delta'.$$

Proof. — We have

$$\left|\int_{\frac{1}{2}}^{1} \cos(2\pi br)\, dr - \frac{1}{2}\right| = \left|\int_{\frac{1}{2}}^{1} (1 - \cos(2\pi br))\, dr\right|.$$

The estimate of the lemma is obtained by bounding the integrand by $1 - \cos(2\pi b(1 - \delta'))$ on $[\frac{1}{2}, 1-\delta']$ and by 1 on $[1-\delta', 1]$. □

Having the lemma, an argument similar to the one used to bound the last three error terms gives, using (28) and (6.7.5),

$$\sum_{a\in K_n\cup K_{-n}} \left|\int_{\frac{1}{2}}^{1} \cos(2\pi(|a|\beta' - n)r)\, dr - \frac{1}{2}\right|$$
$$\le \frac{1}{2}g(1-\delta)(1+o(\epsilon)) + \bar{k}\delta \le \delta(u(1+o(\epsilon)) + \bar{k})$$

Altogether, the four error terms are bounded by $5.8u\sqrt{\delta}+\delta(u(2.5+o(\epsilon))+\bar{k})$. Because $\delta << 1$ this bound is $\le 5.9u\sqrt{\delta}$. Now multiply (26) and (24) by 4 and compare them. One gets (7.4.2) up to noticing that we can neglect the δu term by increasing a bit the constant on the $u\sqrt{\delta}$ term. □

Corollary 7.8. — *We have*

$$|k_n - d_n| \le 2(d_0 - \bar{k}) + \frac{2u^2}{k} + 24u\sqrt{\delta}.$$

Proof. — This is because (7.4.2) implies immediately

$$\mp k_n \pm d_n \le 2(d_0 - \bar{k}) + \frac{2u^2}{k} + 24u\sqrt{\delta}.$$

□

8. Small perturbations in k_0 and u

In this section we prove bounds on $\beta - \beta'$ and $c_n - d_n$. Recall that

$$\beta = \beta(m_0, w), \quad \beta' = \beta(k_0, u), \quad c_n = c_n(m_0, w), \text{ and } d_n = c_n(k_0, u).$$

Therefore, the bounds will depend on m_0, k_0, w and u and all we need to do is to bound the perturbation of the functions β and c_n in terms of the parameters. Since we are assuming that $K \in \mathbb{K}_{\mu_{K_{ex}}(u)}$ and since $E_{K,u} \subseteq [-\beta', \beta']$ by (22) where as $E_{K_{ex},u} \supseteq [-\beta, \beta]$ by (6.12.3), we find $\beta' \ge \beta$. By the definition 7.1 of the parameter δ we have

$$\delta \le \frac{\beta' - \beta}{\beta'} \tag{29}$$

Proposition 8.1. — *Suppose we are given m_0, k_0, w and u that satisfy*

$$|k_0 - m_0| \le u, \tag{8.1.1}$$

$$w \le u \le w + \left(\frac{w}{k}\right)^3 w + 2\left(\frac{w}{k}\right)^{3/2}. \tag{8.1.2}$$

Let $\beta = \beta(m_0, w)$, $\beta' = \beta(k_0, u)$ and assume that $\beta' \ge \beta$. Then,

$$\beta' - \beta \le \left[\frac{1}{2}\left(\frac{u-w}{w}\right) + \frac{3(m_0 - k_0)}{2m_0}\left(1 \pm \frac{2w}{m_0}\right)\right]\left(1 \pm 250\frac{u}{k_0}\right)\beta'$$

and

$$\beta' - \beta \ge \left[\frac{1}{2}\left(\frac{u-w}{w}\right) + \frac{3(m_0 - k_0)}{2m_0}\left(1 \pm \frac{2w}{m_0}\right)\right]\left(1 \pm 250\frac{u}{m_0}\right)\beta.$$

Here, the sign depends on the sings of $m_0 - k_0$. The correct signs are those for which the bound is weakest possible.

Proof. — We will assume that $m_0 > k_0$, the other case being similar. By the mean value theorem there exists some $\beta'' \in [\beta, \beta']$ such that

$$s'_{k_0}(\beta'') = \frac{s_{k_0}(\beta) - s_{k_0}(\beta')}{\beta - \beta'}.$$

Therefore,

$$\begin{aligned} \beta' - \beta &= \frac{s_{k_0}(\beta) - s_{k_0}(\beta')}{-s'_{k_0}(\beta'')} \\ &= \frac{m_0 - w - s_{m_0}(\beta) + s_{k_0}(\beta) - (k_0 - u)}{-s'_{k_0}(\beta'')} \\ &= \frac{u - w + (m_0 - s_{m_0}(\beta)) - (k_0 - s_{k_0}(\beta))}{-s'_{k_0}(\beta'')}. \end{aligned}$$

By using the first expression for s_k in (3) we find

$$\begin{aligned} (m_0 - s_{m_0}(\beta)) - (k_0 - s_{k_0}(\beta)) &= \left(\sum_{n=(1-m_0)/2}^{(m_0-1)/2} - \sum_{n=(1-k_0)/2}^{(k_0-1)/2}\right)(1 - \cos(2\pi n\beta)) \\ &\le (m_0 - k_0)(1 - \cos(\pi m_0 \beta)) \le \frac{1}{2}(m_0 - k_0)(\pi m_0 \beta)^2. \end{aligned}$$

By proposition 3.4 we find

$$(\pi m_0 \beta)^2 = 6\left(\frac{w}{m_0}\right)\left(1 + \frac{3w}{20m_0} + O(1)\left(\frac{w}{m_0}\right)^2\right)^2 \le 6\left(\frac{w}{m_0}\right)\left(1 + \frac{2w}{m_0}\right),$$

which implies

$$(m_0 - s_{m_0}(\beta)) - (k_0 - s_{k_0}(\beta)) \le (m_0 - k_0)\frac{3w}{m_0}\left(1 + \frac{2w}{m_0}\right).$$

Using (6.7.4) one gets

$$-s'_{k_0}(\beta'') = -\frac{f'_{k_0,u}\left(\frac{\beta''}{\beta'}\right)}{\beta'} \geq \frac{2u}{\beta'}\left(1 - 225\frac{u}{k_0}\right)\frac{\beta''}{\beta'} \geq \frac{2w}{\beta'}\left(1 - 225\frac{u}{k_0}\right)\frac{\beta}{\beta'}$$

and therefore

$$\frac{\beta' - \beta}{\beta'} \leq \left[\frac{1}{2}\left(\frac{u-w}{w}\right) + \frac{3(m_0 - k_0)}{2m_0}\left(1 + \frac{2w}{m_0}\right)\right] \times \left[\left(1 - 225\frac{w}{k_0}\right)\left(1 - \frac{\beta' - \beta}{\beta'}\right)\right]^{-1}.$$

To get the first inequality we iterate some trivial estimate on $\beta - \beta'$. Start with $\beta - \beta' \leq \beta'/2$. Then we can derive an inequality of the form $\frac{\beta' - \beta}{\beta} \leq \frac{4u}{m_0}$ (see the derivation of (8.2.2) below). This in turn implies

$$\left[\left(1 - 225\frac{w}{k_0}\right)\left(1 - \frac{\beta' - \beta}{\beta'}\right)\right]^{-1} \leq \left(1 - 225\frac{w}{k_0} - \frac{4u}{m_0}\right)^{-1} \leq 1 + 250\frac{u}{k_0}.$$

This gives the first inequality. The second is similar. □

Corollary 8.2. — *We have the following inequalities.*

$$\bar{k} \geq \bar{m} - \frac{w^3}{k^2} - \left(\frac{w}{k}\right)^{1/2}. \tag{8.2.1}$$

$$\frac{\beta' - \beta}{\beta} \leq \frac{2u}{m_0}. \tag{8.2.2}$$

$$\delta \leq \left(\frac{w}{k}\right)^3 + \frac{2}{k}\left(\frac{w}{k}\right)^{1/2} + \frac{3|\bar{k} - \bar{m}|}{2k}\left(1 + 300\frac{u}{k}\right). \tag{8.2.3}$$

Proof. — For the first inequality we only need to check the case $\bar{k} < \bar{m}$. In this case $m_0 < k_0$. From proposition 8.1 and the assumption $\beta' \geq \beta$ we find

$$0 \leq \frac{1}{2}\left(\frac{u-w}{w}\right) + \frac{3(\bar{k} - \bar{m})}{2m_0}\left(1 - \frac{2w}{m_0}\right).$$

Therefore,

$$\begin{aligned}
\bar{m} - \bar{k} &\leq \frac{1}{2}\left(\frac{w-u}{w}\right)\left(\frac{3}{2m_0}(1 - o(\epsilon))\right)^{-1} \\
&= \frac{1}{3}m_0\left(\frac{w-u}{w}\right)(1 + o(\epsilon)) \\
&\leq \frac{1}{3}m_0\left(\frac{w}{k}\right)^3(1 + o(\epsilon)) + \frac{2}{3}\frac{m_0}{w}\left(\frac{w}{k}\right)^{3/2}(1 + o(\epsilon)) \quad \text{by (8.1.2)} \\
&\leq \frac{w^3}{k^2} + \left(\frac{w}{k}\right)^{1/2}.
\end{aligned}$$

Next, by proposition 8.1 and by (8.1.2) and (8.1.1) we get

$$\frac{\beta' - \beta}{\beta} \leq \left[\frac{1}{2}\left(\frac{w}{k}\right)^3 + 2w^{-1}\left(\frac{w}{k}\right)^{3/2} + \frac{3u}{2m_0}(1 + o(\epsilon))\right](1 + o(\epsilon)) \leq \frac{2u}{m_0},$$

proving (8.2.2). The third inequality follows easily from the proposition and (29). □

Lemma 8.3. — *We have the following inequalities.*

(8.3.1)
$$|d_n - c_n| \leq \frac{u^2}{m_0(\pi n)^2}.$$

(8.3.2)
$$|d_0 - c_0| \leq \frac{u^2}{3m_0}.$$

Proof. — We follow the proofs of lemma 6.6 and proposition 6.8. Let

$$\sin(k_0 z)/\sin(z) = \sum_{j=0}^{\infty} b'_j z^j$$

be a Taylor expansion around 0, and let

$$g(r) = u + \sum_{j=1}^{\infty} a'_j r^{2j}$$

be the expansion for g converging on $[1/2, 1]$. We have $a'_j = b'_{2j}(\pi\beta')^{2j}$. The identity

$$\frac{\sin(m_0 z)}{\sin(z)} - \frac{\sin(k_0 z)}{\sin(z)} = 2\sin\left(\frac{m_0 - k_0}{2}z\right)\cos\left(\frac{m_0 + k_0}{2}z\right) \bigg/ \sin(z)$$

easily implies, together with lemma 6.5, that on a circle C_{m_0} of radius $1/m_0$ we have

$$\left|\frac{\sin(m_0 z)}{\sin(z)} - \frac{\sin(k_0 z)}{\sin(z)}\right| \leq 2|m_0 - k_0|.$$

Therefore, using the Cauchy integral formula, we have

$$|b_j - b'_j| \leq 2{m_0}^j|m_0 - k_0|.$$

Recall how the constant c was defined in lemma 6.6 and used in its proof. We find

$$\begin{aligned}
|a_j - a'_j| &= |b_{2j}(\pi\beta)^{2j} - b'_{2j}(\pi\beta')^{2j}| \\
&\leq |b_{2j} - b'_{2j}|(\pi\beta)^{2j} + |b'_{2j}|\pi^{2j}|\beta'^{2j} - \beta^{2j}| \\
&\leq 2c^j|m_0 - k_0| + 2m_0 c^j\left(\frac{\beta'^{2j} - \beta^{2j}}{\beta^{2j}}\right) \\
&= 2c^j|m_0 - k_0| + 2m_0 c^j\left(\left(1 + \frac{\beta' - \beta}{\beta}\right)^{2j} - 1\right).
\end{aligned}$$

By (8.1.1) and (8.2.2) we get

$$\begin{aligned}
|a_j - a'_j| &\leq 2c^j u + 2m_0 c^j\left(u + m_0((1 + 2u/m_0)^{2j} - 1)\right) \\
&\leq 2c^j\left(u + m_0((1 + 2u/m_0)^{2j} - 1)\right).
\end{aligned}$$

When $j = 2$ we use the bound

$$\left(1 + 2\frac{u}{m_0}\right)^4 - 1 \leq 9\frac{u}{m_0}$$

to get

$$|a_2 - a_2'| \leq 20c^2u \leq 20 \cdot 6^2(1+o(\epsilon))\left(\frac{u}{m_0}\right)^2 u \leq 800\left(\frac{u}{m_0}\right)^2 u.$$

When $j > 2$ we use the bound

$$\left(1+2\frac{u}{m_0}\right)^{2j} - 1 < \left(1+2\frac{u}{m_0}\right)^{2j} = (1+o(\epsilon))^{2j}.$$

This gives

$$|a_j - a_j'| \leq 2\left(c(1+o(\epsilon))^2\right)^j (u+m_0) \leq 2 \cdot 7^j \left(\frac{u}{m_0}\right)^{j-1} u.$$

Since $f(1) = g(1) = 0$ we see that

$$\begin{aligned} |a_1 - a_1'| &\leq |w-u| + \sum_{j=2}^{\infty} |a_j - a_j'| \\ &\leq \left(\frac{w}{k}\right)^3 w + 800\left(\frac{u}{m_0}\right)^2 u + 2\cdot 7^3\left(\frac{u}{m_0}\right)^2 u \\ &\quad + \quad \text{lower order terms} \quad \leq 1500\left(\frac{u}{m_0}\right)^2 u. \end{aligned}$$

Then, imitating the proof of proposition 6.8, we find

$$\begin{aligned} |c_n - d_n| &\leq \frac{8}{(2\pi n)^2}\frac{d}{dr}\left(\sum_{j=0}^{\infty}|a_j - a_j'|r^{2j}\right)|_{r=1} \\ &\leq \frac{2}{(\pi n)^2}(2\cdot|a_1 - a_1'| + 4\cdot|a_2 - a_2'| + 6\cdot|a_3 - a_3'| + \text{l.o.t.}) \\ &\leq \frac{2}{(\pi n)^2}\left(\frac{u}{m_0}\right)^2 u\ (2\cdot 1500 + 4\cdot 800 + 6\cdot 2\cdot 7^3 + \text{l.o.t.}) \\ &\leq 21000\frac{1}{(\pi n)^2}\left(\frac{u}{m_0}\right)^2 u \leq \frac{1}{(\pi n)^2}\left(\frac{u}{m_0}\right) u. \end{aligned}$$

The second estimate is similar. □

9. The main theorems

In the last two sections we gathered many conditional estimates. It is now easy to make these absolute.

Lemma 9.1. — *We have*

$$|\bar{k} - \bar{m}| \leq 2\frac{u^2}{k} + 6u\sqrt{\delta} + 5.2\sqrt{u}.$$

Proof. — If $\bar{k} < \bar{m}$, then (8.2.1) implies that $|\bar{k} - \bar{m}| \le w^3/k^2 + (w/k)^{1/2}$, which is certainly within the bound. Otherwise we have

$$|\bar{k} - \bar{m}| = \bar{k} - \bar{m} = (\bar{k} - d_0) + (d_0 - c_0) + (c_0 - \bar{m}),$$

which, by (6.12.4),(7.4.1) and (8.3.2), is

$$\begin{aligned}
&\le \left(\frac{u^2}{k} + 6u\sqrt{\delta}\right) + \left(\frac{u^2}{3m_0}\right) + (5.2\sqrt{u}) \\
&\le 2\frac{u^2}{k} + 6u\sqrt{\delta} + 5.2\sqrt{u}.
\end{aligned}$$

□

Proposition 9.2. — *We have the inequality*

$$\sqrt{\delta} \le 10\left(\frac{u}{k}\right) + \sqrt{8}\left(\frac{u}{k}\right)^{1/4} k^{-1/4}.$$

Proof. — The last lemma, together with (8.2.3), gives the inequality

$$\begin{aligned}
\delta &\le \left(\frac{w}{k}\right)^3 + \frac{2}{k}\left(\frac{w}{k}\right)^{1/2} + \frac{3}{2k}\left(2\frac{u^2}{k} + 6u\sqrt{\delta} + 5.2\sqrt{u}\right)(1 + o(\epsilon)) \\
&\le 4\left(\frac{u}{k}\right)^2 + 9\left(\frac{u}{k}\right)(1 + o(\epsilon))\sqrt{\delta} + \frac{8\sqrt{u}}{k}.
\end{aligned}$$

Here, notice that we were able to swallow the first two terms on the first line, the $(w/k)^3$ term in the $(u/k)^2$ term and the $(2/k)(w/k)^{1/2}$ term by the $\sqrt{u}/k$ term. We treat the last inequality as a quadratic inequality in $x = \sqrt{\delta}$, which reads

$$x^2 \le 4\left(\frac{u}{k}\right)^2 + 9\left(\frac{u}{k}\right)(1 + o(\epsilon))x + \frac{8\sqrt{u}}{k}.$$

The variable x should lie between the roots of the corresponding equation,

$$x^2 - 9\left(\frac{u}{k}\right)(1 + o(\epsilon))x - \left(4\left(\frac{u}{k}\right)^2 + \frac{8\sqrt{u}}{k}\right) = 0.$$

In particular, it is smaller than the bigger root. This gives

$$\sqrt{\delta} \le \frac{1}{2}\left(\frac{9u}{k}(1 + o(\epsilon)) + \sqrt{(81 + 12)\left(\frac{u}{k}\right)^2(1 + o(\epsilon)) + \frac{32\sqrt{u}}{k}}\right).$$

Using the fact that for any two positive reals a and b we have $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, we easily get the required bound. □

Corollary 9.3. — *We have* $\delta < 200\left(\frac{u}{k}\right)^2 + \frac{16\sqrt{u}}{k}$.

Proof. — This just uses the inequality $(a + b)^2 \le 2(a^2 + b^2)$. □

We now begin to state the main theorems. Recall our assumptions from the beginning of section 7: We have $u > 1$, $k > 30000u$ and k and u satisfy assumption 5.4. We have a set $K \in \mathbb{K}_{\mu_{K_{ex}}(u)}$, written as $K = K_0 \cup \overline{K}$. Recall also that K_0 is contained inside an arithmetic progression. Let q be the difference of this sequence. By translation we can normalize things so that elements of K_0 are divisible by q. Recall that we have a compression procedure for K_0 to an arithmetic progression with at most one gap. Assume also that this progression is symmetric around 0 (as best possible). This fixes K up to shift. With these assumption we associated certain parameters k_n to K, counting roughly the number of elements near $\pm n/\beta(k_0, u)$. When $q \neq 1$ we count only those elements which are divisible by q. We have analogous parameters m_n for our "test set" K_{ex}. The theorem will show that these parameters are pretty close to each other.

Theorem 9.4. — *We have the following inequalities.*

$$m_0 + \frac{u^3}{k^2} + \frac{1}{k}\left(\frac{u}{k}\right)^{1/2} \geq k_0 \geq m_0 - 62\frac{u^2}{k} - 6\sqrt{u} - 20\left(\frac{u}{k}\right)^{1/4} k^{-1/4}u. \tag{9.4.1}$$

$$|k_n - m_n| \leq 250\frac{u^2}{k} + 11\sqrt{u} + 70\left(\frac{u}{k}\right)^{1/4} k^{-1/4}u + 4. \tag{9.4.2}$$

$$|\{a \in K : \quad |a|\beta \geq t \quad \text{or} \quad q \nmid a\}| \leq \frac{4u}{t} + tu\left(400\left(\frac{u}{k}\right)^2 + \frac{32\sqrt{u}}{k}\right), \quad \text{for} \quad t > 0. \tag{9.4.3}$$

Proof. — As discussed at the beginning of section 7, we may assume that K_0 is an arithmetic progression with at most one gap. Continue first to assume that it is a progression, has difference 1 and is symmetric around 0. We now write out the results we have so far.

The left inequality in (9.4.1) is just (8.2.1). The right inequality follows since

$$\begin{aligned}
|\bar{k} - \bar{m}| &\leq 2\frac{u^2}{k} + 6u\sqrt{\delta} + 5.2\sqrt{u} \quad \text{by lemma 9.1} \\
&\leq 2\frac{u^2}{k} + 5.2\sqrt{u} + 6u\left(10\left(\frac{u}{k}\right) + \sqrt{8}\left(\frac{u}{k}\right)^{1/4} k^{-1/4}\right) \quad \text{by proposition 9.2} \\
&\leq 62\frac{u^2}{k} + 6\sqrt{u} + 20\left(\frac{u}{k}\right)^{1/4} k^{-1/4}u.
\end{aligned}$$

We next establish

$$d_0 - \bar{k} \leq 5.5\sqrt{u} + \frac{u^2}{2k}. \tag{30}$$

Indeed

$$\begin{aligned}
\bar{k} &\geq \bar{m} - \frac{u^3}{k^2} - \left(\frac{u}{k}\right)^{1/2} \geq c_0 - 5.2\sqrt{u} - \frac{u^3}{k^2} - \left(\frac{u}{k}\right)^{1/2} \\
&\geq d_0 - \frac{u^2}{3m_0} - 5.2\sqrt{u} - \frac{u^3}{k^2} - \left(\frac{u}{k}\right)^{1/2} \geq d_0 - \frac{u^2}{2k} - 5.5\sqrt{u}
\end{aligned}$$

by (6.12.4), (8.2.1) and (8.3.2). Here again, like in the proof of proposition 9.2, we are able to swallow the terms $(u/k)^{1/2}$ and u^3/k^2. It now follows that

$$\begin{aligned}
|k_n - d_n| &\le 2(d_0 - \bar{k}) + \frac{2u^2}{k} + 24u\sqrt{\delta} \quad \text{by corollary 7.8} \\
&\le 11\sqrt{u} + \frac{u^2}{k} + \frac{2u^2}{k} + 24u\sqrt{\delta} \quad \text{by (30)} \\
&\le 11\sqrt{u} + \frac{3u^2}{k} + 24u\left(10\left(\frac{u}{k}\right) + \sqrt{8}\left(\frac{u}{k}\right)^{1/4} k^{-1/4}\right) \quad \text{by proposition 9.2} \\
&\le 243\frac{u^2}{k} + 11\sqrt{u} + 70\left(\frac{u}{k}\right)^{1/4} k^{-1/4}u.
\end{aligned}$$

From (8.3.2) we get

$$|d_n - c_n| \le \frac{u^2}{m_0\pi^2} \le \frac{u^2}{9k}.$$

By the definition of the m_n in (6.1) (taking into account the possible modifications at the proof of (6.12)), we see that $|c_n - m_n| \le 4$. This gives (9.4.2).

To get (9.4.3) we integrate the inequality of proposition 7.2 on the interval $[1-\frac{1}{t}, 1]$. Recalling that there is a subset of size δ on which the inequality fails to hold, and on which bound $\bar{k} - \operatorname{Re} S_{\overline{K}}$ trivially by $2\bar{k} \le u$, we get

$$\int_{1-\frac{1}{t}}^{1} (\bar{k} - \operatorname{Re} S_{\overline{K}}(\beta r)) \le \int_{1-\frac{1}{t}}^{1} g(r)\left(1 + \frac{u}{k}\right) dr + u\delta. \tag{31}$$

The left hand side is

$$\int_{1-\frac{1}{t}}^{1} (\bar{k} - \operatorname{Re} S_{\overline{K}}(\beta r))\, dr = \sum_{a\in\overline{K}} \int_{1-\frac{1}{t}}^{1} (1 - \cos(2\pi a\beta r))\, dr.$$

One finds that if $b \ge t$, then

$$\begin{aligned}
\int_{1-\frac{1}{t}}^{1} (1-\cos(2\pi b r))\, dr &= \frac{1}{t} - \frac{1}{2\pi b}\left(\sin\left(2\pi b\left(1-\frac{1}{t}\right)\right) - \sin(2\pi b)\right) \\
&\ge \frac{1}{t} - \frac{2}{2\pi b} \ge \frac{1}{2t}.
\end{aligned}$$

It follows that the left hand side of (31) is greater than

$$\frac{1}{2t}|\{a \in K : \quad |a|\beta \ge t\}|.$$

Using (6.7.5) the right hand side of (31) is smaller than

$$\left(1 + 225\frac{u}{k_0}\right) u \int_0^{\frac{1}{t}} 2r\, dr + u\delta \le \frac{2u}{t^2} + u\delta.$$

Therefore,

$$\frac{1}{2t}|\{a \in K : \quad |a|\beta \ge t\}| \le \frac{2u}{t^2} + \frac{1}{2}u\delta.$$

Thus,

$$|\{a \in K : \quad |a|\beta \geq t\}| \leq \frac{4u}{t} + 2u\delta,$$

which gives the result after substituting the estimate for δ given in corollary 9.3.

To see that the everything continues to hold for the case $q > 1$, we consider the structure of the proof and see what modifications one must make. Most of the time we have been integrating certain identities on certain intervals. Whenever there is an integral on some interval I involved for the case $q = 1$, take the corresponding integral on the set $\langle q\rangle^{-1}(I)$ and make the obvious adaptations. It is easy to see that all terms in the integration which involve elements that are not congruent to $0 \mod q$ vanish. There was one place, in deriving (28), where we used the value of the sum at a certain point. But there note that elements congruent to $0 \mod q$ behave the same on all intervals and the claim made there about the existence of δ' is certainly true for at least one interval. With these remarks the proof goes through unchanged. □

Now comes a second "compression argument". It is based on the following lemma.

Lemma 9.5. — *Let x_1, x_2, y_1, y_2 and z be vectors in an Euclidean space with scalar product "$\cdot$" and norm $|\ |$, such that $|x_1| = |x_2| = |y_1| = |y_2| = 1$. Let $x = x_1 + x_2$, $y = y_1 + y_2$. Put $x_1 \cdot x_2 = \cos\theta_1$, $y_1 \cdot y_2 = \cos\theta_2$, $x \cdot z = (\cos\theta)|z|$. If*

$$\cos\theta_1 - \cos\theta_2 > 4(1 - \cos\theta),$$

then

$$|x + z| > |y + z|.$$

Proof

$$x \cdot x - y \cdot y = 2 + 2x_1 \cdot x_2 - 2 - 2y_1 \cdot y_2 = 2(\cos\theta_1 - \cos\theta_2) > 8(1 - \cos\theta).$$

Since $|x| + |y| < 4$,

$$|x| - |y| > 2(1 - \cos\theta) \geq (1 - \cos\theta)|x|,$$

and therefore $|x|\cos\theta > |y|$. Thus

$$|z + x|^2 - |z + y|^2 \geq |x|^2 - |y|^2 + 2|z|(|x|\cos\theta - |y|) > 0.$$

□

Corollary 9.6. — *Suppose K is as before, with K_0 and arithmetic progression symmetric around 0. Suppose a, $b \in K$, $q|a$ and $q|b$, and a and b satisfy the inequalities*

$$\beta^{-1} - 2k \geq |a - b|/q \geq 2k.$$

Then, if K' is the set obtained from K by replacing a and b with two elements c and d on the sides of K_0, i.e., $c \approx k_0/2$ and $d = -c$, we get $\mu_{K'}(u) \geq \mu_K(u)$.

Proof. — Assume again that $q = 1$ for simplicity. Observe first that if $\alpha = \beta_{k_0,u} r$, with $r < \frac{1}{\sqrt{6}}$, then

$$|S_{K_0}(\alpha)| - \bar{k} \approx k_0 - ur^2 - \bar{k} \approx k - \frac{5}{6}u - ur^2 > k - \frac{5}{6}u - u\frac{1}{\sqrt{6}}^2 = k - u,$$

because $\bar{k} \approx \frac{5}{12}u$. This implies that such an α belongs to both $E_{K,u}$ and $E_{K',u}$.

Now assume $\alpha > \frac{1}{10}\beta_{k_0,u}$. Set $x_1 = e^{2\pi i c\alpha}$, $x_2 = e^{2\pi i d\alpha}$, $y_1 = e^{2\pi i a\alpha}$, $y_2 = e^{2\pi i b\alpha}$ and $z = S_K(\alpha) - (y_1 + y_2)$, so that $z + y$, $z + x$ are the values of $S_K(\alpha)$ and $S_{K'}(\alpha)$ respectively. In our case the conditions of the lemma hold. To see this, notice that we have

$$4\pi k\alpha \leq 2\pi(a-b)\alpha \leq 2\pi(a-b)\beta \leq 2\pi - 4\pi k\beta \leq 2\pi - 4\pi k\alpha.$$

It follows that $\cos(\theta_2) = \cos(2\pi(a-b)\alpha) \leq \cos(4\pi k\alpha)$. Therefore,

$$\begin{aligned}\cos\theta_1 - \cos\theta_2 &\geq \cos(\theta_1) - \cos(4\pi k\alpha) \approx \cos(2\pi k\alpha) - \cos(4\pi k\alpha)\\ &\approx (4\pi\alpha k)^2 - (2\pi\alpha k)^2 = 3(2\pi\alpha k)^2 \geq 3\left(2\pi\frac{1}{10}\beta_{k_0,u}k\right)^2 \geq \frac{u}{2k}.\end{aligned}$$

On the other hand, since x is on the real line, θ is almost the angle between S_K and the real line so

$$|\sin\theta| = \left|\frac{\operatorname{Im} S_K}{\operatorname{Re} S_K}\right| < \frac{\frac{1}{2}u}{k - \frac{3}{2}u} < \frac{u}{k}$$

and therefore $1 - \cos\theta \leq \sin^2\theta \leq (u/k)^2$. Since the lemma applies we see that $\alpha \in E_{K,u}$ implies $\alpha \in E_{K',u}$, hence the result. □

The following two theorems are immediate consequences.

Theorem 9.7. — *Suppose $K \in \mathbb{K}_{\mu_{K_{ex}}(u)}$. Then around each of $\pm n/\beta$, $n \neq 0$, there exists an interval of length $2kq$ such that the total number of elements of $\bigcup_{n=1}^{\infty}(K_n \cup K_{-n})$ not in any of the intervals is $\leq 63\frac{u^2}{k} + 7\sqrt{u} + 20\left(\frac{u}{k}\right)^{1/4}k^{-1/4}u$.*

Proof. — Indeed, corollary 9.6 implies that any two elements in any K_n that are of distance $\geq 2kq$ apart can be "pushed" into K_0, increasing $\mu_K(u)$ and in particular keeping the set in $\mathbb{K}_{\mu_{K_{ex}}(u)}$. But by (9.4.1) the number of elements one can move into K_0 keeping K in $\mathbb{K}_{\mu_{K_{ex}}(u)}$ is $\leq 63\frac{u^2}{k} + 7\sqrt{u} + 20\left(\frac{u}{k}\right)^{1/4}k^{-1/4}u$ (again we have swallowed some terms). □

Theorem 9.8. — *The following is a $G_{\mu_{K_{ex}}(u)}$ sub-collection of $\mathbb{K}_{\mu_{K_{ex}}(u)}$. It consists of all sets in which K_0 is an arithmetic progression with at most one gap, and where each of the sets K_n, for $n \neq 0$, is contained in an interval of length $2kq$.*

Proof. — Immediate from corollary 9.6. □

We end with an improved bound on $\mu_{\max}$.

Theorem 9.9. — *We have the following inequalities*

$$\mu_{\max}(k,u) \geq \frac{2\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}\left(1+\left(\frac{3}{20}+\frac{5}{8}\right)\frac{u}{k}-300\left(\frac{u}{k}\right)^2-\frac{8\sqrt{u}}{k}\right)$$
$$\mu_{\max}(k,u) \leq \frac{2\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}\left(1+\left(\frac{3}{20}+\frac{5}{8}\right)\frac{u}{k}+400\left(\frac{u}{k}\right)^2+30\left(\frac{u}{k}\right)^{5/4}k^{-1/4}\right).$$

Proof. — We clearly have $2\beta_{m_0,w} \leq \mu_{\max}(k,u) \leq 2\beta_{k_0,u}$, so the theorem is about estimating the terms on both sides. We show the second inequality only. By (22) and proposition 8.1,

$$\mu_{\max}(k,u) \leq 2\beta_{k_0,u} \leq 2\beta_{k,u}\left[1+\frac{3(k-k_0)}{2k}\left(1+\frac{2u}{k}\right)\left(1+250\frac{u}{k_0}\right)\right].$$

By proposition 3.4,

$$\beta_{k,u} \leq \frac{\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}\left(1+\frac{3u}{20k}+\left(\frac{u}{k}\right)^2\right).$$

By (7.4.1),

$$k-k_0=\bar{k}\leq d_0+\frac{u^2}{k}+6u\sqrt{\delta}.$$

It follows from (6.8.1) that

$$d_0 \leq \frac{5}{12}u+75\frac{u^2}{k_0}\leq\frac{5}{12}u+79\frac{u^2}{k}.$$

Hence, using proposition 9.2 we find

$$\begin{aligned} k-k_0 &\leq \frac{5}{12}u+80\frac{u^2}{k}+6u\left(10\left(\frac{u}{k}\right)+\sqrt{8}\left(\frac{u}{k}\right)^{1/4}k^{-1/4}\right)\\ &\leq \frac{5}{12}u+140\frac{u^2}{k}+17u\left(\frac{u}{k}\right)^{1/4}k^{-1/4}. \end{aligned}$$

It follows that

$$\begin{aligned} &\left(1+\frac{3u}{20k}+\left(\frac{u}{k}\right)^2\right)\left[1+\frac{3}{2k}(k-k_0)\left(1+\frac{2u}{k}\right)\left(1+250\frac{u}{k_0}\right)\right]\\ &\leq 1+\left(\frac{3}{20}+\frac{5}{8}\right)\frac{u}{k}+400\left(\frac{u}{k}\right)^2+30\left(\frac{u}{k}\right)^{5/4}k^{-1/4}. \end{aligned}$$

The upper bound is now clear. □

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