MELVYN B. NATHANSON
GÉRALD TENENBAUM

Inverse theorems and the number of sums and products

Astérisque, tome 258 (1999), p. 195-204
<http://www.numdam.org/item?id=AST_1999__258__195_0>

Astérisque **258**, 1999, p. 195–204

INVERSE THEOREMS AND THE NUMBER OF SUMS AND PRODUCTS

by

Melvyn B. Nathanson & Gérald Tenenbaum

Abstract. — Let $\epsilon > 0$. Erdős and Szemerédi conjectured that if A is a set of k positive integers which large k, there must be at least $k^{2-\varepsilon}$ integers that can be written as the sum or product of two elements of A. We shall prove this conjecture in the special case that the number of sums is very small.

1. A conjecture of Erdős and Szemerédi

Let A be a nonempty, finite set of positive integers, and let $|A|$ denote the cardinality of the set A. Let

$$2A = \{a + a' : a, a' \in A\}$$

denote the 2-fold *sumset* of A, and let

$$A^2 = \{aa' : a, a' \in A\}$$

denote the 2-fold *product set* of A. We let

$$E_2(A) = 2A \cup A^2$$

denote the set of all integers that can be written as the sum or product of two elements of A. If $|A| = k$, then

$$|2A| \leqslant \binom{k+1}{2}$$

and

$$|A^2| \leqslant \binom{k+1}{2},$$

1991 ***Mathematics Subject Classification.*** — Primary 11B05, 11B13, 11B75, 11P99, 05A17.
Key words and phrases. — Additive number theory, sumsets, product sets, inverse theorems, Freiman's theorem, sums and products of integers, divisors.

M.B. N.: This work was supported in part by grants from the PSC-CUNY Research Award Program and the National Security Agency Mathematical Sciences Program.

and so the number of sums and products of two elements of A is

$$|E_2(A)| \leqslant k^2 + k.$$

Erdős and Szemerédi [**3**, p. 60] made the beautiful conjecture that a finite set of positive integers cannot have simultaneously few sums and few products. More precisely, they conjectured that for every $\varepsilon > 0$ there exists an integer $k_0(\varepsilon)$ such that, if A is a finite set of positive integers and

$$|A| = k \geqslant k_0(\varepsilon),$$

then

$$|E_2(A)| \gg_\varepsilon k^{2-\varepsilon}.$$

Very little is known about this question. Erdős and Szemerédi [**4**] have shown that there exists a real number $\delta > 0$ such that

$$|E_2(A)| \gg k^{1+\delta},$$

and Nathanson [**11**] proved that

$$|E_2(A)| \geqslant ck^{32/31},$$

where $c = 0.00028\ldots$.

Erdős and Szemerédi [**4**] also remarked that, in the special case that $|2A| \leqslant ck$, "perhaps there are more than $k^2/(\log k)^\varepsilon$ elements in A^2 ". This cannot be true for arbitrary finite sets of positive integers and arbitrarily small $\varepsilon > 0$. For example, if A is the set of all integers from 1 to k, then Tenenbaum [**16, 17**], improving a result of Erdős [**2**], proved that

$$\frac{k^2}{(\log k)^{\varepsilon_0}} \mathrm{e}^{-c\sqrt{\log_2 k \log_3 k}} \ll |A^2| \ll \frac{k^2}{(\log k)^{\varepsilon_0}\sqrt{\log_2 k}}, \tag{1}$$

where $\log_r$ denotes the r-fold iterated logarithm, and

$$\varepsilon_0 = 1 - \Big(\frac{1+\log_2 2}{\log 2}\Big) \geqslant 0.08607 \tag{2}$$

(cf. Hall and Tenenbaum [**8**, Theorem 23]).

Using an inverse theorem of Freiman, we shall prove that if A is a set of k positive integers such that $|2A| \leqslant 3k-4$, then

$$|A^2| \gg (k/\log k)^2.$$

We obtain a similar result for the sumset and product set of two possibly different sets of integers. Let A_1 and A_2 be nonempty, finite sets of positive integers, and let

$$A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$$

and

$$A_1A_2 = \{a_1a_2 : a_1 \in A_1, a_2 \in A_2\}.$$

Let $|A_1| = |A_2| = k$. We prove that whenever $|A_1 + A_2| \leqslant 3k-4$, then we have $|A_1A_2| \gg (k/\log k)^2$.

2. Product sets of arithmetic progressions

A set Q of positive integers is an *arithmetic progression* of length ℓ and difference q if there exist positive integers r, q, and ℓ such that

$$Q = \{r + uq : 0 \leqslant u < \ell\}.$$

We shall always assume that

$$\ell \geqslant 2.$$

For any sets A and B of positive integers, let $\varrho_{A,B}(m)$ denote the number of representations of m in the form $m = ab$, where $a \in A$ and $b \in B$. Let $\varrho_A(m) = \varrho_{A,A}(m)$. Let $\tau(m)$ denote the number of positive divisors of m. Clearly, for every integer m,

$$\varrho_{A,B}(m) \leqslant \tau(m).$$

If $A_1 \subseteq Q_1$ and $A_2 \subseteq Q_2$, then $\varrho_{A_1,A_2}(m) \leqslant \varrho_{Q_1,Q_2}(m)$.

***Lemma 1* (Shiu).** — *Let $0 < \alpha < 1/2$ and let $0 < \beta < 1/2$. Let x and y be real numbers and let s and q be integers such that*

$$0 < s \leqslant q \text{ and } (s, q) = 1, \tag{3}$$

$$q < y^{1-\alpha}, \tag{4}$$

and

$$x^{\beta} < y \leqslant x. \tag{5}$$

Then

$$\sum_{\substack{w \equiv s \pmod q \\ x - y < w \leqslant x}} \tau(w) \ll_{\alpha,\beta} \frac{\varphi(q) y \log x}{q^2}.$$

Proof. This is a special case of Theorem 2 in Shiu [**14**] (see also Vinogradov and Linnik [**18**] and Barban and Vehov [**1**]).

Lemma 2. — *Let s, q, h, and ℓ be integers such that $h \geqslant 0$, $\ell \geqslant 2$, $0 < s \leqslant q$, and $(s, q) = 1$. Let Q be the arithmetic progression*

$$Q = \{s + vq : h \leqslant v < h + \ell\}.$$

If $(h+1)q < \ell^5$, then

$$\sum_{w \in Q} \tau(w) \ll \ell \log \ell.$$

Proof. We apply Lemma 1 with $\alpha = \beta = 1/6$, $x = (h + \ell)q$, and $y = \ell q$. The integers s and q satisfy (3). Since $q \leqslant (h + 1)q < \ell^5$, we have $q^{1/6} < \ell^{5/6}$, and so

$$q = q^{1/6} q^{5/6} < (\ell q)^{5/6} = y^{1-\alpha}.$$

This shows that (4) is satisfied.

To obtain (5), we consider two cases. If $h \leqslant \ell$, then, since $2 \leqslant \ell \leqslant \ell q$, we have

$$x^\beta = ((h+\ell)q)^\beta \leqslant (2\ell q)^\beta \leqslant (\ell q)^{2\beta} = (\ell q)^{1/3} < \ell q = y \leqslant x.$$

If $h > \ell$, then, since $hq < \ell^5$, we have

$$x^\beta = \{(h+\ell)q\}^\beta < (\ell h q)^\beta < \ell^{6\beta} = \ell \leqslant \ell q = y \leqslant x.$$

This shows that (5) holds.

Applying Lemma 1, we obtain

$$\begin{aligned}\sum_{w\in Q} \tau(w) &= \sum_{\substack{w\equiv s \,(\mathrm{mod}\, q)\\ hq<w\leqslant(h+\ell)q}} \tau(w) \ll \frac{\varphi(q)(\ell q)\log((h+\ell)q)}{q^2}\\ &\ll \ell\log(\ell(h+1)q) \ll \ell\log\ell^6 \ll \ell\log\ell.\end{aligned}$$

This completes the proof.

Lemma 3. — *Let Q_1 and Q_2 be two arithmetic progressions of length $\ell \geqslant 2$, and let $m \in Q_1Q_2$. Then*

$$\varrho_{Q_1,Q_2}(m) \ll_\varepsilon \ell^\varepsilon \tag{6}$$

for every $\varepsilon > 0$, and

$$\sum_{m\in Q_1Q_2} \varrho_{Q_1,Q_2}(m)^2 \ll (\ell\log\ell)^2. \tag{7}$$

Proof. Let $Q_i = \{r_i + uq_i : 0 \leqslant u < \ell\}$ for $i = 1, 2$. We may assume without loss of generality that $(r_i, q_i) = 1$. We write $r_i = s_i + h_i q_i$, where $0 < s_i \leqslant q_i$ and $h_i \geqslant 0$. Then

$$Q_i = \{s_i + vq_i : h_i \leqslant v < h_i + \ell\}.$$

If $w_1 \in Q_1$ and $w_2 \in Q_2$, then, for suitable $v_1 \in [h_1, h_1+\ell[$, $v_2 \in [h_2, h_2+\ell[$, we have

$$h_1q_1 < w_1 = s_1 + v_1q_1 \leqslant (h_1+\ell)q_1 \leqslant \ell(h_1+1)q_1 \tag{8}$$

and

$$h_2q_2 < w_2 = s_2 + v_2q_2 \leqslant (h_2+\ell)q_2 \leqslant \ell(h_2+1)q_2. \tag{9}$$

We can assume that

$$(h_2+1)q_2 \leqslant (h_1+1)q_1.$$

There are two cases. In the first case,

$$(h_1+1)q_1 < \ell^5.$$

By (8) and (9), we deduce that

$$w_1 \leqslant \ell(h_1+1)q_1 < \ell^6, \quad \text{and} \quad w_2 \leqslant \ell(h_2+1)q_2 \leqslant \ell(h_1+1)q_1 < \ell^6.$$

If $m \in Q_1Q_2$, then m is of the form $m = w_1w_2$, and so $m < \ell^{12}$. Since, by a classical estimate, $\tau(m) \ll_\varepsilon m^{\varepsilon/12}$, it follows that

$$\varrho_{Q_1,Q_2}(m) \leqslant \tau(m) \ll_\varepsilon m^{\varepsilon/12} \ll_\varepsilon \ell^\varepsilon.$$

This proves (6).

To prove (7), we use the submultiplicativity of the divisor function, that is, $\tau(uv) \leqslant \tau(u)\tau(v)$ for all positive integers u, v. Then

$$\begin{aligned}
\sum_{m\in Q_1Q_2} \varrho_{Q_1,Q_2}(m)^2 &= \sum_{w_1\in Q_1}\sum_{w_2\in Q_2} \varrho_{Q_1,Q_2}(w_1w_2) \\
&\leqslant \sum_{w_1\in Q_1}\sum_{w_2\in Q_2} \tau(w_1w_2) \\
&\leqslant \sum_{w_1\in Q_1} \tau(w_1) \sum_{w_2\in Q_2} \tau(w_2) \ll \ell^2(\log \ell)^2,
\end{aligned}$$

where the last upper bound follows from Lemma 2.

Consider now the second case

$$(h_1+1)q_1 \geqslant \ell^5.$$

We shall prove that

$$\varrho_{Q_1,Q_2}(m) \leqslant 3 \tag{10}$$

for all $m \geqslant 1$. Suppose that $w_1 = r_1 + uq_1 \in Q_1$ and $w_1' = r_1 + u'q_1 \in Q_1$ are distinct divisors of m, and that $w_1 < w_1'$. Then $(r_1, q_1) = 1$ implies that $(w_1, q_1) = (w_1', q_1) = 1$, and so $((w_1, w_1'), q_1) = 1$. Since (w_1, w_1') divides

$$w_1' - w_1 = (u' - u)q_1,$$

it follows that (w_1, w_1') divides $u' - u$, and so

$$1 \leqslant (w_1, w_1') \leqslant u' - u < \ell.$$

Suppose that $\varrho_{Q_1,Q_2}(m) \geqslant 4$. Then m has at least four distinct representations in the form $m = w_1w_2$ with $w_1 \in Q_1$ and $w_2 \in Q_2$, and so m has at least four different divisors in Q_1, that is, at least four divisors of the form

$$r_1 + uq_1 = s_1 + (h_1 + u)q_1$$

with $0 \leqslant u < \ell$. At most one of these divisors is $s_1 + h_1q_1$, and so m has at least three different divisors, which we shall denote by w_1, w_1', and w_1'', such that

$$\min\{w_1, w_1', w_1''\} \geqslant s_1 + (h_1+1)q_1 > (h_1+1)q_1 \geqslant \ell^5.$$

Let $[w_1, w_1', w_1'']$ denote the least common multiple of w_1, w_1', and w_1''. Since each of these three numbers is a divisor of m, we have

$$\begin{aligned}
m &\geqslant [w_1, w_1', w_1''] \geqslant \frac{w_1w_1'w_1''}{(w_1, w_1')(w_1, w_1'')(w_1', w_1'')} \\
&> \left(\frac{(h_1+1)q_1}{\ell}\right)^3 = \frac{(h_1+1)q_1}{\ell^3}(h_1+1)^2q_1^2 \\
&\geqslant \ell^2\Big((h_1+1)q_1\Big)^2 \geqslant \ell(h_1+1)q_1 \cdot \ell(h_2+1)q_2 \geqslant w_1w_2 = m,
\end{aligned}$$

which is impossible. This proves (10), and inequalities (6) and (7) follow immediately.

Lemma 4. — *Let Q be an arithmetic progression of length $\ell \geqslant 2$, and let $m \in Q^2$. Then*

$$\varrho_Q(m) \ll_\varepsilon \ell^\varepsilon \tag{11}$$

for every $\varepsilon > 0$, and

$$\sum_{m \in Q^2} \varrho_Q(m)^2 \ll (\ell \log \ell)^2. \tag{12}$$

Proof. This follows immediately from Lemma 3 with $Q_1 = Q_2 = Q$.

Lemma 5. — *Let Q_1 and Q_2 be arithmetic progressions of length $\ell \geqslant 2$. Then*

$$|Q_1 Q_2| \gg \Big(\frac{\ell}{\log \ell}\Big)^2.$$

Proof. Let $\varrho_{Q_1,Q_2}(m)$ denote the number of representations of m in the form $m = q_1 q_2$, where $q_1 \in Q_1$ and $q_2 \in Q_2$. By the Cauchy-Schwarz inequality and inequality (7) of Lemma 3,

$$\begin{aligned} \ell^2 &= \sum_{m \in Q_1 Q_2} \varrho_{Q_1,Q_2}(m) \leqslant |Q_1 Q_2|^{1/2} \Big(\sum_{m \in Q_1 Q_2} \varrho_{Q_1,Q_2}(m)^2 \Big)^{1/2} \\ &\ll |Q_1 Q_2|^{1/2} \ell \log \ell. \end{aligned}$$

Therefore,

$$|Q_1 Q_2| \gg \Big(\frac{\ell}{\log \ell}\Big)^2.$$

This completes the proof.

Lemma 6. — *Let Q be an arithmetic progression of length $\ell \geqslant 2$. Then*

$$|Q^2| \gg \Big(\frac{\ell}{\log \ell}\Big)^2.$$

Proof. This follows immediately from Lemma 5 with $Q_1 = Q_2 = Q$.

3. Application of some inverse theorems

We shall use the following two inverse theorems of Freiman.

***Lemma 7* (Freiman).** — *Let A be a nonempty set of k positive integers. If*

$$|2A| \leqslant 3k - 4,$$

then A is a subset of an arithmetic progression of length $\ell < 2k$.

Proof. See **[5, 7, 10, 12]**.

***Lemma 8* (Freiman).** — *Let A_1 and A_2 be nonempty finite sets of positive integers, and let $|A_i| = k_i$ for $i = 1, 2$. If*

$$|A_1 + A_2| \leqslant k_1 + k_2 + \min\{k_1, k_2\} - 4,$$

then A_1 and A_2 are subsets of arithmetic progressions Q_1 and Q_2, respectively, where Q_1 and Q_2 have the same difference and the same length $\ell < k_1 + k_2$.

Proof. See **[6, 9, 12, 15]**.

Theorem 1. — *Let A be a finite set of positive integers, and let $|A| = k \geqslant 2$. If*

$$|2A| \leqslant 3k - 4,$$

then

$$|A^2| \gg \left(\frac{k}{\log k}\right)^2.$$

Proof. By Lemma 7, if $|2A| \leqslant 3k - 4$, then there exists an arithmetic progression Q of length $\ell < 2k$ such that $A \subseteq Q$. Since

$$\varrho_A(m) \leqslant \varrho_Q(m),$$

it follows from (12) that

$$\begin{aligned} k^2 &= \sum_{m \in A^2} \varrho_A(m) \leqslant |A^2|^{1/2} \Big(\sum_{m \in A^2} \varrho_A(m)^2 \Big)^{1/2} \\ &\leqslant |A^2|^{1/2} \Big(\sum_{m \in Q^2} \varrho_Q(m)^2 \Big)^{1/2} \\ &\ll |A^2|^{1/2} \ell \log \ell \ll |A^2|^{1/2} k \log k. \end{aligned}$$

Therefore,

$$|A^2| \gg \left(\frac{k}{\log k}\right)^2. \tag{13}$$

This completes the proof.

Theorem 2. — *Let $\lambda \geqslant 1$. Let A_1 and A_2 be finite sets of positive integers such that $|A_i| = k_i \geqslant 2$ for $i = 1, 2$ and*

$$\frac{1}{\lambda} \leqslant \frac{k_2}{k_1} \leqslant \lambda. \tag{14}$$

If

$$|A_1 + A_2| \leqslant k_1 + k_2 + \min\{k_1, k_2\} - 4,$$

then

$$|A_1 A_2| \gg_\lambda \frac{k_1 k_2}{\Big(\log(k_1 k_2)\Big)^2}.$$

Proof. It follows from (14) that

$$(k_1 + k_2)^2 \leqslant (1+\lambda)^2 k_1^2 = (1+\lambda)^2 \lambda k_1 (k_1/\lambda) \leqslant (1+\lambda)^2 \lambda k_1 k_2,$$

and so

$$k_1 + k_2 \ll_\lambda (k_1 k_2)^{1/2}.$$

By Lemma 8, if $|A_1+A_2| \leqslant k_1+k_2+\min\{k_1,k_2\}-4$, there exist arithmetic progressions Q_1 and Q_2, each of length $\ell < k_1 + k_2$, such that $A_1 \subseteq Q_1$ and $A_2 \subseteq Q_2$. Since

$$\varrho_{A_1,A_2}(m) \leqslant \varrho_{Q_1,Q_2}(m),$$

it follows from (7) that

$$\begin{aligned} k_1 k_2 &= \sum_{m \in A_1 A_2} \varrho_{A_1,A_2}(m) \\ &\leqslant |A_1A_2|^{1/2} \Big(\sum_{m \in A_1 A_2} \varrho_{A_1,A_2}(m)^2 \Big)^{1/2} \\ &\leqslant |A_1A_2|^{1/2} \Big(\sum_{m \in Q_1 Q_2} \varrho_{Q_1,Q_2}(m)^2 \Big)^{1/2} \\ &\ll |A_1A_2|^{1/2} \ell \log \ell \ll |A_1A_2|^{1/2} (k_1+k_2) \log(k_1+k_2) \\ &\ll_\lambda |A_1A_2|^{1/2} (k_1k_2)^{1/2} \log(k_1k_2). \end{aligned}$$

Therefore,

$$|A_1A_2| \gg_\lambda \frac{k_1k_2}{\big(\log(k_1k_2)\big)^2}. \tag{15}$$

This completes the proof.

Theorem 3. — *Let A_1 and A_2 be finite sets of positive integers such that $|A_1| = |A_2| = k \geqslant 2$. If*

$$|A_1 + A_2| \leqslant 3k - 4,$$

then

$$|A_1A_2| \gg \Big(\frac{k}{\log k}\Big)^2.$$

Proof. This follows immediately from Theorem 2 with $k_1 = k_2 = k$ and $\lambda = 1$.

4. Open problems

By Theorem 1, if $|A| = k$ and $|2A| \leqslant 3k-4$, then $|A^2| \gg k^{2-\varepsilon}$. This gives the first general case in which we know that the conjecture of Erdős and Szemerédi is true. It would be nice to prove that if $c \geqslant 3$ and if A is a finite set of k positive integers such that

$$|2A| \leqslant ck, \tag{16}$$

then

$$|A^2| \gg_{c,\varepsilon} k^{2-\varepsilon}.$$

By a general inverse theorem of Freiman **[7, 12, 13]**, a finite set of integers whose sumset satisfies inequality (16) is a "large" subset of what is called an n-dimensional arithmetic progression. This is a set Q with the following structure: For $n \geqslant 1$, there exist positive integers $r, q_1, \ldots, q_n, \ell_1, \ldots, \ell_n$ such that

$$Q = \{r + u_1 q_1 + \cdots + u_n q_n : 0 \leqslant u_i < \ell_i \text{ for } i = 1, \ldots, n\}. \tag{17}$$

The *length* of Q is defined as $\ell(Q) = \ell_1 \cdots \ell_n$. Clearly,

$$|Q| \leqslant \ell(Q)$$

for every n-dimensional arithmetic progression. Freiman's inverse theorem should be applicable to the Erdős-Szemerédi conjecture for sets satisfying the additive condition (16).

Let Q be an n-dimensional arithmetic progression of the form (17). If j is such that $\ell_j = \max\{\ell_i : i = 1, \ldots, n\}$ in (17), then

$$Q \supseteq Q_j = \{r + u_j q_j : 0 \leqslant u_j < \ell_j\}.$$

It follows from Lemma 6 that

$$|Q^2| \geqslant |Q_j^2| \gg \left(\frac{\ell_j}{\log \ell_j}\right)^2. \tag{18}$$

The following example shows that this inequality is almost best possible. Fix $n \geqslant 2$. For $\ell \geqslant 2$, consider the n-dimensional arithmetic progression Q with $r = 1$, $q_i = i$ and $\ell_i = \ell$ for $i = 1, \ldots, n$. Then

$$Q = \left\{1 + \textstyle\sum_{i=1}^n i u_i : 0 \leqslant u_i < \ell\right\} \subseteq \left[1, 1 + \tfrac{1}{2} n(n+1)(\ell - 1)\right] \subseteq [1, n^2 \ell].$$

We apply the lower bound (18) with $\ell = \max\{\ell_i : i = 1, \ldots, n\}$, and we apply the upper bound (1) with $k = n^2 \ell$. For sufficiently large ℓ we obtain

$$\left(\frac{\ell}{\log \ell}\right)^2 \ll |Q^2| \ll \frac{k^2}{(\log k)^{\varepsilon_0}} \ll_n \frac{\ell^2}{(\log \ell)^{\varepsilon_0}},$$

where ε_0 is defined by (2). Since $\ell(Q) = \ell^n$, it is clearly not true that

$$|Q^2| \gg_{n,\varepsilon} \ell(Q)^{2-\varepsilon}.$$

It would be interesting to obtain sufficient conditions for an n-dimensional arithmetic progression Q to satisfy

$$|Q^2| \gg_{n,\varepsilon} |Q|^{2-\varepsilon}.$$

Let A be a set of k positive integers. For $h \geqslant 3$, let $E_h(A)$ denote the set of all numbers that can be written as the sum or product of h elements of A. Erdős and Szemerédi [4] also conjectured that

$$|E_h(A)| \gg_\varepsilon k^{h-\varepsilon}$$

for all $\varepsilon > 0$. Nothing is known about this.

References

[1] Barban M. B. and Vehov P. P., *Summation of multiplicative functions of polynomials* (in Russian), Mat. Zametki, 5(6):669–680, 1969. English translation: Math. Notes Acad. Sci. USSR, **5**, 1969, 400–407.

[2] Erdős P., *An asymptotic inequality in the theory of numbers* (in Russian), Vestnik Leningrad Univ., Serija Mat. Mekh. i Astr., **15(13)**, 1960, 41–49.

[3] Erdős P., *Problems and results on combinatorial number theory III*, In M. B. Nathanson, editor, Number Theory Day, New York 1976, volume 626 of Lecture Notes in Mathematics, Berlin, Springer-Verlag, 1977, 43–72.

[4] Erdős P. and Szemerédi E., *On sums and products of integers*, In P. Erdős, L. Alpár, G. Halász, and A. Sárközy, editors, Studies in Pure Mathematics, To the Memory of Paul Turán, Birkhäuser Verlag, Basel, 1983, 213–218.

[5] Freiman G. A., *On the addition of finite sets. I*, Izv. Vysh. Zaved. Matematika, **13(6)**, 1959, 202–213.

[6] Freiman G. A., *Inverse problems of additive number theory. VI on the addition of finite sets. III* Izv. Vysh. Ucheb. Zaved. Matematika, **28(3)**, 1962, 151–157.

[7] Freiman G. A., *Foundations of a Structural Theory of Set Addition*, volume 37 of Translations of Mathematical Monographs, American Mathematical Society, Providence, 1973.

[8] Hall R. R. and Tenenbaum G., *Divisors*, Number 90 in Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1988.

[9] Lev V. F. and Smeliansky P. Y., *On addition of two different sets of integers*, Preprint, 1994.

[10] Nathanson M. B., *The simplest inverse problems in additive number theory*, In A. Pollington and W. Moran, editors, *Number Theory with an Emphasis on the Markoff Spectrum*, Marcel Dekker, 1993, 191–206.

[11] Nathanson M. B., *On sums and products of integers*, submitted, 1994.

[12] Nathanson M. B., *Additive Number Theory: 2. Inverse Theorems and the Geometry of Sumsets*, Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, to appear.

[13] Ruzsa I. Z., *Generalized arithmetic progressions and sumsets*, to appear.

[14] Shiu P., *A Brun-Titchmarsh theorem for multiplicative functions*, J. reine angew. Math., **313**, 1980, 161–170.

[15] Steinig J., *On G. A. Freiman's theorems concerning the sum of two finite sets of integers*, In *Conference on the Structure Theory of Set Addition*, CIRM, Marseille, 1993, 173–186.

[16] Tenenbaum G., *Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné*, In Séminaire de Théorie des Nombres, Paris 1981-1982, volume **38** of Progress in Math., Birkhäuser, Boston, 1983, 303–312.

[17] Tenenbaum G., *Sur la probabilité qu'un entier possède un diviseur dans un intervalle donné*, Compositio Math., **51**, 1984, 243–263.

[18] Vinogradov A. I. and Linnik Yu. V., *Estimate of the sum of the number of divisors in a short segment of an arithmetic progression*, Uspekhi Mat. Nauk (N.S.), **12**, 1957, 277–280.

M.B. Nathanson, Department of Mathematics, Lehman College (CUNY), Bronx, New York 10468, USA • *E-mail :* `nathansn@dimacs.rutgers.edu`

G. Tenenbaum, Institut Élie Cartan, Université Henri-Poincaré–Nancy 1, 54506 Vandœuvre lès Nancy Cedex, France • *E-mail :* `tenenb@ciril.fr`