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NON-COMMUTATIVE VECTOR VALUED L_p-SPACES AND COMPLETELY p-SUMMING MAPS

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Abstract. — We introduce a non-commutative analog of Banach space valued L_p-spaces in the category of operator spaces. Thus, given a von Neumann algebra M equipped with a faithful normal semi-finite trace φ and an operator space E, we introduce the space $L_p(M, \varphi; E)$, which is an E-valued version of non-commutative L_p, and we prove the basic properties one should expect of such an extension (*e.g.* Fubini, duality, ...). There are two important restrictions for the theory to be satisfactory: first M should be injective, secondly E cannot be just a Banach space, it should be given with an operator space structure and all the stability properties (*e.g.* duality) should be formulated in the category of operator spaces.

This leads naturally to a theory of "completely p-summing maps" between operator spaces, analogous to the Grothendieck-Pietsch-Kwapień theory (*i.e.* "absolutely p-summing maps") for Banach spaces. As an application, we obtain a characterization of maps factoring through the operator space version of Hilbert space. More generally, we study the mappings between operator spaces which factor through a non-commutative L_p-space (or through an ultraproduct of them) using completely p-summing maps. In this setting, we also discuss the factorization through subspaces, or through quotients of subspaces of L_p-spaces.

***Résumé* (Espaces L_p non-commutatifs à valeurs vectorielles et applications complètement p-sommantes).** — Nous introduisons un analogue non-commutatif de la notion d'espace L_p à valeurs vectorielles dans la catégorie des espaces d'opérateurs. Plus précisément, étant donnés une algèbre de von Neumann M, munie d'une trace normale semie-finie et fidèle et un espace d'opérateurs E, nous introduisons l'espace $L_p(M, \varphi; E)$ qui est une version E-valuée d'espaces L_p non commutatif et nous prouvons les propriétés fondamentales que l'on est en droit d'attendre d'une telle extension (e.g. Fubini, dualité...). Il y a deux restrictions importantes pour que cette théorie tourne bien : d'abord M doit être injective, ensuite E ne peut pas être simplement un espace de Banach, il doit être muni d'une structure d'espace d'opérateurs et toutes les propriétés structurelles (e.g. la dualité) doivent être formulées dans la catégorie des espaces d'opérateurs.

Cela conduit naturellement à une théorie des applications « complètement p-sommantes » entre espaces d'opérateurs, analogue à la théorie de Grothendieck-Pietsch-Kwapień (*i.e.* les applications absolument p-sommantes) pour les Banach. Comme application, nous obtenons une caractérisation des applications qui se factorisent par la version « espace d'opérateurs » de l'espace de Hilbert (= l'espace OH). Plus généralement, nous étudions les applications entre espaces d'opérateurs qui se factorisent à travers un espace L_p-non commutatif (ou bien à travers un ultraproduit de tels espaces) dans le langage des applications complètement p-sommantes. Dans ce cadre, nous considérons aussi les factorisations (complètement bornées) à travers un sous-espace (ou un quotient de sous-espace) d'un espace L_p non commutatif.

Contents

INTRODUCTION

In standard Lebesgue integration, for any measure space (Ω, μ) and any Banach space E, we know how to define the Banach space $L_p(\Omega, \mu; E)$ of E-valued L_p-functions (for $1 \le p \le \infty$) using a well known construction attributed to Bochner. When $\Omega = \mathbb{N}$ (resp. $\Omega = \{1, \dots, n\}$) equipped with the counting measure $\mu = \sum_{k \in \Omega} \delta_k$, then $L_p(\Omega, \mu; E)$ is simply the space $\ell_p(E)$ (resp. $\ell_p^n(E)$) formed of all the sequences (x_k) with $x_k \in E$ such that $\sum \|x_k\|_E^p < \infty$, equipped with the norm

$$\|(x_k)\|_{\ell_p(E)} = \left(\sum \|x_k\|_E^p\right)^{1/p} \quad \left(\text{resp.} \quad \|(x_k)\|_{\ell_p^n(E)} = \left(\sum_1^n \|x_k\|_E^p\right)^{1/p}\right).$$

The case of any discrete measure space is analogous.

The non-commutative analog of ℓ_p is the Schatten class S_p which is defined for $1 \le p < \infty$ as the space of all compact operators T on ℓ_2 such that $\operatorname{tr}|T|^p < \infty$ and is equipped with the norm

$$\|T\|_{S_p} = (\operatorname{tr}|T|^p)^{1/p}$$

with which it is a Banach space. We will often denote this simply by $\|T\|_p$. For $p = \infty$, we denote by S_∞ the space of all compact operators on ℓ_2 equipped with the operator norm.

If H is any Hilbert space (resp. if $H = \ell_2^n$) we will denote by $S_p(H)$ (resp. S_p^n) the space of all operators $T\colon\ H \to H$ such that $\operatorname{tr}|T|^p < \infty$ and we equip it with the norm $(\operatorname{tr}|T|^p)^{1/p}$. If $p = \infty$, $S_\infty(H)$ (resp. S_∞^n) is the space of all compact operators on H, equipped with the operator norm.

More generally, given a von Neumann algebra M equipped with a faithful normal semi-finite trace φ, one can define a non-commutative version of L_p which we denote by $L_p(M, \varphi)$. When φ is finite, $L_p(M, \varphi)$ can be described simply as the completion of M equipped with the norm $x \to \varphi(|x|^p)^{1/p}$. In the special case $M = B(\ell_2)$ equipped with its classical (infinite but semi-finite) trace $x \to \operatorname{tr}(x)$, $L_p(M, \varphi)$ can be identified with S_p.

There is an extensive literature about these spaces, following the pioneering work of Segal, Dixmier, Kunze and Stinespring in the fifties (**[S]**, **[Di]**, **[Ku]**, **[St]**). (See *e.g.* **[N]**, **[FaK]**, **[H2]**, **[Ko]**, **[Te1]**-**[Te2]**, **[Hi]**).

Consider in particular the so-called hyperfinite factor R. This is the infinite tensor product of M_2 ($= 2 \times 2$ matrices) equipped with its normalized trace. This object is the non-commutative analog of the probability space $\Omega = \{-1,+1\}^{\mathbb{N}}$ equipped with its usual probability P (P is the infinite product of $(1/2)\delta_1 + (1/2)\delta_{-1}$). When $M = R$, the space $L_p(M,\varphi)$ appears as the non-commutative analog of $L_p(\Omega, P)$, or equivalently of $L_p([0,1], dt)$. In non-commutative integration theory, there seems to be no analog (as far as we know) of vector valued integration, and while S_p and $L_p(M,\varphi)$ appear as the "right" non-commutative counterpart to ℓ_p and $L_p([0,1], dt)$, there is a priori no analog for $\ell_p(E)$ and $L_p([0,1], dt; E)$ when E is a Banach space. The main goal of the present volume is to fill this gap. We will show that if M is hyperfinite (=injective by **[Co]**) and if E is an operator space, *i.e.* E is given as a closed subspace of $B(H)$ (for some Hilbert space H), then using complex interpolation (see below for more on this), we can define in a very natural way the space $L_p(M,\varphi; E)$ for $1 \le p < \infty$. When $(M,\varphi) = (B(\ell_2), \mathrm{tr})$, we obtain the space $S_p[E]$ which is a non-commutative analog of $\ell_p(E)$. Our theory of these spaces has all the properties one should expect, such as duality, Fubini's theorem, injectivity and projectivity with respect to E, and so on...But the crucial point is that we must always work with operator spaces and not only Banach spaces. The theory of operator spaces emerged rather recently (with its specific duality) in the works of Effros-Ruan **[ER1]**-**[ER7]** and Blecher-Paulsen **[BP]**, **[B1]**-**[B3]**. In this theory, bounded linear maps are replaced by *completely bounded* ones, isomorphisms by *complete isomorphisms* and isometric maps by *completely isometric ones.* In particular, given an operator space E, the spaces $S_p[E]$ and $L_p(M,\varphi; E)$ will be constructed not only as Banach spaces but as operator spaces. Moreover, all identifications will have to be "completely isometric" (as defined below) rather than just isometric.

For instance, the classical (isometric) duality theorem

$$\ell_p(E)^* = \ell_{p'}(E^*)$$

becomes in our theory the completely isometric identity

$$S_p[E]^* = S_{p'}[E^*]$$

where on both sides the dual is meant in the operator space sense: when E is an operator space, the dual Banach space E^* can be realized in a specific manner as a closed subspace of some $B(H)$, this is what we call the dual "in the operator space sense" (called the standard dual in **[BP]**); see below for background on this.

In a different direction, let (N,ψ) be another hyperfinite von Neumann algebra equipped with a faithful normal semi-finite trace. We will obtain completely isometric identities

$$L_p(M,\varphi; L_p(N,\psi)) = L_p(M \otimes N, \varphi \times \psi) = L_p(N,\psi; L_p(M,\varphi)).$$

Actually, the first one holds even if N is not assumed hyperfinite, see (3.6) and (3.6)'.

In addition, the resulting functor $E \to L_p(M,\varphi;E)$ is both injective and projective. By this we mean that if $F \subset E$ is a closed subspace (=operator subspace) then the inclusion $L_p(M,\varphi;F) \subset L_p(M,\varphi;E)$ is completely isometric and we have a completely isometric identification

$$L_p(M,\varphi;E/F) = L_p(M,\varphi;E)/L_p(M,\varphi;F).$$

To some extent our theory works in the non-hyperfinite case (see the discussion in chapter 3) but then the preceding injectivity (resp. projectivity) no longer holds if $p = 1$ (resp. $p = \infty$).

In the case $p = 1$ our results are essentially contained in the works of Effros-Ruan **[ER2, ER8]** on the operator space version of the projective tensor product, see also **[BP]**. Indeed, these authors introduced the operator space version of the projective tensor product $E \otimes^\wedge F$ of two operator spaces E, F. Then if X is a non-commutative L_1-space, the E-valued version of X can be defined simply as $X \otimes^\wedge E$. (Warning: In general this is not the Grothendieck projective product of X and E, but its analog in the category of operator spaces.) The case $p = \infty$ is also known: if E is finite dimensional (for simplicity) and if M is any von Neumann algebra, then the minimal tensor product $M \otimes_{\min} E$ is the natural non-commutative analog of $L_\infty(\Omega,\mu;E)$. What we do in this volume is simply to use the complex interpolation method (an approach that has already proved very efficient in the study of non-commutative L_p-spaces, *cf.* **[Ko]**, **[Te1]**) to define the non-commutative "E-valued" L_p-spaces for the intermediate values, i.e. for $1 < p < \infty$.

The first part of this volume (chapters 1 to 4) is devoted to the theory of the spaces $L_p(M,\varphi;E)$. We first concentrate on the discrete case in chapter 1, then in chapter 2, we describe the operator space structure of the usual (=commutative) L_p-spaces and its relation to the discrete non-commutative case. We consider the general case in chapter 3 and the duality in chapter 4.

The second part (chapters 5 to 7) is devoted mainly to "completely p-summing maps". These are a natural extension in our new setting of the "absolutely p-summing maps" studied by Pietsch and Kwapień (**[Pi]**, **[Kw1]**-**[Kw2]**), following Grothendieck's fundamental work on Banach space tensor products **[G]**.

In the third and final part (chapter 8), we try to illuminate our new theory in the light of numerous concrete examples linked with analysis. The main emphasis there is on Khintchine's inequalities for the Rademacher functions (which we denote by (ε_n)), and numerous variants of them involving Gaussian random variables or their analog in Voiculescu's "free" probability theory. If we identify (ε_n) with the sequence of coordinate functions on Ω, the classical Khintchine inequalities provide a remarkable isomorphic embedding

$$\ell_2 \subset L_p(\Omega,P),$$

taking the canonical basis of ℓ_2 to (ε_n) (here $0 < p < \infty$). This is very often used in analysis through the resulting isomorphic embedding

$$L_p([0,1];\ell_2) \subset L_p([0,1]\times\Omega, dt\times dP).$$

A great deal of chapter 8 is devoted to non-commutative analogs of the preceding two embeddings.

We will now describe the contents in more detail chapter by chapter.

In chapter 1, we introduce for any operator space E the space $S_p[E]$ and we construct $S_p[E]$ as an operator space. It turns out that this definition of $S_p[E]$ has all the natural properties of an "E-valued" ℓ_p-space. We review its properties in chapter 1. To some extent, the definition of $S_p[E]$ is already implicit in our previous work **[P1]** where we introduce and study the complex interpolation method in the category of operator spaces.

In chapter 2, we describe in detail the meaning of the preceding definitions in the case of the usual (*i.e.* commutative) L_p-spaces associated to a measure space. We give several formulae which allow to "compute" the operator space structure of these spaces as well as of their vector valued versions. These are used repeatedly in the next chapters.

In chapter 3, we discuss non-commutative vector valued L_p-spaces in the case of a continuous trace. We should emphasize that to have a satisfactory theory we must assume that the underlying von Neumann algebra M is injective. This is required to have the non-commutative analog of the fact that if F is a closed subspace of E then $L_p(\mu; F)$ is a closed subspace of $L_p(\mu; E)$. See Proposition 3.3 in **[ER2]** for the case $p = 1$.

Then, given a faithful normal semi-finite trace φ and an operator space E, we define the operator space $L_p(M, \varphi; E)$ using interpolation as before. The resulting space can alternately be viewed as an inductive limit of a family $L_p(M_\alpha, \varphi_\alpha; E)$ associated to an increasing net of finite dimensional (hence essentially matricial) subalgebras (M_α) equipped with finite traces φ_α which are the restrictions of φ to M_α. The spaces $L_p(M_\alpha, \varphi_\alpha; E)$ can be treated as direct sums of spaces of the kind we study in chapter 1.

We also discuss briefly the possible extensions of our definitions to non hyperfinite (*i.e.* non injective, by **[Co]**) von Neumann algebras.

In chapter 4, we address the duality problem for vector valued non-commutative L_p-spaces. In the Lebesgue-Bochner theory of the spaces $L_p(\Omega, \mu; E)$ (with E Banach), it is well known that duality poses a problem. The dual of the space $L_p(\Omega, \mu; E)$ is not in general the space $L_{p'}(\Omega, \mu; E^*)$ $(1 < p < \infty,\ 1/p + 1/p' = 1)$, however it is so when the dual E^* possesses the Radon Nikodym property (in short the RNP). See *e.g.* **[DU]** for more on this topic. Naturally, a similar problem arises in our new setting, and we have to introduce an operator space analog of the RNP, which we call the ORNP. Now, let E be an operator space. Then assuming that its dual has the ORNP, we obtain the duality theorem, namely the dual of $L_p(M, \varphi; E)$ is completely isometric to $L_{p'}(M, \varphi; E^*)$. Note that the ORNP of an operator space implies the RNP of the underlying Banach space, but the converse is false. We give a simple example of a Hilbertian operator space E (*i.e.* the underlying Banach space is ℓ_2) for which the space $L_2(M, \varphi; E)$ (and also $L_p(M, \varphi; E)$ for all p) contains an isomorphic copy of the Banach space c_0, hence fails the classical RNP (see example 4.2). In particular, E

fails the ORNP, although (being Hilbertian) it clearly has the RNP. Concerning the ORNP, several natural questions remain open. For instance we do not know whether E has the ORNP iff $L_2(M, \varphi; E)$ has the classical RNP, when (M, φ) is the classical hyperfinite factor.

As is well known, the RNP for a Banach space E is closely related to the martingale convergence theorem for bounded E-valued martingales (see **[DU]**) on a probability space (Ω, μ). Moreover, the "super-property" associated to the RNP is equivalent to the validity of certain martingale inequalities in $L_p(\Omega, \mu; E)$, and these in turn are equivalent to the existence of an equivalent uniformly convex norm on E (see **[P7]**). Here again it is natural to look for analogous results for operator spaces: we introduce the notion of uniform $OS-$convexity, and we prove some basic facts, namely it implies the ORNP and all non-commutative L_p-spaces are uniformly $OS-$convex when $1 < p < \infty$. Note however that many questions remain open. We also introduce the operator space analog of the UMD property (UMD stands for "unconditional martingale differences") in Burkholder's sense **[Bu2]**. The recent paper **[PX2]** allows to embark in this direction, but very little is known. We list a few natural questions which we feel should be answered before pursuing further.

In chapter 5, we introduce the notion of "completely p-summing map" $u : E \to F$ between two operator spaces. Our notion coincides with a notion introduced by Effros-Ruan **[ER7]** in the particular case $p = 1$. We say that $u\colon E \to F$ is completely p-summing if $I_{S_p} \otimes u$ defines a bounded mapping from $S_p \otimes_{\min} E$ into $S_p[F]$, and we denote by $\pi_p^o(u)$ the norm of this mapping. We prove a natural analog of the Pietsch factorization for such maps, extending the case $p = 1$ treated in **[ER7]**. This is new already if $p = 2$, although this case is closely related to the $(2, oh)$-summing maps considered in **[P1]**. This new framework allows us in chapter 6 to give a characterization of "operators factoring through OH" (in the sense of **[P1]**) entirely analogous to the Grothendieck-Kwapień **[G]**, **[Kw1]**-**[Kw2]** characterization of operators factoring through a Hilbert space.

In §7.2, we use completely p-summing maps to characterize the mappings $u\colon E \to F$ between operator spaces which factor (completely boundedly) through a quotient of a subspace of an ultraproduct of S_p. This is the analog for operator spaces of a result due to Kwapień **[Kw2]** in the Banach space setting, which we recall in §7.1. We also include in §7.1 several basic perturbation arguments relevant to ultraproducts of operator spaces.

Note that the non-commutative version of the stability of L_p-spaces under ultraproducts is unclear (see however **[Gr]** for the case $p = 1$). This leads us to replace the class of non-commutative L_p-spaces by that of ultraproducts of non-commutative L_p-spaces (based as above on a hyperfinite semi-finite von Neumann algebra) or equivalently by the class of ultraproducts of S_p.

We show that $u\colon E \to F$ factors as above iff for any c.b. map $T : S_p \to S_p$, the mapping $T \otimes u$ defines a bounded map from $S_p[E]$ to $S_p[F]$. Moreover, the factorization constant of u is equal to the smallest constant C such that we have $\| T \otimes u \| \leq C \| T \|_{cb}$ (or equivalently $\| T \otimes u \|_{cb} \leq C \| T \|_{cb}$) for all T as above.

We also discuss factorization through an ultraproduct of S_p or through one of its subspaces. The proofs follow the principles of the duality theory for ideals of operators or tensor products as developed by Kwapień [**Kw2**] and Pietsch [**Pi**] following Grothendieck's fundamental work [**G**]. See [**DF**] for an exposition. There are however some specific difficulties which arise, because in general operator spaces lack local reflexivity in the sense of ([**EH**]) or "exactness", a notion introduced by Kirchberg for C^*-algebras and studied for operator spaces in [**P6**]. As commented in §7, the above mentioned difficulties have now been resolved by Marius Junge [**Ju**], and we briefly explain how his ideas allow to complete our results at the end of §7.

In §8, we try to illustrate the preceding theory in the light of "concrete" situations. This is mostly expository, *i.e.* the results there are essentially known but many facts are formulated and interpreted in a manner not available elsewhere in print. For instance, we show that, if $1 \leq p < \infty$, the closed span in L_p of a sequence of standard independent Gaussian variables is the same operator space (up to complete isomorphism) as that spanned in non-commutative L_p by a (countable) free semi-circular family in Voiculescu's sense (*cf.* Theorem 8.6.5). Moreover, in both cases the orthogonal projection (onto the subspace spanned) is completely bounded on the L_p-space under consideration for all $1 < p < \infty$ (and in the semi-circular case even for $p = 1$ and $p = \infty$).

In §8.1, we discuss completely bounded Schur multipliers on the Schatten class S_p and closely related questions on Fourier multipliers.

In §8.2, we briefly explain the connection between l_1 or L_1 as an operator space and the natural generators of the ("full") C^*-algebra of the free group with countably infinitely many (resp. n) generators, denoted by $\mathbf{F}_\infty$ (resp. $\mathbf{F}_n$).

In §8.3, we turn to the reduced C^*-algebra again for the free group $\mathbf{F}_\infty$, and examine the span of the generators in the associated non-commutative L_p-space.

In §8.4, we discuss at length the consequences of F. Lust-Piquard's "non-commutative Khintchine inequalities" (*cf.* [**Lu**], [**LuP**]) for our theory.

In §8.5, we briefly discuss the $\Lambda(p)_{cb}$-property for a subset of a discrete (possibly non-commutative) group, introduced in Asma Harcharras's recent thesis [**Ha**]. In particular, for each even integer $k \geq 4$, we describe a sufficient combinatorial property for the subset to satisfy an analog of the Lust-Piquard inequality for $p = 2k$.

Finally, in §8.6, after a brief introduction to Voiculescu's "free" probability theory, we describe the operator space structure of the span of a free semi-circular (or circular) family, *i.e.* the "free" analog of real (or complex) Gaussian random variables.

Note. — The main results of this volume were announced in [**P5**]. The first six chapters reproduce (in a different ordering) the contents of the preprint which circulated in the interval, while the last two chapters were added more recently.

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CHAPTER 0

BACKGROUND AND NOTATION

Unless explicitly specified otherwise, we only consider complex Banach (or Hilbert) spaces in this volume. We will denote by $H \otimes_2 K$ the Hilbertian tensor product of two Hilbert spaces H, K. (Note the identities $\ell_2^n \otimes_2 H = \ell_2^n(H)$ and $\ell_2 \otimes_2 H = \ell_2(H)$.) We denote by $B(H)$ (resp. $B(H,K)$) the Banach space of all bounded operators on H (resp. from H to K). When H is n-dimensional $B(H)$ can be identified with the space M_n of all $n \times n$ matrices with complex entries. By an operator space we mean a closed subspace of $B(H)$ for some Hilbert space H. When $E \subset B(H)$ is an operator space, we denote by $M_n(E)$ the space of all $n \times n$ matrices with entries in E, equipped with the norm induced by the space $B(\ell_2^n \otimes_2 H)$ (or equivalently $B(\ell_2^n(H))$).

Two basic examples play a fundamental role in the theory: these are the row and column Hilbert spaces, which are subspaces of $B(\ell_2)$. Let e_{ij} be the element of $B(\ell_2)$ corresponding to the matrix with coefficients equal to one at the i,j entry and zero elsewhere. The "column Hilbert space" C is defined as

$$C = \overline{\mathrm{span}}\{e_{i1} \mid i \in \mathbb{N}\}$$

and the "row Hilbert space" R is defined as

$$R = \overline{\mathrm{span}}\{e_{1j} \mid j \in \mathbb{N}\}.$$

Both are isometric (as Banach spaces) to ℓ_2, but they are quite different as operator spaces. We will also need their finite dimensional versions

$$C_n = \mathrm{span}\{e_{i1} \mid 1 \le i \le n\} \quad \text{and} \quad R_n = \mathrm{span}\{e_{1j} \mid 1 \le j \le n\}.$$

We denote by $E_1 \otimes E_2$ the linear tensor product of two vector spaces. If $E_1 \subset B(H_1)$, $E_2 \subset B(H_2)$ are operator spaces, we will denote by $E_1 \otimes_{\min} E_2$ their minimal (or spatial) tensor product equipped with the minimal (or spatial) tensor norm induced by the space $B(H_1 \otimes_2 H_2)$.

Let H, K be Hilbert spaces. Let $E \subset B(H)$ and $F \subset B(K)$ be operator spaces. A map $u\colon\ E \to F$ is called completely bounded (in short c.b.) if the maps $u_m = I_{M_m} \otimes u\colon\ M_m(E) \to M_m(F)$ are uniformly bounded when $m \to \infty$, i.e. if we have

$\sup_{m\geq 1} \|u_m\| < \infty$. The *c.b.* norm of u is defined as

$$\|u\|_{cb} = \sup_{m\geq 1} \|u_m\|.$$

We denote by $cb(E,F)$ the Banach space of all *c.b.* maps from E to F equipped with this norm. It is known (*cf.* e.g. **[DCH]** or **[Pa1]**, p. 158-159) that we also have

$$\|u\|_{cb} = \|I_{B(\ell_2)} \otimes u\|_{B(\ell_2)\otimes_{\min} E \to B(\ell_2)\otimes_{\min} F}.$$

We will use this repeatedly in the sequel with no further reference.

We will say that u is completely isometric (resp. completely contractive) or is a complete isometry (resp. a complete contraction) if the maps u_m are isometries (resp. of norm ≤ 1) for all m.

We will frequently invoke an abstract characterization of operator spaces due to Ruan **[Ru]** (see also **[ER3]** for a simpler proof) which uses the notion of "matricial structure". By a matricial structure on a vector space E we simply mean that for any integer n we are given a norm on the space $M_n(E)$ of all $n \times n$ matrices with entries in E. So in particular for $n = 1$ we have a norm on E. We will say that it is complete if all the norms are complete (=Banach). We say that we have an L_∞-matricial structure if these norms satisfy the following

$$\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\} \tag{0.1}$$

$$\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\| \tag{0.2}$$

for all $x \in M_n(E), y \in M_m(E)$ and $\alpha, \beta \in M_n(\mathbb{C})$. Ruan proved that for any L_∞-matricial structure on a vector space E, there is a Hilbert space H and an embedding of E into $B(H)$ such that the norm (from the matricial structure) on $M_n(E)$ coincides with the norm induced by the space $M_n(B(H))$. Clearly if the structure is complete the subspace of $B(H)$ will be closed. Conversely it is easy to see that every subspace E of $B(H)$ is equipped with a natural L_∞-matricial structure by simply giving to $M_n(E)$ the norm induced on it by $M_n(B(H))$. Thus operator spaces can be viewed ("abstractly") as vector spaces equipped with a complete L_∞-matricial structure. Therefore, by an "operator space structure" (in short o.s.s.) on a vector space, we will mean a complete L_∞-matricial structure. For instance, this allows to introduce the quotient (**[Ru]**) and the dual (**[BP]**, **[ER2]**) within the category of operator spaces, as we now recall.

Given an operator space E and a subspace $S \subset E$, we equip the quotient space E/S with the matricial structure obtained by giving to $M_n(E/S)$ the norm of the space $M_n(E)/M_n(S)$. It is easy to check that this is an L_∞-matricial structure with which E/S can (and will always in this volume) be viewed as an operator space.

A completely bounded surjective linear map $u\colon E \to F$ between two operator spaces is called a "complete metric surjection" if the associated map from $E/\ker(u)$ onto F

is completely isometric (when the quotient $E/\ker(u)$ is equipped with the structure just defined).

We now turn to the dual. In **[BP]**, **[ER2]**, it was proved that $cb(E,F)$ can be equipped with an operator space structure by giving to $M_n(cb(E,F))$ the norm of the space $cb(E,M_n(F))$. In particular, this defines an operator space structure on the dual $E^* = cb(E,\mathbb{C})$ so that we have isometrically

$$M_n(E^*) = cb(E, M_n).$$

Then, by construction, we have the following very important fact (*cf.* **[BP]**, **[ER2]**): The tensor product $E^* \otimes_{\min} F$ is isometrically embedded into the space $cb(E,F)$ by the natural embedding. This shows that the minimal tensor product is the analog for operator spaces of the injective tensor product of Banach spaces.

The usual rules of the Banach space duality remain valid in the category of operator spaces, for instance the dual of a subspace $S \subset E$ (resp. of a quotient space E/S) is the quotient space $E^*/S^\perp$ (resp. is the subspace $S^\perp \subset E^*$). Also, for any map $u\colon E \to F$ we have $\|u\|_{cb} = \|u^*\|_{cb}$. Moreover, the inclusion $E \subset E^{**}$ is a complete isometry.

In particular, let A be a C^*-algebra. We equip A with its natural o.s.s. (coming from its Gelfand embedding into $B(H)$). Then, by the preceding definition, the successive duals A^*, A^{**}, A^{***} and so on, can now be viewed as operator spaces. We will refer to these operator space structures on A^*, A^{**}, A^{***} and so on, as the "natural" ones.

Now assume that A is a von Neumann algebra with predual A_*. Then, the inclusion $A_* \subset (A_*)^{**} = A^*$ allows to equip the predual A_* with the o.s.s. induced by the one just defined on the dual A^*, so we obtain an operator space, denoted by A_*^{os}, having A_* as its underlying Banach space. Here a natural question arises: if we now consider the dual operator space to the one just defined, namely $(A_*^{os})^*$, do we recover the same operator space structure on A? Fortunately, the answer is affirmative (**[B2]**, Theorem 2.9): we have $(A_*^{os})^* = A$ completely isometrically.

This allows to define an operator space structure on A_*, which we will again call the *natural one.*

For all these fundamental results due to Blecher, Effros, Paulsen and Ruan, which we will use freely in the sequel, we refer the reader to **[BP]**, **[ER2]**, **[B1]**, **[B2]**.

The notion of direct sum of C^*-algebras or of operator spaces is defined in the obvious way. Let $(E_i)_{i\in I}$ be a family of operator spaces. Assume $E_i \subset B(H_i)$. Let $H = \oplus_{i\in I} H_i$ be the Hilbertian direct sum. We will denote by $\oplus_{i\in I} E_i$ the operator space included in $B(H)$ formed of all operators on H of the form $x = \oplus_{i\in I} x_i$ with $x_i \in E_i$ and $\sup_{i\in I}\|x_i\| < \infty$. It is easy to check that $\|x\| = \sup_{i\in I}\|x_i\|$. More generally let $\mathcal{X} \in M_n(\oplus_{i\in I} E_i)$ and let $(\mathcal{X}_i)_{i\in I}$ be the family naturally associated to $\mathcal{X}$, with $\mathcal{X}_i \in M_n(E_i)$, then it is easy to check that

$$\|\mathcal{X}\|_{M_n(\oplus_{i\in I} E_i)} = \sup_{i\in I} \|\mathcal{X}_i\|_{M_n(E_i)}.$$

When the family is reduced to two operator spaces E, F, we should denote the preceding direct sum by $E \oplus F$, but it will be worthwhile, as explained below and in §2, to denote it by $E \oplus_\infty F$ to emphasize the specific choice of norm (and o.s.s.) on the algebraic direct sum.

This notion of direct sum is the natural one when the spaces E_i are C^*-algebras. However, in Banach space theory, there are many other possible direct sums. For instance, given two Banach spaces E_0, E_1 one defines $E_0 \oplus_p E_1$ as $E_0 \oplus E_1$ equipped with the norm $\|(x_0, x_1)\| = (\|x_0\|^p_{E_0} + \|x\|^p_{E_1})^{1/p}$. When E_0, E_1 are given with an o.s.s. it is possible to also equip $E_0 \oplus_p E_1$ with a natural o.s.s. (see §2). For the moment, we will describe this only for $p = 1$.

Let $\mathcal{P}$ be the family of all possible pairs $u = (u_0, u_1)$ of completely contractive mappings $u_0\colon\ E_0 \to B(H_u)$, $u_1\colon\ E_1 \to B(H_u)$ (H_u Hilbert). We define an embedding

$$J\colon\ E_0 \oplus_1 E_1 \longrightarrow \bigoplus_{u \in \mathcal{P}} B(H_u) \subset B\left(\bigoplus_{u \in \mathcal{P}} H_u\right)$$

by setting $J(x_0 \oplus x_1) = \bigoplus\limits_{u \in \mathcal{P}} [u_0(x_0) + u_1(x_1)]$. It can be checked that J is an isometric embedding, and since $\bigoplus\limits_{u \in \mathcal{P}} B(H_u)$ is equipped with a natural o.s.s. (as a C^*-direct sum) we obtain a natural o.s.s. on $E_0 \oplus_1 E_1$.

It is easy to verify that this o.s.s. is characterized by the following universal property: for any operator space E, for any complete contractions $u_0\colon\ E_0 \to E$ and $u_1\colon\ E_1 \to E$, the mapping $(x_0, x_1) \to u_0(x_0) + u_1(x_1)$ is a complete contraction from $E_0 \oplus_1 E_1$ to E.

It is rather easy to check that we have completely isometric identities

$$(E_0 \oplus E_1)^* = E_0^* \oplus_1 E_1^* \quad \text{and} \quad (E_0 \oplus_1 E_1)^* = E_0^* \oplus E_1^*.$$

We have restricted ourselves to the sum of two spaces, but everything we said extends to ℓ_1-direct sums of an arbitrary family $(E_i)_{i \in I}$ of operator spaces. We will denote by $\ell_1(\{E_i \mid i \in I\})$ the resulting space.

Let $j_i\colon E_i \to \ell_1(\{E_i\})$ be the natural completely isometric inclusion map. It is easy to check that the following property characterizes the operator space $\ell_1(\{E_i\})$ (given with the inclusions (j_i)), up to complete isometry: for any family $(u_i)_{i \in I}$ with $u_i \in cb(E_i, B(H))$ such that $\|u_i\|_{cb} \leq 1$ for all i, there is a unique completely contractive $U\colon\ \ell_1(\{E_i\}) \to B(H)$ such that $Uj_i = u_i$ for all i.

Moreover, we have completely isometrically

$$\oplus_{i \in I} E_i^* = (\ell_1(\{E_i\}))^*.$$

More generally, if we are given a family of positive "weights" $\mu = (\mu_i)_{i \in I}$, we can form the Banach space $\ell_1(\mu; \{E_i \mid i \in I\})$ (or briefly $\ell_1(\mu; \{E_i\})$) of all families $x = (x_i)$ such that $\sum_{i \in I} \mu_i \|x_i\| < \infty$, equipped with the norm $x \to \sum_{i \in I} \mu_i \|x_i\|$. We denote by $\mathcal{P}$ the class of all systems $u = (u_i)_{i \in I}$ with $u_i \in cb(E_i, B(H_u))$ such that $\|u_i\|_{cb} \leq \mu_i$ for all i, and we introduce the embedding $J\colon\ \ell_1(\mu; \{E_i\}) \to \oplus_{u \in \mathcal{P}} B(H_u)$ defined by

$J(x) = \oplus_{u\in\mathcal{P}}[\sum_{i\in I} u_i(x_i)]$. The resulting operator space structure will be referred to as the natural one on $\ell_1(\mu; \{E_i \mid i \in I\})$.

Note that we can define the "multiple" of an operator space E by a positive scalar μ. We will denote by $\mu \cdot E$ the resulting operator space. This is the same space but equipped with the operator space structure associated to the following sequence of norms

$$\forall\, a = (a_{ij}) \in M_n(E) \quad \|a\|_n = \mu\|a\|_{M_n(E)}.$$

The space $\mu \cdot E$ is trivially completely isometric to E.

It is then easy to check that we have a completely isometric identity

$$\ell_1(\mu; \{E_i \mid i \in I\}) = \ell_1(\{\mu_i.E_i \mid i \in I\}).$$

The present volume can be viewed as a sequel to **[P1]**. While **[P1]** is mainly devoted to the operator Hilbert space, this paper deals with the "operator L_p-spaces" (and their vector valued versions) which can be defined using interpolation. Let us briefly recall a few basic facts from **[P1]** that we will use:

For any index set I, there is a Hilbert space $\mathcal{H}$ (separable if I is at most countable) and an operator space $OH(I)$ included in $B(\mathcal{H})$ such that

(i) $OH(I)$ is isometric to $\ell_2(I)$ as a Banach space,
(ii) the canonical identification between $OH(I)$ and $\overline{OH(I)^*}$ (corresponding to the canonical identification between $\ell_2(I)$ and $\overline{\ell_2(I)^*}$) is a complete isometry.

Moreover, the space $OH(I)$ is the unique operator space (up to complete isometry) possessing these properties (i) and (ii). Furthermore we have

(iii) Let $(\theta_i)_{i\in I}$ be any orthonormal basis of $OH(I)$. Then for any Hilbert space K and any finite sequence (a_i) in $B(K)$ we have

$$\Big\|\sum\nolimits_{i\in I} \theta_i \otimes a_i\Big\|_{B(\mathcal{H}\otimes K)} = \Big\|\sum\nolimits_{i\in I} a_i \otimes \bar{a}_i\Big\|^{1/2}_{B(K\otimes\overline{K})}.$$

Following **[P1]**, we will denote by OH the space $OH(\mathbb{N})$ and by OH_n the space $OH(I)$ corresponding to $I = \{1, 2, \ldots, n\}$.

In **[P1]**, we introduced complex interpolation for operator spaces. Let (E_0, E_1) be a compatible couple of Banach spaces in the sense of interpolation theory (*cf.* **[BL]**, **[Ca]**). Assume E_0, E_1 each equipped with an operator space structure (in the form of norms on $M_n(E_0)$ and $M_n(E_1)$ for all n). Let $E_\theta = (E_0, E_1)_\theta$ and $E^\theta = (E_0, E_1)^\theta$. Then, we can define an operator space structure on E_θ (resp. E^θ) by setting

$$(0.3) \qquad M_n(E_\theta) = (M_n(E_0), M_n(E_1))_\theta \text{ (resp. } M_n(E^\theta) = (M_n(E_0), M_n(E_1))^\theta).$$

In **[P1]** we observed that these norms verify Ruan's axioms and hence they define an operator space structure on E_θ (resp. E^θ).

In particular, it is well known that

$$\ell_p = (\ell_\infty, \ell_1)_\theta \quad \text{and} \quad S_p = (S_\infty, S_1)_\theta$$

with $\theta = 1/p$. Note that S_1 and ℓ_1 are preduals of von Neumann algebras, hence can be equipped with their natural operator space structure as described above. Thus, we can now use (0.3) to equip ℓ_p and S_p with an operator space structure, which we again call the *natural* one.

More generally, given a von Neumann algebra M equipped with a normal faithful semi-finite trace φ, the predual of M can be viewed as the non-commutative L_1-space associated to the trace φ, denoted by $L_1(\varphi)$, and we may consider the pair $(M, L_1(\varphi))$ as compatible in the sense of interpolation theory. The Banach space $L_p(\varphi)$ is then usually defined for $1 < p < \infty$ as the interpolation space $(M, L_1(\varphi))_\theta$ with $\theta = 1/p$. Using (0.3), here again we may now view the space $L_p(\varphi)$ as an operator space equipped with an operator space structure which we call the natural one.

We will sometimes invoke the following elementary fact.

Lemma 0.1. — *Let (A_0, A_1) be a compatible couple of complex Banach spaces. Let $C \subset A_0$ be a closed subspace. Assume that there is a net (T_α) such that*

(i) $\|T_\alpha\|_{A_0 \to A_0} \le 1,\ \|T_\alpha\|_{A_1 \to A_1} \le 1,$
(ii) $T_\alpha(A_0) \subset C$
(iii) $\forall\, x \in A_1 \quad \|x - T_\alpha(x)\|_{A_1} \to 0$. *Then $(C, A_1)_\theta = (A_0, A_1)_\theta$ isometrically for any $0 < \theta < 1$.*

Proof. — Let $A_\theta = (A_0, A_1)_\theta$ for $0 < \theta < 1$. Let us denote by $\|\ \|_\theta$ the norm in the space A_θ when $0 \le \theta \le 1$. Note that $\|y\|_\theta \le \|y\|_0^{1-\theta} \|y\|_1^\theta$ for all y in A_θ. In particular, we have $\|x - T_\alpha(x)\|_\theta \to 0$ for any x in A_θ. Applying this to the pair (C, A_1), we find that, for all x in $C \cap A_1$, we have $\|x - T_\alpha(x)\|_{(C,A_1)_\theta} \to 0$ and (by interpolation) $\|T_\alpha(x)\|_{(C,A_1)_\theta} \le \|x\|_\theta$. This implies that, for all x in $C \cap A_1$ we have $\|x\|_{(C,A_1)_\theta} \le \|x\|_\theta$. Note that the converse inequality is trivial. Hence to conclude, it suffices to know that $C \cap A_1$ is dense both in $(C, A_1)_\theta$ and in A_θ. It is a classical fact that $A_0 \cap A_1$ is dense in A_θ (see **[BL]**, **[Ca]**), hence $\bigcup_\alpha T_\alpha(A_0 \cap A_1)$ is dense in A_θ and a fortiori $C \cap A_1$ is dense in A_θ. On the other hand $C \cap A_1$ is dense in $(C, A_1)_\theta$ by the same classical fact. This shows that $(C, A_1)_\theta = A_\theta$ isometrically. □

For example, the preceding statement implies that, for any Radon measure μ on a locally compact space Ω, we have isometrically $(C_0(\Omega), L_1(\mu))_\theta = (L_\infty(\mu), L_1(\mu))_\theta$ for any $0 < \theta < 1$. (Here $C_0(\Omega)$ denotes the Banach space of all complex valued continuous functions on Ω which tend to zero at infinity.) Obviously, if we equip all the spaces involved with their "natural" operator space structure as described above, then this equality becomes a completely isometric one.

Let E be an arbitrary operator space. Assume that there is a bounded linear map

$$v\colon\ OH(I) \to E$$

injective and with dense range so that the map

$$v\,\overline{v^*}\colon\ \overline{E^*} \to E$$

(here we identify $OH(I)$ and $\overline{OH(I)^*}$) also is a bounded injective map with dense range. This injection allows us to consider $(\overline{E^*}, E)$ as a compatible couple of operator spaces included into E, and to view $OH(I) \approx \overline{OH(I)^*}$ as also included naturally into E. With these conventions we have

$$OH(I) = (\overline{E^*}, E)_{1/2} \tag{0.4}$$

completely isometrically. See **[Wa]** for an extension.

For example, we can view the couple (R, C) as compatible for interpolation using the transposition $x \to {}^t x$ as the inclusion map of R into C (and, of course, we use the identity to embed C into itself). Then, (0.4) yields (using standard identifications) a completely isometric identity

$$(R, C)_{1/2} = OH.$$

We refer to **[P1]** for more details. Note that some of these results have been extended to the real interpolation method in **[X]**.

Ultraproducts are a tool often used in the sequel. We refer to **[Hei]** for background on ultraproducts of Banach spaces and to **[P1]** for the operator space case.

We only recall the main definitions below.

Let $\mathcal{U}$ be a nontrivial ultrafilter on a set I. Let $(E_i)_{i\in I}$ be a family of Banach spaces. We denote by ℓ the space of all families $x = (x_i)_{i\in I}$ with $x_i \in X_i$ such that $\sup_{i\in I} \|x_i\| < \infty$. We equip this space with the norm $\|x\| = \sup_{i\in I} \|x_i\|$. Let $n_{\mathcal{U}} \subset \ell$ be the subspace formed of all families such that $\lim_{\mathcal{U}} \|x_i\| = 0$. The quotient $\ell/n_{\mathcal{U}}$ is a Banach space called the ultraproduct of the family $(E_i)_{i\in I}$ with respect to $\mathcal{U}$. We denote it by $\Pi_{i\in I} E_i/\mathcal{U}$.

For every element $\dot{x}$ in $\ell/n_{\mathcal{U}}$ admitting $x \in \ell$ as its representative modulo $n_{\mathcal{U}}$, we have

$$\|\dot{x}\| = \lim_{\mathcal{U}} \|x_i\|.$$

Hence, the ultraproduct $\Pi_{i\in I} E_i/\mathcal{U}$ appears as "the limit" of the spaces $(E_i)_{i\in I}$ along $\mathcal{U}$.

Now assume that each space E_i is equipped with an operator space structure. It is very easy to extend the notion of ultraproduct to the operator space setting. We simply equip $\Pi_{i\in I} X_i/\mathcal{U}$ with the matricial structure obtained by giving to $M_n(\Pi_{i\in I} X_i/\mathcal{U})$ the norm of the space $\Pi_{i\in I} M_n(X_i)/\mathcal{U}$. It is easy to check that this is a complete L_∞-matricial structure (=an operator space structure).

We will use as our starting point the operator space version of the projective tensor product, introduced in **[BP]**, **[ER2]**. The more explicit description of **[ER2]** is as follows.

Consider an element u in the linear tensor product $E \otimes F$. Clearly u admits a representation (actually many such representations) of the form

$$u = \sum_{ijk\ell \le n} \alpha_{ik} x_{ij} \otimes y_{k\ell} \beta_{j\ell} \tag{0.5}$$

where n is an integer and where $x \in M_n(E)$, $y \in M_n(F)$ and $\alpha, \beta \in M_n$. Then the operator space/projective tensor norm $\|u\|_{E\otimes^\wedge F}$ is defined as

$$\|u\|_{E\otimes^\wedge F} = \inf\{\|\alpha\|_{S_2^n}\|x\|_{M_n(E)}\|y\|_{M_n(F)}\|\beta\|_{S_2^n}\} \tag{0.6}$$

where the infimum runs over all possible such representations as in (0.5).

We denote by $E \otimes^\wedge F$ the completion of $E \otimes F$ with respect to this norm.

More generally, this space can be equipped with the operator space structure corresponding to the norm defined (for each n) on $M_n(E \otimes^\wedge F)$ as follows: consider $u = (u_{ij}) \in M_n(E \otimes F)$ and assume

$$u = \alpha \cdot (x \otimes y) \cdot \beta$$

where the dot denotes the matrix product, and where $x \in M_\ell(E)$, $y \in M_m(F)$ and α (resp. β) is a matrix of size $n \times (\ell m)$ (resp. $(\ell m) \times n$). Note that $x \otimes y$ is considered here as an element of the space $M_{\ell m}(E \otimes F)$. Then (following **[ER2]**) we can define

$$\|u\|_{M_n(E\otimes^\wedge F)} = \inf\{\|\alpha\|_{n,\ell m}\|x\|_{M_\ell(E)}\|y\|_{M_m(F)}\|\beta\|_{\ell m,n}\}.$$

Then (*cf.* **[ER2]**) these norms define an operator space structure in $E\otimes^\wedge F$. Moreover, we have $(E\otimes^\wedge F)^* = cb(E, F^*)$ completely isometrically. We refer the reader to **[BP]**, **[ER2]** for more information.

We will use repeatedly the Haagerup tensor product. The Haagerup tensor norm was introduced by Effros and Kishimoto in **[EK]**, who, in view of its previous use by Haagerup in **[H3]**, called it this way. They only considered the resulting *Banach spaces*, but the *operator space* structure of the Haagerup tensor product was introduced in **[PS]**, extending the fundamental work of Christensen-Sinclair on multilinear mappings in the C^*-algebra case. We briefly recall the main definitions.

Given an operator space E, we denote by $M_{p,q}(E)$ the space of all matrices with p lines, q columns and with entries in E. We equip it with the obvious norm (for instance, by adding zeros, we can turn it into a square matrix of which we take the norm).

Let E_1, E_2 be operator spaces. Let $x_1 \in M_{p,m}(E_1)$, $x_2 \in M_{m,q}(E_2)$. We will denote by $x_1 \odot x_2$ the matrix x in $M_{p,q}(E_1 \otimes E_2)$ defined by

$$\forall i = 1, \ldots, p \; \forall j = 1, \ldots, q \qquad x(i,j) = \sum_{k=1}^{m} x_1(i,k) \otimes x_2(k,j)$$

Note that $M_n(E)$ is of course the same as $M_{n,n}(E)$. Then for any x in $M_n(E_1 \otimes E_2)$ we define

$$\|x\|_n = \inf\left\{\|x_1\|_{M_{n,m}(E_1)}\|x_2\|_{M_{m,n}(E_2)}\right\} \tag{0.7}$$

where the infimum runs over all m and possible decompositions of x as a "product"

$$x = x_1 \odot x_2$$

with

$$x_1 \in M_{n,m}(E_1), \quad x_2 \in M_{m,n}(E_2).$$

It can be checked that this sequence of norms satisfies the axioms (0.1) and (0.2) of Ruan's theorem. Hence after completion we obtain an operator space denoted by $E_1 \otimes_h E_2$ and called the Haagerup tensor product.

By an entirely similar process we can define the Haagerup tensor product of an N-tuple $E_1, \ldots, E_N$ of operator spaces. Once again for any x in $M_n(E_1 \otimes E_2 \otimes \cdots \otimes E_N)$ we define

$$\|x\|_n = \inf\{\|x_1\|_{M_{n,m_1}(E_1)}\|x_2\|_{M_{m_1,m_2}(E_2)} \cdots \|x_N\|_{M_{m_{N-1},n}(E_N)} \mid x = x_1 \odot x_2 \odot \cdots \odot x_N\}. \tag{0.8}$$

Again this satisfies Ruan's axioms so that we obtain an operator space denoted by

$$E_1 \otimes_h E_2 \otimes \cdots \otimes_h E_N.$$

The very definition of the norm (0.8) clearly shows that this tensor product is associative, i.e. for instance we have

$$(E_1 \otimes_h E_2) \otimes_h E_3 = E_1 \otimes_h (E_2 \otimes_h E_3) = E_1 \otimes_h E_2 \otimes_h E_3.$$

However, it is important to underline that it is *not* commutative (i.e. $E_1 \otimes_h E_2$ can be very different from $E_2 \otimes_h E_1$).

It is immediate from the definition that $E_1 \otimes_h E_2$ enjoys the classical "tensorial" properties required of a decent tensor product, i.e. for any operator spaces F_1, F_2 and any c.b. maps $u_i \colon E_i \to F_i$ $(i = 1, 2)$ the mapping $u_1 \otimes u_2$ extends to a c.b. map from $E_1 \otimes_h E_2$ into $F_1 \otimes_h F_2$ with $\|u_1 \otimes u_2\|_{cb} \leq \|u_1\|_{cb}\|u_2\|_{cb}$. Moreover, this remains valid with N factors instead of 2.

The main properties of the Haagerup tensor product are its "self-duality" and the fact that, in addition to being associative, it is *both* injective and projective. We refer the reader to **[PS]**, **[ER4]**, **[BS]** for details on all these facts.

CHAPTER 1

NON-COMMUTATIVE VECTOR VALUED L_p-SPACES (DISCRETE CASE)

Let E be an operator space. We will define the space $S_p[E]$ for $1 \le p \le \infty$. We start with the known cases $p = 1$ and $p = \infty$. For $p = \infty$, recall that we denote by S_∞ (resp. $S_\infty(K)$) the space of all compact operators on ℓ_2 (resp. on K) equipped with the usual operator norm. Clearly S_∞ (resp. $S_\infty(K)$) is an operator space.

When $p = \infty$, we define $S_\infty[E] = S_\infty \otimes_{\min} E$ (resp. $S_\infty[K; E] = S_\infty(K) \otimes_{\min} E$), as operator spaces.

When $p = 1$, we define $S_1[E]$ (resp. $S_1[K; E]$) as the " projective operator space tensor product" of S_1 (resp. $S_1(K)$) with E, which (following **[ER5]**) we will denote by $S_1 \otimes^\wedge E$ (resp. $S_1(K) \otimes^\wedge E$). This notion was introduced in **[ER2]** and **[BP]**. In **[ER2]** (resp. **[B1]**), these spaces are denoted by $S_1 \otimes_\mu E$ and $S_1(K) \otimes_\mu E$ (resp. by $S_1 \otimes_{\max} E$ and $S_1(K) \otimes_{\max} E$).

Here S_1 (resp. $S_1(K)$) is viewed as the dual of S_∞ (resp. $S_\infty(K)$) with its *dual operator space structure*, as defined in **[ER2]**, equivalently this is the *standard dual*, as introduced and studied in **[BP]**, **[B2]**. In our special case, it is easy to check that the definition of **[ER2]** (or that of **[BP]**) can be rephrased as follows.

Let us denote by $M_\infty(E)$ the space of all matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with entries in E. Assume $E \subset B(H)$. We view $M_\infty(E)$ as a subspace of $B(\ell_2 \otimes_2 H)$ and we equip it with the induced operator space structure. Consider $u \in S_1 \otimes E$ as a linear subspace of $M_\infty(E)$. Then let (u_{ij}) be the associated element of $M_\infty(E)$. We let $|||u||| = \inf\{\|a\|_{S_2}\|v\|_{S_\infty(E)}\|b\|_{S_2}\}$ where the infimum runs over all the representations of u of the form

$$u = (a \otimes I_E)(v)(b \otimes I_E)$$

with $a, b \in S_2$ and $v \in S_\infty(E)$. Then $S_1 \otimes^\wedge E$ coincides isometrically with the completion of $S_1 \otimes E$ with respect to this norm. This description of the norm in $S_1 \otimes^\wedge E$ (or $S_1 \otimes_\mu E$ in the notation of **[ER2]**) corresponds to the fact, proved in **[ER4]**, that this space can be identified with the space $R \otimes_h E \otimes_h C$. In particular, it is known (see **[BS]** for more in this direction) that

$$M_\infty(E^*) = (R \otimes_h E \otimes_h C)^*$$

completely isometrically. We will define $S_1[E]$ to be the space $S_1 \otimes^\wedge E$ equipped with the operator space structure corresponding to the identification between $R \otimes_h E \otimes_h C$ and $S_1 \otimes^\wedge E$ ($= S_1 \otimes_\mu E$ as defined in [**ER2**]). By definition, we may write

$$S_1[E] \approx R \otimes_h E \otimes_h C \tag{1.1}$$

completely isometrically, so that we have as expected

$$M_\infty(E^*) = (S_1[E])^* \tag{1.1*}$$

completely isometrically.

Similarly, in the case $p = \infty$, it is known (*cf.* [**BP**]) that $M_n(E) = C_n \otimes_h E \otimes_h R_n$ or more generally $S_\infty \otimes_{\min} E = C \otimes_h E \otimes_h R$ so that we can write

$$S_\infty[E] \approx C \otimes_h E \otimes_h R, \tag{1.2}$$

completely isometrically.

In particular (1.1) (resp. (1.2)) allows to identify S_1 (resp. S_∞) with $R \otimes_h C$ (resp. $C \otimes_h R$), via the correspondence $e_{ij} \to e_{1i} \otimes e_{j1}$ (resp. $e_{ij} \to e_{i1} \otimes e_{1j}$).

Clearly, we have a contractive injection

$$S_1[E] \longrightarrow S_\infty[E] \subset M_\infty(E).$$

This allows to consider the pair $(S_\infty[E], S_1[E])$ as a compatible couple of operator spaces, to which we can apply the complex interpolation method, following [**P1**]. More precisely, we introduce the following definition

$$S_p[E] = (S_\infty[E], S_1[E])_\theta \tag{1.3}$$

where $\theta = 1/p$.

By section 2 in [**P1**], this defines an operator space structure on $S_p[E]$. Note that when $\dim(E) = 1$, we obviously recover the natural structure on S_p as defined in the introduction. We will now exploit (1.1) and (1.2) to derive a similar description of $S_p[E]$ using the Haagerup tensor product.

Recall that in [**P1**], the spaces R and C are viewed as a compatible couple (in the sense of interpolation) by identifying elements in R and C if they define the same vector in ℓ_2. With this convention, we can interpolate between R and C (see [**P1**] section 8). We will denote

$$R(\theta) = (R, C)_\theta = (C, R)_{1-\theta},$$

and we set $R(0) = R$, $R(1) = C$. (Recall that $R^{op} \approx C$, hence this notation is coherent with the one in the remark before Theorem 3.4 in [**P1**].) We recall that $\overline{R^*} \approx C$ and $\overline{C^*} \approx R$ so that we have (by Th. 2.2, p. 23 in [**P1**], see also p. 83-87 in [**P1**])

$$\overline{R(\theta)^*} \approx (\overline{R^*}, \overline{C^*})_\theta = (C, R)_\theta \approx (R, C)_{1-\theta}$$

and these are all complete isometries. Furthermore we have

Theorem 1.1. — *Let $1 < p < \infty$ and $\theta = 1/p$. We have a completely isometric isomorphism*

$$S_p[E] \approx R(1-\theta) \otimes_h E \otimes_h R(\theta).$$

Proof. — By Th. 2.3, p. 24 in **[P1]**, by (1.1), (1.2) and our definition (1.3) we have a complete isometry

$$S_p[E] = R(1-\theta) \otimes_h (E \otimes_h R, E \otimes_h C)_\theta$$

hence by Theorem 2.3 in **[P1]** again

$$= R(1-\theta) \otimes_h (E \otimes_h (R, C)_\theta)$$

and since the Haagerup tensor product is associative (*cf.* **[BP]**), we obtain Theorem 1.1. □

Remark. — Clearly the preceding definition (1.3) can be imitated when ℓ_2 is replaced by an arbitrary Hilbert space K. In that case we will denote by

$$S_p[K; E]$$

the resulting operator space defined as in (1.3). When $K = \ell_2^n$ we will denote by $S_p^n[E]$ the corresponding operator space. Note in particular that $S_\infty^n[E] = M_n(E)$ and $S_p^n[E]$ is equal to $M_n(E)$ but equipped with a different norm and a different operator space structure if $1 \le p < \infty$ and $n > 1$.

Corollary 1.2. — *If $u\colon E \to F$ is a c.b. map between operator spaces, then $I_{S_p} \otimes u$ extends to a c.b. map $\widetilde{u}\colon S_p[E] \to S_p[F]$ with $\|\widetilde{u}\|_{cb} = \|u\|_{cb}$. Moreover, if u is a complete isometry from E into F then $\widetilde{u}$ is a complete isometry of $S_p[E]$ into $S_p[F]$.*

Proof. — The first part is clear either by interpolation, or by Theorem 1.1. The second part follows from the fact that the Haagerup tensor product is injective in the category of operator spaces (*cf.* **[PS]**, **[BP]**, **[B1]**). □

Corollary 1.3. — *Consider x in $S_p[E]$. Assume that x is a block-diagonal matrix with blocks x_n in $S_p[E]$. Then we have*

$$\|x\|_{S_p[E]} = \Big(\sum \|x_n\|_{S_p[E]}^p\Big)^{1/p}.$$

In particular, if x is a diagonal matrix with entries x_n in E we have

$$\|x\|_{S_p[E]} = \Big(\sum \|x_n\|_E^p\Big)^{1/p}.$$

Moreover, let $P\colon S_p \to S_p$ be the usual projection onto the diagonal matrices (defined by $P(e_{ij}) = e_{ij}$ if $i = j$ and $= 0$ otherwise). Then $P \otimes I_E$ is a complete contraction on $S_p[E]$.

Proof. — The case $p = \infty$ is clear, $p = 1$ follows by duality and the general case follows by interpolation. □

Corollary 1.4. — *If $1 \le p_0, p_1 \le \infty$ and $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$ then we have completely isometrically*

$$S_{p_\theta}[E] = (S_{p_0}[E], S_{p_1}[E])_\theta. \tag{1.4}$$

More generally, if (E_0, E_1) is a compatible couple of operator spaces and if $E_\theta = (E_0, E_1)_\theta$ (in the sense of section 2 in **[P1]***) then we have completely isometrically*

$$S_{p_\theta}[E_\theta] = (S_{p_0}[E_0], S_{p_1}[E_1])_\theta. \tag{1.5}$$

Proof. — The identity (1.4) can be viewed as a consequence of the classical reiteration theorem (*cf.* **[BL]**, **[Ca]**). Note that (1.5) implies (1.4). To check (1.5) we use Th. 1.1, p. 12 in **[P1]** and Th. 2.3, p. 24 in **[P1]**, together with the associativity of the Haagerup tensor product (*cf.* **[BP]**). Indeed let $\theta_0 = 1/p_0$, $\theta_1 = 1/p_1$, by reiteration we have $R(1-\theta) = (R(1-\theta_0), R(1-\theta_1))_\theta$ and $R(\theta) = (R(\theta_0), R(\theta_1))_\theta$, hence we can write by Theorem 1.1

$$S_{p_\theta}[E_\theta] = R(1-\theta) \otimes_h (E_\theta \otimes_h R(\theta))$$

hence by Theorem 2.3 in **[P1]** applied twice

$$\begin{aligned} &= R(1-\theta) \otimes_h (E_0 \otimes_h R(\theta_0), E_1 \otimes_h R(\theta_1))_\theta \\ &= (R(1-\theta_0) \otimes_h (E_0 \otimes_h R(\theta_0)), R(1-\theta_1) \otimes_h (E_1 \otimes_h R(\theta_1)))_\theta \end{aligned}$$

and by Theorem 1.1 again

$$= (S_{p_0}[E_0], S_{p_1}[E_1])_\theta.$$

□

We must clarify the identifications that we are using. First the completely isometric embedding

$$C \otimes_h E \otimes_h R \longrightarrow S_\infty \otimes E \subset M_\infty(E)$$

is the map J which maps

$$e_{i1} \otimes x \otimes e_{1j} \quad \text{to} \quad e_{ij} \otimes x.$$

This maps allows to identify $C \otimes_h E \otimes_h R$ with the image of J. Now to interpolate, we wish to also "identify" $R \otimes_h E \otimes_h C$ with a subset of $M_\infty(E)$. For that purpose we use the map

$$k\colon\ R \otimes_h E \otimes_h C \longrightarrow M_\infty(E)$$

defined by

$$k(e_{1i} \otimes x \otimes e_{j1}) = e_{ij} \otimes x.$$

This is compatible with the "identification" of e_{i1} and e_{1i} (resp. e_{j1} and e_{1j}) needed to define the interpolation space $(R, C)_\theta$.

This amounts to the following identification of $R(1-\theta) \otimes_h E \otimes_h R(\theta)$ with a subset of $M_\infty(E)$: Let (ξ_i) be the orthonormal basis of $R(1-\theta)$ (corresponding to e_{i1} in C and e_{1i} in R) and let (η_j) be the analogous orthonormal basis of $R(\theta)$. Then the mapping

$$J_p\colon\ R(1-\theta) \otimes_h E \otimes_h R(\theta) \longrightarrow M_\infty(E)$$

which maps $\xi_i \otimes x \otimes \eta_j$ to $e_{ij} \otimes x$ is the "natural" inclusion mapping used in Theorem 1.1 to identify $R(1-\theta) \otimes_h E \otimes_h R(\theta)$ with the subset $S_p[E]$ of $M_\infty(E)$.

Notation. — For any x in $M_\infty(E)$ and any a in M_∞ let us denote by $a\cdot x$ (resp. $x\cdot a$) the matrix product, i.e.

$$(a\cdot x)_{ij} = \sum_k a_{ik}x_{kj} \qquad \left(\text{resp. } (x\cdot a)_{ij} = \sum_k x_{ik}a_{kj}\right).$$

We also denote $a\cdot x\cdot b = a\cdot(x\cdot b)$ (equivalently $= (a\cdot x)\cdot b$) if $b\in M_\infty$.

We can now "compute" the norm in the space $S_p[E]$, as follows.

Theorem 1.5. — *Let $1\le p<\infty$. Let $u\in S_p[E]$ (resp. $u\in S_p^n[E]$) and let $(u_{ij})\in M_\infty(E)$ (resp. $(u_{ij})\in M_n(E)$) be the corresponding matrix with $u_{ij}\in E$. Then $\|u\|_{S_p[E]}$ (resp. $\|u\|_{S_p^n[E]}$) is equal to*

$$\inf\{\|a\|_{S_{2p}}\|v\|_{M_\infty(E)}\|b\|_{S_{2p}}\}$$

where the infimum runs over all representations of the form

$$(u_{ij}) = a\cdot v\cdot b$$

with $a,b\in S_{2p}$ and $v\in S_\infty[E]\subset M_\infty(E)$ (resp. with $a,b\in S_{2p}^n$ and $v\in M_n(E)$).

Proof. — Note that if $p=1$, this is essentially the definition we chose for $S_1[E]$. To check the general case, it is easy to reduce to the case when only finitely many entries (u_{ij}) are nonzero. Then the definition of the Haagerup tensor product and Theorem 1.1 give that $\|u\|_{S_p[E]}$ is the infimum of

$$\left\|\sum_1^n a_i\otimes e_{1i}\right\|_{R(1-\theta)\otimes_{\min}R} \|(v_{ij})\|_{M_n(E)} \left\|\sum_1^n b_j\otimes e_{j1}\right\|_{R(\theta)\otimes_{\min}C}$$

where the infimum runs over all representations of u of the form

$$u=\sum_{i,j=1}^n a_i\otimes v_{ij}\otimes b_j,$$

when u is viewed as an element of $R(1-\theta)\otimes_{\min}E\otimes_{\min}R(\theta)$. Then we note that

$$\begin{aligned} R(1-\theta)\otimes_{\min}R &= (R\otimes_{\min}R, C\otimes_{\min}R)_{1-\theta}\\ &= (C\otimes_{\min}R, R\otimes_{\min}R)_\theta = S_{2p}\end{aligned}$$

and similarly $R(\theta)\otimes_{\min}C = S_{2p}$ isometrically.
Indeed, this follows from the definition of $R(1-\theta)$ (resp. $R(\theta)$), from the fact that R (resp. C) is an injective operator space and finally from the cases $\theta=0$ and $\theta=1$. In these cases, we have isometrically $R\otimes_{\min}R\approx S_2$ and $C\otimes_{\min}C\approx S_2$ on one hand, and $C\otimes_{\min}R\approx S_\infty$ and $R\otimes_{\min}C\approx S_\infty$ on the other (*cf.* **[BP]**, **[ER4]**). The general case $0<\theta<1$ follows by interpolation. See Remark 2.11, p. 37 in **[P1]** and Th. 8.4, p. 83 in **[P1]** for more details.

This implies that $\left\|\sum_1^n a_i \otimes e_{1i}\right\|_{R(1-\theta)\otimes_{\min} R} = \|a\|_{S_{2p}}$ where a is the matrix which admits $a_1, \ldots, a_n$ as its columns. Similarly we have

$$\left\|\sum b_j \otimes e_{j1}\right\|_{R(\theta)\otimes_{\min} C} = \|b\|_{S_{2p}}$$

where b is the matrix admitting $b_1, \ldots, b_n$ as its lines. Finally recalling the correspondence between u and $J_p(u)$, as explained before Theorem 1.5, let $a_i = \sum_k a_i(k)\xi_k$ and $b_j = \sum_\ell b_j(\ell)\eta_\ell$. Then we have

$$\begin{aligned}
(u_{ij}) &= J_p\left(\sum a_i \otimes v_{ij} \otimes b_j\right) \\
&= \sum_{ijk\ell} a_i(k)b_j(\ell)e_{k\ell} \otimes v_{ij} \\
&= \sum_{ijk\ell} a_i(k)b_j(\ell)e_{ki}e_{ij}e_{j\ell} \otimes v_{ij} \\
&= \left(\sum_{ik} a_i(k)e_{ki}\right) \cdot \left(\sum_{ij} e_{ij} \otimes v_{ij}\right) \cdot \left(\sum_{j\ell} e_{j\ell}b_j(\ell)\right).
\end{aligned}$$

Hence since $a = \sum_{ik} a_i(k)e_{ki}$, $b = \sum_{j\ell} e_{j\ell}b_j(\ell)$ we obtain the announced result. □

Let us record the following simple facts.

Lemma 1.6. — *Let $1 \le p \le \infty$.*

(i) *For all x in $S_p[E]$ and all a, b in M_∞ we have*

$$\|a \cdot x \cdot b\|_{S_p[E]} \le \|a\|_{M_\infty} \|x\|_{S_p[E]} \|b\|_{M_\infty}.$$

(ii) *More generally if $a \in S_{2q}$, $b \in S_{2q}$ and if $\frac{1}{t} = \frac{1}{p} + \frac{1}{q} \le 1$ then we have*

$$\|a \cdot x \cdot b\|_{S_t[E]} \le \|a\|_{S_{2q}} \|x\|_{S_p[E]} \|b\|_{S_{2q}}.$$

(iii) *Let P_n be a sequence of mutually orthogonal projections in M_∞ decomposing the identity. Let $q = 2p/p+1$ and let $x_n = P_n \cdot x$ (resp. $x_n = x \cdot P_n$). Then*

$$\left(\sum \|x_n\|_{S_p[E]}^{2p}\right)^{1/2p} \le \|x\|_{S_p[E]} \le \left(\sum \|x_n\|_{S_p[E]}^{q}\right)^{1/q}. \tag{1.6}$$

Proof. — (i) and (ii) are clear either by interpolation or as a consequence of Theorem 1.5. Similarly the left side of (1.6) can be proved by interpolation after checking separately the cases $p = 1$ and $p = \infty$. (Alternatively, it is easy to deduce the left side of (1.6) from Theorem 1.5 and the elementary inequality

$$\forall\, a \in S_{2p} \qquad \left(\sum \|aP_n\|_{S_{2p}}^{2p}\right)^{1/2p} \le \|a\|_{S_{2p}},$$

which itself can be checked by interpolation between $p = 1$ and $p = \infty$.)

Finally, to check the right side of (1.6) we first rewrite it as follows: for any y_n in $S_p[E]$ let $x_n = y_n \cdot P_n$ (resp. $P_n \cdot y_n$) and $x = \sum x_n$. Note that $\|x_n\|_{S_p[E]} \le \|y_n\|_{S_p[E]}$. Then $\|x\|_{S_1[E]} \le \sum \|x_n\|_{S_1[E]} \le \sum \|y_n\|_{S_1[E]}$ by the triangle inequality and by a well known estimate

$$\|x\|_{S_\infty[E]} \le \left(\sum \|x_n\|^2_{S_\infty[E]}\right)^{1/2} \le \left(\sum \|y_n\|^2_{S_\infty[E]}\right)^{1/2}.$$

Hence by interpolation applied to the mapping $(y_n)_n \to \sum y_n \cdot P_n$ (resp. $\sum P_n \cdot y_n$) we obtain the right side of (1.6). □

We will need later the following very useful fact.

Lemma 1.7. — *Let F by any operator space, $n \ge 1$ and let $(y_{ij}) \in M_n(F)$. Then for all $1 \le p \le \infty$*

$$\|(y_{ij})\|_{M_n(F)} = \sup\{\|a \cdot (y_{ij}) \cdot b\|_{S_p^n[F]}, \quad a, b \in B_{S_{2p}^n}\}. \tag{1.7}$$

Consequently, a map $u\colon\ E \to F$ is c.b. iff $\sup_n \|I_{S_p^n} \otimes u\colon\ S_p^n[E] \to S_p^n[F]\| < \infty$, and we have

$$\|u\|_{cb} = \sup_n \|I_{S_p^n} \otimes u\|_{S_p^n[E] \to S_p^n[F]}. \tag{1.7$'$}$$

Proof. — Since $M_n(F)^* = S_1^n(F^*)$ we have

$$\|(y_{ij})\|_{M_n(F)} = \sup\{|\langle y, \xi\rangle| \mid \|\xi\|_{S_1^n[F^*]} \le 1\}$$

hence by Theorem 1.5

$$= \sup\{|\langle y, a \cdot z \cdot b\rangle| \mid a, b \in B_{S_2^n} \quad z \in B_{M_n(F^*)}\}.$$

This yields $\left(\text{note } \langle y, z\rangle = \sum\limits_{ij} \langle y_{ij}, z_{ji}\rangle \text{ hence } \langle y, a \cdot z \cdot b\rangle = \langle b \cdot y \cdot a, z\rangle\right)$

$$\|(y_{ij})\|_{M_n(F)} = \sup\{\|b \cdot y \cdot a\|_{S_1^n[F]} \mid a, b \in B_{S_2^n}\}.$$

This yields (1.7) for $p = 1$. The general case is easy to deduce from Lemma 1.6 (ii), and the fact that any a, b in $B_{S_2^n}$ can be written $a = a''a'$ and $b = b'b''$ with $a', b' \in B_{S_{2p'}^n}$ and $a'', b'' \in B_{S_{2p}^n}$. Indeed, using the last identity, we have

$$\|(y_{ij})\|_{M_n(F)} \le \sup\{\|b'' \cdot y \cdot a''\|_{S_p^n[F]} \mid a'', b'' \in B_{S_{2p}^n}\},$$

and the converse inequality follows from Lemma 1.6 (ii). This proves (1.7). The second assertion is an obvious consequence of (1.7). □

Remark. — Let (A_0, A_1) be a compatible couple of complex Banach spaces. It is well known (*cf.* **[BL]**, **[Ca]**) that $A_0 \cap A_1$ is dense in $(A_0, A_1)_\theta$. Hence, it follows from (1.3) that $S_1[E]$ is dense in $S_p[E]$, or more generally that $\bigcup\limits_n M_n(E)$ is dense in $S_p[E]$.

Corollary 1.8. — *Let $1 < p \le \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then*

$$S_p[E]^* = S_{p'}[E^*]$$

completely isometrically.

Proof. — We first check this for finite dimensional S_p-spaces. Let $R_n(\theta)$ be the n-dimensional version of $R(\theta)$ (equivalently, let $R_n(\theta) = (R_n, C_n)_\theta$). Then Theorem 1.1 yields

$$S_p^n[E] = R_n(1-\theta) \otimes_h E \otimes_h R_n(\theta).$$

Therefore by the self duality of the Haagerup tensor product (*cf.* **[ER4]**, **[B1]**)

$$S_p^n[E]^* \approx R_n(1-\theta)^* \otimes_h E^* \otimes_h R_n(\theta)^*$$

completely isometrically.

Now the standard identification $(S_p^n)^* = S_{p'}^n$ corresponds to the completely isometric identifications $R_n(1-\theta)^* = R_n(\theta)$, $R_n(\theta)^* = R_n(1-\theta)$ and hence $(R_n(1-\theta) \otimes_h R_n(\theta))^* = R_n(\theta) \otimes_h R_n(1-\theta)$. These identifications lead to write

$$S_p^n[E]^* = R_n(\theta) \otimes_h E^* \otimes_h R_n(1-\theta)$$

completely isometrically, and hence by Theorem 1.1

$$= S_{p'}^n[E^*].$$

To complete the proof, it clearly suffices to show that the subspace of $S_p[E]^*$ formed by all matrices $(\xi_{ij})_{i,j\in\mathbb{N}}$, $\xi_{ij} \in E^*$ with only finitely many nonzero entries is dense in $S_p[E]^*$. (Note that this subspace is clearly dense in $S_{p'}[E^*]$ by the remark preceding Corollary 1.8.) We will now justify that this is indeed the case.

For any ξ in $S_p[E]^*$ and a, b in M_∞ we denote by $a \cdot \xi$ and $\xi \cdot b$ the elements of $S_p[E]^*$ defined as usual by

$$(a \cdot \xi)(x) = \xi(x \cdot a) \quad \text{and} \quad (\xi \cdot b)(x) = \xi(b \cdot x).$$

Moreover we denote $a \cdot \xi \cdot b = (a \cdot \xi) \cdot b = a \cdot (\xi \cdot b)$. Then to conclude it suffices to check that, if P_n is the orthogonal projection onto the span of the first n basis vectors in ℓ_2, for all ξ in $S_p[E]^*$ we have

$$P_n \cdot \xi \cdot P_n \to \xi \quad \text{in} \quad S_p[E]^* \quad \text{when} \quad n \to \infty.$$

By successive approximations, it suffices to show that $P_n \cdot \xi \to \xi$ and $\xi \cdot P_n \to \xi$ in $S_p[E]^*$. But now by dualizing (1.6) we obtain (note that if $q = \dfrac{2p}{p+1}$, $\dfrac{1}{q} + \dfrac{1}{2p'} = 1$)

$$\|P_n \cdot \xi\|_{S_p[E]^*}^{2p'} + \|(1-P_n) \cdot \xi\|_{S_p[E]^*}^{2p'} \le \|\xi\|_{S_p[E]^*}^{2p'}. \tag{1.8}$$

On the other hand it is easy to see that

$$\|\xi\|_{S_p[E]^*} = \sup_n \|P_n \cdot \xi\|_{S_p[E]^*} = \lim_{n\to\infty} \|P_n \cdot \xi\|_{S_p[E]^*}$$

hence by (1.8) $\|\xi - P_n \cdot \xi\|_{S_p[E]^*} \to 0$ and by a similar argument we have $\|\xi - \xi \cdot P_n\|_{S_p[E]^*} \to 0$. □

We now turn to an extension of Fubini's theorem to our setting. Let E be an operator space. Recall that when K is an arbitrary Hilbert space, we define $S_\infty[K; E] = S_\infty(K) \otimes_{\min} E$ (viewed as a subspace of $B(K) \otimes_{\min} E$) and $S_1[K; E] =$

$S_1(K)\otimes^\wedge E$ in the sense of **[ER5]** or **[ER4]**. Equivalently, if we denote $K_c = B(\mathbb{C}, K)$, $K_r = B(K^*, \mathbb{C})$ and also $(K^*)_r = B(K, \mathbb{C})$ then we have completely isometrically

$$S_1[K;E] = K_r \otimes_h E \otimes_h (K^*)_c \quad \text{and} \quad S_\infty[K;E] = K_c \otimes_h E \otimes_h (K^*)_r.$$

We define the operator space $S_p[K;E]$ as above by setting

$$S_p[K;E] = (S_\infty[K;E], S_1[K;E])_\theta$$

with $\theta = 1/p$. Clearly all the preceding results extend without any difficulty to this setting. Now let H be another Hilbert space. It is known (*cf.* **[ER4]**) that completely isometrically

$$H_r \otimes_h K_r = (H \otimes_2 K)_r \quad \text{and} \quad K_c \otimes_h H_c = (K \otimes_2 H)_c$$

and also

$$(K^*)_c \otimes_h (H^*)_c = ((K \otimes_2 H)^*)_c \quad \text{and} \quad (K^*)_r \otimes_h (H^*)_r = ((K \otimes_2 H)^*)_r$$

so that we have completely isometrically

$$S_1[H; S_1[K;E]] = S_1[H \otimes_2 K; E]$$
$$S_\infty[H; S_\infty[K;E]] = S_\infty[H \otimes_2 K; E].$$

This allows in particular to "exchange the order of integration", i.e. permute the rôles of H and K. This operation induces again a complete isometry on the preceding spaces. Hence by interpolation, we obtain

Theorem 1.9. — *Let $1 \le p \le \infty$. Let H, K be arbitrary Hilbert spaces and let E be an operator space. We have completely isometrically*

$$S_p[H; S_p[K;E]] \simeq S_p[H \otimes_2 K; E] \simeq S_p[K; S_p[H;E]].$$

Proof. — Using Corollary 1.4, this follows by interpolation from the preceding remarks on the cases $p = 1$ and $p = \infty$. □

More generally, we have

Corollary 1.10. — *In the same situation as in Theorem 1.9, if $1 \le p \le q \le \infty$ we have a complete contraction*

$$S_p[H; S_q(K)] \to S_q[K; S_p(H)].$$

Proof. — This is easy to prove by interpolation between the cases $q = p$ (given by Theorem 1.9) and the case $q = \infty$ (which itself can be checked by interpolation). It then suffices to prove that $S_1[H; S_\infty(K)] \to S_\infty[K; S_1(H)]$ is a complete contraction.

To see this, simply recall that by **[BP]**, **[ER6]** the canonical map from the projective tensor product of two operator spaces into their minimal tensor product is a complete contraction. □

Remark 1.11. — In the particular case $E = \mathbb{C}$, our definition (1.3) reduces to $S_2 = (S_\infty, S_1)_{1/2}$. Hence by **[P1**, Th. 1.1, p. 12**]**, S_2 is completely isometric to $OH(\mathbb{N} \times \mathbb{N})$.

Remark. — Let E be an operator space and let L, K be Hilbert spaces. Let $1 \leq p \leq \infty$. Then Lemma 1.6 (i) can clearly be extended as follows. For any u in $S_p[K, E]$ and for any bounded linear operators $a : K \to L$ and $b : L \to K$, we have

$$\|a \cdot u \cdot b\|_{S_p[L,E]} \leq \|a\| \, \|u\|_{S_p[K,E]} \, \|b\|. \tag{1.9}$$

It will be useful to record here the following two facts.

Lemma 1.12. — *Consider orthogonal projections $P_n \colon \ell_2 \to \ell_2$ with $P_1 \leq P_2 \leq \cdots \leq P_n \leq \cdots$ with $\overline{\cup P_n(\ell_2)} = \ell_2$. Let u be an element of $M_\infty(E)$ such that $\sup_n \|P_n \cdot u \cdot P_n\|_{S_p[E]} < \infty$. Then $u \in S_p[E]$ and $P_n \cdot u \cdot P_n \to u$ in $S_p[E]$. (Note in particular that, as we already saw, if V denotes the subspace of $S_p[E]$ formed of all the matrices (x_{ij}) with only finitely many nonzero entries in E, then V is dense in $S_p[E]$.)*

Proof. — Clearly by Lemma 1.6 (i) we have

$$\|P_n \cdot u \cdot P_n\|_{S_p[E]} \leq \|P_{n+1} \cdot u \cdot P_{n+1}\|_{S_p[E]}$$

for all n. Assume $\sup_n \|P_n \cdot u \cdot P_n\|_{S_p[E]} = 1$. Let $\varepsilon > 0$. Choose N such that

$$\|P_N \cdot u \cdot P_N\|^{2p}_{S_p[E]} > 1 - \varepsilon^{2p}. \tag{1.10}$$

Then for all $n, m \geq N$ we have by (1.10)

$$\|P_n \cdot u \cdot P_m\|^{2p}_{S_p[E]} \geq \|P_N \cdot u \cdot P_N\|^{2p}_{S_p[E]} \geq 1 - \varepsilon^{2p}.$$

In particular by (1.6) we have for all $m \geq n \geq N$

$$\begin{aligned} \|P_n \cdot u \cdot P_n\|^{2p}_{S_p[E]} + \|(P_m - P_n) \cdot u \cdot P_n\|^{2p}_{S_p[E]} &\leq \|P_m \cdot u \cdot P_n\|^{2p}_{S_p[E]} \\ &\leq \|P_m \cdot u \cdot P_m\|^{2p}_{S_p[E]} \leq 1 \end{aligned}$$

hence by (1.10) $\|(P_m - P_n) \cdot u \cdot P_n\|^{2p}_{S_p[E]} \leq \varepsilon^{2p}$ for all $n \geq N$. Similarly we find

$$\text{for } \ m \geq n \geq N \qquad \|P_m \cdot u \cdot (P_m - P_n)\|^{2p}_{S_p[E]} \leq \varepsilon^{2p}.$$

Hence $\|P_m \cdot u \cdot P_m - P_n \cdot u \cdot P_n\|_{S_p[E]} \leq 2\varepsilon$, and $P_n \cdot u \cdot P_n$ is a Cauchy sequence in $S_p[E]$, therefore $P_n \cdot u \cdot P_n$ converges in $S_p[E]$ to a limit which has to be u. □

Lemma 1.13. — *Let H be a Hilbert space. Let $x_i, y_i \in B(H)$ $(i = 1, \ldots, N)$. Assume*

$$\left\|\sum x_i^* x_i\right\| \leq 1 \quad \textit{and} \quad \left\|\sum y_i y_i^*\right\| \leq 1.$$

Then for all x in $S_1(H)$

$$\sum \|x_i x y_i\|_{S_1(H)} \leq \|x\|_{S_1(H)}.$$

Proof. — Consider x in the unit ball of $S_1(H)$. We can write $x = x'x''$ with x', x'' in the unit ball of $S_2(H)$. Then, by Cauchy-Schwarz we have

$$\begin{aligned}\sum \|x_i x y_i\|_1 &= \sum \|x_i x' x'' y_i\|_1 \le \sum \|x_i x'\|_2 \|x'' y_i\|_2 \\ &\le \left(\sum \|x_i x'\|_2^2\right)^{1/2} \left(\sum \|x'' y_i\|_2^2\right)^{1/2} \\ &\le \left(\operatorname{tr}\left(\sum x'^* x_i^* x_i x'\right) \quad \operatorname{tr}\left(\sum x'' y_i y_i^* x''^*\right)\right)^{1/2} \\ &\le \|x'\|_2 \|x''\|_2.\end{aligned}$$

□

The next result will be quite useful in chapter 5. It expresses the concavity of a certain functional, which seems closely related (at least if $p = 2$) to the Wigner-Yanase-Dyson/Lieb inequalities **[L]** (See also **[PW]**).

Lemma 1.14. — *Let H be a Hilbert space. Let $p \ge 1$. Consider $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ in $S_{2p}(H)$ with $a_i \ge 0$, $b_i \ge 0$. Then for all x in $B(H)$ and for all $\lambda_i \ge 0$ with $\sum \lambda_i = 1$ we have*

$$\sum \lambda_k \|a_k x b_k\|_p^p \le \left\| \left(\sum \lambda_k a_k^{2p}\right)^{1/2p} x \left(\sum \lambda_k b_k^{2p}\right)^{1/2p} \right\|_p^p. \tag{1.11}$$

More generally, for any matrix $X = (x_{ij})$ in $S_p(\ell_2) \otimes B(H)$, with entries x_{ij} in $B(H)$, we have

$$\begin{aligned}&\sum_k \lambda_k \|(a_k x_{ij} b_k)\|_{S_p(\ell_2 \otimes H)}^p \\ &\qquad \le \left\| \left(\left(\sum \lambda_k a_k^{2p}\right)^{1/2p} x_{ij} \left(\sum \lambda_k b_k^{2p}\right)^{1/2p} \right) \right\|_{S_p(\ell_2 \otimes H)}^p.\end{aligned}$$

Proof. — The second part is easy to deduce from the first one: we can replace ℓ_2 by ℓ_2^N and then apply (1.11) with $I \otimes a_k \in S_p(\ell_2^N \otimes H)$ and $I \otimes b_k \in S_p(\ell_2^N \otimes H)$ instead of a_k and b_k. Therefore it suffices to prove (1.11).

We first assume that a_i, b_i are all of finite rank so that there is a finite rank orthogonal projection P on H satisfying $Pa_i = a_i P = a_i$, $Pb_i = b_i P = b_i$ for all i. Equivalently we may as well assume that $H = \ell_2^n$, that a_i, b_i are all in S_{2p}^n for some integer $n \ge 1$, and that x is in $B(\ell_2^n) = M_n$. Fix $\varepsilon > 0$. Let

$$s = \sum \lambda_i a_i^{2p} + \varepsilon I \quad \text{and} \quad t = \sum \lambda_i b_i^{2p} + \varepsilon I.$$

Clearly (since $\varepsilon > 0$ is arbitrary), it suffices to show that

$$\forall x \in M_n \qquad \sum_i \lambda_i \|a_i s^{-1/2p} x t^{-1/2p} b_i\|_p^p \le \|x\|_p^p. \tag{1.12}$$

This can be checked by interpolation as follows. Let $\ell_p(\lambda; S_p^n)$ be the space of all sequences $(x_i)_{i \le N}$ with $x_i \in S_p^n$ equipped with the norm $\left(\sum \lambda_i \|x_i\|_p^p\right)^{1/p}$. Let $\varphi_i =$

a_i^{2p} and $\psi_i = b_i^{2p}$. Consider $z \in \mathbb{C}$ with $0 \leq Re(z) \leq 1$. Let $p(z) = 1/Re(z)$. Let

$$T(z)\colon\ S^n_{p(z)} \longrightarrow \ell_{p(z)}(\lambda; S^n_{p(z)})$$

be the linear operator which maps x to the sequence $(\varphi_i^{z/2} s^{-z/2} x t^{-z/2} \psi_i^{z/2})_{i \leq N}$. Observe that $z \to T(z)$ is an analytic function. We claim that $T(z)$ is a contraction for $Re(z) = 0$ and $Re(z) = 1$. Indeed, if $Re(z) = 0$ then $p(z) = \infty$ and this is clear. Moreover, if $Re(z) = 1$ then $p(z) = 1$ and Lemma 1.13 implies that $T(z)$ is a contraction from S_1^n to $\ell_1(\lambda; S_1^n)$. By the Stein interpolation principle for analytic families of operators (*cf.* *e.g.* §10.3, p. 119 in **[Ca]**), it follows that $T(z)$ is a contraction for all z with $0 < Re\ z < 1$.

In particular if $1 < p < \infty$ and $z = 1/p$ we obtain (1.12) and hence (1.11) at least in the finite dimensional case. We now extend (1.11) to the general case. We may clearly assume for notational simplicity that $H = \ell_2$. Consider $a_i, b_i \in S_{2p}$ with $a_i \geq 0$, $b_i \geq 0$. For any $\varepsilon > 0$, we can find (by a simple truncation) α_i, β_i Hermitian of finite rank such that $0 \leq \alpha_i \leq a_i$, $0 \leq \beta_i \leq b_i$, $a_i\alpha_i = \alpha_i a_i$, $b_i\beta_i = \beta_i b_i$ and $\|a_i - \alpha_i\|_{2p} < \varepsilon$, $\|b_i - \beta_i\|_{2p} < \varepsilon$. Let x and λ_i be fixed as in Lemma 1.14. Since the map $(a, b) \to axb$ is continuous from $S_{2p} \times S_{2p}$ to S_p we have

$$\sum \lambda_i \|\alpha_i x \beta_i\|_p^p \to \sum \lambda_i \|a_i x b_i\|_p^p \text{ when } \varepsilon \to 0. \tag{1.13}$$

By the first part of the proof, we have

$$\left(\sum \lambda_i \|\alpha_i x \beta_i\|_p^p\right)^{1/p} \leq \left\| \left(\sum \lambda_i \alpha_i^{2p}\right)^{1/2p} x \left(\sum \lambda_i \beta_i^{2p}\right)^{1/2p} \right\|_p .$$

But $\sum \lambda_i \alpha_i^{2p} \leq \sum \lambda_i a_i^{2p}$ (recall a_i and α_i commute), hence (*cf.* **[Ped]**, p. 8) $\left(\sum \lambda_i \alpha_i^{2p}\right)^{1/p} \leq \left(\sum \lambda_i a_i^{2p}\right)^{1/p}$. Similarly $\left(\sum \lambda_i \beta_i^{2p}\right)^{1/p} \leq \left(\sum \lambda_i b_i^{2p}\right)^{1/p}$. Hence we can write $\left(\sum \lambda_i \alpha_i^{2p}\right)^{1/2p} = u\left(\sum \lambda_i a_i^{2p}\right)^{1/2p}$ and $\left(\sum \lambda_i \beta_i^{2p}\right)^{1/2p} = \left(\sum \lambda_i b_i^{2p}\right)^{1/2p} v$ for some u, v with $\|u\| \leq 1$, $\|v\| \leq 1$. As a consequence we have

$$\left\| \left(\sum \lambda_i \alpha_i^{2p}\right)^{1/2p} x \left(\sum \lambda_i \beta_i^{2p}\right)^{1/2p} \right\|_p \leq \left\| \left(\sum \lambda_i a_i^{2p}\right)^{1/2p} x \left(\sum \lambda_i b_i^{2p}\right)^{1/2p} \right\|_p .$$

Therefore using (1.13) we conclude that (1.11) holds in the infinite dimensional case. □

Remark. — Let X, Y be Banach spaces, let $\|\ \|_\wedge$ be the projective norm and let $X \widehat{\otimes} Y$ be the projective tensor product. Consider an element $u = \sum_1^n x_i \otimes y_i$ in $X \otimes Y$. As is well known, the projective norm $\|\ \|_\wedge$ can be written in many equivalent ways, such as for instance (the infimum being over all possible representation of u)

$$\|u\|_\wedge = \inf \left\{ \sum \|x_i\| \max \|y_i\| \right\}$$

or more generally for any $1 \leq p < \infty$

$$\|u\|_\wedge = \inf \left\{ \left(\sum \|x_i\|^p\right)^{1/p} \left(\sum \|y_i\|^{p'}\right)^{1/p'} \right\} .$$

It is interesting to observe that a similar formula holds for the operator space version of the projective tensor product, as follows.

Proposition 1.15. — *Let E, F be operator spaces. Consider u in $E \otimes F$ of the form*

$$u = \sum_{i,j \leq n} x_{ij} \otimes y_{ij} \tag{1.14}$$

with $x \in M_n(E)$, $y \in M_n(F)$. Then we can write for any $1 \leq p < \infty$

$$\|u\|_{E \otimes^\wedge F} = \inf\{\|x\|_{S_p^n[E]} \|y\|_{S_{p'}^n[F]}\}.$$

Proof. — Consider u satisfying (1.14) with $\|x\|_{S_p^n[E]} < 1$, $\|y\|_{S_{p'}^n[F]} < 1$. Then we can write $x = \alpha \cdot \hat{x} \cdot \beta$, $y = \gamma \cdot \hat{y} \cdot \delta$ with α, β (resp. γ, δ) in the open unit ball of S_{2p}^n (resp. $S_{2p'}^n$) and with $\hat{x}$ (resp. $\hat{y}$) in the open unit ball of $M_n(E)$ (resp. $M_n(F)$). Then by a simple computation

$$u = \sum_{ij} x_{ij} \otimes y_{ij} = \sum_{\ell q r s} ({}^t\alpha\gamma)_{\ell r} \hat{x}_{\ell q} \otimes \hat{y}_{rs} (\beta^t\delta)_{qs}$$

(where ${}^t\alpha$ is the transposed of α, i.e. $({}^t\alpha)_{\ell i} = \alpha_{i\ell}$) hence since

$$\|{}^t\alpha\gamma\|_{S_2^n} \leq \|{}^t\alpha\|_{S_{2p}^n} \|\gamma\|_{S_{2p'}^n} < 1$$

and similarly $\|\beta^t\delta\|_{S_2^n} < 1$, we conclude by (0.6) that $\|u\|_{E \otimes^\wedge F} < 1$.

Conversely, if $\|u\|_{E \otimes^\wedge F} < 1$, by (0.6) we can find for some n a representation

$$u = \sum_{\ell, q, r, s \leq n} a_{\ell r} \hat{x}_{\ell q} \otimes \hat{y}_{rs} b_{qs}$$

with $\hat{x}, \hat{y}$ and $(a_{\ell r})$, (b_{qs}) in the unit ball respectively of $M_n(E)$, $M_n(F)$ and S_2^n. If we now factorize a (resp. b) in the form $a = {}^t\alpha\gamma$ (resp. $b = \beta^t\delta$) with α, β (resp. γ, δ) in the unit ball of S_{2p}^n (resp. $S_{2p'}^n$), then we find conversely x, y as in the first part of the proof with $\|x\|_{S_p^n[E]} \leq 1$, $\|y\|_{S_{p'}^n[F]} \leq 1$. By homogeneity, this completes the proof. □

CHAPTER 2

THE OPERATOR SPACE STRUCTURE OF THE COMMUTATIVE L_p-SPACES

In this section, we wish to explicitly describe the operator space structure (o.s.s. in short) of the usual (= commutative) L_p-spaces. This is somewhat implicit in the preceding section 1 (and in **[P1]**).

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We will denote by $L_p(\mu)$ the L_p-space of complex valued functions. If X is a complex Banach space, we will denote by $L_p(\mu; X)$ the L_p-space of X-valued functions in Bochner's sense. It is well known that

$$L_p(\mu) = (L_\infty(\mu), L_1(\mu))_\theta \quad \text{and} \quad L_p(\mu; X) = (L_\infty(\mu; X), L_1(\mu; X))_\theta$$

with $\theta = 1/p$. This is an *isometric* identity, i.e. it is only valid in the category of Banach spaces. We will introduce a specific operator space structure on $L_p(\mu)$ which we will call the *natural* operator space structure on $L_p(\mu)$.

Firstly, if A is a C^*-algebra, it is clearly equipped with a *privileged* operator space structure (associated to any C^*-embedding of A into $B(H)$). We will call this structure the *natural* one on A. In particular, if $p = \infty$, this selects an operator space structure on $L_\infty(\mu)$ which we will call the natural one.

If $p = 1$, again the choice is clear, the natural structure is defined as the one induced on $L_1(\mu)$ by the dual space $L_\infty(\mu)^*$, equipped with its dual operator space structure. Explicitly, this means that the norm of $M_n(L_1(\mu))$ is by definition the norm induced by $cb(L_\infty(\mu), M_n)$. (Note that $M_n(L_1(\mu))$ can be identified with the $\sigma(L_\infty(\mu), L_1(\mu))$-continuous linear maps from $L_\infty(\mu)$ to M_n.) By a known result (see **[B2]**), this yields an operator space structure on $L_1(\mu)$ such that $L_1(\mu)^* = L_\infty(\mu)$ completely isometrically. For the general case, we use interpolation. We consider the operator space structure on $L_p(\mu)$ which corresponds to the structure of $(L_\infty(\mu), L_1(\mu))_\theta$ as defined in **[P1]**. This means that the norm in $M_n(L_p(\mu))$ is by definition the norm of the space $(M_n(L_\infty(\mu)), M_n(L_1(\mu)))_\theta$ with $\theta = 1/p$.

We will call this structure the *natural* operator space structure on $L_p(\mu)$. Similarly, if $E \subset B(H)$ is an operator space then $L_\infty(\mu; E)$ embeds isometrically into the C^*-algebra $L_\infty(\mu; B(H))$, and we will call *natural* the operator space structure induced by the natural one on $L_\infty(\mu; B(H))$. If $p = 1$, by **[ER8]** we have an isometry $L_1(\mu; E) =$

$L_1(\mu)\otimes^\wedge E$ hence we can equip $L_1(\mu; E)$ with the operator space structure of $L_1(\mu)\otimes^\wedge E$. We will call this the *natural* structure on $L_1(\mu; E)$.

Finally, if $1 < p < \infty$ we again use interpolation and define (following **[P1]**) the natural o.s.s. on $L_p(\mu; E)$ as the one corresponding to $(L_\infty(\mu; E), L_1(\mu; E))_\theta$ with $\theta = 1/p$. In the particular case where $(\Omega, \mathcal{A}, \mu)$ is $\mathbb{N}$ equipped with the counting measure, the spaces $L_p(\mu)$ and $L_p(\mu; E)$ are denoted by ℓ_p and $\ell_p(E)$. The preceding definitions apply of course to this case, so that we have defined the natural o.s.s. on ℓ_p or on $\ell_p(E)$.

Moreover, consider the subspace $c_0(E) \subset \ell_\infty(E)$ of all sequences tending to zero. We can equip it with the structure induced by the natural one on $\ell_\infty(E)$. Equivalently this is the o.s.s. of the space $c_0 \otimes_{\min} E$. More generally for any locally compact (resp. compact) set K, let $C_0(K)$ (resp. $C(K)$) be the C^*-algebra of all complex valued continuous functions on K which tend to zero at infinity. We denote $C_0(K; E) = C_0(K)\otimes_{\min} E$ (resp. $C(K; E) = C(K)\otimes_{\min} E$) and we equip it with the corresponding o.s.s. We will say that these o.s.s. on $c_0(E)$, $C_0(K; E)$ and $C(K; E)$ are the *natural* ones. Note that by Lemma 0.1, for any Radon measure μ on K, we have (completely isometrically) $(C_0(K; E), L_1(\mu; E))_\theta = L_p(\mu; E)$ with $\theta = 1/p$.

The next result allows to "compute" these natural structures more explicitly.

Proposition 2.1. — *Let $1 \leq p < \infty$. Let E be an operator space.*

(i) *Let $a = (a_{ij}) \in M_n \otimes L_p(\mu; E)$. We have*
$$\|(a_{ij})\|_{M_n(L_p(\mu;E))} = \sup\{\|\alpha \cdot (a) \cdot \beta\|_{S_p^n[L_p(\mu;E)]} \mid \alpha, \beta \in B_{S_{2p}^n}\}.$$

(ii) *The spaces $L_p(\mu; S_p)$ and $S_p[L_p(\mu)]$ are completely isometric. More generally, $L_p(\mu; S_p[E])$ and $S_p[L_p(\mu; E)]$ are completely isometric.*

(iii) *In particular, $L_2(\mu)$ is completely isometric to $OH(I)$, where I is the cardinal of an orthonormal basis of $L_2(\mu)$.*

Proof

(i) is but an immediate application of Lemma 1.7.

(ii) It clearly suffices to prove this with S_p^n in the place of S_p. Using the isometric identity $L_p(\mu; S_p^n) = (L_1(\mu; S_1^n), L_\infty(\mu; S_\infty^n))_\theta$ ($\theta = 1/p$) and using Corollary 1.4, we are reduced by interpolation to the cases $p = 1$ and $p = \infty$. Since these cases are clear by our definitions (for $p = 1$, see **[ER8]**) this shows that $L_p(\mu; S_p^n)$ and $S_p^n[L_p(\mu)]$ are isometric. The same argument applies for $L_p(\mu; S_p^n[E])$ and $S_p^n[L_p(\mu; E)]$. Using Lemma 1.7 and (i) we then easily obtain that this is a complete isometry.

(iii) This follows either from (i) by a direct calculation or by Corollary 2.4 in **[P1]**.

□

Remark. — The proof of (ii) is more transparent if one first proves that for any compatible couple of operator spaces (E_0, E_1) with $E_\theta = (E_0, E_1)_\theta$ $(0 < \theta < 1)$ one has a completely isometric identity

$$L_p(\mu; E_\theta) = (L_{p_0}(\mu; E_0), L_{p_1}(\mu; E_1))_\theta \tag{2.1}$$

where $1 \le p,\ p_0,\ p_1 \le \infty$ (assuming that $p_0,\ p_1$ are not both infinite), $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$. Indeed, using (1.5) and (2.1) it is easy to derive (ii) in the preceding statement. To prove (2.1), one way is to reduce it by elementary approximations to the complete isometry

$$\ell_p(E_\theta) = (\ell_{p_0}(E_0), \ell_{p_1}(E_1))_\theta. \tag{2.2}$$

Then using Corollary 1.3 one can check that, for any operator space E, $\ell_p(E)$ is completely isometric to the subspace of $S_p[E]$ formed of all the diagonal matrices and moreover the usual projection onto this subspace is a complete contraction. Then (2.2) is a simple consequence of (1.5).

Notation. — Let $E \subset L_p(\mu)$ be a closed subspace and let X be a Banach space. We will denote by $E \otimes_p X$ the closure of $E \otimes X$ in $L_p(\mu; X)$. If X is an operator space, we equip $E \otimes_p X$ with the operator space structure induced by the natural o.s. structure on $L_p(\mu; X)$. We will again refer to this structure on $E \otimes_p X$ as the natural o.s.s. on $E \otimes_p X$.

The following is an immediate consequence of Proposition 2.1. (ii).

Corollary 2.2. — *Let $E \subset L_p(\mu)$ be a closed subspace. Then $E \otimes_p S_p = S_p[E]$ completely isometrically. More generally, for any operator space X we have $E \otimes_p S_p[X] = S_p[E \otimes_p X]$ completely isometrically.*

Proposition 2.3. — *Consider two measure spaces (Ω, μ) and (Ω', μ') and the associated L_p-spaces $L_p(\mu)$ and $L_p(\mu')$. Let $E \subset L_p(\mu)$ and $F \subset L_p(\mu')$ be closed subspaces. We equip E and F with the operator space structures induced by the natural ones on $L_p(\mu)$ and $L_p(\mu')$. Let $u\colon\ E \to F$ be a linear map. Then u is c.b. iff for each $n \ge 1$ the map $u \otimes I_{S_p^n}\colon\ E \otimes_p S_p^n \to F \otimes_p S_p^n$ is bounded and we have $\sup\limits_{n\ge 1} \|u \otimes I_{S_p^n}\| < \infty$. Moreover,*

$$\|u\|_{cb} = \sup_{n \ge 1} \|u \otimes I_{S_p^n}\|_{E\otimes_p S_p^n \to F \otimes_p S_p^n}. \tag{2.3}$$

Proof. — By (1.7)' and Corollary 2.2 we obviously have (2.3). □

For emphasis, we spell out a particular case:

Proposition 2.4. — *A linear map $u\colon\ L_p(\mu) \to L_p(\mu)$ is completely bounded (on $L_p(\mu)$ equipped with its natural o.s.s.) iff the mapping $u \otimes I_{S_p}$ is bounded on $L_p(\mu; S_p)$. Moreover, we have*

$$\|u\|_{cb(L_p(\mu), L_p(\mu))} = \|u \otimes I_{S_p}\|_{L_p(\mu; S_p) \to L_p(\mu; S_p)}.$$

Furthermore, u is a complete isometry (resp. isomorphism) iff $u \otimes I_{S_p}$ is an isometry (resp. isomorphism).

Remark 2.5. — Assume now $E \subset S_p$ and $F \subset S_p$. Let $E \otimes_p X$ be the closure of $E \otimes X$ in $S_p[X]$, when X is an operator space. Then clearly (2.3) remains true with the same proof. Moreover by Theorem 1.9, Corollary 2.2 also extends with the same proof.

To illustrate these remarks, let E be the subspace of $B(\ell_2) \oplus_\infty B(\ell_2)$ which is the closed linear span of the vectors $\delta_i = e_{i1} \oplus e_{1i}$, $i \in \mathbb{N}$. This space is denoted by $R \cap C$ in **[P1]** and its operator space dual E^* is denoted there by $R + C$. As Banach spaces E and E^* are clearly isometric to ℓ_2.

Now let $\mathcal{R}_p$ (resp. $\mathcal{G}_p$) be the subspace of $L_p([0,1], dt)$ spanned by the classical Rademacher functions (r_i) (resp. by a sequence (g_i) of i.i.d. standard Gaussian random variables).

Then $\mathcal{R}_1$ and $\mathcal{G}_1$ are each completely isomorphic to E^*. This is but a reformulation of the main result of **[LuP]**. More generally, when $1 < p < \infty$, using Lemma 1.7 and the results of **[LuP]**, one can describe the natural operator space structures of the spaces $\mathcal{R}_p$ and $\mathcal{G}_p$. The cases $1 < p < 2$ and $2 \le p < \infty$ have to be treated separately. See §8.3 and 8.4 for a detailed presentation of these examples. In particular, this shows (see §8.4) that the orthogonal projection from L_2 onto $\mathcal{R}_2$ is completely bounded on L_p for all $1 < p < \infty$. Similarly the fact that the Hilbert transform on the circle is bounded on $L_p(S_p)$ means that it defines a completely bounded map on L_p. Equivalently, the usual (orthogonal) projection is a *c.b.* map from L_p onto the Hardy space H^p (of the circle) for any $1 < p < \infty$. See §8.1 for more on this.

It will be useful in chapter 5 to introduce the notion of direct sum in the sense of ℓ_p of a family $\{E_i \mid i \in I\}$ of operator spaces. Let $\mu = \{\mu_i \mid i \in I\}$ be a family of positive numbers and let $1 \le p < \infty$. We will denote by $\ell_p(\mu; \{E_i\})$ the space of all families $x = (x_i)_{i \in I}$ with $x_i \in E_i$ such that $\sum \mu_i \|x_i\|^p < \infty$, and we equip it with the norm

$$\|x\| = \left(\sum_{i \in I} \mu_i \|x_i\|^p \right)^{1/p}.$$

If $p = \infty$, we will denote (for convenience) again by $\ell_\infty(\mu; \{E_i\})$ the space of bounded families $x = (x_i)_{i \in I}$ equipped with the norm $\|x\| = \sup_{i \in I} \|x_i\|$.

If $\mu_i = 1$ for all i in I, we will denote the space $\ell_p(\mu; \{E_i\})$ simply by $\ell_p(\{E_i\})$. We will also consider the subspace $c_0(\{E_i\}) \subset \ell_\infty(\{E_i\})$ formed of all $x = (x_i)_{i \in I}$ such that $\|x_i\| \to 0$ when $i \to \infty$ in the discrete topology on I. Clearly $\ell_\infty(\{E_i\})$, and a fortiori $c_0(\{E_i\})$, are operator spaces in an obvious way: if $E_i \subset B(H_i)$ (completely isometric embedding) then we simply use the block-diagonal isometric embedding $\ell_\infty(\{E_i\}) \subset B\left(\bigoplus_{i \in I} H_i\right)$ and we equip $\ell_\infty(\{E_i\})$ with the operator space structure induced by $B\left(\bigoplus_{i \in I} H_i\right)$. Equivalently, we have isometrically

$$\forall n \ge 1 \qquad M_n(\ell_\infty(\{E_i\})) = \ell_\infty(\{M_n(E_i)\}).$$

When the family $\{E_i\}$ is reduced to two operator spaces E, F, the space $\ell_p(\{E_i\})$ will be denoted simply by

$$E \oplus_p F.$$

When $p = \infty$, the operator space $E \oplus_\infty F$ reduces to the direct sum denoted previously by $E \oplus F$ in chapter 0.

The spaces $\ell_p(\mu; \{E_i\})$ can be equipped with a specific operator space structure, which we will call again their *natural* o.s.s., and which we now briefly describe. (This is entirely analogous to the preceding discussion.)

For $p = 1$, we use duality: since the dual of $\ell_1(\mu; \{E_i\})$ is isometric to $\ell_\infty(\mu; \{E_i^*\})$ we have $\ell_1(\mu; \{E_i\}) \subset \ell_\infty(\mu; \{E_i^*\})^*$ (isometric embedding) and we equip $\ell_1(\mu; \{E_i\})$ with the o.s.s. induced by $(\ell_\infty(\mu; \{E_i^*\}))^*$ (the latter space being equipped with its dual o.s.s.).

For the general case $1 < p < \infty$ we use complex interpolation: we note that $\ell_1(\mu; \{E_i\})$ and $\ell_\infty(\mu; \{E_i\})$ form a compatible couple of operator spaces continuously injected into the topological vector space $\prod_{i \in I} E_i$ and we define

$$\ell_p(\mu; \{E_i\}) = (\ell_\infty(\mu; \{E_i\}), \ell_1(\mu; \{E_i\}))_\theta \tag{2.4}$$

with $\theta = 1/p$.

We now summarize the main properties of these direct sums for $1 \leq p < \infty$. We leave the proofs to the reader. They are all easy adaptations of the corresponding arguments in section 1.

(2.5) If $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have completely isometrically

$$\ell_p(\mu; \{E_i\})^* = \ell_{p'}(\mu; \{E_i^*\}).$$

(2.6)' Let $F_i \subset E_i$ be a family of closed subspaces of E_i. Then $\ell_p(\mu; \{F_i\})$ embeds completely isometrically into $\ell_p(\mu; \{E_i\})$. Moreover, $\ell_p(\mu; \{E_i/F_i\})$ is completely isometric to $\ell_p(\mu; \{E_i\})/\ell_p(\mu; \{F_i\})$.

(2.6)" If $\{G_i \mid i \in I\}$ is another family of operator spaces and if $u_i\colon E_i \to G_i$ are c.b. maps with $\sup_{i \in I} \|u_i\| < \infty$, then the direct sum $u = \bigoplus_{i \in I} u_i$ (which maps $(x_i)_{i \in I}$ to $(u_i(x_i))_{i \in I}$) is c.b. from $\ell_p(\mu; \{E_i\})$ to $\ell_p(\mu; \{G_i\})$ with $\|u\|_{cb} = \sup_{i \in I} \|u_i\|_{cb}$.

(2.7) Let $1 \leq p_0, p_1 \leq \infty$, assume that p_0, p_1 are not both infinite, and let

$$\{(E_i^0, E_i^1) \mid i \in I\}$$

be a family of compatible couples of interpolation spaces. Then $\ell_{p_0}(\mu; \{E_i^0\})$ and $\ell_{p_1}(\mu; \{E_i^1\})$ form a compatible couple in the obvious way. Let $E_i^\theta = (E_i^0, E_i^1)_\theta$ for $0 < \theta < 1$. Then we have a completely isometric identity

$$(\ell_{p_0}(\mu; \{E_i^0\}), \ell_{p_1}(\mu; \{E_i^1\}))_\theta = \ell_p(\mu; \{E_i^\theta\})$$

where $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$.

(2.8) If $\{\nu_j \mid j \in J\}$ is another family of positive numbers, let $\mu \otimes \nu$ denote the family $\{\mu_i \nu_j \mid i \in I, j \in J\}$. For each j in J, assume given a family $\{E_i^j \mid i \in I\}$ of operator spaces. We have then a completely isometric identity

$$\ell_p(\nu; \{\ell_p(\mu; \{E_i^j \mid i \in I\})\}) = \ell_p(\mu \otimes \nu; \{E_i^j \mid (i,j) \in I \times J\}).$$

(2.9) For any Hilbert space K we have an isometry (actually a complete isometry)

$$S_p[K; \ell_p(\mu; \{E_i\})] = \ell_p(\mu; \{S_p[K; E_i]\}). \tag{2.10}$$

In particular, if $K = \ell_2^n$, then the isometric identity (2.10) allows to compute by Lemma 1.7 the operator space structure of the space $\ell_p(\mu; \{E_i\})$.

CHAPTER 3

NON-COMMUTATIVE VECTOR VALUED L_p-SPACES (CONTINUOUS CASE)

Consider now a von Neumann algebra M equipped with a faithful normal semi-finite trace φ. Let M_* be the predual of M. It will be convenient to use the notation

$$L_1(\varphi) = M_*.$$

Recall (see e.g. [**Ta**], p. 321) that $L_1(\varphi)$ can be described more concretely as the completion of the linear space of all elements x in M such that $\varphi(|x|) < \infty$ with respect to the norm $\|x\|_1 = \varphi(|x|)$.

Let E be an operator space. We define

$$L_1(\varphi; E) = L_1(\varphi) \otimes^\wedge E$$

where $\otimes^\wedge$ denotes the operator space version of the projective tensor product in the sense of [**ER2**] and [**BP**].

If M is finite and injective, we have by [**ER6**] a c.b. inclusion

$$M \otimes_{\min} E \to L_1(\varphi; E),$$

therefore we can consider $(M \otimes_{\min} E, L_1(\varphi; E))$ as a compatible couple. To justify this, note that by [**ER7**] a finite algebra M is injective iff the canonical inclusion $v\colon\ M \to M_* = L_1(\varphi)$ is integral in the sense of [**ER6**], i.e. v is a point-norm limit of mappings $v_i\colon\ M \to M_*$ which are matricially nuclear in the sense of [**ER6**], with matricially nuclear norm majorized by a fixed constant C. This implies that for any finite dimensional subspace $F \subset M$, the restriction $v_{|F}$ is matricially nuclear with matricially nuclear norm $\leq C$. Therefore for any t in $F \otimes E \subset M \otimes E$ we have

$$\|(v \otimes I_E)(t)\|_{L_1(\varphi)\otimes^\wedge E} \leq C\|t\|_{\min}.$$

In other words, $v \otimes I_E$ defines a bounded map $\widetilde{v}$ from the completion $M \otimes_{\min} E$ into $L_1(\varphi) \otimes^\wedge E$. It is then rather easy (left to the reader) to check that $\widetilde{v}$ is one to one. With some additional effort it can be checked that $\widetilde{v}$ is actually c.b.

If M is semi-finite and injective, let J be the set of all finite nonzero projections in M. For each σ in J, let φ_σ be the restriction of φ to $\sigma M \sigma$. The map $j_\sigma\colon\ x \to \sigma x \sigma$ defines a c.b. map from $L_1(\varphi) \otimes^\wedge E$ (resp. $M \otimes_{\min} E$) to $L_1(\varphi_\sigma) \otimes^\wedge E$ (resp.

$(\sigma M \sigma) \otimes_{\min} E)$. Then considering $J(x) = (j_\sigma(x))_{\sigma \in J}$ we obtain continuous injections of both $L_1(\varphi) \otimes^\wedge E$ and $M \otimes_{\min} E$ into the topological vector space $\prod_{\sigma \in J}(L_1(\varphi_\sigma) \otimes^\wedge E)$. This allows us to consider $(M \otimes_{\min} E, L_1(\varphi) \otimes^\wedge E)$ as a compatible couple for interpolation, and we can formulate the following.

Definition 3.1. — Let φ be a semi-finite normal faithful trace on an injective von Neumann algebra M and let E be an operator space. If $1 < p < \infty$, we define

$$L_p(\varphi; E) = (M \otimes_{\min} E, L_1(\varphi; E))_\theta \tag{3.1}$$

where $\theta = 1/p$. If $p = \infty$, we denote $L^0_\infty(\varphi; E) = M \otimes_{\min} E$.

We do not attempt to define $L_\infty(\varphi; E)$. We will work instead with $L^0_\infty(\varphi; E)$ which behaves equally well with respect to interpolation.

Note that when $\dim(E) = 1$, we obviously recover the natural structure on $L_p(\varphi)$ as defined in the introduction.

Remark 3.2. — By a standard reasoning, one can prove that the linear tensor product $L_p(\varphi) \otimes E$ is dense in $L_p(\varphi; E)$ for $1 \leq p < \infty$. (Recall that $A_0 \cap A_1$ is dense in $(A_0, A_1)_\theta$ if $0 < \theta < 1$, for any compatible couple (A_0, A_1) of Banach spaces.)

We will now show that, *if M is hyperfinite*, this definition has all the nice properties developed in section 1 in the discrete case.

Unless specified otherwise, we assume in this section that $1 \leq p < \infty$.

First we consider the finite dimensional case.

Lemma 3.3. — *Let M be a finite dimensional von Neumann algebra equipped with a finite faithful trace φ. Then (as is well known) there is a decomposition of the unit as a sum $I = \sum_{i \in I} p_i$ where I is a finite set, each p_i is a central projection in M, and each algebra $p_i M p_i$ (with unit p_i) is isomorphic to a matrix algebra M_{n_i} for some $n_i \geq 1$. Let $\mu_i = \varphi(p_i)$. Then the space $L_p(\varphi; E)$ is completely isometric to $\ell_p(\mu; \{S_p^{n_i}[E]\})$.*

Proof. — By well known facts, this is true if $p = \infty$ and $p = 1$, hence, by interpolation using (3.1) and (2.4), it holds for any $1 < p < \infty$. □

Theorem 3.4. — *Let M be a hyperfinite von Neumann algebra, i.e. we have $M = \overline{\cup M_\alpha}$ (weak-$*$ closure) where M_α is a net of finite dimensional $*$-subalgebras directed by inclusion. Let φ be a faithful, normal semi-finite trace on M. Let φ_α be its restriction to M_α. We assume that φ_α is finite for all α, so that $M_\alpha \subset L_p(\varphi)$. Then, for each α, $L_p(\varphi_\alpha; E)$ can be identified completely isometrically with the subspace $M_\alpha \otimes E \subset L_p(\varphi; E)$ and the union $\bigcup_\alpha L_p(\varphi_\alpha; E)$ is dense in $L_p(\varphi; E)$. Moreover, for each α, there is a completely contractive projection Q_α from $L_p(\varphi; E)$ onto $L_p(\varphi_\alpha; E)$, so that we have $Q_\alpha = Q_\alpha Q_\beta$ whenever $\alpha \leq \beta$, and for any x in $L_p(\varphi; E)$, $Q_\alpha(x)$ tends to x, when α tends to infinity.*

Proof. — We have simultaneously a natural completely contractive inclusion map $J_\alpha\colon\ M_\alpha \to M$ and $J_\alpha\colon\ L_1(\varphi_\alpha) \to L_1(\varphi)$. On the other hand, we have a normal projection (actually a "conditional expectation", see the next remark) $P_\alpha\colon\ M \to M_\alpha$ which is simultaneously a complete contraction from M to M_α and from $L_1(\varphi)$ to $L_1(\varphi_\alpha)$, cf. **[Ta]**, p. 332. Therefore, by interpolation we have a diagram

$$L_p(\varphi_\alpha; E) \xrightarrow{K_\alpha} L_p(\varphi; E) \xrightarrow{Q_\alpha} L_p(\varphi_\alpha; E)$$

where the maps $K_\alpha = J_\alpha \otimes I_E, Q_\alpha = P_\alpha \otimes I_E$ are complete contractions and the composition $Q_\alpha K_\alpha$ is the identity on $L_p(\varphi_\alpha; E)$. In particular, this guarantees that K_α is a completely isometric embedding of $L_p(\varphi_\alpha; E)$ into $L_p(\varphi; E)$ and its image is $M_\alpha \otimes E$, whence the first assertion. Moreover, it is well known that $\bigcup\limits_{\alpha\in A} M_\alpha$ is dense in $L_p(\varphi)$, hence by Remark 3.2 $\bigcup\limits_{\alpha\in A} L_p(\varphi_\alpha; E)$ is dense in $L_p(\varphi; E)$. It is clear that $Q_\alpha(x)$ tends to x for any x in $\bigcup\limits_{\alpha\in A} L_p(\varphi_\alpha; E)$, hence for any x in $L_p(\varphi; E)$ by density. This concludes the proof. □

Remark. — Let p_α be the self-adjoint projection which is the unit element in M_α. When φ is infinite, we do not have $p_\alpha = 1_M$ in the preceding proof, hence $P_\alpha\colon\ M \to M_\alpha$ cannot preserve the unit. In this case, the projection P_α can be written as $P_\alpha = P'_\alpha P''_\alpha$ where $P''_\alpha(x) = p_\alpha x p_\alpha$ and where P'_α is the usual (=unit preserving) conditional expectation from $p_\alpha M p_\alpha$ onto M_α (which both admit p_α as their unit). Although this is a slight abuse, we still refer to P_α as a conditional expectation in this case.

Remark. — In the situation of Theorem 3.4, let $C = \overline{\cup M_\alpha}$ (norm closure). By Lemma 0.1, if $1 < p < \infty$, we can identify completely isometrically $L_p(\varphi; E)$ with $(C \otimes_{\min} E, L_1(\varphi; E))_\theta$ for $\theta = 1/p$.

Moreover, again by Lemma 0.1, if E is a von Neumann algebra, and if $M\bar{\otimes}E$ denotes the von Neumann tensor product (which by **[ER2]** can be identified with the dual of $L_1(\varphi; E_*)$), then we can identify completely isometrically $L_p(\varphi; E)$ with $(M\bar{\otimes}E, L_1(\varphi; E))_\theta$ for $\theta = 1/p$.

Remark. — Note that for any hyperfinite von Neumann algebra equipped with a faithful normal semi-finite trace, there is a directed net $(M_\alpha)_{\alpha\in A}$ satisfying the properties of Theorem 3.4. Hence, it is now easy to extend to the hyperfinite case most of the properties of $S_p[E]$ to the case of $L_p(\varphi; E)$.

More explicitly we have, if M is hyperfinite the following properties:

(3.1) If $u\colon\ E \to F$ is a c.b. map between operator spaces, then $I_{L_p(\varphi)} \otimes u$ extends to a c.b. map $\widetilde{u}$ from $L_p(\varphi; E)$ into $L_p(\varphi; F)$ with $\|\widetilde{u}\|_{cb} \leq \|u\|_{cb}$.

(3.2) If u is a complete isometry (resp. a completely isomorphic embedding), then $\widetilde{u}$ also is.

(3.3) If u is surjective and such that the canonical map $E/Ker(u) \to F$ is a complete isometry (resp. a complete isomorphism), then $\widetilde{u}$ is surjective and the associated map $L_p(\varphi; E)/Ker(\widetilde{u}) \to L_p(\varphi; F)$ is a complete isometry (resp. a complete isomorphism).

(3.4) In particular, for any closed subspace $S \subset E$, $L_p(\varphi; S)$ can be identified (completely isometrically) with a closed subspace of $L_p(\varphi; E)$ and we have

$$L_p(\varphi; E/S) \approx L_p(\varphi; E)/L_p(\varphi; S)$$

completely isometrically.

(3.5) Assume $0 < \theta < 1$, $1 \leq p_0,\ p_1 < \infty$, $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$. Let (E_0, E_1) be a compatible couple of operator spaces and let $E_\theta = (E_0, E_1)_\theta$. Then there is a completely isometric identity

$$(L_{p_0}(\varphi; E_0), L_{p_1}(\varphi; E_1))_\theta = L_p(\varphi; E_\theta).$$

Moreover, when $p_0 = \infty$ and $p_1 < \infty$, this becomes

$$(L^0_\infty(\varphi; E_0), L_{p_1}(\varphi; E_1))_\theta = L_p(\varphi; E_\theta),$$

and when $p_0 = p_1 = \infty$, it becomes

$$(L^0_\infty(\varphi; E_0), L^0_\infty(\varphi; E_1))_\theta = L^0_\infty(\varphi; E_\theta).$$

(3.6) Let (N, ψ) be another hyperfinite von Neumann algebra with a faithful normal semi-finite (in short f.n.s.) trace ψ. Then the von Neumann tensor product $M\overline{\otimes}N$ admits $\varphi \otimes \psi$ as a f.n.s. trace and clearly is hyperfinite. For any operator space E, we have completely isometric isomorphisms

$$L_p(\varphi; L_p(\psi; E)) \approx L_p(\varphi \otimes \psi; E) \approx L_p(\psi; L_p(\varphi; E)).$$

(3.6)' Let (N, ψ) be an arbitrary von Neumann algebra with a faithful normal semi-finite (in short f.n.s.) trace ψ. Then, we have a completely isometric identity

$$L_p(\varphi; L_p(\psi)) \approx L_p(\varphi \otimes \psi).$$

To check (3.6)', recall that $L_p(\varphi \otimes \psi) = (L_\infty(\varphi \otimes \psi), L_1(\varphi \otimes \psi))_\theta$ with $\theta = 1/p$. Furthermore, by a simple application of Lemma 0.1 using the hyperfiniteness of M, we find $L_p(\varphi \otimes \psi) = (M \otimes_{\min} N, L_1(\varphi \otimes \psi))_\theta$, which can be rewritten as $L_p(\varphi \otimes \psi) = (L^0_\infty(\varphi; E_0), L_1(\varphi; E_1))_\theta$ with $E_0 = L_\infty(\psi)$ and $E_1 = L_1(\psi)$. Since $E_\theta = L_p(\psi)$, we can deduce (3.6)' from the second part of (3.5).

We will now extend the formulae proved in section 1 to the nondiscrete case. To some extent, the definition of the vector valued non-commutative L_p-spaces makes sense for a general semi-finite von Neumann algebra M. This corresponds to the spaces $\Lambda_p(M, \varphi; E)$ which we introduce below. However, we will quickly show that only in the hyperfinite case does this definition have the natural properties to be expected.

Let M be an arbitrary semi-finite von Neumann algebra with a f.n.s. trace φ and let E be an operator space. Consider y in $M \otimes E$. We can write

$$y = \sum\nolimits_1^n y_i \otimes e_i \quad \text{with} \quad y_i \in M, e_i \in E.$$

Let $a, b \in L_{2p}(\varphi)$. To shorten the notation we denote simply

$$a \cdot y = \sum a y_i \otimes e_i \qquad y \cdot b = \sum y_i b \otimes e_i$$

(these are elements of $L_{2p}(\varphi) \otimes E$) and

$$a \cdot y \cdot b = \sum a y_i b \otimes e_i \in L_p(\varphi) \otimes E.$$

Let us denote by $V \subset L_p(\varphi)$ the subspace

$$V = \bigcup QMQ$$

where the union runs over all projections Q in M such that $\varphi(Q) < \infty$. Note that the set of all such projections forms a lattice (by e.g. **[Ta]**, V.1.6) so that this union is directed by inclusion. Clearly, by the semi-finiteness of φ, V is dense in $L_p(\varphi)$.

Consider an element x in $V \otimes E$. Clearly x can be written as $x = a \cdot y \cdot b$ with $a, b \in V$ and $y \in M \otimes E$. We define

$$\|x\|_{\Lambda_p(E)} = \inf\{\|a\|_{L_{2p}(\varphi)} \|b\|_{L_{2p}(\varphi)} \|y\|_{M \otimes_{\min} E}\}$$

where the infimum runs over all possible such representations.

Lemma 3.5. — *For $1 \leq p < \infty$, $\| \ \|_{\Lambda_p(E)}$ is a norm on $V \otimes E$.*

Proof. — Consider x_1, x_2 in $V \otimes E$. Let $\varepsilon > 0$. We can write $x_1 = a_1 \cdot y_1 \cdot b_1$, $x_2 = a_2 \cdot y_2 \cdot b_2$ with

$$\|a_1\|_{L_{2p}(\varphi)} = \|b_1\|_{L_{2p}(\varphi)} = \|x_1\|^{1/2}_{\Lambda_p(E)}, \quad \|a_2\|_{L_{2p}(\varphi)} = \|b_2\|_{L_{2p}(\varphi)} = \|x_2\|^{1/2}_{\Lambda_p(E)} \tag{3.7}$$

and

$$\|y_1\|_{M \otimes_{\min} E} < 1 + \varepsilon, \quad \|y_2\|_{M \otimes_{\min} E} < 1 + \varepsilon. \tag{3.8}$$

Moreover, we can assume that a_1, b_1, a_2, b_2 belong to QMQ for some projection Q in M with $\varphi(Q) < \infty$. Then we define

$$a = (a_1 a_1^* + a_2 a_2^* + \varepsilon Q)^{1/2} \qquad b = (b_1^* b_1 + b_2^* b_2 + \varepsilon Q)^{1/2}$$

and (for $j = 1, 2$)

$$c_j = a^{-1} a_j, \qquad d_j = b_j b^{-1}$$

where the inverses a^{-1} and b^{-1} are meant in the finite von Neumann algebra QMQ with unit Q. Note that

$$c_1 c_1^* + c_2 c_2^* \leq I \quad \text{and} \quad d_1^* d_1 + d_2^* d_2 \leq I. \tag{3.9}$$

We have

$$x_1 + x_2 = a \cdot Y \cdot b \tag{3.10}$$

where $Y = c_1 \cdot y_1 \cdot d_1 + c_2 \cdot y_2 \cdot d_2$. Let $c = \left(\begin{smallmatrix} c_1 & c_2 \\ 0 & 0 \end{smallmatrix}\right)$ and $d = \left(\begin{smallmatrix} d_1 & 0 \\ d_2 & 0 \end{smallmatrix}\right)$ in $M_2(M)$. We then observe that in $M_2(M) \otimes E$ we have

$$\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} = c \cdot \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \cdot d.$$

Hence by the characteristic property of operator spaces (see (0.1) and (0.2)) we have

$$\|Y\|_{\min} = \|Y\|_{M\otimes_{\min}E} \le \|c\|_{M_2(M)} \max\{\|y_1\|_{\min}, \|y_2\|_{\min}\}\|d\|_{M_2(M)};$$

hence by (3.8) and (3.9)

$$\le 1+\varepsilon.$$

Finally we have by (3.10)

$$\|x_1 + x_2\|_{\Lambda_p(E)} \le \|a\|_{L_{2p}(\varphi)}\|Y\|_{\min}\|b\|_{L_{2p}(\varphi)}. \tag{3.11}$$

Moreover by the triangle inequality in $L_p(\varphi)$

$$\begin{aligned}\|a\|_{L_{2p}(\varphi)} &= \|a_1a_1^* + a_2a_2^* + \varepsilon Q\|_{L_p(\varphi)}^{1/2}\\ &\le (\|a_1a_1^*\|_{L_p(\varphi)} + \|a_2a_2^*\|_{L_p(\varphi)} + \varepsilon\varphi(Q)^{1/p})^{1/2}\\ &\le (\|a_1\|_{L_{2p}(\varphi)}^2 + \|a_2\|_{L_{2p}(\varphi)}^2 + \varepsilon\varphi(Q)^{1/p})^{1/2}\end{aligned}$$

hence by (3.7)

$$\le (\|x_1\|_{\Lambda_p(E)} + \|x_2\|_{\Lambda_p(E)} + \varepsilon\varphi(Q)^{1/p})^{1/2}.$$

A similar upper bound holds for $\|b\|_{L_{2p}(\varphi)}$. Therefore we deduce from (3.11)

$$\|x_1 + x_2\|_{\Lambda_p(E)} \le (1+\varepsilon)(\|x_1\|_{\Lambda_p(E)} + \|x_2\|_{\Lambda_p(E)} + \varepsilon\varphi(Q)^{1/p})$$

and since $\varepsilon > 0$ is arbitrary, we obtain

$$\|x_1 + x_2\|_{\Lambda_p(E)} \le \|x_1\|_{\Lambda_p(E)} + \|x_2\|_{\Lambda_p(E)}.$$

Finally the fact that $\|x\|_{\Lambda_p(E)} = 0 \Rightarrow x = 0$ follows from the next lemma. □

Lemma 3.6. — *For any x in $V \otimes E$ we have*

$$\|x\|_{L_p(\varphi)\overset{\vee}{\otimes} E} \le \|x\|_{\Lambda_p(E)} \le \|x\|_{L_p(\varphi)\overset{\wedge}{\otimes} E} \tag{3.12}$$

where $L_p(\varphi)\overset{\vee}{\otimes} E$ (resp. $L_p(\varphi)\overset{\wedge}{\otimes} E$) is the injective (resp. projective) tensor product in the Banach space sense, corresponding to the smallest (largest) cross norm.

Proof. — Let $x = \sum ay_ib \otimes e_i$ with $y_i \in M \otimes E$, $a, b \in V$. For any ξ in the unit ball of E^*, we have clearly

$$\begin{aligned}\left\|\sum ay_ib\ \xi(e_i)\right\|_{L_p(\varphi)} &\le \|a\|_{L_{2p}(\varphi)}\|b\|_{L_{2p}(\varphi)} \left\|\sum y_i\xi(e_i)\right\|_M\\ &\le \|a\|_{L_{2p}(\varphi)}\|b\|_{L_{2p}(\varphi)} \left\|\sum y_i \otimes e_i\right\|_{M\otimes_{\min}E}.\end{aligned}$$

Therefore taking the supremum over all ξ's, we obtain the left side of (3.12). To prove the right side, note that if $x \in V \otimes E$ it is easy to check that the norm of x in $L_p(\varphi)\widehat{\otimes}E$ coincides with the projective norm of $V \otimes E$ i.e. we have

$$\|x\|_{L_p(\varphi)\widehat{\otimes}E} = \inf\left\{\sum_1^n \|v_i\|_{L_p(\varphi)}\|e_i\|\right\}$$

where the infimum runs over all representation of the form $x = \sum_1^n v_i \otimes e_i$ with $v_i \in V$, $e_i \in E$. The rest of the proof is then clear. We leave the details to the reader. $\square$

We can now define the space
$$\Lambda_p(M, \varphi; E)$$
as the completion of $V \otimes E$ for the norm $\| \ \|_{\Lambda_p(E)}$. For simplicity, we will sometimes abbreviate this by $\Lambda_p(\varphi; E)$ or by $\Lambda_p(E)$. It is rather easy to check that the space $\Lambda_p(\varphi; E)$ satisfies the properties (3.1) and (3.2). However, in general it does not satisfy (3.3) and (3.4), which implies that in general it cannot satisfy the duality theorem of the kind proved in the next section in the hyperfinite case.

The next lemma is elementary.

Lemma 3.7. — *Let (M_α) and M be as in Theorem 3.4. Let $P_\alpha\colon\ M \to M_\alpha$ be the conditional expectation (as in the proof of Theorem 3.4). Then for all y in M and all b in $L_{2p}(\varphi)$ we have*
$$\|(y - P_\alpha y)b\|_{L_{2p}(\varphi)} \to 0$$
when $\alpha \to \infty$ along the directed net.

Proof. — Let $b_\beta = P_\beta b$. Clearly $b_\beta \to b$ in $L_{2p}(\varphi)$ when $\beta \to \infty$. Fix $\varepsilon > 0$ and β such that $\|b - b_\beta\|_{L_{2p}(\varphi)} < \varepsilon$. Then by the conditional expectation property we have for $\alpha > \beta$
$$(y - P_\alpha y)b = (y - P_\alpha y)(b - b_\beta) + [yb_\beta - P_\alpha(yb_\beta)].$$
Note that $P_\alpha(yb_\beta) \to yb_\beta$ in $L_{2p}(\varphi)$ when $\alpha \to \infty$, hence
$$\overline{\lim_{\alpha\to\infty}}\ \|(y - P_\alpha y)b\|_{L_{2p}(\varphi)} \le 2\varepsilon\|y\|_M,$$
and since $\varepsilon > 0$ is arbitrary, we obtain the announced result. $\square$

Theorem 3.8. — *Let $1 \le p < \infty$ and let E be any operator space. Then, if M is hyperfinite, we have an isometric identity*
$$\Lambda_p(\varphi; E) = L_p(\varphi; E).$$

Proof. — We use the notation of Theorem 3.4. Note that $\bigcup M_\alpha \subset V$. Using Theorem 3.4, Theorem 1.5 and Lemma 3.3, it is easy to check that for any x in $\left(\bigcup M_\alpha\right) \otimes E$ we have

$$\|x\|_{\Lambda_p(\varphi;E)} \le \|x\|_{L_p(\varphi;E)}. \tag{3.13}$$

By Lemma 3.6 (and the similar property for $L_p(\varphi; E)$), since $\bigcup M_\alpha$ is dense in $L_p(\varphi)$, to conclude it suffices to prove the converse to (3.13), which we now proceed to do. Assume that $x \in \left(\bigcup M_\alpha\right) \otimes E$ and $\|x\|_{\Lambda_p(E)} < 1$, so that $x = a \cdot y \cdot b$ with $a, b \in V$, $y \in M \otimes E$ such that
$$\|a\|_{L_{2p}(\varphi)} < 1, \quad \|b\|_{L_{2p}(\varphi)} < 1, \quad \|y\|_{M\otimes_{\min}E} < 1.$$
Let $P_\alpha\colon\ M \to M_\alpha$ be the conditional expectation. Let $a_\alpha = P_\alpha a$, $b_\alpha = P_\alpha b$. Clearly $a_\alpha \to a$ and $b_\alpha \to b$ in $L_{2p}(\varphi)$. Let $y_\alpha = (P_\alpha \otimes I_E)y$. Finally let $x_\alpha = a_\alpha \cdot y_\alpha \cdot b_\alpha$. Clearly

by Lemma 3.3 and Theorem 1.5 $\|x_\alpha\|_{L_p(\varphi_\alpha;E)} < 1$, hence a fortiori $\|x_\alpha\|_{L_p(\varphi;E)} < 1$ by Theorem 3.4.

We have

$$x - x_\alpha = (a - a_\alpha) \cdot y \cdot b + a_\alpha \cdot (y - y_\alpha) \cdot b + a_\alpha \cdot y_\alpha \cdot (b - b_\alpha).$$

Clearly $(a-a_\alpha)\cdot y\cdot b$ and $a_\alpha \cdot y_\alpha \cdot (b-b_\alpha)$ tend to zero in $L_p(\varphi)\widehat{\otimes}E$, and by Lemma 3.7, the same is true for $a_\alpha \cdot (y - y_\alpha) \cdot b$. Therefore $\|x - x_\alpha\|_{L_p(\varphi)\widehat{\otimes}E} \to 0$ when $\alpha \to \infty$. Clearly we have $\|z\|_{L_p(\varphi;E)} \le \|z\|_{L_p(\varphi)\widehat{\otimes}E}$ for any z in $\left(\bigcup M_\alpha\right) \otimes E$, hence we have $\|x - x_\alpha\|_{L_p(\varphi;E)} \to 0$ when $\alpha \to \infty$.

Finally, we conclude that

$$\|x\|_{L_p(\varphi;E)} \le \|x_\alpha\|_{L_p(\varphi;E)} + \|x - x_\alpha\|_{L_p(\varphi;E)}$$

hence

$$\|x\|_{L_p(\varphi;E)} \le 1.$$

By homogeneity, this proves the converse to (3.13).

Therefore, the completion of $\left(\bigcup M_\alpha\right) \otimes E$ under the two norms appearing in (3.13) can be isometrically identified. □

It will be convenient to record here the following simple consequence of (3.6).

Proposition 3.9. — *Let $E = OH(I)$ for some set I. Let (M, φ) be as in definition 3.1. Then, for any bounded map v: $L_2(\varphi) \to L_2(\varphi)$, the map $v \otimes I_E$ extends to a bounded map on $L_2(\varphi; E)$ with the same norm as v.*

Proof. — We may identify (by **[P1]**) E with $\ell_2(I)$, completely isometrically. Then, (3.6) gives us an isometric isomorphism

$$L_2(\varphi; E) = \ell_2(\{E_i, i \in I\})$$

where the family appearing on the right is simply defined by $E_i = L_2(\varphi)$ for all i in I. Using the norm of the space $\ell_2(\{E_i, i \in I\})$, the content of Proposition 3.9 is obvious. □

CHAPTER 4

DUALITY, NON-COMMUTATIVE RNP AND UNIFORM CONVEXITY

Before we discuss the duality, we need to introduce martingales in our setting. Let (M, φ) be any finite and hyperfinite von Neumann algebra equipped with a faithful normal trace φ with $\varphi(1) = 1$. Let (M_n) be an increasing sequence of von Neumann subalgebras and let $E^{M_n}\colon\ M \to M_n$ denote the conditional expectation operator (see **[Ta]**, p. 332). Let $\varphi_n = \varphi_{|M_n}$. We can identify $L_1(\varphi_n)$ with a closed subspace of $L_1(\varphi)$ and E^{M_n} defines a completely contractive projection from $L_1(\varphi)$ onto $L_1(\varphi_n)$. Let $1 \le p < \infty$ and let E be an arbitrary operator space. As explained in the proof of Theorem 3.4, we can identify $L_p(\varphi_n; E)$ with the closure of $L_p(\varphi_n) \otimes E$ in $L_p(\varphi; E)$. Moreover, the conditional expectation defines a complete contraction from $L_p(\varphi; E)$ onto $L_p(\varphi_n; E)$ which we will still denote by E^{M_n}.

Then a sequence (x_n) in $L_1(\varphi; E)$ is called a martingale if, for some sequence (M_n) as above, we have $x_n \in L_1(\varphi_n; E)$ and $x_n = \mathbf{E}^{M_n}(x_{n+1})$ for all $n = 0, 1, \ldots$.

Let M_∞ be the von Neumann algebra generated by the union of the M_n's. It can be shown by routine arguments (as in Theorem 3.4) that, for any x in $L_1(\varphi; E)$, the sequence defined by $x_n = \mathbf{E}^{M_n}(x)$ is a martingale which converges in norm in $L_1(\varphi; E)$ to $\mathbf{E}^{M_\infty}(x)$ when n tends to infinity. Moreover, when $x \in L_p(\varphi; E)$ $(p < \infty)$, then the convergence holds in $L_p(\varphi; E)$.

When M is commutative, so that we are back to the classical probability situation, it is well known (see *e.g.* **[DU]**) that, for any $1 < p < \infty$, a Banach space E has the Radon Nikodym property (in short RNP) iff every martingale (x_n) which is bounded in $L_p(\varphi; E)$, actually converges in $L_p(\varphi; E)$.

We will now examine this topic in the non-commutative case. We first turn to the duality. Let (M_α) be a net of finite dimensional $*$-subalgebras directed by inclusion, with the same notation as in Theorem 3.4. Let $1 < p < \infty$ and let $p' = \dfrac{p}{p-1}$. Let E be an arbitrary operator space. Applying Theorem 3.4 to $L_{p'}(\varphi; E^*)$, we find a completely contractive projection

$$Q_\alpha\colon\ L_{p'}(\varphi; E^*) \longrightarrow L_{p'}(\varphi_\alpha; E^*)$$

associated to the conditional expectation from $L_1(\varphi)$ onto $L_1(\varphi_\alpha)$.

By (2.5) we know that

$$L_{p'}(\varphi_\alpha; E^*) = L_p(\varphi_\alpha; E)^* \tag{4.1}$$

completely isometrically.

Consider now ξ in $L_{p'}(\varphi; E^*)$. Using (4.1), we regard $Q_\alpha(\xi)$ as an element of $L_p(\varphi_\alpha; E)^*$. Consider now x in $\bigcup\limits_\alpha L_p(\varphi_\alpha; E) \subset L_p(\varphi; E)$. We have $x \in L_p(\varphi_\alpha; E)$ for some α and the value of $\langle Q_\alpha(\xi), x\rangle$ is independent of the choice of α (since (M_α) is directed by inclusion). Therefore we can unambiguously define (for any such α)

$$\langle \xi, x\rangle = \langle Q_\alpha(\xi), x\rangle.$$

We have

$$|\langle \xi, x\rangle| \leq \|Q_\alpha(\xi)\|_{L_{p'}(\varphi_\alpha; E^*)} \|x\|_{L_p(\varphi_\alpha; E)} \leq \|\xi\|_{L_{p'}(\varphi; E^*)} \|x\|_{L_p(\varphi; E)}$$

hence by density ξ defines a linear form $\widetilde{\xi}$ in $L_p(\varphi; E)^*$ with $\|\widetilde{\xi}\|_{L_p(\varphi; E)^*} \leq \|\xi\|_{L_{p'}(\varphi; E^*)}$. Actually by (4.1) we have $\|\widetilde{\xi}\|_{L_p(\varphi; E)^*} = \|\xi\|_{L_{p'}(\varphi; E^*)}$ for all ξ in $L_{p'}(\varphi_\alpha; E^*)$, hence by density this remains true for all ξ in $L_{p'}(\varphi; E^*)$. Moreover, since (4.1) is a complete isometry, we can now state

Theorem 4.1. — *The correspondence $\xi \to \widetilde{\xi}$ is a completely isometric embedding from $L_{p'}(\varphi; E^*)$ into $L_p(\varphi; E)^*$.*

If M is commutative and if E^* has the Radon Nikodym property (in short RNP), then it is well known that $L_p(\varphi; E)^* = L_{p'}(\varphi; E^*)$. See **[DU]** for more information on the RNP.

It is natural to wonder whether this identity remains valid in the non-commutative case. Quite interestingly, it turns out that the answer is no, even in the reflexive case, actually even in the case when E is isometrically Hilbertian, as the following example shows.

Example 4.2. — Consider the operator space E obtained by embedding ℓ_2 isometrically into the commutative C^*-algebra $C(T)$ with $T = B_{\ell_2}$. Following **[Pa2]**, we denote this operator space by $\min(\ell_2)$. Let $\mathcal{M}$ be the hyperfinite II_1 factor equipped with its normalized trace τ. The space $(\mathcal{M}, \tau)$ can be described as an infinite tensor product of M_2 as follows.

Here M_2 means the algebra of all 2×2 matrices equipped with its normalized trace t. For each $k = 1, 2, \ldots$ we set $(A_k, t_k) = (M_2, t)$. Then we have (in the von Neumann sense)

$$(\mathcal{M}, \tau) = \bigotimes_{k=1}^{\infty} (A_k, t_k).$$

Let $\mathcal{M}_n \subset \mathcal{M}$ be the subalgebra corresponding to $A_1 \otimes \cdots \otimes A_n$. By a classical construction, there is a sequence (V_n) in $\mathcal{M}$ with $V_n \in \mathcal{M}_n$ for all n and such that

$$\forall n > 0 \qquad E^{M_n}(V_{n+1}) = 0 \tag{4.2}$$

satisfying the canonical anticommutation relations (CAR) as follows: $\forall i,\, j = 1, 2, \ldots$

$$\left\{\begin{array}{l} V_iV_j^* + V_j^*V_i = \delta_{ij}I \\ V_iV_j + V_jV_i = 0. \end{array}\right. \tag{4.3}$$

For an explicit construction with the "Pauli spin matrices", see page 795 in [**KR**]. As shown by a simple computation, for any finite sequence (α_i) of complex numbers, we have (see e.g. [**BR**], p. 15)

$$\left\|\sum \alpha_i V_i\right\| = \left(\sum |\alpha_i|^2\right)^{1/2}. \tag{4.4}$$

Let (e_i) be the canonical basis of ℓ_2 and let d_n be the element of $\mathcal{M} \otimes \min(\ell_2)$ defined by

$$d_n = V_n \otimes e_n.$$

Then for any N and any z_n in $\mathbb{C}$ with $|z_n| = 1$, we have

$$\left\|\sum_{n=1}^{N} z_n d_n\right\|_{\mathcal{M}\otimes_{\min}\min(\ell_2)} \leq 1, \tag{4.5}$$

and on the other hand

$$\|d_n\|_{L_1(\tau;\min(\ell_2))} = \|V_n\|_{L_1(\tau)} \geq 1/2. \tag{4.6}$$

To prove (4.5), note that (by definition of $\min(\ell_2)$) the left side of (4.5) is the same as $\left\|\sum_1^N z_n V_n \otimes e_n\right\|_\vee$ (where $\|\ \ \|_\vee$ is the norm of the injective Banach space tensor product $\mathcal{M}\check{\otimes}\ell_2$) or equivalently this is the same as the usual norm of the operator $T_N\colon\ \ell_2 \to \mathcal{M}$ taking (α_n) to $\sum_1^N z_n\alpha_n V_n$, and by (4.4) we have $\|T_N\| \leq 1$. This proves (4.5).

To verify (4.6), note that (4.3) implies $\tau(V_n^*V_n + V_nV_n^*) = \tau(I) = 1$, hence $\|V_n\|_{L_2(\mathcal{M},\tau)} = 2^{-1/2}$. Then we have

$$2^{-1/2} = \|V_n\|_{L_2(\tau)} \leq (\|V_n\|_{L_1(\tau)}\|V_n\|_{L_\infty(\tau)})^{1/2} \leq (\|V_n\|_{L_1(\tau)})^{1/2},$$

whence (4.6).

By (4.5) and (4.6), the sequence $x_n = \sum_1^N d_n$ (with say $x_0 = 0$) is bounded in $L_p(\tau;\min(\ell_2))$ for all $1 \leq p \leq \infty$ (and by (4.2), it is a martingale) but it does not converge in $L_p(\tau;\min(\ell_2))$ for any p. Moreover, it is easy to deduce from (4.5) and (4.6) that for all finite sequences of scalars (α_n) we have for all $1 \leq p \leq \infty$

$$\frac{1}{2}\sup|\alpha_n| \leq \left\|\sum \alpha_n d_n\right\|_{L_p(\tau;\min(\ell_2))} \leq \sup|\alpha_n|. \tag{4.7}$$

In particular, we have proved

Proposition 4.3. — *Let $(\mathcal{M},\tau)$ be the hyperfinite II_1 factor as above. The Banach space $L_p(\tau;\min(\ell_2))$ contains a subspace isomorphic to c_0 for all $1 \leq p < \infty$. In particular it is not reflexive.*

The preceding fact suggests to study the non-commutative version of the RNP, which can be introduced as follows.

Definition 4.4. — Let (M, φ) be as above and let (M_n) be an increasing sequence of subalgebras. Let $1 < p < \infty$ (resp. $p = \infty$). We will say that an operator space E has the $ORNP_p$ with respect to (M_n) if every martingale adapted to (M_n) and bounded in $L_p(\varphi; E)$ converges in $L_p(\varphi; E)$ (resp. in $L_1(\varphi; E)$). We will say that E has the $ORNP$ if it has the $ORNP_p$ for all $1 < p \leq \infty$.

It is probably true that the $ORNP_p$ does not depend on p but we have not been able to verify this at the time of this writing. In another direction, it is probably true that if E satisfies the $ORNP_p$ with respect to the hyperfinite II_1 factor $\mathcal{M}$ and its natural subalgebras $(\mathcal{M}_n)$ as above, then it satisfies the $ORNP_p$ in general, but we did not verify this in detail at this time.

Notation. — Let (M, φ) be a hyperfinite finite von Neumann algebra as above with a normalized faithful normal trace φ.

In the sequel we will say that such a pair (M, φ) is a non-commutative probability space (in short n.c.p. space).

The trace φ defines a linear form $L_1(\varphi) \to \mathbb{C}$ which extends to a completely contractive mapping $L_1(\varphi; E) \to E$. We will denote this mapping again by φ so that for any x in $L_1(\varphi; E)$, $\varphi(x)$ is an element of E analogous to "the integral of x with respect to φ".

Let $N \subset M$ be a subalgebra. We will denote by E^N the conditional expectation operator from $L_1(M, \varphi; E)$ onto $L_1(N, \varphi_{|N}; E)$.

There is also a natural non-commutative analog of uniform convexity which we now describe. We will say that an operator space E is uniformly OS-convex if for each $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ with the following property: For any n.c.p. space (M, φ) and for any x in $M \otimes_{\min} E$ with $\|x\|_{\min} \leq 1$ such that $\|x - \varphi(x)\|_{L_1(\varphi;E)} \geq \varepsilon$, we have

$$\|\varphi(x)\| \leq 1 - \delta(\varepsilon).$$

If $E = OH(I)$ for some I, then by Proposition 3.9 we have for all x in $L_2(\varphi; E)$

$$\|\varphi(x)\|^2 + \|x - \varphi(x)\|^2_{L_2(\varphi;E)} \leq \|x\|^2_{L_2(\varphi;E)}, \tag{4.8}$$

hence $OH(I)$ is uniformly OS-convex. More generally, for any subalgebra $N \subset M$, if $E = OH(I)$ we have (by Proposition 3.9 again) for all x in $L_2(\varphi; E)$

$$\|E^N x\|^2_{L_2(\varphi;E)} + \|x - E^N x\|^2_{L_2(\varphi;E)} \leq \|x\|^2_{L_2(\varphi;E)}, \tag{4.9}$$

from which it is easy to deduce that $E = OH(I)$ satisfies the $ORNP_2$ (see Proposition 4.5 below for more details).

On the other hand, if E is an arbitrary operator space, and $1 \leq q \leq \infty$ we have for all x in $L_q(\varphi; E)$

$$\sup\{\|E^N x\|_{L_q(\varphi;E)}, 2^{-1}\|x - E^N x\|_{L_q(\varphi;E)}\} \leq \|x\|_{L_q(\varphi;E)}. \tag{4.10}$$

In particular, this holds if $E = OH(I)$. By interpolation it follows that if $E = OH(I)$ and if $1 < q < \infty$ the following inequality holds:

$$\|E^N x\|^r_{L_q(\varphi;E)} + \delta_q \|x - E^N x\|^r_{L_q(\varphi;E)} \leq \|x\|^r_{L_q(\varphi;E)} \tag{4.11}$$

where $r = \max(q, q')$ and where $\delta_q = 2^{2-r}$. In particular, this shows (see Proposition 4.5 below) that $OH(I)$ has the $ORNP_q$ for all $1 < q \leq \infty$.

Consider now an operator space of the form $E = (A_0, A_1)_\theta$ with $0 < \theta < 1$ with A_0 arbitrary and with A_1 completely isometric to $OH(I)$ for some I. We can "interpolate" between (4.11) for $E = A_1$ and (4.10) with $E = A_0$. The result is as follows. Let $r = \max(2/\theta, q, q')$. Then, for any $1 < q < \infty$ there is a number $\delta = \delta(r, q) > 0$ such that for any x in $L_q(\varphi; E)$ and any $N \subset M$ we have

$$\|E^N x\|^r_{L_q(\varphi;E)} + \delta\|x - E^N x\|^r_{L_q(\varphi;E)} \leq \|x\|^r_{L_q(\varphi;E)}. \tag{4.12}$$

If (4.12) holds with $r = q$ for some $\delta > 0$ we will say that E is q-uniformly OS-convex. Clearly this implies that E is uniformly OS-convex. The preceding discussion shows in particular that any non-commutative L_p-space (equipped with its natural operator space structure) is uniformly OS-convex.

It will be useful to record the following simple fact.

Proposition 4.5. — *Fix $\delta > 0$, $r > 0$ and $1 < q < \infty$. Let E be an operator space. Assume that E satisfies (4.12) for any n.c.p. space M and any subalgebra $N \subset M$. Then E has the $ORNP_q$.*

Proof. — We repeat a classical argument. Let (M_n) be an increasing sequence of subalgebras and let (x_n) be a martingale as in Definition 4.4. Since $n \to \|x_n\|_{L_q(\varphi;E)}$ is bounded and nondecreasing it converges to a limit c. Let $\varepsilon > 0$. Choose n_0 such that $\|x_n\|^r_{L_q(\varphi;E)} > c^r - \varepsilon$ for all $n \geq n_0$. Then we have by (4.12) for all n, m with $n \geq m \geq n_0$

$$\|x_m\|^r_{L_q(\varphi;E)} + \delta\|x_n - x_m\|^r_{L_q(\varphi;E)} \leq \|x_n\|^r_{L_q(\varphi;E)}$$

hence

$$\|x_n - x_m\|_{L_q(\varphi;E)} \leq (\varepsilon/\delta)^{1/r}.$$

Therefore (x_n) converges in $L_q(\varphi; E)$ by the Cauchy criterion. □

By the preceding discussion, we have

Corollary 4.6. — *For $1 < p < \infty$, any non-commutative L_p-space (equipped with its natural operator space structure) has the $ORNP_q$ for all $1 < q < \infty$.*

Finally, we can complete the discussion of the duality, in analogy with the commutative case (see [**DU**]).

Theorem 4.7. — *Let (M, φ) be any n.c.p. space. Let E be an operator space. If E^* has the $ORNP_{p'}$ with $1 < p < \infty$ and $p' = p/p - 1$, then we have a completely isometric identity*

$$L_p(\varphi; E)^* = L_{p'}(\varphi; E^*).$$

Proof. — We simply repeat the well known argument for the commutative case. Let (M_α) be a directed net of finite dimensional subalgebras of M with union weakly dense in M. Then any ξ in $L_p(\varphi; E)^*$ defines by restriction an element ξ_α in $L_{p'}(\varphi_\alpha; E^*) \subset L_{p'}(\varphi; E^*)$. Moreover, $\sup\limits_\alpha \|\xi_\alpha\|_{L_{p'}(\varphi;E^*)} \leq \|\xi\|_{L_p(\varphi;E)^*}$. We claim that the resulting net (ξ_α) converges in $L_{p'}(\varphi; E^*)$. Otherwise there would exist a countable increasing subnet α_n such that (ξ_{α_n}) diverges and this would contradict the assumption that E^* has the $ORNP_{p'}$. Let $\hat{\xi} = \lim_{\alpha\to\infty} \xi_\alpha$. Then $\hat{\xi} \in L_{p'}(\varphi; E^*)$. Clearly the element of $L_p(\varphi; E)^*$ associated to $\hat{\xi}$ by the proof of Theorem 4.1 coincides with the original functional ξ. Hence this proves that the inclusion in Theorem 4.1 is actually surjective. □

In Banach space valued martingale theory, the notion of UMD-space (UMD stands for "unconditional martingale differences") plays an important rôle. A Banach space B is called UMD if, for each $1 < p < \infty$, there is a constant C such that, any martingale (f_n) in $L_p(B) = L_p(\Omega, \mathcal{A}, P; B)$ (on an arbitrary probability space) satisfies

$$\forall N \ \ \forall \varepsilon_n = \pm 1 \qquad \left\| f_0 + \sum\nolimits_1^N \varepsilon_n (f_n - f_{n-1}) \right\|_{L_p(B)} \leq C \|f_N\|_{L_p(B)}.$$

Actually, if this holds for *some* $1 < p < \infty$, then it holds for all $1 < p < \infty$ as above. See **[Bu2]** for more information and references on this. This notion was inspired by the classical work of Burkholder and Gundy for scalar martingales (see **[Bu1]**).

Recently, a non-commutative version of the Burkholder-Gundy inequalities was obtained in **[PX2]**. The results of **[PX2]** naturally suggest the following definitions and a number of related questions.

Definition 4.8. — With the same notation as in Definition 4.4, we will say that an operator space E is UMD_p with respect to (M_n) if there is a constant C such that any martingale (f_n) in $L_p(\varphi; E)$, adapted to (M_n), satisfies

$$(4.13) \quad \forall N \geq 1 \ \ \forall \varepsilon_n = \pm 1 \qquad \left\| f_0 + \sum\nolimits_1^N \varepsilon_n (f_n - f_{n-1}) \right\|_{L_p(\varphi;E)} \leq C \|f_N\|_{L_p(\varphi;E)}.$$

When this holds for every (M, φ) and every filtration, we will say that E is UMD_p.

By the main result of **[PX2]**, (4.13) holds if $E = \mathbb{C}$ or more generally, if E is itself a non-commutative L_p-space, for example if $E = S_p$.

However, very few examples are known at this point and a lot needs to be done. Here are a few natural questions which come to mind (some of them might be quite easy):

4.9. Is (4.13) satisfied when $E = OH$ and $1 < p \neq 2 < \infty$?
4.10. Same question with $E = S_q$ for $1 < q \neq p < \infty$?
4.11. If E is UMD_p for some $1 < p < \infty$, is it UMD_p for all $1 < p < \infty$?
4.12. If an operator space is UMD_p with respect to the standard filtration appearing in Example 4.2, is it UMD_p?
4.13. A necessary condition for E to be UMD_p is that $S_p[E]$ be UMD as a Banach space. Is this condition sufficient?

CHAPTER 5

COMPLETELY p-SUMMING MAPS

Let $1 \leq p < \infty$. Let E, F be operator spaces and let $u\colon\ E \to F$ be a linear map. We will say that u is "completely p-summing" if the mapping

$$U = I_{S_p} \otimes u$$

is bounded from $S_p \otimes_{\min} E$ into $S_p[F]$. We denote by

$$\pi_p^o(u) = \|U\|_{S_p \otimes_{\min} E \to S_p[F]}.$$

We will denote by $\Pi_p^o(E, F)$ the space of all completely p-summing maps and we equip it with the norm π_p^o for which it becomes a Banach space.

To give immediately an example, we will see below in Proposition 5.6 that if a, b are in S_{2p} then the map $M\colon B(\ell_2) \to S_p$, defined by $M(x) = axb$, is completely p-summing. A fortiori, any restriction of this map also is completely p-summing. We will see that the resulting mapping is the prototype of a completely p-summing map.

Clearly the class of "completely p-summing maps" is an "ideal" in Pietsch's sense. By this we mean that if E_1, F_1, E, F are operator spaces if $v\colon\ E_1 \to E$, $w\colon\ F \to F_1$ are $c.b.$ maps and if $u\colon\ E \to F$ is completely p-summing, then the composition wuv is completely p-summing and

$$\pi_p^o(wuv) \leq \|w\|_{cb}\ \pi_p^o(u)\ \|v\|_{cb}. \tag{5.1}$$

This is clear from the definition and from Corollary 1.2.

Let E, F be operator spaces and let $u\colon\ E \to F$ be a linear map. Then

$$\pi_p^o(u) = \sup\{\pi_p^o(uT) \mid T\colon\ S_p^{n*} \to E, \quad n \geq 1, \quad \|T\|_{cb} \leq 1\}. \tag{5.2}$$

Indeed, clearly $\pi_p^o(uT) \leq \pi_p^o(u)\|T\|_{cb}$ by the ideal property (5.1). Conversely, we have $\pi_p^o(u) = \sup\{\|(I_{S_p^n} \otimes u)(\theta)\|_{S_p^n[F]}\}$ where the supremum runs over all n and all θ in the unit ball of $S_p^n \otimes_{\min} E$. For such a θ let $T\colon\ (S_p^n)^* \to E$ be the associated linear map. Then $(I_{S_p^n} \otimes u)(\theta) = (I_{S_p^n} \otimes uT)(i)$ where $i \in S_p^n \otimes (S_p^n)^*$ corresponds to the identity map on S_p^n. Hence

$$\pi_p^o(u) \leq \sup\{\pi_p^o(uT) \mid \|T\|_{cb} \leq 1\},$$

which completes the proof of (5.2).

The main result is the following extension of the "Pietsch factorization" for completely p-summing maps.

Theorem 5.1. — *Assume $E \subset B(H)$. Let $u\colon\ E \to F$ be a completely p-summing map $(1 \leq p < \infty)$ and let $C = \pi_p^o(u)$. Then there is an ultrafilter $\mathcal{U}$ over an index set I and families $(a_\alpha)_{\alpha \in I}$, $(b_\alpha)_{\alpha\in I}$ in the unit ball of $S_{2p}(H)$ such that for all n and all (x_{ij}) in $M_n(E)$ we have*

$$(5.3)\qquad \|(u(x_{ij}))\|_{S_p^n[F]} \leq C \lim_{\mathcal{U}} \|(a_\alpha\ x_{ij}\ b_\alpha)\|_{S_p(\ell_2^n\otimes H)}$$

and

$$(5.4)\qquad \|(u(x_{ij}))\|_{M_n(F)} \leq C \lim_{\mathcal{U}} \|(a_\alpha\ x_{ij}\ b_\alpha)\|_{M_n(S_p(H))}.$$

Conversely, if an operator u satisfies (5.4) then it is completely p-summing with $\pi_p^o(u) \leq C$.

For the proof we will use the following well known fact

Lemma 5.2. — *Let S be a set and let $\mathcal{F} \subset \ell_\infty(S)$ be a convex cone of real valued functions on S such that*

$$\forall\, f\in\mathcal{F} \qquad \sup_{s\in S} f(s) \geq 0.$$

Then there is a net (λ_α) of finitely supported probability measures on S such that

$$\forall\, f\in\mathcal{F} \qquad \lim \int f d\lambda_\alpha \geq 0.$$

Proof. — We will identify $\ell_\infty(S)$ with the space $C(\widehat{S})$ of all continuous functions on the Stone-Cech compactification $\widehat{S}$ of S. Then in $C(\widehat{S})$ the set $\mathcal{F}$ is disjoint from the set $\{\varphi \in C(\widehat{S}) \mid \max\varphi < 0\}$. Hence by the Hahn-Banach theorem (we separate a convex set and a convex open set) there is a probability measure λ on $\widehat{S}$ such that $\lambda(f) \geq 0\ \ \forall\ f\in\mathcal{F}$. Since λ is the limit for the topology $\sigma(\ell_\infty(S)^*, \ell_\infty(S))$ of a net of finitely supported probability measures on S we obtain the announced result. □

We will also use

Theorem 5.3. — *Let E be any operator space, with $E \subset B(H)$. Consider $x = (x_{ij})$ in $S_p \otimes_{\min} E$ viewed as an element of $M_\infty(E)$. Then we have*

$$(5.5)\qquad \|x\|_{S_p\otimes_{\min}E} = \sup\{\|(ax_{ij}b)\|_{S_p(\ell_2\otimes H)}\}$$

where the supremum runs over all $a \geq 0$, $b \geq 0$ in the unit ball of $S_{2p}(H)$.More generally, let F be another operator space then for all x in $S_p[F] \otimes_{\min} E$ we have

$$\|x\|_{S_p[F]\otimes_{\min}E} = \sup\{\|(I_{S_p[F]}\otimes a)\ x\ (I_{S_p[F]}\otimes b)\|_{S_p[\ell_2\otimes H;F]}\}$$

where the supremum again runs over all $a \geq 0$, $b \geq 0$ in the unit ball of $S_{2p}(H)$.

Proof. — It clearly suffices to prove this for $E = B(H)$. Furthermore, taking the appropriate supremum as usual, it even suffices if we wish to assume $E = M_n$. Then the result is an obvious consequence of Lemma 1.7 and Theorem 1.9. Note that, by the polar factorization of a and b, we can restrict the supremum to Hermitian non-negative operators. □

Let $(E_\alpha)_{\alpha\in I}$ be a family of operator spaces, let $\mathcal{U}$ be an ultrafilter on I and let $(E_\alpha)_\mathcal{U}$ be the Banach space which is the ultraproduct of (E_α) with respect to $\mathcal{U}$ (cf. e.g. **[Hei]**). In **[P1]** we observed that $(E_\alpha)_\mathcal{U}$ can be equipped naturally with an operator space structure by setting $M_n((E_\alpha)_\mathcal{U}) = (M_n(E_\alpha))_\mathcal{U}$. Equivalently, we have an isometric identity

$$F \otimes_{\min} (E_\alpha)_\mathcal{U} = (F \otimes_{\min} E_\alpha)_\mathcal{U}, \text{ valid if } F = M_n.$$

However, the reader should be warned that this identity fails to be isometric in general, for instance when $F = OH_n$ or S_2^n (see **[P6]**). This explains certain precautions that we take below. This phenomenon is closely related to the absence of "local reflexivity" for a general operator space (cf. **[EH]**).

Lemma 5.4. — *Let $(E_\alpha)_{\alpha\in I}$ be a family of operator spaces and let $\mathcal{U}$ be a nontrivial ultrafilter on I. Let $\widehat{E}$ be the corresponding ultraproduct. Let $n \geq 1$ be a fixed integer. Consider a matrix $(\hat{x}_{ij})$ in $M_n(\widehat{E})$. Let $(x_{ij}^\alpha)_{\alpha\in I}$ be a representative of $\hat{x}_{ij}$, with $x_{ij}^\alpha \in E_\alpha$. Then we have*

$$\|(\hat{x}_{ij})\|_{S_p^n[\widehat{E}]} = \lim_\mathcal{U} \|(x_{ij}^\alpha)\|_{S_p^n[E_\alpha]}.$$

Therefore, we have a completely isometric identity

$$S_p^n[\widehat{E}] = (S_p^n[E_\alpha])_\mathcal{U}.$$

Proof. — Assume that $\lim_\mathcal{U} \|(x_{ij}^\alpha)\|_{S_p^n[E_\alpha]} < 1$. Then we can write $(x_{ij}^\alpha) = a_\alpha \cdot (y_{ij}^\alpha) \cdot b_\alpha$ with $a_\alpha, b_\alpha \in S_{2p}^n$ and $(y_{ij}^\alpha) \in M_n(E_\alpha)$ such that

$$\lim_\mathcal{U} \|a_\alpha\|_{2p} \|(y_{ij}^\alpha)\|_{M_n(E_\alpha)} \|b_\alpha\|_{2p} < 1.$$

By homogeneity, we may assume that $\|a_\alpha\|_{2p} = \|b_\alpha\|_{2p} = 1$. Let $a = \lim_\mathcal{U} a_\alpha$, $b = \lim_\mathcal{U} b_\alpha$ (these limits exist by the compactness of the unit ball of S_{2p}^n). Then clearly $(\hat{x}_{ij}) = a \cdot (\hat{y}_{ij}) \cdot b$ where $\hat{y}_{ij}$ denotes the element of $\widehat{E}$ associated to $(y_{ij}^\alpha)_{\alpha\in I}$. Therefore we conclude

$$\begin{aligned}\|(\hat{x}_{ij})\|_{S_p^n[\widehat{E}]} &\leq \|a\|_{2p} \|(\hat{y}_{ij})\|_{M_n(\widehat{E})} \|b\|_{2p} \\ &\leq \lim_\mathcal{U} \|(y_{ij}^\alpha)\|_{M_n(E_\alpha)} \leq 1.\end{aligned}$$

Conversely, if $\|(\hat{x}_{ij})\|_{S_p^n[\widehat{E}]} < 1$, we can write $(\hat{x}_{ij}) = a \cdot (\hat{y}_{ij}) \cdot b$ with $\|a\|_{S_{2p}^n} = \|b\|_{S_{2p}^n} = 1$ and $\|(\hat{y}_{ij})\|_{M_n(\widehat{E})} < 1$. Let $(y_{ij}^\alpha)_{\alpha\in I}$ be a representative of $\hat{y}_{ij}$.

Define z_{ij}^{α} by the matricial identity $(z_{ij}^{\alpha}) = a \cdot (y_{ij}^{\alpha}) \cdot b$. Then $(z_{ij}^{\alpha})_{\alpha \in I}$ represents $\hat{x}_{ij}$. Hence we have $\lim_{\mathcal{U}} \|z_{ij}^{\alpha} - x_{ij}^{\alpha}\| = 0$ for all i, j and we conclude

$$\lim_{\mathcal{U}} \|(x_{ij}^{\alpha})\|_{S_p^n[E_\alpha]} = \lim_{\mathcal{U}} \|(z_{ij}^{\alpha})\|_{S_p^n[E_\alpha]} \leq 1.$$

By homogeneity this completes the proof of the first part. The last assertion is then easy to deduce from Lemma 1.7. □

Proof of Theorem 5.1. — Let $S = \{(a,b) \in B_{S_{2p}(H)} \times B_{S_{2p}(H)} \mid a \geq 0,\ b \geq 0\}$. Consider the set $\mathcal{F}$ of all functions on S of the form

$$\forall\, (a,b) \in S \qquad f(a,b) = C^p \sum_m \|(a x_{ij}^m b)\|_{S_p(\ell_2 \otimes H)}^p - \|(u(x_{ij}^m))\|_{S_p[F]}^p$$

where (x^m) is a finite sequence with $x^m \in S_p \otimes E$ for each m (so that $x_{ij}^m \in E$).

Observe that by (5.5) and by Corollary 1.3, we have $\sup_{s \in S} f(s) \geq 0$. Hence by Lemma 5.2 there is an ultrafilter $\mathcal{U}$ on a set (λ_α) of finitely supported probability measures on S such that

$$\forall\, f \in \mathcal{F} \qquad \lim_{\mathcal{U}} \int f(s) d\lambda_\alpha(s) \geq 0.$$

Now consider a finitely supported probability measure λ on S, say

$$\lambda = \sum_{k=1}^{N} \lambda_k \delta_{(a_k, b_k)}$$

with $\lambda_k \geq 0$, $\sum_1^N \lambda_k = 1$. Then we can write by Lemma 1.14

$$\sum_k \lambda_k \|(a_k x_{ij} b_k)\|_{S_p(\ell_2 \otimes H)}^p \leq \|(\widetilde{a} x_{ij} \widetilde{b})\|_{S_p(\ell_2 \otimes H)}^p$$

where $\widetilde{a}$ and $\widetilde{b}$ are Hermitian, non-negative in the unit ball of $S_{2p}(H)$. Hence applying this to each λ_α, we obtain nets (a_α), (b_α) in the unit ball of $S_{2p}(H)$ for which (5.3) holds. We can now check (5.4) easily using Lemma 1.7. Indeed, we deduce from (5.3) and (1.7) $\forall\, a, b \in B_{S_{2p}^n}$

$$\begin{aligned}\|U(a \cdot (x_{ij}) \cdot b)\|_{S_p^n[F]} &\leq C \lim_{\mathcal{U}} \|(a \cdot (a_\alpha x_{ij} b_\alpha) \cdot b)\|_{S_p(\ell_2^n \otimes H)} \\ &\leq C \lim_{\mathcal{U}} \|(a_\alpha x_{ij} b_\alpha)\|_{M_n(S_p(H))}\end{aligned}$$

hence by Lemma 1.7 taking the supremum over all a, b in the unit ball of S_{2p}^n (and observing $U(a \cdot (x_{ij}) \cdot b) = a \cdot (u(x_{ij})) \cdot b$) we obtain (5.4).

Conversely assume u satisfies (5.4). Let $S_\alpha = S_p(H)$ and let $\widehat{S}_p$ be the ultraproduct of $(S_\alpha)_{\alpha \in I}$ associated to $\mathcal{U}$. Let $E_p \subset \widehat{S}_p$ be the closure in $\widehat{S}_p$ of the subspace spanned in $\widehat{S}_p$ by the elements of $\widehat{S}_p$ associated to families of the form $(a_\alpha x b_\alpha)_{\alpha \in I}$ with x in E. Then by (5.4) there is a (uniquely defined) map $\widetilde{u}\colon\ E_p \to F$ with $\|\widetilde{u}\|_{cb} \leq C$ which takes the element of E_p associated to $(a_\alpha x b_\alpha)_{\alpha \in I}$ to $u(x)$. (See Remark 5.7 below for more details.) By Corollary 1.2, we have $\|I_{S_p^n} \otimes \widetilde{u}\|_{S_p^n[E_p] \to S_p^n[F]} \leq 1$ so that using the second part of Corollary 1.2 and Lemma 5.4 (and recalling Theorem 1.9), we obtain (5.3). From (5.3) it is easy to deduce that $\pi_p^o(u) \le C$. See Remark 5.7 below for more details on this point. □

Corollary 5.5. — *Let $u\colon E \to F$ be a completely p-summing map ($1 \le p < \infty$) and let $U\colon S_p \otimes_{\min} E \to S_p[F]$ be the corresponding map. Then*

$$\|U\|_{cb} = \|U\| = \pi_p^o(u).$$

In particular, we have

$$\|u\|_{cb} \le \pi_p^o(u).$$

Proof. — Since $\ell_2^n \otimes_2 \ell_2 \approx \ell_2$, we have a map $v\colon S_p(\ell_2^n \otimes_2 \ell_2) \otimes_{\min} E \to S_p[\ell_2^n \otimes_2 \ell_2; F]$ associated to u with $\|v\| = \|U\| = \pi_p^o(u)$. Now let $(x_{ij}) \in M_n(S_p \otimes_{\min} E)$ with $x_{ij} \in S_p \otimes_{\min} E$. We have by Lemma 1.7

$$\|(U(x_{ij}))\|_{M_n(S_p[F])} = \sup\{\|a \cdot (U(x_{ij})) \cdot b\|_{S_p^n[S_p[F]]} \mid a, b \in S_{2p}^n,\ \|a\|_{S_{2p}^n} \le 1,\ \|b\|_{S_{2p}^n} \le 1\}.$$

Hence this is

$$\le \|v\| \sup\{\|a \cdot (x_{ij}) \cdot b\|_{S_p(\ell_2^n \otimes \ell_2) \otimes_{\min} E}\}$$

where the supremum is the same as above. Hence by Theorem 5.3 this is

$$\le \|v\| \sup\{\|(a \otimes I_{S_p} \otimes a')(x_{ij})(b \otimes I_{S_p} \otimes b')\|_{S_p(\ell_2^n \otimes_2 \ell_2 \otimes_2 H)}\}$$

where the supremum runs over all a, b, a', b' in the unit ball of S_{2p}^n and $S_{2p}(H)$ respectively. By Theorem 5.3 again this is

$$\le \|v\| \|(x_{ij})\|_{S_p \otimes_{\min} (M_n \otimes_{\min} E)} = \|v\| \|(x_{ij})\|_{M_n(S_p \otimes_{\min} E)}.$$

Hence we conclude $\|U\|_{cb} \le \|v\| = \|U\| = \pi_p^o(u)$. Since the inequalities $\|U\| \le \|U\|_{cb}$ and $\|u\|_{cb} \le \|U\|_{cb}$ are obvious, this concludes the proof. □

Remark. — In **[BP]**, **[ER2]**, a natural operator space structure is defined on the space $cb(E, F)$. The preceding corollary allows to equip the space $\Pi_p^o(E, F)$ with the operator space structure corresponding to $cb(S_p \otimes_{\min} E, S_p[F])$.

Proposition 5.6. — *Let K be any Hilbert space. Consider a, b in $S_{2p}(K)$ and let $M(a, b)\colon B(K) \to S_p(K)$ be the operator defined by $M(a, b)x = axb$ for all x in $B(K)$. Then*

$$\|M(a, b)\|_{cb} \le \pi_p^o(M(a, b)) \le \|a\|_{S_{2p}(K)} \|b\|_{S_{2p}(K)}. \tag{5.6}$$

Proof. — By Theorem 5.3 if a, b are in the unit ball of $S_{2p}(K)$, then $M(a, b)$ is a contraction from $S_p \otimes_{\min} B(K)$ into $S_p(\ell_2 \otimes K)$. But by Theorem 1.9, $S_p(\ell_2 \otimes K) = S_p[S_p(K)]$. Hence we obtain $\pi_p^o(M(a, b)) \le 1$. The inequality (5.6) follows by Corollary 5.5 and by homogeneity. □

Remark 5.7. — We will now reinterpret Theorem 5.1 as a factorization theorem. Let us denote briefly by $\widehat{B}$ the ultraproduct of $(B_\alpha)_{\alpha\in I}$ with $B_\alpha = B(H)$ for all α in I. As above, we let $\widehat{S}_p = (S_\alpha)_\mathcal{U}$ where $S_\alpha = S_p(H)$ for all α in I. Let $M\colon\ \widehat{B} \to \widehat{S}_p$ be the operator associated to the family $(M_\alpha)_{\alpha\in I}$ where $M_\alpha\colon\ B_\alpha \to S_\alpha$ is defined by $M_\alpha(x) = a_\alpha x b_\alpha$. Let $\hat{\imath}\colon\ E \to \widehat{B}$ be the map which takes $x \in E$ to the element of $\widehat{B}$ corresponding to $(x_\alpha)_{\alpha\in I}$ with $x_\alpha = x$ for all α. Clearly $\hat{\imath}$ is a complete isometry. Let $E_\infty = \hat{\imath}(E)$. Similarly let $E_p = \overline{M\hat{\imath}(E)}$ where the closure is in $\widehat{S}_p$. Applying (5.4) first with $n = 1$, we can define a map $\widetilde{u}\colon\ M\hat{\imath}(E) \to F$ by setting

$$\widetilde{u}(M\hat{\imath}(x)) = u(x).$$

By (5.4), $\widetilde{u}$ is unambiguously defined and

$$\|\widetilde{u}(M\hat{\imath}(x))\|_F \le C\|M\hat{\imath}(x)\|_{\widehat{S}_p} \quad \text{for all} \quad x \text{ in } E.$$

Hence $\widetilde{u}$ can be extended to the closure E_p of $M\hat{\imath}(E)$ in $\widehat{S}_p$, and this extension– still denoted by $\widetilde{u}$ – satisfies $\|\widetilde{u}\| \le C$. Actually, applying (5.4) in general, we find

$$\|\widetilde{u}\|_{cb} \le C.$$

This gives a factorization diagram as follows:

$$\begin{array}{ccc} \widehat{B} & \xrightarrow{M} & \widehat{S}_p \\ \cup & & \cup \\ E_\infty & \xrightarrow{\mathcal{M}} & E_p \\ \hat{\imath}\uparrow & & \downarrow\widetilde{u} \\ E & \xrightarrow{u} & F \end{array}$$

where $\mathcal{M}\colon\ E_\infty \to E_p$ is the restriction of M to E_∞. We now claim that

$$\pi_p^o(\mathcal{M}\hat{\imath}) \le 1. \tag{5.7}$$

Equivalently, this claim means that for all n and all (x_{ij}) in $S_p^n \otimes_{\min} E$ we have

$$\|(x_{ij})\|_{S_p^n[E_p]} \le \|(x_{ij})\|_{S_p^n\otimes_{\min} E}. \tag{5.8}$$

Now by Lemma 5.4 (and Corollary 1.2) we have

$$\|(x_{ij})\|_{S_p^n[E_p]} = \lim_{\mathcal{U}} \|(a_\alpha x_{ij} b_\alpha)\|_{S_p^n[S_p(H)]}$$

but by (5.5) we have

$$\|(a_\alpha x_{ij} b_\alpha)\|_{S_p^n[S_p(H)]} \le \|(x_{ij})\|_{S_p^n\otimes_{\min} B(H)} = \|(x_{ij})\|_{S_p^n\otimes_{\min} E}$$

hence we obtain (5.8). This proves our claim (5.7).

We summarize the content of the preceding remark in the next statement.

Corollary 5.8. — *If $u\colon\ E \to F$ is a completely p-summing map, then there is a subspace X of an ultraproduct of spaces of the form $S_p(H)$ for which u admits a factorization $u = AB$ through X with $B\colon\ E \to X$, $A\colon\ X \to F$ such that*

$$\|A\|_{cb}\|B\|_{cb} \le \|A\|_{cb}\pi_p^o(B) \le \pi_p^o(u).$$

Proof. — The factorization follows from the preceding discussion. We let $X = E_p$, $B = \mathcal{M}\hat{\imath}$ and $A = \tilde{u}$. Note $\|A\|_{cb} \leq \pi_p^o(u)$. Moreover, by (5.7) and Corollary 5.5, we have $\|B\|_{cb} \leq \pi_p^o(B) \leq 1$. □

In the case $E \subset M_N$ (or $E = M_N$), we note the following simpler variant of Theorem 5.1.

Theorem 5.9. — *Assume $E \subset M_N$ for some integer $N \geq 1$. Let F be an arbitrary operator space and let $u\colon\ E \to F$ be completely p-summing with $C = \pi_p^o(u)$. Then there are a, b Hermitian non-negative in the unit ball of $B_{S_{2p}^N}$ such that for all n and for all (x_{ij}) in $S_p^n \otimes E$ we have*

$$\|(u(x_{ij}))\|_{S_p^n[F]} \leq C\|(ax_{ij}b)\|_{S_p^n(S_p^N)}. \tag{5.9}$$

Proof. — By Theorem 1.1, we have (5.3) but since the set $S = B_{S_{2p}^N} \times B_{S_{2p}^N}$ is compact, (a_α) and (b_α) norm-converge along $\mathcal{U}$ to elements a and b in S_{2p}^N, so that (5.9) follows. □

Remark 5.10. — In the situation of Theorem 5.9 we have a factorization of u of the form

$$\begin{array}{ccc} M_N & \xrightarrow{\quad M \quad} & S_p^N \\ \cup & & \cup \\ E & \xrightarrow{\quad \mathcal{M} \quad} & E_p \xrightarrow{\tilde{u}} F \end{array}$$

where the arrows are defined as follows: for any $x \in M_N$ we have $Mx = axb$, E_p coincides with $M(E) \subset S_p^N$ and finally $\mathcal{M}$ is the restriction of M. Then, by Lemma 1.7, (5.9) implies that $\tilde{u}$ is c.b. with $\|\tilde{u}\|_{cb} \leq C$.

Note that if $E = M_N$, we can replace a, b respectively by $(1+\varepsilon)^{-1}(a+\varepsilon I)$ and by $(1+\varepsilon)^{-1}(b+\varepsilon I)$ with $\varepsilon > 0$ arbitrarily small. We then obtain $E_p = S_p^N$ and $\|\tilde{u}\|_{cb} \leq C(1+\varepsilon)$. This is the operator space version of the p-integral factorization of u.

Remark 5.11. — M. Junge (personal communication) observed that one can develop a variant of p-summing operator which is intermediate between the Banach space case and the completely p-summing one of this paper. Junge's original motivation was to generalize to any $1 \leq p < \infty$ the notion of $(2, oh)$-summing operator introduced in **[P1]**. This idea has interesting applications to the factorization theory. For instance it yields characterizations of maps which factor through a *commutative* L_p-space, say $L_p(\Omega, \mu)$, equipped with the operator space structure defined by interpolation, as explained in section 2 above.

More generally, let $n \geq 1$ be a *fixed* integer. One can then characterize the operators which factor through an operator space of the form

$$L_p(\Omega, \mu; S_p^n)$$

for some measure space (Ω, μ). Note that, since n is fixed, this class of spaces is stable by ultraproduct. To handle this kind of factorization, Junge observed that the following notion is the "right" one. (Actually he originally considered only the case $n = 1$.) Let E, F be operator spaces and let $u\colon E \to F$ be a linear map. Recall that $n \geq 1$ is fixed. We will say that u is $\ell_p(S_p^n)$-summing if u induces a bounded linear map $U^{(n)}$ from $\ell_p(S_p^n) \otimes_{\min} E$ into $\ell_p(S_p^n[F])$. We denote

$$\pi_p^{(n)}(u) = \|U^{(n)}\|.$$

Let $C = \pi_p^{(n)}(u)$. Then, by the same proof as in Theorem 5.1, it is easy to check that there are I, a_α, b_α and $\mathcal{U}$ as in Theorem 5.1 such that (for the specific integer n) u satisfies (5.4). Conversely (again by the same proof) if u satisfies (5.4) with respect to n then $\pi_p^{(n)}(u) \leq C$.

Equivalently this can be reformulated as a factorization: a map $u\colon E \to F$ is $\ell_p(S_p^n)$-summing with $\pi_p^{(n)}(u) \leq C$ iff u admits a factorization of the form

$$E \xrightarrow{v} E_p \xrightarrow{\widetilde{u}} F$$

with $\pi_p^o(v) \leq 1$ and

$$\|\widetilde{u}\|_{M_n(E_p) \to M_n(F)} \leq C.$$

When $n = 1$ and $p = 2$, this notion reduces to the $(2, oh)$-summing operators and

$$\pi_2^{(1)}(u) = \pi_{2,oh}(u).$$

We refer the reader to a forthcoming paper of M. Junge for more details.

We now compare the notions of completely p-summing and absolutely p-summing.

Proposition 5.12. — *Let (Ω, μ) be any probability space and $J\colon L_\infty(\mu) \to L_p(\mu)$ be the inclusion map. Let $X \subset L_\infty(\mu)$ be a subspace, let X_p be its closure in $L_p(\mu)$ and let $j\colon X \to X_p$ be the restriction of J. Then j is completely p-summing and $\pi_p^o(j) \leq 1$.*

Proof. — By the preceding results we have contractive inclusions

$$L_\infty(\mu) \otimes_{\min} S_p^n \to L_\infty(\mu; S_p^n) \to L_p(\mu; S_p^n) \to S_p^n[L_p(\mu)].$$

It follows that $\pi_p^o(J) \leq 1$ hence a fortiori $\pi_p^o(j) \leq 1$. □

Remark 5.13. — Let E, F be operator spaces. Assume that F is equipped with the so-called minimal operator space structure in the sense of **[BP]**, i.e. the structure induced by any isometric embedding of F into a commutative C^*-algebra. Then it is clear that for any operator space G and any map $v\colon G \to F$ we have $\|v\| = \|v\|_{cb}$. Then if a linear map $u\colon E \to F$ is absolutely p-summing in Pietsch's original sense **[Pi]**, it is completely p-summing. (The converse is obviously false in general.) Indeed, by Pietsch's factorization theorem **[Pi]** there are (Ω, μ), S, S_p and j as in Proposition 5.12 for which there is a factorization of u of the form $E \xrightarrow{i} S \xrightarrow{j} S_p \xrightarrow{\widetilde{u}} F$ with $\|\widetilde{u}\| = \pi_p(u)$

and $\|i\| = 1$. Clearly $\|i\|_{cb} \le \|i\|$ and $\|\widetilde{u}\|_{cb} \le \|\widetilde{u}\|$ by our assumption on F, hence by Proposition 5.12 u is completely p-summing and we have

$$\pi_p^o(u) \le \|i\|_{cb}\pi_p^o(j)\|\widetilde{u}\|_{cb} \le \pi_p(u).$$

If F is an arbitrary operator space, then we can only conclude that u is $\ell_p(S_p^1)$-summing with $\pi_p^{(1)}(u) \le \pi_p(u)$ in the sense of Remark 5.11 above.

The following Lemma will be useful in the sequel.

Lemma 5.14. — *Let K be an arbitrary Hilbert space. Let $1 \le p < \infty$. Consider $u \in S_p(K) \otimes_{\min} E$. Let $u_1\colon\ S_p(K)^* \to E$ and $u_2\colon\ E^* \to S_p(K)$ be the associated linear maps. Then*

(i) *If u_1 is completely p-summing, then $u \in S_p[K;E]$ and $\|u\|_{S_p[K;E]} \le \pi_p^o(u_1)$.*
(ii) *If $u \in S_p[K;E]$ then u_2 is completely p-summing and $\pi_p^o(u_2) \le \|u\|_{S_p[K;E]}$.*

Proof. — Using Lemma 1.12, it is easy to reduce to the case when K is finite dimensional, so that we may assume $K = \ell_2^n$ and $S_p(K) = S_p^n$. Consider then $u \in S_p^n[E]$. Let i be the element of $S_p^n \otimes (S_p^n)^*$ corresponding to the identity map on S_p^n. We have

$$\|(I_{S_p^n} \otimes u_1)(i)\|_{S_p^n[E]} \le \pi_p^o(u_1)\|i\|_{S_p^n \otimes_{\min} (S_p^n)^*}$$

hence since $(I_{S_p^n} \otimes u_1)(i)$ can be identified with u, this implies $\|u\|_{S_p^n[E]} \le \pi_p^o(u_1)$, whence the first part.

To prove the second part, assume $\|u\|_{S_p^n[E]} < 1$. Then by Theorem 1.5 there are x in the unit ball of $M_n(E)$ a and b in the unit ball of S_{2p}^n such that $u = a \cdot x \cdot b$, hence $u_2 = M(a,b)v$ where $v\colon\ E^* \to M_n$ is the map associated to x in the natural way. Therefore by (5.1) we have $\pi_p^o(u_2) \le \pi_p^o(M(a,b))\|v\|_{cb} \le 1$. This proves the second part. □

Remark. — For a recent application of the notion of completely p-summing map to "split inclusions" of (von Neumann) factors, see **[Fi1]**-**[Fi2]** (more precisely, what is used there is a notion of completely p-nuclear map).

CHAPTER 6

OPERATORS FACTORING THROUGH OH

We recall that we say that an operator $u\colon\ E \to F$ factors through OH if for some index set I we have a (completely bounded) factorization of u through the operator Hilbert space $OH(I)$ which is introduced and studied in **[P1]**. Moreover, we denote by $\gamma_{oh}(u)$ the infimum of $\|a\|_{cb}\|b\|_{cb}$ over all I and all factorizations of u of the form $u = ab$ with c.b. maps $b\colon\ E \to OH(I)$ and $a\colon\ OH(I) \to F$.

Note that by Corollary 2.4 in **[P1]** we know that if H is isometric to $\ell_2(I)$ then $S_2(H)$ is completely isometric to $OH(I \times I)$. Moreover, since the class of operator spaces of the form $OH(I)$ is stable by ultraproduct, by Remark 5.7 we have

Proposition 6.1. — *Every completely 2-summing map $u\colon\ E \to F$ factors through OH and satisfies $\gamma_{oh}(u) \le \pi_2^o(u)$. More precisely, assume $E \subset B(H)$. Then there is a set J and maps $V\colon\ B(H) \to OH(J)$ and $T\colon\ OH(J) \to F$ with $\pi_2^o(V) \le 1$ and $\|T\|_{cb} \le \pi_2^o(u)$ such that*

$$u = TV_{|E}.$$

In particular, u admits an extension $v\colon\ B(H) \to F$ satisfying $\pi_2^o(v) = \pi_2^o(u)$.

Proof. — With the notation of Remark 5.7, let $P\colon\ \widehat{S}_2 \to E_2$ be the orthogonal projection. Define $T\colon\ \widehat{S}_2 \to F$ as $T = \widetilde{u}P$. Since $\widehat{S}_2$ is completely isometric to $OH(J)$ for some J, it is homogeneous in the sense of **[P1]**, so that $\|P\| = \|P\|_{cb}$. Hence $\|T\|_{cb} \le \|\widetilde{u}\|_{cb} \le \pi_2^o(u)$, and we clearly have $u = TM\hat{\imath}$. This settles the first assertion. Now, let $\hat{\jmath}\colon\ B(H) \to \widehat{B}$ be the map which takes $x \in B(H)$ to the element of $\widehat{B}$ corresponding to $(x_\alpha)_{\alpha\in I}$ with $x_\alpha = x$ for all α. Clearly, $\hat{\jmath}$ is a complete isometry. Let $V\colon\ B(H) \to \widehat{S}_2$ be the composition $V = M\hat{\jmath}$. Then, by (5.7) applied in the case $E = B(H)$, we have $\pi_2^o(V) \le 1$. Note that $\hat{\jmath}_{|E} = \hat{\imath}$, hence $TV_{|E} = TM\hat{\imath} = u$. Clearly, we can replace $\widehat{S}_2$ by $OH(J)$ in the factorizations if we wish and we obtain (taking $v = TV$) the second part of Proposition 6.1. □

In **[P1]**, we introduced the class of $(2, oh)$-summing maps as follows. An operator $u\colon\ E \to F$ is called $(2, oh)$-summing if there is a constant C such that

$\forall\ n\ \forall\ x_1,\ldots,x_n \in E$ we have

$$\left(\sum \|u(x_i)\|^2\right)^{1/2} \leq C \left\|\sum x_i \otimes \bar{x}_i\right\|^{1/2}.$$

We denote by $\pi_{2,oh}(u)$ the smallest constant C for which this holds. It turns out that if $F = OH$ this notion and the notion of "completely 2-summing" map coincide:

Proposition 6.2. — *Let $u\colon\ E \to F$ be a linear map between operator spaces. If u is completely 2-summing then u is $(2, oh)$-summing and $\pi_{2,oh}(u) \leq \pi_2^o(u)$. Moreover, if $F = OH(I)$ for some set I then the converse also holds and we have*

$$\pi_{2,oh}(u) = \pi_2^o(u).$$

Proof. — Let $x_1,\ldots,x_n \in E$. Let $x_{ij} = x_i$ if $i = j$ and $x_{ij} = 0$ if $i \neq j$. We have by Corollary 1.3

$$\begin{aligned}\left(\sum \|u(x_i)\|^2\right)^{1/2} &= \|(u(x_{ij}))\|_{S_2^n[F]} \leq \pi_2^o(u)\|(x_{ij})\|_{S_2^n \otimes_{\min} E}\\ &\leq \pi_2^o(u) \left\|\sum x_i \otimes \bar{x}_i\right\|^{1/2},\end{aligned}$$

hence $\pi_{2,oh}(u) \leq \pi_2^o(u)$ and the first assertion follows. Conversely, let us assume $F = OH(I)$. By Corollary 6.8 in **[P1]**, every $(2, oh)$-summing map $u\colon\ E \to F$ admits a factorization of the form $u = \widetilde{u}\mathcal{M}\hat{i}$ with $\mathcal{M}, \hat{i}$ as above and with $\|\widetilde{u}\| \leq \pi_{2,oh}(u)$. But now if $F = OH(I)$, since $\widehat{S}_2$ is itself completely isometric to $OH(J)$ for some set J, it follows that for $\widetilde{u}\colon\ \widehat{S}_2 \to F$ we have $\|\widetilde{u}\|_{cb} = \|\widetilde{u}\| \leq \pi_{2,oh}(u)$. Hence, we conclude that u is a completely 2-summing map and we have

$$\pi_2^o(u) \leq \|\widetilde{u}\|_{cb}\ \ \pi_2^o(\mathcal{M}\hat{i}) \leq \pi_{2,oh}(u)\pi_2^o(\mathcal{M}\hat{i})$$

hence by (5.7)

$$\leq \pi_{2,oh}(u).$$

□

In particular, we have obviously

Proposition 6.3. — *Let I, J be arbitrary sets. Let $u\colon\ OH(I) \to OH(J)$ be a linear map. Then the Hilbert-Schmidt norm of u, denoted by $\|u\|_{HS}$ satisfies*

$$\|u\|_{HS} = \pi_2^o(u) = \pi_{2,oh}(u).$$

Proposition 6.4. — *Let I be any set, let F be any operator space and let $v\colon\ OH(I) \to F$ be a linear map. Then v is a completely 2-summing map iff v admits a factorization $v = AB$ with $B\colon\ OH(I) \to OH$, $A\colon\ OH \to F$ such that B is Hilbert-Schmidt and A is c.b. Moreover we have*

$$\pi_2^o(v) = \inf\{\|B\|_{HS}\|A\|_{cb}\}$$

where the infimum runs over all possible factorizations.

Proof. — First assume that $v = AB$ as above. Then by (5.1)

$$\pi_2^o(v) \leq \pi_2^o(B)\|A\|_{cb}$$

hence by Proposition 6.3 we have $\pi_2^o(v) \leq \|B\|_{HS}\|A\|_{cb}$. Conversely, assume that v is completely 2-summing. Then by Proposition 6.1, v admits a factorization of the form $OH(I)\xrightarrow{B}\widehat{S}_2\xrightarrow{A}F$, with $\|A\|_{cb} \leq \pi_2^o(v)$ and $\pi_2^o(B) \leq 1$. But since $\widehat{S}_2$ is completely isometric to $OH(J)$ for some set J, we again have by Proposition 6.3 $\|B\|_{HS} = \pi_2^o(B)$, and since a Hilbert-Schmidt map has separable range, we can replace $OH(J)$ by OH in the factorization. □

Using Propositions 6.2 and 6.4, we can reformulate Theorem 7.7 in **[P1]** in a fashion entirely analogous to a result of Kwapień **[Kw1]** in the Banach space setting.

Theorem 6.5. — *Let E, F be operator spaces and let C be a constant. The following properties of a linear map $u\colon\ E \to F$ are equivalent:*

(i) $u \in \Gamma_{oh}(E,F)$ *and* $\gamma_{oh}(u) \leq C$.

(ii) *For any completely 2-summing map $v\colon\ F \to OH$, the map $(vu)^*$ is completely 2-summing and*

$$\pi_2^o((vu)^*) \leq C\pi_2^o(v).$$

(iii) *For any n and any $v\colon\ F \to OH_n$ we have*

$$\pi_2^o((vu)^*) \leq C\pi_2^o(v).$$

(iii') For any n and any $v\colon\ F \to S_2^n$ we have

$$\pi_2^o((vu)^*) \leq C\pi_2^o(v).$$

(iv) *For any operator space G and any completely 2-summing map $v\colon\ F \to G$, the map $(vu)^*$ is completely 2-summing and*

$$\pi_2^o((vu)^*) \leq C\pi_2^o(v).$$

Proof. — The equivalence of (i) and (ii) is clear from Theorem 7.7 in **[P1]** and the preceding Corollaries. (ii) ⇔ (iii) is easy, (ii) ⇒ (iv) follows from the factorization in Proposition 6.1 and (iv) ⇒ (ii) is trivial.

Finally the equivalence (iii) ⇔ (iii)' is obvious since S_2^n is completely isometric to OH_{n^2}. □

Lemma 6.6. — *Let $X = OH(I)$ for some set I. Let K be an arbitrary Hilbert space. Then*

$$S_2[K;X] = S_2(K) \otimes_h X$$

completely isometrically.

Proof. — Assume $K = \ell_2$ for simplicity of notation. By Theorem 1.1 (case $\theta = 1/2$) we have

$$S_2[K;X] = OH \otimes_h X \otimes_h OH.$$

Now if $X = OH(I)$, by Corollary 2.12 in **[P1]** we have

$$\begin{aligned} OH \otimes_h X \otimes_h OH &\approx OH(\mathbb{N} \times I \times \mathbb{N}) \\ &\approx OH(\mathbb{N} \times \mathbb{N} \times I) \end{aligned}$$

hence again by Corollary 2.12 in **[P1]**

$$\approx OH(\mathbb{N} \times \mathbb{N}) \otimes_h OH(I)$$

hence

$$\approx S_2 \otimes_h X.$$

□

Lemma 6.7. — *Let $u\colon\ E \to F$ be a linear map between operator spaces. Then*

$$\gamma_{oh}(u) = \sup\{\gamma_{oh}(Tu) \mid T\colon\ F \to M_n, \quad \|T\|_{cb} \le 1, \quad n \in \mathbb{N}\}.$$

Proof. — Clearly this supremum is at most $\gamma_{oh}(u)$. To show the equality, let $j\colon\ F \to B(H)$ be a completely isometric embedding. Clearly $\gamma_{oh}(u) = \gamma_{oh}(ju)$. Hence we may as well assume that $F = B(H)$. But then there is a family of matrix spaces $(M_{n_i})_{i\in I}$ and an ultraproduct of $(M_{n_i})_{i\in I}$ which contains $B(H)$ completely isometrically. Let $T_i\colon\ B(H) \to M_{n_i}$ be the corresponding mappings with $\|T_i\|_{cb} \le 1$ so that the associated operator $(T_i)_{\mathcal{U}}\colon\ B(H) \to (M_{n_i})_{\mathcal{U}}$ is a complete isometry. Then we have by the stability of the class of spaces $OH(I)$ by ultraproduct

$$\gamma_{oh}(u) \le \lim_{\mathcal{U}} \gamma_{oh}(T_i u).$$

This yields Lemma 6.7. □

It will be useful to record here the following finite dimensional version of Theorem 5.1.

Theorem 6.8. — *Let $N \ge 1$. For any operator space F and any $u\colon\ M_N \to F$ there are a, b in the unit ball of S_4^N such that u admits a factorization as follows*

$$M_N \xrightarrow{M} S_2^N \xrightarrow{T} F,$$

i.e. $u = TM$ where $T\colon\ S_2^N \to F$ satisfies $\|T\|_{cb} \le \pi_2^o(u)$ and where

$$M(x) = axb.$$

Conversely, any operator admitting such a factorization satisfies $\pi_2^o(u) \le \|T\|_{cb}$.

Proof. — This follows immediately from Theorem 5.9 and Remark 5.10. □

Theorem 6.9. — *Let $u\colon\ E \to F$ be as in Theorem 6.5. The properties considered in Theorem 6.5 are equivalent to each of the following*

(v) *For any bounded linear map* $A\colon S_2 \to S_2$, *the operator* $A \otimes u$ *extends to a c.b. map from* $S_2[E]$ *to* $S_2[F]$, *with c.b. norm* $\leq C\|A\|$.

(vi) *For any* n *and any* $A\colon S_2^n \to S_2^n$ *we have*

$$\|A \otimes u\|_{cb(S_2^n[E], S_2^n[F])} \leq C\|A\|.$$

(vii) *For any* n *and any* $A\colon S_2^n \to S_2^n$ *we have*

$$\|A \otimes u\|_{S_2^n[E] \to S_2^n[F]} \leq C\|A\|.$$

(viii) *For any* n *and any* $A\colon S_2^n \to S_2^n$ *we have*

$$\|A \otimes u^*\|_{S_2^n[F^*] \to S_2^n[E^*]} \leq C\|A\|.$$

Proof. — (i) $\Rightarrow$ (v): To show this it clearly suffices by Corollary 1.2 to show that, if $X = OH(I)$ for some set I, then $A \otimes I_X$ defines a complete contraction on $S_2[X]$ when $\|A\| \leq 1$. This is an immediate consequence of Lemma 6.6 and of the homogeneity of $S_2 = OH(\mathbb{N} \times \mathbb{N})$. (This also follows from Proposition 3.9. Alternate proofs can be given using Theorem 1.9, or (2.10) for $p = 2$ and E_i one dimensional, and Corollary 1.2.) Then (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) are trivial and (vii) $\Rightarrow$ (viii) is easy by duality using Corollary 1.8. It remains to show (viii) $\Rightarrow$ (i).

By Lemma 6.7, we can assume that $F = M_N$ for some N and it suffices to show that (iii)' holds in that particular case. Let $v\colon F \to S_2^n$ be such that $\pi_2^o(v) = 1$. We claim that (iii)' holds, i.e. that

$$\pi_2^o((vu)^*) \leq C\pi_2^o(v) = C. \tag{6.1}$$

To check that, since $F = M_N$ we may assume by Theorem 6.8 that $v = TM$ as in Theorem 6.8 with $\|T\|_{cb} \leq 1$. Since the presence of T clearly does not affect (6.1) we may as well assume that v takes values into S_2^N and that v is of the form $v(x) = axb$ where a, b are in the unit ball of S_4^N. In other words (see Theorem 1.5) $v\colon F \to S_2^N$ is associated to an element $\widetilde{v}$ in $S_2^N[F^*]$ with $\|\widetilde{v}\|_{S_2^N[F^*]} \leq 1$. Now to check that $(vu)^*\colon S_2^{N*} \to E^*$ satisfies (6.1) note that by definition of the norm $\pi_2^o(\cdot)$ we have

$$\pi_2^o((vu)^*) = \sup\{\|(I_{S_2^n} \otimes (vu)^*)(\beta)\|_{S_2^n[E^*]}\} \tag{6.2}$$

where the supremum runs over all $n \geq 0$ and all β in the unit ball of $S_2^n \otimes_{\min} S_2^{N*}$. Clearly such a β can be viewed as a linear map $B\colon S_2^N \to S_2^n$ with $\|B\| \leq 1$. Then we have

$$(I_{S_2^n} \otimes (vu)^*)\beta = (B \otimes u^*)\widetilde{v}$$

so that by our assumption (viii) we have (we may clearly assume $n = N$ if we wish by adding zeros)

$$\|(I_{S_2^n} \otimes (vu)^*)\beta\|_{S_2^n[E^*]} \leq C\|B\| \, \|\widetilde{v}\|_{S_2^N[F^*]} \leq C.$$

By (6.2) we conclude that (6.1) holds and this completes the proof of (viii) $\Rightarrow$ (i) and hence of Theorem 6.9. $\square$

Remark. — The preceding result implies that for all $u\colon E \to F$ we have

$$\gamma_{oh}(u) = \sup\{\|A \otimes u\|_{cb(S_2[E],S_2[F])} \mid A\colon S_2 \to S_2 \quad \|A\| \leq 1\}. \tag{6.3}$$

This allows to equip $\Gamma_{oh}(E, F)$ with a natural operator space structure, by defining for all (u_{ij}) in $M_n(\Gamma_{oh}(E, F))$

$$\|(u_{ij})\|_{M_n(\Gamma_{oh}(E,F))} = \sup\{\|A \otimes (u_{ij})\|_{cb(S_2[E],M_n(S_2[F]))} \mid A\colon S_2 \to S_2 \quad \|A\| \leq 1\}.$$

Equivalently, by (6.3) we can view $\Gamma_{oh}(E, F)$ as subspace of $\bigoplus\limits_{A \in I} E_A$ where $I = \{A\colon S_2 \to S_2 \mid \|A\| \leq 1\}$ and $E_A = cb(S_2[E], S_2[F])$.
The embedding $J\colon \Gamma_{oh}(E, F) \to \bigoplus\limits_{A \in I} E_A$ is defined by $J(u) = (A \otimes u)_{A \in I}$. Since the spaces E_A have a natural operator space structure (cf. **[BP]**, **[ER2]**) the same is true for $\bigoplus\limits_{A \in I} E_A$ and a fortiori for the image of $\Gamma_{oh}(E, F)$ under J.

Remark 6.10. — We recall that when E, F are Banach spaces, we denote by $\Gamma_2(E, F)$ the space of all operators $u\colon E \to F$ which can be factorized through a Hilbert space, i.e. there is a Hilbert space H and a factorization of u of the form $E \xrightarrow{a} H \xrightarrow{b} F$. We denote below

$$\|u\|_{\Gamma_2(E,F)} = \inf\{\|a\| \, \|b\|\}$$

where the infimum runs over all such factorizations.

In **[Kw1]**, Kwapień proved that we have

$$\|u\|_{\Gamma_2(E,F)} \leq C$$

iff for all N and all operators $t\colon \ell_2^N \to \ell_2^N$ we have

$$\|t \otimes u\|_{\ell_2^N(E) \to \ell_2^N(F)} \leq C\|t\|.$$

We will now prove the operator space analog of his result.

Theorem 6.11. — *Let $u\colon E \to F$ be an operator between two operator spaces. Then the properties considered in Theorems 6.5 and 6.9 are all equivalent to*

(ix) *For any N and any $t\colon \ell_2^N \to \ell_2^N$ we have*

$$\|t \otimes u\|_{\Gamma_{oh}(\ell_2^N(E), \ell_2^N(F))} \leq C\|t\|.$$

(x) *For any N and any $t\colon \ell_2^N \to \ell_2^N$ we have*

$$\|t \otimes u\|_{cb(\ell_2^N(E), \ell_2^N(F))} \leq C\|t\|.$$

(xi) *For any measure space (Ω, μ) and any bounded operator $t\colon L_2(\mu) \to L_2(\mu)$ we have*

$$\|t \otimes u\|_{cb(L_2(\mu;E), L_2(\mu;F))} \leq C\|t\|.$$

(xii) *For any n, we have*

$$\|I_{S_2^n} \otimes u\|_{\Gamma_2(S_2^n[E], S_2^n[F])} \leq C.$$

(xiii) *For any n and any A: $S_2^n \to S_2^n$ we have*

$$\|A \otimes u\|_{\Gamma_2(S_2^n[E], S_2^n[F])} \le C\|A\|.$$

Proof. — By Proposition 2.1 (iii), $\ell_2^n(OH(I))$ and $L_2(\mu; OH(I))$ are completely isometric to $OH(J)$ for some set J. Therefore it is easy to show that (i) $\Rightarrow$ (ix), and (ix) $\Rightarrow$ (x) is obvious. Similarly we have (i) $\Rightarrow$ (xi) and (xi) $\Rightarrow$ (x) is trivial. We will now show (x) $\Rightarrow$ (xii).

Assume (x). Fix an integer n. By Proposition 2.1 we have a complete isometry $S_2^n[\ell_2^N(E)] \approx \ell_2^N(S_2^n[E])$. Hence for all t: $\ell_2^N \to \ell_2^N$ if (x) holds we have by Corollary 1.2

$$\|t \otimes I_{S_2^n} \otimes u\|_{\ell_2^N(S_2^n[E]) \to \ell_2^N(S_2^n[F])} \le C\|t\|.$$

Therefore by Kwapień's result (see Remark 6.10)

$$\|I_{S_2^n} \otimes u\|_{\Gamma_2(S_2^n[E], S_2^n[F])} \le C.$$

This proves (x) $\Rightarrow$ (xii).

Now assume (xii). Then for any $\varepsilon > 0$ we have a factorization through some Hilbert space H as follows

$$I_{S_2^n} \otimes u\colon\ S_2^n[E] \xrightarrow{a} H \xrightarrow{b} S_2^n[F]$$

with $\|a\| \le C(1+\varepsilon)$, $\|b\| \le 1$.

Equivalently, we have maps $a_{ij}\colon\ E \to H$ such that for all $x = (x_{ij})$ in $S_2^n[E]$ we have

$$\|(u(x_{ij}))\|_{S_2^n[F]} \le \Big\|\sum a_{ij}(x_{ij})\Big\|_H \le C(1+\varepsilon)\|x\|_{S_2^n[E]}. \tag{6.4}$$

Now consider arbitrary choices of signs $\varepsilon_i' = \pm 1$, $\varepsilon_j'' = \pm 1$ and permutations σ_1, σ_2 of $\{1, \ldots, n\}$. Replacing (x_{ij}) by

$$(\varepsilon_i'\varepsilon_j'' x_{\sigma_1(i)\sigma_2(j)})$$

and averaging (6.4) (after squaring it) over ε_i', ε_j'', σ_1, σ_2 with respect to the uniform measure, we obtain

$$\|(u(x_{ij}))\|_{S_2^n[F]} \le \left(\frac{1}{n^2}\sum_{ijk\ell}\|a_{ij}(x_{k\ell})\|^2\right)^{1/2} \le C(1+\varepsilon)\|x\|_{S_2^n[E]}. \tag{6.5}$$

Let $K = \ell_2^{n^2}(H)$. Let $\alpha\colon\ E \to K$ be the map defined by

$$\alpha(x) = \frac{1}{n}(a_{ij}(x))_{i,j\le n}.$$

Then (6.5) can be rewritten as

$$\|(u(x_{ij}))\|_{S_2^n[F]} \le \left(\sum_{k\ell}\|\alpha(x_{k\ell})\|^2\right)^{1/2} \le C(1+\varepsilon)\|x\|_{S_2^n[E]}. \tag{6.6}$$

Let $U\colon S_2^n \to \ell_2^{n^2}$ be the standard isometry. Then, by (6.6) we have a map $\beta\colon K \to F$ such that $I_{S_2^n} \otimes u$ admits a factorization of the following form

$$I_{S_2^n} \otimes u\colon\ S_2^n[E] \xrightarrow{U\otimes\alpha} \ell_2^{n^2}(K) \xrightarrow{U^{-1}\otimes\beta} S_2^n[F],$$

with $\|U \otimes \alpha\| \le C(1+\varepsilon)$ and $\|U^{-1} \otimes \beta\| \le 1$.
Then consider any $A\colon S_2^n \to S_2^n$. We have

$$A \otimes u = (U^{-1} \otimes \beta)(UAU^{-1} \otimes I_K)(U \otimes \alpha)$$

hence

$$\begin{aligned}\|A \otimes u\|_{\Gamma_2(S_2^n[E],S_2^n[F])} &\le C(1+\varepsilon)\|UAU^{-1} \otimes I_K\|_{\ell_2^{n^2}(K)\to\ell_2^{n^2}(K)}\\ &\le C(1+\varepsilon)\|A\|.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary this proves that (xii) $\Rightarrow$ (xiii). Clearly (xiii) $\Rightarrow$ (vii) hence by Theorem 6.9 we also have (xiii) $\Rightarrow$ (i) and this completes the proof. $\square$

It is well known that Hilbert spaces are characterized among Banach spaces by the parallelogram inequality:

$$\forall\, x, y \in E \qquad \frac{\|x+y\|^2 + \|x-y\|^2}{2} \le \|x\|^2 + \|y\|^2.$$

In other words, given a Banach space E, let

$$T\colon\ \ell_2^2(E) \to \ell_2^2(E)$$

be the operator defined by

$$T(x,y) = \left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right).$$

Then E is isometric to a Hilbert space iff $\|T\| \le 1$.

In the category of operator spaces, we have an analogous result:

Theorem 6.12. — *Let E be an operator space. Then $\|T\|_{cb} \le 1$ iff there is a set I such that E is completely isometric to $OH(I)$.*

Proof. — If $E = OH(I)$, then the map $u = I_E$ satisfies property (x) in Theorem 6.11 with $C = 1$. Hence we have $\|T\|_{cb} \le 1$. To prove the converse, let $v\colon \ell_2^2 \to \ell_2^2$ be the map taking (x,y) to $\left(\dfrac{x+y}{\sqrt{2}}, \dfrac{x-y}{\sqrt{2}}\right)$ (rotation by $\pi/4$) so that $T = v \otimes I_E$. Recall that, by Proposition 2.1, we have isometrically

$$S_2^n[\ell_2^2(E)] = \ell_2^2(S_2^n[E]).$$

Hence if $\|T\|_{cb} \le 1$, we have (by Corollary 1.2)

$$\|I_{S_2^n} \otimes T\|_{cb(S_2^n[\ell_2^2(E)],S_2^n[\ell_2^2(E)])} \le 1,$$

therefore

$$\|v \otimes I_{S_2^n[E]}\|_{\ell_2^2(S_2^n[E])\to\ell_2^2(S_2^n[E])} \le 1.$$

But this means that the normed space $S_2^n[E]$ satisfies the parallelogram inequality, hence that it is isometric to a Hilbert space. It follows that $u = I_E$ satisfies the property (xii) in Theorem 6.11 with $C = 1$, whence $\gamma_{oh}(I_E) \leq 1$, or equivalently E is isometric to $OH(I)$ for some set I. □

As an application, we obtain a new approach to the results of section 9 in **[P1]**.

Theorem 6.13. — *For any n-dimensional operator space E we have*

$$\pi_2^o(I_E) = n^{1/2}. \tag{6.7}$$

Therefore, there is an isomorphism $u\colon\ E \to OH_n$ such that $\|u\|_{cb}\|u^{-1}\|_{cb} \leq \sqrt{n}$, and if $E \subset B(H)$ there is a projection $P\colon\ B(H) \to E$ such that $\|P\|_{cb} \leq n^{1/2}$.

Proof. — The proof is identical in structure to Kwapień's well known argument for the analogous result in the normed space case (cf. e.g. p.15-17 in **[P2]**).
By (5.2) we have

$$\pi_2^o(I_E) = \sup\{\pi_2^o(T) \mid T\colon\ S_2^* \to E \quad \|T\|_{cb} \leq 1\}.$$

Now since S_2^* is completely isometric to OH, (cf. Remark 1.11) it follows that any $T\colon\ S_2^* \to E$ with $\|T\|_{cb} \leq 1$ factors as $T = T_1T_2$ with $T_1\colon\ OH_n \to E, T_2\colon\ S_2^* \to OH_n$ such that $\|T_1\|_{cb} \leq 1$, $\|T_2\|_{cb} \leq 1$. Hence by (5.1) we have

$$\pi_2^o(I_E) = \sup\{\pi_2^o(T) \mid T\colon\ OH_n \to E \quad \|T\|_{cb} \leq 1\}.$$

By (5.1) again this yields $\pi_2^o(I_E) \leq \pi_2^o(I_{OH_n})$, and by Proposition 6.3 we get $\pi_2^o(I_{OH_n}) = n^{1/2}$.
Conversely, by Proposition 6.1 we have a factorization of I_E of the form

$$E \xrightarrow{u} OH_n \xrightarrow{\widetilde{u}} E \quad \text{with} \quad \|\widetilde{u}\|_{cb} \leq \pi_2^o(I_E)$$

and $\pi_2^o(u) \leq 1$. Indeed, using a suitable orthogonal projection we can factor through an n-dimensional subspace of E_2. Thus $\widetilde{u} = u^{-1}$, hence $I_{OH_n} = u\widetilde{u}$ so that we have by Proposition 6.3

$$n^{1/2} = \pi_2^o(I_{OH_n}) = \pi_2^o(u\widetilde{u}) \leq \pi_2^o(u)\|\widetilde{u}\|_{cb} \leq \pi_2^o(I_E).$$

This concludes the proof of (6.7).
We have, by Corollary 5.5, $\|u\|_{cb} \leq \pi_2^o(u) \leq 1$ and $\|u^{-1}\|_{cb} = \|\widetilde{u}\|_{cb} \leq \sqrt{n}$. Hence $\|u\|_{cb}\|u^{-1}\|_{cb} \leq \sqrt{n}$. Moreover, by Proposition 6.1, u admits an extension $v\colon\ B(H) \to OH_n$ with $\pi_2^o(v) = \pi_2^o(u) \leq 1$, hence (by Corollary 5.5) $\|v\|_{cb} \leq 1$. Then $P = u^{-1}v$ is a projection from $B(H)$ onto E with $\|P\|_{cb} \leq \|u^{-1}\|_{cb}\|v\|_{cb} \leq \sqrt{n}$. □

CHAPTER 7

COMPLETELY BOUNDED FACTORIZATION THROUGH L_p, S_p AND ULTRAPRODUCTS

7.1. Factoring through L_p. Perturbations and ultraproducts of operator spaces

In this chapter, we will extend to the operator space setting, a collection of results due to Kwapień **[Kw2]**, characterizing the mappings between Banach spaces which factor through L_p or through one of its subspaces or through a subspace of one of its quotients.

We start by a brief review of this theory. Let $1 \leq p \leq \infty$. We will say that a Banach space B is an L_p-space, if it is isometric to $L_p(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ). We will say that B is an SL_p-space (resp. an QL_p-space) if B is isometric to a subspace (resp. a quotient) of an L_p-space. Moreover, we will say that B is an SQL_p-space if B is isometric to a subspace of a quotient of an L_p-space. (Note that a subspace of a quotient is automatically also a quotient of a subspace, so the QSL_p-spaces are the same as the SQL_p-spaces and there is no need to iterate further.) Note that the class of SQL_p-spaces seems to appear naturally in analysis (*cf.* **[Her]**). Perhaps the most striking result in **[Kw2]** is the following one.

Theorem 7.1.1 (**[Kw2]**). — *A Banach space B is isomorphic to an SQL_p-space iff one of the following equivalent properties hold:*

(i) *Any bounded operator $T\colon\ \ell_p \to \ell_p$ extends naturally to a bounded operator on $\ell_p(B)$.*
(ii) *There is a constant C such that, for any measure space (Ω, Σ, μ), for any bounded operator $T\colon\ L_p(\mu) \to L_p(\mu)$, the operator $T \otimes I\colon\ L_p(\mu; B) \to L_p(\mu; B)$ is bounded with norm $\leq C\|T\|$.*
(iii) *There is a constant C such that, for any n and any $T\colon\ \ell_p^n \to \ell_p^n$ we have*

$$\|T \otimes I_B\|_{\ell_p^n(B) \to \ell_p^n(B)} \leq C\|T\|_{\ell_p^n \to \ell_p^n}.$$

Moreover, the smallest constant C appearing in (ii) *or* (iii) *is equal to the minimal Banach-Mazur distance of B to an SQL_p-space.*

Kwapień also characterized more generally the linear mappings $u\colon B_1 \to B_2$ which can be factored through an L_p-space (or a subspace or a subspace of a quotient of one). Actually there is a technical difficulty which appears here and we must consider factorizations going into the bidual of B_2. That is to say: we denote by $i\colon B_2 \to B_2^{**}$ the canonical inclusion and we consider all the commuting diagrams below:

$$\begin{tikzcd} & B \arrow[dr, "u_2"] & \\ B_1 \arrow[ur, "u_1"] \arrow[r, "u"] & B_2 \arrow[r, "i"] & B_2^{**} \end{tikzcd}$$

Then we let

$$\gamma_{L_p}(u) = \inf\{\|u_1\|\,\|u_2\|\}$$

where the infimum runs over all factorizations of this form where B is an L_p-space. We will denote by $\gamma_{SL_p}(u)$ the infimum when B runs over all SL_p-spaces, and by $\gamma_{SQL_p}(u)$ the infimum when B runs over all possible SQL_p-spaces. Note that obviously

$$\gamma_{SQL_p}(u) \le \gamma_{SL_p}(u) \le \gamma_{L_p}(u).$$

The basic result in **[Kw2]** is the following one.

Theorem 7.1.2 (**[Kw2]**). — *Let $1 \le p \le \infty$. Let $u\colon B_1 \to B_2$ be a continuous linear map between Banach spaces and let C be a constant. The following are equivalent.*

(i) $\gamma_{L_p}(u) \le C$.

(ii) *For any finite dimensional Banach space Y and for any composition $B_2 \xrightarrow{v_1} Y \xrightarrow{v_2} B_1$ with v_1 p'-summing and v_2^* p-summing the composition $v_2 v_1 u$ satisfies*

$$|\operatorname{tr}(v_2 v_1 u)| \le C \pi_{p'}(v_1)\pi_p(v_2^*).$$

From this result it is easy to derive (by routine arguments) characterizations of maps factoring through an SL_p or through an SQL_p-space. Indeed, if $j\colon B_2 \to \ell_\infty(I)$ is an isometric embedding (I being a suitable set) then $\gamma_{SL_p}(u) = \gamma_{L_p}(ju)$ and if $q\colon \ell_1(I) \to B_1$ is a metric surjection then $\gamma_{SQL_p}(u) = \gamma_{L_p}(juq)$.

The general method, used by Kwapień, is the duality theory for ideals of Banach space operators developed by Pietsch following Grothendieck's fundamental work **[G]** on tensor products. Roughly the modern viewpoint can be briefly described like this: Firstly one observes that one can reduce to the case when both B_1 and B_2 are finite dimensional. More precisely, for any $u\colon B_1 \to B_2$ we have

$$\gamma_{L_p}(u) = \sup\{\gamma_{L_p}(qu|_S)\}$$

where the supremum runs over all finite dimensional subspaces S of B_1 and all finite dimensional quotient spaces Q of B_2, with $q\colon B_2 \to Q$ denoting the quotient map. This first point depends on the fact that for $1 \le p < \infty$, the class of L_p-spaces is stable under ultraproducts. (This point does not seem to have a perfect analog in the operator space setting, see below.)

Secondly, when B_1, B_2 are both finite dimensional then factorization through L_p is the same as factorization through ℓ_p or through ℓ_p^n for some n. This point is extended to the operator space setting in Lemma 7.1.5 below. But now the connection with p-summing operators appears: indeed for instance it is easy to see that if we use the trace duality

$$\langle u, v\rangle = \mathrm{tr}(vu)$$

(where $u\colon\ B_1 \to B_2$ and $v\colon\ B_2 \to B_1$ are linear maps) then the dual norm to $\gamma_{L_\infty}(u)$ coincides with the absolutely summing norm of v, denoted by $\pi_1(v)$. More generally, as Kwapień showed, if we define (for $1 \le p \le \infty$ and $1/p + 1/p' = 1$)

$$\gamma^*_{L_p}(v) = \sup\{|\,\mathrm{tr}(vu)|\,\big|\, u\colon\ B_1 \to B_2 \quad \gamma_{L_p}(u) \le 1\}$$

then we have (here by convention π_∞ stands for the operator norm)

$$\gamma^*_{L_p}(v) = \inf\{\pi_{p'}(v_1)\pi_p(v_2^*)\} \tag{7.1.1}$$

where the infimum runs over all possible factorizations of v of the form

$$B_2 \xrightarrow{v_1} Y \xrightarrow{v_2} B_1,$$

Y being an arbitrary Banach space.

Thus we can describe the dual norm $\gamma^*_{L_p}$ in terms of p-summing operators, and by the bipolar theorem (recall B_1, B_2 are finite dimensional in the present discussion) we obtain a new description of γ_{L_p} by identifying it with $(\gamma^*_{L_p})^*$. In other words, we can write

$$\gamma_{L_p}(u) = \sup\{|\,\mathrm{tr}(uv)|\,\big|\, v\colon\ B_2 \to B_1 \quad \gamma^*_{L_p}(v) \le 1\} \tag{7.1.2}$$

and this is now a significant result because, by (7.1.1) we have a specific description of $\gamma^*_{L_p}$. The preceding identities (7.1.1) and (7.1.2) imply essentially all of Kwapień 's results stated above. In the next section, we will follow essentially the same program, and discuss the difficulties as they appear.

We end this section with several simple facts from the Banach space folklore which can be easily transferred to the operator space setting. We start by a well known fact (the proof is the same as for ordinary norms of operators).

Lemma 7.1.3. — *Let $v\colon\ E \to F$ be a complete isomorphism between operator spaces. Then clearly any map $w\colon\ E \to F$ with $\|v - w\|_{cb} < \|v^{-1}\|_{cb}^{-1}$ is again a complete isomorphism and if we let $\Delta = \|v - w\|_{cb}\|v^{-1}\|_{cb}$ we have*

$$\|w^{-1}\|_{cb} \le \|v^{-1}\|_{cb}(1-\Delta)^{-1} \quad \textit{and} \quad \|w^{-1} - v^{-1}\|_{cb} \le \|v^{-1}\|_{cb}(1-\Delta)^{-1}.$$

Recall that the cb-distance between two n-dimensional operator spaces E_1, E_2 is defined as follows

$$d_{cb}(E_1, E_2) = \inf\{\|w\|_{cb}\|w^{-1}\|_{cb}\},$$

where the infimum runs over all possible isomorphisms $w\colon\ E_1 \to E_2$.

Lemma 7.1.4. — *Fix $0 < \varepsilon < 1$. Let E be an operator space. Consider a biorthogonal system (x_i, x_i^*) $(i = 1, 2, \ldots, n)$ with $x_i \in E$, $x_i^* \in E^*$ and let $y_1, \ldots, y_n \in E$ be such that*

$$\sum \|x_i^*\| \, \|x_i - y_i\| < \varepsilon.$$

Then there is a complete isomorphism $w\colon\ E \to E$ such that $w(x_i) = y_i$ and

$$\|w\|_{cb} \le 1 + \varepsilon \qquad \|w^{-1}\|_{cb} \le (1-\varepsilon)^{-1}.$$

In particular, if $E_1 = \operatorname{span}(x_1, \ldots, x_n)$ and $E_2 = \operatorname{span}(y_1, \ldots, y_n)$, we have

$$d_{cb}(E_1, E_2) \le (1+\varepsilon)(1-\varepsilon)^{-1}.$$

Proof. — Recall that any rank one linear map $v\colon\ E \to E$ satisfies $\|v\| = \|v\|_{cb}$. Let $\delta\colon\ E \to E$ be the map defined by setting $\delta(x) = \sum x_i^*(x)(y_i - x_i)$ for all x in E. Then $\|\delta\|_{cb} \le \sum \|x_i^*\| \, \|y_i - x_i\| < \varepsilon$. Let $w = I + \delta$. Note that $w(x_i) = y_i$ for all $i = 1, 2, \ldots, n$, $\|w\|_{cb} \le 1 + \|\delta\|_{cb} \le 1 + \varepsilon$ and by the preceding lemma we have $\|w^{-1}\|_{cb} \le (1-\varepsilon)^{-1}$. □

Lemma 7.1.5. — *Consider an operator space E and a family of subspaces $E_\alpha \subset E$ directed by inclusion and such that $\overline{\cup E_\alpha} = E$. Then for any $\varepsilon > 0$ and any finite dimensional subspace $S \subset E$, there exists α and $\widetilde{S} \subset E_\alpha$ such that $d_{cb}(S, \widetilde{S}) < 1 + \varepsilon$. Let $u\colon\ F_1 \to F_2$ be a linear map between two operator spaces. Assume that u admits the following factorization $F_1 \xrightarrow{a} E \xrightarrow{b} F_2$ with c.b. maps a, b such that a is of finite rank. Then for each $\varepsilon > 0$ there exists α and a factorization $F_1 \xrightarrow{\widetilde{a}} E_\alpha \xrightarrow{\widetilde{b}} F_2$ with $\|\widetilde{a}\|_{cb}\|\widetilde{b}\|_{cb} < (1+\varepsilon)\|a\|_{cb}\|b\|_{cb}$, and $\widetilde{a}$ of finite rank.*

Proof. — For the first part let $x_1, \ldots, x_n$ be a linear basis of S and let x_i^* be the dual basis extended (by Hahn-Banach) to elements of E^*. Fix $\varepsilon' > 0$. Choose α large enough and $y_1, \ldots, y_n \in E_\alpha$ such that $\sum \|x_i^*\| \, \|x_i - y_i\| < \varepsilon'$. Let $\widetilde{S} = \operatorname{span}(y_1, \ldots, y_n)$. Then, by the preceding lemma, there is a complete isomorphism $w\colon\ E \to E$ with $\|w\|_{cb}\|w^{-1}\|_{cb} < (1+\varepsilon')(1-\varepsilon')^{-1}$ such that $w(S) = \widetilde{S} \subset E_\alpha$. In particular, $d_{cb}(S, \widetilde{S}) \le (1+\varepsilon')(1-\varepsilon')^{-1}$ so it suffices to adjust ε' to obtain the first assertion.

Now consider a factorization $F_1 \xrightarrow{a} E \xrightarrow{b} F_2$ and let $S = a(F_1)$. Note that S is finite dimensional by assumption. Applying the preceding to this S, we find α and a complete isomorphism $w\colon\ E \to E$ with $\|w\|_{cb}\|w^{-1}\|_{cb} < 1 + \varepsilon$ such that $w(S) \subset E_\alpha$. Thus, if we take $\widetilde{a} = wa\colon\ F_1 \to E_\alpha$ and $\widetilde{b} = bw^{-1}_{|E_\alpha}$, we obtain the announced factorization. □

Convention. — Whenever we are discussing an ultraproduct $\prod_{i \in I} E_i/\mathcal{U}$ of a family of Banach spaces or operator spaces, it will be convenient to identify abusively a bounded family $(x_i)_{i \in I}$ with $x_i \in E_i$ for all i in I with the corresponding equivalence class modulo $\mathcal{U}$ which it determines in $\prod_{i \in I} E_i/\mathcal{U}$. Thus when we speak of $(x_i)_{i \in I}$ as an element of $\prod_{i \in I} E_i/\mathcal{U}$, we really are referring to the equivalence class it determines. This abuse is consistent with one routinely done in standard measure theory.

Remark 7.1.6. — Let $(E_\alpha)_{\alpha\in I}$ and $(F_\alpha)_{\alpha\in I}$ be families with $E_\alpha = E$ and $F_\alpha = F$ for all i in the index set I equipped with an ultrafilter $\mathcal{U}$. Let $\widehat{E}$ and $\widehat{F}$ be the associated ultraproducts (= ultrapowers). Let $\varphi_E\colon E \to \widehat{E}$ be the canonical (completely isometric) inclusion and let $\psi_F\colon \widehat{F} \to F^{**}$ be the canonical (completely contractive) map obtained by compactness of $(B_{F^{**}}, \sigma(F^{**}, F^*))$, and defined by $\psi((x_\alpha)) = \lim_{\mathcal{U}} x_\alpha$. Then, let $T\colon E \to F$ be a bounded linear map with associated "ultraproduct map" $\widehat{T}\colon \widehat{E} \to \widehat{F}$. It is an easy exercise to check that

$$\psi_F \widehat{T} \varphi_E = i_F T\colon\ E \to F^{**}.$$

More generally, we have

$$\psi_F \widehat{T} = T^{**}\psi_E. \tag{7.1.3}$$

Note in passing that $i_F T = T^{**} i_E$.

Let $(E_\alpha)_{\alpha\in I}$ be a net of subspaces of E directed by inclusion and such that $\overline{\bigcup_{\alpha\in I} E_\alpha} = E$. We can then still define $\varphi\colon \bigcup_\alpha E_\alpha \to \prod E_\alpha/\mathcal{U}$ by setting $\varphi(x) = (\varphi_\alpha(x))_\alpha$ where we set $\varphi_\alpha(x) = x$ if $x \in E_\alpha$ and $= 0$ (say) otherwise. By density, φ extends to a (completely isometric) map $\varphi\colon E \to \prod_{\alpha\in I} E_\alpha/\mathcal{U}$. Moreover, we again have a canonical complete contraction $\psi\colon \prod_{\alpha\in I} E_\alpha/\mathcal{U} \to E^{**}$ defined by $\psi((x_\alpha)) = \lim_{\mathcal{U}} x_\alpha$ (the limit is relative to $\sigma(E^{**}, E^*)$).

Remark 7.1.7. — In particular, if $1 < p < \infty$, the identity of the space S_p factors (completely contractively) through an ultraproduct of the family $\{S_p^n \mid n \geq 1\}$. Conversely, the identity of the space S_p^n obviously factors (completely contractively) through S_p.

Remark 7.1.8. — It will be convenient in the sequel to use the fact that an ultraproduct of ultraproducts is again an ultraproduct. Let us briefly recall why this is true. Let I_1, I_2 be two sets equipped with respective ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$. Then the set $I_1 \times I_2$ can be equipped with a "product ultrafilter" $\mathcal{W} = \mathcal{U}_1 \times \mathcal{U}_2$ defined simply as follows: $\mathcal{W}$ is the collection of all subsets $A \subset I_1 \times I_2$ with the property that

$$\{i \in I_1 | \{j \in I_2 | (i,j) \in A\} \in \mathcal{U}_2\} \in \mathcal{U}_1.$$

Using the fact that an ultrafilter $\mathcal{U}$ on a set I is characterized as a filter such that, for any arbitrary subset $A \subset I$, either A or its complement $I - A$ must belong to $\mathcal{U}$, it is easy to verify that $\mathcal{W} = \mathcal{U}_1 \times \mathcal{U}_2$ is indeed an ultrafilter when $\mathcal{U}_1$ and $\mathcal{U}_2$ are ultrafilters. Moreover, if we are given a doubly indexed family $\{E_{ij} \mid i \in I_1, j \in I_2\}$ of Banach spaces (resp. operator spaces), then it is easy to check that we have

$$\prod_{i\in I_1} \left(\prod_{j\in I_2} E_{ij}/\mathcal{U}_2 \right) /\mathcal{U}_1 = \prod_{(i,j)\in I_1\times I_2} E_{ij}/\mathcal{W}$$

isometrically (resp. completely isometrically).

In particular, this, with the preceding remark, implies that, when $1 < p < \infty$, it is the same for a map to factorize contractively (resp. completely contractively) through an ultraproduct of S_p or through an ultraproduct of the family $\{S_p^n \mid n \geq 1\}$.

7.2. Factorization through S_p

Let E, F be operator spaces. We will say that a linear map $u\colon\ E \to F$ factors through S_p if u admits a factorization of the form

$$E \xrightarrow{a} S_p \xrightarrow{b} F$$

with c.b. maps a, b. Given such a map u, we let

$$\gamma_{S_p}(u) = \inf\{\|a\|_{cb}\|b\|_{cb}\}$$

where the infimum runs over all possible factorizations as above.

It is easy to transfer the Banach space arguments to the present setting in order to check that γ_{S_p} is a norm, with which the space $\Gamma_{S_p}(E, F)$ is a Banach space.

Actually, we will need to work first with tensor products rather that with Γ_{S_p}. Consider an element T in the algebraic tensor product $E \otimes F$. As usual, T defines a weak-$*$ continuous finite rank linear operator $\widetilde{T}\colon\ E^* \to F$. We define

$$\nu_{S_p}(T) = \inf\{\|a\|_{cb}\|b\|_{cb}\}$$

where the infimum runs over all possible factorizations of $\widetilde{T}$ of the form

$$E^* \xrightarrow{a} S_p^n \xrightarrow{b} F$$

with the first map a weak-$*$continuous and n arbitrary.

Equivalently, a simple perturbation argument (see Lemma 7.1.5) shows that this definition is unchanged if we let the infimum run over all possible factorizations of the form $E^* \xrightarrow{a} S_p \xrightarrow{b} F$ with a, b of finite rank and a weak-$*$ continuous.

We will denote by $E \otimes_{S_p} F$ the completion of $E \otimes F$ equipped with this norm ν_{S_p}. Note that we have equivalently

$$\nu_{S_p}(T) = \inf\left\{\left\|\sum e_{ij} \otimes a_{ij}\right\|_{S_p^n \otimes_{\min} E} \left\|\sum e_{ij} \otimes b_{ij}\right\|_{S_{p'}^n \otimes_{\min} F}\right\} \tag{7.2.1}$$

where the infimum runs over $a_{ij} \in E$, $b_{ij} \in F$ such that $T = \sum_{ij} a_{ij} \otimes b_{ij}$. Indeed, $\sum e_{ij} \otimes a_{ij}$ (resp. $\sum e_{ij} \otimes b_{ij}$) can be identified with a weak-$*$ continuous c.b. map $a\colon\ E^* \to S_p^n$ (resp. $b\colon\ S_p^n \to F$).

We will now describe the dual space $(E \otimes_{S_p} F)^*$.

Theorem 7.2.1. — *Let $1 \leq p \leq \infty$. Let $\varphi\colon\ E \otimes F \to \mathbb{C}$ be a linear form. The following are equivalent*

(i) *For any T in $E \otimes F$, we have the inequality $|\varphi(T)| \leq \nu_{S_p}(T)$, or equivalently $\|\varphi\|_{(E \otimes_{S_p} F)^*} \leq 1$.*

(ii) *For some operator space G there is a completely p-summing map $u\colon\ E \to G$ and a completely p'-summing map $v\colon\ F \to G^*$ with $\pi_p^o(u)\pi_{p'}^o(v) \le 1$ such that*

$$\forall\, x \in E\ \forall\, y \in F \qquad \varphi(x \otimes y) = \langle v(y), u(x)\rangle.$$

Warning. — In the sequel, it will be convenient to make the convention that a completely ∞-summing map $u\colon\ E \to F$ is simply a c.b. map and $\pi_\infty^o(u) = \|u\|_{cb}$, wherever it appears.

Proof. — Once the proof of Theorem 5.1 is understood, this can be proved by a routine adaptation of the corresponding Banach space result (cf. [**Kw2**]). Note that, in the case $p = 2$, this is closely related to Theorem 6.11 above. We merely sketch the argument. We will use the elementary identity

$$\text{(7.2.2)} \qquad \forall\, x, y \ge 0 \qquad x^{1/p}y^{1/p'} = \inf_{t>0}\left\{\frac{1}{p}xt^p + \frac{1}{p'}yt^{-p'}\right\}.$$

Assume (i). Assume moreover $E \subset B(H)$ and $F \subset B(K)$. Let $\mathcal{S} = S \times S'$ where

$$S = \{(a,b) \in B_{S_{2p}(H)} \times B_{S_{2p}(H)} \mid a \ge 0, b \ge 0\}$$

and

$$S' = \{(c,d) \in B_{S_{2p'}(K)} \times B_{S_{2p'}(K)} \mid c \ge 0, d \ge 0\}.$$

We will show that there exists families $(a_\alpha, b_\alpha)_{\alpha\in I}$ in S, $(c_\alpha, d_\alpha)_{\alpha\in I}$ in S' and an ultrafilter $\mathcal{U}$ on the set I such that, for any $T = \sum\limits_{ij=1}^{n} a_{ij} \otimes b_{ij}$ in $E \otimes F$ (with n arbitrary) we have

$$\text{(7.2.3)} \qquad \left|\varphi\left(\sum a_{ij} \otimes b_{ij}\right)\right| \le \lim_{\mathcal{U}} \|(a_\alpha a_{ij} b_\alpha)\|_{S_p(\ell_2 \otimes H)} \lim_{\mathcal{U}} \|(c_\alpha b_{ij} d_\alpha)\|_{S_{p'}(\ell_2 \otimes K)}.$$

Consider a finite sequence

$$T_m = \sum_{ij=1}^{n_m} a_{ij}^m \otimes b_{ij}^m \text{ in } E \otimes F,$$

and consider the associated function f defined on $\mathcal{S}$ as follows

$$f((a,b),(c,d)) = -\sum_m \left|\varphi\left(\sum_{ij} a_{ij}^m \otimes b_{ij}^m\right)\right| + p^{-1}\sum_m \|(a a_{ij}^m b)\|_{S_p(\ell_2\otimes H)}^p + p'^{-1}\sum_m \|(c b_{ij}^m d)\|_{S_{p'}(\ell_2\otimes K)}^{p'}.$$

Let $\mathcal{F}$ be the cone of all functions of this form.

By (5.5), (7.2.1) and (7.2.2) we have $\sup\limits_{\mathcal{S}} f \ge 0$ for any f in $\mathcal{F}$. (Note that $\ell_p\{S_p\}$ can be block-diagonally embedded into S_p to take into account the summation over m.) Hence, by Lemma 5.2, there is an ultrafilter $\mathcal{U}$ on a set of finitely supported probability measures (Λ_α) on $\mathcal{S}$ such that $\lim\limits_{\mathcal{U}} \int f d\Lambda_\alpha \ge 0$ for any f in $\mathcal{F}$. Taking the images of this probability on the two coordinates of the product $\mathcal{S} = S \times S'$ we obtain a net of finitely supported probability measures (λ_α) (resp. (λ'_α)) on S (resp. S') such that, for any $T = \sum_{ij=1}^{n} a_{ij} \otimes b_{ij}$ in $E \otimes F$, we have

$$\left|\varphi\left(\sum a_{ij} \otimes b_{ij}\right)\right| \leq p^{-1} \lim_{\mathcal{U}} \int \|(aa_{ij}b)\|^p_{S_p(\ell_2 \otimes H)} d\lambda_\alpha(a,b) + p'^{-1} \lim_{\mathcal{U}} \int \|(cb_{ij}d)\|^{p'}_{S_{p'}(\ell_2 \otimes K)} d\lambda'_\alpha(c,d).$$

Then arguing as in the proof of Theorem 5.1 we obtain nets (a_α, b_α) in S and (c_α, d_α) in S' such that

$$\left|\varphi\left(\sum a_{ij} \otimes b_{ij}\right)\right| \leq p^{-1} \lim_{\mathcal{U}} \|(a_\alpha a_{ij} b_\alpha)\|^p_{S_p(\ell_2 \otimes H)} + p'^{-1} \lim_{\mathcal{U}} \|(c_\alpha b_{ij} d_\alpha)\|^p_{S_{p'}(\ell_2 \otimes K)}.$$

Finally, using (7.2.2), we obtain the announced result (7.2.3).

We now use Remark 5.7 for both u and v.

Let $\widehat{S}_p = (S_\alpha)_{\mathcal{U}}$ where $S_\alpha = S_p(H)$ for all α and $\widehat{S}_{p'} = (S'_\alpha)_{\mathcal{U}}$ where $S'_\alpha = S_{p'}(K)$ for all α. As in Remark 5.7, there is a natural map $E \to \widehat{S}_p$ which takes $x \in E$ to $(a_\alpha x b_\alpha)_\alpha$, and another map $F \to \widehat{S}_{p'}$ which takes $y \in F$ to $(c_\alpha y d_\alpha)_\alpha$. We denote the closures of their ranges respectively by E_p and $F_{p'}$. Let us denote respectively by

$$B_1\colon\ E \to E_p \quad \text{and} \quad B_2\colon\ F \to F_{p'}$$

the resulting mappings. By Remark 5.7, we know that

$$\pi^o_p(B_1) \leq 1 \quad \text{and} \quad \pi^o_{p'}(B_2) \leq 1.$$

Then (recalling Lemma 5.4) we deduce from (7.2.3) that $\forall\,(a_{ij}) \in M_n(E)$, $\forall\,(b_{ij}) \in M_n(F)$

$$\left|\varphi\left(\sum a_{ij} \otimes b_{ij}\right)\right| \leq \|(B_1(a_{ij}))\|_{S^n_p[E_p]}\ \|(B_2(b_{ij}))\|_{S^n_{p'}[F_{p'}]}. \tag{7.2.4}$$

Therefore, φ defines a linear mapping $w\colon\ F_{p'} \to (E_p)^*$ such that

$$\varphi(a \otimes b) = \langle w(B_2(b)), B_1(a)\rangle \quad \text{for} \quad a \in E, b \in F,$$

and moreover, by (7.2.4) (recall Lemma 1.7 and Corollary 1.8) we have

$$\|w\|_{cb} \leq 1.$$

Let then $G = E_p$, $u = B_1$ and let $v\colon\ F \to E^*_p$ be defined as the composition wB_2. With these choices, it is now clear that (ii) holds. This shows that (i) $\Rightarrow$ (ii).

Conversely, assume (ii) and consider $T = \sum_{ij} a_{ij} \otimes b_{ij} \in E \otimes F$. We have (by Corollary 1.8)

$$|\varphi(T)| = \left|\sum_{ij} \langle v(b_{ij}), u(a_{ij})\rangle\right| \leq \|(v(b_{ij}))\|_{S^n_{p'}[G^*]} \|(u(a_{ij}))\|_{S^n_p[G]}$$

hence by definition of $\pi_p^o(u)$ and $\pi_{p'}^o(v)$

$$\leq \pi_{p'}^o(v)\|(b_{ij})\|_{S_{p'}^n\otimes_{\min}F}\ \pi_p^o(u)\|(a_{ij})\|_{S_p^n\otimes_{\min}E}$$

from which we immediately deduce (i) using (7.2.1). □

Remark. — Let $\tau\colon\ E_1 \to E_2$ be a finite rank continuous linear map between operator spaces and assume E_1 finite dimensional. Let $T \in E_1^* \otimes E_2$ be the associated tensor. Then it is easy to check (using Lemma 7.1.5) that

$$\gamma_{S_p}(\tau) = \nu_{S_p}(T).$$

Corollary 7.2.2. — *Let $1 \leq p \leq \infty$. Let E_1, E_2 be finite dimensional operator spaces. The following properties of a linear map $\tau\colon\ E_1 \to E_2$ are equivalent.*

(i) *For any $\varepsilon > 0$, there is an integer n and a factorization of τ of the form*

$$E_1 \xrightarrow{a} S_p^n \xrightarrow{b} E_2$$

with $\|a\|_{cb}\|b\|_{cb} < 1+\varepsilon$. (Equivalently, this means that $\gamma_{S_p}(\tau) \leq 1$).

(ii) *For any operator space Y and for any maps $u_2\colon\ E_2 \to Y$ and $u_1\colon\ Y \to E_1$ the composition $u_1u_2\colon\ E_2 \to E_1$ satisfies*

$$|\operatorname{tr}(u_1u_2\tau)| \leq \pi_{p'}^o(u_2)\pi_p^o(u_1^*).$$

(iii) *Same as* (ii) *for any finite dimensional space Y.*

Proof. — Let $T \in E_1^* \otimes E_2$ be the tensor associated to τ. Clearly, we have $\gamma_{S_p}(\tau) = \nu_{S_p}(T) \leq 1$ iff

$$\sup\{|\varphi(T)| \mid \varphi \in (E_1^* \otimes E_2)^*\ \|\varphi\|_{(E_1^*\otimes_{S_p}E_2)^*} \leq 1\} \leq 1.$$

By Theorem 7.2.1, for any φ with $\|\varphi\|_{(E_1^*\otimes_{S_p}E_2)^*} \leq 1$, there are G and maps $u\colon\ E_1^* \to G$, $v\colon\ E_2 \to G^*$ with $\pi_p^o(u)\pi_{p'}^o(v) \leq 1$ such that

$$\varphi(T) = \operatorname{tr}(\tau u^* v) = \operatorname{tr}(u^* v\tau).$$

Given a pair u, v as above, then let $Y = G^*$, $u_2 = v$ and let $u_1 = u^*\colon\ G^* \to E_1$. It then becomes clear that (ii) $\Rightarrow$ (i). Conversely given u_1, u_2 as in (ii), the preceding theorem shows that the linear form φ defined by $\varphi(T) = \operatorname{tr}(u_1u_2\tau)$ satisfies $\|\varphi\|_{(E_1^*\otimes_{S_p}E_2)^*} \leq 1$. Thus, we obtain conversely that (i) $\Rightarrow$ (ii).

Finally, let u_1, u_2 be as in (ii). Let $\widetilde{Y} \subset Y$ be the (finite dimensional) range of u_2. Then, replacing Y by $\widetilde{Y}$, we easily check that (iii) $\Rightarrow$ (ii) and the converse is trivial. □

Remark 7.2.3. — Let us now assume that $1 < p < \infty$ and there are integers N_1 and N_2 and subspaces $G_1 \subset M_{N_1}$ and $G_2 \subset M_{N_2}$ such that $E_1 = G_1^*$ and $E_2 = G_2$. Then the equivalent conditions in Corollary 7.2.2 are also equivalent to

(iv) There is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that τ can be factorized as

$$E_1 \xrightarrow{a} \widehat{S}_p \xrightarrow{b} E_2$$

where $\|a\|_{cb}\|b\|_{cb} \leq 1$, and $\widehat{S}_p = \Pi S_p^n/\mathcal{U}$.

Indeed (i) $\Rightarrow$ (iv) is obvious. To check the converse, assume (iv). By our special assumption on E_1 and E_2, we know that

$$cb(E_1, \widehat{S}_p) = E_1^* \otimes_{\min} \widehat{S}_p = \Pi E_1^* \otimes S_p^n/\mathcal{U}. \tag{7.2.5}$$

Indeed, since we assume $E_1^* = G_1 \subset M_{N_1}$ this follows from the identity $M_{N_1} \otimes_{\min} \widehat{S}_p = \Pi M_{N_1} \otimes_{\min} S_p^n/\mathcal{U}$ which is the very definition of $\widehat{S}_p$. Similarly, we have

$$cb(\widehat{S}_p, E_2) = (\widehat{S}_p)^* \otimes_{\min} E_2 = \Pi S_p^{n*} \otimes_{\min} E_2/\mathcal{U}. \tag{7.2.5)''}$$

This can be verified as follows. First we observe that we may assume that $E_2 = M_N$ for some integer N (say $N = N_2$), since (7.2.5)$''$ is inherited by subspaces of E_2. But then we have $\widehat{S}_p^* \otimes_{\min} M_N = (S_1^N[\widehat{S}_p])^*$. Now by Lemma 5.4, $S_1^N[\widehat{S}_p] = \Pi S_1^N[S_p^n]/\mathcal{U}$ and since $S_1^N[S_p]$ is super-reflexive, it is known (cf. Cor. 7.2 in **[Hei]**) that the dual of $\Pi S_1^N[S_p^n]/\mathcal{U}$ coincides isometrically with the ultraproduct of the duals, i.e. with $\Pi(S_1^N[S_p^n])^*/\mathcal{U}$. This gives us

$$\begin{aligned}(\widehat{S}_p)^* \otimes_{\min} M_N &= \Pi(S_1^N[S_p^n])^*/\mathcal{U}\\ &= \Pi M_N(S_p^{n*})/\mathcal{U}\\ &= \Pi S_p^{n*} \otimes_{\min} M_N.\end{aligned}$$

This completes the verification of (7.2.5)$''$.
Now using (7.2.5)$'$ and (7.2.5)$''$, the condition (iv) implies the existence of nets (a_n) and (b_n) with $a_n\colon\ E_1 \to S_p^n$, $b_n\colon\ S_p^n \to E_2$ such that $\lim_{\mathcal{U}} \|a_n\|_{cb} = \|a\|_{cb}$, $\lim_{\mathcal{U}} \|b_n\|_{cb} = \|b\|_{cb}$ and such that $\tau = \lim_{\mathcal{U}} b_n a_n$. Then, by an easy perturbation argument, this implies (i) in Corollary 7.2.2.

Let us distinguish the case $p = \infty$ which is of special interest.

Corollary 7.2.4. — *Let $c > 0$ be a constant. The following properties of a map $\tau\colon\ E_1 \to E_2$ between finite dimensional operator spaces are equivalent.*

(i) *For any $\varepsilon > 0$, there is an integer n and a factorization of τ of the form*

$$E_1 \xrightarrow{a} M_n \xrightarrow{b} E_2 \quad \text{with} \quad \|a\|_{cb}\|b\|_{cb} < c(1+\varepsilon).$$

(ii) *For any $u\colon\ E_2 \to E_1$, we have*

$$|\operatorname{tr}(u\tau)| \le c\pi_1^o(u).$$

(iii) *For any $u\colon\ E_2 \to E_1$, we have*

$$\|u\tau\|_{E_1^* \otimes^\wedge E_1} \le c\pi_1^o(u).$$

Proof. — The preceding statement with $p = \infty$ contains the equivalence (i) $\Leftrightarrow$ (ii) (with the above convention $\pi_\infty^o(\cdot) = \|\cdot\|_{cb}$). The o.s. projective tensor product $E_1^* \otimes^\wedge E_1$ is by construction (see **[BP]** and **[ER2]**) the dual and predual of the space $cb(E_1, E_1)$. Hence, for any $v \in E_1^* \otimes E_1$ corresponding to an operator $v\colon\ E_1 \to E_1$ we have

$$\|v\|_{E_1^* \otimes^\wedge E_1} = \sup\{|\operatorname{tr}(wv)| \mid w\colon\ E_1 \to E_1\ \|w\|_{cb} \le 1\}.$$

Using this, it is easy to show (ii) $\Leftrightarrow$ (iii). □

It is perhaps worthwhile to reformulate Corollary 7.2.2 as a duality theorem. In order to do that, we first introduce some more notation.

Notation. — Let $u\colon E_2 \to E_1$ be a linear map between operator spaces. Assume that u can be written as factorized through some operator space Y as $E_2 \xrightarrow{u_2} Y \xrightarrow{u_1} E_1$ where u_2 is completely p'-summing and where the adjoint u_1^* of u_1 is completely p-summing. We define

$$\alpha_{p'}(u) = \inf\{\pi^o_{p'}(u_2)\pi^o_p(u_1^*)\}$$

where the infimum runs over all possible such factorizations. Moreover, we will denote by $\alpha_{p'}(E_2, E_1)$ the space of all such mappings u equipped with the norm $\alpha_{p'}$.

Corollary 7.2.5. — *Let $1 < p < \infty$. Let E_1, E_2 be two finite dimensional operator spaces. Then the dual of the space $\Gamma_{S_p}(E_1, E_2)$ coincides isometrically with the space $\alpha_{p'}(E_2, E_1)$, with respect to the trace duality*

$$\langle u, \tau\rangle = \mathrm{tr}(u\tau) \qquad \forall\, u \in \alpha_{p'}(E_2, E_1) \quad \forall\, \tau \in \Gamma_{S_p}(E_1, E_2).$$

When $p = \infty$, the dual of $\Gamma_{S_\infty}(E_1, E_2)$ can be identified isometrically with the space $\prod^o_1(E_2, E_1)$. (The case $p = 1$ can be treated by transposition from the case $p = \infty$.)

We will now turn to the factorization of operators through quotients of subspaces. The following notation will be convenient.

Notation. — Given an operator space G we denote by $QS(G)$ the class of all quotients of a subspace of G, i.e. $Z \in QS(G)$ means that there are subspaces $G_2 \subset G_1 \subset G$ such that $Z = G_1/G_2$. Note that this class coincides with the class of all subspaces of quotients of G (since $G_1/G_2 \subset G/G_2$), so that there is no need to consider the classes $SQ(G)$ or $SQS(G)\ldots$

Theorem 7.2.6. — *Let n, m be integers. Let E_1 be a quotient of S_1^m and let E_2 be a subspace of $M_n(= S_\infty^n)$. Let $1 \le p \le \infty$. The following properties of a linear map $\tau\colon E_1 \to E_2$ are equivalent.*

(i) *For any $\varepsilon > 0$, there is an integer N and Z in $QS(S_p^N)$ for which τ admits a factorization of the form $E_1 \xrightarrow{a} Z \xrightarrow{b} E_2$ with $\|a\|_{cb}\|b\|_{cb} < 1 + \varepsilon$.*

(ii) *For any integers n', m' and any linear map $v\colon S_p^{m'} \to S_p^{n'}$, we have*

$$\|v \otimes \tau\|_{S_p^{m'}[E_1] \to S_p^{n'}[E_2]} \le \|v\|_{cb}.$$

(iii) *Same as* (ii) *with $m' = m$ and $n' = n$.*

Proof. — First observe that (i) $\Rightarrow$ (ii) is easy. Indeed, take first $\tau = I_{S_p^N}$, then the result follows from the identity $S_p^n[S_p^N] = S_p^N[S_p^n]$ valid for all n (*cf.* Corollary 1.10). By a routine argument, (ii) remains valid if $\tau = I_Z$ with Z as in (i). But it is then easy to show that (ii) holds when τ is factorized as indicated in (i).

Thus it suffices to prove (iii) $\Rightarrow$ (i). Assume (iii). Let $j\colon\ E_2 \subset M_n$ be the inclusion mapping and let $q\colon\ S_1^m \to E_1$ be the quotient map. To show (iii) $\Rightarrow$ (i), it clearly suffices to prove that (iii) implies $\gamma_{S_p}(j\tau q) \le 1$. The latter is a consequence of Corollary 7.2.2. Indeed, consider a composition $u_1u_2\colon\ M_n \to S_1^m$ formed of $u_2\colon\ M_n \to Y$, $u_1\colon\ Y \to S_1^m$ with Y finite dimensional, and satisfying

$$\pi^o_{p'}(u_2)\pi^o_p(u_1^*) \le 1.$$

Then, by Theorem 5.9 (applied twice) we can rewrite this composition u_1u_2 as follows

$$M_n \xrightarrow{\alpha} S^n_{p'} \xrightarrow{v^*} S^m_{p'} \xrightarrow{\beta^*} S^m_1$$

where $\alpha = M(a,b)$, $\beta = M(c,d)$ with a,b,c,d and v satisfying:

$$\|a\|_{2p'}\|b\|_{2p'} \le 1, \quad \|c\|_{2p}\|d\|_{2p} \le 1 \quad \text{and} \quad \|v\|_{cb} \le 1. \tag{7.2.6}$$

It will be useful to consider $\alpha\colon\ M_n \to S^n_{p'}$ (resp. $\beta\colon\ (S_1^m)^* \to S_p^m$) as an element of the unit ball of $(S_p^n[M_n])^*$ (resp. $S_p^m[S_1^m]$). (Indeed, note that $S_p^n[M_n] = S_p^n \otimes M_n$ and $S_p^m[S_1^m] = S_p^m \otimes S_1^m$ as vector spaces, and for instance β can be identified with $c \cdot y \cdot d$ where $y \in M_m \otimes S^1_m$ is the tensor associated to the identity map on S^1_m, which has *cb*-norm 1, so that $\|y\|_{\min} = 1$. Using Theorem 1.5, we find $\|\beta\|_{S_p^m[S_1^m]} \le 1$. By duality, a similar argument applies to α.) But then we have

$$\begin{aligned}\operatorname{tr}(u_1u_2j\tau q) &= \operatorname{tr}(\beta^*v^*\alpha j\tau q)\\ &= \langle v \otimes j\tau q(\beta), \alpha\rangle\end{aligned}$$

hence

$$|\operatorname{tr}(u_1u_2j\tau q)| \le \|v \otimes j\tau q\|_{S_p^m[S_1^m]\to S_p^n[M_n]} \cdot \|\beta\|_{S_p^m[S_1^m]}\|\alpha\|_{(S_p^n[M_n])^*}$$

hence by (7.2.6) and assuming (iii) we find

$$|\operatorname{tr}(u_1u_2\tau)| \le \|v \otimes \tau\|_{S_p^m[E_1]\to S_p^n[E_2]} \le \|v\|_{cb} \le 1.$$

This shows that $j\tau q$ satisfies the second condition in Corollary 7.2.2, whence $\gamma_{S_p}(j\tau q) \le 1$, which clearly implies (i). □

Corollary 7.2.7. — *Let $T\colon\ E_1 \to E_2$ be a linear map between arbitrary operator spaces. Let $1 < p < \infty$. The following are equivalent.*

(i) *There is an ultraproduct $G = \prod_{i\in I} G_i/\mathcal{U}$ with $G_i = S_p$ for all i in I and a factorization of T*

$$E_1 \xrightarrow{a} Z \xrightarrow{b} E_2$$

through a quotient of a subspace of G (i.e. we have $Z \in QS(G)$) such that $\|a\|_{cb}\|b\|_{cb} \le 1$.

(i)' *Same as* (i) *with $G_i = S_p^{n_i}$ for some $n_i < \infty$.*

(ii) *For any n and any linear map $v\colon\ S_p^n \to S_p^n$ we have*

$$\|v \otimes T\|_{S_p^n[E_1]\to S_p^n[E_2]} \le \|v\|_{cb}.$$

(ii)' *Same as* (ii) *with S_p instead of S_p^n.*

Proof. — The equivalence (ii) $\Leftrightarrow$ (ii)' is obvious and (i) $\Leftrightarrow$ (i)' follows immediately from Remark 7.1.8. The implication (i) $\Rightarrow$ (ii) is easy. Indeed, one first proves (ii) when $T = I_{S_p}$ (in that case it follows from the identity $S_p^n[S_p] = S_p[S_p^n]$), *cf.* Corollary 1.10), then using Lemma 5.4 one deduces that (ii) is also true when $T = I_G$ or when $T = I_Z$. It is then easy to show that (i) $\Rightarrow$ (ii). We leave the details to the reader.

Thus it remains only to show that (ii) $\Rightarrow$ (i). Assume (ii). We will use the notation introduced in Remark 7.1.6. Since E_2 (resp. E_1) embeds into $B(H)$ (resp. is completely isometric to a quotient of $S_1(H)$, *cf.* [**B2**]) for some Hilbert space H, we can find families of complete contractions $q_i\colon\ S_1^{n_i} \to E_1$ and $j_i\colon\ E_2 \to M_{n_i}$ indexed by some set I equipped with an ultrafilter $\mathcal{U}$ such that $\hat{\jmath} = \prod j_i/\mathcal{U}$ (resp. $\hat{q} = \prod_{i\in I} q_i/\mathcal{U}$) is a complete isometry when restricted to $\varphi_{E_2}(E_2)$ (resp. a complete metric surjection when restricted to the inverse image of $\varphi_{E_1}(E_1)$). By applying Theorem 7.2.6 to the mappings $\tau_i = j_i T q_i\colon\ S_1^{n_i} \to M_{n_i}$ we find factorizations $\tau_i = b_i a_i$ through S_p with $\|a_i\|_{cb}\|b_i\|_{cb} \le 1 + \varepsilon_i$, where $\varepsilon_i > 0$ and $\varepsilon_i \to 0$. Let $G = \prod_{i\in I} G_i/\mathcal{U}$ with $G_i = S_p$ for all i. We can then form the completely contractive mappings $\hat{a} = \prod_{i\in I} a_i/\mathcal{U}\colon\ \prod_{i\in I} S_1^{n_i}/\mathcal{U} \to G$ and $\hat{b}\colon\ G \to \prod_{i\in I} M_{n_i}/\mathcal{U}$.

This gives us a completely contractive factorization through G for the mapping

$$\hat{\jmath}\hat{T}\hat{q} = \hat{b}\hat{a}.$$

But then, recalling that $\hat{\jmath}$ and $\hat{q}$ are respectively a complete isometry and a complete quotient map when suitably restricted, we obtain by doubly restricting the last factorization that T factors completely contractively through a quotient of a subspace of G. □

Remark. — At this point, we have reached the limit of what we knew roughly at the time of the announcement [**P5**]. Note that, although they were not included in the privately circulated preprint, the results of this chapter up to now were clear to me as direct consequences of chapter 5, following the Banach space model treated in [**Kw2**].

However, I had serious difficulties to characterize the maps $T\colon\ E_1 \to E_2$ which factor through an ultraproduct of S_p (when viewed as maps into E_2^{**}, as usual). Except for subspaces of quotients as above, I could not obtain a satisfactory "if and only if" statement without any "exactness" assumption on E_1 or E_2. This (as well as being kept busy by other tasks) probably explains why the completion of the present manuscript was delayed.

Since then however, Marius Junge found a way to resolve all the above mentioned difficulties and the reader is referred to his habilitationsschrift for more details. After reading part of the latter thesis, I finally could see what I had been missing, namely Theorem 7.2.10 below (implicit in Junge's work) which greatly clarifies the study of the factorization through ultraproducts, by reducing it to the "exact" case, or more precisely the case when $E_1^* \subset M_{N_1}$ and $E_2 \subset M_{N_2}$.

It seems convenient for our exposition to introduce the following two definitions.

Definition 7.2.8. — Let F be an arbitrary operator space and let $i_F\colon\ F \to F^{**}$ be the canonical inclusion map. By an "injective presentation" of F, we will mean the following: an ultrafilter $\mathcal{U}$ on a set I, a family $(F_\alpha)_{\alpha\in I}$ of finite dimensional matricial operator spaces $F_\alpha \subset M_{n_\alpha}$ (with $n_\alpha < \infty$ for each α), a family of complete contractions $j_\alpha\colon\ F \to F_\alpha$, and a complete contraction $k\colon\ \Pi F_\alpha/\mathcal{U} \to F^{**}$ such that the mapping $j\colon\ F \to \Pi F_\alpha/\mathcal{U}$ associated to the family (j_α) satisfies $i_F = kj$, as expressed by the following commuting diagram.

$$\begin{tikzcd} & \Pi F_\alpha/\mathcal{U} \arrow[dr, "k"] & \\ F \arrow[ur, "j"] \arrow[rr, "i_F"] & & F^{**} \end{tikzcd}$$

Now let E be another arbitrary operator space. By a "projective presentation" of E, we will mean a family $(Q_\beta)_{\beta\in I}$ of finite dimensional operator spaces such that Q_β is a quotient of $S_1^{n_\beta}$ (for some $n_\beta < \infty$) together with a family of complete contractions $q_\beta\colon\ Q_\beta \to E$ and a complete contraction $r\colon\ E \to \Pi Q_\beta/\mathcal{U}$ such that the map $q\colon\ \Pi Q_\beta/\mathcal{U} \to E^{**}$ associated to (q_β) satisfies $i_E = qr$, as expressed by the following diagram

$$\begin{tikzcd} & \Pi Q_\alpha/\mathcal{U} \arrow[dr, "q"] & \\ E \arrow[ur, "r"] \arrow[rr, "i_E"] & & E^{**} \end{tikzcd}$$

Remark 7.2.9. — It is *essential* to have *each* j_α (and each q_α) completely contractive and not only j and q.

Theorem 7.2.10. — *Every operator space admits both an injective presentation and a projective one.*

To prove this result the following very simple lemma will be useful.

Lemma 7.2.11. — *Any finite dimensional operator space E possesses the following two properties.*

(i) *There is a sequence of subspaces $E_n \subset M_n$ and completely contractive maps $a_n\colon\ E \to E_n$ such that for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$ the mapping $\hat{a}\colon\ E \to \prod E_n/\mathcal{U}$ associated to (a_n) is a completely isometric isomorphism.*
(ii) *There is a sequence (Q_n), where, for each n, Q_n is a quotient of S_1^n and a sequence of complete contractions $b_n\colon\ Q_n \to E$ such that the associated map $\hat{b}\colon\ \prod Q_n/\mathcal{U} \to E$ is a completely isometric isomorphism.*

Proof. — This is entirely elementary, so we merely sketch the argument. As is well known, since E is separable, we can assume $E \subset B(\ell_2)$ completely isometrically. Let then $P_n\colon\ B(\ell_2) \to M_n$ be the usual projection ($P(e_{ij}) = e_{ij}$ if $i, j \leq n$ and $P(e_{ij}) = 0$ otherwise), let $P_n(E) = E_n$, and let $a_n\colon\ E \to E_n$ be the restriction of P_n to E. Clearly when n is large enough a_n becomes a linear isomorphism and it is easy to check that $\hat{a}$ is completely isometric. This yields (i). To prove (ii) we simply apply (i) to E^* and transpose the resulting diagram. (Note that by Lemma 14 in **[P6]** we have $(\prod E_n/\mathcal{U})^* = \prod E_n^*/\mathcal{U}$ completely isometrically since the dimension of E_n is essentially constant.) □

Remark. — It should be emphasized that in the preceding lemma the maps

$$\hat{a}^{-1}\colon\ \prod E_n/\mathcal{U} \to E \quad \text{and} \quad \hat{b}^{-1}\colon\ E \to \prod Q_n/\mathcal{U}$$

cannot in general be written as associated to a sequence of complete contractions (contrary to their inverses which can). Indeed, if it is the case then we have necessarily with the notation of **[P5]** either $d_{SK}(E) = 1$ or $d_{SK}(E^*) = 1$.

Proof of Theorem 7.2.10. — It will be shorter to use the notion of product ultrafilter, described above in Remark 7.1.8. Let F be an arbitrary operator space. We will show that F admits an injective presentation. We first use the set I_1 of all finite dimensional subspaces of F^* directed by inclusion and we let $\mathcal{U}_1$ be an ultrafilter refining this net. Then, for any α in I_1 (so $\alpha \subset F^*$ with $\dim \alpha < \infty$, a fortiori α is weak-$*$ closed) we define G_α to be the finite dimensional quotient space of F such that $\alpha = (G_\alpha)^*$, moreover we denote by $c_\alpha\colon\ F \to G_\alpha$ the canonical (completely contractive) quotient map. It is then easy to see that even though G_α is not necessarily matricial, the other requirements of an injective presentation are satisfied. Indeed, we clearly have a canonical map $\varphi\colon\ F^* \to \prod G_\alpha^*/\mathcal{U}_1$ defined as follows: $\varphi(\xi) = (\varphi_\alpha(\xi))_\alpha$ where $\varphi_\alpha(\xi) = \xi$ if $\xi \in \alpha$ and $\varphi_\alpha(\xi) = 0$ (say) otherwise. Note that φ is completely isometric.

Let $c\colon\ F^{**} \to \prod G_\alpha/\mathcal{U}_1$ be the completely contractive map taking $x'' \in F^{**}$ to $(c_\alpha^{**}(x''))_\alpha$. We have a natural mapping

$$\chi\colon\ \prod G_\alpha/\mathcal{U}_1 \to \left(\prod G_\alpha^*/\mathcal{U}_1\right)^*$$

defined by: $\forall\, \xi = (\xi_\alpha)_\alpha \in \prod G_\alpha^*/\mathcal{U}_1$

$$\forall\, x = (x_\alpha) \in \prod G_\alpha/\mathcal{U}_1$$

$$\langle \chi(x), \xi\rangle = \lim_{\mathcal{U}_1} \langle x_\alpha, \xi_\alpha\rangle.$$

Let then $d\colon\ \prod G_\alpha/\mathcal{U}_1 \to F^{**}$ be defined as $d = \varphi^*\chi$.

We claim that $dc = I_{F^{**}}$. Indeed, for any x'' in F^{**} and any ξ in F^* we have

$$\begin{aligned}\langle dcx'', \xi\rangle &= \langle \varphi^*\chi cx'', \xi\rangle = \langle \chi cx'', \varphi\xi\rangle \\ &= \lim_{\mathcal{U}_1} \langle c_\alpha^{**}(x''), \varphi_\alpha(\xi)\rangle \\ &= \lim_{\mathcal{U}_1} \langle x'', c_\alpha^*\varphi_\alpha(\xi)\rangle\end{aligned}$$

but $c_\alpha^*\colon\ G_\alpha^* \to F^*$ is the canonical inclusion so when α is large enough we have $\xi \in G_\alpha^* = \alpha$ and $c_\alpha^*\varphi_\alpha(\xi) = \xi$ hence we obtain $\langle dcx'', \xi\rangle = \langle x'', \xi\rangle$, which proves that $dc = I_{F^{**}}$. Thus we have proved that the identity of F^{**} (and a fortiori of course the inclusion of F into F^{**}) factors completely contractively through an ultraproduct of finite dimensional spaces (G_α).

Now using Lemma 7.2.11, we can find for each α in I_1 a sequence $G_{\alpha n}$ of finite dimensional *matricial* (*i.e.* each embeddable into M_N for some N) o.s. and completely contractive maps $a_{\alpha n}\colon\ G_\alpha \to G_{\alpha n}$ such that $\hat{a}_\alpha\colon\ G_\alpha \to \prod\limits_{n\in\mathbb{N}} G_{\alpha n}/\mathcal{U}_1$ is a completely isometric isomorphism. Then (see Remark 7.1.8) if we let $I = I_1 \times \mathbb{N}$ and $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ where $\mathcal{U}_2$ is a nontrivial ultrafilter on $\mathbb{N}$, and if we let $j_{\alpha n} = a_{\alpha n}c_\alpha$ and $F_{\alpha n} = G_{\alpha n}$ we immediately obtain an injective presentation of F.

Now let E be an arbitrary operator space. For the projective case, we let J_1 be the set of all finite dimensional subspaces of E directed by inclusion and let $\mathcal{V}_1$ be an ultrafilter refining this net. For β in J_1, we denote by $G_\beta \subset E$ the subspace of index β (the purist will write $G_\beta = \beta$). Then, by Remark 7.1.6 the (completely contractive) inclusions $b_\beta\colon\ G_\beta \to E$ induce a map $\hat{b}\colon\ \prod G_\beta/\mathcal{V}_1 \to E^{**}$. Moreover, we have a completely contractive map $\varphi\colon\ E \to \prod G_\beta/\mathcal{V}_1$ associated to (φ_β) as in the first part of this proof. Clearly the composition $\hat{b}\varphi$ coincides with $i_E\colon\ E \to E^{**}$. It remains to replace G_β by $G_{\beta n}$, as above: using the second part of Lemma 7.2.11 and a product ultrafilter we immediately obtain a projective presentation of E. □

Theorem 7.2.12 **(Junge [Ju]).** — *Let $T\colon\ E \to F$ be a linear map between arbitrary operator spaces and let $i_F\colon\ F \to F^{**}$ be the canonical inclusion. Let $1 < p < \infty$. The following are equivalent.*

(i) *There is an ultraproduct $G = \prod\limits_{\alpha\in I} G_\alpha/\mathcal{U}$ with $G_\alpha = S_p$ for all α in I and a factorization of $i_F T$*
$$E \xrightarrow{a} G \xrightarrow{b} F^{**}$$
through G with $\|a\|_{cb}\|b\|_{cb} \le 1$.

(ii) *For any integers N_1, N_2, subspaces $G_1 \subset M_{N_1}$, $G_2 \subset M_{N_2}$ and completely contractive maps $a_1\colon\ G_1^* \to E$, $a_2\colon\ F \to G_2$ we have*
$$\gamma_{S_p}(a_2 T a_1) \le 1.$$

Remark. — Note that Corollary 7.2.2 allows to "dualize" a bit further the formulation of (ii) above.

Proof of Theorem 7.2.12. — Assume (i). Consider a_1, a_2 as in (ii). Extend a_2 to $a_2^{**}\colon\ F^{**} \to G_2$. Then (i) $\Rightarrow$ (ii) follows from the implication (iv) $\Rightarrow$ (i) in Remark 7.2.3.

Conversely, assume (ii). By Theorem 7.2.10, E admits a projective presentation $(Q_\beta)_{\beta\in J}$ and F admits an injective one $(F_\alpha)_{\alpha\in I}$. Then by (ii) we have (using the notation in Definition 7.2.8)
$$\gamma_{S_p}(j_\alpha T q_\beta) \le 1.$$

Actually, taking the product set $I \times J$ and the product ultrafilter, we can assume for simplicity of notation that (Q_β) and (F_α) are relative to the same set with the same ultrafilter. Then the mapping

$$\Pi j_\alpha T q_\alpha/\mathcal{U}\colon\ \Pi Q_\alpha/\mathcal{U} \to \Pi F_\alpha/\mathcal{U}$$

obviously factors through $G = \prod\limits_{\alpha \in I} G_\alpha/\mathcal{U}$ with $G_\alpha = S_p$ for all α in I, via complete contractions. Let $\check{T}$ be this mapping. Then it is easy to check, using (7.1.3), that $k\check{T}r = i_F T$, whence (i). □

CHAPTER 8

ILLUSTRATIONS IN CONCRETE SITUATIONS

8.1. Completely bounded Fourier and Schur multipliers on L_p and S_p

In this section, we study the Fourier multipliers which are completely bounded on L_p and the c.b. Schur multipliers on S_p. There are very strong anologies between these two classes of multipliers, although many interesting questions remain open specifically for the Schur multipliers.

Let $\psi\colon \mathbb{N}\times\mathbb{N}\to\mathbb{C}$ be any function. We will say that ψ is a bounded Schur multiplier on S_p (resp. $B(\ell_2)$) if for any $x=(x_{ij})$ in S_p, the matrix $(\psi(i,j)x_{ij})$ represents an element of S_p (resp. $B(\ell_2)$). Here $1\le p\le\infty$. We will denote by $\mathcal{M}_\psi\colon S_p\to S_p$ (resp. $\mathcal{M}_\psi\colon B(\ell_2)\to B(\ell_2)$) the corresponding bounded linear map. When the latter is c.b., we say that ψ is a c.b. Schur multiplier on S_p (resp. $B(\ell_2)$). Note that by definition

$$\mathcal{M}_\psi e_{ij}=\psi(i,j)e_{ij}.$$

Note that $\mathcal{M}_\psi$ is a bounded (resp. c.b.) Schur multiplier on S_p iff the same is true on $S_{p'}$ with $\dfrac{1}{p}+\dfrac{1}{p'}=1$, and we have

$$\|\mathcal{M}_\psi\|_{B(S_p)}=\|\mathcal{M}_\psi\|_{B(S_{p'})}\text{ and }\|\mathcal{M}_\psi\|_{cb(S_p,S_p)}=\|\mathcal{M}_\psi\|_{cb(S_{p'},S_{p'})}.$$

Moreover, it is easy to check in the case $p=\infty$ that the norm (resp. c.b. norm) of $\mathcal{M}_\psi$ is the same when acting on S_∞ or acting on $B(\ell_2)$. In this case, a characterization is known, we describe it in Proposition 8.1.11 below.

Let G be a compact Abelian group with normalized Haar measure m. We will denote by $L_p(G)$ the space $L_p(G,m)$. Let Γ be the dual group formed of all the continuous characters on G, as usual (*cf.* e.g. **[Rud1]**). We view Γ as a discrete Abelian group. Given a function f in $L_p(G)$ $(1\le p\le\infty)$ we define its Fourier transform $\hat f\colon \Gamma\to\mathbb{C}$ as follows

$$\forall\gamma\in\Gamma\qquad \hat f(\gamma)=\int f(t)\overline{\gamma(t)}m(dt). \tag{8.1.1}$$

Then, at least say for $p = 2$, for any f in $L_p(G)$ we have a "Fourier series" expansion

$$f = \sum_{\gamma \in \Gamma} \hat{f}(\gamma)\gamma.$$

Let X be a Banach space. We denote by $L_p(G; X)$ the space $L_p(G, m; X)$. Note that, as usual, by definition the space $L_p(G) \otimes X$ is dense in $L_p(G; X)$ when $p < \infty$.

The definition (8.1.1) of the Fourier transform clearly remains valid for any f in $L_p(G; X)$, but this time $\hat{f}$ takes values in X. The subset of $L_p(G; X)$ formed of all the sums

$$f = \sum_{\gamma \in \Gamma} \hat{f}(\gamma)\gamma$$

with $\hat{f}\colon\ \Gamma \to X$ finitely supported is dense in $L_p(G; X)$.

Let $\Lambda \subset \Gamma$ be a subset. We denote by $L_p(G)_\Lambda$ (resp. $L_p(G; X)_\Lambda$) the subset of $L_p(G)$ (resp. $L_p(G; X)$) formed of all the functions f in $L_p(G)$ (resp. $L_p(G; X)$) such that the support of $\hat{f}$ is included in Λ. When $p < \infty$, $L_p(G)_\Lambda$ (resp. $L_p(G; X)_\Lambda$) coincides with the closure in $L_p(G)$ (resp. $L_p(G; X)$) of the subset of all the functions f of the form

$$f = \sum_{\gamma \in A} x_\gamma \gamma$$

where $x_\gamma \in \mathbb{C}$ (resp. $x_\gamma \in X$) and $A \subset \Lambda$ is a *finite* subset of Λ. Given an arbitrary function $\varphi\colon\ \Lambda \to \mathbb{C}$, we define a multiplier M_φ on the *linear* span of Λ by setting

$$\forall\, f \in \operatorname{span}(\Lambda) \qquad M_\varphi f = \sum_{\gamma \in \Lambda} \varphi(\gamma)\hat{f}(\gamma)\gamma.$$

Similarly, for any f in $L_p(G; X)_\Lambda$ with $\hat{f}$ finitely supported we denote again

$$M_\varphi f = \sum_{\gamma \in \Lambda} \varphi(\gamma)\hat{f}(\gamma)\gamma.$$

We will say that φ defines a bounded multiplier on $L_p(G)_\Lambda$ (resp. on $L_p(G; X)_\Lambda$) when the linear map just defined is bounded, and hence uniquely extends by density to a bounded linear map on $L_p(G)_\Lambda$ (resp. on $L_p(G; X)_\Lambda$). Note that if T is the operator M_φ acting on $L_p(G)_\Lambda$, then $T \otimes I_X$ corresponds to M_φ acting on $L_p(G; X)_\Lambda$, but for simplicity we will abusively denote T and $T \otimes I_X$ by M_φ in this section. There should be no confusion. When X is an operator space and the resulting map M_φ on $L_p(G)_\Lambda$ (resp. on $L_p(G; X)_\Lambda$) is actually completely bounded, then of course we will say that φ defines a c.b. multiplier on $L_p(G)_\Lambda$ (resp. on $L_p(G; X)_\Lambda$). Naturally, when $\Lambda = \Gamma$ we will omit the subscript Λ for all these notions. The next statement spells out the meaning of complete boundedness for a Fourier multiplier of $L_p(G)_\Lambda$.

Proposition 8.1.1. — *With the above notation, let $\varphi\colon\ \Lambda \to \mathbb{C}$ be any function and let $c \geq 0$ be a constant. The following are equivalent.*

(i) *The multiplier φ is completely bounded on $L_p(G)_\Lambda$ with*

$$\|M_\varphi\|_{cb(L_p(G)_\Lambda, L_p(G)_\Lambda)} \leq c.$$

(ii) *The multiplier φ is bounded on $L_p(G; S_p)_\Lambda$ with $\|M_\varphi\|_{B(L_p(G;S_p)_\Lambda)} \leq c$.*

(iii) *For any n and for any finitely supported family $(x_\gamma)_{\gamma\in\Lambda}$ of coefficients with $x_\gamma \in S_p^n$ we have*

$$\left\|\sum_{\gamma\in\Lambda} \varphi(\gamma)x_\gamma\gamma\right\|_{L_p(G;S_p^n)} \leq c\left\|\sum_{\gamma\in\Lambda} x_\gamma\gamma\right\|_{L_p(G;S_p^n)}.$$

Proof. — This is a particular case of the above Proposition 2.3. □

Example. — Let $G = \mathbf{T}$, $\Gamma = \mathbb{Z}$ and $\Lambda = \mathbb{N} \subset \mathbb{Z}$. Then the space $L_p(T)_\Lambda$ can be identified with the classical Hardy space H_p. It is well known that the orthogonal projection $L_2 \to H_2$ is also bounded from L_p to H_p if $1 < p < \infty$. Equivalently, the indicator function of $\mathbb{N}$ is a bounded Fourier multiplier on $L_p(\mathbf{T})$ for any $1 < p < \infty$. It has been known for a long time (*cf.* e.g. **[Bo1]**, **[Bo2]**,...]) that this particular multiplier remains bounded from $L_p(\mathbf{T}; S_p)$ into itself (and its norm is $O(p)$ when $p \to \infty$), in other words this multiplier (or equivalently the Hilbert transform) is completely bounded on $L_p(\mathbf{T})$ for any $1 < p < \infty$ (and its *c.b.*-norm is $O(p)$ when $p \to \infty$). More generally, the Riesz transforms on $L_p(\mathbb{R}^n)$ (equipped either with the Lebesgue measure or with the standard Gaussian measure) are completely bounded when $1 < p < \infty$ (with *cb*-norms bounded by a constant independent on n). Their boundedness is a classical result due to Elias Stein (and to P.A. Meyer in the Gaussian case). The complete boundedness can be seen for instance from the proof in **[P13]**.

Remark. — It is well known that a Fourier multiplier φ is bounded on $L_1(G)$ (or equivalently on $C(G)$) iff there is a complex Radon measure μ on G such that $\varphi = \hat{\mu}$ and the norm as a multiplier is equal to the total variation norm of μ, $\|\mu\|_M$. Then, for any Banach space X, φ defines a bounded Fourier multiplier on $L_p(G; X)$ with norm $\leq \|\mu\|_M$ for all $1 \leq p \leq \infty$. Note that the case $p = 2$ is trivial: a multiplier φ is bounded on $L_2(G)$ iff it is bounded and

$$\|M_\varphi\|_{B(L_2(G))} = \sup_{\gamma\in\Gamma} |\varphi(\gamma)|.$$

Remark 8.1.2. — It follows from the preceding remark that when $\Lambda = \Gamma$ and $p = 1, 2$ or ∞, boundedness and complete boundedness are equivalent for M_φ and

$$\text{if } p \in \{1, 2, \infty\} \qquad \|M_\varphi\|_{cb(L_p(G),L_p(G))} = \|M_\varphi\|_{B(L_p(G))}.$$

However, as the next result shows this is no longer valid for other values of p.

Proposition 8.1.3. — *Let G be any infinite compact Abelian group with dual group Γ. Then, for any $1 < p \neq 2 < \infty$, there is a bounded Fourier multiplier of $L_p(G)$ which is not completely bounded.*

The proof will use the following simple observation.

Lemma 8.1.4. — *Let $1 \leq p \leq \infty$. Let*

$$\Lambda' = \{\gamma_1', \gamma_2', \dots\} \quad \textit{and} \quad \Lambda'' = \{\gamma_1'', \gamma_2'', \dots\}$$

be two countable subsets of Γ *and let*

$$\Lambda = \Lambda' + \Lambda'' = \{\gamma' + \gamma'' \mid \gamma' \in \Lambda', \gamma'' \in \Lambda''\}.$$

Then, if φ *is a c.b. Fourier multiplier of* $L_p(G)_\Lambda$, *the function* $\widetilde{\varphi}\colon \mathbb{N}\times\mathbb{N} \to \mathbb{C}$ *defined by*

$$\forall\, i,j \in \mathbb{N} \qquad \widetilde{\varphi}(i,j) = \varphi(\gamma'_i + \gamma''_j)$$

is a c.b. Schur multiplier on S_p, *with*

$$\|\mathcal{M}_{\widetilde{\varphi}}\|_{cb(S_p,S_p)} \le \|M_\varphi\|_{cb(L_p(G)_\Lambda, L_p(G)_\Lambda)}. \tag{8.1.2}$$

Proof. — Let (x_{ij}) be a finitely supported family with $x_{ij} \in S_p$. Let $f(t) = \sum\limits_{ij}(\gamma'_i + \gamma''_j)(t)x_{ij} = \sum\limits_{ij} \gamma'_i(t)\gamma''_j(t)x_{ij}$. Then

$$M_\varphi(f)(t) = \sum_{ij} \varphi(\gamma'_i + \gamma''_j)\gamma'_i(t)\gamma''_j(t)x_{ij}.$$

Now assume that x_{ij} is of the following special form: $x_{ij} = \alpha_{ij}e_{ij}$ with $\alpha_{ij} \in \mathbb{C}$. We then have, for any t in G,

$$\|f(t)\|_{S_p} = \left\|\sum \alpha_{ij}e_{ij}\right\|_{S_p},$$

and

$$\|M_\varphi(f)(t)\|_{S_p} = \left\|\sum \varphi(\gamma'_i + \gamma''_j)\alpha_{ij}e_{ij}\right\|_{S_p}.$$

From this it is easy to deduce (by integration in t) that

$$\|\mathcal{M}_{\widetilde{\varphi}}\|_{S_p\to S_p} \le \|M_\varphi\|_{cb(L_p(G)_\Lambda, L_p(G)_\Lambda)}. \tag{8.1.3}$$

Now, if we use, instead of the scalar coefficients (α_{ij}), matrix coefficients $a_{ij} \in S_p(H)$ and we set

$$x_{ij} = e_{ij} \otimes a_{ij} \in S_p(\ell_2 \otimes_2 H),$$

we obtain by a similar reasoning that (8.1.2) holds. □

We will also use the following well known fact.

Lemma 8.1.5. — *The canonical "basis"* (e_{ij}) *is* **not** *an unconditional basis of* S_p *when* $1 \le p \ne 2 < \infty$. *More precisely, for any* $n \ge 1$, *there exists an element* $x = \sum\limits_{i,j=1}^{n} x_{ij}e_{ij}$ *in the unit ball of* S_p *and complex scalars* z_{ij} *with* $|z_{ij}| = 1$ *such that*

$$\left\|\sum_{i,j=1}^{n} z_{ij}x_{ij}e_{ij}\right\|_{S_p} = n^{|1/2-1/p|}.$$

Proof. — Let $U = (U_{ij})$ be an $n \times n$ unitary matrix such that $|U_{ij}| = n^{-1/2}$ for any (i,j). For instance, we can take the matrix representing the Fourier transform on the group of n-th roots of unity, i.e.

$$U_{pq} = n^{-1/2} \exp(2\pi i pq/n).$$

Note that if $z'_{ij} = \overline{U}_{ij} n^{1/2}$ then $|z'_{ij}| = 1$ and we have

$$\left\| \sum_{i,j=1}^{n} z'_{ij} U_{ij} e_{ij} \right\|_{S_p} = n^{-1/2} \left\| \sum_{i,j=1}^{n} e_{ij} \right\|_{S_p} = n^{-1/2} \left\| \sum_{1}^{n} e_i \right\|^2 = n^{1/2}.$$

On the other hand, since U is unitary, $\left\| \sum_{i,j=1}^{n} U_{ij} e_{ij} \right\|_{S_p} = n^{1/p}$. Thus, if $p > 2$, we can take $x_{ij} = n^{-1/p} U_{ij}$ and $z_{ij} = z'_{ij}$, and if $1 \le p < 2$, we take $x_{ij} = n^{-1/2} z'_{ij} U_{ij}$ and $z_{ij} = \overline{z'_{ij}}$. □

Remark 8.1.6. — Let G be a compact Abelian group, equipped with its normalized Haar measure m, and let Γ be the dual (discrete) group. Consider a subset $\Lambda \subset \Gamma$. When $2 < p < \infty$, a subset $\Lambda \subset \Gamma$ is called a $\Lambda(p)$-set if $L_p(G,m)_\Lambda = L_2(G,m)_\Lambda$ with equivalent norms. In other words, there is a constant C such that for any f in $L_2(G,m)$ with Fourier transform supported in Λ, we have

$$(\|f\|_2 \le) \quad \|f\|_p \le C\|f\|_2.$$

When this holds, any bounded function on Λ extends to a bounded multiplier on $L_p(G,m)$, which vanishes outside Λ. In particular, the indicator function of Λ is a bounded multiplier on $L_p(G,m)$.

It is now known that, for any $p > 2$, there is a $\Lambda(p)$-set which is not "better", *i.e.* which is a $\Lambda(q)$-set for *no* $q > p$. This was established by Rudin **[Rud2]** when p is an even integer (with explicit examples), and it remained open for a long time for the intermediate values of p until Bourgain **[Bo3]** settled the general case, by a probabilistic argument.

A subset $\Lambda \subset \Gamma$ is called a Sidon set (*cf.* **[LoR]**) if $L_\infty(G,m)_\Lambda = \ell_1(\Lambda)$ with equivalent norms, or equivalently if there exists a positive constant C such that any function f with Fourier transform $\hat{f}$ supported in a finite subset of Λ satisfies

$$C^{-1} \sum_{n\in\Lambda} |\hat{f}(n)| \le \|f\|_\infty \quad (\le \sum_{n\in\Lambda} |\hat{f}(n)|).$$

Proof of Proposition 8.1.3. — The idea of this proof goes back to **[P12]**. By transposition, it clearly suffices to treat the case $2 < p < \infty$. We start by the case $G = \mathbf{T}$ (one dimensional torus), and $\Gamma = \mathbb{Z}$. We will apply the preceding lemma to the case $\Lambda' = \{3^{2i} \mid i \in \mathbb{N}\}$ and $\Lambda'' = \{3^{2j+1} \mid j \in \mathbb{N}\}$. We will use the fact that the map $(i,j) \to 3^{2i} + 3^{2j+1}$ is one to one. Moreover, it is well known that, in the present case, the set $\Lambda = \Lambda' + \Lambda''$ is a $\Lambda(p)$-set (in the sense of Remark 8.1.6) for any $2 < p < \infty$ (*cf.* **[LoR]**, p. 65).

In particular, let $z = (z_{ij}) \in \mathbf{T}^{\mathbb{N}\times\mathbb{N}}$ be an arbitrary family of unimodular complex scalars and let φ_z be the Fourier multiplier defined by

$$\forall n \in \mathbb{Z} \qquad \varphi_z(n) = \begin{cases} z_{ij} & \text{if } n = 3^{2i} + 3^{2j+1} \\ 0 & \text{if } n \notin \Lambda = \Lambda' + \Lambda''. \end{cases}$$

Since Λ is a $\Lambda(p)$ set, there is a constant c_p such that, for any $z = (z_{ij})$ as above we have

$$\|M_{\varphi_z}\|_{B(L_p(G))} \leq c_p.$$

Note that we have trivially $\|M_{\varphi_z}\|_{cb(L_p(G),L_p(G))} \geq \|M_{\varphi_z}\|_{cb(L_p(G)_\Lambda,L_p(G)_\Lambda)}$. We claim that

$$\sup_z \|M_{\varphi_z}\|_{cb(L_p(G)_\Lambda,L_p(G)_\Lambda)} = \infty. \tag{8.1.4}$$

Indeed, note that with the notation of Lemma 8.1.4 we have

$$\widetilde{\varphi}_z(i,j) = z_{ij}.$$

Hence, if (8.1.4) failed, there would exist by (8.1.2) a constant c'_p such that, for all $z = (z_{ij})$, we would have

$$\|\mathcal{M}_{\widetilde{\varphi}_z}\|_{S_p \to S_p} \leq c'_p,$$

but this would contradict Lemma 8.1.5. This contradiction establishes the above claim (8.1.4). Using (8.1.4) it is easy by routine arguments to complete the proof of Proposition 8.1.3 with $\Gamma = \mathbb{Z}$. Now, when Γ is an arbitrary infinite discrete group, it is well known that it contains an infinite sequence $\{\gamma_j \mid j = 1, 2, \dots\}$ which forms a Sidon set (as defined in Remark 8.1.6) and is such that the map $(i,j) \to \gamma_{2i} + \gamma_{2j+1}$ is one to one and its range is a $\Lambda(p)$-set for any $p < \infty$. The preceding argument can then be repeated with $\Lambda' = \{\gamma_{2i} \mid i \geq 1\}$ and $\Lambda'' = \{\gamma_{2j+1} \mid j \geq 1\}$. □

Remark. — Fix an integer N and $1 < p \neq 2 < \infty$. Let

$$\lambda(p, N) = \sup\{\|M_\varphi\|_{cb(L_p(T),L_p(T))}\}$$

where the supremum runs over all functions $\varphi\colon\ \mathbb{Z} \to \mathbb{C}$ with support in $[0, 1, \dots, N]$ and such that $\|M_\varphi\|_{L_p(T)\to L_p(T)} \leq 1$. The preceding argument shows that there is $\delta_p > 0$ such that, for all $N = 1, 2, \dots$

$$\lambda(p, N) \geq \delta_p (\mathrm{Log}\, N)^{|\frac{1}{2} - \frac{1}{p}|}.$$

On the other hand it is not difficult to check that there is a constant C_p such that $\lambda(p, N) \leq C_p N^{|1/2 - 1/p|}$. It would be interesting to find sharper bounds for $\lambda(p, N)$ when $N \to \infty$.

Remark 8.1.7. — By known results on Sidon sets, the following fact holds: if Λ is any Sidon subset in a discrete Abelian group Γ then any *bounded* function $\varphi\colon\ \Lambda \to \mathbb{C}$ can be extended to a bounded Fourier multiplier on $L_p(G)$. Moreover, this holds for any $1 \leq p \leq \infty$. When $p = 1$ or $p = \infty$ this property characterizes Sidon sets (*cf.* **[Rud1]**, p. 121). When $p = 2$, this property is trivially valid for *any* set Λ. When $2 < p < \infty$, the above property is known to characterize $\Lambda(p)$-sets.

The natural "c.b. version" of this property is the following.

Definition 8.1.8. — Let $2 < p < \infty$. A subset $\Lambda \subset \Gamma$ of a discrete Abelian group Γ is called a $\Lambda(p)_{cb}$-set if any bounded function $\varphi\colon\ \Lambda \to \mathbb{C}$ can be extended to a c.b. Fourier multiplier of $L_p(G)$.

This notion is extensively studied in Asma Harcharras's recent thesis [**Ha**] to which we refer the interested reader (see also §8.5 below). Note that when Λ is a $\Lambda(p)_{cb}$-set the extension in the preceding definition can always be made by setting $\varphi \equiv 0$ outside Λ. Moreover, there are $\Lambda(p)$-sets which are not $\Lambda(p)_{cb}$-sets. Indeed, the set $\Lambda = \{3^i + 3^j \mid i,j = 1,2,\dots\}$ appearing in the proof of Proposition 8.1.3 is $\Lambda(p)$ for all $2 < p < \infty$, but, by (8.1.4), it is $\Lambda(p)_{cb}$ for none of these values of p. On the other hand, it is proved in [**Ha**] that for any even integer $p > 2$, there are $\Lambda(p)_{cb}$-sets which are not "better", *i.e.* which are $\Lambda(p+\varepsilon)_{cb}$-sets for *no* $\varepsilon > 0$.

The anologous notion for Schur multipliers is the following.

Definition 8.1.9. — Let $2 < p < \infty$. A subset $A \subset \mathbb{N} \times \mathbb{N}$ is called a $\sigma(p)$-set (resp. $\sigma(p)_{cb}$-set) if every bounded function $\psi\colon\ A \to \mathbb{C}$ extends to a bounded (resp. c.b.) Schur multiplier on S_p.

It can be shown (see [**Ha**]) that if $\Lambda \subset \mathbb{N}$ is a $\Lambda(p)_{cb}$-set in $\mathbb{Z}$, then the set

$$A = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i + j \in \Lambda\}$$

is a $\sigma(p)_{cb}$-set.

In particular, this (together with the construction of "large" $\Lambda(p)_{cb}$-sets) yields the following interesting result.

Theorem 8.1.10 ([**Ha**]). — *Let $2 < p < \infty$ and assume that p is an even integer. Then there are positive constants α_p and β_p for which the following holds: for any $n \geq 1$, there is a subset $A_n \subset \{1,\dots,n\}^2$ with $|A_n| \geq \alpha_p n^{1+2/p}$ such that, for any function $\psi\colon\ \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ with support in A_n, we have*

$$\|\mathcal{M}_\psi\|_{cb(S_p,S_p)} \leq \beta_p \sup\{|\psi(i,j)| \mid (i,j) \in A_n\}.$$

Remark. — The preceding result can also be used to show (see [**Ha**]) that for any even integer $p > 2$, there is an idempotent Schur multiplier Ψ (*i.e.* $\Psi(i,j)$ is equal to zero or one) which is *c.b.* on S_p but is not bounded on S_q for any $q > p$. (All the preceding statements restricted to even integers ≥ 4 are probably valid for any $p > 2$, but this seems out of reach at the moment.)

We now return to Schur multipliers. We first recall the following well known result due to Haagerup (but, in some form, it is already in Grothendieck's "Résumé" [**G**]).

Proposition 8.1.11. — *Let $\psi\colon\ \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be a function and let $c \geq 0$ be a constant. The following are equivalent:*

(i) *$\mathcal{M}_\psi$ is a bounded Schur multiplier on $B(\ell_2)$ with norm $\leq c$.*

(ii) *$\mathcal{M}_\psi$ is a c.b. Schur multiplier on $B(\ell_2)$ with c.b. norm $\leq c$.*

(iii) *There are bounded functions* $x\colon \mathbb{N} \to \ell_2$ *and* $y\colon \mathbb{N} \to \ell_2$ *such that*

$$\forall\, i,j \in \mathbb{N} \qquad \psi(i,j) = \langle x_i, y_j\rangle \quad \text{and} \quad \sup_i \|x_i\| \sup_j \|y_j\| \le c.$$

Proof. — We refer the reader to e.g. p. 92 in **[P9]**. □

Remark. — As already observed these properties are also equivalent to the inequality $\|\mathcal{M}_\psi\|_{B(S_1,S_1)} \le c$ and to $\|\mathcal{M}_\psi\|_{B(S_\infty,S_\infty)} \le c$, and moreover we have

$$\|\mathcal{M}_\psi\|_{B(S_1,S_1)} = \|\mathcal{M}_\psi\|_{cb(S_1,S_1)} \text{ and } \|\mathcal{M}_\psi\|_{B(S_\infty,S_\infty)} = \|\mathcal{M}_\psi\|_{cb(S_\infty,S_\infty)}.$$

The preceding statement shows that for $p = 1$ and $p = \infty$, all bounded Schur multipliers on S_p are completely bounded. For $p = 2$, this is trivially also true: indeed we have clearly

$$\|\mathcal{M}_\psi\|_{B(S_2,S_2)} = \|\mathcal{M}_\psi\|_{cb(S_2,S_2)} = \sup_{i,j} |\psi(i,j)|.$$

For $1 < p \neq 2 < \infty$, we conjecture that this is no longer true:

Conjecture 8.1.12. — For any $1 < p \neq 2 < \infty$, there is a Schur multiplier which is bounded on S_p but not c.b. on S_p.

Remark. — Any c.b. map on $B(H)$ is a linear combination of completely positive maps. This property no longer holds on S_p. The class of maps $u\colon S_p \to S_p$ which are linear combinations of bounded completely positive maps are called "completely regular" in **[P10]** and studied extensively there.

Let us denote by $\mathcal{S}_p$ (resp. $\mathcal{S}_p^{cb}$) the Banach space of all bounded (resp. c.b.) Schur multipliers on S_p equipped with its natural norm. Since we have a nice description of $\mathcal{S}_p = \mathcal{S}_p^{cb}$ in the cases $p = 1, 2, \infty$, it was natural to wonder whether the general case could be obtained by interpolation between these particular cases (the question was raised by V. Peller). Indeed, by routine arguments, we have contractive inclusions

$$(\mathcal{S}_\infty, \mathcal{S}_2)_\theta \subset \mathcal{S}_p \text{ and } (\mathcal{S}_\infty^{cb}, \mathcal{S}_2^{cb})_\theta \subset \mathcal{S}_p^{cb}$$

when $2 < p < \infty$ and $1/p = \theta/2$. So the question arose whether these inclusions were actually equalities. The negative answer was given (for both cases) in **[Ha]** using $\Lambda(p)_{cb}$-sets.

We now turn to the Hankelian subspace of S_p, i.e. the subspace of S_p corresponding to all matrices (x_{ij}) in S_p such that x_{ij} depends only on $i+j$. This subspace is spanned by a natural system $\{D_n\}$ defined as follows

$$D_n = \sum_{i+j=n} e_{ij}.$$

The next result due to V. Peller is fundamental to study Hankel operators in S_p. To state it, we use the dyadic partition (I_n) of the integers, as follows

$$I_0 = \{0\}, I_n = [2^{n-1}, 2^n[\qquad \forall\, n \ge 1.$$

Theorem 8.1.13. — *Let $1 < p < \infty$. There are positive constants α_p, β_p such that, for any finitely supported scalar sequence (x_k), we have*

$$\alpha_p \left(\sum_n 2^n \left\| \sum_{k \in I_n} x_k e^{ikt} \right\|_{L_p(dt)}^p \right)^{1/p} \leq \left\| \sum_{k \geq 0} x_k D_k \right\|_{S_p} \leq \beta_p \left(\sum_n 2^n \left\| \sum_{k \in I_n} x_k e^{ikt} \right\|_{L_p(dt)}^p \right)^{1/p}.$$

More generally, for any finitely supported sequence x_k with $x_k \in S_p(H)$ we have

$$\alpha_p \left(\sum_n 2^n \left\| \sum_{k \in I_n} x_k e^{ikt} \right\|_{L_p(dt;S_p(H))}^p \right)^{1/p} \leq \left\| \sum_{k \geq 0} D_k \otimes x_k \right\|_{S_p(\ell_2 \otimes H)} \leq \beta_p \left(\sum_n 2^n \left\| \sum_{k \in I_n} x_k e^{ikt} \right\|_{L_p(dt;S_p(H))}^p \right)^{1/p}.$$

Proof. — We refer the reader to Peller's papers **[Pe1]** (for the first part) and **[Pe2]** (for the second one). □

Remark. — In particular, this theorem characterizes the Hankel matrices (x_{ij}) in $S_p(\ell_2 \otimes H)$ as those such that

$$\sum_n 2^n \left\| \sum_{k \in I_n} x_{0k} e^{ikt} \right\|_{L_p(dt;S_p(H))}^p < \infty.$$

(Note that $x_{ij} = x_{0i+j}$ if the matrix is assumed Hankelian.)

In the language of operator spaces, it has the following striking interpretation.

Corollary 8.1.14. — *For each $n \geq 0$, let $\mu_n = 2^n$ and let $E_n \subset L_p(\mathbf{T}, dt)$ be the subspace spanned by the functions $\{e^{ikt} \mid k \in I_n\}$. Let us denote by B_p the space $\ell_p(\mu; \{E_n\})$. (This space coincides with a "Besov space".) We equip $B_p = \ell_p(\mu; \{E_n\})$ with an operator space structure as defined at the end of §2. For each $k \geq 0$, we denote by Φ_k the element of $\ell_p(\mu; \{E_n\})$ which has its n-th coordinate equal to the function e^{ikt} when $k \in I_n$ and to zero otherwise. Then, if $1 < p < \infty$, the linear mapping which takes Φ_k to D_k extends to a complete isomorphism between B_p and the subspace of S_p formed of all the Hankel matrices.*

Remark. — Let SH_p denote the subspace of S_p formed of all the Hankel matrices. Let $\psi\colon \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be a function of Hankelian type, i.e. such that $\psi(i,j)$ depends only on $i + j$. Let us write $\psi(i,j) = \varphi(i + j)$. Then, the preceding result has the following interesting application: the restriction of $\mathcal{M}_\psi$ to SH_p is c.b. iff the sequence

$\{1_{I_n}\varphi \mid n \geq 0\}$ is uniformly bounded in the space of c.b. Fourier multipliers of $L_p(\mathbf{T})$ $(1 < p < \infty)$.

To conclude this section, we mention another connection between Fourier multipliers of H_1 and Schur multipliers, which was observed in **[P9]**, page 109.

Theorem 8.1.15. — *Let $G = \mathbf{T}$, $\Gamma = \mathbb{Z}$ and $\Lambda = \mathbb{N}$ so that $L_1(\mathbf{T})_\Lambda$ can be identified with the Hardy space H_1. Let $c \geq 0$ be a constant. The following properties of a function $\varphi\colon\ \mathbb{N} \to \mathbb{C}$ are equivalent.*

(i) *φ defines a c.b. Fourier multiplier on H_1 with c.b. norm $\leq c$.*
(ii) *The function $\psi(i,j) = \varphi(i+j)$ defines a bounded Schur multiplier on $B(\ell_2)$ (or equivalently on S_1) with norm $\leq c$ (see Proposition 8.1.11 for a further description).*

8.2. The space L_1 and the full C^*-algebra of the free group

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Recall that the natural operator space structure on $L_1(\mu)$ is defined as the one induced on $L_1(\mu)$ by the dual space $L_\infty(\mu)^*$, equipped with its dual operator space structure. In particular, in the case $\Omega = \mathbb{N}$, we have a natural o.s.s. on ℓ_1. It is not hard to verify that this natural o.s.s. on ℓ_1 also coincides with the one obtained by considering ℓ_1 as the dual of c_0. See **[B2]** for details on all this.

We can describe the associated norm $\|\ \|_{\min}$ on $\mathcal{K} \otimes \ell_1$ in the following manner: let (e_n) be the canonical basis of ℓ_1. For any finite sequence (a_n) in $\mathcal{K}$ (or in $B(\ell_2)$) we have

$$\|\sum a_n \otimes e_n\|_{B(\ell_2)\otimes_{\min}\ell_1} = \sup\{\|\sum a_n \otimes b_n\|_{B(\ell_2)\otimes_{\min}B(\ell_2)}\}, \tag{8.2.1}$$

where the supremum runs over all sequences (b_n) in the unit ball of $B(\ell_2)$ (equivalently, the supremum over all sequences (b_n) in the unit ball of $\mathcal{K}$ is actually the same). Indeed, (8.2.1) is easy to check by introducing the linear map $u\colon\ c_0 \to B(\ell_2)$ corresponding to $\sum a_n \otimes e_n$ and by expressing that $\|\sum a_n \otimes e_n\|_{B(\ell_2)\otimes_{\min}\ell_1} = \|u\|_{cb}$ using the definition of the *c.b.* norm.

Now, applying the factorization theorem of *c.b.* maps to this mapping u, we can easily prove that, for any finite sequence (a_n) in $\mathcal{K}$ (resp. in $B(\ell_2)$), we have

$$\|\sum a_n \otimes e_n\|_{B(\ell_2)\otimes_{\min}\ell_1} = \inf\{\|\sum b_n b_n^*\|^{1/2}\|\sum c_n^* c_n\|^{1/2}\}, \tag{8.2.2}$$

where the infimum runs over all possible decompositions $a_n = b_n c_n$ in $\mathcal{K}$ (resp. $B(\ell_2)$).

Analogously, we can describe the natural o.s.s. of $L_1(\mu)$ as follows. Let $f \in \mathcal{K} \otimes L_1(\mu)$. We may consider f as a $\mathcal{K}$-valued function on Ω. We have then

$$\|f\|_{\mathcal{K}\otimes_{\min}L_1(\mu)} = \inf\left\{\left\|\int g(t)g(t)^* d\mu(t)\right\|^{1/2}\left\|\int h(t)^* h(t) d\mu(t)\right\|^{1/2}\right\}, \tag{8.2.3}$$

where the infimum runs over all possible decompositions $f = gh$ as a product of (measurable, and say square integrable) $\mathcal{K}$-valued functions.

The formula (8.2.2) (resp. (8.2.3)) is the *quantum* version of the fact that the unit ball of ℓ_1 (resp. $L_1(\mu)$) coincides with the set of all products of two elements in the unit ball of ℓ_2 (resp. $L_2(\mu)$). They both can be deduced from the fundamental factorization of *c.b.* maps. This can also be related to the Haagerup tensor product as follows: if we denote by $L_2(\mu)_c$ (resp. $L_2(\mu)_r$) the space $L_2(\mu)$ equipped with the column (resp. row) o.s.s., then (8.2.3) says that the pointwise product map from $L_2(\mu) \otimes L_2(\mu)$ to $L_1(\mu)$ extends to a complete metric surjection from $L_2(\mu)_r \otimes_h L_2(\mu)_c$ onto $L_1(\mu)$ (the latter equipped with its natural o.s.s.).

Using Lemma 1.7 (with $p = 1$ and $F = L_1(\mu)$), it is easy to prove yet another formula: for any $f \in \mathcal{K} \otimes L_1(\mu)$ (viewed as a $\mathcal{K}$-valued measurable function), then we have

$$\|f\|_{\mathcal{K}\otimes_{\min} L_1(\mu)} = \sup\{\|a\ f\ b\|_{L_1(\mu;S_1)}\} \tag{8.2.4}$$

where the supremum runs over all a, b in the unit ball of S_2.

We will show that the operator space structure of ℓ_1 (resp. ℓ_1^n) described above is closely related to the unitary generators in the "full" C^*-algebra of the free group with infinitely (resp. n) generators. We first recall some classical notation from *non-commutative* Abstract Harmonic Analysis on an arbitrary discrete group Γ.

Let $\pi\colon\ \Gamma \to B(\mathcal{H})$ be a unitary representation on Γ. We denote by $C^*_\pi(\Gamma)$ the C^*-algebra generated by the range of π. Equivalently, $C^*_\pi(\Gamma)$ is the closed linear span of $\pi(\Gamma)$.

In particular, this applies to the so-called universal representation of Γ, a notion which we now recall. Let $(\pi_j)_{j\in I}$ be a family of unitary representations of Γ, say $\pi_j\colon\ \Gamma \to B(H_j)$, in which every equivalence class of a cyclic unitary representation of Γ has an equivalent copy. Now one can define the "universal" representation $\pi_u\colon\ \Gamma \to B(\mathcal{H})$ of Γ by setting

$$\pi_u = \oplus_{j\in I}\pi_j \quad \text{on} \quad \mathcal{H} = \oplus_{j\in I} H_j.$$

Then the associated C^*-algebra $C^*_{\pi_u}(\Gamma)$ is simply denoted by $C^*(\Gamma)$ and is called the C^*-algebra of the group Γ. (Note that this is the closed linear span of $\{\pi_u(t) \mid t \in \Gamma\}$.) It is often called the "full" C^*-algebra of Γ to distinguish it from the "reduced" one which is discussed in the next section §8.3.

Let $\mathbf{F}_n$ (resp. $\mathbf{F}_\infty$) be the free group with n (resp. countably many) generators, and let $\{g_1, g_2, \dots\}$ be the generators. Let $\pi\colon\ \mathbf{F}_\infty \to B(H)$ be a unitary representation of the free group. We will see that, in several instances, the operator space $E(\pi)$ spanned in $B(H)$ by $\{\pi(g_i) \mid i = 1, 2, \dots\}$ has interesting properties.

We will illustrate this with the "universal" representation $\pi_u\colon\ \mathbf{F}_\infty \to B(\mathcal{H})$, which generates the "full" C^*-algebra $C^*(\mathbf{F}_\infty)$, as introduced above (see §8.3 for the reduced case).

We let

$$E_u^n = \operatorname{span}[\pi_u(g_i) \mid i = 1, 2, \dots, n]$$

and

$$E_u = \overline{\text{span}}[\pi_u(g_i) \mid i \geq 1].$$

It is easy to see that for any finite sequence (a_i) in $B(\ell_2)$ we have

$$\left\|\sum a_i \otimes \pi_u(g_i)\right\|_{\min} = \sup_{j\in I}\left\{\left\|\sum a_i \otimes \pi_j(g_i)\right\|_{\min}\right\}.$$

Now since $(\pi_j(g_i))_{i\geq 1}$ runs over all possible choices of families of unitary operators (when j runs over I), we have

$$\left\|\sum a_i \otimes \pi_u(g_i)\right\|_{\min} = \sup\left\{\left\|\sum a_i \otimes u_i\right\|_{\min}\right\} \tag{8.2.5}$$

where the supremum runs over all sequences (u_i) of unitary operators in $B(H)$ and over all possible Hilbert spaces H. Since the unit ball of $B(H)$ is the closed convex hull of its unitaries (by the Russo-Dye theorem, p. 4 in **[Ped]**), it follows that the supremum over sequences (u_i) in the unit ball of $B(H)$ is the same. Actually, the supremum remains unchanged if we restrict ourselves to $H = \ell_2$ or to H finite dimensional with arbitrary dimension. But then, the formula defining the dual operator space shows that, if we denote by (e_i^*) the dual basis to the canonical basis of c_0 (equipped with its natural o.s.s.), we also have

$$\left\|\sum a_i \otimes \pi_u(g_i)\right\|_{\min} = \left\|\sum a_i \otimes e_i^*\right\|_{B(\ell_2)\otimes_{\min} c_0^*}.$$

Therefore, the mapping $u\colon\ c_0^* \to E_u$ which takes e_i^* to $\pi_u(g_i)$ is a complete isometry. Hence, we have proved:

Theorem 8.2.1. — *The operator space E_u spanned by the generators in $C^*(\mathbf{F}_\infty)$ is completely isometric to ℓ_1 equipped with its natural operator space structure (or equivalently its o.s.s. as the dual of c_0). Similarly, E_u^n is completely isometric to ℓ_1^n.*

Remark. — It is easy to check that, in $C^*(\mathbf{F}_n)$, the linear span of the unit and E_u^n is completely isometric to E_u^{n+1}, via the natural isomorphism (which takes, say, the unit to $\pi_u(g_1)$ and takes $\pi_u(g_i)$ to $\pi_u(g_{i+1})$ for $i = 1, 2, \ldots, n$).

The formula (8.2.5) can be viewed as the "quantum" analog of the classical formula

$$\forall\,(\lambda_i) \in \mathbb{C}^{(I)} \quad \|(\lambda_i)\|_{\ell_1} = \sum |\lambda_i| = \sup\left\{\left|\sum \lambda_i\ z_i\right| \mid z_i \in \mathbb{C},\ |z_i| = 1\right\}. \tag{8.2.6}$$

The space E_u gives us a "concrete realization" of the space ℓ_1 as an operator space. More generally, for any measure space (Ω, μ), one can describe the natural operator space structure of $L_1(\Omega, \mu)$ (induced by $L_\infty(\Omega, \mu)^*$) as follows. For all f in $L_1(\Omega, \mu) \otimes B(\ell_2)$, we have

$$\|f\|_{L_1(\Omega,\mu)\otimes_{\min} B(\ell_2)} = \sup\left\{\left\|\int f(\omega) \otimes g(\omega) d\mu(\omega)\right\|_{B(\ell_2)\otimes_{\min} B(\ell_2)}\right\}, \tag{8.2.7}$$

where the supremum runs over all functions g in the unit ball of the space of L_∞-functions with values in $B(\ell_2)$.

8.3. The non-commutative L_p-space and the reduced C^*-algebra of the free group with n generators

Let Γ be a discrete group. We denote by

$$\lambda_\Gamma\colon\ \Gamma \to B(\ell_2(\Gamma)),$$

(sometimes denoted simply by λ when the relevant group is clear) the left regular representation of Γ, which means that $\lambda_\Gamma(t)$ is the unitary operator of left translation by t on $\ell_2(\Gamma)$. Explicitly, if we denote by $(\delta_t)_{t\in\Gamma}$ the canonical basis of $\ell_2(\Gamma)$, we have $\lambda_\Gamma(t)\delta_s = \delta_{ts}$ for all t, s in Γ.

We denote by $C^*_\lambda(\Gamma)$ the C^*-algebra generated in $B(\ell_2(\Gamma))$ by $\{\lambda_\Gamma(t) \mid t \in \Gamma\}$ or equivalently, $C^*_\lambda(\Gamma) = \overline{\text{span}}\{\lambda_\Gamma(t) \mid t \in \Gamma\}$. Clearly, we have a C^*-algebra morphism

$$Q\colon\ C^*(\Gamma) \to C^*_\lambda(\Gamma)$$

which takes $\pi_u(t)$ to $\lambda_\Gamma(t)$. By elementary properties of C^*-algebras, it is onto and we have

$$C^*_\lambda(\Gamma) \approx C^*(\Gamma)/\ker(Q).$$

In general $\ker(Q) \neq \{0\}$, but one can show that $C^*_\lambda(\Gamma) = C^*(\Gamma)$ (*i.e.* $\ker(Q) = \{0\}$) iff Γ is amenable. The free groups are typical examples of non-amenable groups. The fact that the algebras $C^*_\lambda(\Gamma)$ and $C^*(\Gamma)$ are distinct in this case, is manifestly visible on the generators. Indeed, when $\Gamma = \mathbf{F}_n$ or $\Gamma = \mathbf{F}_\infty$, if we let

$$E^n_\lambda = \text{span}[\lambda(g_i) \mid i = 1, \ldots, n]$$

$$E_\lambda = \overline{\text{span}}[\lambda(g_i) \mid i \geq 1]$$

we can see that E_λ is a very different space from its analog in the full C^*-algebra, namely the space E_u studied in §8.2. Indeed, as Banach spaces, we have $E_u \approx \ell_1$ and $E_\lambda \approx \ell_2$. The first isomorphism is elementary (see Theorem 8.2.1 above), while the second one is due to Leinert **[Le]**. Using Haagerup's ideas from **[H4]**, one can describe the operator space structure of E_λ, as follows (see **[HP2]** for more details).

Consider the space $B(\ell_2)\oplus B(\ell_2)$, equipped with the norm $\|(x\oplus y)\| = \max\{\|x\|, \|y\|\}$. In the subspace $R\oplus C \subset B(\ell_2)\oplus B(\ell_2)$, we consider the vectors δ_i $(i = 1, 2, \ldots)$ defined by setting

$$\delta_i = e_{1i} \oplus e_{i1}.$$

We denote by $R \cap C$ the closed subspace spanned in $R \oplus C$ by the sequence $\{\delta_i\}$.

(This notation is consistent with the notion of "intersection" used in interpolation theory, provided we view the pair (R, C) as a compatible pair using the transposition mapping $x \to {}^t x$ as a way to "embed" R into C. This means we let $\mathcal{X} = C$, we use $x \to {}^t x$ to inject R into $\mathcal{X}$, and the identity map of C to inject C into $\mathcal{X}$.)

Similarly, we will denote by $R_n \cap C_n$ the subspace of $R \cap C$ spanned by $\{\delta_i \mid i = 1, 2, \ldots, n\}$. It is easy to verify that, for any Hilbert space H and for any finite sequence (a_i) in $B(H)$ we have

$$\left\|\sum a_i \otimes \delta_i\right\|_{\min} = \max\left\{\left\|\sum a_i^* a_i\right\|^{1/2}, \left\|\sum a_i a_i^*\right\|^{1/2}\right\}.$$

Then, we can state (see **[HP2]**).

Theorem 8.3.1. — *The space E_λ is completely isomorphic to $R \cap C$. More precisely, for any finite sequence (a_i) in $B(H)$, we have*

$$\Big\|\sum a_i \otimes \delta_i\Big\|_{\min} \le \Big\|\sum a_i \otimes \lambda(g_i)\Big\|_{\min} \le 2\Big\|\sum a_i \otimes \delta_i\Big\|_{\min},$$

so that the mapping $u\colon\ R \cap C \to E_\lambda$ defined by $u(\delta_i) = \lambda(g_i)$ is a complete isomorphism satisfying $\|u\|_{cb} = 2$ and $\|u^{-1}\|_{cb} = 1$. Moreover, the map $P\colon\ C^*_\lambda(\mathbf{F}_\infty) \to E_\lambda$, defined by $P\lambda(t) = \lambda(t)$ if t is a generator and $P\lambda(t) = 0$ otherwise, is a c.b. projection from $C^*_\lambda(\mathbf{F}_\infty)$ onto E_λ with norm $\|P\|_{cb} \le 2$. (Similar results hold a fortiori for E^n_λ and $R_n \cap C_n$.)

We now return to an arbitrary group Γ. Let us denote by τ_Γ the standard trace on the von Neumann algebra M_Γ generated by λ_Γ and defined by $\tau_\Gamma(T) = \langle T\delta_e, \delta_e\rangle$. Let $L_p(\tau_\Gamma)$ denote for $1 \le p < \infty$ the associated non-commutative L_p-space. The space $L_1(\tau_\Gamma)$ is the predual of the von Neumann algebra M_Γ, which we will also denote sometimes by $L_\infty(\tau_\Gamma)$. As explained before Lemma 0.1, when $1 < p < \infty$ and $\theta = 1/p$, we will view the space $L_p(\tau_\Gamma)$ as equipped with its natural o.s.s. for which the isometric identity $L_p(\tau_\Gamma) = (L_\infty(\tau_\Gamma), L_1(\tau_\Gamma))_\theta$ becomes completely isometric.

We now turn to the case $\Gamma = \mathbf{F}_\infty$ for the rest of this section. We will drop the index Γ and write simply λ, τ,... instead of λ_Γ, τ_Γ,... with $\Gamma = \mathbf{F}_\infty$.

We will denote simply by $L_p(\tau)$ the non-commutative L_p-space ($1 \le p < \infty$) associated to $\Gamma = \mathbf{F}_\infty$. Note that, by Lemma 0.1 and **[H4]**, we have (completely isometrically)

$$L_p(\tau) = (L_\infty(\tau), L_1(\tau))_\theta = (C^*_\lambda(\mathbf{F}_\infty), L_1(\tau))_\theta$$

whenever $0 < \theta = 1/p < 1$.

We wish to describe the operator space generated by the free unitary generators $\{\lambda(g_i) \mid i = 1, 2, \dots\}$ in $L_p(\tau)$.

Let us denote by E_p the closed subspace of $L_p(\tau)$ generated by $\{\lambda(g_i) \mid i = 1, 2, \dots\}$. Note that $E_\infty = E_\lambda$. We may view E_p as an operator space with the o.s.s. induced by $L_p(\tau)$. Clearly, the orthogonal projection P from $L_2(\tau)$ to E_2 is completely contractive (since, by (0.4), $L_2(\tau)$ can be identified with $OH(I)$ for a suitable set I). On the other hand, by Theorem 8.3.1, that "same" projection P is completely bounded from $C^*_\lambda(\mathbf{F}_\infty)$ onto E_λ. Actually, the proof of Theorem 8.3.1 shows that P extends to a weak$-*$ continuous projection from $L_\infty(\tau)$ onto E_λ. By transposition, P also defines a *c.b.* projection from $L_1(\tau)$ onto E_1. Therefore, by interpolation, P defines a completely bounded projection from $L_p(\tau)$ onto E_p for any $1 < p < \infty$.

It is natural to expect (as our notation suggests) that E_p can be identified with $(E_\infty, E_1)_\theta$ with $\theta = 1/p$. However, the complex interpolation functor has a very important "defect" which is well known to specialists, but is often overlooked by non-specialists: it is *not* injective. By this we mean that, given a compatible Banach couple (A_0, A_1), if we interpolate between two closed subspaces of A_0 and A_1, we do not get a closed subspace of the interpolation space $(A_0, A_1)_\theta$. More precisely, if we give ourselves a linear subspace $\mathcal{S} \subset A_0 \cap A_1$ and if we define $\mathcal{S}_0 = \overline{\mathcal{S}}^{A_0}$, $\mathcal{S}_1 = \overline{\mathcal{S}}^{A_1}$, it is (in general) *not true* that $\overline{\mathcal{S}}^{A_\theta}$ coincides with $(\mathcal{S}_0, \mathcal{S}_1)_\theta$, and the corresponding norms are *not* equivalent. A simple and classical counterexample is provided by the Rademacher functions (see §8.4 below) which span ℓ_1 inside L_∞, and span ℓ_2 inside L_1, but nevertheless still span ℓ_2 $(\neq (\ell_1, \ell_2)_\theta)$ in the intermediate spaces $L_p = (L_\infty, L_1)_\theta$ for $1 < p < \infty$, $\theta = 1/p$.

There is however a classical instance where this difficulty disappears, when we have a linear projection which is *simultaneously* bounded from A_0 to $\mathcal{S}_0$ and from A_1 to $\mathcal{S}_1$. In that case, it is easy to check that we do have $(\mathcal{S}_0, \mathcal{S}_1)_\theta \simeq \overline{\mathcal{S}}^{A_\theta}$ with equivalent norms. We will need the following extension to operator spaces, which is immediate:

Proposition 8.3.2. — *Let (A_0, A_1) be a compatible couple of operator spaces. Let $\mathcal{S}_0, \mathcal{S}_1$ be as above. Assume that there is a c.b. linear projection $P\colon\ A_0 \to \mathcal{S}_0$ which also extends completely boundedly to a projection from A_1 to $\mathcal{S}_1$. Then we have a completely isomorphic identification*

$$(\mathcal{S}_0, \mathcal{S}_1)_\theta \simeq \overline{\mathcal{S}}^{A_\theta},$$

and P defines a c.b. projection from A_θ to $\overline{\mathcal{S}}^{A_\theta}$.

Thus, in contrast with the Rademacher case, the existence of this simultaneous *c.b.* projection ensures that the space E_p can be identified completely isomorphically to $(E_\lambda, E_1)_\theta$ with $\theta = 1/p$. In addition, $E_1 \simeq (E_\lambda)^*$. By Theorem 8.3.1, we have $E_\lambda \simeq R \cap C$ and by duality $E_1 \simeq (R \cap C)^*$. We can describe the operator space $(R \cap C)^*$ as follows.

Consider the direct sum $R \oplus_1 C$ (as defined in §0), and its subspace $\Delta \subset R \oplus_1 C$ defined by

$$\Delta = \{(x, -{}^t x) \mid x \in R\}.$$

We will denote by $R + C$ the quotient space $(R \oplus_1 C)/\Delta$. Since $R \oplus_1 C$ is equipped with a natural o.s.s., the space $R + C$ itself is thus equipped with a natural o.s.s. as a quotient space (see §0). It is easy to see (*cf.* §0) that

$$(R \cap C)^* = R + C \quad \text{completely isometrically.}$$

Thus, $E_1 \simeq R + C$. Hence we have (completely isomorphically) $E_p \simeq (R \cap C, R + C)_\theta$. We will "compute" the latter space more explicitly in Theorem 8.4.8 below, but let us state what we just proved.

Corollary 8.3.3. — *Let $L_p(\tau)$ denote the non-commutative L_p-space of the free group and let E_p be the closed subspace generated by the free generators $\{\lambda(g_i) \mid i = 1, 2, \dots\}$, with $1 \le p \le \infty$. Then we have, completely isomorphically (with $\theta = 1/p$)*

$$E_p \simeq (R \cap C, R + C)_\theta, \tag{8.3.1}$$

where as before, we use the transposition mapping as the continuous injection from R to C which allows us to view (R, C) as a compatible couple. In addition, the orthogonal projection from $L_2(\tau)$ onto E_2 defines a c.b. projection from $L_p(\tau)$ onto E_p for all

$1 \leq p \leq \infty$. *Moreover, the equivalence constants in (8.3.1) as well as* $\|P\|_{cb(L_p(\tau),E_p)}$ *remain bounded when* p *runs over the whole interval* $[1,\infty]$.

Remark 8.3.4. — By Cor. 2.4, p. 26 in **[P1]**, we know that $(R \cap C, R + C)_{1/2} = OH$ completely isometrically. Hence, by the reiteration theorem (*cf.* **[BL]**, p. 101) we have

$$(R \cap C, R + C)_\theta = (R \cap C, OH)_{2\theta} \quad \text{if} \quad \theta < 1/2$$

and

$$(R \cap C, R + C)_\theta = (OH, R + C)_{2\theta - 1} \quad \text{if} \quad \theta > 1/2$$

Remark. — In Theorems 8.4.8 and 8.4.10 below, we give a very explicit description of the operator space structures of $(R \cap C, R + C)_\theta$ and E_p.

8.4. Operator space spanned in L_p by standard Gaussian random variables or by the Rademacher functions

Let $(\Omega, \mathcal{A}, P)$ be a probability space. We will say that a real-valued Gaussian random variable (in short r.v.) γ is standard if $E\gamma = 0$ and $E\gamma^2 = 1$. We will say that a complex valued Gaussian r.v. $\widetilde{\gamma}$ is Gaussian standard if we can write $\widetilde{\gamma} = 2^{-1/2}(\gamma' + i\gamma'')$ with γ', γ'' real-valued, independent, standard Gaussian r.v.'s.

Let $\{\gamma_n \mid n = 1, 2, \ldots\}$ (resp. $\{\widetilde{\gamma}_n \mid n = 1, 2, \ldots\}$) be a sequence of real (resp. complex) valued independent standard Gaussian r.v.'s on $(\Omega, \mathcal{A}, P)$. As is well known, for any finite sequence of real (resp. complex) scalars (α_n), the r.v. $S = \sum \alpha_i \gamma_i$ (resp. $\sum \alpha_i \widetilde{\gamma}_i$), has the same distribution as the variable $\widetilde{S} = \left(\sum |\alpha_i|^2\right)^{1/2} \gamma_1$ (resp. $\left(\sum |\alpha_i|^2)^{1/2} \widetilde{\gamma}_1\right)$. In particular, we have for any finite sequence of complex scalars

$$\left\| \sum \alpha_i \widetilde{\gamma}_i \right\|_p = \|\widetilde{\gamma}_1\|_p \left(\sum |\alpha_i|^2 \right)^{1/2}. \tag{8.4.1}$$

Let $\mathcal{G}_p$ be the subspace of $L_p(\Omega, \mathcal{A}, P)$ generated by $\{\widetilde{\gamma}_n \mid n = 1, 2, \ldots\}$. Then, as a Banach space, $\mathcal{G}_p$ is isometric to ℓ_2 for all $1 \leq p < \infty$. (To simplify, we will discuss mostly the complex case in the sequel, although the real case is entirely similar provided we restrict ourselves to $\mathbb{R}$-linear transformations.) Moreover, for any isometry $U : \mathcal{G}_p \to \mathcal{G}_p$ the sequence $\{U(\widetilde{\gamma}_i) \mid i = 1, 2, \ldots\}$ has the same distribution as the sequence $\{\widetilde{\gamma}_i \mid i = 1, 2, \ldots\}$. If we equip $\mathcal{G}_p$ with the o.s.s. induced by $L_p(\Omega, \mathcal{A}, P)$, it follows (see Proposition 2.4 and Remark 2.5) that U is a complete isometry from $\mathcal{G}_p$ to $\mathcal{G}_p$.

Let $\{\varepsilon_n \mid n = 1, 2, \ldots\}$ be a sequence of independent, identically distributed r.v.'s on $(\Omega, \mathcal{A}, P)$ with ± 1 values and such that $P\{\varepsilon_n = +1\} = P\{\varepsilon_n = -1\} = 1/2$. The reader who so wishes can replace $\{\varepsilon_n \mid n = 1, 2, \ldots\}$ by the classical Rademacher functions (r_n) on the Lebesgue interval, this does not make any difference in the sequel.

Let $\mathcal{R}_p$ be the subspace generated in $L_p(\Omega, \mathcal{A}, P)$ by the sequence (ε_n).

The analog of (8.4.1) for the variables (ε_n) (or equivalently for the Rademacher functions) is given by the classical Khintchine inequalities (*cf.* e.g. [**LT1**], p. 66 or [**DJT**] p. 10), which say that, for $1 \le p < \infty$, there are positive constants A_p and B_p such that for any scalar sequence of coefficients (α_n) in ℓ_2 we have

$$(8.4.2) \qquad A_p \left(\sum |\alpha_n|^2\right)^{1/2} \le \left\|\sum \alpha_n \varepsilon_n\right\|_p \le B_p \left(\sum |\alpha_n|^2\right)^{1/2}.$$

(Note that we have trivially $B_p = 1$ if $p \le 2$ and $A_p = 1$ if $p \ge 2$.) This implies that, as a Banach space, $\mathcal{R}_p$ is isomorphic to ℓ_2 for any $1 \le p < \infty$. A fortiori $\mathcal{R}_p$ and $\mathcal{G}_p$ are isomorphic Banach spaces if $1 \le p < \infty$.

By Proposition 2.4, we can deduce from the known general results on Gaussian and Rademacher series in Banach spaces (*cf.* [**MaP**], Corollaire 1.3) that the spaces $\mathcal{G}_p$ and $\mathcal{R}_p$ are completely isomorphic for any $1 \le p < \infty$. Of course, in the case $p = 2$, $\mathcal{G}_2$ and $\mathcal{R}_2$ are completely isometric to OH since the space $L_2(\Omega, \mathcal{A}, P)$ itself is completely isometric to $OH(I)$, where the cardinal I is its Hilbertian dimension.

We now wish to describe the operator space structure induced by L_p on $\mathcal{G}_p$ (resp. $\mathcal{R}_p$). By Proposition 2.1 (applied with $\dim(E) = 1$), this can be reduced to the knowledge of the norm

$$\left\|\sum \widetilde{\gamma}_n x_n\right\|_{L_p(\Omega,P;S_p)}$$

(resp. $\|\sum \varepsilon_n x_n\|_{L_p(\Omega,P;S_p)}$) when (x_n) is an arbitrary finite sequence of elements of S_p. In other words, to describe the o.s.s. of $\mathcal{G}_p$ (resp. $\mathcal{R}_p$) up to complete isomorphism, it suffices to produce two-sided inequalities analogous to (8.4.1) and (8.4.2) but with coefficients in S_p instead of scalar ones. The non-commutative versions of Khintchine's inequalities proved in [**Lu**] and [**LuP**] are exactly what is needed here. The case $1 < p < \infty$ is a remarkable result due to F. Lust-Piquard ([**Lu**]). The case $p = 1$ comes from the later paper [**LuP**], which also contains an alternate proof of the other cases.

Theorem 8.4.1
(i) *Assume* $2 \le p < \infty$. *Then there is a constant* B'_p *such that for any finite sequence* (x_n) *in* S_p, *we have*

$$(8.4.3) \qquad \max\left\{\left\|\left(\sum x_n^* x_n\right)^{1/2}\right\|_{S_p}, \left\|\left(\sum x_n x_n^*\right)^{1/2}\right\|_{S_p}\right\}$$
$$\le \left\|\sum \varepsilon_n x_n\right\|_{L_p(\Omega,P;S_p)}$$
$$\le B'_p \max\left\{\left\|\left(\sum x_n^* x_n\right)^{1/2}\right\|_{S_p}, \left\|\left(\sum x_n x_n^*\right)^{1/2}\right\|_{S_p}\right\}.$$

(ii) *Assume* $1 \le p \le 2$. *Then there is a positive constant* A' *(independent of* p*) such that, for any finite sequence* (x_n) *in* S_p, *we have*

$$(8.4.4) \qquad A' |||(x_n)|||_p \le \left\|\sum \varepsilon_n x_n\right\|_{L_p(\Omega,P;S_p)} \le |||(x_n)|||_p$$

where we have set

$$|||(x_n)|||_p = \inf\left\{\left\|\left(\sum y_n^* y_n\right)^{1/2}\right\|_{S_p} + \left\|\left(\sum z_n z_n^*\right)^{1/2}\right\|_{S_p} \,\middle|\, x_n = y_n + z_n\right\}.$$

Moreover, similar inequalities are valid with a real or complex Gaussian i.i.d. sequence (γ_n) *or* $(\widetilde{\gamma}_n)$ *in the place of* (ε_n). *Finally, the same inequalities are valid when* S_p *is replaced by any non-commutative* L_p *space associated to a semi-finite faithful normal trace on a von Neumann algebra.*

Remark. — The fact that the constant A' appearing in (8.4.4) can be taken independent of $1 \le p \le 2$ is proved in [**LuP**] (see Cor. III.4, p. 254).

Remark. — (Independently observed by Marius Junge in [**Ju**].) We claim that there is a numerical constant C such that, for all $2 \le p < \infty$

$$B'_p \le C\sqrt{p}. \tag{8.4.5}$$

Since this is only implicitly contained in [**LuP**] and it might be of independent interest, we will give the details explicitly. Let us denote by $P_1\colon\ L_2 \to \mathcal{R}_2$ the orthogonal projection. Recall that the K-convexity constant of a Banach space X is defined as follows

$$K(X) = \|P_1 \otimes I_X\|_{L_2(X)\to L_2(X)}.$$

By a standard averaging technique, one easily verifies that

$$\|P_1 \otimes I_X\|_{L_p(X)\to L_p(X)} \le K(L_p(X)).$$

When $X = S_p$, then since $\ell_p(S_p)$ embeds isometrically into S_p, we have

$$K(L_p(S_p)) = K(S_p).$$

By duality, it follows from the preceding estimate of $\|P_1 \otimes I_X\|_{L_p(X)\to L_p(X)}$ for $X = S_p$, using (8.4.4), that for all $2 \le p < \infty$,

$$B'_p \le (A')^{-1} K(L_p(S_p)) = (A')^{-1} K(S_p).$$

By Remark 2.10 in [**MaP**], the latter constant is dominated by the type 2 constant, and by [**TJ2**] the type 2 constant of S_p is equal to the best constant in the classical (scalar) Khintchine inequalities B_p (at least when p is an even integer) which is of order $p^{1/2}$ when $p \to \infty$. Thus we obtain our claim that there is a numerical constant C such that, for all $2 \le p < \infty$

$$B'_p \le C\sqrt{p}.$$

The following result, essentially due to Mark Rudelson, has applications to the geometry of convex bodies (*cf.* [**Rn**]). A simple proof of it can be given using the preceding remark.

Proposition 8.4.2. — *There is a numerical constant* C *such that for any* n, m *and any* m*-tuple of rank one orthogonal projections* $(P_1, \ldots, P_m)$ *in* M_n, *we have*

$$\forall\, (\alpha_j) \in \mathbb{C}^m \qquad \int_\Omega \|\sum_1^m \varepsilon_j \alpha_j P_j\|_{M_n} dP \le C \|\sum_1^m |\alpha_j|^2 P_j\|^{1/2} (\log(n+1))^{1/2}.$$

Proof. — We apply (8.4.3) with $p = 2+\log(n)$ and $x_j = \alpha_j P_j$, so that $x_j x_j^* = x_j^* x_j = |\alpha_j|^2 P_j$. Then for any x in M_n, we have

$$\|x\|_{S_p^n} \leq n^{1/p}\|x\|_{M_n} \leq e\|x\|_{M_n}.$$

Hence

$$\|(\sum_1^m |\alpha_j|^2 P_j)^{1/2}\|_{S_p^n} \leq e\|\sum_1^m |\alpha_j|^2 P_j\|_{M_n}^{1/2}.$$

Therefore the result follows from (8.4.3) and (8.4.5). □

Let us now identify $\mathcal{G}_p$ and $\mathcal{R}_p$ as operator spaces. We start by the case $p = 1$ which is particularly interesting. As mentioned before stating Corollary 8.3.3, we have

$$(R \cap C)^* = R + C \quad \text{completely isometrically.} \tag{8.4.6}$$

In particular, $R + C$ is isomorphic to ℓ_2 as a Banach space. We will denote by (σ_i) the natural basis of $R + C$ which is biorthogonal to the basis (δ_i) of $\Delta^\perp \approx R \cap C$. Equivalently, if we denote by $q : R \oplus_1 C \to R + C$ the canonical surjection, then we have $\sigma_n = q(e_{1n} \oplus e_{n1})$. Similarly, we will denote by $R_n + C_n$ the quotient operator space $(R_n \oplus_1 C_n)/\Delta_n$, with $\Delta_n = \{(x, -{}^t x) \mid x \in R_n\}$. We also can identify $R_n + C_n$ with the subspace spanned in $R + C$ by $(\sigma_1, \ldots, \sigma_n)$. Moreover, we have $(R_n \cap C_n)^* = R_n + C_n$ completely isometrically.

Now we can reformulate the main result of **[LuP]** in the language of operator spaces as follows:

Theorem 8.4.3. — *The space $\mathcal{R}_1$ (resp. $\mathcal{G}_1$) is completely isomorphic to $R + C$, via the isomorphism which takes ε_i (resp. $\widetilde{\gamma}_i$) to σ_i.*

Proof. — It is easy to verify that the norm $|||.|||_1$ appearing in Theorem 8.4.1 is the dual norm of the natural norm from the space $\mathcal{K} \otimes_{\min} (R \cap C)$. Equivalently, it coincides with the norm in the space $S_1[(R\cap C)^*] = S_1[R + C]$. Therefore, the linear mapping which takes ε_i to σ_i defines an isomorphism from $S_1[\mathcal{R}_1]$ onto $S_1[R + C]$. Thus, the announced result for $\mathcal{R}_1$ follows from Proposition 2.4 applied with $p = 1$. The case of $\mathcal{G}_1$ is analogous. □

By combining (8.4.6) with Theorem 8.3.1 we obtain a surprising connection between the standard Gaussian (or ± 1 valued) independent random variables and the generators of $C_\lambda^*(\mathbf{F}_\infty)$:

Corollary 8.4.4. — *We have $(E_\lambda)^* \approx \mathcal{G}_1 \approx \mathcal{R}_1$ completely isomorphically. More precisely, let us denote by $(\lambda_*(g_i))$ the system in $(E_\lambda)^*$ which is biorthogonal to $(\lambda(g_i))$. Then the mapping $u : \mathcal{R}_1 \to (E_\lambda)^*$ (resp. $u : \mathcal{G}_1 \to (E_\lambda)^*$) defined by $u(\varepsilon_i) = \lambda_*(g_i)$ (resp. $u(\widetilde{\gamma}_i) = \lambda_*(g_i)$) is a complete isomorphism.*

More generally, for all $1 \leq p \leq \infty$, let us denote by $R[p]$ (resp. $C[p]$) the operator space generated by the sequence $\{e_{1j} \mid j = 1, 2, \ldots\}$ (resp. $\{e_{i1} \mid i = 1, 2, \ldots\}$) in the operator space S_p equipped with its natural o.s.s. defined by interpolation. Here again we set $S_\infty = \mathcal{K}$. Note that $R[\infty]$ (resp. $C[\infty]$) obviously coincides with the "row" (resp. "column") space R (resp. C). A moment of thought shows that $R[1] \approx R^*$ and $C[1] \approx C^*$ completely isometrically, therefore we may identify $R[1]$ with C on one hand, and $C[1]$ with R on the other. Moreover, we obviously have a natural projection *simultaneously* completely contractive from S_1 to $R[1]$ and S_∞ to $R[\infty]$ (and similarly for columns). This implies, by Proposition 8.3.2 that if we make the couple (R, C) into a compatible one (as we did in §0, see **[P1]** for more on this theme) by using transposition to inject R into C, then we have completely isometric identifications (with $\theta = 1/p$)

$$R[p] = (R, C)_\theta \qquad C[p] = (C, R)_\theta .$$

In other words, the space $R[p]$ is exactly the same as the space denoted by $R(1/p)$ in Theorem 1.1.

Now if we use (1.5) we find

$$S_p[R[p]] = (S_\infty[R], S_1[C])_\theta \quad \text{and} \quad S_p[C[p]] = (S_\infty[C], S_1[R])_\theta . \tag{8.4.7}$$

If we view $S_\infty[R]$, (resp. $S_1[C]$) as a space of sequences of elements of S_∞ (resp. S_1), then the norm in $S_\infty[R]$ (resp. $S_1[C]$) is easily seen to be $(x_j) \to \left\| \left(\sum x_j x_j^*\right)^{1/2} \right\|_{S_\infty}$ (resp. $(x_j) \to \left\| \left(\sum x_j x_j^*\right)^{1/2} \right\|_{S_1}$). Therefore, since we have simultaneously contractive projections (see the discussion before Proposition 8.3.2) onto the corresponding subspaces of $S_\infty[S_\infty]$ and $S_1[S_1]$ we find that the norm in $(S_\infty[R], S_1[C])_\theta$ coincides with $(x_j) \to \left\| \left(\sum x_j x_j^*\right)^{1/2} \right\|_{S_p}$. In other words, for any sequence (x_j) in S_p we have by (8.4.7)

$$\left\| \sum x_j \otimes e_{1j} \right\|_{S_p[R[p]]} = \left\| \left(\sum x_j x_j^*\right)^{1/2} \right\|_{S_p} . \tag{8.4.8}$$

Similarly, we have

$$\left\| \sum x_i \otimes e_{i1} \right\|_{S_p[C[p]]} = \left\| \left(\sum x_i^* x_i\right)^{1/2} \right\|_{S_p} . \tag{8.4.9}$$

We denote by $R[p] \cap C[p]$ the subspace of $R[p] \oplus_\infty C[p]$ formed of all couples of the form $(x, {}^t x)$. On the other hand, we denote by $R[p] + C[p]$ the operator space which is the quotient of $R[p] \oplus_1 C[p]$ modulo the subspace formed of all couples of the form $(x, -{}^t x)$. Then, by (8.4.8) and (8.4.9), the norm appearing on the left in (8.4.3) is equivalent to the natural norm of the space $S_p[C[p]] \cap S_p[R[p]]$ or equivalently $S_p[R[p] \cap C[p]]$. Similarly, the norm $|||\ |||_p$ in Theorem 8.4.1 (case $p \leq 2$) is equivalent to the natural norm of the space $S_p[C[p]] + S_p[R[p]]$ or equivalently $S_p[R[p] + C[p]]$. This allows us to state:

Theorem 8.4.5. — *Let $1 < p < \infty$. The space $\mathcal{G}_p$ (or the space $\mathcal{R}_p$) is completely isomorphic to $R[p] + C[p]$ if $p \leq 2$ and to $R[p] \cap C[p]$ if $p \geq 2$.*

Proof. — First observe that the natural norm in the space $S_p[\mathcal{R}_p]$ is equal to the norm induced by $L_p(\Omega, \mathcal{A}, P; S_p)$, by Corollary 2.2. Then, by (8.4.3), (8.4.4) and the preceding discussion, the latter norm is equivalent to the natural norm of either the space $S_p[R[p] \cap C[p]]$ if $p \geq 2$ or the space $S_p[R[p] + C[p]]$ if $p \leq 2$. Whence the announced complete isomorphisms by Proposition 2.4. $\square$

Remark. — Note that Theorem 8.4.1 is nothing but an extension of Theorem 8.4.5 to the case $p = 1$.

Remark 8.4.6. — In the Banach space setting, it is well known that the orthogonal projection $P_2\colon L_2 \to \mathcal{G}_2$ (resp. $Q_2\colon L_2 \to \mathcal{R}_2$) extends to a bounded linear projection $P_p\colon L_p \to \mathcal{G}_p$ (resp. $Q_p\colon L_p \to \mathcal{R}_p$), provided $1 < p < \infty$, and this fails if $p = 1$ or $p = \infty$. (Warning: it is customary in Harmonic Analysis to consider that P_p and P_2 are the "same" operator, since they coincide on simple functions.)

In the operator space setting, the situation is analogous: for any $1 < p < \infty$, P_p (resp. Q_p) is a c.b. projection from L_p onto $\mathcal{G}_p$ (resp. onto $\mathcal{R}_p$). This can be seen easily using Proposition 2.4 and the fact that, when $1 < p < \infty$, S_p is a "K-convex" Banach space in the sense of **[P11]**. (See also **[TJ1]**, p. 86 or **[DJT]**, p. 258.)

This can also be viewed as a corollary of Theorem 8.4.5, since the latter result implies that $(\mathcal{G}_p)^* \simeq \mathcal{G}_{p'}$ and $(\mathcal{R}_p)^* \simeq \mathcal{R}_{p'}$ (completely isomorphically) when $1 < p, p' < \infty$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Indeed, the complete boundedness of the natural mapping

$$(\mathcal{G}_p)^* = L_{p'}/\mathcal{G}_p^{\perp} \to \mathcal{G}_{p'}$$

is clearly equivalent to the complete boundedness of P_p, and similarly for Q_p.

Notation. — We will denote (albeit abusively) by (δ_i) the natural basis of both spaces $R[p] \cap C[p]$ and $(R \cap C, R + C)_\theta$. This simpler notation should not create any confusion.

Corollary 8.4.7. — *Let $2 \leq p \leq \infty$. Consider the space $\mathcal{Q}_p = (\mathcal{R}_{p'})^*$, i.e. we denote $\mathcal{Q}_p = L_p(\Omega, P)/(\mathcal{R}_{p'})^{\perp}$. Let $\hat{\varepsilon}_i$ be the equivalence class of ε_i in $\mathcal{Q}_p$. Then the mapping $T_1\colon R[p] \cap C[p] \to \mathcal{Q}_p$, which takes δ_i to $\hat{\varepsilon}_i$, satisfies*

$$\|T_1\|_{cb} \leq C \quad \text{and} \quad \|T_1^{-1}\|_{cb} \leq 1,$$

where C is a constant independent of $2 \leq p \leq \infty$.

Proof. — This follows from an essentially trivial dualization of (8.4.4). $\square$

Theorem 8.4.8. — *Let $2 \leq p < \infty$ and $\theta = 1/p$. Then the natural identification induces an isomorphism*

$$T\colon R[p] \cap C[p] \to (R \cap C, R + C)_\theta,$$

with $\|T\|_{cb} \leq C$ and $\|T^{-1}\|_{cb} \leq 1$ where C is a constant independent of $2 \leq p < \infty$. Consequently, for any $1 < p < \infty$ and $\theta = 1/p$ we have completely isomorphic identities (with isomorphy constants bounded independently of p)

$$(R \cap C, R + C)_\theta \simeq R[p] \cap C[p] \ \textit{if}\ p \geq 2$$
$$(R \cap C, R + C)_\theta \simeq R[p] + C[p] \ \textit{if}\ p \leq 2.$$

Proof. — Let $2 \le p < \infty$. Let $\widetilde{E}_p = (R \cap C, R + C)_\theta$. We will first prove that $\|T^{-1}\|_{cb(\widetilde{E}_p, R[p]\cap C[p])} \le 1$ or equivalently that

$$\|T^{-1}\|_{cb(\widetilde{E}_p, R[p])} \le 1 \quad \text{and} \quad \|T^{-1}\|_{cb(\widetilde{E}_p, C[p])} \le 1.$$

By interpolation, it suffices to prove this for $\theta = 0$ and $\theta = 1/2$. But the case $\theta = 0$ is trivial, and $\theta = 1/2$ follows from Remark 8.3.4.

To prove the converse, we write T as a composition $T = T_2T_1$ where T_1 is as in Corollary 8.4.7 and where $T_2\colon\ \mathcal{Q}_p \to \widetilde{E}_p$ is the map taking $\hat{\varepsilon}_i$ to δ_i. To conclude it is enough to show that $\|T_2\|_{cb(\mathcal{Q}_p, \widetilde{E}_p)} \le 1$. Equivalently, it suffices to show that the map $T_3\colon\ L_p(\Omega, P) \to \widetilde{E}_p$ which is the composition of T_2 with the canonical quotient map from $L_p(\Omega, P)$ onto $\mathcal{Q}_p$ satisfies

$$\|T_3\|_{cb(L_p, \widetilde{E}_p)} \le 1. \tag{8.4.10}$$

But now, note that, for any von Neumann algebra M, we have obviously

$$\forall\, f \in L_\infty(\Omega, P; M), \qquad \left\| \int f^* f dP \right\|^{1/2} \le \|f\| \quad \text{and} \quad \left\| \int f f^* dP \right\|^{1/2} \le \|f\|$$

hence, it is easy to show that (8.4.10) holds for $p = \infty$. On the other hand, (8.4.10) clearly holds also for $p = 2$ by Remark 8.3.4. Thus, by interpolation, we obtain (8.4.10) for any $2 \le p < \infty$. This completes the proof since the last assertion can be obtained by duality. □

Remark. — The preceding statement might be a very particular case of a general phenomenon (yet to be proved) in complex interpolation theory. See the discussion in Maligranda's paper **[Ma]** about the "real" interpolation between sums and intersections.

Using Corollary 8.3.3, we obtain another striking isomorphism.

Corollary 8.4.9. — *Let $1 \le p < \infty$. Let E_p be as in Corollary 8.3.3. Then the correspondence $\varepsilon_i \to \lambda(g_i)$ (resp. $\widetilde{\gamma}_i \to \lambda(g_i)$) is a complete isomorphism between the spaces $\mathcal{R}_p$ (resp. $\mathcal{G}_p$) and E_p (and here the isomorphism constants remain bounded when p runs over the interval $[1, 2]$).*

Proof. — We just combine Corollary 8.3.3 with the preceding statement. □

Theorem 8.4.10. — *There is a constant C' such that, for any $2 \le p \le \infty$, for any von Neumann algebra N equipped with a standard trace ψ, and for any finite sequence (x_n) in $L_p(N, \psi)$, we have*

$$\left\| \sum \lambda(g_n) \otimes x_n \right\|_{L_p(\tau \times \psi)} \le C' \max\left\{ \left\| \left(\sum x_n^* x_n\right)^{1/2} \right\|_{L_p(N,\psi)}, \left\| \left(\sum x_n x_n^*\right)^{1/2} \right\|_{L_p(N,\psi)} \right\}.$$

Proof. — With S_p in the place of $L_p(N,\psi)$, this is an immediate consequence of Theorem 8.4.8 (recalling (8.4.7)). The general case can be proved exactly as above for Theorem 8.4.8, but taking into account the last assertion in Theorem 8.4.1 (which follows from Remark 3.6, p. 255 in **[LuP]**). □

Remark 8.4.11. — Let $k \geq 1$ be a fixed integer. Let $(\varepsilon_n^1)_{n\geq 1}, (\varepsilon_n^2)_{n\geq 1}, \ldots, (\varepsilon_n^k)_{n\geq 1}$ be independent copies of the original sequence (ε_n) as above, on a suitable probability space (Ω, A, P). Let us denote by $\mathcal{R}_p^k$ the subspace of $L_p(\Omega, A, P)$ spanned by the functions of the form $\varepsilon_{n_1}^1 \varepsilon_{n_2}^2 \ldots \varepsilon_{n_k}^k$ $(n_1 \geq 1, n_2 \geq 1, \ldots)$. Then, modulo a simple reformulation, the results of the paper **[HP2]** describe the operator space structure of the space $\mathcal{R}_1^k$ for any $k = 1, 2, \ldots$ and its dual. (The Gaussian case is similar by general arguments.)

The paper **[HP2]** also describes the space $E_\lambda \otimes_{\min} \cdots \otimes_{\min} E_\lambda$ (k-times) and proves that $\mathcal{R}_1^k$ is completely isomorphic to $(E_\lambda \otimes_{\min} \cdots \otimes_{\min} E_\lambda)^*$. Here of course the isomorphism constants depend on k.

Concerning $\mathcal{R}_p^k$ for $1 < p < \infty$, it is easy to iterate the inequalities appearing in Theorem 8.4.1 to obtain (after successive integrations) two-sided inequalities describing the operator space structure of $\mathcal{R}_p^k$. To describe these iterated inequalities, assume for simplicity that $k = 2$. Let (x_{ij}) be a matrix with entries in S_p, with only finitely many of them non-zero. Then both $x = (x_{ij})$ and the transposed matrix ${}^t x = (x_{ji})$ can be viewed as elements of S_p on the Hilbert space $\ell_2 \oplus \ell_2 \oplus \cdots$ and we denote the corresponding norms simply by $\|x\|_{S_p}$ and $\|{}^t x\|_{S_p}$.

Then after iteration (8.4.3) becomes when $2 \leq p < \infty$

$$
\text{(8.4.11)} \quad \max\left\{ \|x\|_{S_p}, \|{}^t x\|_{S_p}, \left\| \left(\sum_{ij} x_{ij}^* x_{ij} \right)^{1/2} \right\|_{S_p}, \left\| \left(\sum_{ij} x_{ij} x_{ij}^* \right)^{1/2} \right\|_{S_p} \right\}
\leq \left\| \sum_{ij} \varepsilon_i^1 \varepsilon_j^2 x_{ij} \right\|_{L_p(\Omega, P; S_p)}
$$

$$
\leq (B_p')^2 \max\left\{ \|x\|_{S_p}, \|{}^t x\|_{S_p}, \left\| \left(\sum_{ij} x_{ij}^* x_{ij} \right)^{1/2} \right\|_{S_p}, \left\| \left(\sum_{ij} x_{ij} x_{ij}^* \right)^{1/2} \right\|_{S_p} \right\}.
$$

Moreover, the orthogonal projection induces a c.b. projection from $L_p(\Omega, P)$ onto $\mathcal{R}_p^k$, for any $k = 1, 2, \ldots$, so that (8.4.11) can be dualized to treat the case $1 < p < 2$. We do not spell out the corresponding inequality. Again, when $2 \leq p < \infty$, these inequalities can be interpreted as describing $\mathcal{R}_p^2$ as completely isomorphic to the intersection of four operator spaces, as follows. First recall that, for any Hilbert space H, we denote by H_c (resp. H_r) the operator space obtained by equipping H with the o.s.s. of the column (resp. row) Hilbert space. Let us define $H_r[p] = (H_r, H_c)_{1/p}$ and

$H_c[p] = (H_c, H_r)_{1/p} \simeq H_r[p']$ for $1 < p < \infty$, with $p^{-1} + p'^{-1} = 1$. (Also set, by convention, $H_r[\infty] = H_r$, $H_c[\infty] = H_c$, $H_r[1] = H_c$, $H_c[1] = H_r$.)

Then (8.4.11) can be interpreted as saying that $\mathcal{R}_p^2$ is completely isomorphic to the intersection $S_p \cap S_p^{op} \cap (S_2)_c[p] \cap (S_2)_r[p]$ (the opposite E^{op} of an operator space E is defined e.g. in **[BP]**). Thus we can extend essentially all the preceding discussion of $\mathcal{R}_p$ to the spaces $\mathcal{R}_p^k$. In particular, here is what becomes of Theorem 8.4.8 in the case $k = 2$:

Let $\theta = 1/p$, $1 < p < \infty$ and let us denote $\mathcal{K}$ by S_∞. Then the interpolation space

$$((S_2)_r \cap (S_2)_c \cap S_\infty \cap S_\infty^{op}, (S_2)_r + (S_2)_c + S_1 + S_1^{op})_\theta$$

is completely isomorphic to the intersection

$$(S_2)_r[p] \cap (S_2)_c[p] \cap S_p \cap S_p^{op} \quad \text{if} \quad p \geq 2$$

and to the sum

$$(S_2)_r[p] + (S_2)_c[p] + S_p + S_p^{op} \quad \text{if} \quad p \leq 2.$$

In the case $p = 1$, the results of **[HP2]** show that $\mathcal{R}_1^2$ is completely isomorphic to the sum $(S_2)_r + (S_2)_c + S_1 + S_1^{op}$. The case of a general $k > 2$ can be handled similarly, and we obtain for $p \geq 2$ (resp. $p \leq 2$) the intersection (resp. the sum) of a family of 2^k operator spaces. We leave the details to the reader (see **[HP2]** for the cases $p = 1$ and $p = \infty$).

Remark. — By a well known symmetrization procedure, one can deduce from the Khintchine inequalities that, for any sequence $(Z_n)_{n \geq 1}$ of independent mean zero random variables in L_p ($1 \leq p < \infty$), we have (for any n)

$$\frac{1}{2} A_p \left\| \left(\sum_{i=1}^n |Z_i|^2 \right)^{1/2} \right\|_p \leq \left\| \sum_{i=1}^n Z_i \right\|_p \leq 2B_p \left\| \left(\sum_{i=1}^n |Z_i|^2 \right)^{1/2} \right\|_p .$$

Note that the partial sums $S_n = \sum_1^n Z_i$ form a very special class of martingales. These inequalities were extended to the case of general martingales by Burkholder, Davis and Gundy (see [Bu]).

For a non-commutative version of the Burkholder-Gundy inequalities, with an application to Clifford martingales and stochastic integrals, see **[PX1]**-**[PX2]**.

8.5. Non-commutative $\Lambda(p)$-sets

In this section, we will briefly describe some results from Asma Harcharras's recent thesis [Ha]. Let Γ be an arbitrary discrete group with unit element e. Let M be the von Neumann algebra generated by the left regular representation λ_Γ, equipped with its standard trace τ_Γ. We denote by $L_p(\tau_\Gamma)$ the associated (non-commutative) L_p-space. A linear map $T\colon\ L_p(\tau_\Gamma) \to L_p(\tau_\Gamma)$ is called a multiplier if there is a function $\varphi\colon\ \Gamma \to \mathbb{C}$ such that $T(\lambda_\Gamma(t)) = \varphi(t)\lambda_\Gamma(t)$ for all $t \in \Gamma$. In that case, we write $T = T_\varphi$. We say that $\varphi\colon\ \Gamma \to \mathbb{C}$ is a bounded (resp. c.b.) multiplier on $L_p(\tau_\Gamma)$ if

$T_\varphi\colon\ L_p(\tau_\Gamma) \to L_p(\tau_\Gamma)$ is bounded (resp. c.b.). Then in analogy with the commutative case (*cf.* Remarks 8.1.6 and 8.1.7), one can introduce the following definitions.

Definition 8.5.1. — Let $2 < p < \infty$. A subset $\Lambda \subset \Gamma$ of a discrete group Γ is called a $\Lambda(p)$-set (resp. $\Lambda(p)_{cb}$-set) if every bounded function $\varphi\colon\ \Lambda \to \mathbb{C}$ can be extended to a bounded (resp. c.b.) multiplier on $L_p(\tau_\Gamma)$.

Remark. — For more information on $\Lambda(p)$-sets, we refer the reader to **[Rud2]**, **[Bon1]**-**[Bon2]** and **[LoR]** in the Abelian case, and to **[Boz1]**-**[Boz3]** for the non-Abelian one.

Using the results of **[TJ2]**, it is easy to show that a subset $\Lambda \subset \Gamma$ is a $\Lambda(p)$-set iff there is a constant C such that, for any finitely supported function $x\colon\ \Lambda \to \mathbb{C}$, we have

$$\Big\|\sum_{t\in\Lambda} x(t)\lambda_\Gamma(t)\Big\|_{L_p(\tau_\Gamma)} \le C\Big(\sum_{t\in\Lambda} |x(t)|^2\Big)^{1/2}.$$

Note that conversely we have, for any Λ,

$$\Big\|\sum x(t)\lambda_\Gamma(t)\Big\|_{L_p(\tau_\Gamma)} \ge \Big(\sum |x(t)|^2\Big)^{1/2} = \Big\|\sum x(t)\lambda_\Gamma(t)\Big\|_{L_2(\tau_\Gamma)}.$$

In particular, this shows that, in the Abelian case, the preceding definition is equivalent to the one given in Remark 8.1.6.

To state the analogous fact for $\Lambda(p)_{cb}$-sets, it will be convenient to denote by $L_p(\tau_\Gamma \times \mathrm{tr})$ the (non-commutative) L_p-space associated to the product trace $\tau_\Gamma \times \mathrm{tr}$ on the (von Neumann) tensor product $M\overline{\otimes}B(\ell_2)$. This space can be equivalently identified with the space $S_p[L_p(\tau_\Gamma)]$, by (3.6)'.

Proposition 8.5.2. — *Let $2 < p < \infty$. A subset $\Lambda \subset \Gamma$ is a $\Lambda(p)_{cb}$-set iff there is a constant C such that, for any finitely supported function $x\colon\ \Lambda \to S_p$, we have*

$$\Big\|\sum_{t\in\Lambda} \lambda_\Gamma(t)\otimes x(t)\Big\|_{L_p(\tau_\Gamma\times\mathrm{tr})} \le C\max\left\{\Big\|\Big(\sum_{t\in\Lambda} x(t)^*x(t)\Big)^{1/2}\Big\|_{S_p},\ \Big\|\Big(\sum_{t\in\Lambda} x(t)x(t)^*\Big)^{1/2}\Big\|_{S_p}\right\}. \tag{8.5.1}$$

In other words, $\Lambda(p)_{cb}$-sets are exactly the subsets of Γ which satisfy the analog of (8.4.3).

Remark 8.5.3. — Note that the inverse of (8.5.1) holds for any set Λ with constant 1; in particular, taking $\Lambda = \Gamma$, we see that the indicator function of a $\Lambda(p)_{cb}$-set is a c.b. multiplier on $L_p(\tau_\Gamma)$. Equivalently, the orthogonal projection from $L_2(\tau_\Gamma)$ onto $\overline{\mathrm{span}}[\lambda_\Gamma(t) \mid t \in \Lambda]$ is c.b. on $L_p(\tau_\Gamma)$ $(1 < p < \infty)$.

Remark. — Let $L_p(\tau_\Gamma)_\Lambda$ be the closed span in $L_p(\tau_\Gamma)$ of $\{\lambda_\Gamma(t) \mid t \in \Lambda\}$. For simplicity of notation, assume Λ countable and let $\{t_n \mid n \in \mathbb{N}\}$ be an enumeration of Λ. Then Proposition 8.5.2 says that Λ is a $\Lambda(p)_{cb}$-set iff the correspondence $\lambda_{\mathbf{F}_\infty}(g_n) \to \lambda_\Gamma(t_n)$ extends to a complete isomorphism from the space E_p (described in Corollary 8.3.3) to $L_p(\tau_\Gamma)_\Lambda$. Equivalently, the same is true with $(\varepsilon_n)_{n\in\mathbb{N}}$ and $\mathcal{R}_p$ in the place of $(\lambda(g_n))$ and E_p.

Just as in the classical case of **[Rud2]**, it turns out that there are nice combinatorial conditions which imply the $\Lambda(p)_{cb}$-property, when $p = 2k$ is an even integer.

Definitions. — Let $k \geq 2$ be an integer. A subset $\Lambda \subset \Gamma$ is called a $B(k)$-set if whenever two k-tuples (s_i) and (t_i) in Λ satisfy

$$s_1 t_1^{-1} s_2 t_2^{-1} \dots s_k t_k^{-1} = e$$

we have necessarily $\{s_1, \dots, s_k\} = \{t_1, \dots, t_k\}$ with multiplicity (meaning that if an element is repeated on the left, it is repeated an equal number of times on the right). Let $\varepsilon(k) = (-1)^{k-1}$. A subset $\Lambda \subset \Gamma$ is called a $Z(k)$-set if there is a constant C such that, for any x in Γ, the number of possibilities to write $x = x_1 x_2^{-1} x_3 \cdots x_k^{\varepsilon(k)}$, with $x_i \in \Lambda$ and $x_i \neq x_j$ for all $i \neq j$, is bounded by C.

In the Abelian case, these notions are classical (especially for $k = 2$): see **[HR]**, p. 85 for a number theoretical discussion of $B(k)$-sets (called B_k-sequences in **[HR]**) and Zygmund's paper **[Z]** for applications to Fourier analysis.

Fix $k \geq 2$. It is easy to check that any infinite subset of Γ contains a further subset which is a $B(k)$-set. Rudin in **[Rud2]** exhibited rather large subsets of $\mathbb{Z}$ which are $B(k)$-sets. On the other hand, it is easy to check that if $\Gamma = \mathbb{Z}^{(\mathbb{N})}$, i.e. Γ is the free Abelian group with infinitely many generators denoted by $(Z_n)_{n\in\mathbb{N}}$, then the set $\Lambda = \{Z_n \mid n \in \mathbb{N}\}$ is a $B(k)$-set for all $k \geq 2$ (*cf.* **[Bon1]**-**[Bon2]**).
Furthermore, in a free (non-Abelian) group, any free subset is easily seen to be a $B(k)$-set for all $k \geq 2$.

It is easy to give examples of $\Lambda(p)$-sets which are not $\Lambda(p)_{cb}$-sets, for instance the proof of Proposition 8.1.3 shows that the set $\Lambda = \{3^i + 3^j \mid i, j \in \mathbb{N}\}$ is $\Lambda(p)$ for all $p > 2$, but $\Lambda(p)_{cb}$ for *no* $p > 2$. However, the following result holds. It is the main source of examples of $\Lambda(p)_{cb}$-sets.

Theorem 8.5.4. — *(***[Ha]***). Let $p = 2k$ be an even integer > 2. Then, any $B(k)$-set is a $Z(k)$-set, and any $Z(k)$-set is a $\Lambda(p)_{cb}$-set.*

We refer the interested reader to [Ha] for the proof, and for related results. The computations, which are fairly easy for $p = 4$, become increasingly complicated as k grows.

Remark. — The sets Λ which satisfy the property in Proposition 8.5.2 for $p = \infty$ were studied in **[P8]**, under the name of "L-sets".

8.6. Semi-circular systems in Voiculescu's free probability theory

In this section, we describe the operator space generated in (non-commutative) L_p by the "free" analog of independent Gaussian variables, namely a free semicircular family.

In his recent and very beautiful theory of "free probability", Voiculescu discovered a "free" analog of Gaussian random variables (see **[VDN]**). This discovery gives a new insight into a remarkable limit theorem for random matrices, due to Wigner (1955). In Wigner's result, a particular probability distribution plays a crucial rôle, namely the probability measure on $\mathbb{R}$ (actually supported by $[-2,2]$) defined as follows.

$$\mu_W(dt) = 1_{[-2,2]}\sqrt{4-t^2}\ dt/2\pi.$$

We will call it the standard Wigner distribution. We have

$$\int t\mu_W(dt) = 0 \qquad \int t^2\mu_W(dt) = 1.$$

In classical probability theory, Gaussian random variables play a prominent rôle. They usually can be discussed in the framework attached to a family $(\gamma_i)_{i\in I}$ (resp. $(\widetilde{\gamma}_i)_{i\in I}$) of independent identically distributed (i.d.d. in short) real (resp. complex) valued Gaussian variables with mean zero and L_2-norm equal to 1. When (say) $I = \{1,2,\ldots,n\}$ the distribution of $(\gamma_i)_{i\in I}$ (resp. $(\widetilde{\gamma}_i)_{i\in I}$) is invariant under the orthogonal (resp. unitary) group $O(n)$ (resp. $U(n)$).

In Voiculescu's theory, stochastic independence of random variables is replaced by freeness of C^*-random variables. We will review the basic definitions below. After that, we will introduce a free family $(W_i)_{i\in I}$ of C^*-random variables, each distributed according to the standard Wigner distribution. These are called "free semi-circular" variables. The family $(W_i)_{i\in I}$ is the free analog of $(\gamma_i)_{i\in I}$ in classical probability; it satisfies a similar distributional invariance under the orthogonal group. But actually, since we work mostly with *complex* coefficients, we will also introduce a free family $(\widetilde{W}_i)_{i\in I}$ which is the free analog of $(\widetilde{\gamma}_i)_{i\in I}$; their "joint distribution" satisfies an analogous unitary invariance. Such variables are called "free circular" variables.

We now start reviewing the precise definitions of the basic concepts of "free probability", following **[VDN]**.

Definitions. — A C^*-probability space is a unital C^*-algebra A equipped with a state φ (a state is a positive linear form of norm 1). We will say that an element x of A is a C^*-random variable (in short C^*-r.v.). If x is self-adjoint, we will say that it is a real C^*-r.v. By definition, the distribution of a real C^*-r.v. x is the probability measure μ_x on $\mathbb{R}$ such that

$$\forall\, k \geq 0 \quad \varphi(x^k) = \int t^k\mu_x(dt).$$

It follows that, for any continuous function $f\colon\ \mathbb{R}\to\mathbb{R}$, we have

$$\varphi(f(x)) = \int f(t)\mu_x(dt), \tag{8.6.1}$$

Indeed, we can approximate f by a sequence of polynomials uniformly on every compact subset. Hence, in particular for all $0 < p < \infty$

$$\varphi(|x|^p) = \int |t|^p \mu_x(dt). \tag{8.6.2}$$

Moreover, if φ is "faithful" on the C^*-algebra A_x generated by x (meaning that $\varphi(y) = 0$ for $y \geq 0$ implies $y = 0$) then the support of μ_x is exactly the spectrum of x, denoted by $\sigma(x)$. Therefore, we can record here the following fact:

(8.6.3) *Let (A, φ) and (B, ψ) be two C^*-probability spaces with φ and ψ faithful. Let $x \in A$ and $y \in B$ be two real C^*-r.v. with the same distribution, i.e. such that $\mu_x = \mu_y$. Then we necessarily have $||x|| = ||y||$ (where $||x||$ is the norm in A and $||y||$ the norm in B).*

This property is immediate, since

$$||x|| = \sup\{|\lambda| \mid \lambda \in \sigma(x)\}.$$

It can also be obtained by letting p tend to infinity in (8.6.2). Note that it suffices that φ (resp. ψ) be faithful on the C^*-algebra generated by x (resp. y).
A probabilist will legitimately object that this theory is restricted to *bounded* variables and that the usual probability distributions (Gaussian, Poisson,...) have unbounded support. But, by a truncation, one can easily extend this viewpoint to the unbounded real case. Besides, it turns out that the free analog of Gaussian variables happens to be bounded (see below), although it is not so for the "free" stable distributions (see [**BV**]).

Example. — Let $\omega \to a(\omega) \in M_n$ be a random $n \times n$-matrix defined on a standard probability space $(\Omega, \mathcal{A}, P)$. Then the space $A = L_\infty(\Omega, \mathcal{A}, P; M_n)$ can be viewed as a C^*-probability space once we equip it with the state φ defined by

$$\varphi(a) = \int \frac{1}{n} \operatorname{tr}(a(\omega)) dP(\omega).$$

Assume moreover that $a(\omega) = a(\omega)^*$ almost surely. Let $(\lambda_1(\omega), \ldots, \lambda_n(\omega))$ be the eigenvalues of the matrix $a(\omega)$. Then the distribution μ_a of the real C^*-r.v. a is nothing but

$$\mu_a = \int \frac{1}{n} \sum_1^n \delta_{\lambda_i(\omega)} dP(\omega).$$

Definitions. — Let (A_n, φ_n) be a sequence of C^*-probability spaces and let $x_n \in A_n$ be a sequence of real C^*-r.v.'s. We will say that x_n tends to x in distribution if the distributions μ_{x_n} tend weakly to μ_x. By a classical criterion, this is equivalent to

$$\forall\, k \geq 0 \quad \varphi_n(x_n^k) \to \varphi(x^k) \quad \text{when } n \to \infty.$$

More generally, we can define the joint distribution of a family $x = (x_i)_{i\in I}$ of real C^*-r.v.'s, but it is no longer a measure: we consider the set $\mathcal{P}(I)$ of all polynomials in a family of non-commuting variables $(X_i)_{i\in I}$. First we define

$$F(X_{i_1}X_{i_2}\dots X_{i_k}) = \varphi(x_{i_1}x_{i_2}\dots x_{i_k}),$$

then we extend F linearly to a linear form on $\mathcal{P}(I)$. We will say that F is the "joint distribution" of the family $x = (x_i)_{i\in I}$. If we give ourselves for each n such a family $(x_i^n)_{i\in I}$ with distribution F^n, we say that $(x_i^n)_{i\in I}$ converges in distribution to $(x_i)_{i\in I}$ if F^n converges pointwise to F.

Let (A,φ) be a C^*-probability space and let $(A_i)_{i\in I}$ be a family of subalgebras of A. We say that $(A_i)_{i\in I}$ is free if $\varphi(a_1a_2\dots a_n) = 0$ every time we have $a_j \in A_{i_j}, i_1 \neq i_2 \neq \cdots \neq i_n$ and $\varphi(a_j) = 0\ \forall j$.

Let $(x_i)_{i\in I}$ be a family of C^*-r.v.'s in A. Let A_i be the unital algebra (resp. C^*-algebra) generated by x_i inside A. We say that the family $(x_i)_{i\in I}$ is free (resp. $*$-free) if $(A_i)_{i\in I}$ is free.

We can now reformulate Wigner's Theorem in Voiculescu's language. Fix $n \geq 1$. We introduce the random (real symmetric) $n\times n$-matrix

$$G^n = (g_{ij})_{1\leq i,j\leq n}$$

with entries defined as follows: $\{g_{ij} \mid i \leq j\}$ is a collection of independent Gaussian real-valued r.v.'s with distribution $N(0,1/n)$ (*i.e.* $E(g_{ij}) = 0$ and $E|g_{ij}|^2 = 1/n$) and $g_{ij} = g_{ji}\ \forall i > j$. We assume these (classical sense) random variables defined on a sufficiently rich probability space (for instance the Lebesgue interval). Let $A_n = L_\infty(\Omega,\mathcal{A},P;M_n)$ and let φ_n be the state defined on A_n by setting

$$\forall\, x \in A_n \quad \varphi_n(x) = \int \frac{1}{n}\operatorname{tr}(x(\omega))dP(\omega).$$

Then, Voiculescu's reformulation of Wigner's Theorem is:

Theorem 8.6.1. — *If we consider G^n as a real C^*-r.v. relative to (A_n,φ_n), then we have the weak convergence of probability measures:*

$$\mu_{G_n} \to \mu_W \quad \text{when } n \to \infty.$$

More generally, Voiculescu showed that if $(G_i^n)_{i\in I}$ is a family of independent copies (in the usual sense) of the random variable G^n, then the family $(G_i^n)_{i\in I}$ converges in distribution to a free family $(W_i)_{i\in I}$ of real C^*-r.v.'s each with the same distribution equal to μ_W.

We will say that a real C^*-r.v. x is semi-circular if there exists $\lambda > 0$ such that the distribution of λx is equal to μ_W. If $\lambda = 1$, if x admits exactly μ_W for its distribution, then we will say that x is semi-circular standard. We have then $\varphi(x) = 0,\ \varphi(x^2) = 1$. (We should warn the reader that our standard normalization differs from that of **[VDN]**.)

In Voiculescu's theory, the analog of an *independent* family of standard real Gaussian variables is a *free* family of standard semi-circular C^*-r.v.'s. Such a family can be realized on the "full" Fock space, as follows. Let $H = \ell_2(I)$. We denote by $\mathcal{F}(H)$(or simply by $\mathcal{F}$) the "full" Fock space associated to H, that is to say we set $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_n = H^{\otimes n}$ (Hilbertian tensor product) and finally

$$\mathcal{F} = \oplus_{n \geq 0} \mathcal{H}_n.$$

We consider from now on $\mathcal{H}_n$ as a subspace of $\mathcal{F}$. For every $h \in H$, we denote by $\ell(h)\colon\ \mathcal{F} \to \mathcal{F}$ the operator defined by:

$$\ell(h)x = h \otimes x.$$

More precisely, if $x = \lambda 1 \in \mathcal{H}_0 = \mathbb{C}1$, we have $\ell(h)x = \lambda h$ and if $x = x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \mathcal{H}_n$ we have $\ell(h)x = h \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n$. We will denote by Ω the unit element in $\mathcal{H}_0 = \mathbb{C}1$. The C^*-algebra $B(\mathcal{F})$ is equipped with the state φ defined by

$$\varphi(T) = < T\Omega, \Omega > .$$

Let $(e_i)_{i \in I}$ be an orthonormal basis of H.

The pair $(B(\mathcal{F}), \varphi)$ is an example of a C^*-probability space. Moreover, φ is tracial on the C^*-algebra generated by the operators $\ell(e_i) + \ell(e_i)^*$ $(i \in I)$,*i.e.* we have $\varphi(xy) = \varphi(yx)$ for all x, y in this subalgebra. (Note that $\varphi(\ell(h)^*\ell(h)) = \langle h, h \rangle$ and $\varphi(\ell(h)\ell(h)^*) = 0$, so that φ is not tracial on the whole of $B(\mathcal{F})$.)

In this subalgebra, let

$$W_i = \ell(e_i) + \ell(e_i)^*.$$

Then the family $(W_i)_{i \in I}$ is an example of a free family of standard semi-circular C^*-r.v.'s, or in short a standard semi-circular free family. This family enjoys properties very much analogous to those of a standard independent Gaussian family $(g_i)_{i \in I}$. Indeed, for every family $(\alpha_i)_{i \in I} \in \mathbb{R}^{(I)}$ with $\sum \alpha_i^2 = 1$ the real C^*-r.v. $S = \sum_{i \in I} \alpha_i W_i$ admits μ_W as its distribution. This is analogous to the rotational invariance of the usual Gaussian distributions. More explicitly, this means that for every continuous function $f\colon\ \mathbb{R} \to \mathbb{R}$ we have

$$\varphi(f(S)) = \int f(t) \mu_W(dt).$$

In particular, by (8.6.3), for all finitely supported families of real scalars we have

$$\left\| \sum_{i \in I} \alpha_i W_i \right\| = 2 \left(\sum_{i \in I} \alpha_i^2 \right)^{1/2} .$$

Thus, the operator space $\mathbb{R}$-linearly generated by $(W_i)_{i \in I}$ is isometric to a real Hilbert space.

We now pass to the complex case. Let $(Z_i)_{i \in I}$ be a family of (not necessarily self-adjoint) C^*-r.v.'s. We can then consider the distribution F of the family of real C^*-r.v.'s obtained by forming the disjoint union of the family of real parts and that of imaginary parts of $(Z_i)_{i \in I}$. We will say that F is the joint $*$-distribution of the family $(Z_i)_{i \in I}$. Of course if the family is reduced to one variable Z, we will say that

F is the $*$-distribution of Z. Note that the data of the $*$-distribution of $(Z_i)_{i\in I}$ is equivalent to that of all possible moments of the form

$$\varphi(X_{i_1}X_{i_2}\dots X_{i_n})$$

where $X_i =$ either Z_i or Z_i^* and where $i_1, i_2, \dots, i_n$ are arbitrary in I.

We now come to the analog of complex Gaussian random variables. Let (W', W'') be a standard semi-circular free family (with two elements). We set $\widetilde{W} = \dfrac{1}{\sqrt{2}}(W'+iW'')$.

Every C^*-r.v. having the same $*$-distribution as $\widetilde{W}$ (resp. as $\lambda\widetilde{W}$ for some $\lambda > 0$) will be called "standard circular" (resp. "circular").

Suppose given a (partitioned) orthonormal basis $\{e_i \mid i \in I\} \cup \{f_i \mid i \in I\}$ of H. Then, one can show that $\widetilde{W}_i = \ell(e_i) + \ell(f_i)^*$ is a $*$-free family of standard circular C^*-r.v.'s (in short a standard circular $*$-free family).

Now, let $(\widetilde{W}_i)_{i\in I}$ be any $*$-free family formed of standard circular variables. Then, for any finitely supported family $(\alpha_i)_{i\in I}$ of complex scalars with $\sum |\alpha_i|^2 = 1$, the variable $\widetilde{S} = \sum_{i\in I} \alpha_i \widetilde{W}_i$ has the same $*$-distribution as $\widetilde{W}$. As above, we have

$$\left\|\sum \alpha_i \widetilde{W}_i\right\| = 2\left(\sum |\alpha_i|^2\right)^{1/2} \tag{8.6.4}$$

(one can verify that $\|\widetilde{W}\| = 2$).

Let $\mathcal{V}_I$ be the operator space spanned by this family $\{\widetilde{W}_i \mid i \in I\}$. By (8.6.4), $\mathcal{V}_I$ is isometrically Hilbertian and $(\widetilde{W}_i)_{i\in I}$ is an orthonormal basis. Moreover (see **[VDN]**, p. 56) for any isometric transformation $U\colon \mathcal{V}_I \to \mathcal{V}_I$ the family $(U(\widetilde{W}_i))_{i\in I}$ has the same $*$-distribution as $(\widetilde{W}_i)_{i\in I}$.

Lemma 8.6.2. — *Let (A,φ) and (B,ψ) be two C^*-probability spaces with φ and ψ faithful. Let $(Z_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be two families of C^*-r.v.'s in A and in B respectively, admitting the same joint $*$-distribution. Let A_Z (resp. B_Y) be the C^*-algebra generated by $(Z_i)_{i\in I}$ (resp. $(Y_i)_{i\in I}$) and let $E_Z \subset A_Z$ (resp. $E_Y \subset B_Y$) be the operator space spanned by the families. Then the linear mapping U defined by $U(Z_i) = Y_i$ extends to a complete isometry from E_Z onto E_Y and actually to an isometric C^*-representation from A_Z onto B_Y.*

Proof. — Without restricting the generality, we may replace the family $(Z_i)_{i\in I}$ by the disjoint union of the families$(Z_i)_{i\in I}$ and $(Z_i^*)_{i\in I}$, and similarly for the family $(Y_i)_{i\in I}$. Let then $P = \sum \alpha_{i_1 i_2 \cdots k} Z_{i_1} \cdots Z_{i_k}$ be a polynomial with complex coefficients in the non-commutative variables $(Z_i)_{i\in I}$. We set

$$\pi(P) = \sum \alpha_{i_1 i_2 \cdots k} Y_{i_1} \cdots Y_{i_k}$$

Then, as (Z_i) and (Y_i) have the same joint $*$-distribution, P^*P and $\pi(P)^*\pi(P)$ have the same distribution, hence, by (8.6.3), since φ and ψ are faithful, $\|P\| = \|\pi(P)\|$. In particular, π extends to an isometric C^*-representation from A_Z onto B_Y. As is well known, the latter is automatically completely isometric. A fortiori, the restriction U of π to E_Z is completely isometric. □

Actually, it is very easy to identify the operator space $\mathcal{V}_I$ (up to complete isomorphism) as the next result shows (see **[HP2]** for some refinements).

Theorem 8.6.3. — *The operator space $\mathcal{V}_I$ generated by a standard circular $*$-free family $(\widetilde{W}_i)_{i\in I}$ is isometric to $\ell_2(I)$. Similarly, the closed span of a free semi-circular family $\{(W_i)_{i\in I}\}$ is 2-isomorphic to $\ell_2(I)$. Moreover, if (say) $I = \mathbb{N}$, each of these spaces is completely isomorphic to $R \cap C$ or equivalently to E_λ.*

Proof. — We already saw that $\mathcal{V}_I$ is Hilbertian.

Let $(a_i)_{i\in I}$ be a finitely supported family in $B(H)$. The identity $W_i = \ell(e_i) + \ell(e_i)^*$ together with $\sum \ell(e_i)\ell(e_i)^* \le I$ yields

$$\begin{aligned}\left\|\sum a_i \otimes W_i\right\|_{\min} &\le \left\|\sum a_i \otimes \ell(e_i)\right\|_{\min} + \left\|\sum a_i \otimes \ell(e_i)^*\right\|_{\min}\\ &\le \left\|\sum a_i^* a_i\right\|^{1/2} + \left\|\sum a_i a_i^*\right\|^{1/2}\end{aligned}$$

whence

$$\left\|\sum a_i \otimes W_i\right\|_{\min} \le 2\left\|\sum a_i \otimes \delta_i\right\|_{\min}. \tag{8.6.5}$$

Conversely, it is easy to check that $\varphi(W_i^* W_j) = \varphi(W_j W_i^*) = 0$ if $i \ne j$ and $= 1$ otherwise. Hence, letting $T = \sum a_i \otimes W_i$, we have $\|\sum a_i^* a_i\| = \|(I \otimes \varphi)(T^*T)\| \le \|T\|^2_{\min}$, and similarly we have $\|\sum a_i a_i^*\| \le \|T\|^2_{\min}$. It follows that

$$\max\left\{\left\|\sum a_i^* a_i\right\|^{1/2}, \left\|\sum a_i a_i^*\right\|^{1/2}\right\} \le \left\|\sum a_i \otimes W_i\right\|_{\min}. \tag{8.6.6}$$

The inequalities (8.6.5) and (8.6.6) imply that $\overline{\mathrm{span}}[W_i \mid i \in I]$ is 2-isomorphic to $\ell_2(I)$.

For simplicity, we assume $I = \mathbb{N}$ in the rest of the proof. By Theorem 8.3.1, the last two inequalities imply that the closed span of $(W_i)_{i\in I}$ is completely isomorphic to E_λ or equivalently to $R \cap C$. Finally, as the variables $\widetilde{W}_j = (W_j' + iW_j'')2^{-1/2}$ appear as a sequence of "blocks" (normalized in ℓ_2) on a standard semi-circular system, the same inequalities (8.6.5) and (8.6.6) remain valid if we replace $(W_i)_{i\in I}$ by $(\widetilde{W}_i)_{i\in I}$. Therefore, we conclude that $\mathcal{V}_I$ itself is completely isomorphic to E_λ or to $R \cap C$. This last point can also be deduced from the concrete realization $\widetilde{W}_i = \ell(e_i) + \ell(f_i)^*$ already mentioned above for a standard circular $*$-free system. □

Remark. — In a recent preprint (Computing norms of free operators with matrix coefficients), Franz Lehner has given an explicit exact formula for the left side of 8.6.5 and several related *equalities* refining Theorem 8.3.1.

Remark 8.6.4. — Let M be the von Neumann algebra generated by a free semi-circular family $(W_i)_{i\in I}$. We assume $I = \mathbb{N}$ for simplicity. Recall a classical notation: for any x in M, we define $x\varphi \in M_*$ by $x\varphi(y) = \varphi(yx)$ for all y in M. Thus we obtain a continuous injection $M \to M_*$ which allows us to consider the interpolation spaces $(M, M_*)_\theta$ for $0 < \theta < 1$. Let us denote for simplicity $L_\infty(\varphi) = M$, $L_1(\varphi) = M_*$ and $L_p(\varphi) = (M, M_*)_\theta$ with $\theta = 1/p$.

Let us denote by $\mathcal{W}_p$ the closed linear span of $(W_i)_{i\in I}$ in $L_p(\varphi)$. In analogy with Corollary 8.3.3, we claim that the orthogonal projection $\mathcal{P}$ from $L_2(\varphi)$ onto $\mathcal{W}_2$ defines a completely bounded projection from $L_p(\varphi)$ onto $\mathcal{W}_p$ for any $1 \le p \le \infty$. Here again, the case $p = 2$ is clear since, by (0.4), $L_2(\varphi)$ is $OH(I)$ for some set I. Therefore, by interpolation and transposition, it suffices to prove this claim for $p = \infty$. The latter case can be justified as follows: given a Hilbert space H, let us denote by H_r (resp. H_c) the space H equipped with the row (resp. column) operator space structure associated to $B(H^*, \mathbb{C})$ (resp. $B(\mathbb{C}, H)$). It is easy to check that the natural inclusion map $M \to L_2(\varphi)$ is completely contractive from M to $L_2(\varphi)_r$ (resp. $L_2(\varphi)_c$). Hence, (recall that W_i is normalized in $L_2(\varphi)$) $\mathcal{P}$ induces a completely contractive mapping $T\colon\ M \to R \cap C$ defined by

$$\forall x \in M \qquad T(x) = \sum_i \delta_i\ \varphi(xW_i^*).$$

Let $V\colon\ R \cap C \to M$ be the mapping defined by $V(\delta_i) = W_i$. By (8.6.5), the composition $VT\colon\ M \to M$ satisfies $\|VT\|_{cb} \le \|V\|_{cb} \le 2$. Moreover, VT is the adjoint of an operator on M_* and VT "coincides" with $\mathcal{P}$ on the $*$-algebra generated by $(W_i)_{i\in I}$. Therefore, VT is a completely bounded projection from M onto $\mathcal{W}_\infty$, which naturally "extends" $\mathcal{P}$. By transposition, we obtain a *c.b.* projection from $L_1(\varphi)$ onto $\mathcal{W}_1$ and by interpolation from $L_p(\varphi)$ onto $\mathcal{W}_p$ for all $1 \le p \le \infty$. This establishes the above claim.

Therefore, exactly as in Corollary 8.3.3, we conclude that $\mathcal{W}_1 \simeq \mathcal{W}_\infty^*$ and $\mathcal{W}_p \simeq (\mathcal{W}_\infty, \mathcal{W}_1)_\theta$ (completely isomorphically) with $\theta = 1/p$. But by Theorem 8.6.3 we already know that $\mathcal{W}_\infty \simeq R \cap C$, hence by duality $\mathcal{W}_1 \simeq R + C$ and consequently, by Theorem 8.4.8, we obtain again (with equivalence constants bounded independently of p)

$$\mathcal{W}_p \simeq R[p] + C[p] \quad \text{if} \quad p \le 2 \quad \text{and} \quad \mathcal{W}_p \simeq R[p] \cap C[p] \quad \text{if} \quad p \ge 2.$$

Moreover, Theorem 8.4.10 remains valid with (W_i) in the place of $(\lambda(g_i))$. The case of circular variables can be treated by the same argument, thus, to recapitulate, we can state

Theorem 8.6.5. — *For simplicity let $I = \mathbb{N}$. Let $(W_i)_{i\in I}$ (resp. $(\widetilde{W}_i)_{i\in I}$) be a standard free semi-circular (resp. $*$-free circular) family. For $1 \le p \le \infty$, let $\mathcal{W}_p$ (resp. $\widetilde{\mathcal{W}}_p$) be the closed span of $(W_i)_{i\in I}$ (resp. $(\widetilde{W}_i)_{i\in I}$) in $L_p(\varphi)$. Then, for any $p < \infty$, $\mathcal{W}_p$ and $\widetilde{\mathcal{W}}_p$ are completely isomorphic to the Gaussian subspace $\mathcal{G}_p$ (or to the space $\mathcal{R}_p$) considered in §8.4. The correspondences $W_i \to \gamma_i$, $\widetilde{W}_i \to \widetilde{\gamma}_i$ (and also $W_i \to \widetilde{W}_i$) or $W_i \to \varepsilon_i$ all define complete isomorphisms between the corresponding L_p-subspaces. Moreover, the orthogonal projection defines a c.b. projection from $L_p(\varphi)$ onto $\mathcal{W}_p$ (or onto $\widetilde{\mathcal{W}}_p$) for any $1 \le p \le \infty$, with cb-norm bounded independently of $1 \le p \le \infty$.*

We refer to **[VDN]** for a description of the various forms of Voiculescu's central limit theorem which is a generalization of Theorem 8.6.1. On the other hand, the reader will find in **[Sk]** a description of the applications of Voiculescu's theory to von Neumann algebras.

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