Astérisque

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Astérisque, tome 236 (1996), p. 227-247
<http://www.numdam.org/item?id=AST_1996__236__227_0>

Maassen Kernels and Self-Similar Quantum Fields

K.R. Parthasarathy

Abstract. — In his Lecture Notes [Maj] P. Major has outlined a theory of multiple Wiener-Itô integrals with respect to a stationary Gaussian random field ξ over the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing smooth functions in $\mathbb{R}^d$. Furthermore, he has exploited the same to construct self-similar random fields subordinate to ξ. Here, we observe that the Hilbert space of functions square integrable with respect to the probability measure P of ξ can be identified in a natural way with the Hilbert space of functions square integrable with respect to the symmetric Guichardet measure [Gui] constructed from the spectrum of ξ. Under such an identification, multiplication of random variables on the probability space of ξ becomes the twisted convolution of Lindsay and Maassen [Li M 1,2] for Maassen kernels [Maa], [Mey]. The multiple Wiener-Itô integral of Major is described neatly by a twisted version of Meyer's multiplication formula (see (IV.4.1 in [Mey]). Following Lindsay and Parthasarathy [Li P] we introduce the weighted and twisted convolution of Maassen kernels, present a generalization of Meyer's formula and exploit it to construct a family of operator fields whose expectations in the vacuum state exhibit a simultaneous self-similarity property. Such a construction includes Major's examples and at the same time yields a self-similar Clifford field.

1 An involutive Gaussian random field and the Lindsay-Maassen twisted convolution algebra

Let $(X, \mathcal{F}, m)$ be a σ-finite measure space equipped with an m-preserving involution $x \to \tilde{x}$ on X satisfying $(\tilde{x})^{\sim} \equiv x$. For any measure μ, denote by $L^2_{\mathbb{R}}(\mu)$ and $L^2(\mu)$ respectively the real and complex Hilbert spaces of functions square integrable with respect to μ. Then the following holds:

Theorem 1.1 There exists a probability space $(\Omega, \mathcal{F}_m, P_m)$ and a linear map $\xi : L^2_{\mathbb{R}}(m) \to L^2(P_m)$ satisfying the following:

(a) For each $f \in L^2_{\mathbb{R}}(m)$, $\xi(f)$ is a complex-valued Gaussian random variable of mean 0.

(b) For any $f, g \in L^2_{\mathbb{R}}(m)$,

$$\mathbb{E}\,\overline{\xi(f)}\xi(g) = \int f(x)g(x)dm(x).$$

(c) If $\tilde{f}(x) \equiv f(\tilde{x})$ and $f \in L^2_{\mathbb{R}}(m)$ then $\xi(\tilde{f}) = \overline{\xi(f)}$.

(d) The σ algebra generated by $\{\xi(f), f \in L^2_{\mathbb{R}}(m)\}$ is $\mathcal{F}_m$.

Proof: For any $f, g \in L^2_{\mathbb{R}}(m)$ define

$$K_{\pm}(f,g) = \int \frac{1}{2}(f(x) \pm f(\tilde{x}))g(x)dm(x). \tag{1.1}$$

From the $\sim$ - invariance of m and Schwarz's inequality we have $K_{\pm}(f,g) = K_{\pm}(g,f)$,

$$|\int f(\tilde{x})f(x)dm(x)| \leq \int f^2(x)dm(x)$$

and therefore

$$K_{\pm}(f,f) = \frac{1}{2}\int (f^2(x) \pm f(\tilde{x})f(x))dm(x) \geq 0.$$

In other words K_+ and K_- are non-negative definite bilinear forms on $L^2_{\mathbb{R}}(m)$ with non-trivial kernel (consisting of odd functions for K_+ and even functions for K_-). Hence there exist two independent real Gaussian random fields ξ_+ and ξ_- over $L^2_{\mathbb{R}}(m)$ on some probability space $(\Omega, \mathcal{F}_m, P_m)$ for which

$$\mathbb{E}\xi_{\pm}(f) = 0, \quad \mathbb{E}\xi_+(f)\xi_+(g) = K_+(f,g), \mathbb{E}\xi_-(f)\xi_-(g) = K_-(f,g) \tag{1.2}$$

and $\mathcal{F}_m$ is generated by $\{\xi_+(f), \xi_-(f), f \in L^2_{\mathbb{R}}(m)\}$. Elementary algebra using (1.1), (1.2) and $\sim$-invariance of m yields

$$\mathbb{E}(\xi_+(f) - \xi_+(\tilde{f}))^2 = \mathbb{E}(\xi_-(f) + \xi_-(\tilde{f}))^2 = 0 \tag{1.3}$$

where $\tilde{f}(x) = f(\tilde{x})$. Define

$$\xi(f) = \xi_+(f) + i\xi_-(f).$$

Clearly, ξ is a linear map satisfying (a) and (c). Furthermore

$$\mathbb{E}\,\overline{\xi(f)}\,\xi(g) = K_+(f,g) + K_-(f,g) = \int f(x)g(x)dm(x)$$

proving (b). Property (d) is immediate. ■

Corollary 1.2 Let $\{\xi(f), f \in L^2_{\mathbb{R}}(m)\}$ be as in Theorem 1.1. For any f in the complex Hilbert space $L^2(m)$ with $f = f_1 + if_2$, where f_1 and f_2 are respectively the real and imaginary parts of f, let $\xi(f) = \xi(f_1) + i\xi(f_2)$. Then $\{\xi(f), f \in L^2(m)\}$ satisfies the following:

(a) The correspondence $f \to \xi(f)$ is complex linear.

(b) For each $f, \xi(f)$ is a complex-valued Gaussian random variable of mean 0.

(c) $\mathbb{E}\,\overline{\xi(f)}\,\xi(g) = \int \overline{f}(x)g(x)dm(x)$.

(d) If $\tilde{f}(x) \equiv \overline{f(\tilde{x})}$, then $\xi(\tilde{f}) = \overline{\xi(f)}$.

(e) $\mathbb{E}e^{\xi(f)} = \exp\, \frac{1}{2} \int f(\tilde{x})f(x)dm(x)$.

Proof: The first four parts (a) - (d) are immediate from Theorem 1.1. The last part follows from the $\sim$-invariance of m and the relation

$$\xi(f) = \xi_+(f_1) + i\xi_+(f_2) + i(\xi_-(f_1) + i\xi_-(f_2))$$

where ξ_+ and ξ_- are the independent real Gaussian random fields over $L^2_{\mathbb{R}}(m)$ with respective covariance kernels K_+ and K_- in the proof of Theorem 1.1. ∎

Remark 1.3 In Corollary 1.2 define the normalised exponential random variable $e_\xi(f)$ by

$$e_\xi(f) = \exp(\xi(f) - \frac{1}{2}\int f(\tilde{x})f(x)dm(x)) \tag{1.4}$$

for $f \in L^2(m)$. Then $\{e_\xi(f), f \in L^2(m)\}$ is a linearly independent and total set in $L^2(P_m)$. Furthermore

$$\mathbb{E}\,\overline{e_\xi(f)}\,e_\xi(g) = \exp\int \overline{f(x)}\,g(x)dm(x), \tag{1.5}$$

$$e_\xi(f)e_\xi(g) = e_\xi(f+g)\exp\int f(\tilde{x})g(x)dm(x) \tag{1.6}$$

for all $f, g \in L^2(m)$.

We shall denote by $\mathcal{E}_\xi \subset L^2(P_m)$ the dense linear manifold generated by $\{e_\xi(f), f \in L^2(m)\}$. Then (1.6) implies that $\mathcal{E}_\xi$ is an algebra of random variables on $(\Omega, \mathcal{F}_m, P_m)$. Owing to property (d) in Corollary 1.2 we may call ξ an *involutive Gaussian random field.*

From now on we assume that $(X, \mathcal{F}, m)$ is a separable, nonatomic and σ-finite measure space. Our aim is to identify $L^2(P_m)$ in Theorem 1.1 with $L^2(m_\Gamma)$ where m_Γ is the symmetric measure of Guichardet [Gui] in the space $\Gamma(X)$ of all finite subsets of X, constructed from m. We denote the Guichardet symmetric measure space by $(\Gamma(X), \mathcal{F}_\Gamma, m_\Gamma)$ so that integration with respect to m_Γ is determined by

$$\int_{\Gamma(x)} f(\sigma)dm_\Gamma(\sigma) = f(\emptyset) + \sum_{n=1}^{\infty}\frac{1}{n!}\int f(\{x_1, x_2, \cdots, x_n\})m(dx_1)\cdots m(dx_n) \tag{1.7}$$

for any $f \in L^1(m_\Gamma)$ where, on the right hand side, $f(\{x_1, x_2, \cdots, x_n\})$ is viewed as a symmetric measurable function of n variables $x_1, x_2, ..., x_n$ with all the $x_i'^s$ distinct. It is to be noted that the n-fold product of the nonatomic measure m has its support in the subset $\{(x_1, x_2, ..., x_n) : x_i \in X \text{ and } x_i \neq x_j \text{ if } i \neq j\}$. Denote by $\Gamma^n(X)$ the n-fold cartesian product of $\Gamma(X)$ and by $\Gamma^{(n)}(X) \subset \Gamma^n(X)$ the subset

$$\{\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_n) | \sigma_i \in \Gamma(X), \sigma_i \cap \sigma_j = \emptyset \text{ if } i \neq j\}.$$

Then the product measure m_Γ^n satisfies $m_\Gamma^n(\Gamma^n(X) \backslash \Gamma^{(n)}(X)) = 0$. For simplicity we write $d\sigma = dm_\Gamma(\sigma)$ in $\Gamma(X)$. If $\sigma_1, \sigma_2, ..., \sigma_n$ are disjoint elements of $\Gamma(X)$ we write $\sigma_1 + \sigma_2 + \cdots + \sigma_n$ or $\sum_{i=1}^n \sigma_i$ to denote $\bigcup_{i=1}^n \sigma_i$. Then one has the following Maassen's sum-integral formula for $f \in L^1(m_\Gamma^n)$:

$$\int_{\Gamma^{(n)}(X)} f(\sigma_1, \sigma_2, ..., \sigma_n) d\sigma_1 d\sigma_2 ... d\sigma_n = \int_{\Gamma(X)} \{ \sum_{\sigma_1 + \cdots + \sigma_n = \sigma} f(\sigma_1, ..., \sigma_n)\} d\sigma. \quad (1.8)$$

For a proof see [Mey], [Li P]. Following [Maa] we introduce the space $\mathcal{K}(X) = \mathcal{K}(X, m, \sim) \subset L^2(m_\Gamma)$ of Massen kernels:

$$\mathcal{K}(X) = \{f | \int a^{\#\sigma} |f(\sigma)|^2 d\sigma < \infty \ \forall \ a > 1\}. \quad (1.9)$$

The Lindsay-Maassen twisted convolution $f * g$ between any two Maassen kernels f and g is defined by

$$(f * g)(\sigma) = \sum_{\sigma_1 + \sigma_2 = \sigma} \int f(\sigma_1 + \tilde{\omega}) g(\omega + \sigma_2) d\omega \quad (1.10)$$

where the summation on the right hand side is over all partitions of σ into a pair σ_1, σ_2 of subsets (which can be empty). Then $f * g \in \mathcal{K}(X)$ and satisfies the inequality

$$\int |a^{\#\sigma}(f * g)(\sigma)|^2 d\sigma \leq \int |(a\sqrt{3})^{\#\sigma} f(\sigma)|^2 d\sigma \cdot \int |(a\sqrt{3})^{\#\sigma} g(\sigma)|^2 d\sigma \text{ for all } a \geq 1. \quad (1.11)$$

For a proof see Proposition 3.2 in [Li P]. The $\sim$ - invariance of m implies the invariance of the associated Guichardet measure m_Γ on $\Gamma(X)$ under the involution transformation $\omega \to \tilde{\omega} = \{\tilde{x} | x \in \omega\}$ and hence it is clear from (1.10) that $f * g = g * f$. It follows from the sum-integral formula (1.8) that $*$ is even associative. This will also follow from our Theorem 1.4. Thus $\mathcal{K}(X)$ becomes a commutative and associative algebra equipped with the involution $f \to \tilde{f}$ where $\tilde{f}(\sigma) = \overline{f(\tilde{\sigma})}$. A simple computation shows that $(f * g)^\sim = \tilde{f} * \tilde{g}$.

For any $\varphi \in L^2(m)$ define the associated exponential kernel $e(\varphi) \in \mathcal{K}(X)$ by

$$e(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \sigma = \emptyset, \\ \prod_{x \in \sigma} \varphi(x) & \text{otherwise.} \end{cases} \quad (1.12)$$

Then

$$e(\varphi) * e(\psi) = e(\varphi + \psi) \exp \int \varphi(\tilde{x}) \psi(x) dm(x), \quad (1.13)$$

$$e(\varphi)^{\sim} = e(\tilde{\varphi}). \tag{1.14}$$

for any $\varphi, \psi \in L^2(m)$. The set $E = \{e(\varphi), \varphi \in L^2(m)\}$ is linearly independent and total in $L^2(m_\Gamma)$. The linear manifold $\mathcal{E} \subset \mathcal{K}(X)$ generated by E is an involutive subalgebra of $\mathcal{K}(X)$. A comparison of (1.12) - (1.14) with (1.4) - (1.6) leads to the following theorem.

Theorem 1.4: Let ξ be the complex Gaussian random field over $L^2(m)$ in the probability space $(\Omega, \mathcal{F}_m, P_m)$ satisfying the properties (a) - (e) of Corollary 1.2 and property (d) of Theorem 1.1. Then there exists a unique unitary isomorphism $V : L^2(P_m) \to L^2(m_\Gamma)$ satisfying the following:

(a) $V\, e_\xi(\varphi) = e(\varphi)$ for all $\varphi \in L^2(m)$;

(b) $V\bar{g} = (Vg)^{\sim}$ for all $g \in L^2(P_m)$;

(c) $V(e_\xi(\varphi)e_\xi(\psi)) = e(\varphi) * e(\psi)$ for all $\varphi, \psi \in L^2(m)$;

where $e_\xi(\varphi)$ and $e(\varphi)$ are defined by (1.4) and (1.12) respectively.

Proof: First observe that $\langle e(\varphi), e(\psi)\rangle = \exp\langle \varphi, \psi\rangle = \langle e_\xi(\varphi), e_\xi(\psi)\rangle$ for all $\varphi, \psi \in L^2(m)$. The totality of $\mathcal{E}_\xi$ in $L^2(P_m)$ and $\mathcal{E}$ in $L^2(m_\Gamma)$ yields the existence of a unique unitary operator V satisfying (a). Now (b) and (c) are immediate. ■

Remark 1.5: The map V^{-1} identifies the Lindsay-Maassen twisted convolution algebra $\mathcal{K}(X) = \mathcal{K}(X, m, \sim)$ with the ordinary multiplication algebra of random variables on a Gaussian random field ξ satisfying the involutive property $\xi(\tilde{\varphi}) = \overline{\xi(\varphi)}, \varphi \in L^2(m)$. The involution $\sim$ of $\mathcal{K}(X, m, \sim)$ is then carried over to the complex conjugation of random variables.

We now describe a topology on $\mathcal{K}(X)$.To this end consider the selfadjoint number operator N in $L^2(m_\Gamma)$ defined by

$$(Nf)(\sigma) = (\#\sigma)f(\sigma), \quad f \in L^2(m_\Gamma)$$

with maximal domain. Define

$$\|f\|^{(a)} = \|a^N f\|, \quad a > 1, \quad f \in \mathcal{K}(X). \tag{1.15}$$

With the family $\{\|\cdot\|^{(a)},\ a > 1\}$ of norms $\mathcal{K}(X)$ becomes a topological vector space.

Theorem 1.6 The twisted convolution operator $*$ is continuous. The subalgebra $\mathcal{E}$ is dense in $\mathcal{K}(X)$.

Proof: By Proposition 3.2 in [Li P] we have the inequality

$$\|a^N f * g\| \le \|(a\sqrt{3})^N f\|\ \|(a\sqrt{3})^N g\| \ \text{ for all } \ f, g \in \mathcal{K}(X), a > 1.$$

If $\lim_{n\to\infty}(\|f_n - f\|^{(a)} + \|g_n - g\|^{(a)}) = 0$ for every $a > 1$ then the inequality

$$\begin{aligned}\|a^N(f_n * g_n - f * g)\| &\le \|a^N((f_n - f) * g_n)\| + \|a^N(f * (g_n - g))\| \\ &\le \|(a\sqrt{3})^N(f_n - f)\|\ \|(a\sqrt{3})^N g_n\| + \|(a\sqrt{3})^N f\|\ \|(a\sqrt{3})^N(g_n - g)\|\end{aligned}$$

implies that

$$\lim_{n\to\infty} \|a^N(f_n * g_n - f * g\| = 0.$$

This proves the first part.

To prove the second part consider an element $(f, a^N f)$ in the graph of the operator a^N with $f \in \mathcal{K}(X)$. Suppose that this element is orthogonal to every element of the form $(e(\varphi), a^N e(\varphi))$. Then

$$\begin{aligned}\langle (e(\varphi), a^N e(\varphi)),\ (f, a^N f)\rangle &= \langle e(\varphi), f\rangle + \langle e(\varphi), a^{2N} f\rangle \\ &= \langle e(\varphi), f + a^{2N} f\rangle \\ &= 0 \ \text{ for all } \ \varphi \in L^2(m).\end{aligned}$$

The totality of exponential kernels implies that $f + a^{2N} f = 0$. Since the spectrum of N is $\{0, 1, 2, \ldots\}$ it follows that $f = 0$. This enables us to conclude that for any fixed $a > 1, \varepsilon > 0$ and $f \in \mathcal{K}(X)$ there exists a $g \in \mathcal{E}$ such that

$$\|g - f\| + \|a^N g - a^N f\| < \varepsilon.$$

Choose $a = n, \varepsilon = \frac{1}{n}$ and denote the corresponding g by g_n. Since a^N is monotonic increasing in a for $a > 1$ it follows that

$$\lim_{n\to\infty} \|g_n - f\| + \|a^N g_n - a^N f\| = 0 \text{ for every } a > 1.$$

∎

2 Weighted and twisted convolution of Maassen kernels

Following Lindsay and Parthasarathy [Li P] we shall now investigate deformations of the twisted convolution operator $*$ in (1.10) by introducing a *weight function* or *multiplier* p inside the integral on the right hand side of (1.10). To this end we introduce the space $\mathcal{M}(X)$ of all complex-valued bounded measurable functions defined on $\Gamma^{(3)}(X)$ and call any element $p \in \mathcal{M}(X)$ a multiplier. Thus p is a function of three arguments $\sigma_1, \sigma_2, \sigma_3$ which are disjoint finite subsets of X. For any two Maassen kernels $f, g \in \mathcal{K}(X)$ and any multiplier $p \in \mathcal{M}(X)$ define the *weighted* and *twisted convolution* $f *_p g$ by

$$(f *_p g)(\sigma) = \sum_{\sigma_1+\sigma_2=\sigma} \int p(\omega, \sigma_1, \sigma_2) f(\sigma_1 + \tilde{\omega}) g(\omega + \sigma_2) d\omega \tag{2.1}$$

where $d\omega = dm_\Gamma(\omega)$ as in Section 1. It is to be noted that for any fixed $\sigma \in \Gamma(X)$, the complement of the set $\{\omega | \omega \cap \sigma = \emptyset\}$ in $\Gamma(X)$ has m_Γ - measure 0.

Proposition 2.1 In the Hilbert space $L^2(m_\Gamma)$, for any $p \in \mathcal{M}(X), f, g \in \mathcal{K}(X)$ the following inequality holds:

$$\|a^N f *_p g\| \le \sup\ |p|\ \|(a\sqrt{3})^N f\|\ \|(a\sqrt{3})^N g\| \ \text{ for all } \ a > 1. \tag{2.2}$$

In particular, $\mathcal{K}(X)$ is closed under the multiplication operation $*_p$.

Proof: This is the same as the first part of Proposition 3.2 in [Li P]. ∎

The next proposition is a twisted and weighted version of the Wiener product for Maassen kernels. (See IV.4.1 in [Mey]).

Theorem 2.2 For any multiplier p and Maassen kernel f define the operator $B_p(f)$ in $L^2(m_\Gamma)$ with domain $\mathcal{K}(X)$ and

$$B_p(f)g = f *_p g$$

where the right hand side is given by (2.1). Then, for any given $f_i \in \mathcal{K}(X), p_j \in \mathcal{M}(X),\ 1 \le i \le n,\ 1 \le j \le n-1,$

$$\begin{aligned}(B_{p_1}(f_1)&B_{p_2}(f_2)\cdots B_{p_{n-1}}(f_{n-1})f_n)(\delta)\\ &= \sum_{\Sigma\delta_i=\delta}\int\prod_{k=1}^{n} f_k((\sum_{i<k}\sigma_{ik})+\delta_k+\sum_{j>k}\tilde{\sigma}_{kj})\\ &\times\prod_{\ell=1}^{n-1} p_\ell(\sum_{j>\ell}\sigma_{\ell j},(\sum_{i<\ell}\sigma_{i\ell})+\delta_\ell,(\sum_{i<\ell<j}\sigma_{ij})+\sum_{j>\ell}\delta_j)\\ &\times\prod_{1\le i<j\le n} d\sigma_{ij} \qquad (2.3)\end{aligned}$$

where all the sets $\sigma_{ij}, \delta_k, 1 \le i,j,k \le n,\ i<j$ are disjoint and the indices k,ℓ are kept fixed under the Σ-signs inside f_k and p_ℓ.

Proof When $n = 2$, (2.3) is same as (2.1) if we put $\sigma_{12} = \omega,\quad \delta_1 = \sigma_1, \delta_2 = \sigma_2$. We prove (2.3) inductively. Assume (2.3) for n. To prove the same for $n+1$ put $g_n = f_n *_{p_n} f_{n+1}$. Then

$$\begin{aligned}(B_{p_1}(f_1)&\ldots B_{p_n}(f_n)f_{n+1})(\delta) = (B_{p_1}(f_1)\cdots B_{p_{n-1}}(f_{n-1})g_n)(\delta)\\ &= \sum_{\Sigma\delta_i=\delta}\int\prod_{k=1}^{n-1} f_k((\sum_{i<k}\sigma_{ik})+\delta_k+\sum_{j>k}\tilde{\sigma}_{kj})\\ &\times\prod_{\ell=1}^{n-1} p_\ell(\sum_{j>\ell}\sigma_{\ell j},(\sum_{i<\ell}\sigma_{i\ell})+\delta_\ell,(\sum_{i<\ell<j}\sigma_{ij})+\sum_{j>\ell}\delta_j)\\ &\times\sum_{\varepsilon_1+\varepsilon_2=(\sum_{i=1}^{n-1}\sigma_{in})+\delta_n} p_n(\sigma_{n\,n+1},\varepsilon_1,\varepsilon_2)f_n(\varepsilon_1+\tilde{\sigma}_{n\,n+1})f_{n+1}(\sigma_{n\,n+1}+\varepsilon_2)\\ &\times\left(\prod_{1\le i<j\le n} d\sigma_{ij}\right)d\sigma_{n\,n+1}. \qquad (2.4)\end{aligned}$$

Introduce new disjoint set variables $\sigma'_{in}, \sigma'_{in+1}, \delta'_n, \delta'_{n+1}, 1 \le i \le n-1$ by putting

$$\sigma'_{in} = \varepsilon_1 \cap \sigma_{in}, \quad \sigma'_{i\,n+1} = \varepsilon_2 \cap \sigma_{in},$$

$$\delta'_n = \varepsilon_1 \cap \delta_n, \quad \delta'_{n+1} = \varepsilon_2 \cap \delta_n.$$

Then $\sigma'_{in} + \sigma'_{i\,n+1} = \sigma_{in} \cap (\varepsilon_1 + \varepsilon_2) = \sigma_{in}$ and $\delta'_n + \delta'_{n+1} = \delta_n \cap (\varepsilon_1 + \varepsilon_2) = \delta_n$. Substituting these new variables in (2.4) and using the sum-integral formula (1.8) we get

$$\begin{aligned}
&(B_{p_1}(f_1)\cdots B_{p_n}(f_n)f_{n+1})(\delta) = \\
&= \sum_{(\Sigma_{i=1}^{n-1}\delta_i)+\delta'_n+\delta'_{n+1}=\delta} \int \prod_{k=1}^{n-1} f_k(\sum_{i<k}\sigma_{ik} + \delta_k + \sum_{k<j\le n-1} \tilde{\sigma}_{kj} + \tilde{\sigma}'_{kn} + \tilde{\sigma}'_{k\,n+1}) \\
&\times \prod_{\ell=1}^{n-1} p_\ell((\sum_{\ell<j\le n-1} \sigma_{\ell j}) + \sigma'_{\ell n} + \sigma'_{\ell n+1}, (\sum_{i<\ell}\sigma_{i\ell}) + \delta_\ell, \\
&\sum_{i<\ell<j\le n-1} \sigma_{ij} + \sum_{i<l}(\sigma'_{in} + \sigma'_{in+1}) + \sum_{\ell\le j\le n-1} \delta_j + \delta'_n + \delta'_{n+1}) \\
&\times f_n(\sum_{i=1}^{n-1}\sigma'_{in} + \delta'_n + \tilde{\sigma}_{n\,n+1}) f_{n+1}(\sum_{i=1}^{n-1}\sigma'_{i\,n+1} + \sigma_{n\,n+1} + \delta'_{n+1}) \\
&\times p_n(\sigma_{n\,n+1}, \sum_{i=1}^{n-1}\sigma'_{in} + \delta'_n, \sum_{i=1}^{n-1}\sigma'_{i\,n+1} + \delta'_{n+1}) \\
&\times (\prod_{1\le i<j\le n-1} d\sigma_{ij})(\prod_{i=1}^{n-1} d\sigma'_{in} d\sigma'_{i\,n+1}) d\sigma_{n\,n+1}.
\end{aligned}$$

If we now drop the primes $'$ in the expression above it is the same as (2.3) with n replaced by $n+1$. ■

Proposition 2.3 For any $f, g, h \in \mathcal{K}(X)$ and $p \in \mathcal{M}(X)$

$$\langle f, g *_p h\rangle = \langle \tilde{g} *_q f, h\rangle$$

where $\tilde{g}(\sigma) = \overline{g(\tilde{\sigma})}$ and $q(\sigma_1, \sigma_2, \sigma_3) = \overline{p(\sigma_2, \sigma_1, \sigma_3)}$.

Proof: By (2.1) and an application of (1.8) twice for the case $n = 2$ we have

$$\begin{aligned}
\langle f, g *_p h\rangle &= \int \bar{f}(\sigma)\{\sum_{\sigma_1+\sigma_2=\sigma} \int p(\omega, \sigma_1, \sigma_2) g(\sigma_1 + \tilde{\omega}) h(\omega + \sigma_2) d\omega\} d\sigma \\
&= \int \bar{f}(\sigma_1 + \sigma_2) p(\omega, \sigma_1, \sigma_2) g(\sigma_1 + \tilde{\omega}) h(\omega + \sigma_2) d\sigma_1 d\sigma_2 d\omega
\end{aligned}$$

$$= \int \{ \sum_{\omega+\sigma_2=\gamma} \int \bar{q}(\sigma_1,\omega,\sigma_2)\tilde{g}(\omega+\tilde{\sigma}_1)\bar{f}(\sigma_1+\sigma_2)d\sigma_1\} h(\gamma)d\gamma$$
$$= \langle \tilde{g} *_q f, h\rangle.$$

■

Corollary 2.4 For any multiplier p and Maassen kernel f the operators $B_p(f)$ and $B_q(\tilde{f})$ are adjoint to each other on the domain $\mathcal{K}(X)$, where $q(\sigma_1,\sigma_2,\sigma_3) = \overline{p(\sigma_2,\sigma_1,\sigma_3)}$.

Proof : Immediate . ■

Proposition 2.5 Let $f_i, 1 \leq i \leq k$ be Maassen kernels. Then

$$|(f_1 * f_2 * \cdots * f_k)(\emptyset)| \leq \prod_{i=1}^{k} \|(k-1)^{N/2} f_i\|.$$

Proof: When $k=2, (f_1 * f_2)(\emptyset) = \langle f_1, f_2\rangle$ and hence the required inequality coincides with Schwarz's inequality. To deal with the general case introduce the operation A by

$$(Af)(\sigma) = \int f(\sigma+\omega)d\omega.$$

Then, by Theorem 2.2, putting $p_i = 1$ for all i and $n = k$ we get from a repeated application of Schwartz's inequality in the integrating variables $\sigma_{12}, \sigma_{13}, ..., \sigma_{1n}$,

$$\begin{aligned}
&|f_1 * \cdots * f_k(\emptyset)| \\
&= |\int f_1(\tilde{\sigma}_{12}+\tilde{\sigma}_{13}+\cdots+\tilde{\sigma}_{1k}) f_2(\sigma_{12}+\tilde{\sigma}_{23}+\cdots+\tilde{\sigma}_{2k}) \\
&\quad \cdots f_k(\sigma_{1k}+\sigma_{2k}+\cdots+\sigma_{k-1\,k}) d\sigma_{1k}\cdots d\sigma_{k-1\,k}| \\
&\leq \int (A|f_1|^2)^{1/2}(\tilde{\sigma}_{13}+\cdots+\tilde{\sigma}_{1k})(A|f_2|^2)^{1/2}(\tilde{\sigma}_{23}+\cdots+\tilde{\sigma}_{2k}) \\
&\quad \times |f_3|(\sigma_{13}+\sigma_{23}+\tilde{\sigma}_{34}+\cdots+\tilde{\sigma}_{3k}) \\
&\quad \cdots |f_k|(\sigma_{1k}+\sigma_{2k}+\cdots+\sigma_{k-1\,k}) \prod_{\substack{1\leq i<j\leq n \\ (i,j)\neq(1,2)}} d\sigma_{ij} \\
&\leq \int (A^2|f_1|^2)^{1/2}(\tilde{\sigma}_{14}+\cdots+\tilde{\sigma}_{1k})(A|f_2|^2)^{1/2}(\tilde{\sigma}_{23}+\cdots+\tilde{\sigma}_{2k}) \\
&\quad \times (A|f_3|^2)^{1/2}(\sigma_{23}+\tilde{\sigma}_{34}+\cdots+\tilde{\sigma}_{3k})
\end{aligned}$$

$$\times \quad |f_4|(\sigma_{14} + \sigma_{24} + \sigma_{34} + \tilde{\sigma}_{45} + \cdots + \tilde{\sigma}_{4k}) \cdots |f_k|(\sigma_{1k} + \cdots + \sigma_{k-1\,k})$$
$$\times \quad \prod_{\substack{1 \le i < j \le n \\ (i,j) \notin \{(1,2),(1,3)\}}} d\sigma_{ij}$$
$$\cdots$$
$$\le \left(\int (A^{k-2}|f_1|^2)(\sigma)d\sigma\right)^{1/2} \{(A|f_2|^2)^{1/2} * (A|f_3|^2)^{1/2} * \cdots * (A|f_k|^2)^{1/2}(\emptyset)\}.$$

A repeated application of the inequality above yields

$$|f_1 * \cdots * f_k(\emptyset)| \le \prod_{j=1}^{k} \{\int (A^{k-2}|f_j|^2)(\sigma)d\sigma\}^{1/2}. \tag{2.5}$$

For any Maassen kernel f we have from (1.8)

$$\begin{aligned}
\int (A^k|f|^2)(\sigma)d\sigma &= \int |f|^2(\sigma_1 + \sigma_2 + \cdots + \sigma_{k+1})d\sigma_1 \cdots d\sigma_{k+1} \\
&= \int \sum_{\sigma_1 + \cdots + \sigma_{k+1} = \sigma} |f|^2(\sigma)d\sigma \\
&= \int (k+1)^{\#\sigma}|f|^2(\sigma)d\sigma \\
&= \|(k+1)^{N/2}f\|^2.
\end{aligned}$$

Now the proposition follows from (2.5). ■

Proposition 2.6 Let $f, f_i, 1 \le i \le k$ be Maassen kernels and let $p_i, 1 \le i \le k$ be multipliers. Then

$$\|B_{p_1}(f_1) \cdots B_{p_k}(f_k)f\| \le \left(\prod_{i=1}^{k} (\sup |p_i|)\|(2k+1)^{N/2}f_i\|\right)\|(2k+1)^{N/2}f\|.$$

Proof: From Theorem 2.2 we have

$$\|B_{p_1}(f_1) \cdots B_{p_k}(f_k)f\| \le \left(\prod_{i=1}^{k} \sup |p_i|\right)\| \, |f_1| * \cdots * |f_n| * |f| \, \|. \tag{2.6}$$

From the $\sim$ - invariance of m_Γ we have for any Maassen kernel g

$$\|g\|^2 = (g * \tilde{g})(\emptyset).$$

By Proposition 2.5 and commutativity of the operation $*$ we have

$$\begin{aligned} &\| \, |f_1| * \cdots * |f_k| * |f| \, \|^2 \\ &\quad = (|f_1| * |f_1|^{\sim} * |f_2| * |f_2|^{\sim} * \cdots * |f_k| * |f_k|^{\sim} * |f| * |f|^{\sim})(\emptyset) \\ &\quad \leq \{\prod_{j=1}^{k} \|(2k+1)^{N/2}(|f_j|)\|\}^2 \, \|(2k+1)^{N/2}(|f|)\|^2. \end{aligned}$$

From (2.6) and Proposition 2.1 with $p = 1$ we now have

$$\|B_{p_1}(f_1) \cdots B_{p_k}(f_k) f\|^2 \leq \{\prod_{i=1}^{k} \sup |p_i|^2 \prod_{i=1}^{k} \|(2k+1)^{N/2} f_i\|^2\} \|(2k+1)^{N/2} f\|^2.$$

■

Corollary 2.7 Let $f, f_i, 1 \leq i \leq k$ be Maassen kernels and let $p_i, 1 \leq i \leq k$ be multipliers. Then, for any $a \geq 1$,

$$\|a^N B_{p_1}(f_1) \cdots B_{p_k}(f_k) f\| \leq (\prod_{i=1}^{k} (\sup |p_i|) \|(a\sqrt{2k+1})^N f_i\|) \|(a\sqrt{2k+1})^N f\|.$$

Proof: From (2.3) it is clear that

$$\begin{aligned} &|a^N B_{p_1}(f_1) \cdots B_{p_{k-1}}(f_{k-1}) f|(\delta) \\ &\quad \leq (\prod_{i=1}^{k} \sup |p_i|) \, (a^N |f_1|) * \cdots * (a^N |f_k|) * (a^N |f|)(\delta) \end{aligned}$$

for any $a \geq 1$. The required inequality is immediate from Proposition 2.6. ■

Proposition 2.8: Let $f \in \mathcal{K}(X)$ have support in $\{\sigma | \#\sigma = 1\}$. Define the symmetric operator $A(f)$ with domain $\mathcal{K}(X)$ by $A(f) = B_p(f) + B_q(\tilde{f})$ where p and q are as in Corollary 2.4. Suppose $g \in \mathcal{K}(X)$ is such that either it has support in $\{\sigma | \#\sigma \leq n\}$ for some positive integer n or $g \in \mathcal{E}$. Then

$$\sum_{k=0}^{\infty} \frac{\|A(f)^k g\|}{k!} < \infty.$$

Proof: Let g be an n-particle element in $\mathcal{K}(X)$, in the sense that its support is contained in $\{\sigma | \#\sigma = n\}$. It follows from Corollary 2.7 that

$$\|A(f)^k g\| \leq (2 \sup |p|)^k (2k+1)^{\frac{k+n}{2}} \|f\|^k \|g\|$$

for all $k = 0, 1, 2, \ldots$ On the other hand, if $g = e(\varphi)$ for some $\varphi \in L^2(m)$, we have

$$\|A(f)^k g\| \leq (2 \sup |p|)^k (2k+1)^{\frac{k}{2}} \|f\|^k e^{\frac{2k+1}{2}\|\varphi\|^2}.$$

Thus, in either case,

$$\|A(f)^k g\| \leq C^k k^{\frac{k}{2}}, \quad k = 0, 1, 2, \ldots$$

for some positive constant C. Now the required result follows from Stirling's formula. ■

Remark: The symmetric operator $A(f)$ of Proposition 2.8 is essentially selfadjoint on the domain of finite particle vectors as well as the exponential domain $\mathcal{E}$.

3 Covariance properties of the family $\{B_p(f)\}$ under a group action

Let $(X, m, \sim)$ be a nonatomic, separable and σ-finite measure space equipped with an m-preserving involution as in Section 2. Suppose G is a group of transformations acting as measurable automorphisms of X, leaving m quasi-invariant and satisfying the relation $g\tilde{x} = (gx)^{\sim}$ for all $x \in X$. Let

$$\rho(g, x) = \{\frac{dm}{dmg}(x)\}^{1/2}. \tag{3.1}$$

Let $\alpha = \alpha(g, x)$ be a measurable complex-valued 1-cocycle of unit modulus in the sense of Mackey [Mac] for the G-action with quasi-invariant measure m. Then

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 x)\alpha(g_2, x) \text{ a.e. } x(m) \tag{3.2}$$

for each $g_1, g_2 \in G$. We assume that

$$\alpha(g, \tilde{x}) \equiv \overline{\alpha(g, x)}, \quad g \in G, \quad x \in X. \tag{3.3}$$

Extend the G-action to $\Gamma(X)$ by putting

$$g\sigma = \begin{cases} \emptyset & \text{if } \sigma = \emptyset, \\ \{gx, x \in \sigma\} & \text{otherwise.} \end{cases}$$

Then the Guichardet measure m_Γ is quasi-invariant under the extended G action on $\Gamma(X)$ and

$$\rho(g, \sigma) := \left\{ \left(\frac{dm_\Gamma}{dm_\Gamma g} \right) (\sigma) \right\}^{1/2} \tag{3.4}$$

is given by

$$\rho(g, \sigma) = \begin{cases} 1 & \text{if } \sigma = \emptyset, \\ \prod_{x \in \sigma} \rho(g, x) & \text{otherwise.} \end{cases} \tag{3.5}$$

Define $\alpha(g,\sigma)$ by

$$\alpha(g,\sigma) = \begin{cases} 1 & \text{if } \sigma = \emptyset, \\ \prod_{x\in\sigma} \alpha(g,x) & \text{otherwise.} \end{cases} \tag{3.6}$$

Then we have the relations:

$$\begin{aligned} \alpha(g,\sigma_1+\sigma_2) &= \alpha(g,\sigma_1)\alpha(g,\sigma_2), \\ \alpha(g_1g_2,\sigma) &= \alpha(g_1,g_2\sigma)\alpha(g_2,\sigma), \\ \alpha(g,\tilde{\sigma}) &= \overline{\alpha(g,\sigma)}, \\ \rho(g,\sigma_1+\sigma_2) &= \rho(g,\sigma_1)\rho(g,\sigma_2), \\ \rho(g_1g_2,\sigma) &= \rho(g_1,g_2\sigma)\rho(g_2,\sigma), \\ \rho(g,\tilde{\sigma}) &= \rho(g,\sigma). \end{aligned}$$

Consider the unitary representation $g \to U_g$ of G in $L^2(m)$ defined by

$$(U_gf)(x) = \alpha(g,g^{-1}x)\rho(g,g^{-1}x)f(g^{-1}x), \quad f \in L^2(m) \tag{3.7}$$

and its second quantization $g \to \Gamma(U_g)$ defined by

$$(\Gamma(U_g)h)(\sigma) = \alpha(g,g^{-1}\sigma)\rho(g,g^{-1}\sigma)h(g^{-1}\sigma), h \in L^2(m_\Gamma) \tag{3.8}$$

where $\rho(g,\sigma)$ and $\alpha(g,\sigma)$ are given by (3.5) and (3.6). Then $g \to \Gamma(U_g)$ is a unitary representation of G in $L^2(m_\Gamma)$.

Theorem 3.1: Let $p \in \mathcal{M}(X)$ and let $\{B_p(f), f \in \mathcal{K}(X)\}$ be defined as in Theorem 2.2. Then $\Gamma(U_g)$ leaves $\mathcal{K}(X)$ invariant and

$$\Gamma(U_g)B_p(f)\Gamma(U_g)^{-1}h = B_{pg^{-1}}(\Gamma(U_g)f)h$$

for all $f, h \in \mathcal{K}(X), g \in G$, where

$$pg^{-1}(\sigma_1,\sigma_2,\sigma_3) \equiv p(g^{-1}\sigma_1, g^{-1}\sigma_2, g^{-1}\sigma_3).$$

Proof: Straightforward substitution from (3.7) and (3.8) using (2.1) yields

$$\begin{aligned} &(\Gamma(U_g)B_p(f)\Gamma(U_g)^{-1}h)(\sigma) \\ &= \alpha(g,g^{-1}\sigma)\rho(g,g^{-1}\sigma) \sum_{\sigma_1+\sigma_2=g^{-1}\sigma} \int p(\omega,\sigma_1,\sigma_2)\alpha(g^{-1},g(\omega+\sigma_2)) \\ &\times \rho(g^{-1},g(\omega+\sigma_2)) \ f(\sigma_1+\tilde{\omega})h(g(\omega+\sigma_2))d\omega. \end{aligned}$$

Writing $\delta_i = g\sigma_i, g\omega = \omega'$ and using the relations satisfied by α and ρ we get

$$
\begin{aligned}
&(\Gamma(U_g)B_p(f)\Gamma(U_g)^{-1}h)(\sigma) \\
&= \sum_{\delta_1+\delta_2=\sigma} \int pg^{-1}(\omega',\delta_1,\delta_2)\alpha(g,g^{-1}\delta_1)\rho(g,g^{-1}\delta_1)f(g^{-1}(\delta_1+\tilde{\omega}')) \\
&\qquad\qquad \times\ h(\omega'+\delta_2)\alpha(g^{-1},\omega')\rho(g^{-1},\omega')^{-1}d\omega' \\
&= \sum_{\delta_1+\delta_2=\sigma} \int pg^{-1}(\omega',\delta_1,\delta_2)\alpha(g,g^{-1}(\delta_1+\tilde{\omega}'))\rho(g,g^{-1}(\delta_1+\tilde{\omega}'))f(g^{-1}(\delta_1+\tilde{\omega}')) \\
&\qquad\qquad \times h(\omega'+\delta_2)d\omega' \\
&= (B_{pg^{-1}}(\Gamma(U_g)f)h)(\sigma).
\end{aligned}
$$

■

4 Construction of self-similar operator fields

We shall now describe how the covariance property of the operator fields $B_p(\cdot)$ under the group action G can be exploited to construct a family of simultaneously self-similar fields. To this end consider a topological vector space $\mathcal{S}$ equipped with a homomorphism $g \to \pi(g)$ of the group G into the group of all bicontinuous linear isomorphisms of $\mathcal{S}$. Let $\mathcal{M}_G(X) \subset \mathcal{M}(X)$ be the subset of all G-invariant multipliers and let $\mathcal{M}_0 \subset \mathcal{M}_G(X)$ be a fixed subset. Suppose that for every $p \in \mathcal{M}_0$ there exists a continuous linear map $L_p : \mathcal{S} \to \mathcal{K}(X)$ satisfying the relation

$$L_p\,\pi(g)\varphi = \tau_p(g)\Gamma(U_g)L_p\,\varphi,\ \ p \in \mathcal{M}_0,\ \ g \in G,\ \ \varphi \in \mathcal{S} \tag{4.1}$$

where τ_p is a homomorphism from G into the multiplicative group of all nonzero real scalars and $\Gamma(U_g)$ is defined by (3.8). Recall that $\mathcal{K}(X)$ is equipped with the topology induced by the family of norms given by (1.15). Define the operators $A_p(\varphi), A_p^\dagger(\varphi),\ \varphi \in \mathcal{S}$ by

$$A_p(\varphi) = B_p(L_p\varphi),\ \ A_p^\dagger(\varphi) = B_{\tilde{p}}((L_p\varphi)^\sim), p \in \mathcal{M}_0, \varphi \in \mathcal{S} \tag{4.2}$$

where $B_p(\cdot)$ is as in Theorem 2.2 and $\tilde{p}(\sigma_1,\sigma_2,\sigma_3) = \overline{p(\sigma_2,\sigma_1,\sigma_3)}$. From Corollary 2.4 we know that $A_p(\varphi)$ and $A_p^\dagger(\varphi)$ are adjoint to each other on the domain $\mathcal{K}(X)$. With these notations we have the following proposition.

Proposition 4.1: For any $p \in \mathcal{M}_0, \varphi \in \mathcal{S}$ let $A_p^\#(\varphi)$ denote either of the operators $A_p(\varphi), A_p^\dagger(\varphi)$ defined by (4.2). Then the following holds:

(i) For any fixed Maassen kernel f and multipliers $p_i \in \mathcal{M}_0,\ \ 1 \le i \le n$ the correspondence $(\varphi_1,\varphi_2,...,\varphi_n) \to A_{p_1}^\#(\varphi_1)A_{p_2}^\#(\varphi_2)...A_{p_n}^\#(\varphi_n)f$ from $\mathcal{S}\times\mathcal{S}\times\cdots\times\mathcal{S}$ (n-fold) into $\mathcal{K}(X)$ is real multilinear and continuous;

(ii) If $\delta_\emptyset$ denotes the Maassen kernel defined by $\delta_\emptyset(\sigma) = 0$ or 1 according as $\sigma = \emptyset$

or $\neq \emptyset$ then

$$\langle \delta_\emptyset , A^{\#}_{p_1}(\pi(g)\varphi_1)A^{\#}_{p_2}(\pi(g)\varphi_2)...A^{\#}_{p_n}(\pi(g)\varphi_n)\delta_\emptyset \rangle > \\ = \{\prod_{i=1}^{n} \tau_{p_i}(g)\}\langle \delta_\emptyset, A^{\#}_{p_1}(\varphi_1)A^{\#}_{p_2}(\varphi_2)...A^{\#}_{p_n}(\varphi_n)\delta_\emptyset \rangle$$

for all $g \in G$, $\varphi_i \in \mathcal{S}$, $p_i \in \mathcal{M}_0$.

Proof: The first part is immediate from Corollary 2.7. To prove the second part observe that (4.1) and Theorem 3.1 together with the G-invariance of the p_i's imply

$$A^{\#}_{p_i}(\pi(g)\varphi_i) = \tau_{p_i}(g)\Gamma(U_g)A^{\#}_{p_i}(\varphi_i)\Gamma(U_g)^{-1}$$

and $\Gamma(U_g)\delta_\emptyset = \delta_\emptyset$. ∎

Remark Property (ii) of the fields $\{A_p(\cdot), p \in \mathcal{M}_0\}$ may be interpreted as the simultaneous self-similarity of all their expectation values in the state $\delta_\emptyset$ where the self-similarity parameter for $A_p(\cdot)$ under the action of the group G is described by the homomorphism τ_p of G into the multiplicative group $\mathbb{R}\backslash\{0\}$.

We shall now illustrate Proposition 4.1 when $X = \mathbb{R}^d$, $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$, the Schwartz's space of rapidly decreasing C^∞ functions in $\mathbb{R}^d$ and $G = \mathbb{R}^d o(0,\infty)$, the semidirect product of the additive group $\mathbb{R}^d$ and the multiplicative group of positive real scalars with the group operation

$$(x,a)(x',a') = (x + a^{-1}x', aa'), \quad x, x' \in \mathbb{R}^d, \quad a, a' > 0.$$

Put $\tilde{x} = -x$ and define the measure m in X by

$$dm(x) = |x|^\mu \ h(\frac{x}{|x|})dx \tag{4.3}$$

where $|x|$ is the Euclidean norm of x and h is a nonnegative bounded measurable function on the unit sphere in $\mathbb{R}^d$ satisfying $h(y) = h(-y), |y| = 1$. Then $(\mathbb{R}^d, m, \sim)$ is a nonatomic separable and σ-finite measure space with m-preserving involution. Define the G-action on this measure space by $(x,a)y = ay$ for all $x, y \in \mathbb{R}^d, a > 0$. Then

$$\rho(g,y) = \left\{\frac{dm}{dmg}(y)\right\}^{1/2} = a^{-\frac{1}{2}(\mu+d)} \text{ if } g = (x,a). \tag{4.4}$$

Define

$$\alpha((x,a),y) = e^{iax.y} \tag{4.5}$$

where $x \cdot y$ is the scalar product between x, y in $\mathbb{R}^d$. Then α is a 1-cocycle of modulus unity for the G-action in $\mathbb{R}^d$ with quasi-invariant measure m given by (4.3) and furthermore $\alpha((x,a),-y) = \overline{\alpha((x,a),y)}$. Following the notations in (3.5) - (3.8) we have

$$\begin{aligned}\alpha((x,a),\sigma) &= \exp\ ia\, x \cdot \Sigma_{y\in\sigma} y,\\ \rho((x,a),\sigma) &= a^{-\frac{1}{2}(\mu+d)\#\sigma},\\ (U_{(x,a)}f)(y) &= e^{i\,x\cdot y} a^{-\frac{1}{2}(\mu+d)} f(a^{-1}y), f \in L^2(m),\\ \{\Gamma(U_{(x,a)})g\}(\sigma) &= e^{ix\cdot\Sigma_{y\in\sigma} y} a^{-\frac{1}{2}(\mu+d)\#\sigma} g(a^{-1}\sigma),\ \ g \in L^2(m_\Gamma).\end{aligned}$$

Let

$$(\pi(x,a)\varphi)(y) = \varphi(a(y-x)),\ \ (x,a) \in G,\ \ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

With these notations we have the following proposition.

Proposition 4.2: For each G-invariant multiplier $p \in \mathcal{M}_G(\mathbb{R}^d)$ let $L_p : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{K}(\mathbb{R}^d, m, \sim)$ be a map satisfying

$$(L_p\varphi)(\sigma) = \hat{\varphi}(\sum_{y\in\sigma} y) F_p(\sigma), \sigma \in \Gamma(X)$$

where $\hat{\varphi}$ is the Fourier transform of φ and F_p satisfies the relation

$$F_p(a\sigma) = a^{d-\beta_p-\frac{1}{2}(\mu+d)\#\sigma} F_p(\sigma),\ \ \sigma \in \Gamma(X), \tag{4.6}$$

β_p being a real scalar. Then

$$L_p\pi(x,a)\varphi = a^{-\beta_p}\ \Gamma(U_{(x,a)})L_p\varphi \tag{4.7}$$

for all $(x,a) \in G, \varphi \in \mathcal{S}(\mathbb{R}^d), p \in \mathcal{M}_G(\mathbb{R}^d)$.

Proof: We have

$$(\pi(x,a)\varphi)^\wedge(y) = e^{ix\cdot y} a^{-d} \hat{\varphi}(a^{-1}y).$$

By the definition of L_p we obtain

$$(L_p\pi(x,a)\varphi)(\sigma) = a^{-d} e^{ix.\Sigma_{y\in\sigma} y} \hat{\varphi}(a^{-1}\sum_{y\in\sigma} y) F_p(\sigma).$$

Now (4.7) follows from (4.6) and the definition of $\Gamma(U_{(x,a)})$. ■

In order to construct functions $F_p, p \in \mathcal{M}_G(\mathbb{R}^d)$ satisfying the properties of Proposition 4.2 we shall make use of the following inequality.

Proposition 4.3 (P. Major [Maj]) Let $\theta_i > 0,\ 1 \le i \le n,\ n \ge 2, \theta_1 + \cdots + \theta_n < d$. Then

$$\int_{x_1+x_2+\cdots+x_n=x} \prod_{i=1}^{n} |x_i|^{\theta_i - d} dx_1 dx_2 \cdots dx_{n-1}$$
$$\le C(\theta_1, \theta_2, ..., \theta_n)|x|^{\theta_1+\cdots+\theta_n-d} \text{ for all } x \in \mathbb{R}^d,$$

where $|x|$ denotes the Euclidean norm in $\mathbb{R}^d, dx_j$ indicates integration with respect to the d-dimensional Lebesgue measure and $C(\theta_1, \theta_2, ..., \theta_n)$ is a positive constant.

Proof: This is done by straightforward induction in n. (For details see the proof of Proposition 6.3 in [Maj].). ■

Proposition 4.4 Let $r_j > 0, 1 \le j \le n, \sum_{j=1}^{n} r_j < \frac{d}{2}, -\infty < \beta < d$ and let

$$G_n(x_1, x_2, .., x_n) = |x_1 + \cdots + x_n|^{d-\beta-\sum_{j=1}^{n} r_j} \prod_{j=1}^{n} |x_j|^{r_j - \frac{1}{2}(\mu+d)}, x_j \in \mathbb{R}^d.$$

Then the following holds:
(i) $G_n(ax_1, ax_2, ..., ax_n) = a^{d-\beta-\frac{1}{2}(\mu+d)n} G_n(x_1, x_2, ..., x_n)$.
(ii) For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\int |\hat{\varphi}(\sum_{i=1}^{n} x_i) G_n(x_1, ..., x_n)|^2 dm(x_1)...dm(x_n)$$
$$\le C \sup_x (1 + |x|^N)|\hat{\varphi}(x)|^2 \text{ for } N > 2(d - \beta)$$

where C is a constant independent of φ.

Proof (i) is immediate from definitions. To prove (ii) we denote by $C_1, C_2, ...$ constants independent of φ and observe that the boundedness of h in (4.3) together with Proposition 4.3 implies that, for $N > 2(d - \beta)$,

$$\int |\hat{\varphi}(\sum_{i=1}^{n} x_i) G_n(x_1, ..., x_n)|^2 \, dm(x_1)...dm(x_n)$$
$$\le C_1 \int |\hat{\varphi}(x)|^2 |x|^{2(d-\beta-\sum_{j=1}^{n} r_j)} \Big(\int_{x_1+\cdots+x_n=x} \prod_{j=1}^{n} |x_j|^{2r_j - d} dx_1 ... dx_{n-1}\Big) dx$$
$$\le C_2 \int |\hat{\varphi}(x)|^2 \ |x|^{d-2\beta} dx$$
$$\le C_2 \sup_{|x|\le 1} |\hat{\varphi}(x)|^2 \int_{|x|\le 1} |x|^{d-2\beta} dx + C_2 \sup_{|x|>1} |x|^N |\hat{\varphi}(x)|^2 \int_{|x|>1} |x|^{d-2\beta-N} dx$$
$$\le C \sup_x (1 + |x|^N)|\hat{\varphi}(x)|^2.$$
■

Proposition 4.5: For each G-invariant multiplier $p \in \mathcal{M}_G(\mathbb{R}^d)$ let F_p be a function on $\Gamma(\mathbb{R}^d)$ given by

$$F_p(\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ c_p(n)\sum_{\pi \in \mathcal{G}_n} G_{n,p}(x_{\pi(1)}, ..., x_{\pi(n)}) & \text{if } \sigma = \{x_1, ..., x_n\} \end{cases}$$

where $c_p(n)$ is a scalar, $c_p(n) = 0$ for $n > n_p$,

$$G_{n,p}(x_1, ..., x_n) = |x_1 + \cdots + x_n|^{d-\beta_p - \sum_{j=1}^n r(j,n,p)} \prod_{j=1}^n |x_j|^{r(j,n,p) - \frac{1}{2}(\mu+d)},$$

$-\infty < \beta_p < d, r(j,n,p) > 0, \sum_{j=1}^n r(j,n,p) < \frac{d}{2}$ and $\mathcal{G}_n$ is the group of all permutations on the set $\{1, 2, ..., n\}$. Define the linear map L_p on $\mathcal{S}(\mathbb{R}^d)$ by

$$(L_p\varphi)(\sigma) = \hat{\varphi}(\sum_{x \in \sigma} x) F_p(\sigma), \quad p \in \mathcal{M}_G(\mathbb{R}^d).$$

Then the following are fulfilled:

(i) L_p is a continuous map from $\mathcal{S}(\mathbb{R}^d)$ into the space $\mathcal{K}(\mathbb{R}^d, m, \sim)$ of all Maassen kernels, satisfying

$$\|(L_p\varphi)\| \leq C_p(N) \sup_x (1 + |x|^N)^{\frac{1}{2}} |\hat{\varphi}(x)| \text{ for } N > 2(d - \beta_p).$$

(ii) $L_p \pi(x,a)\varphi = a^{-\beta_p} \Gamma(U_{(x,a)}) L_p \varphi$ for all $(x,a) \in G, \varphi \in \mathcal{S}(\mathbb{R}^d)$.

(iii) If $A_p(\varphi), A_p^\dagger(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^d)$ are defined by (4.2) then properties (i) and (ii) of Proposition 4.1 are fulfilled with

$$\tau_p((x,a)) = a^{-\beta_p} \text{ for all } (x,a) \in G, \ p \in \mathcal{M}_G(\mathbb{R}^d).$$

Proof: This is immediate from Proposition 4.2 and 4.4. ■

Remark 4.6 Note that $p \in \mathcal{M}_G(\mathbb{R}^d)$ simply means that p is a bounded measurable function of disjoint triplets $(\sigma_1, \sigma_2, \sigma_3)$ of finite subsets of $\mathbb{R}^d$ satisfying the identity $p(a\sigma_1, a\sigma_2, a\sigma_3) \equiv p(\sigma_1, \sigma_2, \sigma_3)$ for all $a > 0$. Thus Proposition 4.5 yields explicit examples of families of simultaneously self-similar fields $\{A_p(\cdot), p \in \mathcal{M}_G(\mathbb{R}^d)\}$ in the vacuum state $\delta_\emptyset$. If the measure m defined by (4.3) has the additional property that the function h on the unit sphere is a constant then the self-similarity property extends to the group of orthogonal transformations also. It follows from Proposition 4.5, Corollary 2.7 and the Schwartz's kernel theorem that the real multilinear functionals $\langle \delta_\emptyset, A_{p_1}^\#(\varphi_1)...A_{p_n}^\#(\varphi_n)\delta_\emptyset \rangle, (\varphi_1, ..., \varphi_n) \in \mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$ are, indeed, restrictions of tempered distributions on $(\mathbb{R}^d)^n$ with self-similarity property under the G-action: $(x,a)(y_1, ..., y_n) = (ay_1 + x, ..., ay_n + x)$. It should be interesting to find out, under what conditions on p, the operators $A_p(\varphi) + A_p^\dagger(\varphi)$ are essentially selfadjoint on the domain $\mathcal{K}(\mathbb{R}^d)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Remark 4.7 Consider the special case when $p \equiv 1$. Then for any two Maassen kernels $f, g, B_p(f)g = f * g$. By Theorem 1.4 the operators $A_p(\varphi) = A(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^d)$ of Proposition 4.5 can be identified through the unitary conjugation by V as multiplication operators by random variables on a Gaussian random field with involution. Then the example in Proposition 4.5 yields a self-similar classical random field subordinate to a Gaussian random field. This is just a translation of the construction by P. Major [Maj] in terms of Maassen kernels.

We now conclude our discussion with the construction of a self-similar Clifford field by choosing an appropriate weight p. To this end we begin with a general proposition concerning *one-particle Maassen kernels*, i.e., kernels with support in $\{\sigma | \#\sigma = 1\}$.

Proposition 4.8 Let $(X, m, \sim)$ be as in Section 3, $p_1, p_2 \in \mathcal{M}(X)$ and let q be a scalar. For any two one-particle Maassen kernels f, g, the operator $B_{p_1}(f)B_{p_2}(g) + qB_{p_2}(g)B_{p_1}(f)$ is also the operator of multiplication by a bounded measurable function b on $\Gamma(X)$ if, for any $x, y \in X, \sigma \in \Gamma(X)$ such that $x \neq y, \{x, y\} \cap \sigma = \emptyset$ the following four relations hold:

(i) $p_1(\emptyset, \{x\}, \sigma + \{y\})p_2(\emptyset, \{y\}, \sigma) + qp_1(\emptyset, \{x\}, \sigma)p_2(\emptyset, \{y\}, \sigma + \{x\}) = 0$;
(ii) $p_1(\{x\}, \emptyset, \sigma)p_2(\{y\}, \emptyset, \sigma + \{x\}) + qp_1(\{x\}, \emptyset, \sigma + \{y\})p_2(\{y\}, \emptyset, \sigma) = 0$;
(iii) $p_1(\emptyset, \{x\}, \sigma)p_2(\{y\}, \emptyset, \sigma) + qp_1(\emptyset, \{x\}, \sigma + \{y\})p_2(\{y\}, \emptyset, \sigma + \{x\}) = 0$;
(iv) $p_1(\{x\}, \emptyset, \sigma + \{y\})p_2(\emptyset, \{y\}, \sigma + \{x\}) + qp_1(\{x\}, \emptyset, \sigma)p_2(\emptyset, \{y\}, \sigma) = 0$.

In such a case b is given by

$$b(\sigma) = \int p_1(\{x\}, \emptyset, \sigma)p_2(\emptyset, \{x\}, \sigma) + qp_1(\emptyset, \{\tilde{x}\}, \sigma)p_2(\{\tilde{x}\}, \emptyset, \sigma)f(\tilde{x})g(x)dm(x).$$

Proof: This is a consequence of somewhat tedious but straightforward verification by using the definition of $B_p(\cdot)$ in Theorem 2.2 and (2.1). ■

Proposition 4.9: In Proposition 4.8 let $p_1 = p_2 = p$ and $q = 1$. Suppose that $p(\emptyset, \{x\}, \sigma) \equiv \overline{p(\{x\}, \emptyset, \sigma)}$ and

$$p(\emptyset, \{x\}, \sigma) = \prod_{y \in \sigma} k(x, y)$$

where $k(x, y)$ is a complex-valued function of modulus unity on $\{(x, y) : x \neq y\} \subset X \times X$ satisfying the relation

$$k(x, y) + k(y, x) \equiv 0.$$

Then, for any two one-particle Maassen kernels f, g, the following holds:

(i) $B_p(f)B_p(g) + B_p(g)B_p(f) = 2 \int f(\tilde{x})g(x)dm(x)$;
(ii) $B_p^\dagger(f) = B_p(\tilde{f})$

where $f(x) \equiv f(\{x\})$.

Proof: (i) is an immediate consequence of Proposition 4.8 whereas (ii) follows from Corollary 2.4. ∎

Remark 4.10 It is clear from Proposition 4.9 that $B_p(f)$ extends to a bounded operator in $L^2(m_\Gamma)$. Indeed,

$$B_p(f)B_p^\dagger(f) + B_p^\dagger(f)B_p(f) = 2\int |f(x)|^2 dm(x).$$

Thus the closure of the operators $B_p(f), f \in L^2(m)$ yields a Clifford field.

Remark 4.11: Choose $X = \mathbb{R}^d, \quad \tilde{x} = -x$,

$$dm(x) = |x|^\mu \, h(\frac{x}{|x|})dx$$

where μ is a real scalar, h is a bounded, nonnegative and measurable function on the unit sphere in $\mathbb{R}^d$ and the multiplier p in Proposition 4.9 is chosen with

$$k(x,y) = \begin{cases} e^{i\theta} & \text{if } x > y \;, \\ -e^{-i\theta} & \text{if } x < y \end{cases}$$

where $\mathbb{R}^d$ is equipped with the lexicographic order and θ is a fixed real scalar. Then p is invariant under the action of the group $G = \mathbb{R}^d o(0,\infty)$ in $\Gamma(\mathbb{R}^d)$. Define, for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$(L_p\varphi)(\sigma) = \begin{cases} \hat{\varphi}(x)|x|^{\frac{1}{2}(d-\mu)-\beta} & \text{if } \sigma = \{x\}, \;\; x \in \mathbb{R}^d, \\ 0 & \text{if } \#\sigma \neq 1,. \end{cases}$$

where $\hat{\varphi}$ is the Fourier transform of φ and β is a real scalar satisfying $-\infty < \beta < d$. Let $A(\varphi)$ denote the closure of $B_p(L_p\varphi)$. Then it follows from Proposition 4.5, 4.9 and Remark 4.6, 4.10 that the family $\{A(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^d)\}$ satisfies the following:
(i) $A(\varphi)$ is a bounded operator in $L^2(m_\Gamma)$ and $A(\varphi)^* = A(\bar{\varphi})$;
(ii) $A(\varphi)A(\psi) + A(\psi)A(\varphi) = 2\int \hat{\varphi}(-x)\hat{\psi}(x)|x|^{d-2\beta}h(\frac{x}{|x|})dx$;
(iii) The correspondence $(\varphi_1, ..., \varphi_n) \to A(\varphi_1)...A(\varphi_n)$ is strongly continuous;
(iv) $\langle \delta_\emptyset, A(\pi(x,a)\varphi_1)...A(\pi(x,a)\varphi_n)\delta_\emptyset\rangle = a^{-n\beta}\langle \delta_\emptyset, A(\varphi_1)...A(\varphi_n)\delta_\emptyset\rangle$, where

$$(\pi(x,a)\varphi)(y) = \varphi(a(y-x)), \;\; x,y \in \mathbb{R}^d, \;\; a > 0.$$

Thus we have constructed a self-similar Clifford field with self-similarity parameter β. If β is varied in the interval $(-\infty, d)$ the fields constructed above are jointly self-similar. If, in addition, h is a constant then, in the vacuum state, the expectation values of the Clifford field thus constructed are invariant under the action of the orthogonal group.

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