

Astérisque

ETIENNE BLANCHARD

Tensor products of $C(X)$ -algebras over $C(X)$

Astérisque, tome 232 (1995), p. 81-92

http://www.numdam.org/item?id=AST_1995__232__81_0

© Société mathématique de France, 1995, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Tensor products of $C(X)$ -algebras over $C(X)$

Etienne Blanchard

0 Introduction

Tensor products of C^* -algebras have been extensively studied over the last decades (see references in [12]). One of the main results was obtained by M. Takesaki in [15] where he proved that the spatial tensor product $A \otimes_{\min} B$ of two C^* -algebras A and B always defines the minimal C^* -norm on the algebraic tensor product $A \otimes_{alg} B$ of A and B over the complex field \mathbb{C} .

More recently, G.G. Kasparov constructed in [10] a tensor product over $C(X)$ for $C(X)$ -algebras. The author was also led to introduce in [3] several notions of tensor products over $C(X)$ for $C(X)$ -algebras and to study the links between those objects.

Notice that E. Kirchberg and S. Wassermann have proved in [11] that the subcategory of continuous fields over a Hausdorff compact space is not closed under such tensor products over $C(X)$ and therefore, in order to study tensor products over $C(X)$ of continuous fields, it is natural to work in the $C(X)$ -algebras framework.

Let us introduce the following definition:

DEFINITION 0.1 *Given two $C(X)$ -algebras A and B , we denote by $\mathcal{I}(A, B)$ the involutive ideal of the algebraic tensor product $A \otimes_{alg} B$ generated by the elements $(fa) \otimes b - a \otimes (fb)$, where $f \in C(X)$, $a \in A$ and $b \in B$.*

Our aim in the present article is to study the C^* -norms on the algebraic tensor product $(A \otimes_{alg} B)/\mathcal{I}(A, B)$ of two $C(X)$ -algebras A and B over $C(X)$ and to see how one can enlarge the results of Takesaki to this framework.

We first define an ideal $\mathcal{J}(A, B) \subset A \otimes_{alg} B$ which contains $\mathcal{I}(A, B)$ such that every C^* -semi-norm on $A \otimes_{alg} B$ which is zero on $\mathcal{I}(A, B)$ is also zero on $\mathcal{J}(A, B)$ and we prove that there always exist a minimal C^* -norm $\| \cdot \|_m$ and a maximal C^* -norm $\| \cdot \|_M$ on the quotient $(A \otimes_{alg} B)/\mathcal{J}(A, B)$.

We then study the following question of G.A. Elliott ([5]): when do the two ideals $\mathcal{I}(A, B)$ and $\mathcal{J}(A, B)$ coincide?

The author would like to express his gratitude to C. Anantharaman-Delaroche and G. Skandalis for helpful comments. He is also very indebted to S. Wassermann for sending him a preliminary version of [11] and to J. Cuntz who invited him to the Mathematical Institute of Heidelberg.

1 Preliminaries

We briefly recall here the basic properties of $C(X)$ -algebras.

Let X be a Hausdorff compact space and $C(X)$ be the C^* -algebra of continuous functions on X . For $x \in X$, define the morphism $e_x : C(X) \rightarrow \mathbf{C}$ of evaluation at x and denote by $C_x(X)$ the kernel of this map.

DEFINITION 1.1 ([10]) *A $C(X)$ -algebra is a C^* -algebra A endowed with a unital morphism from $C(X)$ in the center of the multiplier algebra $M(A)$ of A .*

We associate to such an algebra the unital $C(X)$ -algebra \mathcal{A} generated by A and $u[C(X)]$ in $M[A \oplus C(X)]$ where $u(g)(a \oplus f) = ga \oplus gf$ for $a \in A$ and $f, g \in C(X)$.

For $x \in X$, denote by A_x the quotient of A by the closed ideal $C_x(X)A$ and by a_x the image of $a \in A$ in the fibre A_x . Then, as

$$\|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\},$$

the map $x \mapsto \|a_x\|$ is upper semi-continuous for all $a \in A$ ([14]).

Note that the map $A \rightarrow \bigoplus A_x$ is a monomorphism since if $a \in A$, there is a pure state ϕ on A such that $\phi(a^*a) = \|a\|^2$. As the restriction of ϕ to $C(X) \subset M(A)$ is a character, there exists $x \in X$ such that ϕ factors through A_x and so $\phi(a^*a) = \|a_x\|^2$.

Let $S(A)$ be the set of states on A endowed with the weak topology and let $\mathcal{S}_X(A)$ be the subset of states φ whose restriction to $C(X) \subset M(A)$ is a character, i-e such that there exists an $x \in X$ (denoted $x = p(\varphi)$) verifying $\varphi(f) = f(x)$ for all $f \in C(X)$. Then the previous paragraph implies that the set of pure states $P(A)$ on A is included in $\mathcal{S}_X(A)$.

Let us introduce the following notation: if \mathcal{E} is a Hilbert A -module where A is a C^* -algebra, we will denote by $\mathcal{L}_A(\mathcal{E})$ or simply $\mathcal{L}(\mathcal{E})$ the set of bounded A -linear operators on \mathcal{E} which admit an adjoint ([9]).

DEFINITION 1.2 ([3]) *Let A be a $C(X)$ -algebra.*

A $C(X)$ -representation of A in the Hilbert $C(X)$ -module \mathcal{E} is a morphism $\pi : A \rightarrow \mathcal{L}(\mathcal{E})$ which is $C(X)$ -linear, i.e. such that for every $x \in X$, the representation $\pi_x = \pi \otimes e_x$ in the Hilbert space $\mathcal{E}_x = \mathcal{E} \otimes_{e_x} \mathbf{C}$ factors through a representation of A_x . Furthermore, if π_x is a faithful representation of A_x for every $x \in X$, π is said to be a field of faithful representations of A .

A continuous field of states on A is a $C(X)$ -linear map $\varphi : A \rightarrow C(X)$ such that for any $x \in X$, the map $\varphi_x = e_x \circ \varphi$ defines a state on A_x .

If π is a $C(X)$ -representation of the $C(X)$ -algebra A , the map $x \mapsto \|\pi_x(a)\|$ is lower semi-continuous since $\langle \xi, \pi(a)\eta \rangle \in C(X)$ for every $\xi, \eta \in \mathcal{E}$. Therefore, if A admits a field of faithful representations π , the map $x \mapsto \|a_x\| = \|\pi_x(a)\|$ is continuous for every $a \in A$, which means that A is a continuous field of C^* -algebras over X ([4]).

The converse is also true ([3] théorème 3.3): given a separable $C(X)$ -algebra A , the following assertions are equivalent:

1. A is a continuous field of C^* -algebras over X ,

2. the map $p : \mathcal{S}_X(A) \rightarrow X$ is open,
3. A admits a field of faithful representations.

2 C^* -norms on $(A \otimes_{alg} B)/\mathcal{J}(A, B)$

DEFINITION 2.1 *Given two $C(X)$ -algebras A and B , we define the involutive ideal $\mathcal{J}(A, B)$ of the algebraic tensor product $A \otimes_{alg} B$ of elements $\alpha \in A \otimes_{alg} B$ such that $\alpha_x = 0$ in $A_x \otimes_{alg} B_x$ for every $x \in X$.*

By construction, the ideal $\mathcal{I}(A, B)$ is included in $\mathcal{J}(A, B)$.

PROPOSITION 2.2 *Assume that $\|\cdot\|_\beta$ is a C^* -semi-norm on the algebraic tensor product $A \otimes_{alg} B$ of two $C(X)$ -algebras A and B .*

If $\|\cdot\|_\beta$ is zero on the ideal $\mathcal{I}(A, B)$, then

$$\|\alpha\|_\beta = 0 \text{ for all } \alpha \in \mathcal{J}(A, B).$$

Proof: Let D_β be the Hausdorff completion of $A \otimes_{alg} B$ for $\|\cdot\|_\beta$. By construction, D_β is a quotient of $A \otimes_{\max} B$. Furthermore, if C_Δ is the ideal of $C(X \times X)$ of functions which are zero on the diagonal, the image of C_Δ in $M(D_\beta)$ is zero.

As a consequence, the map from $A \otimes_{\max} B$ onto D_β factors through the quotient $A \otimes_{C(X)}^M B$ of $A \otimes_{\max} B$ by $C_\Delta \times (A \otimes_{\max} B)$.

But an easy diagram-chasing argument shows that $(A \otimes_{C(X)}^M B)_x = A_x \otimes_{\max} B_x$ for every $x \in X$ ([3] corollaire 3.17) and therefore the image of $\mathcal{J}(A, B) \subset A \otimes_{\max} B$ in $A \otimes_{C(X)}^M B$ is zero. \square

2.1 The maximal C^* -norm

DEFINITION 2.3 *Given two $C(X)$ -algebras A_1 and A_2 , we denote by $\|\cdot\|_M$ the C^* -semi-norm on $A_1 \otimes_{alg} A_2$ defined for $\alpha \in A_1 \otimes_{alg} A_2$ by*

$$\|\alpha\|_M = \sup\{\|(\sigma_1^x \otimes_{\max} \sigma_2^x)(\alpha)\|, x \in X\}$$

where σ_i^x is the map $A_i \rightarrow (A_i)_x$.

As $\|\cdot\|_M$ is zero on the ideal $\mathcal{J}(A_1, A_2)$, if we identify $\|\cdot\|_M$ with the C^* -semi-norm induced on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$, we get:

PROPOSITION 2.4 *The semi-norm $\|\cdot\|_M$ is the maximal C^* -norm on the quotient $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.*

Proof: By construction, $\|\cdot\|_M$ defines a C^* -norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. Moreover, as the quotient $A_1 \otimes_{C(X)}^M A_2$ of $A_1 \otimes_{\max} A_2$ by $C_\Delta \times (A_1 \otimes_{\max} A_2)$ maps injectively in

$$\bigoplus_{x \in X} (A_1 \otimes_{C(X)}^M A_2)_x = \bigoplus_{x \in X} ((A_1)_x \otimes_{\max} (A_2)_x),$$

the norm of $A_1 \overset{M}{\otimes}_{C(X)} A_2$ coincides on the dense subalgebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$ with $\|\cdot\|_M$. But we saw in proposition 2.2 that if $\|\cdot\|_\beta$ is a C*-norm on the algebra $(A \otimes_{alg} B)/\mathcal{J}(A, B)$, the completion of $(A \otimes_{alg} B)/\mathcal{J}(A, B)$ for $\|\cdot\|_\beta$ is a quotient of $A \overset{M}{\otimes}_{C(X)} B$. \square

2.2 The minimal C*-norm

DEFINITION 2.5 *Given two C(X)-algebras A_1 and A_2 , we define the semi-norm $\|\cdot\|_m$ on $A_1 \otimes_{alg} A_2$ by the formula*

$$\|\alpha\|_m = \sup\{\|(\sigma_1^x \otimes_{\min} \sigma_2^x)(\alpha)\|, x \in X\}$$

where σ_i^x is the map $A_i \rightarrow (A_i)_x$ and we denote by $A_1 \overset{m}{\otimes}_{C(X)} A_2$ the Hausdorff completion of $A_1 \otimes_{alg} A_2$ for that semi-norm.

Remark: In general, the canonical map $(A_1 \overset{m}{\otimes}_{C(X)} A_2)_x \rightarrow (A_1)_x \otimes_{\min} (A_2)_x$ is not a monomorphism ([11]).

By construction, $\|\cdot\|_m$ induces a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. We are going to prove that this C*-norm defines the minimal C*-norm on the involutive algebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.

Let us introduce some notation.

Given two unital C(X)-algebras A_1 and A_2 , let $P(A_i) \subset \mathcal{S}_X(A_i)$ denote the set of pure states on A_i and let $P(A_1) \times_X P(A_2)$ denote the closed subset of $P(A_1) \times P(A_2)$ of couples (ω_1, ω_2) such that $p(\omega_1) = p(\omega_2)$, where $p : P(A_i) \rightarrow X$ is the restriction to $P(A_i)$ of the map $p : \mathcal{S}_X(A_i) \rightarrow X$ defined in section 1.

LEMMA 2.6 *Assume that $\|\cdot\|_\beta$ is a C*-semi-norm on the algebraic tensor product $A_1 \otimes_{alg} A_2$ of two unital C(X)-algebras A_1 and A_2 which is zero on the ideal $\mathcal{J}(A_1, A_2)$ and define the closed subset $S_\beta \subset P(A_1) \times_X P(A_2)$ of couples (ω_1, ω_2) such that*

$$|(\omega_1 \otimes \omega_2)(\alpha)| \leq \|\alpha\|_\beta \text{ for all } \alpha \in A_1 \otimes_{alg} A_2.$$

If $S_\beta \neq P(A_1) \times_X P(A_2)$, there exist self-adjoint elements $a_i \in A_i$ such that $a_1 \otimes a_2 \notin \mathcal{J}(A_1, A_2)$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

Proof: Define for $i = 1, 2$ the adjoint action ad of the unitary group $\mathcal{U}(A_i)$ of A_i on the pure states space $P(A_i)$ by the formula

$$[(ad_u)\omega](a) = \omega(u^* a u).$$

Then S_β is invariant under the product action $ad \times ad$ of $\mathcal{U}(A_1) \times \mathcal{U}(A_2)$ and we can therefore find non empty open subsets $U_i \subset P(A_i)$ which are invariant under the action of $\mathcal{U}(A_i)$ such that $(U_1 \times U_2) \cap S_\beta = \emptyset$.

Now, if K_i is the complement of U_i in $P(A_i)$, the set

$$K_i^\perp = \{a \in A_i \mid \omega(a) = 0 \text{ for all } \omega \in K_i\}$$

is a non empty ideal of A_i and furthermore, if $\omega \in P(A_i)$ is zero on K_i^\perp , then ω belongs to K_i ; ([8] lemma 8,[15]).

As a consequence, if (φ_1, φ_2) is a point of $U_1 \times_X U_2$, there exist non zero self-adjoint elements $a_i \in K_i^\perp$ such that $\varphi_i(a_i) = 1$. If $x = p(\varphi_i)$, this implies in particular that $(a_1)_x \otimes (a_2)_x \neq 0$, and hence $a_1 \otimes a_2 \notin \mathcal{J}(A_1, A_2)$. \square

LEMMA 2.7 ([15] theorem 1) *Let A_1 and A_2 be two unital $C(X)$ -algebras.*

If the algebra A_1 is an abelian algebra, there exists only one C^ -norm on the quotient $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.*

Proof: Let $\|\cdot\|_\beta$ be a C^* -semi-norm on $A_1 \otimes_{alg} A_2$ such that for all $\alpha \in A_1 \otimes_{alg} A_2$, $\|\alpha\|_\beta = 0$ if and only if $\alpha \in \mathcal{J}(A_1, A_2)$.

If $\rho \in P(A_\beta)$ is a pure state on the Hausdorff completion A_β of $A_1 \otimes_{alg} A_2$ for the semi-norm $\|\cdot\|_\beta$, then for every $a_1 \otimes a_2 \in A_1 \otimes_{alg} A_2$,

$$\rho(a_1 \otimes a_2) = \rho(a_1 \otimes 1)\rho(1 \otimes a_2)$$

since $A_1 \otimes 1$ is included in the center of $M(A_\beta)$. Moreover, if we define the states ω_1 and ω_2 by the formulas $\omega_1(a_1) = \rho(a_1 \otimes 1)$ and $\omega_2(a_2) = \rho(1 \otimes a_2)$, then ω_2 is pure since ρ is pure, and $(\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)$. It follows that $P(A_\beta)$ is isomorphic to S_β .

In particular, if $a_1 \otimes a_2 \in A_1 \otimes_{alg} A_2$ verifies

$$(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0 \text{ for all couples } (\omega_1, \omega_2) \in S_\beta,$$

the element $a_1 \otimes a_2$ is zero in A_β and therefore belongs to the ideal $\mathcal{J}(A_1, A_2)$. Accordingly, the previous lemma implies that $P(A_1) \times_X P(A_2) = S_\beta = P(A_\beta)$.

As a consequence, we get for every $\alpha \in A_1 \otimes_{alg} A_2$

$$\begin{aligned} \|\alpha\|_\beta^2 &= \sup\{\rho(\alpha^*\alpha), \rho \in P(A_\beta)\} \\ &= \sup\{(\omega_1 \otimes \omega_2)(\alpha^*\alpha), (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)\} \end{aligned}$$

But that last expression does not depend on $\|\cdot\|_\beta$, and hence the unicity. \square

PROPOSITION 2.8 ([15] theorem 2) *Let A_1 and A_2 be two unital $C(X)$ -algebras.*

If $\|\cdot\|_\beta$ is a C^ -semi-norm on $A_1 \otimes_{alg} A_2$ whose kernel is $\mathcal{J}(A_1, A_2)$, then*

$$\forall \alpha \in A_1 \otimes_{alg} A_2, \quad \|\alpha\|_\beta \geq \|\alpha\|_m.$$

Proof: If we show that $S_\beta = P(A_1) \times_X P(A_2)$, then for every $\rho \in S_\beta$ and every α in $A_1 \otimes_{alg} A_2$, we have $\rho(s^*\alpha^*\alpha s) \leq \rho(s^*s)\|\alpha\|_\beta^2$ for all $s \in A_1 \otimes_{alg} A_2$. Therefore

$$\begin{aligned} \|\alpha\|_m^2 &= \sup\{\|(\sigma_1^x \otimes_{\min} \sigma_2^x)(\alpha)\|^2, x \in X\} \\ &= \sup\left\{\frac{(\omega_1 \otimes \omega_2)(s^*\alpha^*\alpha s)}{(\omega_1 \otimes \omega_2)(s^*s)}, (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2) \text{ and} \right. \\ &\quad \left. s \in A_1 \otimes_{alg} A_2 \text{ such that } (\omega_1 \otimes \omega_2)(s^*s) \neq 0\right\} \\ &\leq \|\alpha\|_\beta^2. \end{aligned}$$

Suppose that $S_\beta \neq P(A_1) \times_X P(A_2)$. Then there exist thanks to lemma 2.6 self-adjoint elements $a_i \in A_i$ and a point $x \in X$ such that $(a_1)_x \otimes (a_2)_x \neq 0$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

Let B be the unital abelian $C(X)$ -algebra generated by $C(X)$ and a_1 in A_1 . The preceding lemma implies that $B \overset{m}{\otimes}_{C(X)} A_2$ maps injectively into the Hausdorff completion A_β of $A_1 \otimes_{alg} A_2$ for $\| \cdot \|_\beta$.

Consider pure states $\rho \in P(B_x)$ and $\omega_2 \in P((A_2)_x)$ such that $\rho(a_1) \neq 0$ and $\omega_2(a_2) \neq 0$ and extend the pure state $\rho \otimes \omega_2$ on $B \overset{m}{\otimes}_{C(X)} A_2$ to a pure state ω on A_β . If we set $\omega_1(a) = \omega(a \otimes 1)$ for $a \in A_1$, then ω_1 is pure and $\omega(\alpha) = (\omega_1 \otimes \omega_2)(\alpha)$ for all $\alpha \in A_1 \otimes_{alg} A_2$ since ω and ω_2 are pure ([15] lemma 4). As a consequence, $(\omega_1, \omega_2) \in S_\beta$, which is absurd since $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = \rho(a_1)\omega_2(a_2) \neq 0$. \square

PROPOSITION 2.9 *Given two $C(X)$ -algebras A_1 and A_2 , the semi-norm $\| \cdot \|_m$ defines the minimal C^* -norm on the involutive algebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.*

Proof: Let $\| \cdot \|_\beta$ be a C^* -norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. Thanks to the previous proposition, all we need to prove is that one can extend $\| \cdot \|_\beta$ to a C^* -norm on $(\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2)/\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$, where \mathcal{A}_1 and \mathcal{A}_2 are the unital $C(X)$ -algebras associated to the $C(X)$ -algebras A_1 and A_2 (definition 1.1).

Consider the Hausdorff completion D_β of $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$ and denote by π_i the canonical representation of A_i in $M(D_\beta)$ for $i = 1, 2$. Let us define the representation $\tilde{\pi}_i$ of \mathcal{A}_i in $M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X))$ by the following formulas:

$$\begin{aligned} \tilde{\pi}_1(b_1 + u(f))(\alpha \oplus a_1 \oplus a_2 \oplus g) &= (\pi_1(b_1) + g)\alpha \oplus (b_1 + f)a_1 \oplus fa_2 \oplus fg \\ \tilde{\pi}_2(b_2 + u(f))(\alpha \oplus a_1 \oplus a_2 \oplus g) &= (\pi_2(b_2) + g)\alpha \oplus fa_1 \oplus (b_2 + f)a_2 \oplus fg \end{aligned}$$

For $i = 1, 2$, let $\varepsilon_i : \mathcal{A}_i \rightarrow C(X)$ be the map defined by

$$\varepsilon_i[a + u(f)] = f \text{ for } a \in A_i \text{ and } f \in C(X).$$

Then using the maps $(\varepsilon_1 \otimes \varepsilon_2)$, $(\varepsilon_1 \otimes id)$ and $(id \otimes \varepsilon_2)$, one proves easily that if $\alpha \in \mathcal{A}_1 \otimes_{alg} \mathcal{A}_2$, $(\tilde{\pi}_1 \otimes \tilde{\pi}_2)(\alpha) = 0$ if and only if α belongs to $\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$.

Therefore, the norm of $M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X))$ restricted to the subalgebra $(\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2)/\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$ extends $\| \cdot \|_\beta$. \square

Remark: As the $C(X)$ -algebra A is nuclear if and only if every fibre A_x is nuclear ([12]), $A \overset{M}{\otimes}_{C(X)} B \simeq A \overset{m}{\otimes}_{C(X)} B$ for every $C(X)$ -algebra B if and only if A is nuclear.

3 When does the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$ hold?

Given two $C(X)$ -algebras A and B , Giordano and Mingo have studied in [6] the case where the algebra $C(X)$ is a von Neumann algebra: their theorem 3.1 and lemma 1.5 of [10] imply that in that case, we always have the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.

Our purpose in this section is to find sufficient conditions on the $C(X)$ -algebras A and B in order to ensure this equality and to present a counter-example in the general case.

PROPOSITION 3.1 *Let X be a second countable Hausdorff compact space and let A and B be two $C(X)$ -algebras.*

If A is a continuous field of C^ -algebras over X , then $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.*

Proof: Let us prove by induction on the non negative integer n that if

$$s = \sum_{1 \leq i \leq n} a_i \otimes b_i \in \mathcal{J}(A, B),$$

then s belongs to the ideal $\mathcal{I}(A, B)$.

If $n = 0$, there is nothing to prove. Consider therefore an integer $n > 0$ and suppose the result has been proved for any $p < n$.

Fix an element $s = \sum_{1 \leq i \leq n} a_i \otimes b_i \in \mathcal{J}(A, B)$ and define the continuous positive function $h \in C(X)$ by the formula $h(x)^{10} = \sum \|(a_k)_x\|^2$.

The element $a'_k = h^{-4} a_k$ is then well defined in A for every k since $a_k^* a_k \leq h^{10}$. Consequently, the function $f_k(x) = \|(a'_k)_x\|$ is continuous.

For $1 \leq k \leq n$, let D_k denote the separable $C(X)$ -algebra generated by 1 and the $a'_k{}^* a'_j$, $1 \leq j \leq n$, in the unital $C(X)$ -algebra \mathcal{A} associated to A (definition 1.1). Then D_k is a unital continuous field of C^* -algebras over X (see for instance [3] proposition 3.2).

Consider the open subset $\mathcal{S}^k = \{\psi \in \mathcal{S}_X(D_k) / \psi[a'_k{}^* a'_k] > \psi(f_k^2/2)\}$. If we apply lemma 3.6 b) of [3] to the restriction of $p : \mathcal{S}_X(D_k) \rightarrow X$ to \mathcal{S}^k , we may construct a continuous field of states ω_k on D_k such that $\omega_k[a'_k{}^* a'_k] \geq f_k^2/2$.

Now, if we set $s' = \sum_i a'_i \otimes b_i$, as $(a'_k{}^* \otimes 1)s'$ belongs to $\mathcal{J}(D_k, B)$,

$$(\omega_k \otimes id)[(a'_k{}^* \otimes 1)s'] = \omega_k[a'_k{}^* a'_k] b_k + \sum_{j \neq k} \omega_k[a'_k{}^* a'_j] b_j = 0.$$

Noticing that f_k^3 is in the ideal of $C(X)$ generated by $\omega_k[a'_k{}^* a'_k]$, we get that $f_k^3 b_k$ belongs to the $C(X)$ -module generated by the b_j , $j \neq k$, and thanks to the induction hypothesis, it follows that $(f_k^3 \otimes 1)s' \in \mathcal{I}(A, B)$ for each k .

But $h^2 = \sum_k f_k^2$ and so $h^4 \leq n \sum_k f_k^4$ is in the ideal of $C(X)$ generated by the f_k^3 , which implies

$$s = (h^4 \otimes 1)s' \in \mathcal{I}(A, B). \quad \square$$

Remarks: 1. As a matter of fact, it is not necessary to assume that the space X satisfies the second axiom of countability thanks to the following lemma of [11]: if $P(a) \in C(X)$ denotes the map $x \mapsto \|a_x\|$ for $a \in \mathcal{A}$, there exists a separable C^* -subalgebra $C(Y)$ of $C(X)$ with same unit such that if D_k is the separable unital C^* -algebra generated by $C(Y).1 \subset \mathcal{A}$ and the $a'_k{}^* a'_j$, $1 \leq j \leq n$, then $P(D_k) = C(Y)$. Furthermore, if $\Phi : X \rightarrow Y$ is the transpose of the inclusion map $C(Y) \hookrightarrow C(X)$ restricted to pure states, the map $D/(C_{\Phi(x)}(Y)D) \rightarrow A_x$ is a monomorphism for every $x \in X$ since \mathcal{A} is continuous.

Consider now a continuous field of states $\omega_k : D_k \rightarrow C(Y)$ on the continuous field D_k over Y ; if $\sum c^j \otimes d^j \in D_k \otimes_{alg} B$ is zero in $A_x \otimes_{alg} B_x$ for $x \in X$, then $\sum \omega_k(c^j)(x)d_x^j = 0$ in B_x , which enables us to conclude as in the separable case.

2. J. Mingo has drawn the author's attention to the following result of Glimm ([7] lemma 10): if $C(X)$ is a von Neumann algebra and A is a $C(X)$ -algebras, then $P(a)^2 = \min\{z \in C(X)_+, z \geq a^*a\}$ is continuous for every $a \in A$. Therefore we always have in that case the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$ thanks to the previous remark.

COROLLARY 3.2 *Let A and B be two $C(X)$ -algebras and assume that there exists a finite subset $F = \{x_1, \dots, x_p\} \subset X$ such that for all $a \in A$, the function $x \mapsto \|a_x\|$ is continuous on $X \setminus F$.*

Then the ideals $\mathcal{I}(A, B)$ and $\mathcal{J}(A, B)$ coincide.

Proof: Fix an element $\alpha = \sum_{1 \leq i \leq n} a_i \otimes b_i \in A \otimes_{alg} B$ which belongs to $\mathcal{J}(A, B)$. In particular, we have $\sum_i (a_i)_x \otimes (b_i)_x = 0$ in $A_x \otimes_{alg} B_x$ for each $x \in F$.

As a consequence, thanks to theorem III of [13], we may find complex matrices $(\lambda_{i,j}^m)_{i,j} \in M_n(\mathbb{C})$ for all $1 \leq m \leq p$ such that, if we define the elements $c_k^m \in A$ and $d_k^m \in B$ by the formulas

$$c_k^m = \sum_i \lambda_{i,k}^m a_i \text{ and } d_k^m = b_k - \sum_j \lambda_{k,j}^m b_j,$$

we have $(c_k^m)_{x_m} = 0$ and $(d_k^m)_{x_m} = 0$ for all k and all m .

Consider now a partition $\{f_l\}_{1 \leq l \leq p}$ of $1 \in C(X)$ such that for all $1 \leq l, m \leq p$, $f_l(x_m) = \delta_{l,m}$ where δ is the Kronecker symbol and define for all $1 \leq k \leq n$ the elements $c_k = \sum_m f_m c_k^m$ and $d_k = \sum_m f_m d_k^m$. Thus,

$$\begin{aligned} \alpha &= (\sum_i a_i \otimes d_i) + (\sum_{i,j,m} \lambda_{i,j}^m a_i \otimes f_m b_j) \\ &= (\sum_i a_i \otimes d_i) + (\sum_j c_j \otimes b_j) + (\sum_{i,j,m} \lambda_{i,j}^m (a_i \otimes f_m b_j - f_m a_i \otimes b_j)) \end{aligned}$$

and there exists therefore an element $\beta \in \mathcal{I}(A, B)$ such that $\alpha - \beta$ admits a finite decomposition $\sum_i a'_i \otimes b'_i$ with $a'_i \in C_0(X \setminus F)A$ and $b'_i \in C_0(X \setminus F)B$.

But $C_0(X \setminus F)A$ is a continuous field. Accordingly, proposition 3.1 implies that $\alpha - \beta \in \mathcal{I}(C_0(X \setminus F)A, C_0(X \setminus F)B) \subset \mathcal{I}(A, B)$. \square

Remark: If $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of \mathbb{N} and if A and B are two $C(\widetilde{\mathbb{N}})$ -algebras, the corollary 3.2 implies the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.

Let us now introduce a counter-example in the general case.

Consider a dense countable subset $X = \{a_n\}_{n \in \mathbb{N}}$ of the interval $[0, 1]$. The C^* -algebra $C_0(\mathbb{N})$ of sequences with values in \mathbb{C} vanishing at infinity is then endowed with the $C([0, 1])$ -algebra structure defined by:

$$\forall f \in C([0, 1]), \forall \alpha = (\alpha_n) \in C_0(\mathbb{N}), \quad (f.\alpha)_n = f(a_n)\alpha_n \text{ for } n \in \mathbb{N}.$$

If we call A this $C([0, 1])$ -algebra, then $A_x = 0$ for all $x \notin X$.

Indeed, assume that $x \notin X$ and take $\alpha \in A$. If $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\alpha_n| < \varepsilon$ for all $n \geq N$. Consider a continuous function $f \in C([0, 1])$ such that $0 \leq f \leq 1$, $f(x) = 0$ and $f(a_i) = 1$ for every $1 \leq i \leq N$; we then have:

$$\|\alpha_x\| \leq \|(1-f)\alpha\| < \varepsilon.$$

Let $Y = \{b_n\}_{n \in \mathbf{N}}$ be another dense countable subset of $[0, 1]$ and denote by B the associated $C([0, 1])$ -algebra whose underlying algebra is $C_0(\mathbf{N})$.

Then, if $X \cap Y = \emptyset$, the previous remark implies that for every $x \in [0, 1]$, $A_x \otimes_{alg} B_x = 0$ and hence, $\mathcal{I}(A, B) = A \otimes_{alg} B$. What we therefore need to prove is that the ideal $\mathcal{I}(A, B)$ is strictly included in $A \otimes_{alg} B$.

Let us fix two sequences $\alpha \in A$ and $\beta \in B$ whose terms are all non zero and suppose that $\alpha \otimes \beta$ admits a decomposition $\sum_{1 \leq i \leq k} [(f_i \alpha^i) \otimes \beta^i - \alpha^i \otimes (f_i \beta^i)]$ in $A \otimes_{alg} B$. Then for every $n, m \in \mathbf{N}$,

$$\alpha_n \beta_m = \sum_{1 \leq i \leq k} \alpha_n^i \beta_m^i [f_i(a_n) - f_i(b_m)].$$

Now, if we set $\phi_i(a_n) = \alpha_n^i / \alpha_n$ and $\psi_i(b_n) = \beta_n^i / \beta_n$ for $1 \leq i \leq k$ and $n \in \mathbf{N}$, this equality means that for all $(x, y) \in X \times Y$,

$$1 = \sum_{1 \leq i \leq k} \psi_i(x) \phi_i(y) [f_i(x) - f_i(y)].$$

But this is impossible because of the following proposition:

PROPOSITION 3.3 *Let X and Y be two dense subsets of the interval $[0, 1]$ and let n be a non negative integer.*

Given continuous functions f_i on $[0, 1]$ and numerical functions $\psi_i : X \rightarrow \mathbf{C}$ and $\varphi_i : Y \rightarrow \mathbf{C}$ for $1 \leq i \leq n$, if there exists a constant $c \in \mathbf{C}$ such that

$$\forall (x, y) \in X \times Y, \quad \sum_{1 \leq i \leq n} \psi_i(x) \varphi_i(y) [f_i(x) - f_i(y)] = c$$

then $c = 0$.

Proof: We shall prove the proposition by induction on n .

If $n = 0$, the result is trivial. Take therefore $n > 0$ and assume that the proposition is true for any $k < n$.

Suppose then that the subsets X and Y of $[0, 1]$, the functions f_i , ψ_i and φ_i , $1 \leq i \leq n$, satisfy the hypothesis of the proposition for the constant c .

For $x \in X$, let $p(x) \leq n$ be the dimension of the vector space generated in \mathbf{C}^n by the $(\varphi_i(y) [f_i(x) - f_i(y)])_{1 \leq i \leq n}$, $y \in Y$.

If $p(x) < n$, there exists a subset $F(x) \subset \{1, \dots, n\}$ of cardinal $p(x)$ such that for every $j \notin F(x)$:

$$\varphi_j(y) [f_j(x) - f_j(y)] = \sum_{i \in F(x)} \lambda_i^j(x) \varphi_i(y) [f_i(x) - f_i(y)] \text{ for all } y \in Y,$$

where the $\lambda_i^j(x) \in \mathbf{C}$ are given by the Cramer formulas. As a consequence,

$$\sum_{i \in F(x)} \left(\psi_i(x) + \sum_{j \notin F(x)} [\lambda_i^j(x) \psi_j(x)] \right) \varphi_i(y) [f_i(x) - f_j(y)] = c.$$

Now, if $p(x) < n$ for every $x \in X$, there exists a subset $F \subset \{1, \dots, n\}$ of cardinal $p < n$ such that the interior of the closure of the set of those x for which $F(x) = F$ is not empty and contains therefore a closed interval homeomorphic to $[0, 1]$. The induction hypothesis for $k = p$ implies that $c = 0$.

Assume on the other hand that $x_0 \in X$ verifies $p(x_0) = n$. We may then find y_1, \dots, y_n in Y such that if we set

$$a_{i,j}(x) = \varphi_i(y_j)[f_i(x) - f_i(y_j)],$$

the matrix $(a_{i,j}(x_0))$ is invertible. There exists therefore a closed connected neighborhood I of x_0 on which the matrix $(a_{i,j}(x))$ remains invertible.

But for each $1 \leq j \leq n$, $\sum_i a_{i,j}(x)\psi_i(x) = c$ and therefore the $\psi_i(x)$ extend by the Cramer formulas to continuous functions on the closed interval I .

For $y \in Y \cap I$, let $q(y)$ denote the dimension of the vector space generated in \mathbf{C}^n by the $(\psi_i(x)[f_i(x) - f_i(y)])_{1 \leq i \leq n}$, $x \in X \cap I$.

If $q(y) < n$ for every y , then the induction hypothesis implies $c = 0$. But if there exists y_0 such that $q(y_0) = n$, we may find an interval $J \subset I$ homeomorphic to $[0, 1]$ on which the φ_i extend to continuous functions; evaluating the starting formula at a point $(x, x) \in J \times J$, we get $c = 0$. \square

4 The associativity

Given three $C(X)$ -algebras A_1, A_2 and A_3 , we deduce from [3] corollaire 3.17:

$$[(A_1 \overset{M}{\otimes}_{C(X)} A_2) \overset{M}{\otimes}_{C(X)} A_3]_x = (A_1 \overset{M}{\otimes}_{C(X)} A_2)_x \otimes_{\max} (A_3)_x = (A_1)_x \otimes_{\max} (A_2)_x \otimes_{\max} (A_3)_x,$$

which implies the associativity of the tensor product $\cdot \overset{M}{\otimes}_{C(X)} \cdot$ over $C(X)$.

On the contrary, the minimal tensor product $\cdot \overset{m}{\otimes}_{C(X)} \cdot$ over $C(X)$ is not in general associative. Indeed, Kirchberg and Wassermann have shown in [11] that if $\widetilde{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ is the Alexandroff compactification of \mathbf{N} , there exist separable continuous fields A and B such that

$$(A \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} B)_\infty \neq A_\infty \otimes_{\min} B_\infty.$$

If we now endow the C^* -algebra $D = \mathbf{C}$ with the $C(\widetilde{\mathbf{N}})$ -algebra structure defined by $f.a = f(\infty)a$, then for all $C(\widetilde{\mathbf{N}})$ -algebra D' , we have

$$[D \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} D']_{\hat{n}} = \begin{cases} 0 & \text{if } n \text{ is finite,} \\ (D')_\infty & \text{if } n = \infty. \end{cases}$$

Therefore, $[(A \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} B) \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} D]_\infty \simeq (A \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} B)_\infty$ whereas $[A \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} (B \overset{m}{\otimes}_{C(\widetilde{\mathbf{N}})} D)]_\infty$ is isomorphic to $A_\infty \otimes_{\min} B_\infty$.

However, in the case of (separable) continuous fields, we can deduce the associativity of $\cdot \overset{m}{\otimes}_{C(X)} \cdot$ from the following proposition:

PROPOSITION 4.1 *Let A and B be two $C(X)$ -algebras.*

Assume π is a field of faithful representations of A in the Hilbert $C(X)$ -module \mathcal{E} , then the morphism $a \otimes b \rightarrow \pi(a) \otimes b$ induces a faithful $C(X)$ -linear representation of $A \overset{m}{\otimes}_{C(X)} B$ in the Hilbert B -module $\mathcal{E} \otimes_{C(X)} B$.

Proof: Notice that for all $x \in X$, we have $(\mathcal{E} \otimes_{C(X)} B) \otimes_B B_x = \mathcal{E}_x \otimes B_x$.

Now, as B maps injectively in $B_d = \bigoplus_{x \in X} B_x$, $\mathcal{L}_B(\mathcal{E} \otimes_{C(X)} B)$ maps injectively in $\bigoplus_{x \in X} \mathcal{L}_{B_x}(\mathcal{E}_x \otimes B_x) \subset \mathcal{L}_{B_d}(\mathcal{E} \otimes_{C(X)} B \otimes_B B_d)$ and therefore if $\alpha \in A \otimes_{alg} B$, we have $\|(\pi \otimes id)(\alpha)\| = \sup_{x \in X} \|(\pi_x \otimes id)(\alpha_x)\| = \|\alpha\|_m$. \square

Accordingly, if for $1 \leq i \leq 3$, A_i is a separable continuous field of C^* -algebras over X which admits a field of faithful representations in the $C(X)$ -module \mathcal{E}_i , the $C(X)$ -representations of $(A_1 \overset{m}{\otimes}_{C(X)} A_2) \overset{m}{\otimes}_{C(X)} A_3$ and $A_1 \overset{m}{\otimes}_{C(X)} (A_2 \overset{m}{\otimes}_{C(X)} A_3)$ in the Hilbert $C(X)$ -module $(\mathcal{E}_1 \otimes_{C(X)} \mathcal{E}_2) \otimes_{C(X)} \mathcal{E}_3 = \mathcal{E}_1 \otimes_{C(X)} (\mathcal{E}_2 \otimes_{C(X)} \mathcal{E}_3)$ are faithful, and hence the maps $A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \rightarrow (A_1 \overset{m}{\otimes}_{C(X)} A_2) \overset{m}{\otimes}_{C(X)} A_3$ and $A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \rightarrow A_1 \overset{m}{\otimes}_{C(X)} (A_2 \overset{m}{\otimes}_{C(X)} A_3)$ have the same kernel.

References

- [1] B. BLACKADAR *K-theory for Operator Algebras*, M.S.R.I. Publications **5**, Springer Verlag, New York (1986).
- [2] E. BLANCHARD *Représentations de champs de C^* -algèbres*, C.R.Acad.Sc. Paris **314** (1992), 911-914.
- [3] E. BLANCHARD *Déformations de C^* -algèbres de Hopf*, thèse de Doctorat (Paris VII, nov. 1993).
- [4] J. DIXMIER *Les C^* -algèbres et leurs représentations*, Gauthiers-Villars Paris (1969).
- [5] G.A. ELLIOTT *Finite Projections in Tensor Products von Neumann Algebras*, Trans. Amer. Math. Soc. **212** (1975), 47-60.
- [6] T. GIORDANO and J.A. MINGO *Tensor products of C^* -algebras over abelian subalgebras*, preprint, Queen's University, April 1994.
- [7] J. GLIMM *A Stone-Weierstrass theorem for C^* -algebras*, Ann. of Math. **72** (1960), 216-244.
- [8] J. GLIMM *Type I C^* -algebras*, Ann. Math. **73** (1961), 572-612.
- [9] G.G. KASPAROV *Hilbert C^* -modules: theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), 133-150.
- [10] G.G. KASPAROV *Equivariant KK -theory and the Novikov conjecture*, Invent. Math. **91** (1988), 147-201.

- [11] E. KIRCHBERG and S. WASSERMANN *Operations on continuous bundles of C^* -algebras, preprint* (preliminary version), September 1993.
- [12] C. LANCE *Tensor product and nuclear C^* -algebras*, Operator Algebras and Applications, Proc. Symposia Pure Math **38** (1982) Part I, 379–399.
- [13] F.J. MURRAY and J. von NEUMANN *On rings of Operators*, Ann. of Math. **37** (1936), 116–229.
- [14] M.A. RIEFFEL *Continuous fields of C^* -algebras coming from group cocycles and actions*, Math. Ann. **283** (1989), 631–643.
- [15] M. TAKESAKI *On the cross norm of the direct product of C^* -algebras*, Tôhoku Math J. **16** (1964), 111–122.

Etienne Blanchard
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
D – 69120 Heidelberg
e-mail:
blanchar@mathi.uni-heidelberg.de