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# Lattices in semi-simple Lie groups, and multipliers of group $C^*$ -algebras

Mohammed E. B. BEKKA and Alain VALETTE

## 1 Introduction, and some history.

Let  $G$  be a locally compact group, and  $H$  be a closed subgroup. Viewing  $L^1(G)$  as a two-sided ideal in the measure algebra  $M(G)$ , and viewing elements of  $L^1(H)$  as measures on  $G$  supported inside  $H$ , we obtain an action of  $L^1(H)$  on  $L^1(G)$  as double centralizers. It is easy to check (see e.g. proposition 4.1 in [Rie]) that this action extends to an action of the full group  $C^*$ -algebra  $C^*(H)$  as double centralizers on  $C^*(G)$ ; this corresponds to a  $*$ -homomorphism  $j_H : C^*(H) \rightarrow M(C^*(G))$ , where  $M(C^*(G))$  denotes the multiplier  $C^*$ -algebra of  $C^*(G)$ . We now quote from p. 209 of Rieffel's Advances paper [Rie]:

*It does not seem to be known whether this homomorphism  $j_H$  is injective. It will be injective if and only if every unitary representation of  $H$  is weakly contained in the restriction to  $H$  of some unitary representation of  $G$  [Fe2]. J.M.G. Fell has pointed out to us that the example that he gave in which this appeared to fail (p. 445 of [Fe2]) depended on the completeness of the classification of the irreducible representations of  $SL_3(\mathbb{C})$  given in [GeN], and there is now some doubt that this classification is complete [Ste].*

Probably this quotation requires some word of explanation. In [Fe2], Fell studies extensions to the topological framework of Frobenius reciprocity for finite groups. Thus he introduces a list of weak Frobenius properties, the last and weakest one being (WF3):

The locally compact group  $G$  satisfies property (WF3) if, for any closed subgroup  $H$  of  $G$ , every representation  $\sigma$  in the dual  $\hat{H}$  is weakly contained in the restriction  $\pi|_H$  of some unitary representation  $\pi$  of  $G$ .

Property (WF3) is indeed equivalent to the injectivity of  $j_H$  for any closed subgroup  $H$ ; for completeness, we shall give a proof in Proposition 2.1 below. In §6 of [Fe2], Fell wishes to show that even (WF3) may fail, by taking  $G = SL_3(\mathbb{C})$  and  $H = SL_2(\mathbb{C})$ ; to this end he appeals to the incomplete description of  $\hat{G}$  given in [GeN]; Fell's proof was recently corrected in Remark 1.13(i) of [BLS].

In this paper, we take for  $G$  a semi-simple Lie group with finite centre and without compact factor, and as closed subgroup a lattice  $\Gamma$ . In section 3, we prove:

**THEOREM 1.1** *Let  $G$  be a semi-simple Lie group without compact factors, with finite centre and with Kazhdan's property (T). Let  $\Gamma$  be an irreducible lattice in  $G$ , and let  $\sigma$  be a non-trivial irreducible unitary representation of  $\Gamma$  of finite dimension  $n$ . Then  $\sigma$*

determines a direct summand of  $C^*(\Gamma)$  which is contained in the kernel of  $j_\Gamma : C^*(\Gamma) \rightarrow M(C^*(G))$ ; this direct summand is isomorphic to the algebra  $M_n(\mathbb{C})$  of  $n$ -by- $n$  matrices.

If  $G$  is a non-compact simple Lie group with finite centre, then  $G$  has property (T) unless  $G$  is locally isomorphic either to  $SO_o(n, 1)$  or  $SU(n, 1)$  (see [HaV]). For these two families, we prove in section 4:

**THEOREM 1.2** *Let  $G$  be locally isomorphic either to  $SO_o(n, 1)$  or  $SU(n, 1)$ , for some  $n \geq 2$ . Let  $\Gamma$  be a lattice in  $G$ . Denote by  $\hat{\Gamma}_f$  the set of (classes of) irreducible, finite-dimensional unitary representations of  $\Gamma$ . If the trivial representation  $1_\Gamma$  is not isolated in  $\hat{\Gamma}_f$  (for the induced Fell-Jacobson topology), then infinitely many elements of  $\hat{\Gamma}_f$  are not weakly contained in the restriction to  $\Gamma$  of any unitary representation of  $G$ . In particular  $j_\Gamma : C^*(\Gamma) \rightarrow M(C^*(G))$  is not injective.*

In view of Theorems 1.1 and 1.2, it seems natural to formulate the following

**Conjecture.** If  $\Gamma$  is a lattice in a non-compact semi-simple Lie group  $G$ , then  $j_\Gamma : C^*(\Gamma) \rightarrow M(C^*(G))$  is not injective.

This conjecture means that, if  $\rho$  is a representation of  $G$  which is faithful on  $M(C^*(G))$  (e.g. take for  $\rho$  either the universal representation of  $G$ , or the direct sum of all its irreducible representations), then  $\rho|_\Gamma$  is never faithful on  $C^*(\Gamma)$ ; this has bearing on a question of de la Harpe in his paper in these Proceedings (see immediately after Problem 13 in [Har]). In §5, we give examples of lattices  $\Gamma$  in  $SO_o(n, 1)$  or  $SU(n, 1)$  such that  $1_\Gamma$  is not isolated in  $\hat{\Gamma}_f$ ; this is the case for any lattice in  $SL_2(\mathbb{R})$ , any non-uniform lattice in  $SL_2(\mathbb{C})$ , and any arithmetic lattice in  $SO_o(n, 1)$  for  $n \neq 3, 7$ .

In the final §6, we come back to property (WF3) and show that it always fails for almost connected, non-amenable groups:

**THEOREM 1.3** *Let  $G$  be an almost connected, locally compact group. The following properties are equivalent:*

- (i)  $G$  has Fell's property (WF3);
- (ii)  $G$  is amenable.

Observe that Theorem 1.3 cannot hold for any locally compact group. Indeed, any discrete group  $G$  satisfies property (WF3) since, given a subgroup  $H$  of  $G$ , one checks easily that  $C^*(H)$  is a  $C^*$ -subalgebra of  $C^*(G) = M(C^*(G))$ .

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A word about terminology: as usual, semi-simple Lie groups are assumed to be connected and non-trivial; group representations are assumed to be unitary, strongly continuous, and on non-zero Hilbert spaces.

## 2 On multipliers of $C^*$ -algebras.

For a  $C^*$ -algebra  $B$ , we denote by  $M(B)$  its multiplier algebra.

**PROPOSITION 2.1** *Let  $A, B$  be  $C^*$ -algebras, and let  $j : A \rightarrow M(B)$  be a  $*$ -homomorphism. The following properties are equivalent:*

- (i)  $j$  is one-to-one;
- (ii) for any  $\sigma \in \hat{A}$ , there exists a non-degenerate  $*$ -representation  $\pi$  of  $B$  such that  $\sigma$  is weakly contained in  $\tilde{\pi} \circ j$ , where  $\tilde{\pi}$  denotes the extension of  $\pi$  to  $M(B)$ ;
- (iii) any  $\sigma \in \hat{A}$  is weakly contained in  $\{\tilde{\pi} \circ j \mid \pi \in \hat{B}\}$ .

Fell's property (WF3), mentioned in §1, is deduced from property (ii) above by taking  $B = C^*(G)$  and  $A = C^*(H)$ , for any closed subgroup  $H$  of the locally compact group  $G$ .

**Proof of Proposition 2.1.** (i)  $\Rightarrow$  (ii) Let us assume that  $j$  is injective, so that we may identify  $A$  with a  $C^*$ -subalgebra of  $M(B)$ . Let  $\pi$  be a faithful representation of  $B$ . It is known that the extension  $\tilde{\pi}$  of  $\pi$  to  $M(B)$  is also faithful ([Ped], 3.12.5). Thus any representation of  $A$  is weakly contained in the restriction of  $\tilde{\pi}$  to  $A$ .

(ii)  $\Rightarrow$  (iii) This follows from decomposition theory.

(iii)  $\Rightarrow$  (i) Assume that (iii) holds. Fix a non-zero element  $x$  of  $A$ ; choose  $\sigma \in \hat{A}$  such that  $\sigma(x) \neq 0$ . Our assumption says that  $\text{Ker } \sigma$  contains  $\bigcap_{\pi \in \hat{B}} \text{Ker } \tilde{\pi} \circ j = \text{Ker}(\bigoplus_{\pi \in \hat{B}} \tilde{\pi} \circ j)$ ; in particular  $x \notin \text{Ker}(\bigoplus_{\pi \in \hat{B}} \tilde{\pi} \circ j)$ . It follows that  $j(x) \neq 0$ , i.e. that  $j$  is one-to-one.

## 3 Proof of theorem 1.1

We slightly generalize Theorem 1.1 in the following form:

**THEOREM 3.1** *Let  $G$  be a non-compact semi-simple Lie group with finite centre and with Kazhdan's property (T). Let  $\Gamma$  be an irreducible lattice in  $G$ , and let  $\sigma$  be an irreducible representation of  $\Gamma$  of finite dimension  $n$ , which is not contained in the restriction to  $\Gamma$  of a unitary, finite-dimensional representation of  $G$ . Then  $\sigma$  determines a direct summand of  $C^*(\Gamma)$  isomorphic to the algebra  $M_n(\mathbb{C})$  of  $n$ -by- $n$  matrices, which moreover is contained in the kernel of  $j_\Gamma : C^*(\Gamma) \rightarrow M(C^*(G))$ .*

Observe that Theorem 1.1 is an immediate consequence of Theorem 3.1: indeed, if  $G$  has no compact factor, then any unitary, finite-dimensional representation of  $G$  is trivial.

**Proof of Theorem 3.1.** Since  $G$  has property (T), so has  $\Gamma$  (see [HaV], Théorème 4 in Chapter 3). Let  $\sigma$  be an irreducible representation of  $\Gamma$ , of finite dimension  $n$ . By Theorem 2.1 in [Wan],  $\sigma$  is isolated in the dual  $\hat{\Gamma}$ , hence determines a direct sum decomposition of  $C^*(\Gamma)$ :

$$C^*(\Gamma) = J \oplus M_n(\mathbb{C})$$

where  $J$  is the  $C^*$ -kernel of  $\sigma$ .

We assume from now on that  $\sigma$  is not contained in the restriction to  $\Gamma$  of a unitary, finite-dimensional representation of  $G$ , and wish to prove that the direct summand  $M_n(\mathbb{C})$  lies in the kernel of  $j_\Gamma : C^*(\Gamma) \rightarrow M(C^*(G))$ . Suppose by contradiction that  $j_\Gamma$  is non-zero on  $M_n(\mathbb{C})$ . Choose  $\pi \in \hat{G}$  such that  $\tilde{\pi} \circ j_\Gamma$  is non-zero, hence faithful on  $M_n(\mathbb{C})$  (here  $\tilde{\pi}$  denotes the extension of  $\pi$  to  $M(C^*(G))$ , as in Proposition 2.1). Then the  $C^*$ -kernel of  $\tilde{\pi} \circ j_\Gamma$  is contained in  $J$ , which means that  $\sigma$  is weakly contained in the restriction  $\pi|_\Gamma$ . As  $\sigma$  is isolated in  $\hat{\Gamma}$ , this implies that  $\sigma$  is actually a subrepresentation of  $\pi|_\Gamma$  (see Corollary 1.9 in [Wan]). Our assumption shows that  $\pi$  is infinite-dimensional. Two cases may occur:

- (a)  $\pi$  is a discrete series representation of  $G$  (if any); this would imply that  $\sigma$  is an irreducible subrepresentation of the left regular representation of  $\Gamma$ , which in turn implies that  $\Gamma$  is finite - and this is absurd.
- (b)  $\pi$  is not in the discrete series of  $G$ ; then, by a result of Cowling and Steger (Proposition 2.4 in [CoS]), the restriction  $\pi|_\Gamma$  is irreducible, which contradicts the fact that  $\sigma$  is a finite-dimensional subrepresentation.

With a contradiction reached in both cases, the proof of Theorem 3.1 is complete. We thank G. Skandalis for a helpful conversation that led to a more explicit version of Theorem 3.1.

**Remark.** Let us show that there are countably many finite-dimensional elements  $\sigma \in \hat{\Gamma}$  satisfying the assumptions of Theorem 3.1.

Thus, let  $G/Z(G)$  be the adjoint group of  $G$ ; this is a linear group. Denote by  $\Gamma_1$  the image of  $\Gamma$  in  $G/Z(G)$ ; as a finitely generated linear group,  $\Gamma_1$  is residually finite (see [Mal]); a non-trivial irreducible representation  $\sigma$  of  $\Gamma$  that factors through a finite quotient of  $\Gamma_1$  cannot be contained in the restriction to  $\Gamma$  of a finite-dimensional unitary representation of  $G$ .

This argument shows that  $\text{Ker}[j_\Gamma : C^*(\Gamma) \rightarrow M(C^*(G))]$  contains the  $C^*$ -direct sum of countably many matrix algebras.

## 4 The cases $SO_o(n, 1)$ and $SU(n, 1)$ .

We begin with the following result, which is certainly known to many experts (see [Moo], Proposition 3.6; compare also with [Mar], Chap. III, (1.12), Remark 1).

**PROPOSITION 4.1** *Let  $G$  be a simple Lie group with finite centre, and let  $\Gamma$  be a lattice in  $G$ . Denote by  $\gamma$  the quasi-regular representation of  $G$  on  $L^2(G/\Gamma)$ , and by  $\gamma_0$  the restriction of  $\gamma$  to  $L^2_0(G/\Gamma) = \{f \in L^2(G/\Gamma) \mid \int f = 0\}$ .*

- (a) *There exists  $N \in \mathbb{N}$  such that the  $N$ -fold tensor product  $\gamma_0^{\otimes N}$  is weakly contained in the left regular representation  $\lambda_G$  of  $G$ .*
- (b) *The trivial representation  $1_G$  is not weakly contained in  $\gamma_0$ .*

**Proof.** (a) Suppose first that  $G$  has Kazhdan's property (T). Then, by Theorems 2.4.2 and 2.5.3 in [Cow], there exists  $N \in \mathbb{N}$  such that  $\pi^{\otimes N}$  is weakly contained in

$\lambda_G$  for any unitary representation  $\pi$  of  $G$  which does not contain  $1_G$ . This implies the result.

Suppose now that  $G$  is locally isomorphic either to  $SO_o(n, 1)$  or to  $SU(n, 1)$ . Let  $K$  be a maximal compact subgroup of  $G$ . Let  $\hat{G}_1 = \{\pi \in \hat{G} \mid \pi|_K \text{ contains } 1_K\}$  be the set of all spherical representations of  $G$ . Observe that  $\hat{G}_1$  is open in  $\hat{G}$  (because  $\pi \in \hat{G}_1$  if and only if there exists  $\xi \in H_\pi$  such that  $\int_K \langle \pi(k)\xi | \xi \rangle dk \neq 0$ ). For a unitary representation  $\sigma$  of  $G$ , set  $\text{Supp } \sigma = \{\pi \in \hat{G} \mid \pi \text{ is weakly contained in } \sigma\}$ . By Proposition 3.6 in [Moo], the existence of  $N \in \mathbb{N}$  such that  $\sigma^{\otimes N}$  is weakly contained in  $\lambda_G$  is equivalent to  $1_G \notin \text{Supp } \sigma \cap \hat{G}_1$  (the proof of this uses the explicit description of the unitary duals of  $SO_o(n, 1)$  and  $SU(n, 1)$ ). So we must prove that  $1_G$  is not in  $\text{Supp } \gamma_0 \cap \hat{G}_1$  or, equivalently, that  $1_G$  is isolated in  $\text{Supp } \gamma \cap \hat{G}_1$ .

Recall the standard parametrization of  $\hat{G}_1$ . Let  $\rho$  be half the sum of the positive roots associated with a maximal split torus of  $G$ . Then  $\hat{G}_1$  identifies (topologically) with  $i\mathbb{R}^+ \cup [0, \rho]$ , the representations  $\pi_s$  with  $s \in i\mathbb{R}^+$  being the spherical principal series representations, those  $\pi_s$  with  $s \in ]0, \rho[$  being the spherical complementary series representations, and  $\pi_\rho$  being the trivial representation  $1_G$ .

Let  $X$  be the Riemannian symmetric space associated with  $G$ . The Laplace-Beltrami operator  $\Delta$  on  $X$  is invariant for the left action of  $G$ , so it descends to a positive, unbounded operator on  $L^2(\Gamma \backslash X)$ . It is well-known that  $\pi_s$  is weakly contained in  $\gamma$  if and only if  $\rho^2 - s^2$  belongs to the spectrum of  $\Delta$  on  $L^2(\Gamma \backslash X)$  (see §4 of Chap. I in [GGP] for  $G = SL_2(\mathbb{R})$  and  $\Gamma$  uniform, or Theorem 1.7.10 in [GaV] for the general case; note that this Theorem is stated there for the quasi-regular representation of  $G$  on  $L^2(X)$ , but the proof extends word for word to our representation  $\gamma$ ).

Denoting by  $\lambda_1(\Gamma \backslash X)$  the bottom of the spectrum of the restriction of  $\Delta$  to the orthogonal of constants in  $L^2(\Gamma \backslash X)$ , we see that our result follows from  $\lambda_1(\Gamma \backslash X) > 0$ . In turn, this is a consequence of the facts that the continuous spectrum of  $\Delta$  on  $L^2(\Gamma \backslash X)$  is the half-line  $[\rho^2, \infty[$  (see [OsW]), and that its discrete spectrum is a sequence increasing to  $\infty$  (see Theorem 3 in [BoG]). In our case,  $\lambda_1(G \backslash X) > 0$  can also be deduced from the fact that  $\lambda_1(M) > 0$  for any complete Riemannian manifold  $M$  with finite volume and pinched negative sectional curvature (see [Dod]).

(b) This follows from (a) and non-amenability of  $G$ .

### Proof of Theorem 1.2

We shall use several times Fell's inner hull-kernel topology, which is defined on sets of unitary (not necessarily irreducible) representations of a locally compact group (cf. [Fel], section 2): a net  $(\pi_i)_{i \in I}$  of representations converges to a representation  $\pi$  if and only if  $\pi$  is weakly contained in  $\{\pi_j \mid j \in J\}$  for each subnet  $(\pi_j)_{j \in J}$  of  $(\pi_i)_{i \in I}$ .

Assume that  $G$  and  $\Gamma$  satisfy the assumptions of Theorem 1.2. We are going to show that Fell's property (WF3) fails for the pair  $(G, \Gamma)$ ; i.e., we shall produce some  $\sigma \in \hat{\Gamma}$  such that  $\sigma$  is not weakly contained in the set  $\{\pi|_\Gamma \mid \pi \in \hat{G}\}$ .

Since  $1_\Gamma$  is not isolated in  $\hat{\Gamma}_f$ , there exists a sequence  $(\sigma_n)_{n \in \mathbb{N}}$  in  $\hat{\Gamma}_f - \{1_\Gamma\}$  that converges to  $1_\Gamma$ .

**1st step:** There exists a sequence of integers  $n_1 < n_2 < \dots$ , and spherical complementary series representations  $\pi_{n_k}$  of  $G$  such that  $\pi_{n_k}$  is weakly contained in  $\text{Ind}_\Gamma^G \sigma_{n_k}$  for any  $k$ , and  $\lim_{k \rightarrow \infty} \pi_{n_k} = 1_G$ .

Indeed, by continuity of induction ([Fe1], Theorem 4.1),

$$\lim_{n \rightarrow \infty} \text{Ind}_{\Gamma}^G \sigma_n = \text{Ind}_{\Gamma}^G 1_{\Gamma} = \gamma.$$

Since  $1_G$  is a subrepresentation of  $\gamma$ , we also have

$$\lim_{n \rightarrow \infty} \text{Ind}_{\Gamma}^G \sigma_n = 1_G.$$

This implies that there exists integers  $n_1 < n_2 < \dots$ , and irreducible representations  $\pi_{n_k}$  of  $G$  such that  $\pi_{n_k}$  is weakly contained in  $\text{Ind}_{\Gamma}^G \sigma_{n_k}$  for any  $k$ , and such that  $\lim_{k \rightarrow \infty} \pi_{n_k} = 1_G$  (cf. proof of Lemme 2, §1, in [Bur]). Since the spherical dual  $\hat{G}_1$  is open in  $\hat{G}$ , and since  $G$  is not amenable, we can clearly assume that  $\pi_{n_k}$  is either  $1_G$ , or a spherical complementary series representation. To exclude the case  $\pi_{n_k} = 1_G$ , we are going to show that  $1_G$  is not weakly contained in  $\text{Ind}_{\Gamma}^G \sigma_{n_k}$ ; this can be viewed as a form of Frobenius reciprocity.

Indeed, since  $\sigma_{n_k}$  is finite-dimensional,  $\sigma_{n_k}$  does not contain  $1_G$  weakly. Moreover, we know by Proposition 4.1(b) that  $1_G$  is isolated in  $\text{Supp } \gamma$ . Hence, by a result of Margulis ([Mar], Chap. III, (1.11)(b)),  $1_G$  is not weakly contained in  $\text{Ind}_{\Gamma}^G \sigma_{n_k}$ . This proves the 1st step.

Let  $\pi_{n_k} \in \hat{G}$  be a sequence as above. By Proposition 4.1(a), there exists  $N \in \mathbb{N}$  such that  $\gamma_0^{\otimes N}$  is weakly contained in  $\lambda_G$ . Since  $\lim_{k \rightarrow \infty} \pi_{n_k} = 1_G$ , we see that  $\pi_{n_l}^{\otimes N}$  is not weakly contained in  $\lambda_G$  for  $l \in \mathbb{N}$  big enough. Fix such an  $l$ , and set  $\sigma = \sigma_{n_l}$  and  $\pi = \pi_{n_l}$ .

**2nd step:**  $\sigma$  is not weakly contained in  $\{\rho|_{\Gamma} : \rho \in \hat{G}\}$ . Indeed, assume by contradiction that there exists a sequence  $\rho_n \in \hat{G}$  with  $\lim_{n \rightarrow \infty} \rho_n|_{\Gamma} = \sigma$ .

Then  $\lim_{n \rightarrow \infty} \text{Ind}_{\Gamma}^G \rho_n|_{\Gamma} = \text{Ind}_{\Gamma}^G \sigma$ . Hence, since  $\pi$  is weakly contained in  $\text{Ind}_{\Gamma}^G \sigma$  :

$$\lim_{n \rightarrow \infty} \text{Ind}_{\Gamma}^G (\rho_n|_{\Gamma}) = \pi.$$

But

$$\text{Ind}_{\Gamma}^G (\rho_n|_{\Gamma}) = \rho_n \otimes \text{Ind}_{\Gamma}^G 1_{\Gamma} = \rho_n \oplus (\rho_n \otimes \gamma_0).$$

Since  $\pi$  is irreducible, this implies (upon passing to a subsequence) that either  $\lim_{n \rightarrow \infty} \rho_n \otimes \gamma_0 = \pi$  or  $\lim_{n \rightarrow \infty} \rho_n = \pi$ .

We first exclude the case  $\lim_{n \rightarrow \infty} \rho_n \otimes \gamma_0 = \pi$ . Indeed,  $(\rho_n \otimes \gamma_0)^{\otimes N} = \rho_n^{\otimes N} \otimes \gamma_0^{\otimes N}$  is weakly contained in  $\lambda_G$ . Hence,  $\lim_{n \rightarrow \infty} \rho_n \otimes \gamma_0 = \pi$  would imply that  $\pi^{\otimes N} = \lim_{n \rightarrow \infty} (\rho_n \otimes \gamma_0)^{\otimes N}$  is weakly contained in  $\lambda_G$ ; this would contradict our choice of  $\pi$ .

It remains to exclude the case  $\lim_{n \rightarrow \infty} \rho_n = \pi$ . Since the set  $\hat{G}_1^c = \{\pi_s \mid s \in ]0, r[ \}$  of all spherical complementary series representations is open in  $\hat{G}$  and since  $\pi \in \hat{G}_1^c$ , we can clearly assume that  $\rho_n \in \hat{G}_1^c$  for all  $n$ . Then, there exists  $s_0 \in ]0, r[$  such that, for all  $n$ :

$$\rho_n \in \{\pi_s : 0 < s < s_0\}.$$

Therefore, there exists  $M \in \mathbb{N}$  such that  $\rho_n^{\otimes M}$  is weakly contained in  $\lambda_G$ , for all  $n \in \mathbb{N}$ . Hence

$$\sigma^{\otimes M} = \lim_{n \rightarrow \infty} (\rho_n^{\otimes M})|_{\Gamma}$$

is weakly contained in  $\lambda_\Gamma$ . Since  $\sigma^{\otimes M}$  is finite-dimensional, this contradicts non-amenability of  $G$ . This concludes the proof of Theorem 1.2.

**Remark:** In our previous paper [BeV], Theorem 1.2 was already proved for  $G = PSL_2(\mathbb{R})$  and  $\Gamma$  the fundamental group of a closed Riemann surface of genus 2.

## 5 Some examples of lattices in $SO_o(n, 1)$ and $SU(n, 1)$ .

Let  $\Gamma$  be a lattice in a simple Lie group locally isomorphic either to  $SO_o(n, 1)$  or  $SU(n, 1)$ . Let us denote by  $\hat{\Gamma}_{fq}$  the set of elements of  $\hat{\Gamma}$  that factor through some finite quotient of  $\Gamma$ .

**DEFINITION 5.1** *We say that  $\Gamma$  satisfies property (\*) if the trivial representation  $1_\Gamma$  is not isolated in  $\hat{\Gamma}_{fq}$ , for the induced Fell-Jacobson topology.*

Our property (\*) is precisely the negation of property  $(T; R(\mathfrak{S}))$  in the notation of Lubotzky-Zimmer [LuZ], where a lucid discussion of this property appears on pp. 291-292. Since  $\hat{\Gamma}_{fq}$  is a subset of  $\hat{\Gamma}_f$ , it is clear that, if  $\Gamma$  satisfies (\*), then  $1_\Gamma$  is not isolated in  $\hat{\Gamma}_f$ ; note the question at the bottom of p. 291 of [LuZ] whether or not the converse implication holds.

The purpose of this section is to give examples of lattices with property (\*), i.e. for which Theorem 1.2 is true. We begin with a sufficient condition for property (\*).

**PROPOSITION 5.2** *If  $\Gamma$  has a finite index subgroup  $\Gamma_o$  that maps homomorphically onto  $\mathbb{Z}$ , then  $\Gamma$  has property (\*).*

**Proof.** Let  $(\chi_m)_{m \in \mathbb{N}}$  be a sequence of non-trivial characters of finite order of  $\mathbb{Z}$ , viewed as characters of  $\Gamma_o$ , that converges to the trivial character. Set:

$$\pi_m = \text{Ind}_{\Gamma_o}^{\Gamma} \chi_m.$$

**Claim:**  $\pi_m$  factors through some finite quotient of  $\Gamma$ . Indeed, since  $\chi_m$  has finite order, the subgroup  $\text{Ker } \chi_m$  of  $\Gamma_o$  has finite index in  $\Gamma_o$ , so there exists a normal subgroup  $N_m$  of  $\Gamma$ , of finite index and contained in  $\text{Ker } \chi_m$ . Then  $\pi_m$  factors through the finite group  $\Gamma/N_m$ , which establishes the claim.

The rest of the proof is similar in spirit to the first step of the proof of Theorem 1.2, but considerably easier: by continuity of induction, the sequence  $(\pi_m)_{m \in \mathbb{N}}$  converges to the quasi-regular representation  $\lambda_o$  of  $\Gamma$  on  $l^2(\Gamma/\Gamma_o)$ . Since  $\lambda_o$  contains the trivial representation  $1_\Gamma$ , we may select for any  $m \in \mathbb{N}$  an irreducible component  $\sigma_m$  of  $\pi_m$  in such a way that the sequence  $(\sigma_m)_{m \in \mathbb{N}}$  converges to  $1_\Gamma$  in  $\hat{\Gamma}$ . By the claim, each  $\sigma_m$  lies in  $\hat{\Gamma}_{fq}$ ; finally, no  $\sigma_m$  may be trivial, by Frobenius reciprocity. This shows that  $1_\Gamma$  is not isolated in  $\hat{\Gamma}_{fq}$ .

Because  $\Gamma$  is finitely generated, the condition that  $\Gamma_o$  maps homomorphically onto  $\mathbb{Z}$  is equivalent to the non-vanishing of the first cohomology  $H^1(\Gamma_o, \mathbb{C})$ . This is known to have deep representation-theoretic consequences, as it gives information on the decomposition of  $L^2(G/\Gamma_o)$  into irreducibles (see the whole of Chapter VII in [BoW], and



especially Propositions 4.9 and 4.11). There is a conjecture, sometimes attributed to Thurston (see e.g. [Bor], 2.8), according to which any uniform lattice  $\Gamma$  in  $SO_o(n, 1)$  ( $n \geq 2$ ) admits a finite index subgroup  $\Gamma_o$  such that  $H^1(\Gamma_o, \mathbb{C}) \neq 0$ . Next proposition summarizes what we know about this problem, both in the uniform and non-uniform cases.

**PROPOSITION 5.3** *The following lattices  $\Gamma$  in  $SO_o(n, 1)$  admit a finite index subgroup  $\Gamma_o$  such that  $H^1(\Gamma_o, \mathbb{C}) \neq 0$ , and hence satisfy property (\*):*

- (i) any lattice in  $PSL_2(\mathbb{R}) \simeq SO_o(2, 1)$ ;
- (ii) any non-uniform lattice in  $PSL_2(\mathbb{C}) \simeq SO_o(3, 1)$ ;
- (iii) any uniform lattice  $\Gamma$  in  $PSL_2(\mathbb{C})$  such that, for some  $x$  in the 3-dimensional real hyperbolic space  $H_3(\mathbb{R})$ , the orbit  $\Gamma.x$  is invariant under some orientation-reversing involutive isometry of  $H_3(\mathbb{R})$ ;
- (iv) any arithmetic lattice, provided  $n \neq 3, 7$ ; any non-uniform arithmetic lattice, without restriction on  $n$ .

**Proof.** The proof is compilation; however, it makes constant use of Selberg's lemma asserting that any lattice has a torsion-free subgroup of finite index.

- (i) A torsion-free lattice in  $PSL_2(\mathbb{R})$  is either a surface group (in the uniform case) or a non-abelian free group (in the non-uniform case); in any case, it surjects onto  $\mathbb{Z}$ .
- (ii) Any torsion-free non-uniform lattice in  $PSL_2(\mathbb{C})$  surjects onto  $\mathbb{Z}$ , by Propositions 5.1 and 3.1 in [Lub].
- (iii) See Theorem 3.3 and Corollary 3.4 in [Hem]. Explicit examples of such lattices are given in §4 of [Hem].
- (iv) The first statement is the main result of [LiM]. For the second one, combine the main result in [Mil] with the remarks on p. 365 of [LiM].

Concerning uniform lattices in  $PSL_2(\mathbb{C})$ , it seems appropriate to mention here the connection with a somewhat (in)famous question which is for sure due to Thurston (question 18 in [Thu]): does any complete, finite-volume, hyperbolic 3-manifold have a finite-sheeted cover that fibers over the circle  $S^1$ ? An affirmative answer would imply that any lattice  $\Gamma$  in  $PSL_2(\mathbb{C})$  satisfies property (\*) (indeed, let  $\Gamma_1$  be a torsion-free subgroup of finite index in  $\Gamma$ ; then  $\Gamma_1$  is the fundamental group,  $\pi_1(M)$ , of a complete finite-volume hyperbolic 3-manifold  $M$ ; if  $N$  is a finite-sheeted cover of  $M$  which fibers over  $S^1$ , then  $\Gamma_o = \pi_1(N)$  is a finite-index subgroup of  $\Gamma_1$  that maps onto  $\pi_1(S^1) = \mathbb{Z}$ ). For an example of a compact hyperbolic 3-manifold that does not fiber over  $S^1$  but with a finite-sheeted cover that does, see example 2.1 in [Gab]<sup>1</sup>.

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<sup>1</sup>Clearly, for a 3-dimensional closed hyperbolic manifold  $M$ , fibering over  $S^1$  is a much stronger condition than having non-zero first Betti number. Algebraically, this can be seen by Stallings' fibration theorem [Sta]: if  $N$  is a normal subgroup of  $\pi_1(M)$  such that  $\pi_1(M)/N = \mathbb{Z}$ , then  $N$  comes from a fibration of  $M$  over  $S^1$  if and only if  $N$  is a finitely generated subgroup. Also, surface groups in  $\pi_1(M)$  that come from some finite-sheeted cover of  $M$  fibering over  $S^1$  (so-called virtual fibre groups) have been characterized algebraically in Corollary 1 of [Som].

In contrast with Proposition 5.3, we are not aware of any "large" class of lattices in  $SU(n, 1)$  that satisfies property (\*). Essentially the only result we know is that, for any  $n \geq 2$ , there exists a uniform arithmetic lattice  $\Gamma$  in  $SU(n, 1)$  such that  $H^1(\Gamma, \mathbb{C}) \neq 0$  (see Theorem 1 in [Kaz], or Theorem 1.4(b) in [Li]).

To conclude, let us indicate why, for a given lattice  $\Gamma$  in  $SO_o(n, 1)$ , it is usually difficult to check that  $\Gamma$  satisfies property (\*). Assume that  $\Gamma$  is arithmetic. It is then easy to construct elements of  $\hat{\Gamma}_{fq}$ : take a congruence subgroup  $\Gamma(\wp)$  and consider irreducible representations of  $\Gamma$  that factor through the finite group  $\Gamma/\Gamma(\wp)$  (for  $\Gamma = SL_2(\mathbb{Z})$ , these are representations that factor through some  $SL_2(\mathbb{Z}/n\mathbb{Z})$ ). Denote by  $\hat{\Gamma}_{arith}$  the subset of elements in  $\hat{\Gamma}_{fq}$  that factor through some  $\Gamma/\Gamma(\wp)$ ; then it follows from Selberg's inequality (see [Sel] for  $n = 2$ , and corollary 1.3 in [BuS] for  $n > 2$ ) that the trivial representation  $1_\Gamma$  is isolated in  $\hat{\Gamma}_{arith}$ . Thus, if  $\Gamma$  verifies property (\*), any non-stationary net in  $\hat{\Gamma}_{fq}$  that converges to  $1_\Gamma$  will have to leave  $\hat{\Gamma}_{arith}$  eventually.

We have been informed by M. Burger that, in unpublished work with P. Sarnak, similar phenomena have been obtained for a large class of arithmetic lattices in  $SU(n, 1)$ .

## 6 Proof of Theorem 1.3

We begin with hereditary properties of the class of groups satisfying Fell's property (WF3).

**LEMMA 6.1** *Let  $G$  be a locally compact group with property (WF3).*

- (a) *Any closed subgroup of  $G$  has property (WF3).*
- (b) *Let  $K$  be a compact normal subgroup of  $G$ ; then  $G/K$  has property (WF3).*

**Proof.** (a) is obvious. To see (b), denote by  $p : G \rightarrow G/K$  the quotient map. Let  $L$  be a closed subgroup of  $G/K$ ; fix  $\tau \in \hat{L}$ . Set  $H = p^{-1}(L)$  and  $\sigma = \tau \circ (p|_H)$ . Let  $\pi$  be a representation of  $G$  on a Hilbert space  $\mathcal{H}$  such that  $\pi|_H$  weakly contains  $\sigma$ . Let  $\mathcal{H}^K$  be the space of  $K$ -fixed vectors in  $\mathcal{H}$ . Since  $K$  is a normal subgroup,  $\mathcal{H}^K$  is an invariant subspace of  $\pi$ , and we denote by  $\pi_o$  the restriction of  $\pi$  to  $\mathcal{H}^K$ . Since  $K$  is compact and  $\sigma$  is irreducible, it is easy to see that  $\sigma$  is weakly contained in  $\pi_o|_H$ . But  $\pi_o|_H$  can be viewed as a representation of  $L = H/K$ , that weakly contains  $\tau$ .

Next lemma is probably well-known.

**LEMMA 6.2** *Let  $G$  be a Lie group, and let  $S$  be a semisimple analytic subgroup. The closure  $\bar{S}$  of  $S$  is reductive.*

**Proof.** We begin with a

Claim: Let  $h$  be a finite-dimensional Lie algebra, and let  $s$  be a semisimple ideal; then there exists an ideal  $j$  of  $h$  such that  $h = s \oplus j$ . Indeed, let  $Der(s)$  be the Lie algebra of derivations of  $s$ . Since  $s$  is an ideal in  $h$ , we have a Lie algebra homomorphism:

$$\alpha : h \rightarrow Der(s) : X \rightarrow ad(X)|_s$$

the kernel of which is precisely the centralizer of  $s$  in  $h$ ; set  $j = \text{Ker}\alpha$ . Since  $s$  is semisimple,  $\text{Der}(s)$  is canonically isomorphic to  $s$ , so that  $\alpha$  is onto and  $h = s \oplus j$ ; this establishes the claim.

To prove Lemma 6.2, denote by  $s$  and  $h$  the Lie algebras of  $S$  and  $\bar{S}$  respectively. Clearly  $\text{Ad}(x)(s) = s$  for any  $x$  in  $S$ , so by density the same is true for any  $x$  in  $\bar{S}$ . This shows that  $s$  is an ideal in  $h$ . By the claim, there exists an ideal  $j$  of  $h$  such that  $h = s \oplus j$ . To see that  $h$  is reductive, it is enough to prove that  $j$  is central in  $h$ . But, for  $X \in j$ , we have  $\text{Ad}(x)(X) = X$  for any  $x$  in  $S$ ; again by density, this remains true for any  $x$  in  $\bar{S}$ ; so  $X$  is central in  $h$ .

**Proof of Theorem 1.3** It is easy to see that any amenable group  $G$  satisfies property (WF3); indeed, for a closed subgroup  $H$  of  $G$ , by amenability of  $H$  any representation of  $H$  is weakly contained in the left regular representation of  $H$ , which is itself contained in the restriction to  $H$  of the left regular representation of  $G$ ; see also Corollary 1.5 in [BLS] for another proof.

Let us now prove the converse, namely that any almost connected locally compact group  $G$  with property (WF3) is amenable. In this proof, the stability of amenability under short exact sequences will be used constantly.

**1st step: reduction to the connected case.** Let  $G_0$  be the connected component of the identity of  $G$ . By Lemma 6.1(a),  $G_0$  has property (WF3). If  $G_0$  is amenable, then so is  $G$ , since  $G/G_0$  is compact.

**2nd step: reduction to the Lie group case.** Let  $G$  be a connected group with property (WF3). By the structure theory for connected groups,  $G$  admits a compact normal subgroup  $K$  such that  $G/K$  is a Lie group. By Lemma 6.1(b),  $G/K$  has property (WF3). If  $G/K$  is amenable, then so is  $G$ , since  $K$  is compact.

**3rd step: reduction to the reductive case.** Let  $G$  be a connected Lie group with property (WF3). Let  $G = RS$  be a Levi decomposition, with  $R$  the solvable radical and  $S$  a semisimple analytic subgroup. Then the closure  $\bar{S}$  is reductive with property (WF3), by Lemmas 6.1(a) and 6.2. If  $\bar{S}$  is amenable, then so is  $\bar{S}/(\bar{S} \cap R) = G/R$ , hence so is  $G$ .

**Coda.** Let  $G$  be a connected, reductive Lie group with property (WF3). The adjoint group  $G/Z(G)$  is a semisimple Lie group without centre, so it decomposes as a direct product

$$G/Z(G) = G_1 \times \cdots \times G_n$$

of simple Lie groups without centre. To prove that  $G$  is amenable, we have to show that  $G_j$  is compact for  $j = 1, \dots, n$ . So suppose by contradiction that some  $G_j$ , say  $G_1$ , is not compact. By root theory,  $G_1$  then contains a 3-dimensional analytic subgroup  $L$  which is locally isomorphic to  $SL_2(\mathbb{R})$ . Because  $G_1$  is centreless, hence linear,  $L$  is closed in  $G_1$  (any semisimple analytic subgroup in a linear group is closed, see Theorem 2 in [Got]). By the proof of Theorem 3 in [BeV], there exists a lattice  $\Gamma$  in  $L$  and a representation  $\tau \in \hat{\Gamma}_f$  such that  $\tau \otimes \bar{\tau}$  is not weakly contained in the restriction to  $\Gamma$  of any unitary representation of  $L$ . Denote by  $p : G \rightarrow G_1$  the homomorphism obtained by composing the quotient map  $G \rightarrow G/Z(G)$  with the projection of  $G/Z(G)$  onto  $G_1$ . Set  $H = p^{-1}(\Gamma)$  and  $\sigma = \tau \circ (p|_H)$ . Because  $H$  is closed in  $G$ , we find by property

(WF3) a net  $(\rho_i)_{i \in I}$  in  $\hat{G}$  such that

$$\lim_i \rho_i|_H = \sigma.$$

Then:

$$\lim_i (\rho_i \otimes \bar{\rho}_i)|_H = \sigma \otimes \bar{\sigma}.$$

But, since  $\rho_i$  is irreducible, the representation  $\rho_i \otimes \bar{\rho}_i$  of  $G$  is trivial on  $Z(G)$ , so it factors through a representation  $\pi_i$  of  $G/Z(G)$ . Last formula then reads:

$$\lim_i \pi_i|_\Gamma = \tau \otimes \bar{\tau}$$

and this contradicts our choice of  $G$  and  $\tau$ .

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