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Astérisque, tome 232 (1995), p. 211-230

<http://www.numdam.org/item?id=AST_1995__232__211_0>

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1 Introduction

While entropy has for a third of a century been a central concept in ergodic theory, its non-Abelian counterpart is still in its adolescent stage with only a few signs of mature strength. The signs, however, are promising and show a potential of a subject of importance in operator algebras, so much that I am glad to use this opportunity to take the reader on a guided tour of its ideas and their resulting definitions and theorems. I have also included some open problems with the hope that they may inspire further development of the subject into maturity. In addition to giving the necessary definitions I shall mainly be concerned with explicit formulas for entropy of automorphisms. I shall therefore not discuss entropy of endomorphisms and completely positive maps, nor will I say much about applications to physics.

There is another very promising approach to non-Abelian entropy which we shall not discuss but is presently persued by Voiculescu [32,33]. The definitions are quite different from the ones we shall give, but the values of the entropies are closely related to ours in nice cases, but are essentially different in general, see section 5.

2 Definitions and basic results

Before we embark on the non-Abelian definition of entropy let us recall the classical definition. We are then given a probability space $(X, \mathcal{B}, \mu)$ and a nonsingular measure preserving transformation T of X. If $\mathcal{P} = (P_1, \ldots, P_n)$ is a measurable partition of X we shall often identify it with the finite dimensional Abelian algebra generated by the characteristic functions $\mathcal{X}_{P_i}$. The entropy of $\mathcal{P}$ is

$$H(\mathcal{P}) = \sum_{i=1}^{n} \eta(\mu(P_i)),$$

where η is the real function on the unit interval, $\eta(t) = -t \log t$ for $t \in (0,1]$, and $\eta(0) = 0$. If $\mathcal{P} \vee \mathcal{Q}$ is the partition generated by two partitions $\mathcal{P}$ and $\mathcal{Q}$ then $H(\mathcal{P} \vee \mathcal{Q}) \leq H(\mathcal{P}) + H(\mathcal{Q})$, so we have convergence of the sequence

$$\frac{1}{k} H\left(\bigvee_{i=0}^{k-1} T^{-i} \mathcal{P}\right).$$

We denote by $H(T,\mathcal{P})$ its limit, and define the entropy of T by

$$H(T) = \sup_{\mathcal{P}} H(T,\mathcal{P}), \tag{2.1}$$

where the sup is taken over all finite measurable partitions. The crucial result for computing $H(T)$ is the Kolmogoroff-Sinai Theorem [34, 4.17].

Theorem 2.2. *If $\mathcal{P}$ is a generator, i.e. the σ-algebra generated by $(T^{-i}\mathcal{P})_{i\in\mathbb{Z}}$ equals $\mathcal{B}$, written $\bigvee\limits_{-\infty}^{\infty} T^{-i}\mathcal{P} = \mathcal{B}$, then $H(T) = H(T,\mathcal{P})$.*

There is another version of this theorem which will be of interest in the sequel [34, 4.22].

Theorem 2.3. *If $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is an increasing sequence of partitions of X with $\bigvee\limits_{n=1}^{\infty} \mathcal{P}_n = \mathcal{B}$ then*

$$H(T) = \lim_n H(T,\mathcal{P}_n)\,.$$

If one wants to extend the above definition of entropy to von Neumann algebras one is immediately confronted with a major obstacle. While there is a natural extension of the concept of finite partitions, namely finite dimensional von Neumann algebras, there is no natural candidate for the analogue of the partition $\mathcal{P}\vee\mathcal{Q}$ generated by $\mathcal{P}$ and $\mathcal{Q}$. Remember that the von Neumann algebra generated by two finite dimensional algebras can easily be infinite dimensional. However, if one considers the function $H\left(\bigvee\limits_{i=1}^{k}\mathcal{P}_i\right)$ with $\mathcal{P}_i$ finite partitions as a function $H(\mathcal{P}_1,\dots,\mathcal{P}_k)$ of k-variables, one can try to generalize this function. This will now be done following [8] for a von Neumann algebra M with a faithful normal finite trace τ such that $\tau(1) = 1$. In section 6 we shall see how this definition can be extended to general C^*-algebras and states.

For each $k \in \mathbb{N}$ denote by S_k the set of multiple indexed finite partitions of unity of M^+, $(x_{i_1\dots i_k})_{i_j\in\mathbb{N}}$, i.e. each $x_{i_1\dots i_k} \in M^+$, zero except for a finite number of indices and satisfying $\sum\limits_{i_1,\dots,i_k} x_{i_1\dots i_k} = 1$.

For $x \in S_k$, $\ell \in \{1,\dots,k\}$, $i_\ell \in \mathbb{N}$, we put

$$x^{\ell}_{i_\ell} = \sum_{i_1,\dots,i_{\ell-1},i_{\ell+1},\dots,i_k} x_{i_1\dots i_k}\,.$$

If $N \subset M$ is a von Neumann subalgebra we denote by E_N the unique τ-invariant conditional expectation $E_N : M \to N$ defined by the identity

$$\tau(E_N(x)y) = \tau(xy), \qquad x \in M,\ y \in N\,.$$

Definition 2.4. Let $N_1, \dots, N_k$ be finite dimensional von Neumann subalgebras of M. Then

$$H(N_1, \dots, N_k) = \sup_{x \in S_k} \left\{ \sum_{i_1, \dots, i_k} \eta(\tau(x_{i_1 \dots i_k})) - \sum_{\ell=1}^{k} \sum_{i_\ell} \tau(\eta(E_{N_\ell}(x^\ell_{i_\ell}))) \right\}.$$

Since the trivial partition $x = (1)$ gives the value zero, $H \geq 0$. Also it is clear that H is symmetric in the N's. Furthermore H satisfies the following nice requirements

(A) $H(N_1, \dots, N_k) \leq H(P_1, \dots, P_k)$ when $N_j \subset P_j$, $j = 1, \dots, k$.

(B) $H(N_1, \dots, N_k) \leq H(N_1, \dots, N_j) + H(N_{j+1}, \dots, N_k)$ for $1 \leq j < k$.

(C) $N_1, \dots, N_j \subset N \Rightarrow H(N_1, \dots, N_j, N_{j+1}, \dots, N_k) \leq H(N, N_{j+1}, \dots, N_k)$.

(D) For any family of minimal projections of N, $(e_\alpha)_{\alpha \in I}$ such that $\sum_{\alpha \in I} e_\alpha = 1$ one has $H(N) = \sum_{\alpha \in I} \eta\tau(e_\alpha)$.

(E) If $(N_1 \cup \dots \cup N_k)''$ is generated by pairwise commuting von Neumann subalgebras P_j of N_j then

$$H(N_1, \dots, N_k) = H\big((N_1 \cup \dots \cup N_k)''\big).$$

The crucial technical ingredient in the proof of the above properties, and in particular of (C), is the relative entropy of two states, or rather positive operators in our case, defined by

$$S(x|y) = \tau(x(\log x - \log y)), \qquad x, y \in M^+, \ x \leq \lambda y$$

for some $\lambda > 0$. For general normal states of von Neumann algebras the relative entropy is defined by Araki [1] via the relative modular operator of the two states, and by Pusz and Woronowicz [25] for states of C^*-algebras. The main property of S is that it is a jointly convex function in x and y [16], see also [15] and [25].

Having H it is now an easy matter to extend the classical definition (2.1) of entropy. We look at the measure preserving transformation T on $(X, \mathcal{B}, \mu)$ as an automorphism α_T of the Abelian von Neumann algebra $L^\infty(X, \mathcal{B}, \mu)$ defined by $\alpha_T(f) = f \circ T^{-1}$, and partitions as finite dimensional algebras.

Definition 2.5 Let α be an automorphism of M such that $\tau \circ \alpha = \tau$. If $N \subset M$ is finite dimensional we let

$$H(\alpha, N) = \lim_{k \to \infty} \tfrac{1}{k} H(N, \alpha(N), \dots, \alpha^{k-1}(N)),$$

where as in the classical case the sequence converges by the subadditivity of H, property (B). The *entropy* of α is

$$H(\alpha) = \sup_N H(\alpha, N),$$

where the sup is taken over all finite dimensional subalgebras $N \subset M$.

Remark 2.6. If $P \subset M$ is a von Neumann subalgebra such that $\alpha(P) = P$, it is immediate from the definition that the restriction $\alpha|P$ satisfies $H(\alpha|P) \leq H(\alpha)$.

Remark 2.7. If α is periodic then $H(\alpha, N) = 0$ for all N, hence $H(\alpha) = 0$. More generally, if α is contained in a compact subgroup of $\mathrm{Aut}(M)$ then Besson [2] has shown that we still have $H(\alpha) = 0$.

To compute $H(\alpha)$ it is as in the classical case necessary to reduce the choice of N's. The following concept is helpful for this purpose.

Definition 2.8. If N and P are finite dimensional von Neumann subalgebras of M their *relative entropy* is

$$H(N|P) = \sup_{x \in S_1} \sum_i (\tau\eta E_P(x_i) - \tau\eta E_N(x_i)).$$

$H(N|P)$ has the following nice properties:

(F) $H(N_1, \ldots, N_k) \leq H(P_1, \ldots, P_k) + \sum\limits_{j=1}^{k} H(N_j|P_j)$.

(G) $H(N|Q) \leq H(N|P) + H(P|Q)$.

(H) $H(N|P)$ is increasing in N and decreasing in P.

(I) If N and P commute then

$$H(N|P) = H((N \cup P)''|P) = H(H \cup P)'') - H(P).$$

Properties (F), (G), (H) are easy to prove, while (I) is a consequence of the Lieb-Ruskai second strong subadditivity property [17]. The relative entropy is continuous in the following sense.

Theorem 2.9. *For all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all pairs of von Neumann subalgebras N and P of M with $\dim N = n$, we have*

$$N \overset{\delta}{\subset} P \Rightarrow H(N|P) < \varepsilon .$$

Here $N \overset{\delta}{\subset} P$ means that for all $x \in N$, $\|x\| \leq 1$, there exists $y \in P$ with $\|y\| \leq 1$ such that $\|x - y\|_2 < \delta$, where $\|z\|_2 = \tau(z^*z)^{1/2}$. This result together with property (F) is very useful in restricting the choice of N in the definition of $H(\alpha)$. An example is the proof of the generalization of the Kolmogoroff-Sinai Theorem (2.3).

Theorem 2.10. *Suppose M is hyperfinite with an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of finite dimensional subalgebras with union weakly dense in M. Then if $\alpha \in \mathrm{Aut}(M)$ and $\tau \circ \alpha = \tau$ we have*

$$H(\alpha) = \lim_n H(\alpha, P_n) .$$

Proof. Given N and $\varepsilon > 0$ choose by Theorem 2.9 P_n such that $H(N|P_n) < \varepsilon$. Then for $k \in \mathbb{N}$, by property (F),

$$\begin{aligned} H(\alpha, N) &= \lim_k \tfrac{1}{k} H(N, \alpha(N), \ldots, \alpha^{k-1}(N)) \\ &\le \lim_k \tfrac{1}{k} H(P_n, \alpha(P_n), \ldots, \alpha^{k-1}(P_n)) + \overline{\lim_k} \tfrac{1}{k} \sum_{j=0}^{k-1} H(\alpha^j(N)|\alpha^j(P_n)) \\ &\le H(\alpha, P_n) + \varepsilon, \end{aligned}$$

since $H(\alpha^j(N)|\alpha^j(P_n)) = H(N|P_n)$. The theorem follows.

3 Entropy of shifts

The definition of entropy as given in section 2 had originally as its main model the n-shift. Therefore, to make the results of section 2 easier to understand let me describe the n-shift in some detail.

Let $n \in \mathbb{N}$ and $M^i = M_n(\mathbb{C})$ for $i \in \mathbb{Z}$. Let $\tau_i = \frac{1}{n} Tr$ be the normalized trace on M^i. Let

$$R = \bigotimes_{-\infty}^{\infty} (M^i, \tau_i)$$

be the von Neumann algebra tensor product of the M^i with respect to the traces τ_i. Then R is the hyperfinite II_1-factor with trace $\tau = \otimes \tau_i$. Even though it is not needed for the computation of the entropy of the shift let me as an illustration introduce a partition which will give the right value for $H(N_1, \ldots, N_k)$.

Let e_j be the minimal projection in the diagonal in $M_n(\mathbb{C})$ with matrix which is 0 except in the j^{th} row, where it is 1. Let $k \in \mathbb{N}$ and

$$x_{i_1 \ldots i_k} = \cdots \otimes 1 \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes 1 \otimes \cdots,$$

where $e_{i_j} \in M^j$. Then $(x_{i_1 \ldots i_k}) \in S_k$,and

$$x^{\ell}_{i_\ell} = \cdots \otimes 1 \otimes e_{i_\ell} \otimes 1 \otimes \cdots .$$

Put

$$N_i = \cdots \otimes 1 \otimes M^i \otimes 1 \otimes \cdots \subset R,$$

i.e. N_i is M^i imbedded in R. Let σ be the n-shift on R, i.e. σ is the shift automorphism on $\otimes M^i$ which maps each factor one factor to the right. Since $\eta(e) = 0$ for each projection e, we have

$$\begin{aligned} H(N_1, \sigma(N_1), \ldots, \sigma^{k-1}(N_1)) &= H(N_1, \ldots, N_k) \\ &\ge \sum \eta\tau(x_{i_1 \ldots i_k}) - \sum_\ell \sum_{i_\ell} \tau\eta E_{N_\ell}(x^\ell_{i_\ell}) \\ &= n^k \eta(n^{-k}) - \sum_\ell \sum_{i_\ell} \tau\eta(e_{i_\ell}) \\ &= k \log n - 0, \end{aligned}$$

so that $H(\sigma, N_1) \geq \log n$.

Now $(N_1 \cup N_2 \cup \cdots \cup N_k)''$ is a factor of type I_{n^k}, hence we have by properties (D) and (E)

$$\tfrac{1}{k} H(N_1, \ldots, N_k) = \tfrac{1}{k} \log n^k = \log n\,,$$

so the sup is indeed obtained for our choice $(x_{i_1} \ldots i_k)$ in S_k. To show that $H(\sigma) = \log n$ we put

$$P_q = \cdots \otimes 1 \otimes \bigotimes_{-q}^{q} M^i \otimes 1 \otimes \cdots = \Big(\bigcup_{-q}^{q} N_i\Big)''.$$

Then the sequence (P_q) satisfies the requirements of the Kolmogoroff-Sinai Theorem 2.10. Since

$$(P_q \cup \sigma(P_q) \cup \cdots \cup \sigma^{k-1}(P_q))'' = \Big(\bigcup_{-q}^{q+k-1} N_i\Big)'',$$

we have by property (E) that

$$\tfrac{1}{k} H(P_q, \sigma(P_q), \ldots, \sigma^{k-1}(P_q)) = \tfrac{1}{k} \log n^{2q+k} = \frac{2q+k}{k} \log n$$

Therefore

$$H(\sigma) = \lim H(\sigma, P_q) = \log n\,.$$

It should be noted from the above computation that if D denotes the diagonal in P_q then we have

$$\begin{aligned} H(P_q, \sigma(P_q), \ldots, \sigma^{k-1}(P_q)) &= H(D, \sigma(D), \ldots, \sigma^{k-1}(D)) \\ &= H(D \cup \sigma(D) \cup \cdots \cup \sigma^{k-1}(D))'')\,. \end{aligned}$$

Thus the shift σ is not much different from the classical n-shift. This situation prevails for a large class of non-Abelian extensions of classical shift operators. Using the above notation we define them as follows. Let

$$X = \prod_{-\infty}^{\infty} X_i\,, \qquad X_i = \{1, \ldots, n\} \quad \text{for all } i\,.$$

Let S be the shift on X given by $S(x_i) = (x_{i+1})$, and suppose μ is an invariant probability measure on X. If D_n denotes the diagonal in $M_n(\mathbb{C})$ we identify $C(X_0)$ with D_n, and thus $C(X)$ with the infinite tensor product $D = \bigotimes_{-\infty}^{\infty} D_n^i$, where $D_n^i = D_n$. S induces a shift σ_0 on D in the obvious way, and μ defines likewise a σ_0-invariant state ω on D. Let E_n denote the canonical trace invariant expectation of $M_n(\mathbb{C})$ on D_n, and let $E = \bigotimes_{-\infty}^{\infty} E_n^i$ with $E_n^i : M^i \to D_n^i$ equal to E_n. Let $A = \bigotimes_{-\infty}^{\infty} M^i$ be the C^*-algebraic infinite tensor product. Then E is a conditional expectation of A on D. Extend ω to a state ρ on A by $\rho = \omega \circ E$. If σ_1 denotes the shift on A then $\rho \circ \sigma_1 = \rho$ and $\sigma_0 \circ E = E \circ \sigma_1$. Let (π, ξ, H) be the GNS-representation of ρ, and put $M = \pi(A)''$. Then σ_1 extends to an ω_ξ-invariant automorphism of M, hence restricts to an ω_ξ-invariant automorphism σ of the centralizer M_ρ of ω_ξ in M. If the measure μ is sufficiently ergodic, M is a factor, and even M_ρ is often a factor with trace $\tau = \omega_\xi$. From its construction M_ρ is the hyperfinite II_1-factor R, and σ is an ergodic automorphism of R. In the cases which have been computed it turns out that the entropy $H(\sigma)$ is equal to the classical entropy $H(S)$ of S with respect to μ.

The above construction was first performed by Connes and myself for Bernoulli shifts in [8], then by Besson [3] for Markov shifts, however both results were known to Krieger (unpublished). Later on these results were generalized by Quasthoff [26], who used a different, however equivalent construction to define shifts on R. He also showed such a result for an extension to flows of Markov shifts [27].

There is an interesting shift automorphism on R which arises from the theory of subfactors. For $\lambda \in (0,1]$ let $\{e_0, e_1, e_2, \ldots\}$ be a sequence of projections in R satisfying the axioms

a) $e_i e_{i\pm 1} e_i = \lambda e_i$,

b) $e_i e_j = e_j e_i$ for $|i-j| \geq 2$,

c) $\tau(we_i) = \lambda \tau(w)$ if w is a word on $\{1, e_0, \ldots, e_{i-1}\}$.

It was shown by Jones [14] that such a sequence exists and generates R if and only if $\lambda \in \left(0, \frac{1}{4}\right] \cup \left\{\frac{1}{4}\sec^2\frac{\pi}{n} : n \geq 3\right\}$. Furthermore, if R_λ is the subfactor generated by $\{1, e_1, e_2, \ldots\}$ then the index $[R : R_\lambda] = \lambda^{-1}$.

We now reindex the projections e_i so the index set is $\mathbb{Z}$, and we denote by θ_λ the shift automorphism of R defined by $\theta_\lambda e_i = e_{i+1}$. It follows from the relations a)–c) that θ_λ is mixing with respect to the trace, hence is ergodic. This automorphism was studied by Pimsner and Popa in [22]. They showed by explicit construction of a sequence (e_i) that if $\lambda < 1/4$ then θ_λ is a Bernoulli shift as described above, and $H(\theta_\lambda) = \eta(t) + \eta(1-t)$, where $t(1-t) = \lambda$. If $\lambda > 1/4$ the situation is quite different and they showed $H(\theta_\lambda) = -\frac{1}{2}\log\lambda$. The same result was then showed by Choda [4] when $\lambda = 1/4$.

4 Inner automorphisms of the hyperfinite II_1-factor

In the last section we saw how to extend many shifts to outer automorphisms of the hyperfinite II_1-factor R, and it was remarked that the entropy of the extended shift was the same as that of the original (classical) shift. One can also via crossed products extend ergodic transformations to inner automorphisms of R. Let $(X, \mathcal{B}, \mu)$ be a probability space and T a nonsingular ergodic measure preserving transformation of X. Then T defines an automorphism α_T on $L^\infty(X,\mu)$ by $(\alpha_T f)(x) = f(T^{-1}x)$ for $f \in L^\infty(X,\mu)$, $x \in X$. The crossed product $L^\infty(X,\mu) \times_{\alpha_T} \mathbb{Z}$ equals R, and α_T extends in a natural way to an inner automorphism $\operatorname{Ad} U_T$ on R.

Since the restriction $\operatorname{Ad} U_T | L^\infty(X,\mu) = \alpha_T$, and $H(\alpha_T) = H(T)$, it follows from Remark 2.6 that

$$H(\operatorname{Ad} U_T) \geq H(T), \tag{4.1}$$

where $\operatorname{Ad} U_T$ is considered as an automorphism of R.

Problem 4.2. Do we have equality in (4.1)?

The first result on this problem is due to Besson [2], who showed that if the unitary operator V_T on $L^2(X,\mu)$ defined by $(V_T f)(x) = f(T^{-1}x)$, $f \in L^2(X,\mu)$, $x \in X$, has pure point spectrum then $H(\operatorname{Ad} U_T) = 0$.

Let T be as above and let A denote the Abelian von Neumann algebra generated by U_T. Using the Fourier expansion $\sum_{-\infty}^{\infty} a_n U_T^n$ with $a_n \in L^\infty(X,\mu)$ of an element $x \in A' \cap R$ it follows easily from the ergodicity of T that each a_n is a scalar, hence $x \in A$. Thus A is a maximal Abelian subalgebra of R. It turns out that the entropy $H(\operatorname{Ad} U_T)$ depends essentially on the size of the normalizer of A in R, i.e. the set of unitaries $u \in R$ such that $uAu^* = A$. If the normalizer of A generates R as a von Neumann algebra, A is called a *Cartan* (or regular) *subalgebra*. The next result [30] shows that if $H(T) > 0$ then A is not a Cartan subalgebra.

Theorem 4.3. *Let u be a unitary operator contained in a Cartan subalgebra of R. Then the entropy $H(\operatorname{Ad} u) = 0$.*

In the proof of this result one uses the Connes-Feldman-Weiss Theorem [6], [24] on the uniqueness of Cartan subalgebras, so we may assume $R = \bigotimes_{i=1}^{\infty} (M^i, \tau_i)$ and $u \in \bigotimes_{i=1}^{\infty} D^i$, and use the subalgebras $P_q = \left(\bigotimes_1^q M^i\right) \otimes 1 \otimes \cdots$ in the application of the Kolmogoroff-Sinai Theorem 2.10. One can show that the function $H(P_q, \operatorname{Ad} u(P_q), \ldots, \operatorname{Ad} u^{k-1}(P_q))$ grows as $\log k$, hence $H(\operatorname{Ad} u, P_q) = 0$, and therefore $H(\operatorname{Ad} u) = 0$.

In particular if $T = T_\theta$ is the irrational rotation by an angle θ on the circle $\mathbb{T}$, and V_φ denotes the multiplication operator on $L^2(\mathbb{T})$ by $e^{i\varphi}$, then in the crossed product $L^\infty(\mathbb{T}) \times_{\alpha_T} \mathbb{Z}$ $U_T V_\varphi = e^{i\theta} V_\varphi U_T$. In particular $V_\varphi A V_\varphi^{-1} = A$ with A as before the Abelian von Neumann subalgebra generated by U_T. Since V_φ and U_T generate R, A is a Cartan subalgebra of R, hence by Theorem 4.3 we have $H(\operatorname{Ad} U_T) = 0$. In particular we have a positive solution to Problem 4.2 in this case. It should be remarked that this argument can be generalized greatly. It would be interesting to see what kind of maximal Abelian subalgebra U_T generates when $H(T)$ is large, as it is for example for Bernoulli shifts.

5 The free shift

So far we have only looked at the hyperfinite case, in which the Kolmogoroff-Sinai Theorem is applicable. I'll now discuss a completely different case, that of the II_1-factor $L(\mathbb{F}_\infty)$ obtained from the left regular representation of the free group $\mathbb{F}_\infty$ in infinite number of generators. Let G denote the set of generators of $\mathbb{F}_\infty$. Then each bijection of G gives rise to an automorphism of $\mathbb{F}_\infty$ and hence of $L(\mathbb{F}_\infty)$. We say a bijection α is *free* if each orbit $\{\alpha^n(x) : x \in G\}$ is infinite, or equivalently the map $n \longmapsto \alpha^n(x)$ of $\mathbb{Z}$ into G is injective for all $x \in G$. An example is the *free shift*, which is defined by indexing G by $\mathbb{Z}$ and letting α correspond to the shift $n \longmapsto n+1$ of $\mathbb{Z}$. The free shift is an extremely ergodic automorphism; indeed Popa [22] showed that the only globally invariant injective subalgebra of $L(\mathbb{F}_\infty)$ is the scalars, and recently Gaure [12] showed that each nontrivial globally invariant von Neumann subalgebra is a full

II_1-factor. It is therefore rather surprising that we have, see [29].

Theorem 5.1. *Let α be an automorphism of $L(\mathbb{F}_\infty)$ defined by a free bijection of the set of generators G on itself. Then its entropy $H(\alpha) = 0$.*

This result makes my comment at the beginning of section 3, that the definition of entropy was modelled on the n-shift, more profound. The theorem shows that entropy is a function of independence with respect to commutativity of the translates by the automorphism, rather than the degree of ergodicity.

Since each bijection of G on itself is a combination of free and periodic maps, and each periodic automorphism has entropy zero (Remark 2.7), I would expect that all automorphisms of $L(\mathbb{F}_\infty)$ arising from bijections of G have entrópy zero. Even more may be true.

Problem 5.2. *Suppose α is an automorphism of $L(\mathbb{F}_\infty)$ arising from an automorphism of $\mathbb{F}_\infty$. Is $H(\alpha) = 0$?*

Note that $L(\mathbb{F}_\infty)$ has lots of inner automorphisms with positive entropy. This follows from Remark 2.6 since the hyperfinite II_1-factor R is a subfactor of $L(\mathbb{F}_\infty)$, and R has, as we saw in section 4, many inner automorphisms with positive entropy.

The proof of Theorem 5.1 is quite different from the other proofs we have seen. Since we don't have a Kolmogoroff-Sinai Theorem at our disposal we must use Definition 2.4 of the entropy function $H(N_1, \ldots, N_k)$ directly, hence we have to study the behaviour of partitions of unity in $L(\mathbb{F}_\infty)$. Given a finite dimensional subalgebra $N \subset L(\mathbb{F}_\infty)$ we may use freeness of the automorphism α together with approximation results to assume that for some $p \in \mathbb{N}, N, \alpha^p(N), \alpha^{2p}(N), \ldots, \alpha^{(k-1)p}(N)$ belong to subfactors $L(\mathbb{F}_{S_i})$ corresponding to the free subgroups of $\mathbb{F}_\infty$ obtained from disjoint subsets $S_0, \ldots, S_{k-1}$ of G. Then we show a uniform estimate on the set S_k of all partitions of unity, namely,

Lemma 5.3. *Given $\varepsilon > 0$ there exists $r = r(\varepsilon, N) \in \mathbb{N}$ such that for all partitions $(x_{i_1 \ldots i_k}) \in S_k$ and all k there is a set $J \subset \mathbb{N}$ with card $J \le r$ such that*

$$\sum_{i_\ell} \|E_{N_\ell}(x^\ell_{i_\ell}) - \tau(x^\ell_{i_\ell})1\| < \varepsilon \qquad \text{for } \ell \notin J\,,$$

where $N_\ell = \alpha^{p(\ell-1)}(N)$.

Thus the projection of $x^\ell_{i_\ell}$ on N_ℓ is almost a scalar for all $\ell \notin J$, and this happens for ℓ outside a set of cardinality less than r independently of the partition. This is the crucial idea in the proof of Theorem 5.1, because if we assume $E_{N_\ell}(x^\ell_{i_\ell}) = \tau(x^\ell_{i_\ell}) = E_{\mathbb{C}}(x_{i_\ell} =^\ell)$ for all i_ℓ in Definition 2.4, we get

$$\begin{aligned} &\sum \eta\tau(x_{i_1\ldots i_k}) - \sum_\ell \sum_{i_\ell} \tau\eta E_{N_\ell}(x^\ell_{i_\ell}) \\ &= \sum \eta\tau(x_{i_1\ldots i_k}) - \sum_\ell \sum_{i_\ell} \tau\eta E_{\mathbb{C}}(x^\ell_{i_\ell}) \\ &\le H(\mathbb{C}, \ldots, \mathbb{C}) = H(\mathbb{C}) = 0\,. \end{aligned}$$

As mentioned in the introduction Voiculescu [32, 33] has introduced another concept of entropy, called the *perturbation theoretic* entropy. In the nicer cases like the classical case and Bernoulli shifts this entropy gives values close to ours. However, for $L(\mathbb{F}_\infty)$ it is quite different; indeed he showed [33] that the perturbation theoretic entropy of the free shift is $+\infty$. Thus it is plausible that the perturbation theoretic entropy is a better concept of entropy for studying automorphisms of highly non-Abelian factors like $L(\mathbb{F}_\infty)$.

6 Entropy in C^*-algebras

A natural problem that arose after the introduction of entropy in finite von Neumann algebras, was its extension to general von Neumann algebras and even C^*-algebras. Using the quantum mechanical entropy of states $-S(\rho) = Tr(\eta(\Omega_\rho))$ if p is a normal state on $B(H)$ defined by a trace class operator Ω_ρ - and the delicacy of the relative entropy, an attempt was made in [9]. But all we could do was to show that the definition we came up with, gave the classical entropy in the Abelian case. Then Evans [10] modified Definition 2.4 to AF-algebras and defined the topological entropy by taking the sup over all tracial states. By taking the shift on an AF-algebra defined by an aperiodic $n \times n$ matrix A with entries 0 or 1, he could show that the topological entropy= of the shift was the logarithm of the spectral radius of A. For more recent work on topological entropy see the thesis of Hudetz [13].

Connes [5] was the first who succeeded in giving a general definition for normal states of von Neumann algebras. Essentially what he did was to replace the trace in Definition 2.4 by a given normal state φ, and instead of considering partitions $(x_{i_1 \dots i_k}) \in S_k$ to consider finite sets of positive linear functionals φ_i with $\sum \varphi_i = \varphi$. After this breakthrough Connes together with Narnhofer and Thirring [7] succeeded in extending the definition to general C^*-algebras. This was done as follows.

We are given a C^*-algebra A with a state φ and want to define the analogue of $H(N_1, \dots, N_k)$ as in Definition 2.4. Since A need not have any nontrivial finite dimensional subalgebras we replace the N's by finite dimensional C^*-algebras $C_1, \dots, C_k$ which are not necessarily subalgebras of A, but have unital completely positive maps $\gamma_j : C_j \to A$. As we have seen in the previous sections the entropy considered, has a built in Abelianness in it. This is made more explicit in the new definition. We let B be a finite dimensional Abelian C^*-algebra and $P : A \to B$ a unital positive linear map such that there is a state μ on B with $\mu \circ P = \varphi$. We now define a concept called entropy defect for P, which will later on be used for some maps from the C_j's and not for P.

Let $p_1, \dots, p_k$ be the minimal projections in B. Then there are states $\varphi_1, \dots, \varphi_r$ on A such that

$$P(x) = \sum_{i=1}^{r} \varphi_i(x) p_i ,$$

and

$$\varphi = \sum_{i=1}^{r} \mu(p_i) \varphi_i$$

is φ written as a convex sum of states, cf. Connes' von Neumann algebra definition, where he considered families $\{\varphi_i\}$ with $\varphi = \sum \varphi_i$. We now introduce the important relative entropy of states, and use a general definition of Pusz and Woronwicz [28],so [15]. Put

$$\varepsilon_\mu(P) = \sum_{i=1}^{r} \mu(p_i) S(\varphi|\varphi_i),$$

and let the entropy defect be

$$s_\mu(P) = S(\mu) - \varepsilon_\mu(P)\,,$$

where $S(\mu) = \sum \eta(\mu(p_i))$ is the entropy of μ.

In order to come back to the original finite dimensional algebras C_j and completely positive maps γ_j we let $B_1, \ldots, B_k$ be C^*-subalgebras of B, and $E_j : B \to B_j$ the μ-invariant conditional expectation. Then the quadruple (B, E_j, P, μ) is called an *Abelian model* for $(A, \varphi, \gamma_1, \ldots, \gamma_n)$ and its entropy is defined to be

$$S\Big(\mu \Big| \bigvee_{\gamma=1}^{k} B_j\Big) - \sum_{j=1}^{k} s_\mu(P_j)\,, \tag{6.1}$$

where $P_j = E_j \circ P \circ \gamma_j : C_j \to B_j$, and the definitionof $s_\mu(P_j)$ is the same as for P above, where we replace μ by $\mu|B_j$, P by P_j, and φ by $\varphi \circ \gamma_j$.

We can now define the entropy function H in analogy with Definition 2.4 as

$$H_\varphi(\gamma_1, \ldots, \gamma_k) = \sup \text{ of (6.1) over all Abelian models.}$$

Then we can show similar properties as (A)–(E) in section 2. From here on we continue as before. If $\alpha \in \mathrm{Aut}(A)$ is φ-invariant and C is a finite dimensional C^*-algebra with a unital completely positive map $\gamma : C \to A$ we denote by

$$h_{\varphi,\alpha}(\gamma) = \lim_{k\to\infty} \tfrac{1}{k} H_\varphi(\gamma, \alpha\circ\gamma, \ldots, \alpha^{k-1}\circ\gamma)\,,$$

and define the entropy of α to be

$$h_\varphi(\alpha) = \sup_\gamma h_{\varphi,\alpha}(\gamma)\,,$$

where the sup is taken over all pairs (C, γ). The interested reader will see from [7, Remark III.5.3] how the definition of $H_\varphi(\gamma_1, \ldots, \gamma_k)$ extends the definition in [5], and thus the original definition (2.4).

While in the finite von Neumann algebra case the Kolmogoroff-Sinai Theorem was true for hyperfinite algebras, in the C^*-algebra case it holds true for nuclear C^*-algebras [7, V.2].

Theorem 6.2. *Let A, φ, α be as above. Suppose there exists a sequence $(\theta_n)_{n\in\mathbb{N}}$ of unital completely positive maps $\theta_n : A \to A$ such that for each n there are finite dimensional C^*-algebras A_n and unital completely positive maps $\sigma_n : A \to A_n$, $\tau_n : A_n \to A$ for which*

$\theta_n = \tau_n \circ \sigma_n$, and for all $x \in A$

$$\lim_n \|\theta_n(x) - x\| = 0\,.$$

Then

$$\lim_n h_{\varphi,\alpha}(\tau_n) = h_\varphi(\alpha)\,.$$

In particular for an AF-algebra $A = \overline{\bigcup_{n-1}^{\infty} A_n}$ with $1 \in A_1 \subset A_2 \subset \cdots$, finite dimensional, if we let $\gamma_n : A_n \to A$ be the inclusion map, so we write A_n instead of γ_n, we have

$$h_\varphi(\alpha) = \lim_n h_{\varphi,\alpha}(A_n)\,.$$

This result has been extended to the setting of quasi local algebras in quantum statistical mechanics by Park and Shin [20].

If M is a von Neumann algebra and φ a normal state of M we can do the same as above assuming all functionals and maps to be normal. For states on a C^*-algebra A we get the same entropy as before if we use this approach to the GNS-representation of the state. This is often useful, since it is in many cases easier to compute entropy in the von Neumann algebra setting. The following result is analogous to those we discussed for shifts in section 3, see [7, VIII.8].

Theorem 6.3. *Let M be a von Neumann algebra with a faithful normal state φ. Let M_φ denote the centralizer of φ in M. Suppose $N_1, \ldots, N_k$ are finite dimensional subalgebras of M which contain Abelian subalgebras $A_j \subset N_j \cap M_\varphi$ which are pairwise commuting and generate a maximal Abelian subalgebra $A = \bigvee_{j=1}^{k} A_j$ in the von Neumann algebra $N = (N_1 \cup \cdots \cup N_k)''$. Then*

$$H_\varphi(N_1, \ldots, N_k) = S(\varphi|N)\,.$$

This result tells us in particular that the modular automorphism σ^φ comes into play, because $M_\varphi = \{x \in M : \sigma_t^\varphi(x) = x \text{ for all } t \in \mathbb{R}\}$.

One might believe that we need a large centralizer in order to have positive entropy. This is not so. Connes [5] has exhibited an example of a factor M of type III_1 with a state φ with $M_\varphi = \mathbb{C}$, and $h_\varphi(\alpha) > 0$ for a φ-invariant automorphism α.

In some cases one does not have the nice situation of Theorem 6.3 but only an approximation to it. The following is a result on such a situation [7, IX.1]. It is applicable to shift invariant states which are not necessarily product states, but almost so.

Theorem 6.4. *Let $M^i = M_q(\mathbb{C})$ for $i \in \mathbb{Z}$. Let A be the C^*-algebra $A = \bigotimes_{i=-\infty}^{\infty} M^i$ and σ the shift $\sigma : M^i \to M^{i+1}$. Let φ be a state with $\varphi \circ \sigma = \varphi$. For $I \subset \mathbb{Z}$ let $A(I) = \bigotimes_{i \in I} M^i$ and $\varphi_I = \varphi|A(I)$.= Assume that for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that with $I = [-N, n+N]$ then*

(1) $\|\sigma^{\varphi}_{i/2}(\sigma^{\varphi_I}_{-i/2}(x)) - x\| < \varepsilon \, \|x = \|$ *for* $x \in A([0,n])$,

(2) $\lim_{n \to \infty} \frac{N}{n} = 0$.

Then $h_\varphi(\sigma)$ *equals the* mean entropy $S(\varphi)$, *where per definition*

$$S(\varphi) = \lim_{|I| \to \infty} \frac{1}{|I|} S(\varphi | A(I)) .$$

Here $|I| = 2N + n + 1$ *is the length of* I.

At this point it is appropriate to mention that not everything seems to go smoothly with entropy. In the classical case if T_i is a nonsingular measure preserving transformation on a probability space $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, then the entropy of $T_1 \times T_2$ on $(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2, \mu_1 \times \mu_2)$ satisfies

$$H(T_1 \times T_2) = H(T_1) + H(T_2) .$$

Transformed into the language of C^*-algebras this says that if α_i is an automorphism of an Abelian C^*-algebra A_i with an invariant state φ_i, $i = 1, 2$, then

$$h_{\varphi_1 \otimes \varphi_2}(\alpha_1 \otimes \alpha_2) = h_{\varphi_1}(\alpha_1) + h_{\varphi_2}(\alpha_2) . \tag{6.5}$$

Problem 6.6. Does (6.5) hold if A_1 and A_2 are non-Abelian?

The inequality

$$h_{\varphi_1 \otimes \varphi_2}(\alpha_1 \otimes \alpha_2) \geq h_{\varphi_1}(\alpha_1) + h_{\varphi_2}(\alpha_2)$$

is easy [31, Lem. 3.4], because we have many more choices of Abelian models to estimate the left side than the right, where we have to use tensor products of Abelian models in A_1 and A_2. The technical reason why (6.5) holds for Abelian algebras is that each pure state of $A_1 \otimes A_2$ is a product state $\rho_1 \otimes \rho_2$ with ρ_i pure. This is also true if one of the A_i's is Abelian, but not in general. I thus incline to the view that the answer to Problem 6.6 is negative when both A_1 and A_2 are non-Abelian.

7 The anti-commutation relations

The main example for which the C^*-algebra entropy has been computed, is that of quasifree states of the CAR-algebra and invariant Bogoliubov (or quasifree) automorphisms. Let us recall the definitions.

Let H be a complex Hilbert space. The CAR-algebra $\mathcal{A}(H)$ over H is a C^*-algebra with the property that there is a linear map $f \to a(f)$ of H into $\mathcal{A}(H)$ whose range generates $\mathcal{A}(H)$ as a C^*-algebra and satisfies the canonical anticommutation relations

$$a(f)a(g)^* + a(g)^*a(f) = (f,g)1, \qquad f, g \in H ,$$

$$a(f)a(g) + a(g)a(f) = 0 ,$$

where $(\cdot,\cdot)$ is the inner product on H and 1 the unit of $\mathcal{A}(H)$. If $0 \leq A \leq 1$ is an operator on H, then the quasifree state ω_A on $\mathcal{A}(H)$ is defined by its values on products of the form $a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)$ given by

$$\omega_A(a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)) = \delta_{nm} \det((Ag_i, f_i)).$$

If U is a unitary operator on H then U defines an automorphism α_U on $\mathcal{A}(H)$, called a *Bogoliubov automorphism*, determined by

$$\alpha_U(a(f)) = a(Uf).$$

If U and A commute it is an easy consequence of the above definition of ω_A that α_U is ω_A-invariant.

Problem 7.1. Compute $h_{\omega_A}(\alpha_U)$ when $[U, A] = 0$.

Connes suggested to me that if ω_A is the trace τ the answer should be

$$h_\tau(\alpha_U) = \frac{\log 2}{2\pi} \int_0^{2\pi} m(U)(\theta) d\theta, \tag{7.1}$$

where $m(U)$ is the multiplicity function of the absolutely continuous part U_a of U. Then Voiculescu and I [31] showed this and more by solving the problem when A has pure point spectrum. Later on Narnhofer and Thirring [19] and Park and Shin [21] independently extended the result to more general A. Before I state the results let us look more closely at the concepts and ideas in question.

If A has pure point spectrum there is an orthonormal basis (f_n) of H such that $Af_n = \lambda_n f_n$, $n \in \mathbb{N}$, $0 \leq \lambda_n \leq 1$. Define recursively operators

$$V_0 = 1, \qquad V_n = \prod_{i=1}^{n} \left(1 - 2a(f_i)^* a(f_i)\right), \qquad e_{11}^{(n)} = a(f_n) a(f_n)^*$$

$$e_{12}^{(n)} = a(f_n) V_{n-1}, \quad e_{21}^{(n)} = V_{n-1} a(f_n)^*, \quad e_{22}^{(n)} = a(f_n)^* a(f_n).$$

Then the $e_{ij}^{(n)}$, $i, j = 1, 2$ form a complete set of 2×2 matrix units generating a I_2-factor $M_2(\mathbb{C})_n$, and for distinct n and m $e_{ij}^{(n)}$ and $e_{k\ell}^{(m)}$ commute. Thus $\mathcal{A}(H) \simeq \bigotimes_1^\infty M_2(\mathbb{C})_n$, and ω_A is a product state $\omega_A = \bigotimes_1^\infty \omega_{\lambda_n}^0$ with respect to this factorization, where ω_λ^0 is the state on $M_2(\mathbb{C})$ given by

$$\omega_\lambda^0\left(\begin{pmatrix} a & b \\ c & a \end{pmatrix}\right) = (1-\lambda)a + \lambda d.$$

In case $A = \lambda 1$ we write ω_λ for ω_A. Then α_U is ω_A-invariant for all U. We first consider the entropy $h_{\omega_\lambda}(\alpha_U)$.

Each unitary U is a direct sum $U = U_a \oplus U_s$, where U_a has spectral measure absolutely continuous with respect to Lebesgue measure $d\theta$ on the circle, while U_s has spectral measure singular with respect to $d\theta$. We shall as above denote by $m(U)$ the multiplicity function of U_a. The idea is now to approximate the case when

$$U = U_s \oplus U_1 \oplus \cdots \oplus U_n,$$

where each U_i acts on a Hilbert space H_i, $i = 1, \dots, n$, and U_i is unitarily equivalent to V^{p_i}, where V is a bilateral shift. Let us for simplicity ignore the complications due to the grading of $\mathcal{A}(H)$ as a direct sum of its even and odd parts. Then

$$\alpha_U = \alpha_{U_s} \otimes \alpha_{U_1} \otimes \cdots \otimes \alpha_{U_n}$$

and

$$\omega_\lambda = \omega_\lambda|\mathcal{A}(H_s) \otimes \omega_\lambda|\mathcal{A}(H_1) \otimes \cdots \otimes \omega_\lambda|\mathcal{A}(H_n)\,.$$

Thus we could hope that

$$h_{\omega_\lambda}(\alpha_U) = h_{\omega_\lambda|\mathcal{A}(H_s)}(\alpha_{U_s}) + \sum_{i=1}^{n} h_{\omega_\lambda|\mathcal{A}(H_i)}(\alpha_{U_i}), \tag{7.2}$$

and thus restrict attention to the case when U is singular or a power of a bilateral shift. We do have problems because of Problem 6.6, but life turns out nicely because we can as with the shift in section 3 restrict attention to the diagonal, and the diagonal is contained in the even CAR-algebra, where the tensor product formulas above hold. First we take care of the singular part U_s.

Lemma 7.3. *If U has spectral measure singular with respect to the Lebesgue measure, and α_U is φ-invariant for a state φ, then $h_\varphi(\alpha_U) = 0$.*

Thus in (7.2) we can forget about U_s. If $U = V^p$ with V a bilateral shift and $p \in \mathbb{Z}$, then

$$h_{\omega_\lambda}(\alpha_U) = h_{\omega_\lambda}((\alpha_V)^p) = |p| h_{\omega_\lambda}(\alpha_V)\,.$$

If we write $\mathcal{A}(H) = \bigotimes_{n=-\infty}^{\infty} M_\alpha(\mathbb{C})_n$, then on the diagonal α_V is the shift, so like in section 3 we get

$$h_{\omega_\lambda}(\alpha_V) = \eta(\lambda) + \eta(1-\lambda)\,.$$

Now $|p|$ is the multiplicity $m(U)$ of U, and since $\frac{1}{2\pi} d\theta$ is the normalized Haar measure on the circle, it is not surprising that we have

Theorem 7.4. *Let U be a unitary operator on H and $\lambda \in [0,1]$. Then* $h_{\omega_\lambda}(\alpha_U) = \frac{1}{2\pi}\big(\eta(\lambda) + \eta(1-\lambda)\big) \int_0^{2\pi} m(U)(\theta) d\theta$.

Note that if $\lambda = 1/2$, $\omega_\lambda = \tau$, so we get formula (7.1). For more general A we use direct integral theory with respect to the von Neumann algebra generated by U_a. If A commutes with U, $A = A_a \oplus A_s$, where $A_a = \int_0^{2\pi}{}^{\oplus} A(\theta) d\theta$, where $H = \int^{\oplus} H_\theta d\theta$, and $H_\theta = 0$ if $m(U)(\theta) = 0$, and $A(\theta) \in B(H_\theta)$. If A has pure point spectrum, $A = \sum_{j \in I} \lambda_j e_j$ with J finite or countably infinite, and (e_j) is an orthogonal family of projections with sum 1, $\lambda_j \in [0,1]$. Denote by $U_j = U|e_j(H)$, and let Tr be the usual trace on $B(H_\theta)$. Writing $e_j = \int^{\oplus} e_j(\theta) d\theta$ we get

$$\begin{aligned} Tr\big(\eta(A(\theta)) + \eta(1 - A(\theta))\big) &= \sum_j (\eta(\lambda_j) + \eta(1-\lambda_j)) Tr(e_j(\theta)) \\ &= \sum_j (\eta(\lambda_j) + \eta(1-\lambda_j)) m(U_j)(\theta)\,. \end{aligned}$$

Integrating and using Theorem 7.4 we have

$$\sum_{j\in J} h_{\omega_{\lambda_j}}(\alpha_{U_j}) = \frac{1}{2\pi}\int_0^{2\pi} Tr(\eta(A(\theta)) + \eta(1-A(\theta)))d\theta\,.$$

Some extra work shows that the left side equals $h_{\omega_A}(\alpha_U)$, thus we have

Theorem 7.5. *Let $0 \leq A \leq 1$ have pure point spectrum and U a unitary commuting with A. Then*

$$h_{\omega_A}(\alpha_U) = \frac{1}{2\pi}\int_0^{2\pi} Tr(\eta(A(\theta)) + \eta(1-A(\theta)))d\theta\,.$$

A natural problem is to extend this result to general A. This is nontrivial and was studied already by Narnhofer and Thirring in [18], where they showed, with some gaps in the proof, that the formula is true for a bilateral shift. There are recently two independent papers on the problem, by Narnhofer and Thirring [19] and Park and Shin [21]. They both concentrate attention to the case when U is absolutely continuous. Imposing technical assumptions on A they prove formulas like the one in the theorem. In [21] this was also done for the canonical commutation relations. The mathematics in [21] is quite involved. They use the definition of entropy directly and go through hard analysis to estimate the entropy defects. A perhaps easier approach would be to apply Theorem 6.4 directly. I believe this is done in [19], but I must admit, I have not understood the proof. Narnhofer and Thirring claim [19, Remark 3.4] that the assumptions they impose on A are so weak that they consider Problem 7.1 as settled.

In the above papers it is shown that when α_U is space translation on $\mathcal{A}(H)$ then $h_{\omega_A}(\alpha_U)$ is the mean entropy in the sense of Theorem 6.4. It should, however, be noted that Fannes [11] showed a formula like Theorem 7.5 for this mean entropy.

8 An alternative definition of entropy

Sauvageot and Thouvenot [28] have given an alternative definition of entropy which is very close to that of Connes, Narnhofer, and Thirring [7}, but which is closer in spirit and notation to the classical definition.

Let A be a C^*-algebra together with a state ρ. A *coupling* of (A,ρ) with an Abelian C^*-algebra is a pair (λ, B), where B is an Abelian C^*-algebra, and λ is a state on the C^*-algebra $A \otimes B$ whose restriction to A (identified with $A \otimes 1$) is ρ. We denote by μ the probability measure on B obtained from the restriction of λ to B. If B is finite dimensional then $B = C(X)$ with X a finite set, and for each $x \in X$ the characteristic function $\mathcal{X}_x(= \mathcal{X}_{\{x\}})$ is a minimal projection in B. We then get a state ρ_x on A defined by

$$\rho_x(a) = \mu(\{x\})^{-1}\lambda(a \otimes \mathcal{X}_x)\,,$$

which gives ρ as a convex sum of states,

$$\rho = \sum_{x\in X} \mu(\{x\})\rho_x\,.$$

In analogy with $\varepsilon_\mu(P)$ from section 6 we put

$$\varepsilon_\lambda(A,B) = \sum_{x\in X} \mu(\{x\})S(\rho|\rho_x) = S(\rho\otimes\mu|\lambda)\,.$$

It follows that $\varepsilon_\lambda(A,B)$ measures how far λ is from being a product state by using the distance function relative entropy. The entropy defect s_μ now takes the form of a conditional entropy when we define it as before as

$$H_\lambda(B|A) = H_\mu(B) - \varepsilon_\lambda(A,B) = -\sum_{x\in X} S(\rho|\mu(\{x\})\rho_x)\,.$$

Here $H_\mu(B)$ is the entropy of μ as a probability measure on X. If we identify each finite dimensional subalgebra of B with the partition consisting of its minimal projections we find for $\mathcal{P}$ and $\mathcal{Q}$ partitions of X, that

$$H_\lambda(\mathcal{P}\vee\mathcal{Q}|A) = H_\lambda(\mathcal{P}|A) + H_\lambda(\mathcal{Q}|A\otimes\mathcal{P})\,.$$

In the special case when $A=\mathbb{C}$ this identity reduces to the classical identity for conditional entropy

$$H(\mathcal{P}\vee\mathcal{Q}) = H(\mathcal{P}) + H(\mathcal{Q}|\mathcal{P})\,.$$

Let now α be a ρ-invariant automorphism of A, and (λ,B) a coupling with an Abelian C^*-algebra which is not necessarily finite dimensional. Suppose further that $\sigma\in Aut(B)$ and that $\alpha\otimes\sigma$ is λ-invariant. For a finite partition $\mathcal{P}$ of B denote its past by

$$\mathcal{P}^- = \bigvee_{i=1}^{\infty}\sigma^{-i}\mathcal{P}\,.$$

In the classical case we have

$$h(\sigma,\mathcal{P}) = \lim_n \tfrac{1}{n} H_\mu\Big(\bigvee_{i=0}^{n-1}\sigma^{-1}(\mathcal{P})\Big) = H_\mu(\mathcal{P}|\mathcal{P}^-).$$

Analogously we define two expressions

$$\begin{aligned} h(\lambda,\mathcal{P}) &= H_\mu(\mathcal{P}|\mathcal{P}^-) - H_\lambda(\mathcal{P}|A\otimes\mathcal{P}^-) \\ h'(\lambda,\mathcal{P}) &= H_\mu(\mathcal{P}|\mathcal{P}^-) - H_\lambda(\mathcal{P}|A)\,. \end{aligned}$$

Then $h(\lambda,\mathcal{P})\ge 0$ and $h(\lambda,\mathcal{P})\ge h'(\lambda,\mathcal{P})$.

Sauvageot and Thouvenot now define the entropy of the dynamical system (A,α,ρ) to be

$$H_\rho(\alpha) = \sup h(\lambda,\mathcal{P})\,,$$

where the sup is taken over all couplings (λ,B), partitions $\mathcal{P}$ and automorphisms σ as above. Then they show that we get the same by using h', i.e.

$$H_\rho(\alpha) = \sup h'(\lambda,\mathcal{P})\,.$$

Furthermore they show

Theorem 8.1. *If A is nuclear the entropy $H_\rho(\alpha)$ is the same as the entropy $h_\rho(\alpha)$ defined in section 6.*

In addition to its resemblance to the classical case the above definition has another nice feature. Let A and ρ be as before and (π, H, ξ) the GNS-representation of ρ. If $\alpha \in Aut(A)$ is ρ-invariant it is implemented in the GNS-representation by the unitary U_ρ defined by $U_\rho \pi(x)\xi = \pi(\alpha(x))\xi$. The following result is analogous to Lemma 7.3.

Theorem 8.2. *If the spectral measure of U_ρ is singular with respect to the Lebesgue measure on the circle then $H_\rho(\alpha) = 0$.*

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