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Free-independent sequences in type II_1 factors and related problems

by Sorin Popa

Dedicated to Professor Ciprian Foias, on his 60'th birthday

Introduction

We will show in this paper that, unlike central sequences (i.e., commuting-independent sequences) which in general may or may not exist, free-independent sequences exist in any separable type II_1 factor.

More generally, we will in fact prove the following:

Theorem. *Let $N \subset M_\infty$ be an inclusion of separable type II_1 factors. Assume there exists an increasing sequence of von Neumann subalgebras $N \subset M_n \subset M_\infty$ such that $\overline{\bigcup_n M_n} = M_\infty$ and such that $N' \cap M_n$ is finite dimensional for all n. Then there exists a unitary element $v = (v_n)_n$ in the ultrapower algebra N^ω ([D1]) such that*

$$M_\infty \vee vM_\infty v^* = M_\infty \underset{N'\cap M_\infty}{*} vM_\infty v^*.$$

Here $P_1 \underset{B}{*} P_2$ denotes the finite von Neumann algebra free product with amalgamation, with its free trace $\tau_1 * \tau_2$, where (P_1, τ_1), (P_2, τ_2) are finite von Neumann algebras with their corresponding finite, normal, faithful traces, and with $B \subset P_1$, $B \subset P_2$ a common subalgebra such that $\tau_{1|_B} = \tau_{2|_B}$ ([Po6], [V2]).

In the particular case when $N \subset M = M_\infty$ are factors and $N' \cap M = \mathbb{C}$, for example when $N = M$, the amalgamated free product is a genuine free product ([V1]) and any element of the form vxv^*, $x \in M$ is free with respect to M. Thus we get:

Corollary. *If $N \subset M$ is an inclusion of type II_1 factors with trivial relative commutant then there exist unitary elements $(u_n)_n$ in N that are free independent with respect to M, i.e., such that $\tau(u_n^k) = 0$, $\forall n$, $\forall k \neq 0$, and $\lim_{n\to\infty} \tau(u_n^{k_1} b_1 u_n^{k_2} b_2 \cdots u_n^{k_\ell} b_\ell) = 0$ for any $\ell \geq 1$ and any $b_1, \cdots, b_\ell \in M$, $\tau(b_i) = 0$, $1 \leq i \leq \ell - 1$, $k_1, k_2, \cdots, k_\ell \in \mathbb{Z} \setminus \{0\}$.*

Since the notion of "independent events" in classical probability theory becomes "free independence" in the noncommutative probability of ([V3]), our result on the existence of free-independent sequences can be regarded as the "free" analogue of the results on the existence of nontrivial central sequences in a factor ([D1], [McD], [C2]) or a subfactor ([Bi]). There are thus some notable differences between central and free-independent sequences: Nontrivial central sequences may not exist in general, but they always form an algebra while free sequences always exist, though the set of all such sequences doesn't form an algebra. Also, the existence of noncommuting central sequences in a factor M implies that M splits off the hyperfinite type II_1 factor, i.e., $M \simeq M \otimes R$, but, although all factors have free-independent sequences, neither the hyperfinite nor the property T factors ([C3]) are free products of algebras (cf. [MvN], [Po5]). Along these lines note also that, while taking the free product $M * R$ of a property T factor M by R cancels the property T for $M * R$, the fundamental group of $M * R$ will remain countable (cf. [Po5]), yet $M \otimes R$ will have fundamental group $\mathbb{R}_+^*$. Thus, as also pointed out in ([V1,3]), the analogy between tensor and free products seems, in certain respects, rather limited.

The above theorem was first stated, without a proof, in Sec. 8 of [Po6]. But in fact it was obtained prior to the rest of the results in [Po6]. It was this theorem that led us to the construction of irreducible subfactors of arbitrary index s, $N^s(Q) \subset M^s(Q)$, by using free traces on amalgamated free product algebras. Indeed, when suitably interpreted the theorem shows that such inclusions $N^s(Q) \subset M^s(Q)$ can be asymptotically recovered in any other irreducible inclusion of same index s.

The paper is organized as follows. In Sec. 1 we prove the technical results needed for the proof of the theorem. The proofs are inspired from (2.1 in [Po4]), where a slightly weaker version of the results here were obtained. The proofs rely on the local quantization principle ([Po1, 7]) and on a maximality argument, like in [Po4]. Conversely, the results in [Po1, 7] are immediate consequences of the theorem and its corollary, giving some sharp estimates as a bonus. This fact will be explained in Sec. 2, where the main result of the paper, a generalization of the above stated theorem, is proved, (see 2.1) and some more immediate corollaries are deduced. We expect it in fact to be useful for approaching some other problems as well, an aspect on which we comment in Sec. 3. Thus, we speculate on the possibility of having a functional analytical characterization of the free group algebras, on the indecomposability of such algebras and their possible embedding into other algebras. We also include a construction of separable type II_1 factors M with the fundamental group $\mathcal{F}(M)$ countable but containing any prescribed countable subset of $(0, \infty)$, e.g., with $\mathcal{F}(M) \supset \mathbb{Q}$.

We are grateful to D. Voiculescu for stimulating us to write down the proof of the result announced in Sec. 8 of [Po4], through his constant interest and motivating comments.

1 Some technical results

In what follows all finite von Neumann algebras are assumed given with a normal, finite, faithful trace, typically denoted by τ. For standard notations and terminology, we refer the reader to [Po6, 7].

We will also often use the following:

1.1. Notation. Let B be a von Neumann algebra. If $v \in B$ is a partial isometry with $v^*v = vv^*$, $S \subset B$ is a subset and $k \leq n$ are nonnegative integers then denote $S_v^{0,n} \overset{\text{def}}{=} S$ and $S_v^{k,n} \overset{\text{def}}{=} \Big\{ b_0 \underset{i=1}{\overset{k}{\pi}} v_i b_i \mid b_i \in S,\ 1 \leq i \leq k-1, \quad b_0,\ b_k \in S \cup \{1\}$ and $v_i \in \{v^j | 1 \leq |j| \leq n\} \Big\}$.

The next lemma is the crucial technical result needed to prove the theorem in this paper:

1.2. Lemma. *Let $N \subset M$ be an inclusion of type II_1 von Neumann algebras. Assume $N' \cap M$ is finite dimensional. Let $\varepsilon > 0$, n a positive integer, $F \subset M$ a finite set and $f \in N$ a projection of scalar central trace in N such that $E_{N' \cap M}(b) = 0$, for all $b \in fFf$. Then there exists a partial isometry v in fNf such that:*

a) $v^*v = vv^*$ *and its central trace in N is a scalar* $> \dfrac{\tau(f)}{4}$.

b) $\|E_{N' \cap M}(x)\|_1 \leq \varepsilon, \quad x \in \underset{k=1}{\overset{n}{\cup}} F_v^{k,n}$.

Proof. Let $\delta > 0$. Denote $\varepsilon_0 = \delta$, $\varepsilon_k = 2^{k+1}\varepsilon_{k-1}$, $k \geq 1$. Let $\mathcal{W} = \{v \in fNf | v$ partial isometry, $v^*v = vv^*$, the central trace of v^*v in N is a scalar, and $\|E_{N' \cap M}(x)\|_1 \leq \varepsilon_k \tau(v^*v)$, for all $1 \leq k \leq n$, $x \in F_v^{k,n}\}$. Endow $\mathcal{W}$ with the order $\leq$ in which $v_1 \leq v_2$ iff $v_1 = v_2 v_1^* v_1$. $(\mathcal{W}, \leq)$ is then clearly inductively ordered. Let v be a maximal element in $\mathcal{W}$. Assume $\tau(v^*v) \leq \tau(f)/4$. If w is a partial isometry in pNp, where $p = f - v^*v$, and if $u = v + w$ then for $x = b_0 \underset{i=1}{\overset{k}{\pi}} u_i b_i \in F_u^{k,n}$ we have

$$x = b_0 \prod_{i=1}^{k} v_i b_i + \sum_{\ell} \sum_{i} z_0^i \prod_{j=1}^{\ell} w_{i_j} z_j^i \tag{1}$$

where $k \geq \ell \geq 1$, $i = (i_1, \ldots, i_\ell)$ with $1 \leq i_1 < \cdots < i_\ell \leq k$, $w_{i_j} = w^s$ if $v_{i_j} = v^s$, $z_0^i = b_0 v_1 b_1 \cdots b_{i_1-1} p$, $z_j^i = p b_{i_j} v_{i_j+1} \cdots v_{i_{j+1}} p$, for $1 \leq j < \ell$ and $z_\ell^i = p b_{i_\ell} v_{i_\ell+1} \cdots v_k b_k$ and where the sum is taken over all $\ell = 1, 2, \cdots, k$ and all $i = (i_1, \ldots, i_\ell)$.

By (A.1.4 in [Po7]), given any $\alpha > 0$ there exists a projection q in pNp, of scalar central support in pNp (and thus in N), such that

$$\|qzq - E_{(N'\cap M)_p}(z)q\|_{1,pMp} < \alpha\tau_{pMp}(q) \tag{2}$$

for all z of the form z_j^i, for some $\ell \geq 2$, some $i = (i_1, \cdots, i_\ell)$ and $1 \leq j \leq \ell - 1$.

In the case $\ell = 2$ and $i_1 = 1$, $i_2 = k$, if we take the partial isometry $w \in pNp$ so that $w^*w = ww^* = q$, then we get for $z = pb_1v_2b_2 \cdots v_{k-1}b_{k-1}p$:

$$\begin{aligned} (3)\quad & \|E_{N'\cap M}(b_0w_1b_1v_2b_2\cdots v_{k-1}b_{k-1}w_kb_k)\|_1 \\ &\leq \|w_1b_1v_2b_2\cdots b_{k-1}w_k\|_1 \\ &= \|qb_1v_2b_2\cdots b_{k-1}q\|_1 \\ &= \|qzq\|_1 = \|qzq\|_{1,pMp}\tau(p) \\ &\leq \left(\|E_{(N'\cap M)_p}(z)q\|_{1,pMp} + \alpha\tau_{pMp}(q)\right)\tau(p) \\ &= \left(\|E_{(N'\cap M)_p}(z)\|_{1,pMp}\tau(q)/\tau(p) + \alpha\tau(q)/\tau(p)\right)\tau(p) \\ &= \left(\|E_{(N'\cap M)}(z)\|_1\tau(p)^{-1} + \alpha\right)\tau(q). \end{aligned}$$

But since for $x \in N'\cap M$, v and $p = vv^*$ commute with x we get by taking into account that $vb_1v_2b_2\cdots v_{n-1}b_{n-1}v^* \in F_v^{k,n}$ and $b_1v_2b_2\cdots v_{k-1}b_{k-1} \in F_v^{k-2,n}$ the following estimate:

$$\begin{aligned} (4)\quad & \|E_{N'\cap M}(z)\|_1 \\ &= \sup\{|\tau(zx)| \mid x \in N'\cap M, \ \|x\| \leq 1\} \\ &= \sup\{|\tau(pb_1v_2b_2\cdots v_{k-1}b_{k-1}x)| \mid x \in N'\cap M, \ \|x\| \leq 1\} \\ &\leq \sup\{|\tau(b_1v_2b_2\cdots v_{k-1}b_{k-1}x)| \mid x \in N'\cap M, \ \|x\| \leq 1\} \\ &\quad + \sup\{|\tau(vb_1v_2b_2\cdots v_{k-1}b_{k-1}v^*x)| \mid x \in N'\cap M, \ \|x\| \leq 1\} \\ &= \|E_{N'\cap M}(b_1v_2b_2\cdots v_{k-1}b_{k-1})\|_1 \\ &\quad + \|E_{N'\cap M}(vb_1v_2b_2\cdots b_{k-1}v^*)\|_1 \\ &\leq \varepsilon_{k-2}\tau(v^*v) + \varepsilon_k\tau(v^*v). \end{aligned}$$

By combining (3) and (4) and noting that $\tau(v^*v) \leq \tau(f)/4$ implies $\tau(v^*v)/\tau(p) \leq \tau(f)/3$, it follows that if we take $\alpha \leq \delta/3 < (\varepsilon_k - \varepsilon_{k-2})/3$ then we get:

$$\|E_{N'\cap M}(b_0w_1b_1v_2b_2\cdots v_{k-1}b_{k-1}w_kb_k)\|_1 \leq 2/3\varepsilon_k\tau(q). \tag{5}$$

Note now that $z = pb_1v_2b_2\cdots v_{k-1}b_{k-1}p$ is the only element of the form z_j^i for which $i_2 - i_1 = k - 1$ and that it appears in the sum (1) only once, in the writing of the element $b_0w_1b_1v_2b_2\cdots v_{k-1}b_{k-1}w_kb_k = b_0w_1zw_kb_k$. For all other elements z_j^i with $\ell \geq 2$ we have $i_2 - i_1 < k-1$.

Thus, if the partial isometry $w \in pNp$ is supported on q like before, i.e., $w^*w = ww^* = q$ then we get:

$$
\begin{aligned}
(6)\quad & \|E_{N'\cap M}(z_0w_{i_1}z_1^iw_{i_2}z_2^i\cdots w_{i_\ell}z_\ell^i)\|_1 \\
&\leq \|w_{i_1}z_1^iw_{i_2}\|_1 \\
&= \|qz_1^iq\|_1 = \|qz_1^iq\|_{1,pMp}\tau(p) \\
&\leq \left(\|E_{(N'\cap M)p}(z_1^i)q\|_{1,pMp} + \alpha\tau_{pMp}(q)\right)\tau(p) \\
&= \left(\|E_{N'\cap M}\ (z_1^i)\|_1\tau(p)^{-1} + \alpha\right)\tau(q).
\end{aligned}
$$

Since $b_{i_1}v_{i+1+1}\cdots v_{i_2-1}b_{i_2-1} \in F_v^{i_2-i_1-1,n}$, with $i_2 - i_1 - 1 \leq k-3$, and $vb_{i_1}v_{i_1+1}b_{i_1+1}\cdots v_{i_2-1}b_{i_2-1}v^* \in F_v^{i_2-i_1+1,n}$ with $i_2 - i_1 + 1 \leq k-1$, we obtain like in (3), (4) that

$$(7)\qquad \|E_{N'\cap M}(z_1^i)\|_1 \leq (\varepsilon_{k-1} + \varepsilon_{k-3})\tau(v^*v).$$

Combining (6) and (7) we obtain for $\tau(v^*v) \leq \tau(f)/4$ and for $\alpha \leq \delta/3 \leq (\varepsilon_{k-1} - \varepsilon_{k-3})/3$, the estimate:

$$(8)\qquad \|E_{N'\cap M}(z_0w_{i_1}z_1^iw_{i_2}z_2^i\cdots w_{i_\ell}z_\ell^i)\|_1 \leq 2/3\varepsilon_{k-1}\tau(q).$$

Since $2^{k+1}\varepsilon_{k-1} = \varepsilon_k$ and since there are at most $\sum_{i=2}^{k}\binom{k}{i} = 2^k - k - 1$ elements in the sum in (1) for which $\ell \geq 2$ and $i_2 - i_1 \neq k-1$, we get

$$
\begin{aligned}
(9)\quad & \sum_{\ell\geq 2}\ \sum_{i_2-i_1\neq k-1}\left\|E_{N'\cap M}\left(z_0\prod_{j=1}^{\ell}w_{i_j}z_j^i\right)\right\|_1 \\
&\leq 2/3(2^k - k - 1)\varepsilon_{k-1}\tau(q) \\
&\leq 1/3\varepsilon_k\tau(q) - (2k+2)/3\varepsilon_{k-1}\tau(q).
\end{aligned}
$$

Finally, from the sum on the right hand side of (1) we will now estimate the terms with $\ell = 1$. These are terms which are obtained from $b_0v_1b_1v_2b_2\cdots v_kb_k$ by replacing exactly one v_i by w_i, so there are k of them. Note at this point that for the above estimates we only used the fact that $w^*w = ww^* = q$ and not the form of w. We will make the appropriate choice for w now, to get the necessary estimates for these last terms. To do this, note that for any finite set of elements $Y \subset M$, any integer $n \geq 1$ and any $\beta > 0$ there exists $w \in qNq$, $\quad w^*w = ww^* = q$, such that

$$|\tau(w^i x)| < \beta\tau(q), \quad \forall x \in Y, \quad 0 < |i| \leq n. \tag{10}$$

To see this, take for instance $A \subset qNq$ to be a maximal abelian algebra which since qNq is of type II_1 will be diffuse, thus it will contain a separable subalgebra of the form $L^\infty(\pi, \quad \mu)$ and let w_0 be its generator, so that $\tau(w_0^m) = 0$ for all $m \neq 0$. Thus w_0^m tends weakly to 0 so that $w = w_0^m$ for large enough m will do. If we take $Y = \{b_j v_{j+1} \cdots v_k b_k e_{rs}^p b_0 v_1 b_1 \cdots v_{j-1} b_{j-1} w_j | 1 \leq j \leq k, \quad r, s, p\}$, where $\{e_{rs}^p\}_{p,r,s}$ is a matrix unit for $N' \cap M$, then by (10) we get for $w_i \in \{w^j | 0 < |j| \leq n\}$

$$\begin{aligned} (11) \quad & \|E_{N'\cap M}(b_0 v_1 b_1 \cdots v_{j-1} b_{j-1} w_j b_j v_{j+1} \cdots v_k b_k)\|_1 \\ &= \sup\{|\tau(b_0 v_1 b_1 \cdots v_{j-1} b_{j-1} w_j b_j v_{j+1} \cdots v_k b_k x)| \mid x \in N' \cap M, \quad \|x\| \leq 1\} \\ &\leq \sum_{p,r,s} |\tau(b_0 v_1 b_1 \cdots v_{j-1} b_{j-1} w_j b_j v_{j+1} \cdots v_k b_k e_{rs}^p)| \\ &< (\dim N' \cap M)\beta\tau(q). \end{aligned}$$

Thus, if β is chosen such that

$$k(\dim N' \cap M)\beta < (2k+1)/3\varepsilon_{k-1}, \quad \forall k \leq n$$

then by adding up (5), (9) and (11) we get from (1) the following estimates for $x \in F_{v+w}^{k,n}$, $\quad 1 \leq k \leq n$:

$$\begin{aligned} \|E_{N'\cap M}(x)\|_1 \quad &\leq \left\| E_{N'\cap M}\left(b_0 \prod_{i-1}^{k} v_i b_i\right)\right\|_1 \\ &\quad + \sum_{\ell \geq 2} \sum_i \left\| E_{N'\cap M}\left(z_0^i \prod_{j=1}^{\ell} w_{i_j} z_j^i\right)\right\|_1 \\ &\quad + \sum_j \|E_{N'\cap M}(b_0 v_1 b_1 \cdots w_j \cdots v_k b_k)\|_1 \\ &\leq \varepsilon_k \tau(v^*v) \\ &\quad + 2/3\varepsilon_k\tau(q) + 1/3\varepsilon_k\tau(q) - (2k+2)/3\varepsilon_{k-1}\tau(q) \\ &\quad + (2k+2)/3\varepsilon_{k-1} \\ &= \varepsilon_k\tau(v^*v + q) = \varepsilon_k\tau((v+w)^*(v+w)). \end{aligned}$$

But this contradicts the maximality of $v \in \mathcal{W}$.

We conclude that $\tau(v^*v) > \tau(f)/4$. If δ is taken so that $\varepsilon_n < \varepsilon$ then the statement follows. Q.E.D.

1.3. Lemma. *Let $\{N_m \subset M_m\}_m$ be a sequence of inclusions of type II_1 von Neumann algebras with $\dim(N'_m \cap M_m) < \infty$, for all m. Let ω be a free ultrafilter on $\mathbb{N}$. Given any nonzero projection f in ΠN_m of scalar central support and any countable set of elements $X \subset \prod_\omega M_m$, with $E_{\prod_\omega N'_m \cap \prod_\omega M_m}(b) = 0$, for all $b \in fXf$, there exists a nonzero partial isometry $v \in f(\prod_\omega N_m)f$ such that $v^*v = vv^*$ has scalar central support in ΠN_m and such that for all n, all $k \leq n$ and all $x \in X_v^{k,n}$ one has:*

$$E_{\prod_\omega N'_m \cap \prod_\omega M_m}(x) = 0.$$

Proof. Let $f = (f_m)_m$ be a representation of f in ΠN_m, with $f_m \in N_m$ projections of scalar central support. Let $X = \{x^k\}_k$ and let $x^k = (x_n^k)_n$ be a representation of x^k in $\prod_\omega M_n$, with $E_{N' \cap M_n}(x_n^k) = 0$, for all k. For each n apply Lemma 1.2 for the inclusion $N_n \subset M_n$, the positive element $\varepsilon = 2^{-n}$, the integer n, the projection f_n and the finite set $X_n = \{x_n^k | k \leq n\}$ to get a partial isometry v_n in $f_n N_n f_n$ such that $v_n^* v_n = v_n v_n^*$ has central trace in N_n a scalar $\geq \tau(f_n)/4$ and

$$\left\| E_{N'_n \cap M_n}(x) \right\|_1 \leq 2^{-n}, \qquad x \in \bigcup_{k \leq n} (X_n)_{v_n}^{k,n}.$$

Then $v = (v_n)$ clearly satisfies the conditions. Q.E.D.

1.4. Lemma. *Let $\{N_n \subset M_\infty^n\}_n$ be a sequence of inclusions of type II_1 von Neumann algebras and assume that for each n there exists an increasing sequence of von Neumann subalgebras $\{M_k^n\}_k$ of M_∞^n, containing N_n such that $M_\infty^n = \overline{\bigcup_k M_k^n}$ and $N'_n \cap M_k^n$ is atomic for each k. Let ω be a free ultrafilter on $\mathbb{N}$. Given any countable set $Y \subset \prod_\omega M_\infty^n$ with $E_{\prod_\omega N'_n \cap \prod_\omega M_\infty^n}(b) = 0$, for all $b \in Y$, there exists a unitary element $v \in \prod_\omega N_n$ such that $\tau(w) = 0$ for any word w of alternating letters a_i, b_i (i.e., $w = a_1 b_1 a_2 b_2 \cdots$, or $w = b_1 a_1 b_2 a_2 \cdots$, ending either with b_i or a_i) where $b_i \in Y$ and where each a_i is of the form v^{n_i}, for some $n_i \neq 0$, we have*

$$E_{\prod_\omega N'_n \cap \prod_\omega M_\infty^n}(w) = 0.$$

Proof. Since Y is countable there exist $k_1 < k_2 < \ldots$ and $p_n \in N'_n \cap M_{k_n}^n$ such that $Y \subset \prod_\omega p_n M_{k_n}^n p_n$, and $N'_n p_n \cap p_n M_{k_n}^n p_n$ is finite dimensional for each n. Let $\mathcal{W} = \{v \in \prod_\omega N_n | v$ partial isometry, $v^*v = vv^*$ has scalar central support in $\prod_\omega N_n$, $E_{\prod_\omega N'_n \cap \prod_\omega M_n}(x) = 0$, for all $k \leq n$ and all $x \in Y_v^{k,n}\}$, where $M_n = M_{k_n}^n$. Endow $\mathcal{W}$ with the order given by $v_1 \leq v_2$ iff $v_1 = v_2 v_1^* v_1$. Then $(\mathcal{W}, \leq)$ is clearly inductively ordered. Let v be a maximal element of $\mathcal{W}$. Assume v is not a unitary element and let $f = 1 - v^*v \neq 0$. Then $E_{(\Pi N_n)' \cap (\Pi M_n)}(fxf) = 0$, for all $x \in X \stackrel{\text{def}}{=} \underset{k\leq n}{\cup} Y_v^{k,n}$. Indeed, because for $y \in (\prod_\omega N_n)' \cap (\prod_\omega M_n)$ we have $\tau(fxfy) = \tau(fxy) = \tau((1-v^*v)xy) = \tau(xy) - \tau(vxv^*y) = 0$, since either $x \in Y_v^{k,n}$ begins or ends with a nonzero power of v (when $x = b_0 v_1 b_1 \cdots v_k b_k$ with either b_0 or b_k equal to 1) or $vxv^* \in Y_v^{k+2}$, so both $\tau(xy) = 0$, $\tau(vxv^*y) = 0$ by the fact that $v \in \mathcal{W}$ and by the definition of $\mathcal{W}$.

Thus Lemma 1.3 applies to get a nonzero partial isometry $u \in f \prod_\omega N_n f$ such that $u^*u = uu^*$ and $E_{(\Pi N_n)' \cap (\Pi M_n)}(x) = 0$ for all $x \in \underset{k\leq n}{\cup} X_u^{k,n}$. But then $v+u \in \mathcal{W}$, $v+u \geq v$ and $v + u \neq v$, thus contradicting the maximality of v.

We conclude that v must be a unitary element and it then clearly satisfies the conditions. Q.E.D.

2 The main results

We can now deduce the main result of this paper:

2.1. Theorem. *Let $\{N_n \subset M_\infty^n\}_n$ be a sequence of inclusions of type II_1 von Neumann algebras. Assume that for each n there exists an increasing sequence of subalgebras $\{M_k^n\}_k$ in M_∞^n, generating M_∞^n, containing N_n, and such that $N_n' \cap M_k^n$ is atomic for each k. Let $\{P_j\}_j$ be a countable family of separable von Neumann subalgebras of $\prod_\omega M_\infty^n$ with a common subalgebra $B \subset P_j$ such that $E_{(\prod_\omega N_n)' \cap (\Pi M_n^n)}(P_j) = B$ for all j, i.e., $(\Pi N_n)' \cap P_i = B$, $\forall i$, and such that for each i P_i, $(\Pi N_n)' \cap (\Pi M_\infty^n)$ and B satisfy the commuting square condition of ([Po3]). Then there exists a unitary element v in $\prod_\omega N_n$ such that $\bigvee_j v^j P_j v^{-j} = \ast_B v^j P_j v^{-j}$, i.e. the algebras $v^j P_j v^{-j}$ generate an amalgamated free product over B.*

Proof. Since $\{P_j\}_j$ are all separable, there exists a countable set $Y \subset \underset{j}{\cup} P_j$, with $E_{(\Pi N_n)' \cap (\Pi M_n)}(x) = E_B(x) = 0$ for $x \in Y$, dense in the norm $\| \ \|_2$ in the set $\underset{j}{\cup}(P_j \ominus B)$. Thus, if v satisfies the conditions in the conclusion of Lemma 1.4 for the set Y, then $v^j P_j v^{-j}$ will clearly generate an amalgamated free product over their common subalgebra B. Q.E.D.

Note that Theorem stated in the introduction is just the case $N_n = N$, $M_\infty^n = M_\infty$ for all n of Theorem 2.1 and that its Corollary is then trivial. Another feature of Theorem 2.1, the proof of which relied almost entirely on the local quantization principle of [Po1, 7], is that it can be used to get back that theorem, with some sharp estimate, coming from direct calculation (in the norm $\| \ \|_2$ case) and from the spectral estimates in ([V1]) (in the norm $\| \ \|$ case).

2.2. Corollary. *Let $N \subset M$ be an inclusion of type II_1 factors with trivial relative commutant and let ω be a free ultrafilter on $\mathbb{N}$. Let $B \subset M^\omega$ be a separable von Neumann subalgebra. Given any n there exists a partition of the unity $\{p_i\}_{1\leq i\leq n}$ in N^ω, with projections of trace $1/n$, such that:*

(i) *For any $x \in B$, with $\tau(x) = 0$, and any i, one has:*

$$\|p_i x p_i\|_2^2 = \|x\|_2^2 1/n\tau(p_i).$$

(ii) *For any projection $f \in B$, with $\tau(f) \leq 1 - 1/n$, and any i one has:*

$$\|p_i(f - \tau(f)1)p_i\| = 1/n - 2\tau(f)/n + \sqrt{4\tau(f)(1-\tau(f))1/n(1-1/n)}.$$

(iii) *For any unitary element $v \in B$, with either $\tau(v^k) = 0, \quad \forall k \neq 0$, or $v^m = 1$ for some $m \geq 2$ and $\tau(v^k) = 0, \quad 0 < |k| < m$, and any i one has:*

$$\|p_i v p_i\| = 4/n(1-1/n).$$

Proof. By Theorem A there exists a separable diffuse abelian subalgebra A in N^ω which is free with respect to B. Then [V1] applies to get (ii) and (iii). Direct calculations then give (i), since $\|p_i x p_i\|_2^2 = \tau(p_i x p_i x^* p_i) = \tau(p_i x x^* p_i)\tau(p_i) = \|x\|_2^2 \tau(p_i)^2$. Q.E.D.

2.3. Remark. For each n, $N \subset M$ an inclusion of type II_1 factors with trivial relative commutant and $x \in M$ an element with $\tau(x) = 0$ let $c_n(N \subset M; x) = \inf\left\{ \sum_{i=1}^{n} \|p_i x p_i\|_2^2 \,|\, \{p_i\}_{1\leq i\leq n} \subset N \text{ partition of the unity with projections of trace } 1/n\right\}$. Let $c_n = \sup\{c_n(N \subset M; x) | N \subset M, \quad N' \cap M = \mathbb{C} \text{ and } x \in M \text{ as above}\}$. The above Corollary shows that $c_n(N \subset M; x) \leq 1/n$, for all $N \subset M$ and $x \in M, \quad \|x\|_2 \leq 1$. If one takes $M = N * B$ for some B without atoms then for any $x \in B, \quad \|x\|_2 = 1, \quad \tau(x) = 0$, and any partition $\{p_i\}_{1\leq i\leq n}$ in N, as above, $\sum_{i=1}^{n} \|p_i x p_i\|_2^2 = 1/n$. Thus $c_n = 1/n$. This shows in particular that $1/n + \varepsilon$, with ε arbitrarily small, are the best constants α for which given any inclusion $N \subset M$, with trivial relative commutant and any $x \in M$ as above, one can find projections q of trace $1/n$ so that $\|qxq\|_2^2 < \alpha\tau(q)$.

2.4. Corollary. *If $N \subset M$, with $N' \cap M = \mathbb{C}1$, and ω are as in the preceding corollary, then for any $x \in M^\omega$, $\tau(x)1 \in \overline{\text{co}}^n\{uxu^* | u \in \mathcal{U}(N^\omega)\}$.*

Proof. Since any $x \in M^\omega$ is in the norm closure of the linear combinations of projections in M^ω and since by (i),b) of Corollary 2.2 we have the statement for $x = f$ a projection, by taking $u = \Sigma \lambda^i p_i$, $\lambda = \exp 2\pi i/n$ and by using that $\sum_{i=1}^{n} p_i f p_i = \frac{1}{n} \sum_{k=1}^{n} u^k f u^{-k}$, it follows arguing like in Dixmier's theorem ([D2]) that we get the statement for any $x \in M^\omega$. Q.E.D.

Let us finally deduce here the result stated without proof in (8.1 of [Po6]), which we already mentioned in the introduction.

2.5. Corollary. *Let $N \subset M$ be an inclusion of type II_1 factors of finite index $[M : N] = s$. Assume $N \subset M$ is extremal ([PiPo], [Po7]), i.e., $E_{N' \cap M}(e_0) \in \mathbb{C}1$ for $e_0 \subset M$ a Jones projection (i.e., $E_N(e_0) = s^{-1}1$). Let $N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \subset \cdots$ be the associated Jones tower and $M_\infty = \overline{\bigcup_n M_n}$ be its enveloping algebra. Denote by $R = vN\{e_1, \cdots\}$, $R^s = vN\{e_2, \cdots\}$. If $Q_0 \subset M^\omega$ is a separable von Neumann algebra then there exists a unitary element $v \in N^\omega$ such that $Q = vQ_0v^*$ and R generate the algebra $M^s_\infty(Q) = (Q \otimes R^s) \underset{R^s}{*} R$ in a way that identifies $Q \subset Q \vee R$ with $Q \simeq (Q \otimes \mathbb{C}) * \mathbb{C} \subset M^s_\infty(Q)$ and $R \subset Q \vee R$ with $R \simeq \mathbb{C} * R \subset M^s_\infty(Q)$. Moreover $M^s_\infty(Q)$ makes commuting squares with N^ω, M^ω and M^ω_n, $n \geq 1$, and one has $N^\omega \cap M^s_\infty(Q) = N^s(Q)$, $M^\omega \cap M^s_\infty(Q) = M^s(Q)$, $M^\omega_n \cap M^s_\infty(Q) = M^s_n(Q)$, $n \geq 1$, where $N^s(Q) \subset M^s(Q) \subset M^s_1(Q) \subset \cdots$ are the subalgebras of $M^s_\infty(Q)$ defined in* [Po6].

Proof. Since $N \subset M$ is extremal, $E_{M' \cap M_1}(e_1) \in \mathbb{C}1$, so that $E_{M' \cap M_\infty}(R) = R^s \subset M' \cap M_\infty$ and the conditions of Theorem 2.1 are fulfilled for $P_1 = Q_0 \vee R^s$ and $P_2 = R$ and $B = R^s$. The rest follows by the definitions of $M^s_n(Q)$ in [Po6]. Q.E.D.

3 Some problems and comments

3.1. Embeddings of $L(\mathbb{F}_n)$. The following is a problem for which the above results may be useful. It was first posed in [Po5] and it can be regarded as the operator algebra analogue of von Neumann's problem on the embeddings of free groups into nonamenable groups.

3.1.1. Problem. Does any nonhyperfinite type II_1 factor contain copies of $L(\mathbb{F}_2)$? Do all the non Γ factors, or at least the property T factors, necessarily contain copies of $L(\mathbb{F}_2)$?

Upon inspection of the proof of 2.1 one can see that a main obstruction for getting back in M the asymptotically free sequences in M^ω seems to be to obtain a "rigidity" type result that would insure $\tau(w) = 0$ for any word w in u and v once one has some u', v' for which $\tau(w') = 0$ for words w' in u', v' of length less than some n. The necessary rigidity feature could be provided by the property non Γ (or T) assumption on the factor M. Related to this, note that if $u \in M$ is the generator of a Cartan subalgebra A of M then by 2.1 there are $v \in M^\omega$ which are free with respect to u but there are no such v in M. Indeed, if $N \subset M$ is any von Neumann subalgebra of M that contains A then A is Cartan in N (see e.g. [JPo]]), but if $N = \{u, v\}''$ with v free with respect to u then A is singular in N (see e.g. [Po2]).

3.2. Functional analytical characterization of $L(\mathbb{F}_n)$. We recall that there exists no satisfactory functional analytical characterization of the free group factors.

Related to this, we propose here, the following working conjecture (see also [dlHV] and [dlH] in these Proceedings for related problems and comments):

3.2.1. Problem. Are all separable non Γ (or full [C1]) type II_1 factors which have the Haagerup approximation property mutually isomorphic? Are all non Γ subfactors of $L(\mathbb{F}_\infty)$ isomorphic to $L(\mathbb{F}_\infty)$?

Recall that M has the Haagerup approximation property [H] if the identity on M can be approximated by compact unital trace-preserving completely positive maps, in the point-$\| \quad \|_2$ topology. Note that this property is clearly inherited by subalgebras. One can strengthen this property in the above assumption by requiring the compact maps to have only unitaries from M as eigenvectors.

While at this stage this problem seems hopelessly difficult, due to Voiculescu's noncommutative probability approach, the problem of the mutual isomorphism of the $L(\mathbb{F}_n), 2 \leq n \leq \infty$, which is a particular case of 3.2.1, may soon be resolved. In fact, by the recent results in [Ra1, 2, 3], in order to show that all the $L(\mathbb{F}_\kappa)$ are isomorphic it is necessary and sufficient to prove that for some n the fundamental group of $L(\mathbb{F}_n)$ is nontrivial or that $L(\mathbb{F}_n) \otimes \mathcal{B}(H) \simeq L(\mathbb{F}_\infty) \otimes \mathcal{B}(H)$. A related problem that was not yet solved is whether $L(\mathbb{F}_n) \otimes R$ is isomorphic to $L(\mathbb{F}_\infty) \otimes R$ or not and whether this would be sufficient to insure the isomorphism of $L(\mathbb{F}_n)$ with $L(\mathbb{F}_\infty)$.

3.2.2. Problem. Are all non Γ type II_1 factors with the Haagerup approximation property stably isomorphic?

3.2.3. Problem. If M is non Γ and is approximable by subfactors isomorphic to $L(\mathbb{F}_\infty)$ is then M itself isomorphic to $L(\mathbb{F}_\infty)$? Is at least the stabilized version of this true?

Related to this note that by [C4] there are plenty of examples of nonisomorphic nonhyperfinite type II_1 factors M with $M \simeq M \otimes R$ (and even factorial $M' \cap M^\omega$, i.e., without hypercentral sequences) which have the Haagerup approximation property.

If one seeks to give a negative answer to 3.2.1 then a class of factors that could be tested are the factors $M^s(Q)$ of [Po6], since they are non Γ by [Po6] and have the Haagerup approximation property when Q has it by [Bo]. For each $Q = L(\mathbb{F}_x)$, as defined in [Ra2], [Dy] for $x \in (1, \infty]$, and for each $s \in \{4\cos^2 \pi/n | n \geq 4\}$, the factors $M^s(Q)$ were proved to be of the form $L(\mathbb{F}_y)$, for some $y \in (1, \infty]$ depending on x and s, in [Ra2]. But for $s > 4$ and/or $Q = L(\mathbb{F}_1) = L^\infty(\mathbb{T}, \mu)$, $Q = R$, the problem of identifying $M^s(Q)$ (or its generalizations [Ba], [Ra3]) as some $L(\mathbb{F}_y)$ is still open. If answered positively, this would prove the existence of irreducible subfactors of any index $s > 4$ in $L(\mathbb{F}_x)$, $1 < x \leq \infty$.

3.3. Primeness (or indecomposability) for factors. (see also [dlH]). Besides implying the mutual isomorphism of all $L(\mathbb{F}_n)$, $2 \leq n \leq \infty$, a positive answer to 3.2.1 would imply that $L(\mathbb{F}_\kappa) \otimes \mathbb{L}(\mathbb{F}_>) \simeq \mathbb{L}(\mathbb{F}_\infty)$, $\nvDash \leq \kappa, > \leq \infty$, in particular that the type II_1 factors that are associated to free groups are decomposable i.e., that can be written as a tensor product of type II_1 factors, in other words, that they are not "prime". However, the result in [Po2] showing that for uncountable sets of generators S the type II_1 factors $L(\mathbb{F}_S)$ are prime (or indecomposable) gives an indication that the same result may be true for finitely and countably many generators as well. Thus, the algebras $L(\mathbb{F}_n) \otimes L(\mathbb{F}_m)$ are good candidates for giving a negative answer to Problem 3.2.1 and the following seems important to settle:

3.3.1. Problem. Are $L(\mathbb{F}_n)$, $2 \leq n \leq \infty$, prime factors? Do there exist separable prime factors at all? Do property T factors have a decomposition into a (unique?) product of prime factors?

In case $L(\mathbb{F}_n)$ turns out to be prime, then one probably has to add primeness to the conditions in Problem 3.2.1.

On the other hand, as many situations show, results holding true in nonseparable situations can hardly suggest what the answer to the similar separable situation would be, and vice versa. We will illustrate this by two examples:

For instance, it is showed in [Po1] that if $M \subset P$ are separable type II_1 factors with $M' \cap P = \mathbb{C}$ then M contains maximal abelian subalgebras of P But that if M is non Γ and $P = M^\omega$ then no maximal abelian subalgebra of M is maximal abelian in M^ω (although $M' \cap M^\omega = \mathbb{C}$). So if M^ω is regarded as a union of an increasing net of separable type II_1 factors M_i that contain M then for each i there is a masa of M_i in M but at the limit there is none.

Let us consider a second example when a certain property that holds true in a separable situation "blows" when passing to nonseparability, showing one more time that one could hardly guess whether or not $L(\mathbb{F}_n)$ are prime, just from the fact that $L(\mathbb{F}_S)$ are prime for uncountable sets S.

We first need 2 simple observations:

3.3.2. Lemma. *If $N \subset M$ are type II_1 factors and N has the property T then $N' \cap M^\omega = (N' \cap M)^\omega$.*

Proof. If $x = (x_n) \in N' \cap M^\omega$, then $\lim\limits_{n\to\infty} \|[y, x_n]\|_2 = 0$, for any element $y \in N$, so that by the property T of N this holds true uniformly for $y \in M$, $\|y\| \leq 1$. Thus ([Po5]) $\lim\limits_{n\to\omega} \|x_n - E_{N'\cap M}(x_n)\| = 0$. Q.E.D.

3.3.3. Lemma. *If M is a type II_1 factor then $\overline{\mathcal{F}(M)} \subset \mathcal{F}(M^\omega)$.*

Proof. Let $t \in \overline{\mathcal{F}(M)}$, $0 < t < 1$, and $t_n \in (0,1) \cap \mathcal{F}(M)$ be such that $\lim\limits_n t_n = t$. Let $p_n \in \mathcal{F}(M)$ be projections with $\frac{\tau(p_n)}{\tau(1-p_n)} = t_n$ and $\theta_n : p_n M_{p_n} \simeq (1-p_n)M(1-p_n)$ onto isomorphisms. Then $p = (p_n) \in M^\omega$, $\tau(p) = t$ and $\theta = \prod \theta_n$ is an isomorphism of $pM^\omega p$ onto $(1-p)M^\omega(1-p)$. Thus $t \in \mathcal{F}(M^\omega)$. Q.E.D.

3.3.4. Corollary. *If M is a separable type II_1 factor containing a subfactor Q with the property T such that $Q' \cap M = \mathbb{C}$ then any separable subfactor $P \subset M^\omega$ that contains Q has countable fundamental group, although M^ω itself may have fundamental group $(0, \infty)$.*

Proof. By [Po6], given any $s \in (0,\infty)$ there exists a separable type II_1 factor M^s containing a subfactor Q with the property T and trivial relative commutant, and having s in its fundamental group. If $s_1, s_2 \in (0,\infty)$ are so that the multiplicative group generated by s_1, s_2 in $(0,\infty)$ is dense in $(0,\infty)$ then let $M = M^{s_1} \otimes M^{s_2}$, which is separable, has the subfactor $Q \otimes Q$ with the property T and trivial relative commutant, and has fundamental group containing s_1, s_2, thus dense in $(0,\infty)$. Then the above two lemmas apply, together with the result in [Po5], which shows that separable factors that contain a subfactor with the property T and trivial relative commutant have countable fundamental group. Q.E.D.

At this point it is worth mentioning a trivial consequence of the above construction: while in [Po6] we proved that given any finite set of elements $S \subset (0,\infty)$ there exist separable type II_1 factors with the fundamental group countable but containing that finite set, we can now prescribe countable sets as well.

3.3.5. Corollary. *Given any countable set $S \subset (0, \infty)$ there exist a separable type II_1 factor M with countable fundamental group $\mathcal{F}(M)$ containing the set S. In particular, by taking S to contain all the positive rationals, one gets examples of separable type II_1 factors M with $\mathcal{F}(M)$ countable and such that $M_{n \times n}(M) \simeq M$ for all n.*

Proof. Let Q be a type II_1 factor with the property T, $s_1, s_2 \in (0, \infty)$ generating a dense multiplicative subgroup in $(0, \infty)$ and let $N = M^{s_1}(Q) \otimes M^{s_2}(Q)$ ([Po6]). Since $s_i \in \mathcal{F}(M^{s_i}(Q))$ we have that $s_1, s_2 \in \mathcal{F}(N)$, so that $\mathcal{F}(N^\omega) = (0, \infty)$, by 3.3.3. Let $S = \{t_n\}_n$. For each n there exists a t_n-scaling automorphism θ_n of $N^\omega \otimes \mathcal{B}(H)$. Let $\tilde{M}$ be the von Neumann algebra generated by

$$\cup\{\theta_{k_1}^{i_1}\theta_{k_2}^{i_2}\cdots\theta_{k_j}^{i_j}(N \otimes \mathcal{B}(H)) | j \geq 0, k_1, \ldots, k_j \geq 1, i_1, \ldots, i_j \in \mathbb{Z}\}.$$

Then $\tilde{M}$ is separable and $\theta_n(\tilde{M}) = \tilde{M}$, for all n.

Since $\tilde{M} \supset N \otimes \mathcal{B}(H)$, there exists $M \supset N$, a separable type II_1 factor, such that $\tilde{M} = M \otimes \mathcal{B}(H)$. Since $Q \otimes Q$ has the property T and $(Q \otimes Q)' \cap N = \mathbb{C}$, by 3.3.2 $(Q \otimes Q)' \cap M \subset (Q \otimes Q)' \cap N^\omega = \mathbb{C}$. Thus M has countable $\mathcal{F}(M)$. But since $\theta_n(M \otimes \mathcal{B}(H)) = M \otimes \mathcal{B}(H)$ and θ_n is t_n-scaling the trace, we obtain that $t_n \in \mathcal{F}(M)$, thus $S \subset \mathcal{F}(M)$. Q.E.D.

Added in the proof. It was pointed out to us by D. Bisch and V. Jones that a result similar to Corollary 3.3.5 was already obtained by V. Ya. Golodets and N. I. Nessonov in their paper "T-property and nonisomorphic full factors of type II and III", J. Funct. Analysis, 70 (1987), 80-89, by using a different method.

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