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Appendix to: "Ergodic Actions of Compact Matrix Pseudogroups on C*-algebras" by Florin Boca.

Introduction. For an ergodic covariant system $(\mathcal{M}, \rho, G)$ of a compact group G it was shown in [L2,Theorem 8] and [Wa1,Theorem 15] that the crossed product $\mathcal{M} \times_\rho G$ is a simple C*-algebra (or a factor in the von Neumann algebra case) $\Longleftrightarrow$ the multiplicity of each $\pi \in \widehat{G}$ in ρ equals $\dim(\pi)$, and in this case $\mathcal{M} \times_\rho G$ is isomorphic to the algebra of compact operators on $L^2(G)$. We shall here study the corresponding result for an ergodic coaction $(\mathcal{M}, \sigma, \mathcal{A})$ of a compact matrix pseudogroup $G = (\mathcal{A}, u)$ with faithful Haar measure as defined in F. Boca's article.

The main tool used in the group case is the construction of a fundamental eigenoperator $U \in M(\mathcal{M} \otimes C^*(G))$ satisfying $\rho_x \otimes i(U) = U1 \otimes L_x$ for $x \in G$. We shall construct a similar operator Y in Lemma A1. In the group case the multiplicity of each $\pi \in \widehat{G}$ in ρ is always $\leq \dim(\pi)$, hence U can be considered a partial isometry over $L^2(\mathcal{M}, \omega) \otimes L^2(G)$, with ω the invariant trace on $\mathcal{M}$. Since the bound M_α of the multiplicity obtained in Theorem 17 can be larger than d_α, we have to be more careful with the domain and the range of the eigenoperator Y, it turns out that Y is a partial isometry from a subspace of $L^2(\mathcal{M}, \omega) \otimes L^2(\mathcal{A}, h)$ onto $L^2(\mathcal{M}, \omega) \otimes L^2(\mathcal{M}, \omega)$, here h and ω are the canonical invariant states on $\mathcal{A}$, respectively $\mathcal{M}$. It also has to be taken into account that the invariant state ω is not a trace. It was shown above in Proposition 18 by F. Boca that the modular operator Θ leaves the finite dimensional spaces $\mathcal{M}_\alpha$ invariant and that $\Theta|\mathcal{M}_\alpha \cong \Lambda_\alpha \otimes F_\alpha$, where F_α is the fundamental matrix corresponding to α and Λ_α is a $N \times N$–matrix, N being the multiplicity of α in σ.

The main result, Theorem A, can then be stated as follows: $\mathcal{M} \times_\sigma \widehat{\mathcal{A}}$ is a simple C*-algebra $\Longleftrightarrow$ $\mathrm{Tr}(\Lambda_\alpha) = \mathrm{Tr}(F_\alpha)$ for all $\alpha \in \widehat{G}$, and in this case $\mathcal{M} \times_\sigma \widehat{\mathcal{A}}$ is isomorphic to the algebra of compact operators on $L^2(\mathcal{M}, \omega)$. Therefore, if we define the quantum dimension of α to be $\mathrm{Tr}(F_\alpha)$, it is natural to define the quantum multiplicity of α in σ as $\mathrm{Tr}(\Lambda_\alpha)$. We then get a generalisation of the result for ordinary compact groups.

All unexplained notation and references are as in Boca's article.

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Notation. Let $\mathcal{H}_\alpha = \alpha$-part of $\mathcal{H}_h = L^2(\mathcal{A}, h)$, this is generated by $\{(u_{ij}^\alpha)_h | i, j \leq d\}$ and has dimension d^2. Similarily we have that $\mathcal{M}_\alpha = \alpha$-part of $\mathcal{M}$ has dimension dN.

Define the following partial isometries:

$$\begin{array}{lll} A^\alpha_{ij} : \mathcal{H}_\alpha \to \mathcal{H}_\alpha & A^\alpha_{ij}((u^\alpha_{kl})_h) = \delta_{jl}(u^\alpha_{ki})_h & i,j \le d \\ B^\alpha_{ij} : \mathcal{M}_\alpha \to \mathcal{M}_\alpha & B^\alpha_{ij}((e^{(k)}_l)_\omega) = \delta_{jk}(e^{(i)}_l)_\omega & i,j \le N \\ C^\alpha_{ij} : \mathcal{H}_\alpha \to \mathcal{M}_\alpha & C^\alpha_{ij}((u^\alpha_{kl})_h) = \delta_{jl}(e^{(i)}_k)_\omega & i \le N,\ j \le d. \end{array}$$

The following formulas should then be easy to verify:

$$\begin{array}{lll} A^\alpha_{ij}A^\alpha_{kl} = \delta_{jk}A^\alpha_{il} & B^\alpha_{ij}B^\alpha_{kl} = \delta_{jk}B^\alpha_{il}, & \\ B^\alpha_{ij}C^\alpha_{kl}A^\alpha_{mn} = \delta_{jk}\delta_{lm}C^\alpha_{in} & C^\alpha_{ij}C^{\alpha*}_{kl} = \delta_{jl}B^\alpha_{ik} & C^{\alpha*}_{ij}C^\alpha_{kl} = \delta_{ik}A^\alpha_{jl}. \end{array}$$

With P_α the ortogonal projction $\mathcal{H}_h \to \mathcal{H}_\alpha$ and V as in Remark 15, let $V(\alpha) = P_\alpha \otimes 1 V = V P_\alpha \otimes 1$. Then over $\mathcal{H}_\alpha \otimes \mathcal{H}_h$ we have that $V(\alpha)((u^\alpha_{ij})_h \otimes a_h) = \Delta(u^\alpha_{ij})1_h \otimes a_h = \sum_k (u^\alpha_{ik})_h \otimes u^\alpha_{kj} a_h$. So we have that

$$V(\alpha) = \sum_{jk} A^\alpha_{jk} \otimes u^\alpha_{jk}.$$

Definition. *$\sigma \otimes i$ is the action on $\mathcal{M} \otimes \mathcal{K}$ given by $\sigma \otimes i(m \otimes k) = \sigma(m)_{13} 1 \otimes k \otimes 1$, so $\sigma' = Ad(V_{23})\sigma \otimes i$. Next, let λ_i be as in Proposition 18, i.e. $\sum_i e^{(k)}_i e^{(l)*}_i = \delta_{kl}\lambda_k 1_\mathcal{M}$ and take*

$$Y(\alpha) = \sum_{ik} \lambda_k^{-\frac{1}{2}} e^{(k)}_i \otimes C^\alpha_{ki} \in \mathcal{M} \otimes \mathcal{L}(\mathcal{H}_\alpha, \mathcal{M}_\alpha).$$

Lemma A1.

(1) $Y(\alpha)Y(\alpha)^* = 1_\mathcal{M} \otimes 1_{\mathcal{L}(\mathcal{M}_\alpha)}$
(2) $\sigma \otimes i(Y(\alpha)) = (Y(\alpha) \otimes 1)(1 \otimes V(\alpha))$
(3) *Z satisfies (2) $\iff$ $Z = (1 \otimes D)Y(\alpha)$ for some $D \in \mathcal{L}(\mathcal{M}_\alpha)$*

Proof: (1): $Y(\alpha)Y(\alpha)^* = \sum \lambda_k^{-\frac{1}{2}}\lambda_l^{-\frac{1}{2}} e^{(k)}_i e^{(l)*}_i \otimes B^\alpha_{kl} = 1_\mathcal{M} \otimes 1_{\mathcal{L}(\mathcal{M}_\alpha)}$.
Then for (2):

$$\begin{aligned} \sigma \otimes i(Y(\alpha)) = & \sum \lambda_k^{-\frac{1}{2}} e^{(k)}_j \otimes C^\alpha_{ki} \otimes u^\alpha_{ji} \\ = & \sum \lambda_k^{-\frac{1}{2}} (e^{(k)}_j \otimes C^\alpha_{kj} \otimes 1)(1 \otimes A^\alpha_{si} \otimes u^\alpha_{si}) = (Y(\alpha) \otimes 1)(1 \otimes V(\alpha)). \end{aligned}$$

And for (3): If $Z \in \mathcal{M} \otimes \mathcal{L}(\mathcal{H}_\alpha, \mathcal{M}_\alpha)$ satisfies (2), then the "$\mathcal{M}$-part" of Z must be in $\mathcal{M}_\alpha$, i.e. $Z \in \mathcal{M}_\alpha \otimes \mathcal{L}(\mathcal{H}_\alpha, \mathcal{M}_\alpha)$, so we can write $Z = \sum e^{(l)}_r \otimes E_{lr}$ for some maps $E_{lr} \in \mathcal{L}(\mathcal{H}_\alpha, \mathcal{M}_\alpha)$. If (2) holds we get

$$\sum e^{(l)}_j \otimes E_{lr} \otimes u^\alpha_{jr} = \sum e^{(l)}_j \otimes E_{lj} A^\alpha_{sr} \otimes u^\alpha_{sr},$$

thus $E_{lj}A^\alpha_{sr} = \delta_{js}E_{lr}$. Taking $D = \sum \lambda_j^{\frac{1}{2}} E_{j1} C^{\alpha*}_{j1} \in \mathcal{L}(\mathcal{M}_\alpha)$ we get

$$\begin{aligned} (1 \otimes D)Y(\alpha) = & \sum \lambda_k^{-\frac{1}{2}} e^{(k)}_i \otimes \lambda_j^{\frac{1}{2}} E_{j1} C^{\alpha*}_{j1} C_{ki} \\ = & \sum e^{(k)}_i \otimes E_{k1} A_{1i} = \sum e^{(k)}_i \otimes E_{ki} = Z. \end{aligned}$$

□

An element Z satisfying (2) is called an *α-eigenoperator* for the action. So Lemma A1 tells us that $Y(\alpha)$ generates all α-eigenoperators by the formula (3). We shall also need the universal eigenoperator $Y = \sum^{\oplus} Y(\alpha)$, this is a map from $\mathcal{H}_\omega \otimes \mathcal{H}_h$ to $\mathcal{H}_\omega \otimes \mathcal{H}_\omega$ satisfying

$$\sigma \otimes i(Y) = Y_{12}V_{23} \qquad YY^* = 1_{\mathcal{M}} \otimes 1_{\mathcal{M}}.$$

It then follows that $\sigma'(Y^*aY) = Y_{12}^*\sigma \otimes i(a)Y_{12}$ for all a.

Lemma A2. *Let Θ be as in Proposition 18 and let Θ_α be its restriction to $\mathcal{M}_\alpha$. With Λ_α the matrix given by $(\Lambda_\alpha)_{kl}1_{\mathcal{M}} = \frac{1}{M_\alpha}\sum_j e_j^{(k)}e_j^{(l)^*}$ we have*

$$\begin{aligned}
&(1) \quad Tr(\Theta_{\alpha^c}) = Tr(\Theta_\alpha^{-1})\\
&(2) \quad Tr(\Theta_\alpha) = Tr(\Lambda_\alpha)M_\alpha = \textstyle\sum \lambda_k\\
&(3) \quad Tr(\Theta_\alpha^{-1}) = Tr(\Lambda_\alpha^{-1})M_\alpha = M_\alpha^2 \textstyle\sum \lambda_k^{-1}\\
&(4) \quad Tr(\Lambda_{\alpha^c}) = M_\alpha \textstyle\sum \lambda_k^{-1}
\end{aligned}$$

Proof: (1) follows from the fact that $\Theta(x) = \lambda x \Longrightarrow \Theta(x^*) = \lambda^{-1}x^*$. In Proposition 18 it is proved that $\Theta_\alpha \cong \Lambda_\alpha \otimes F_\alpha$, hence (2) and (3). Combining these three properties with the fact that $\mathcal{M}_{\alpha^c} = \mathcal{M}_\alpha^*$, then $M_{\alpha^c} = M_\alpha$ and (4) follows.

□

We are now ready to prove the main result:

Theorem A. *With the same assumptions as in Theorem 19 and if $\mathcal{M}_\alpha \neq 0$ for all α, then the following conditions are equivalent:*

$$\begin{aligned}
&(1) \quad \mathcal{N} = \mathcal{M} \times_\sigma \hat{\mathcal{A}} \quad \text{is a simple } C^*\text{-algebra}\\
&(2) \quad \mathcal{N} \cong \mathcal{K}(\mathcal{H}_\omega)\\
&(3) \quad Y(\alpha)^*Y(\alpha) = 1_{\mathcal{M}} \otimes 1_{\mathcal{L}(\mathcal{H}_\alpha)} \text{ for all } \alpha \in \hat{G}\\
&(4) \quad Tr(\Lambda_\alpha) = Tr(F_\alpha) \text{ for all } \alpha \in \hat{G}.
\end{aligned}$$

Proof: (3) $\Longrightarrow$ (2): In this case Y is a unitary eigenoperator between $\mathcal{H}_\omega \otimes \mathcal{H}_h$ and $\mathcal{H}_\omega \otimes \mathcal{H}_\omega$, so

$$\begin{aligned}
\mathcal{N} &\cong (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_h))^{\sigma'} = [Y^*(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\omega)Y]^{\sigma'} = Y_{12}^*(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\omega))^{\sigma\otimes i}Y_{12}\\
&= Y_{12}^*\mathcal{M}^\sigma \otimes \mathcal{K}(\mathcal{H}_\omega)Y_{12} = Y_{12}^*1 \otimes \mathcal{K}(\mathcal{H}_\omega)Y_{12} \cong \mathcal{K}(\mathcal{H}_\omega).
\end{aligned}$$

Note that from Lemma 4 there is a conditional expectation from $\mathcal{M}$ onto $\mathcal{M}^\sigma$, so we have that $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\omega))^{\sigma\otimes i} = \mathcal{M}^\sigma \otimes \mathcal{K}(\mathcal{H}_\omega)$.
(2) $\Longrightarrow$ (1) is obvious.
(1) $\Longrightarrow$ (3): If $\mathcal{N}$ is simple, so is $1 \otimes p(\alpha)_*\mathcal{N}1 \otimes p(\alpha)_* = (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\alpha))^{\sigma'}$. Now $\mathcal{J} = Y(\alpha)^*1 \otimes \mathcal{K}(\mathcal{M}_\alpha)Y(\alpha)$ is a 2-sided ideal in $(\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\alpha))^{\sigma'}$:

If $A \in \mathcal{K}(\mathcal{H}_\alpha)$, $B \in (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\alpha))^{\sigma'}$ then $Y(\alpha)B$ satisfies (2) in Lemma A1, so $Y(\alpha)B = 1 \otimes CY(\alpha)$ for some $C \in \mathcal{K}(\mathcal{M}_\alpha)$. Therefore $Y(\alpha)^*1 \otimes AY(\alpha)B = Y(\alpha)^*1 \otimes ACY(\alpha) \in \mathcal{J}$, and since $\mathcal{J}^* = \mathcal{J}$, $\mathcal{J}$ is a 2-sided ideal.

If $\mathcal{J} = \{0\}$ then $\mathcal{M}_\alpha = \{0\}$, so simplicity gives us that $\mathcal{J} = (\mathcal{M} \otimes \mathcal{K}(\mathcal{H}_\alpha))^{\sigma'}$. Thus $1 \in \mathcal{J}$, hence $Y(\alpha)^*Y(\alpha) = 1$.
(3) $\Longleftrightarrow$ (4): Since $Y(\alpha)^*Y(\alpha)$ always is a projection and ω is faithful, we have for all α:

$$(3) \iff \omega \otimes i(Y(\alpha)^* Y(\alpha)) = 1_{\mathcal{L}(\mathcal{H}_\alpha)} \iff \sum \lambda_k^{-1} \omega\left(e_i^{(k)^*} e_j^{(k)}\right) A_{ij}^\alpha = 1_{\mathcal{L}(\mathcal{H}_\alpha)}$$
$$\iff \sum \lambda_k^{-1} = 1 \iff \mathrm{Tr}(\Lambda_{\alpha^c}) = M_\alpha \iff \mathrm{Tr}(\Lambda_{\alpha^c}) = \mathrm{Tr}(F_{\alpha^c}).$$

□

Remark. $\mathrm{Tr}(F_\alpha) = M_\alpha$ is called the quantum dimension of α. ¿From Theorem A it then seems reasonable to call $\mathrm{Tr}(\Lambda_\alpha)$ the quantum multiplicity of α in σ. Since $Y(\alpha)^* Y(\alpha)$ always is a projection we have from the proof of (3) $\iff$ (4) the following:

Corollary. *With $(\mathcal{M}, \sigma, G)$ as in Theorem 19 one has always $Tr(\Lambda_\alpha) \leq Tr(F_\alpha)$ with equality $\iff$ (1)–(4) in Theorem A hold.*

Addtional reference:

[L2] M. B. Landstad, Operator algebras and compact groups. Proc. of the Int. Conf. in Operator Algebras and Group Representations in Neptun (Romania) 1980, Monographs and Studies in Math. 18, vol.II (1984), 33–47, Pitman.

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