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1. Introduction

We apply the machinery of Explicit Formulae to investigate the rank of some traditionally interesting abelian varieties. For any integer N, consider the Jacobian variety $J_0(N)$ of the modular curve $X_0(N)$ classifying isogenies of degree N, whose $\mathbb{C}$-rational points are the Riemann surface $X_0(N)(\mathbb{C}) = \mathbb{H}^*/\Gamma_0(N)$. Here $\mathbb{H} = \{x+iy\,|y>0\}$, $\mathbb{H}^* = \mathbb{H}\cup\mathbb{Q}$ and $\Gamma_0(N)$ is the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices whose lower right corner is divisible by N, acting on the upper half-plane as linear fractional transformations.

These varieties owe much of their appeal to the generalized Shimura-Taniyama conjecture which asserts that any abelian variety A with real multiplications, both defined over $\mathbb{Q}$, is isogenous to a factor of $J_0(N)$ for a suitable N.

Very little seems to be known about the rank of these abelian varieties, except for elliptic modular factors whose ranks have been studied numerically ([4], [12], [22]). For N a prime, Barry Mazur [20] performed descents on the Eisenstein quotient A, the part of $J_0(N)$ with non-trivial torsion over $\mathbb{Q}$, and

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deduced that $A(\mathbb{Q})$ has rank 0. Despite their importance, one knows only that the Eisenstein factor has dimension at least $O(\log N)$ for infinitely many primes N (cf. [20]). We have started a numerical investigation and we report some of our findings in the last section.

Since the abelian varieties A we consider here admit endomorphisms by a ring of rank equal to $\dim A$, we shall occasionally use the *normalized rank* of A, denoted $\operatorname{nrk} A = \operatorname{rank} A / \dim A$.

Since the results of Gross-Zagier [14] and of Kolyvagin [18] provide good information about those factors of $J_0(N)$ whose rank over the Hecke algebra is at most one, it seemed of some interest to determine their prevalence. Our main results are the following *conditional* bounds.

Theorem 3.15. *Assume the Riemann hypothesis for the L-series of newforms. For any positive integer N, let χ be a character of the Atkin-Lehner group $\mathcal{W}(N)$. For each $\epsilon > 0$, the* analytic *rank satisfies*

$$\operatorname{rank} A \leq \left(\frac{3}{2} + \epsilon\right) \dim A,$$

for all sufficiently large N.

This improves the conditional bound obtained implicitly in [21] for a general abelian variety, namely $\operatorname{rank} A = O(\dim A \log N / \log\log N)$.

Theorem 4.9. *Assume the Riemann Hypothesis for L-functions of newforms and Dirichlet L-series of quadratic characters. Then the average* analytic *normalized rank for Jacobians $J_0(N)$ with N* prime *is bounded by $\frac{7}{6}$. More precisely, for any c and any ϵ, we have*

$$\sum_{Y<N\leq cY} \operatorname{rank} J_0(N) \leq \left(\frac{7}{6} + \epsilon\right) \sum_{Y<N\leq cY} \dim J_0(N),$$

where the sum runs over primes N and Y is sufficiently large.

At the end of Section 4, we conclude that under suitable Riemann hypotheses, the Birch-Swinnerton-Dyer conjecture is true for a substantial proportion of the varieties we are considering.

Our methods are fairly traditional: we use explicit formulae. The improvements over [21] are due to the Eichler-Selberg trace formula, which gives an estimate better than the Hasse-Weil bound for small p. Since our estimates are fairly brutal, better techniques may provide substantial improvements.

The approach used here could be used in the function field case to get analogous *unconditional* bounds for the rank of the Jacobian of Drinfeld modular curves (cf. [6], [8]). The missing ingredient is an *explicit* Selberg trace formula in this context.

In section 5, we deduce from conductor bounds, obtained in [9], that "large enough" wild ramification imposes conditions on the endomorphism ring of the variety. For instance, we have the following special case of Theorem 5.5.

Theorem. *Let $f(z) = \sum a_n e^{2\pi i n z}$ be a new form of weight 2 and level N. Let $\mathbb{E} = \mathbb{Q}(\{a_n\})$ be the field of $\mathbb{Q}$-endomorphisms of the associated abelian variety A_f.*

Assume that $p^{r_p} || N$ and define s_p by the following:

$$s_p = \left\lceil \frac{r_p}{2} - 1 - \frac{1}{p-1} \right\rceil$$

where $\lceil x \rceil$ is the least integer $\geq x$. Then $\mathbb{E}$ contains the real subfield of the cyclotomic field of p^{s_p}-th roots of unity.

In contrast to what seems to happen in the tame case, this shows that small factors of $J_0^{\mathrm{new}}(N)$ cannot occur when N is divisible by high powers and we expect that non-trivial factors of the new part of dimension $O(\sqrt{N})$ may occasionally occur.

I wish to thank Ken Ribet for advising me that Saito [27] had obtained by modular methods a weaker result of the same type and for making some of their correspondence available.

We had earlier established upper bounds for the average rank of elliptic curves, first 2.3 in [6], later improved to 2 in [8], *conditionally* for curves over $\mathbb{Q}$ and *unconditionally* for curves over $\mathbf{F}_q(t)$. We make some comments on the lower ranks obtained here at the end of this paper.

In Section 6, we describe some experimental work on the rank of $J_0^-(N)$ for primes N at most 10000, based on work of Birch and Gross. There is a tendency for $J_0^-(N)$ to have a very large simple abelian factor, with non-trivial torsion and hence of rank 0 by [20]. It should be noted that a surprisingly large proportion of the small factors have normalized analytic rank at least 2.

We use the standard notation $f << g$ to mean $|f| = O(g)$. As usual, we write $\omega(n)$ for the number of prime factors of n and $\sigma_k(n)$ for the sum of the k-th powers of the divisors of n. For any divisor m of N, $m||N$ means that m is prime to N/m. The symbol $(\frac{x}{y})$ is the usual Jacobi symbol and $x = \square$ abbreviates the assertion that x be a square. The largest square factor of x will be denoted $Q(x)^2$, that is $Q(x) = a$ if $x = a^2 b$ with b squarefree. Set $\delta(\mathcal{P}) = 1$ or 0 depending on whether the property $\mathcal{P}$ be true or not.

I wish to thank Henri Cohen for mentioning the trace formula in [32]. The key estimates were suggested by calculations made with Pari, the system created by Cohen and his collaborators.

I am grateful to Oisín McGuinness and the Department of Mathematics at Columbia University for allowing some of the calculations to be done on their machines. I also wish to thank him for valuable comments.

When Maple became too slow to compute the characteristic polynomials of Hecke operators, Robert Lewis kindly translated some of my Maple programs into his much faster symbolic language Fermat. This permitted us to complete our table.

Added intellectual stimulation, during the initial stages, was provided by an invitation to a workshop organized by the Centre de Mathématiques de Montreal and Concordia University in February 1992. I owe to J.-P. Serre a number of useful suggestions improving the exposition.

2. The Eichler-Selberg trace formula

We consider the trace of Hecke operators on various pieces of $J_0(N)$. The clearest and cleanest form of these formulae seems to be in [32], whose notation we shall adopt. An introduction to this topic, in case N is a prime, may be found in [13].

For any negative discriminant d, let $h(d)$ be the class number of primitive binary integral quadratic forms of discriminant d and let $w(d)$ be half of the number of units in the corresponding order. The Hurwitz class numbers $H_1(D)$, for *negative* $D \equiv 0$ or $1 \bmod 4$ are defined by

$$H_1(D) = \sum_{df^2=D} \frac{h(d)}{w(d)},$$

$H_1(0) = -1/12$ and $H_1(D) = 0$ otherwise. We need a modification of the Hurwitz class number. Let $(n, D) = a^2b$ with b squarefree and define

$$H_n(D) = \begin{cases} a^2b\left(\frac{D/a^2b^2}{n/a^2b}\right) H_1(D/a^2b^2) \text{ if } D \equiv 0 \bmod a^2b^2, \\ 0 \text{ otherwise.} \end{cases}$$

In particular, $H_n(0) = \frac{-1}{12}n$.

It is easy to check that for a fundamental discriminant d_0, we have

$$
\begin{aligned}
H_1(F^2 d_0) &= \left(\sum_{t|F} \mu(t)\left(\frac{d_0}{t}\right)\sigma_1(F/t)\right)\frac{h(d_0)}{w(d_0)} \\
&\leq F\left(\frac{\sigma_1(F)}{F}\right)^2 \frac{h(d_0)}{w(d_0)}. \qquad (2.1)
\end{aligned}
$$

We deduce the estimate

$$|H_1(D)| \ll h(d_0)\, F \log\log^2(\max\{3,F\}) \ll |D|^{\frac{1}{2}} \max\left(1, \log^2|D|\right)$$

for all $D = F^2 d_0 < 0$ from the well-known upper bound for the class number of imaginary quadratic fields, quoted on page 147 of [20], and the argument determining the size of $\sigma_1(n)$ in §22.9 of [16].

More generally, there is an absolute constant c_1 such that

$$|H_n(D)| \leq c_1\, Q((n,D))|D|^{\frac{1}{2}} \max(1, \log^2|D|) \qquad (2.2)$$

for all $D < 0$.

We denote by $\mathbf{tr}(L|V)$ the trace of the operator L on the vector space V. Let ℓ be prime to N and let T_ℓ be the Hecke operator on the space $S_2^{\text{new}}(N)$ of newforms of weight 2 for $\Gamma_0(N)$. If $m||N$, let W_m be the corresponding Atkin-Lehner involution [1].

We use, in the special case of weight 2, a class number function introduced in [32] for $n||m$:

$$
\begin{aligned}
\mathrm{sz}_m(\ell,n) &= -\frac{1}{2}\sum_{d|n}\sum_{dt^2\leq 4l} H_{\frac{m}{n}}(d^2t^2 - 4dl) \\
&\quad -\frac{1}{2}\sum_{\ell'|\ell} \min(\ell', \frac{\ell}{\ell'})\cdot\left(Q(n), \ell' + \frac{\ell}{\ell'}\right)\cdot\left(Q\left(\frac{m}{n}\right), \ell' - \frac{\ell}{\ell'}\right) \qquad (2.3) \\
&\quad + \delta\left(\frac{m}{n} = \square\right)\sigma_0(n)\sigma_1(\ell),
\end{aligned}
$$

where the sum over the integers t is limited to those for which $(t, Q(n/d)) = 1$. We need an estimate for this function:

Lemma 2.4. *There is an absolute constant c_2 such that for all $u = rs$ with $(r,s) = 1$, and all ℓ prime to u, we have*

$$\left|\mathrm{sz}_u(\ell,s) - \delta(l = \square)\frac{r}{12}\right| \leq c_2\left(\ell + \ell^{\frac{1}{2}}u^{\frac{1}{2}}\right)\sigma_0(l)\,\sigma_0^2(u)\,\log^2(4\ell s)$$

Proof: Since ℓ and rs are relatively prime, terms with $H_r(0)$ can occur in the definition of the Skoruppa-Zagier function (2.3) only if ℓ is a perfect square and both $d=1$, $t=\pm 2\sqrt{\ell}$ when s is not divisible by 4 (resp. $d=4$, $t=\pm\sqrt{\ell}$ when 4 divides s). The bound (2.2) on the class number gives

$$\begin{aligned}\left|\mathrm{sz}_{rs}(\ell,s)-\delta(\ell=\Box)\frac{r}{12}\right| &\le c_1\sum_{d|s}\sum_{t^2<\frac{4\ell}{d}} Q\left((r,dt^2-4\ell)\right)\ell^{\frac12}d^{\frac12}\log^2(4\ell d)\\ &\quad +u^{\frac12}\ell^{\frac12}\sigma_0(\ell)+\sigma_0(s)\,\sigma_1(\ell).\end{aligned}$$

We used $(r,d^2t-4d\ell)=(r,dt-4\ell)$, true because d must be prime to r, since it divides s.

Recall that the number of solutions of $dt^2\equiv 4\ell \bmod k^2$ is $O(2^{\omega(k)})$ to rewrite the inner sum above by summing over divisors k of $Q(r)$. Let t run through values fixing $k=Q((r,dt^2-4\ell))$ to obtain:

$$\begin{aligned}\left|\mathrm{sz}_{rs}(\ell,s)-\delta(\ell=\Box)\frac{r}{12}\right| &\le c_1\sum_{d|s}\sum_{k|Q(r)}\sum_{\substack{t^2<\frac{4\ell}{d}\\ dt^2\equiv 4\ell \bmod k^2}} k\,d^{\frac12}\ell^{\frac12}\log^2(4\ell d)\\ &\quad +u^{\frac12}\ell^{\frac12}\sigma_0(\ell)+\sigma_0(s)\,\sigma_1(\ell)\\ &\le c_3\sum_{d|s}\sum_{k|Q(r)}2^{\omega(k)}\left(1+\frac{\ell^{\frac12}}{k^2d^{\frac12}}\right)k\,d^{\frac12}\ell^{\frac12}\log^2(4\ell d)\\ &\quad +u^{\frac12}\ell^{\frac12}\sigma_0(\ell)+\sigma_0(s)\,\sigma_1(\ell)\\ &\le (1+c_3)\,2^{\omega(r)}\,(\ell+u^{\frac12}\ell^{\frac12})\,\sigma_0(\ell)\,\sigma_0(u)\,\log^2(4\ell s),\end{aligned}$$

after some elementary manipulations using the inequality $Q(r)\le\sqrt{r}$. Finally $2^{\omega(n)}\le\sigma_0(n)$ gives the assertion of the lemma. ∎

From [32], we quote a special case of the Eichler-Selberg trace formula:

Proposition 2.5. *For N, ℓ, m positive integers, ℓ prime to N and $m||N$. Then*

$$\mathbf{tr}(T_\ell W_m|S_2^{\mathrm{new}}(N))=\sum_{d|N}\alpha\left(\frac{N}{d}\right)\,\mathrm{sz}_d(\ell,(m,d)),$$

where $\alpha(n)$ is the multiplicative function with $\alpha(p)=\alpha(p^2)=-1$, $\alpha(p^3)=1$ and $\alpha(p^r)=0$ for $r\ge 4$.

Two multiplicative functions involving α will be used below, namely

$$\beta(m)=\sum_{d|m}\alpha(d)=\begin{cases}\mu(a) & \text{if } m=a^2,\\ 0 & \text{otherwise.}\end{cases}\tag{2.6}$$

and

$$\gamma(m) = m \sum_{d|m} \frac{\alpha(d)}{d}. \tag{2.7}$$

Finally, one obtains the following important estimate of traces:

Proposition 2.8. *For N, ℓ, m positive integers, ℓ prime to N and $m||N$.*

$$\left|\mathbf{tr}(T_\ell W_m | S_2^{\mathrm{new}}(N)) - \frac{1}{12}\delta(\ell = \square)\beta(m)\,\gamma\left(\frac{N}{m}\right)\right|$$
$$\leq c_2\,(\ell + \ell^{\frac{1}{2}} N^{\frac{1}{2}})\,\sigma_0^3(N)\,\sigma_0(\ell)\,\log^2(4\ell N).$$

Proof: In the trace formula (2.5), substitute the estimate (2.4) after writing $N = mm'$, $(d, m) = r$ and $(d, m') = s$ to get

$$\left|\mathbf{tr}(T_\ell W_m | S_2^{\mathrm{new}}(N)) - \frac{1}{12}\delta(\ell = \square) \sum_{\substack{r|m \\ s|m'}} \alpha\left(\frac{m'}{s}\right)\alpha\left(\frac{m}{r}r\right)\right|$$
$$\leq c_2 \sum_{\substack{r|m \\ s|m'}} (\ell + \ell^{\frac{1}{2}}(rs)^{\frac{1}{2}})\,\sigma_0^2(rs)\,\sigma_0(\ell)\,\log^2(4\ell s).$$

The result now follows readily from the definitions and properties of multiplicative functions. ∎

3. The explicit formula

Assume that A is an abelian variety defined over the global field K. Hasse defined an L-series $L(s, A)$ as a Dirichlet series with an Euler product converging for $s > \dim A + \frac{1}{2}$. Shimura [31] proved that $L(s, J_0^{\mathrm{new}}(N))$ has an analytic continuation to an entire function with a functional equation, as a result of the factorization

$$L(s, J_0^{\mathrm{new}}(N)) = \prod_f L(s, f) \tag{3.1}$$

where f runs through a basis of newforms of $S_2^{\mathrm{new}}(N)$, with Fourier series

$$f(z) = \sum_{n=1}^{\infty} a_n(f) \exp(2\pi i n z)$$

and $L(s,f)$ is the Mellin transform

$$
\begin{aligned}
(3.2)\quad L(s,f) &= \sum_{n=1}^{\infty} a_n(f)n^{-s} \\
&= \prod_{p|N}(1-a_p(f)p^{-s})^{-1}\prod_{p\nmid N}(1-a_p(f)p^{-s}+p^{1-2s})^{-1}.
\end{aligned}
$$

Note that for $p||N$, $a_p = \pm 1$ while $a_p = 0$ if $p^2|N$. Hecke had already proved that $L(s,f)$ has a functional equation of the shape

$$\Lambda(2-s,f) = w_f\Lambda(s,f)$$

with $\Lambda(s) = (2\pi)^{-s}N^{s/2}\Gamma(s)L(s,f)$ and $w_f = \pm 1$ is the famous root number. Only when N is squarefree is the formula for w_f simple: $w_f = -\prod_{p|N}(-a_p(f))$.

More precisely, for each newform f, let $\mathcal{O}_f$ be the ring generated over $\mathbb{Z}$ by the coefficients of f. Then $\mathcal{O}_f$ is an order in a totally real algebraic number field $\mathbb{E}_f$ of finite degree d_f, say. Shimura defines a simple abelian variety A_f of dimension d_f, along with an action by $\mathcal{O}_f$, both defined over $\mathbb{Q}$. The cotangent space of A_f is naturally identified with the $\mathbb{Q}$-span of the conjugates f^σ of f, where $f^\sigma = \sum a_n(f)^\sigma \exp(2\pi i n z)$. The Eichler-Shimura relation yields the associated decomposition

$$L(s,A_f) = \prod_{\sigma} L(s,f^\sigma), \tag{3.3}$$

where σ runs through the embeddings of $\mathbb{E}_f$ into $\mathbb{R}$. The Mordell-Weil group $A_f(\mathbb{Q})$ is an $\mathcal{O}_f$-module. Hence $A_f(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Q}$ is an $\mathbb{E}_f$-vector space of dimension $\operatorname{nrk} A = \operatorname{rank} A_f/d_f$, the *normalized rank* of A_f. Finally, $J_0^{\mathrm{new}}(N)$ is isogenous to the product of the $\mathbb{Q}$-simple factors A_f, where f runs through non-conjugate newforms in $S_2^{\mathrm{new}}(N)$.

The nature of A_f is quite mysterious, so we consider only the coarser splitting induced by the action of the elementary 2-group of Atkin-Lehner involutions $\mathcal{W}(N) = \{W_d : d||N\}$, generated by $W_{q^{r(q)}}$ with q running through the prime factors of N and $r(q) = \operatorname{ord}_q(N)$. Note that $|\mathcal{W}(N)| = 2^{\omega(N)}$. To each character $\chi : \mathcal{W}(N) \mapsto \{\pm 1\}$, we may attach the idempotent

$$e_\chi = \frac{1}{|\mathcal{W}(N)|}\sum_{d||N}\chi(W_d)\,W_d$$

and the corresponding abelian subvariety $A_\chi(N) = e_\chi\, J_0^{\mathrm{new}}(N)$.

From Mestre [21], we borrow the following so-called explicit formula, as modified in [6].

Lemma 3.4. *Let A be a modular abelian variety of dimension g defined over $\mathbb{Q}$ and let $h(u) = \max(1-|u|, 0)$ be the triangle function, then under the Riemann Hypothesis for $L(s,A)$, we have*

$$\begin{aligned}\operatorname{rank} A \leq\ & \frac{\log \operatorname{cond} A}{\log X} \\ & - \frac{2}{\log X} \sum_{p^m \leq X} \frac{b_{p^m}(A)}{p^m} \log p\, h\left(\frac{\log p^m}{\log X}\right) \\ & + \frac{\dim A}{\pi} \int_{-\infty}^{\infty} \left(\frac{\Gamma'}{\Gamma}(1+it) - \log 2\pi\right) \widehat{h}(t \log X),\, dt\end{aligned}$$

where $\hat{h}$ is the Fourier transform of h, $\alpha_{p,i}$ for $1 \leq i \leq 2g$ are the eigenvalues of Frobenius at p acting on the Tate module $T_\ell(A) \otimes \mathbb{Q}_\ell = H^1_{et}(A, \mathbb{Q}_\ell)$ and $b_{p^m}(A) = \sum_{1 \leq i \leq 2g} \alpha^m_{p,i}$.

As in [6], we deduce the following inequality from Weil's bound $|\alpha_{p,i}| = \sqrt{p}$:

$$\begin{aligned}\operatorname{rank} A \leq\ & \frac{\log \operatorname{cond} A}{\log X} - \frac{2}{\log X} \sum_{p \leq X} \frac{b_p(A)}{p} \log p\, h\left(\frac{\log p}{\log X}\right) \\ & - \frac{2}{\log X} \sum_{p \leq \sqrt{X}} \frac{b_{p^2}(A)}{p^2} \log p\, h\left(\frac{\log p^2}{\log X}\right) + O\left(\frac{\dim A}{\log X}\right),\end{aligned} \tag{3.5}$$

with an absolute implied constant.

Remark 3.6. In the special case $A = \operatorname{Jac}(C)$, the Jacobian of a curve C defined over $\mathbb{Q}$, we have the well-known interpretation $b_q(A) = q+1-|C(\mathbf{F}_q)|$ for any prime power q.

In the case $A = J_0^{\text{new}}(N)$, we know from Eichler-Shimura-Igusa that the real algebraic numbers $a_{p,i} = \alpha_{p,i} + \overline{\alpha}_{p,i}$ are the eigenvalues of the Hecke operator T_p acting on the space of newforms $S_2^{\text{new}}(N)$. Recall the identity $T_{p^2} = T_p^2 - pT_1$ and that the Hecke operators commute with $\mathcal{W}(N)$, to conclude that

$$\begin{aligned}b_{p^2}(A_\chi(N)) &= \sum_{i=1}^{g} (\alpha^2_{p,i} + \overline{\alpha}^2_{p,i}) = \sum_i a^2_{p,i} - 2p \dim A_\chi(N) \\ &= \mathbf{tr}(T_{p^2}|e_\chi S_2^{\text{new}}(N)) - p \dim A_\chi(N),\end{aligned} \tag{3.7}$$

where the $a_{p,i}$ consist of the eigenvalues of T_p acting on the invariant differential forms of $A_\chi(N)$, identified with $e_\chi S_2^{\text{new}}(N)$. With this identification, we see that

$$(3.8)\qquad \begin{aligned} \dim A_\chi(N) &= \tfrac{1}{|\mathcal{W}(N)|}\textstyle\sum_{d||N}\chi(W_d)\,\mathbf{tr}(T_1W_d|S_2^{\mathrm{new}}(N)),\\ b_p(A_\chi(N)) &= \tfrac{1}{|\mathcal{W}(N)|}\textstyle\sum_{d||N}\chi(W_d)\,\mathbf{tr}(T_pW_d|S_2^{\mathrm{new}}(N)),\\ b_{p^2}(A_\chi(N)) &= \tfrac{1}{|\mathcal{W}(N)|}\textstyle\sum_{d||N}\chi(W_d)\,\mathbf{tr}(T_{p^2}W_d|S_2^{\mathrm{new}}(N))\\ &\quad -p\dim A_\chi(N). \end{aligned}$$

The following estimate of the dimension of the Atkin-Lehner pieces is a consequence of (2.8) with $\ell = 1$:

Lemma 3.9. *For every positive integer N and every character χ of the Atkin-Lehner group $\mathcal{W}(N)$, we have*

$$\left|\dim A_\chi(N) - \frac{1}{12}\frac{\phi_\chi(N)}{|\mathcal{W}(N)|}\right| \le c_2\,\sigma_0^3(N)\,N^{\frac{1}{2}}\log^2 4N,$$

where

$$\phi_\chi(N) = \sum_{d||N}\chi(W_d)\,\beta(d)\,\gamma\left(\frac{N}{d}\right),$$

and β and γ were defined in (2.6) and (2.7). When N is squarefree, $\phi_\chi(N) = \phi(N)$, where $\phi(N)$ is Euler's totient function. In general, there are absolute constants c_4 and c_5 such that

$$c_4\phi(N) \le \phi_\chi(N) \le c_5\phi(N).$$

Proof: Only the assertions about $\phi_\chi(N)$ need comment. Let $N_1 = \prod p^2$ where p runs through the primes such that $p^2||N$ and let $N_2 = N/N_1$. Since all the functions are multiplicative, a calculation gives the factorization

$$\phi_\chi(N) = \gamma(N_2)\sum_{d||N_1}\chi(W_d)\,\beta(d)\,\gamma\left(\frac{N_1}{d}\right) = \prod_{p^r||N}\phi_\chi(p^r).$$

We have the following p-components:

$$\phi_\chi(p^r) = \begin{cases} 1 & \text{if } r=0,\\ p-1 & \text{if } r=1,\\ p^2-p-1-\chi(W_{p^2}) & \text{if } r=2,\\ p^r-p^{r-1}-p^{r-2}+p^{r-3} & \text{if } r\ge 3. \end{cases}$$

The remaining claim is now clear from the convergence of $\sum p^{-2}$. ∎

We control the contribution of $\mathbf{tr}(T_lW_m|S_2^{\mathrm{new}}(N))$, to the explicit formula by the prime number theorem (cf. Cor. 2.4 in [6]). Explicitly, use (2.8) and

(3.8) for $\ell = p \leq X$, where $X \geq 2$ is a parameter to be chosen later as a function of N, to obtain

$$\begin{aligned}
&\left|\sum_{p\leq X} b_p(A_X(N))\,\frac{\log p}{p}\,h\left(\frac{\log p}{\log X}\right)\right| \\
(3.10)\qquad &\leq c_2\,\sigma_0^3(N)\,\log^2(4XN)\sum_{p\leq X}(p+p^{\frac12}N^{\frac12})\,\frac{\log p}{p}\,h\left(\frac{\log p}{\log X}\right) \\
&\leq c_6\,\sigma_0^3(N)\,(X+X^{\frac12}N^{\frac12})\,\log^2(4XN),
\end{aligned}$$

with an absolute constant c_6.
Similarly, the next inequality for $A = A_X(N)$ follows from (3.7):

$$\begin{aligned}
(3.11)\qquad &\left|\sum_{p\leq\sqrt{X}}(b_{p^2}(A)+p\dim A)\,\frac{\log p}{p^2}\,h\left(\frac{\log p^2}{\log X}\right)\right| \\
&\leq c_7\,\sigma_0^3(N)\,(X^{\frac12}+N^{\frac12}\log X)\log^2(4XN).
\end{aligned}$$

In Cor. 2.4 in [6], we saw that

$$(3.12)\qquad \sum_{p\leq\sqrt{X}}\frac{\log p}{p}\,h\left(\frac{\log p^2}{\log X}\right) = \frac{1}{4}\log X + O(1),$$

so the last estimate and (3.11) give

$$\begin{aligned}
&\left|\sum_{p\leq\sqrt{X}} b_{p^2}(A)\,\frac{\log p}{p^2}\,h\left(\frac{\log p^2}{\log X}\right) + \frac{1}{4}\log X\dim A\right| \\
(3.13)\qquad &\leq c_8(\dim A + \sigma_0^3(N)\,(X^{\frac12}+N^{\frac12}\log X)\log^2(4XN)),
\end{aligned}$$

for $A = A_X(N)$.

Putting all this information into the Explicit Formula (3.5) for $A = A_X(N)$, we conclude that

$$\begin{aligned}
\operatorname{rank} A \;\leq\; &\left[\frac{\log N}{\log X}+\frac{1}{2}\right]\dim A \\
(3.14)\qquad &+c_9\left(\sigma_0^3(N)\,(X+X^{\frac12}N^{\frac12})\,\frac{\log^2(4XN)}{\log X}+\frac{\dim A}{\log X}\right),
\end{aligned}$$

for all $X \geq 2$ and positive integers N.

The inequalities $1 \le N/\phi(N) \ll \log\log N$ and $\sigma_0(N) = O(N^{\log 2/\log\log N})$ can be found in §22.9 of [16]. It follows from (3.9) that for any $\epsilon > 0$, we have $\dim A_\chi(N) \ge N^{1-\epsilon/4}$, for all sufficiently large integers N.

Now set $X = N^c$ with $c = (1+\frac{\epsilon}{2})^{-1}$ to obtain:

Theorem 3.15. *For any positive integer N, let χ be a character of the Atkin-Lehner group $\mathcal{W}(N)$. Assume the Riemann Hypothesis for the series $L(s,f)$ belonging to $A = A_\chi(N)$. For each $\epsilon > 0$, the analytic rank satisfies*

$$\operatorname{rank} A \le \left(\frac{3}{2}+\epsilon\right)\dim A,$$

for all sufficiently large N.

4. Averaging over N

In this section, we use the trace formula with more care and avail ourselves of the cancellations afforded by averaging over a range of N. To do this painlessly, we restrict our attention to prime conductors and consider the full Jacobians $J_0(N)$.

We note some special cases of (2.5). The case $N = 1$ gives a classical class number relation due to Glaisher [17], which says that the trace of T_n is zero on the trivial space $S_2(1)$.

Lemma 4.1. *Let $\psi(n) = \sum_{d>\sqrt{n}} d - \sum_{d<\sqrt{n}} d$ be the excess of divisors greater than $\sqrt{n}$ over those smaller. Then*

$$\sum_{s^2 \le 4n} H_1(s^2-4n) = \sigma_1(n) + \psi(n).$$

For N a prime, the trace formula (2.5) simplifies to:

Lemma 4.2. *Let N be prime and let n be prime to N, then*

$$\begin{aligned}\mathbf{tr}(T_n|S_2(N)) \quad &= \quad -\frac{1}{2}\sum_{s^2\le 4n}\left(\frac{s^2-4n}{N}\right)H_1(s^2-4n) - \frac{1}{2}\sum_{d|n}\min\left(d,\frac{n}{d}\right)\\ &\quad -\frac{1}{2}\sum_{s^2\equiv 4n \bmod N^2} NH_1\left(\frac{s^2-4n}{N^2}\right)\end{aligned}$$

and

$$\mathbf{tr}(T_nW_N|S_2(N)) = \sigma_1(n) - \frac{1}{2}\sum_{\substack{s^2\le 4nN\\ N|s}} H_1(s^2-4nN)$$

We also need a consequence of the explicit Cebotarev Density Theorem found, for instance, as Theorem 4 of [28]:

Lemma 4.3. *Assume the Riemann Hypothesis for quadratic fields. Then, for any $x \geq 2$ and any discriminant d of a quadratic number field, we have*

$$\left| \sum_{\substack{N \leq x \\ N\, prime}} \left(\frac{d}{N} \right) \right| \leq c_9\, x^{\frac{1}{2}} \log(|d|x^2),$$

with an absolute constant c_9.

We consider the rank of the abelian varieties $J_0(N)$ over a range $Y < N \leq cY$ of primes N, with a fixed $c > 1$. The parameter X bounding the primes used in the explicit formula, will be chosen later. We shall assume that $4X < Y^2$ in order to simplify the contribution of the third sum in the trace of T_n in (4.2). The estimates below will be valid for such X and sufficiently large Y (depending on c) with absolute implied constants.

Let $\Delta(Y, Z) = \sum_N \dim J_0(N)$ and $\mathrm{Rk}(Y, Z) = \sum_N \mathrm{rank}\, J_0(N)$, where N runs through the primes in the interval $(Y, Z]$. We shall estimate the "average normalized rank", namely $\mathrm{Rk}(Y, Z)/\Delta(Y, Z)$. Recall that the dimension of $J_0(N)$ is $\frac{1}{12}N$, up to an error of at most 1. By the prime number theorem, we have for $Z = cY$

$$\Delta(Y, cY) = \frac{1}{12} \sum_{\substack{Y < N \leq cY \\ N\, prime}} (N + O(1)) = \frac{c^2 - 1}{24} \frac{Y^2}{\log Y} + O\left(c^2 Y^2 \frac{\log c}{\log^2 Y}\right).$$

where the implied constant does not depend on c.

Since the logarithm of the conductor of $J_0(N)$ is $\dim J_0(N) \log N$, the contribution of the conductors is similarly

$$\begin{aligned} \sum_{Y < N \leq Z} \log \mathrm{cond}\, J_0(N) &= \frac{1}{12} \sum_{Y < N \leq Z} N \log N + O\Big(\sum_{Y < N \leq Z} \log N\Big) \\ &= \frac{(c^2 - 1)}{24} Y^2 + O\left(c^2 Y^2 \frac{\log c}{\log Y}\right). \end{aligned} \tag{4.4}$$

We add up, for a fixed $p \leq X < \frac{1}{4}Y^2$, the contributions of $\mathbf{tr}(T_p | S_2(N)) =$

$b_p(J_0(N))$ over *primes* N. We use (4.2) and the bounds (2.2) and (4.3):

$$\begin{aligned}\sum_{Y<N\le Z} \mathbf{tr}(T_p|S_2(N))) &= -\frac{1}{2}\sum_{s^2\le 4p}\sum_{N}\left(\frac{s^2-4p}{N}\right)H_1(s^2-4p)+O\left(\frac{cY}{\log Y}\right)\\ \text{(4.5)}\qquad &\le c_{10}\sqrt{c}\,p\,Y^{\frac12}\log(Xc^2Y^2)\log^2 X+\frac{cY}{\log Y}\end{aligned}$$

and N runs through $Y<N\le cY$ with Y large enough. Similarly, for any fixed $p\le\sqrt{X}<\frac12 Y$, we get

$$\begin{aligned}\sum_{Y<N\le Z} \mathbf{tr}(T_{p^2}|S_2(N)) &= -\frac{1}{2}\sum_{s^2\le 4p^2}\sum_{Y<N\le Z}\left(\frac{s^2-4p^2}{N}\right)H_1(s^2-4p^2)\\ \text{(4.6)}\qquad &\quad+\sum_{Y<N\le Z}\frac{N}{12}+O\left(\frac{pcY}{\log Y}\right)\\ &\le c_{11}\sqrt{c}p^2Y^{\frac12}\log(2cXY)\log^2 X+\frac{cpY}{\log Y}+\Delta(Y,cY).\end{aligned}$$

Next, the weighted sum (4.5) over $p\le X$ gives:

$$\begin{aligned}&\sum_{Y<N\le Z}\sum_{p\le X} b_p(J_0(N))\frac{\log p}{p}h\left(\frac{\log p}{\log X}\right)\\ \text{(4.7)}\qquad &\le c_{12}\sqrt{c}XY^{\frac12}\log(c^2XY^2)\log^2 X+cY\frac{\log X}{\log Y}\end{aligned}$$

Since $\mathbf{tr}(T_p^2|S_2(N))=b_{p^2}(J_0(N))+p\dim J_0(N)$, we get from (4.6), the estimate

$$\begin{aligned}&\sum_{Y<N\le Z}\sum_{p\le\sqrt{X}}[b_{p^2}(J_0(N))+p\dim J_0(N)]\frac{\log p}{p^2}h\left(\frac{\log p^2}{\log X}\right)\\ \text{(4.8)}\qquad &\le c_{13}\sqrt{c}X^{\frac12}Y^{\frac12}\log(2cXY)\log^2 X+cY\frac{\log X}{\log Y}+\Delta(Y,cY).\end{aligned}$$

We recall the explicit formula (3.5) in our context:

$$\begin{aligned}\operatorname{rank} J_0(N)\ \log X &\le \log\operatorname{cond} J_0(N)-2\sum_{p\le X}\frac{b_p(J_0(N))}{p}\log p\ h\left(\frac{\log p}{\log X}\right)\\ &\quad-2\sum_{p\le\sqrt{X}}\frac{b_{p^2}(J_0(N))}{p^2}\log p\ h\left(\frac{\log p^2}{\log X}\right)+O\left(\dim J_0(N)\right).\end{aligned}$$

Add up over primes N in the range $Y < N \leq cY$ and substitute (4.4), (4.7), (4.8) and the estimate for $\Delta(Y, cY)$ to obtain

$$\begin{aligned} \mathrm{Rk}(Y, cY)\log X \ \leq\ & \frac{1}{24}(c^2-1)\,Y^2 + 2\Delta(Y, cY)\sum_{p\leq\sqrt{X}}\frac{\log p}{p}\,h\left(\frac{\log p^2}{\log X}\right) \\ & + O\left(\frac{c^2\,Y^2}{\log Y} + \sqrt{c}\,XY^{\frac{1}{2}}\log(XY)\log^2 X + c\,Y\frac{\log X}{\log Y}\right) \\ & + O\left(\sqrt{c}\,X^{\frac{1}{2}}Y^{\frac{1}{2}}\log(XY)\log^2 X + cY\frac{\log X}{\log Y} + \Delta(Y, cY)\right) \end{aligned}$$

We conclude upon dividing this last result by $\Delta(Y, cY)\log X$, using (3.12), our earlier estimate for $\Delta(Y, cY)$, and taking $X = Y^{\frac{3}{2}-\epsilon}$ which is adequate to give the final claim.

Theorem 4.9. *Assume the Riemann Hypothesis for the L-functions of newforms and Dirichlet L-series for quadratic characters. Then the average analytic normalized rank for Jacobians $J_0(N)$ with N prime is bounded by $\frac{7}{6}$. More precisely, for any fixed c and $\epsilon > 0$, we have*

$$\sum_{Y<N\leq cY}\operatorname{rank} J_0(N) \leq \left(\frac{7}{6}+\epsilon\right)\sum_{Y<N\leq cY}\dim J_0(N),$$

where the sum runs over primes N and Y is sufficiently large.

Even our weak results imply entertaining consequences of the deep results of Kolyvagin, Bump-Friedberg-Hoffstein, Gross-Zagier and Murty-Murty ([18], [11], [14], [23]). For instance, Theorem (3.15) gives

Corollary 4.10. *Let N be any positive integer and let $A = A_\chi(N)$ be an Atkin-Lehner component with $\chi(W_N) = -1$, so that A is of even parity. Under the Riemann hypotheses, A has a quotient B of analytic rank 0 and dimension at least $\frac{1}{4}\dim A$. Moreover $B(\mathbb{Q})$ and $\text{Ш}(B)$ are finite.*

In particular, if N is large and A is simple, it will be of rank 0.

Similarly (4.9) implies that, among Jacobians $J_0(N)$ with N prime with $Y < N \leq cY$, the abelian subvarieties of normalized analytic rank at most one, make up at least two-thirds of the total dimension. These will satisfy a form of the conjecture of Birch-Swinnerton-Dyer. To see the former claim, let r be the proportion of the pieces of rank zero and let s be that of the pieces of rank one. Since the odd and even pieces contribute roughly equal amounts, we conclude that the average normalized rank is at least $2(.5-r)+s+3(.5-s) \leq \frac{7}{6}+\epsilon$, so $r+s$ is at least $\frac{2}{3}-\epsilon$ if Y is large enough.

5. Abelian varieties of $\mathbf{GL_2}$-type and wild ramification

In this section, we show that the endomorphism rings of abelian varieties with real multiplications over $\mathbb{Q}$, whose conductor is divisible by high powers of a prime, contain large cyclotomic rings. This is an application of results obtained in [9], to which the reader is referred for a more leisurely treatment of conductors. We restrict our attention to modular forms of weight 2, since we are concerned here with the associated modular abelian varieties, but our methods apply to higher weights upon replacing the Tate module by the Deligne representation.

Let f be a newform of weight 2 on $\Gamma_0(N)$ of Nebentypus ϵ and let A_f be the corresponding abelian variety, as in [31]. It is defined over $\mathbb{Q}$ and has multiplications by $\mathcal{O}_f = \mathbb{Z}[\{a_n\}]$, the order defined by adjoining the coefficients of f. The quotient field $\mathbb{E}_f$ of $\mathcal{O}_f$ is a totally real field when ϵ is trivial and a CM-field when ϵ is non-trivial. The characteristic property of A_f is that its invariant differentials $\Omega(A_f)$ may be identified with the $\mathbb{Q}$-span of the conjugates $f^\sigma(z) = \sum_n a_n^\sigma q^n$, where σ runs through the isomorphisms of $\mathbb{E}_f$ into $\mathbb{C}$. Thus $\dim A_f = [\mathbb{E}_f : \mathbb{Q}]$.

More generally, let A be a simple abelian variety of dimension d defined over $\mathbb{Q}$ with $\mathbb{E} \subseteq \mathrm{End}^o_{\mathbb{Q}}(A) = \mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$ for some number field $\mathbb{E}$ of dimension d. Ribet [26] calls such abelian varieties "of $\mathbf{GL_2}$-type". Then $\mathbb{E}$ is either totally real or a CM field, and $\mathbb{E}$ is a maximal commutative subfield of the simple algebra $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \otimes \mathbb{Q}$, whose center will be denoted by $\mathbb{F}$. It is expected that, up to isogeny, all such abelian varieties A are modular ([29], [26]).

There is an abelian variety B isogenous to A over $\mathbb{Q}$, with an action of the ring of integers $\mathcal{O}$ of $\mathbb{E}$. Since the questions considered here are insensitive to isogenies, it is convenient to assume $A = B$. Let λ run through the primes above ℓ in $\mathbb{E}$. We have a natural representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the two-dimensional $\mathcal{O}/\ell\mathcal{O}$-module $A[\ell] = \prod A[\lambda]$ of points of order ℓ on A. We obtain representations $\rho_\lambda : \mathrm{Gal}(L/\mathbb{Q}) \to GL_2(\mathbf{F}_\lambda)$ with $\mathbf{F}_\lambda = \mathcal{O}/\lambda$ and $L = \mathbb{Q}(A[\lambda])$, because the Galois action commutes with that of $\mathcal{O}$.

The conductor of A is N^d for some integer N which may be calculated from the Artin conductor of the representation ρ_λ on $V = A[\lambda]$, for any sufficiently large prime ℓ (cf. [15], [29], [26]). Precisely, let p be any prime, let r_p be the exponent of p in N, and let v be any prime above p in L. Let G denote the decomposition group of v and $\{G_i\}$ be its higher decomposition groups, then r_p can be computed from

$$[\mathbf{F}_\lambda : \mathbf{F}_\ell]\, r_p = \epsilon_\lambda(A/\mathbb{Q}_\ell) + \mathrm{sw}(A[\lambda]),$$

where the "tame exponent" $\epsilon_\lambda(A/\mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell}(W/W^{G_0})$, which depends on the Tate module $W = \mathbb{E}_\lambda \otimes T_\lambda$ of A, is known to be at most 2, while the "wild" or Swan exponent is given by

$$\begin{aligned}\mathbf{sw}(V) &= \sum_{i>0} \frac{g_i}{g_0} \dim_{\mathbf{F}_\ell}(V/V^{G_i}) \\ &= [\mathbf{F}_\lambda : \mathbf{F}_\ell] \sum_{i>0} \frac{g_i}{g_0} \dim_{\mathbf{F}_\lambda}(V/V^{G_i}) \qquad (5.1)\end{aligned}$$

with $g_i = |G_i|$. In the modular case, N is known to be the analytic conductor according to deep work of Deligne, Langlands and Carayol, for all sufficiently large ℓ. For higher weight, still unpublished work of Faltings-Jordan-Livné and Ribet would be needed for this assertion, but our applications need only the Swan conductor, to which the work of Carayol already applies.

We quote the following result from [9].

Lemma 5.2. *Let L/K be a Galois extension of local fields with residue characteristic p and Galois group G. Suppose G is a p-group and that χ is the character of an irreducible $\mathbb{C}[G]$-representation V. Then*

$$\mathbf{sw}(V) \leq \chi(1)[v_K(\chi(1)[\mathbb{Q}(\chi):\mathbb{Q}]) + p\, e_K/(p-1)],$$

where $\mathbb{Q}(\chi)$ is the field of values of the character, $\chi(1)$ is its degree and e_K is the absolute ramification index of K.

As a consequence, we deduce a bound on the Swan conductors in our setting:

Corollary 5.3. *Let G be a p-group, let $\mathbf{F} = \mathbf{F}_q$ be the Galois field with q elements and let V be a faithful 2-dimensional $\mathbf{F}[G]$-module. Let L/K be an extension of local fields with Galois group G, whose residue characteristic p is prime to q and let e_K be the absolute ramification index of K. Then we have the bound*

$$\mathbf{sw}(V, L/K) \leq 2\, e_K\, [s + 1/(p-1)],$$

with $s = \mathrm{ord}_p(q^2 - 1)$.

When $G \subset SL_2(\mathbf{F})$, this bound may be refined by taking p^s as the exponent of G.

Proof: Clearly, G is a subgroup of a p-Sylow subgroup S_p of $GL_2(\mathbf{F})$. We shall apply the lemma after lifting the representation on V to characteristic 0. First, note that if G is abelian of exponent p^s, then every faithful absolutely irreducible character χ of G is one-dimensional with $\mathbb{Q}(\chi) = \mathbb{Q}(\mu_{p^s})$. Our claim follows from the lemma since Swan conductors are additive. We proceed with a case by case analysis after recalling that $|GL_2(\mathbf{F})| = q\,(q-1)^2\,(q+1)$.

When p is odd, $S_p = 1$ unless $q \equiv \pm 1 \bmod p$ when we have:

$$S_p \cong \begin{cases} \mathbb{Z}_{p^t} \times \mathbb{Z}_{p^t} & \text{if } p^t || (q-1) \\ \mathbb{Z}_{p^t} & \text{if } p^t || (q+1), \end{cases}$$

so we are done, since these groups are abelian.

When $p = 2$, the story is slightly more complicated. Write $\eta = (-1)^{\frac{q-1}{2}}$ and define $t \geq 2$ by $2^t || (q - \eta)$. If $q \equiv 3 \pmod 4$, then the two-dimensional $\mathbf{F}_q$-space $\mathbf{F}_{q^2}$ contains a primitive 2^{t+1}-st root of unity $\zeta_{2^{t+1}}$ and $S_2 \cong \langle x, y \rangle$ with x induced by multiplication by $\zeta_{2^{t+1}}$ and y induced by $\zeta \mapsto \zeta^q$. We have the relations $x^{2^{t+1}} = 1$, $y^2 = 1$, $yxy^{-1} = x^{-1+2^t}$. This is a semi-dihedral group whose intersection with SL_2 is a generalized quaternion group Q_s of order 2^{t+1}.

On the other hand, if $q \equiv 1 \pmod 4$, denote by ω a primitive 2^t-th root of unity in $\mathbf{F}_q$. Then S_2 is the extension of $\mu_{2^t} \oplus \mu_{2^t}$ along the diagonal by $y = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The element $z = \left(\begin{smallmatrix} 0 & \omega \\ 1 & 0 \end{smallmatrix}\right)$ has the largest order, namely 2^{t+1}. The intersection of S_2 with SL_2 is the dihedral group of order 2^{t+1}.

We verify readily that all the possible non-abelian 2-subgroups of GL_2 with cyclic maximal normal subgroup, and in particular all subgroups of SL_2, have irreducible 2-dimensional $\mathbb{C}$-representations whose field of character values is a subfield of index 2 of $\mathbb{Q}(\mu_{2^s})$, where 2^s is the exponent. The only remaining case occurs when $q \equiv 1 \bmod 4$, and the field of character values is $\mathbb{Q}(\mu_{2^s})$. But in this case $\mathrm{ord}_2(q^2 - 1) = 1 + \mathrm{ord}_2(q - 1)$ and the claim still obtains. ∎

Recall the so-called Eichler-Shimura relation for any q prime to $N\ell$:

$$\begin{aligned} \mathbf{tr}(\rho_\lambda(\mathrm{Frob}_q)) &\equiv a_q \pmod \lambda \\ \det(\rho_\lambda(\mathrm{Frob}_q)) &\equiv \epsilon(q) q \pmod \lambda, \end{aligned} \tag{5.4}$$

where Frob_q is the Frobenius class of any prime extending q. This is of course classical in the modular case, and was extended by Ribet in [26].

Theorem 5.5. *Let A be an abelian variety of $\mathbf{GL_2}$-type of dimension d and let $\mathbb{E}$ be its field of $\mathbb{Q}$-endomorphisms. Then the conductor of A is N^d for some integer N. Suppose that $p^{r_p} || N$ and define s_p by the following:*

$$s_p = \left\lceil \frac{r_p}{2} - 1 - \frac{1}{p-1} \right\rceil$$

where $\lceil x \rceil$ is the least integer $\geq x$. Let ζ be a primitive p^{s_p} root of unity.

(i) $\mathbb{E}$ contains $\mathbb{Q}(\epsilon, \zeta + \zeta^{-1})$ if p is odd (resp. $\mathbb{Q}(\epsilon, \zeta^2)$ if $p = 2$).

(ii) Suppose that p^2 does not divide the conductor of ϵ and that A is not of CM-type. Then the center $\mathbb{F}$ of $\mathrm{End}^o_{\overline{\mathbb{Q}}}(A)$ contains $\mathbb{Q}(\zeta + \zeta^{-1})$ if $p > 2$ (resp. $\mathbb{Q}(\zeta^2 + \zeta^{-2})$ if $p = 2$).

Proof: We consider primes λ of $\mathbb{E}$ above rational primes $\ell \gg 0$ different from p. Let G be the image $\rho_\lambda(G_1)$ in $GL_2(\mathbf{F}_\lambda)$ of the first decomposition subgroup G_1 of $\mathrm{Gal}(\mathbb{Q}(A[\lambda])/\mathbb{Q})$ for a valuation extending p. Then G is a p-group and Corollary 5.3 gives a lower bound for the exponent in terms of the exponent of p in the conductor. We see that the norm of λ satisfies $N(\lambda)^2 \equiv 1 \pmod{p^{s_p}}$ for almost all primes λ, since there must be an element of order p^{s_p} in the image G of the higher ramification group in $GL_2(\mathbf{F}_\lambda$, according to (5.3). Apply the Cebotarev density theorem to conclude that the exponent of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{E} \cap \mathbb{Q}(\zeta))$ is at most 2, this proves (i).

To prove (ii), observe that our hypothesis implies that the abelian extension corresponding to the determinant is tamely ramified and so the group G above is a subgroup of $SL_2(\mathbf{F}_\lambda)$. From (5.3), we must have an element g in G of order p^{s_p} and determinant 1. We work in the field $\mathbb{E}(\zeta)$. Replacing g by a power if necessary, we may suppose that $\mathbf{tr}(g) \equiv \zeta + \zeta^{-1}$. By Cebotarev applied to the extension $\mathbb{Q}(A[\lambda])/\mathbb{Q}$, there is a prime r not dividing lN such that $\mathrm{Frob}_r = g$. It follows that $a_r \equiv \mathbf{tr}(\mathrm{Frob}_r) \equiv \zeta + \zeta^{-1} \bmod \lambda$.

Ribet ([25], [26]) showed that under our assumptions, $\mathbb{F}$ is generated over $\mathbb{Q}$ by $a_r^2/\epsilon(r)$ with r running through almost all primes. Note that that (5.4) implies that for almost all primes λ in $\mathbb{E}$ there is a prime r such that $r \equiv \epsilon(r) \equiv 1 \bmod \lambda$ and so $a_r^2 - 2 \equiv \zeta^2 + \zeta^{-2} \bmod \lambda$. Ribet's result and the congruence above shows that $\beta = \zeta^2 + \zeta^{-2}$ must belong to $\mathbb{F}$, since the irreducible polynomial of β over $\mathbb{F}$ has a root modulo almost every prime. ∎

Remark 5.6. Our result gives an entertaining explanation of the upper bound on the conductor of elliptic curves over $\mathbb{Q}$ in [29].

Remark 5.7. Our argument shows that (i) holds in general, for weight k at least 2 and arbitary Nebentypus ϵ.

Remark 5.8. As a consequence of special twisting operators U_χ he discovered, Saito [27] had given part (i) of the previous Theorem with the weaker bound $s_p = [r_p/3]$ in the modular case. Amusingly, this gives our result for the lowest non-trivial exponents and all p, making a computational discovery unlikely. Ribet, in a letter to Saito in 1980, suggested that Saito's bound might be true of the smaller field $\mathbb{F}$, and verified it in a special case. A modular explanation of our sharper result should be of interest: it is likely that Saito's computations involving U_χ may be refined, but we have not attempted to do this.

6. Discussion of numerical experiments

In order to compare our bounds for the ranks of Jacobians with "reality", we decided to investigate the latter empirically, an elusive goal until we remembered some work of Gross, to be recalled later. Our numerical findings are in the Table at the end of this paper with a summary at the end of this section. In the next section, we mention some problems they suggested. We only hint at the ingredients of the lengthy and time-consuming calculations here.

In this section, N will always denote a prime number and denote by $J_0^-(N)$ the maximal quotient of $J_0(N)$ on which W_N acts as -1, so that the corresponding L-series have even order at $s = 1$. Ogg conjectured and Mazur [20] proved that the torsion subgroup of the abelian variety $J_0^-(N)$ is cyclic of order n equal to the numerator of $\frac{1}{12}(N-1)$. For each p dividing n, Mazur defined the Eisenstein quotient $\tilde{J}_0^p(N)$, which has nontrivial p-torsion. By a descent, he proved that these factors have rank 0.

For most *primes* N, $J_0^-(N)$ seems to be simple, in which case it is *the* Eisenstein quotient, and so has Mordell-Weil rank 0 over $\mathbb{Q}$. Our table gives the splittings of $J_0^-(N)$ for those $N < 10000$ for which it is not a simple variety. We estimate the rank of the comparatively few non-Eisenstein factors by using the following beautiful result due to Gross [13] to establish a lower bound for the analytic rank (presumably the actual analytic rank and, under the Birch-Swinnerton-Dyer conjecture, even the algebraic rank).

Theorem 6.1. *Let N be a prime. Then the number of linear dependence relations between the theta functions of positive definite ternary quadratic forms of discriminant $32N^2$ and level $4N$ is equal to the number of Hecke eigenforms f invariant under the Atkin-Lehner involution W_N for which the associated L-series has a zero of order at least 2 at $s = 1$. In particular, this counts the number of associated abelian varieties A_f with normalized rank at least 2.*

The reader is invited to look at the paper of Birch [3] for an exposition of the rather complex interaction of this theorem with results of Ponomarev and Kitaoka on ternary and quaternary quadratic forms. The reader should also consult [19] and [5][1] which contains a generalization of Gross's theorem to the case of square-free N.

Some parts of our table were computed independently by several people and our results are compatible with those I have had access to. We find as many linear factors as isogeny classes of elliptic of even rank, and get one relation

[1]We found this reference after most of this work was done. To our dismay, we found a table of the number of relations for $p < 5000$, computed by Hashimoto. His table agrees with ours except that he missed the 2 relations for $N = 4483$.

for *each* elliptic curve of rank 2, as given in the unpublished table of Mestre and Oesterlé [22]. There is agreement with partial tables of the splittings for conductors at most 1000 that H. Cohen had given me on the occasion of a visit to Bordeaux in 1990.

I used the algorithm of Birch [3] to find the Hecke graph of classes of positive definite ternary quadratic forms of discriminant $2N$. The characteristic polynomial of the adjacency matrix was obtained at first brutally. Then we used an algorithm based on continued fractions due to Mills and Wiedemann, which I learned from early fragments of the 1990 DEA of Dewisme, a student of Mestre. The relations between the theta functions were obtained by a calculation independent of the splitting, for additional confirmation, using about $N/3$ terms for N at least 1000. The "small vectors" were calculated by an algorithm in the book of Pohst and Zassenhaus. The investigation was carried out in Maple at Fordham and at Columbia.

The first abelian surface of analytic rank 4 appears as *one* of the two 2-dimensional factors of $J_0^-(1061)$. Both have endomorphisms by the maximal order of $\mathbb{Q}(\sqrt{5})$. It may be of some interest to exhibit this example more explicitly.

Consider the three parameter family of hyperelliptic curves $C_{b,c,d}$

$$\begin{aligned} y^2+(x^3+x+1+c(x^2+x))y &= b+(1+3b)x+(1-bd+3b)x^2 \\ &\quad +(b-2bd-d)x^3-bdx^4 \end{aligned} \tag{6.2}$$

It follows from some old work of ours [7] that their Jacobian admits endomorphisms by $\mathbb{Z}[\alpha]$, where $\alpha^2+\alpha-1=0$. If we specialize to the curve $X=C_{0,1,1}$, we find the hyperelliptic curve

$$y^2=x^6+2x^5+5x^4-2x^3+10x^2+8x+1 \tag{6.3}$$

with at least 18 rational points with $x=0,1,3,-1,-2,-1/2,1/3,-10/3,\infty$. Its Jacobian has real multiplication by the maximal order in $\mathbb{Q}(\sqrt{5})$ and conductor 1061^2. In fact, $J_0(1061)$ has 2 factors with that property but only 2 of the L-series vanish at $s=1$, the corresponding pair of conjugate normalized Hecke eigenforms f, f^σ have coefficients given by the list $[p, a_p]$ for p from 2 to 149:

$[2,-1+\alpha]$, $[3,-1+\alpha]$, $[5,-3-\alpha]$, $[7,-3]$, $[11,-3-\alpha]$, $[13,-4-\alpha]$, $[17,-3+\alpha]$, $[19,1]$, $[23,-3]$, $[29,-3+3\alpha]$, $[31,0]$, $[37,-1+2\alpha]$, $[41,-6+2\alpha]$, $[43,-5]$, $[47,-10-\alpha]$, $[53,3+6\alpha]$, $[59,6\alpha]$, $[61,-3+4\alpha]$, $[67,-5-9\alpha]$, $[71,-3+2\alpha]$, $[73,1-6\alpha]$, $[79,-1-6\alpha]$, $[83,8-2\alpha]$, $[89,4\alpha]$, $[97,6-2\alpha]$, $[101,-4-8\alpha]$, $[103,5\alpha+2]$, $[107,-11-10\alpha]$, $[109,-8+\alpha]$, $[113,-9-6\alpha]$, $[127,-1]$, $[131,-9-14\alpha]$, $[137,-11-4\alpha]$, $[139,17-3\alpha]$, $[149,-6+3\alpha]$.

Since we have agreement to many terms with the L-series of the Jacobian of X, we expect that the method of Faltings and Serre should prove that $\mathrm{Jac}(X) = A_f$.

7. Open questions

This investigation began in the hope of better understanding the ranks of modular elliptic curves. While the bounds on ranks we have obtained here are better than those we had obtained earlier for elliptic curves ([6], [8]), the table at the end of this paper shows that we are still far from the truth.

Our experiments with $J_0^-(N)$ for N prime suggest that the Atkin-Lehner subvarieties $A_\chi(N)$ might consist mainly of simple abelian subvarieties whose dimension grows with N. When N is highly divisible by 2 or 3, the additional special Atkin-Lehner operators should also be used, since they provide additional splitting. Such abelian varieties are likely to have normalized rank 0 or 1 according to their parity, since they have "small conductors" compared to their dimension.

We mention a few interesting questions.

Problem 1: Is the average normalized rank of $J_0(N) = \frac{1}{2}$? More precisely, does one have $\mathrm{nrk}(A_\chi(N)) = \eta + o(\dim A_\chi(N))$, with $\eta = \frac{1}{2}(1 + \chi(N))$?
Problem 2: Consider abelian varieties of *fixed* dimension, perhaps even with fixed endomorphism structure. Is there a definite positive proportion of each (normalized) rank? What can be said about the average normalized rank?

Some condition on the endomorphism structure must be imposed to ensure that the moduli space have infinitely many rational points, since families of "twists" are likely to have average normalized rank $\frac{1}{2}$. There are probably very few totally real number fields $\mathbb{E}$ which can be the field of endomorphisms of *infinitely* many isomorphism $\mathbb{C}$-classes of abelian varieties of GL_2-type defined over $\mathbb{Q}$. It may be worthwhile to pursue calculations similar to ours for composite conductors to see whether the behaviour suggested by the splittings of $J_0^-(N)$ in our table prevails. In the case of elliptic curves, our numerical investigations [10] suggested Problem 2 (cf. the table on the next page) while ([6], [8]) addressed the second question.

Problem 4: Assume that N is prime to 6 and that χ is a character of the Atkin-Lehner group. Is $A_\chi(N)$ a simple abelian variety up to a piece of dimension at most $O(\sqrt{N})$?

When N is squarefree, small factors of dimension at most 3 or 4 occasionally appear in $A_\chi(N)$, but note $N = 4751$ in our main Table. In contrast, when all exponents in N are large, we deduce from (5.5) that the dimensions of simple factors have size at least of the order of $\sqrt{N}$.

$\mathbb{E}$	Eis. pieces	Non-Eis. rank 0	nrk≥ 2	Total
$\mathbb{Q}$	23	107	59	189
$\mathbb{Q}(\sqrt{5})$	7	27	33	67
$\mathbb{Q}(\sqrt{2})$	3	12	2	17
$\mathbb{Q}(\sqrt{13})$	2	6	1	9
Other quad.	1	3	0	4
$\mathbb{Q}(\cos(\frac{2\pi}{7}))$	0	2	13	15
Other cubics	6	11	2	19
All quartics	4	5	3	12

The table above gives a summary of our more detailed table in the next section. It highlights the relative frequency of factors of normalized analytic rank at least two when there is *any* splitting at all among the varieties $J_0^-(N)$ with N prime at most 10 000. The first column give the field of endomorphisms, the second lists the number of Eisenstein factors, the third the non-Eisenstein pieces of rank 0, while the fourth contains the number whose analytic rank is at least twice their dimension. We hope the reader will agree that the proportions are quite remarkable given the small size of the conductors involved.

8. Table of splittings of $J_0(N)$

This section contains the table for those primes N for which $J_0^-(N)$ splits non-trivially, up to isogeny. The column after the conductor N contains the dimensions of all the simple factors of $J_0^-(N)$, the "Comments" column contains the *discriminant* D of the corresponding totally real field of multiplications, for those factors of dimension between 2 and 5 and the number of relations between the theta functions of the ternary quadratic forms of determinant $2N$ (cf. [19]). This provides a lower bound for the *analytic* rank of $J_0^-(N)$ and presumably an upper bound for the algebraic rank, as a tedious descent might verify.

The factor of largest dimension is usually the only Eisenstein piece and when that is the case, no special mention is made. Otherwise, we use the convention introduced by Mazur in [20] of indicating in bold face with subscript p quotients of the Eisenstein factor $\tilde{J}^p$.

N	Splitting	Comments	N	Splitting	Comments
67	1+2	$D=5$	797	2+38	$D=8$
71	$\mathbf{3}_5+\mathbf{3}_7$	$D=257;257$	997	1+1+42	2 rels
73	$\mathbf{1}_2+\mathbf{2}_3$	$D=13$	1061	2+2+46	$D=5$; 2 rels
89	$\mathbf{1}_2+\mathbf{5}_{11}$	$D=535120$	1063	2+51	$D=5$
109	1+4	$D=7537$	1093	3+44	$D=49$
113	$\mathbf{1}_2+\mathbf{2}_2+\mathbf{3}_7$	$D=12;321$	1153	$\mathbf{1}_2$+50	
139	1+7		1171	1+1+53	1 rel
151	3+6	$D=257$	1187	1+1+1+55	
179	1+11		1193	3+55	$D=49$
199	2+10	$D=5$	1201	2+51	D=8
211	$\mathbf{2}_5+\mathbf{9}_7$	$D=5$	1223	9+59	
227	2+2+10	$D=5;29$	1259	1+1+65	
233	$\mathbf{1}_2+\mathbf{11}_{29}$		1283	2+62	$D=21$
277	3+9	$D=148$	1289	$\mathbf{1}_2$+61	
307	1+1+1+1	$D=13$	1297	1+55	
	$+\mathbf{2}_3+\mathbf{9}_{17}$		1319	2+2+4+69	$D=5;5;725$
313	2+12	$D=5$	1321	1+3+56	$D=148$
353	$\mathbf{1}_2$+3+14	$D=229$	1361	2+69	$D=8$
389	1+20	1 rel	1373	1+60	
397	$2+\mathbf{5}_{11}+\mathbf{10}_3$	$D=8$; 245992	1409	2+65	$D=8$
431	1+3+24	$D=473$	1433	$\mathbf{1}_2$+2+65	$D=8$
433	1+3+16	$D=404$; 1 rel	1481	3+71	$D=169$
443	1+22		1483	1+67	1 rel
487	2+2+3+16	$D=13;13;257$	1531	1+73	1 rel
503	1+1+3+26	$D=257$	1567	3+69	$D=49$; 3 rels
557	1+26		1613	1+1+75	1 rel
563	1+31	1 rel	1621	1+70	1 rel
571	1+1+2+2	$D=5;37;$	1627	1+73	1 rel
	+4+18	11344; 1 rel	1693	3+72	$D=49$; 3 rels
577	2+2+3+18	$D=5;13;257$	1867	1+81	
593	$\mathbf{1}_2$+2+27	$D=13$	1871	$\mathbf{2}_5+\mathbf{98}_{187}$	$D=5$
643	1+28	1 rel	1873	1+79	1 rel
659	1+37		1901	1+88	
661	2+29	$D=8$	1907	1+1+90	1 rel
673	$\mathbf{2}_2+\mathbf{4}_7+\mathbf{24}_2$	$D=8;8069$	1913	$1+\mathbf{1}_2$	$D=5$; 3 rels
683	2+31	$D=5$		$+2+\mathbf{84}_{239}$	
701	1+36		1933	1+83	1 rel
709	1+30	1 rel	1979	1+104	
733	1+32		1999	2+94	$D=13$
739	1+34		2027	1+94	1 rel

N	Splitting	Comments	N	Splitting	Comments
2029	2+90	$D=5$; 2 rels	3307	2+144	D=8
2039	2+2+4+99	$D=5;5;725$	3313	$\mathbf{1}_2$+142	
2063	2+106	$D=5$	3331	1+152	
2081	2+99	$D=5$; 2 rels	3371	1+1+159	
2089	1+1+1+$\mathbf{1}_2$	D=13; 1 rel	3463	2+151	D=5; 2 rels
	+2+91		3467	1+162	
2143	1+94		3517	2+151	D=5
2203	2+94	$D=5$	3547	1+1+154	
2213	1+101		3583	2+161	D=5; 2 rels
2237	1+99		3623	1+172	
2251	1+99	1 rel	3701	2+174	D=5; 2 rels
2273	$\mathbf{1}_2$+105		3779	1+187	1 rel
2293	2+96	$D=5$; 2 rels	3803	2+171	D=5
2333	4+101	$D=725$	3907	1+168	
2339	2+114	$D=5$	3911	2+ 202	D=5; 2 rels
2341	2+2+100	$D=5;8$	3943	4+173	D=725; 4 rels
2351	1+5+123	$D=442552$	3967	1+180	1 rel
2381	2+106	$D=5$	4013	1+176	
2393	1+3+110	$D=621$	4027	2+174	D=5; 2 rels
2593	4+109	$D=1957$; 4 rels	4093	2+174	D=5; 2 rels
2609	2+127	$D=5$; 2 rels	4139	1+2+188	D=8; 1 rel
2617	2+114	$D=5$; 2 rels	4217	2+186	D=5; 2 rels
2677	1+ 115	1 rel	4229	1+194	
2699	1+1+125		4253	3+184	D=49; 3 rels
2797	1+119	1 rel	4283	1+198	
2837	1+128	1 rel	4289	$\mathbf{1}_2$+205	
2843	1+3+129	D=49; 4 rels	4337	1+188	
2861	2+133	D=5; 2 rels	4339	1+196	
2917	1+125		4357	1+187	1 rel
2939	1+150		4397	2+194	D=8
2953	1+127	1 rel	4451	1+213	
2963	2+134	D=5; 2 rels	4457	1+199	
3001	$\mathbf{2}_5 + \mathbf{132}_{50}$	D=5	4481	1+201	1 rel
3019	2+130	D=5; 2 rels	4483	2+193	D=5; 2 rels[1]
3089	$\mathbf{1}_2$+2+135	D=5; 2 rels	4547	1+205	1 rel
3181	1+144		4621	$\mathbf{2}_5+\mathbf{196}_{77}$	D=5
3203	1+143		4729	2+204	D=5
3257	1+143		4733	1+210	
3259	1+143		4751	18+225	
3271	3+146	D=49; 3 rels	4787	2+222	D=5; 2 rels

N	Splitting	Comments	N	Splitting	Comments
4799	1+230	1 rel	6911	1+330	
4951	2+219	D=5; 2 rels	6949	3+303	D=49; 3 rels
5003	3+220	D=49; 3 rels	7019	1+334	1 rel
5021	1+2+225	D=2	7057	1+1+1+1	D=321
5171	1+249	1 rel		+ 3+295	
5197	1+223		7297	2+312	D=5
5273	2+227	D=5	7451	1+344	1 rel
5303	1+247		7459	2+323	D=5
5323	3+233	D=229; 3 rels	7541	1+337	1 rel
5347	4+231	D=725	7621	2+331	D=5; 2 rels
5351	2+267	D=5; 2 rels	7639	2+331	D=5; 2 rels
5393	$\mathbf{1}_2$+241		7669	1+335	1 rel
5419	1+237		7691	1+362	
5443	1+234		7723	1+329	
5471	3+260	D=49; 3 rels	7753	3+326	D=49; 3 rels
5477	2+244	D=5; 2 rels	7841	1+349	
5651	1+265		7867	1+337	1 rel
5689	$\mathbf{1}_2$+248		7919	2+376	D=5; 2 rels
5737	3+244	D=49; 3 rels	7933	2+337	D=5; 2 rels
5741	1+261	1 rel	8117	3+353	D=49; 3 rels
5749	2+249	D=5; 2 rels	8219	1+376	1 rel
5813	1+260	1 rel	8237	1+361	
5821	2+251	D=8; 2 rels	8243	1+363	
5927	2+280	D=5	8363	1+1+381	1 rel
5987	1+1+262		8369	2+370	D=5; 2 rels
6007	2+261	D=5; 2 rels	8443	1+361	1 rel
6011	1+1+275	1 rel	8513	3+366	D=81; 3 rels
6043	1+259	1 rel	8539	1+371	
6067	1+266		8543	1+403	
6199	1+276	1 rel	8623	1+383	
6211	1+272		8699	1+396	1 rel
6323	1+283		8713	$\mathbf{1}_2$+372	
6337	2+268	D=5; 2 rels	8731	1+379	
6451	1+284		8747	1+1+1+382	1 rel
6571	1+287	1 rel	8803	1+2+372	D=5;
6581	3+290	D=49; 3 rels	8861	1+402	
6691	1+298	1 rel	8999	1+423	
6779	3+318	D=469	9011	1+4+403	D=725; 4 rels
6899	1+321		9049	2+392	D=5

N	Splitting	Comments	N	Splitting	Comments
9127	1+407	1 rel	9551	1+461	
9151	1+415		9661	1+410	
9161	1+416	1 rel	9781	2+417	D=5; 2 rels
9203	1+413	1 rel	9811	1+428	
9277	1+391	1 rel	9829	1+1+417	2 rels
9281	2+424	D=8; 2 rels	9857	3+422	D=49; 3 rels
9323	1+416		9901	1+426	
9341	1+422		9907	2+425	D=5; 2 rels
9467	1+434	1 rel	9923	1+437	
9473	$\mathbf{1}_2$+413		9931	2+434	D=5;
9479	1+444	1 rel	9967	1+2+431	D=13; 2 rels

References

1. A.O.L. Atkin and J. Lehner. *Hecke operators on $\Gamma_0(m)$*. Math. Ann. **185** 134-160 (1970).
2. A.O.L. Atkin and W. Li. *Twists of new forms and pseudo-eigenvalues of W-operators*, Math. Ann. **185** 134-160 (1970).
3. B. Birch. *Hecke actions on classes of ternary quadratic forms.* in Computational Number Theory, a conference (1989).
4. B. Birch and W. Kuyk (eds). *Modular functions of one variable*, Lecture Notes in Mathematics **476**, Springer-Verlag, New-York,1975.
5. S. Böcherer and R. Schulze-Pillot. *Siegel modular forms and theta series attached to quaternion algebras*, Nagoya Jl. of Mathematics **121** 35-96 (1991).
6. A. Brumer. *The average rank of elliptic curves I*, Inv. Math. **109**, 445-472 (1992).
7. A. Brumer. *Curves with real multiplications*, (in preparation).
8. A. Brumer and R. Heath-Brown. *The average rank of elliptic curves II*, (in preparation).
9. A. Brumer and K. Kramer. *The conductor of abelian varieties*, (to appear in Compos. Math.).
10. A. Brumer and O. McGuinness. *The behaviour of the Mordell-Weil group of elliptic curves*, Bull. AMS **23** 375-381 (1990).
11. D. Bump, S. Friedberg and J. Hoffstein, *Nonvanishing theorems for L-functions of modular forms and their derivatives*, Inv. Math. **102** 543-618 (1990).
12. J.E. Cremona. *Algorithms for modular elliptic curves*, Cambridge Univ. Press (1992). (1990).
13. B. Gross. *Heights and the special values of L-series*, in Can. Math. Soc. Conf. Proc. **7** (1989).
14. B. Gross and D. Zagier. *Heegner points and derivatives of L-series*, Inv. Math. **84** 225-320 (1986).
15. A. Grothendieck. *Modèles de Néron et monodromie*, SGA 7, Exposé IX, Springer Lecture Notes **288** 313-523, Springer-Verlag, New-York.
16. G.H. Hardy and E.M. Wright. *Introduction to the theory of numbers*, 3rd ed., Oxford (1954).
17. A. Hurwitz. *Ueber Relationen zwischen Klassenzahlen binärer quadratischer Formen von negativer Determinante*, Math. Ann. **25** 157-196 (1885).

18. V.A. Kolyvagin and D.Y. Logachev. *Finiteness of the Shafarevich-Tate group and the group of rational points for some modular abelian varieties*, Leningrad Math. Jl. **1** (1990).
19. J. Kramer. *On the linear independence of certain theta series*, Math. Ann. **281** 219-228 (1988).
20. B. Mazur. *Modular curves and the Eisenstein ideal*, Publ. Math. I.H.E.S. **47** 33-186 (1977).
21. J.-F. Mestre. *Formules explicites et minorations de conducteurs de variétés algébriques*, Compos. Math. **58** 209-232 (1982).
22. J.-F. Mestre and J.Oesterlé. *Courbes de Weil fortes de conducteur premier* $<$ 14000, (Unpublished).
23. M.R.Murty, V.K.Murty. *Mean values of derivatives of modular L-series*, Preprint (1989).
24. K. Ribet. *Endomorphisms of semi-stable abelian varieties over number fields*, Annals of Math. **101** 555-562 (1975).
25. K. Ribet. *Galois actions on division points of abelian varieties with many real multiplications*, Amer. Jl. of Math. **98** 751-804 (1976).
26. K. Ribet. *Abelian varieties over $\mathbb{Q}$ and modular forms*, preprint (1992).
27. H. Saito. *On a decomposition of spaces of cusp forms and trace formula Hecke operators*, Nagoya Jl. of Math. **80** 129-165 (1980).
28. J.-P. Serre. *Quelques applications du théorème de densité de Chebotarev*, Publ. Math. I.H.E.S. **47** 323-201 (1981).
29. J.-P. Serre. *Sur les représentations modulaires de degré 2 de* $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. Jl. **54** 179-230 (1987).
30. J.-P. Serre. *Sur le nombre des points rationnels d'une courbe algébrique sur un corps fini*, C.R. Acad Sci. Paris **296** 397-402 (1983).
31. G. Shimura. *Introduction to the theory of automorphic functions*. Princeton, 1971.
32. N.-P. Skoruppa and D. Zagier. *Jacobi forms and a certain space of modular forms*, Inv. Math. **94** 113-146 (1988).

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