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COMPARISON THEOREM FOR λ-OPERATIONS IN HIGHER ALGEBRAIC K-THEORY

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Introduction

In his paper [G2], D. Grayson defined a map

$$\Lambda^k : \mathrm{Sub}_k\, G\mathcal{M} \to G^{(k)}\mathcal{M}$$

for every $k = 1, 2\ldots$ and every exact category $\mathcal{M}$ with a suitable notion of exterior and tensor products, where both the domain and the codomain of the map are certain k-fold multisimplicial sets representing the homotopy type of K-theory of the category $\mathcal{M}$. This provides a definition of the operation λ^k on $K.\mathcal{M}$ as induced by the map Λ^k on the homotopy groups.

The G-construction $G\mathcal{M}$ is a simplicial set defined in [GG] whose vertices are in one-to-one correspondence with all pairs (A, B) of objects of $\mathcal{M}$, and an edge from (A, B) to (A', B') in $G\mathcal{M}$ is a pair of short exact sequences $(A \rightarrowtail A' \twoheadrightarrow C, B \rightarrowtail B' \twoheadrightarrow C)$ with equal cokernels. In order for the G-construction to serve as a domain of the map Λ^k, one should subdivide it first by the k-fold edgewise subdivision functor Sub_k : Simp.Sets $\to k$-fold Multisimp.Sets (see [G2], sect. 4). The codomain $G^{(k)}\mathcal{M}$ is the iterated G-construction. Its vertices correspond bijectively to all 2^k-tuples of objects of $\mathcal{M}$ positioned naturally at the vertices of a k-dimensional cube.

Another definition of the operation λ^k in the same fashion was given by the author in [N]. This definition is provided by the map

$$\Lambda^k : \mathrm{Diag}\,\mathrm{Sub}_k\, G\mathcal{M} \to G(k; \mathcal{M})$$

where Diag : k-fold Multisimp.Sets $\to$ Simp.Sets is the total diagonal functor and $G(k; \mathcal{M})$ is a simplicial set whose vertices are in one-to-one correspondence with all $(k+1)$-tuples $(A_0, \ldots, A_k)$ of objects of $\mathcal{M}$, and an edge from (A_i) to (B_i) is a $(k+1)$-tuple of short exact sequences $A_i \rightarrowtail B_i \twoheadrightarrow C_i)$ together with a long exact sequence of cokernels $0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$.

It is not hard (and this is done in [G2] and [N]) to establish independently the equivalence of each of the two definitions of λ-operations to that given by means of Quillen's plus construction in [Hi] and [Kr] in the case of K-theory of a ring. The purpose of this paper is to show directly the equivalence of the above two definitions for any exact category $\mathcal{M}$ with operations, in the very spirit of the definitions, i.e., by means of certain simplicial structures.

We demonstrate the idea in the case $k = 2$. The map Λ^2 of Grayson takes a vertex (V, W) of $G\mathcal{M}$ to the 4-tuple of objects of $\mathcal{M}$

$$\begin{pmatrix} W \wedge W & W \otimes W \\ V \wedge V & V \otimes W \end{pmatrix}$$

regarded as a vertex of $G^{(2)}\mathcal{M}$. We observe that the objects $W \wedge W$ and $W \otimes W$ standing in the 1st row admit the natural maps $W \wedge W \rightarrowtail W \otimes W$ and $W \otimes W \twoheadrightarrow W \wedge W$, which lead to the pair of dual short exact sequences $0 \to W \wedge W \to W \otimes W \to S^2 W \to 0$ and $0 \leftarrow W \wedge W \leftarrow W \otimes W \leftarrow W \circ W \leftarrow 0$, where $S^2 W$ is the symmetric square and $W \circ W = D^2 W$ is the 2nd divided power (invariants of the symmetric group Σ_2 action on $W \otimes W$ in the case of modules). In the sequel, we prefer to use sequences of the second type, i.e. coSchur complexes (*cf.* [ABW], ch. V).

This observation leads to the definition of a bisimplicial set $A(2; \mathcal{M})$ whose vertices are in one-to-one correspondence with all diagrams of the type

$$\begin{pmatrix} A & \twoheadleftarrow & B & \leftarrowtail & C \\ D & & E & & \end{pmatrix}$$

where the sequence $A \twoheadleftarrow B \leftarrowtail C$ is exact. There is a natural map $A(2; \mathcal{M}) \to G^{(2)}\mathcal{M}$ given on vertices by

$$\begin{pmatrix} A & \twoheadleftarrow & B & \leftarrowtail & C \\ D & & E & & \end{pmatrix} \longmapsto \begin{pmatrix} A & B \\ D & E. \end{pmatrix}$$

This map is a homotopy equivalence, and the map $\Lambda^2 : Sub_2 G\mathcal{M} \to G^{(2)}\mathcal{M}$ from [G2] can be lifted to a map $\Lambda^2 : \mathrm{Sub}_2\, G\mathcal{M} \to A(2; \mathcal{M})$,

$$(V, W) \longmapsto \begin{pmatrix} W \wedge W & \leftarrow & W \otimes W & \leftarrow & W \circ W \\ V \wedge V & & V \otimes W & & \end{pmatrix}.$$

On the other hand, there is a natural map $\mathrm{Diag}\, A(2; \mathcal{M}) \to G(2; \mathcal{M})$ given on vertices by

$$\begin{pmatrix} A & \twoheadleftarrow & B & \leftarrowtail & C \\ D & & E & & \end{pmatrix} \longmapsto (D, E, C)\ ,$$

which also proves to be a homotopy equivalence, and the composite map $\operatorname{Diag}\operatorname{Sub}_2 G\mathcal{M} \to \operatorname{Diag} A(2;\mathcal{M}) \to G(2;\mathcal{M})$ given on vertices of $G\mathcal{M}$ by

$$(V,W) \mapsto \begin{pmatrix} W\wedge W & \leftarrow & W\otimes W & \leftarrowtail & W\circ W \\ V\wedge V & & V\otimes W & & \end{pmatrix} \mapsto (V\wedge V,\ V\otimes W,\ W\circ W)$$

is nothing but the map Λ^2 from [N]. Thus, we result with the equivalence of the Λ^2-maps of [G2] and [N].

For an arbitrary k, we define a k-fold multisimplicial set $A(k;\mathcal{M})$ and a map $\Lambda^k : \operatorname{Sub}_k G\mathcal{M} \to A(k;\mathcal{M})$ by means of which the Λ-maps of [G2] and [N] can be linked in the following manner.

MAIN THEOREM. — *The commutative diagram of spaces holds,*

$$\begin{array}{ccccc}
|\operatorname{Sub}_k G\mathcal{M}| & \xrightarrow{\Lambda^k_{[G2]}} & |G^{(k)}\mathcal{M}| & & \\
\| & & \uparrow & & \\
|\operatorname{Sub}_k G\mathcal{M}| & \xrightarrow{\Lambda^k\text{loc.cit.}} & |A(k;\mathcal{M})| & \cong & |\operatorname{Diag} A(k;\mathcal{M})| \\
\|\,\iota & & & & \downarrow \\
|\operatorname{Diag}\operatorname{Sub}_k G\mathcal{M}| & & \xrightarrow{\qquad\Lambda^k_{[N]}\qquad} & & |G(k;\mathcal{M})|
\end{array}$$

where all arrows are given by certain simplicial maps, the vertical arrows on the right are homotopy equivalences, the map Λ^k on the top is that of [G2], and Λ^k at the bottom is defined in [N].

REMARK. The construction of each of the three arrows in the bottom square depends on choice of cokernels for all admissible monomorphisms in $\mathcal{M}$, hence these maps are defined up to natural simplicial homotopy. Given such a choice, the bottom square is strictly commutative.

In section 1 we recall the definition of multidimensional $S.$ and C (mapping cone) constructions given in [G3] , and develop some technique for them. This technique is based mainly on a generalization for multidimensional case of the Theorem C of Grayson [G1]. It enables one to compute the C-construction of a cube of exact categories as a homotopy fibre of the map of $S.$ constructions of the corresponding faces, under a certain assumption on the cube (*cf.* Proposition 1.6). In order to prove such a generalization, we need to restrict the class of dominant functors introduced in [G1] and consider strictly dominant functors (*cf.* Definition 1.1) which prove to be stable under applying the C-construction degreewise (Proposition 1.5).

In section 2, some finite categories of words (actually ordered sets) are introduced. We declare certain sequences of words to be "long exact" and call them "formal Schur complexes". Then we define some exact categories of diagrams in $\mathcal{M}$ in which the positions correspond to those words, and formal Schur complexes give rise to long exact sequences in those diagrams. We define the multisimplicial set $A(k;\mathcal{M})$ by means of these categories of diagrams via the C-construction and show that the natural forgetful map $A(k;\mathcal{M}) \to G^{(k)}\mathcal{M}$ is a homotopy equivalence. Hence, $A(k;\mathcal{M})$ represents the $K.\mathcal{M}$ homotopy type.

In section 3, we construct a chain of k-fold multisimplicial sets $A(k;\mathcal{M}) = A(k,k;\mathcal{M}) \to A(k;k-1;\mathcal{M}) \to \cdots \to A(k,1;\mathcal{M})$ and show that all maps in it are homotopy equivalences. Then we define a map from the total diagonal $\operatorname{Diag} A(k,1;\mathcal{M})$ to $G(k;\mathcal{M})$ which also proves to be a homotopy equivalence. This results with a homotopy equivalence of simplicial sets $\operatorname{Diag} A(k;\mathcal{M}) \to G(k;\mathcal{M})$.

In section 4 we define a map $\Lambda^k : \operatorname{Sub}_k G\mathcal{M} \to A(k;\mathcal{M})$ with "real" coSchur complexes corresponding to formal Schur complexes in the diagrams. We check that this map is compatible with the Λ-maps defined in [G2] and [N], therefore establishing the desired equivalence.

We note that another (with respect to $A(k,\ell;\mathcal{M})$) interesting class of multisimplicial spaces representing the delooping of the K-theory homotopy type was introduced by Grayson in [G3] in order to define the operations of Adams. Further investigation of operations in higher K-theory on the simplicial level is carried out in the preprint of B. Köck [Kö].

§1. Simplicial Technique

We recall the definition of the multidimensional C and $S.$ constructions (*cf.* [G3], sect. 4).

Let Δ denote the category of finite nonempty totally ordered sets and nondecreasing maps. For any partially ordered set P, we denote by $\operatorname{Ar} P$ the category of arows in P, where a map of arrows is an obvious commutative diagram. We use the notation i/j for the arrow $(j \leq i) \in \operatorname{Ar} P$. Given an exact category $\mathcal{M}$ with a distinguished zero object $*$, the S-construction of Waldhausen is a simplicial set $S.\mathcal{M}$ with $S.\mathcal{M}[P] = \operatorname{Exact}(\operatorname{Ar} P, \mathcal{M})$, $P \in \Delta$, with obvious face and degeneracy maps, where "Exact" refers to the set of functors satisfying the condition : $F(i/i) = *$ for any $i \in P$ and $0 \to F(j/i) \to F(k/i) \to F(k/j) \to 0$ is an exact sequence in $\mathcal{M}$ for any $i \leq j \leq k$ in P (*cf.* sect. 1.3 of [W2] or sect. 7 of [W1]).

We regard the set $[1] = \{0 < 1\}$ as a category. By an n-dimensional cube $\mathcal{X} = \mathcal{X}_{(n)}$ of (exact) categories we mean a functor from $([1]^n)^{\mathrm{op}}$ to the category of (exact) categories. For $\varepsilon \in [1]^n$ we denote by $\mathcal{X}(\varepsilon)$ the category standing at the vertice ε. By a map of n-dimensional cubes we mean a natural (exact) transformation of functors. We also consider covariant cubes of certain categories of words in §2; a confusion seems impossible. Given some cubes $\mathcal{X}_{(m)}$ and $\mathcal{Y}_{(n)}$ we define a $(m+n)$-dimensional cube $\mathcal{X} \boxtimes \mathcal{Y}$ by

$$\mathcal{X} \boxtimes \mathcal{Y}(\varepsilon_1, \ldots, \varepsilon_{m+n}) = \mathcal{X}(\varepsilon_1, \ldots, \varepsilon_m) \times \mathcal{Y}(\varepsilon_{m+1}, \ldots, \varepsilon_{m+n}).$$

Let B be a symbol. For $P \in \Delta$ we denote simply by BP the disjoint union $\{B\} \cup P$ with B declared to be less than any element of P. Given $P_1, \ldots, P_n \in \Delta$ and $\varepsilon \in \{0,1\}^n$ we set

$$\Gamma(P_1, \ldots, P_n) = [\operatorname{Ar} P_1 \to \operatorname{Ar}(BP_1)] \boxtimes \cdots \boxtimes [\operatorname{Ar} P_n \to \operatorname{Ar}(BP_n)]$$

and let $\Gamma(P_1, \ldots, P_n; \varepsilon)$ be the category at the ε-vertice of the cube $\Gamma(P_1, \ldots, P_n)$, i.e., $\Gamma(P_1, \ldots, P_n; \varepsilon)$ is the direct product of n categories with i-th factor equal to $\operatorname{Ar} P_i$ or $\operatorname{Ar}(BP_i)$ accordingly to $\varepsilon_i = 1$ or 0.

Suppose we are given an n-dimensional cube of exact categories $\mathcal{X}$. We use the notation of [G3], sect. 4, and define the mapping cone construction $C\mathcal{X}$ to be an n-fold multisimplicial set given by

$$C\mathcal{X}[P_1, \ldots, P_n] = \mathrm{Exact}(\Gamma(\mathrm{P}_1, \ldots, \mathrm{P}_n), \mathcal{X})$$

where "Exact" stands for the set of functors whose components at all vertices are polyexact, i.e. exact in each variable. If $n = 0$, then $\mathcal{X}$ is actually an exact category, and we have $C\mathcal{X} = \mathcal{X}$. Notice that in the case $n = 1$ the same construction was denoted by $F.(\mathcal{A} \to \mathcal{B})$ in [W1, p.182] and by $S.(\mathcal{A} \to \mathcal{B})$ in [W2, p. 343].

Regarding $C\mathcal{X}$ as a multisimplicial exact category in a natural way, we apply the $S.$ construction degreewise to obtain the $(n+1)$-fold multisimplicial set $S.C\mathcal{X}$ which we denote simply by $S.\mathcal{X}$. We write explicitly

$$S.\mathcal{X}[P_0, P_1, \ldots, P_n] = \mathrm{Exact}(\operatorname{Ar} \mathrm{P}_0 \boxtimes \Gamma(\mathrm{P}_1, \ldots, \mathrm{P}_n), \mathcal{X}).$$

The notation $S.\mathcal{X}$ is justified by the case $n = 0$ where we obtain the ordinary $S.$ construction of Waldhausen.

We notice that the G-construction of an exact category $\mathcal{M}$ can be defined as $G\mathcal{M} = C[\mathcal{M} \xrightarrow{\text{diag}} \mathcal{M} \times \mathcal{M}]$ (compare with the original definition in [GG], sect. 3).

Given an exact functor of exact categories $F : \mathcal{A} \to \mathcal{B}$ we denote by $\operatorname{im} \mathrm{F}$ the class of those $B \in \mathcal{B}$ with $B \cong F(A)$ for some $A \in \mathcal{A}$. We call an admissible monomorphism $B \rightarrowtail B'$ in $\mathcal{B}$ an F-mono if its cokernel belongs to $\operatorname{im} \mathrm{F}$.

An exact functor F is called dominant if for each $B \rightarrowtail B'$ in $\mathcal{B}$ there exists another admissible monomorphism $B \rightarrowtail B''$ and a commutative diagram

$$\begin{array}{ccccc} B & \rightarrowtail & B_1 & \rightarrowtail \cdots \rightarrowtail & B_m \\ \downarrow & & & & \| \\ B' \coprod_B B'' & \rightarrowtail & C_1 & \rightarrowtail \cdots \rightarrowtail & C_n \end{array}$$

in which all horizontal arrows are F-monos. By virtue of Theorem 2.1 of Grayson [G1], this property is sufficient for the sequence

$$C[F : \mathcal{A} \to \mathcal{B}] \to S.\mathcal{A} \to S.\mathcal{B}$$

to be a fibration up to homotopy.

When dealing with the multidimensional C-construction, it seems convenient to restrict the class of dominant functors.

DEFINITION 1.1. We call an exact functor F *strictly dominant* if for each $f : B \rightarrowtail B'$ in $\mathcal{B}$ there exists an object B'' in $\mathcal{B}$ and a filtration $B \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n = B' \oplus B''$ in which the composite map is equal to $f \oplus 0$ and all arrows are F-monos.

This means that in the definition of dominant functor the required pushout must be the addition of a direct summand and the lower sequence is reduced to one object. The class of strictly dominant functors contains the following two types of exact functors :

a) surjective functors, i.e. those $F : \mathcal{A} \to \mathcal{B}$ with $\operatorname{im} \mathrm{F} = \operatorname{Obj} \mathcal{B}$;

b) cofinal functors, i.e. those $F : \mathcal{A} \to \mathcal{B}$ that for any B in $\mathcal{B}$ there exists some B' in $\mathcal{B}$ with $B \oplus B' \in \operatorname{im} \mathrm{F}$.

Given an exact category $\mathcal{A}$ and two exact subcategories $\mathcal{A}'$ and $\mathcal{A}''$ of $\mathcal{A}$ together with exact functors $j' : \mathcal{A} \to \mathcal{A}'$, $j'' : \mathcal{A} \to \mathcal{A}''$, we say that the collection $(\mathcal{A}, \mathcal{A}', \mathcal{A}''; j', j'')$ satisfies the condition of the additivity theorem if there exists a short exact sequence of endofunctors of $\mathcal{A} : 0 \to j' \to 1_{\mathcal{A}} \to j'' \to 0$.

PROPOSITION 1.2. — *Suppose $(\mathcal{A}, \mathcal{A}', \mathcal{A}''; j', j'')$ and $(\mathcal{B}, \mathcal{B}', \mathcal{B}''; \ell', \ell'')$ satisfy the condition of the additivity theorem and an exact functor $F : \mathcal{A} \to \mathcal{B}$ is compatible with them, i.e.,*

$$F(\mathcal{A}') \subset \mathcal{B}', F(\mathcal{A}'') \subset \mathcal{B}'', \text{ and } \mathrm{F}(\mathrm{j'A} \to \mathrm{A} \to \mathrm{j''A}) = (\ell'\mathrm{FA} \to \mathrm{FA} \to \ell''\mathrm{FA})$$

for any $A \in \mathcal{A}$. If both of the restricted functors $F : \mathcal{A}' \to \mathcal{B}'$ and $F : \mathcal{A}'' \to \mathcal{B}''$ are strictly dominant, then F is strictly dominant.

LEMMA 1.3. — *Given an exact category and a commutative diagram in it of the form*

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d] & A \arrow[r, two heads] \arrow[d] & A'' \arrow[d] \\
B' \arrow[r, tail] & B \arrow[r, two heads] & B''
\end{tikzcd}$$

in which the horizontal sequences are short exact, we have a natural isomorphism $B' \coprod_{A'} A \xrightarrow{\sim} B \underset{B''}{\times} A''$. In other words, we can pass from $A' \rightarrowtail A \twoheadrightarrow A''$ to $B' \rightarrowtail B \twoheadrightarrow B''$ by the two steps

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d] & A \arrow[r, two heads] \arrow[d] & A'' \arrow[d, equal] \\
B' \arrow[r, tail] \arrow[d, two heads] & C \arrow[r, two heads] \arrow[d, two heads] & A'' \\
B'/A' \arrow[r, "\sim"] & C/A &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
B' \arrow[r, tail] \arrow[d, equal] & C \arrow[r, two heads] \arrow[d] & A'' \arrow[d] \\
B' \arrow[r, tail] & B \arrow[r, two heads] \arrow[d, two heads] & B'' \arrow[d, two heads] \\
 & B/C \arrow[r, "\sim"] & B''/A''
\end{tikzcd}$$

where all short sequences are exact.

Proof of the lemma is trivial □

LEMMA 1.4. — *Given an exact functor $F : \mathcal{A} \to \mathcal{B}$ and admissible monomorphisms $f : A \rightarrowtail C$ and $g : C \rightarrowtail B$ in $\mathcal{B}$, suppose that there exist such objects C', B' in $\mathcal{B}$ and filtrations of F-monos $A \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail C \oplus C'$, $C \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail B \oplus B'$ with the composite maps equal to $f \oplus 0$ and $g \oplus 0$, respectively. Then the same property holds for the arrow $gf : A \rightarrowtail B$.*

Proof. The filtration $A \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail C \oplus C' \rightarrowtail C_1 \oplus C' \rightarrowtail \cdots \rightarrowtail B \oplus B' \oplus C'$ is the desired □

Proof of Proposition 1.2. Given an admissible monomorphism $A \rightarrowtail B$ in the category $\mathcal{B}$, we apply Lemma 1.3 to the diagram

$$\begin{array}{ccccc} \ell' A & \rightarrowtail & A & \twoheadrightarrow & \ell'' A \\ \downarrow & & \downarrow & & \downarrow \\ \ell' B & \rightarrowtail & B & \twoheadrightarrow & \ell'' B. \end{array}$$

By Lemma 1.4 it suffices to show that both of the arrows $A \rightarrowtail C$ and $C \rightarrowtail B$ have the required property, where $C \cong \ell' B \coprod_{\ell' A} A \cong B \underset{\ell'' B}{\times} \ell'' A$. Since $F : \mathcal{A}' \to \mathcal{B}'$ is strictly dominant, there exists a filtration of $F|_{\mathcal{A}'}$-monos $\ell' A \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail \ell' B \oplus B'$ which gives rise, by the cobase change via $\ell' A \rightarrowtail A$, to a filtration $A \rightarrowtail A_1 \coprod_{\ell' A} A \rightarrowtail \cdots \rightarrowtail C \oplus B'$ with the same cokernels. The desired filtration for $C \rightarrowtail B$ is obtained similarly by a base change from that for $\ell'' A \rightarrowtail \ell'' B$, hence we are done □

If $\mathcal{X}_{(n)}$ is a cube of exact categories and $p_1, \ldots, p_n$ are nonnegative integers, we write $C_{p_1 \ldots p_n}(\mathcal{X})$ instead of $C\mathcal{X}([p_1], \ldots, [p_n])$, where $[p]$ denotes the set $\{0 < 1 < \cdots < p\} \in \Delta$. We call $\mathcal{X}$ strictly dominant in the first direction if for every $(\varepsilon_2, \ldots, \varepsilon_n) \in [1]^{n-1}$ the functor $\mathcal{X}(1, \varepsilon_2, \ldots, \varepsilon_n) \to \mathcal{X}(0, \varepsilon_2, \ldots, \varepsilon_n)$ is strictly dominant. We denote by $\mathcal{X}(\varepsilon_1 = 1)$, $\mathcal{X}(\varepsilon_1 = 1, \varepsilon_2 = 0)$, etc., the face of $\mathcal{X}$ determined by the indicated equalities.

PROPOSITION 1.5. — *Let $\mathcal{X}_{(n)}$, with $n \geq 2$, be a cube of exact categories strictly dominant in the first direction. Then for every $p_2, \ldots, p_n \geq 0$ the functor $C_{p_2 \ldots p_n}(\mathcal{X}(\varepsilon_1 = 1)) \to C_{p_2 \ldots p_n}(\mathcal{X}(\varepsilon_1 = 0))$ is strictly dominant.*

Proof. First we treat the case $n = 2$. We do this similarly to the proof of Proposition 1.5.5 in [W2]. We have to show that if both $\mathcal{X}(1,0) \to \mathcal{X}(0,0)$ and $\mathcal{X}(1,1) \to \mathcal{X}(0,1)$ are strictly dominant, then so is $C_p[\mathcal{X}(1,1) \to \mathcal{X}(1,0)] \to C_p[\mathcal{X}(0,1) \to \mathcal{X}(0,0)]$ for every $p \geq 0$. If $p = 0$, we are restricted to the functor $\mathcal{X}(1,0) \to \mathcal{X}(0,0)$. Suppose $p \geq 1$ and set $\mathcal{A} = C_p[\mathcal{X}(1,1) \to \mathcal{X}(1,0)]$. We let $\mathcal{A}' = s_{p-1}(C_{p-1}[\mathcal{X}(1,1) \to \mathcal{X}(1,0)])$ be the image of the degeneracy map, $j' = s_{p-1}d_p$, $\mathcal{A}'' = \{x \in \mathcal{A} \mid d_p x = *\}$, and $j''x = x/j'x$. Then $(\mathcal{A}, \mathcal{A}', \mathcal{A}''; j', j'')$ satisfy the condition of the additivity theorem. The same can be done with $\mathcal{B} = C_p[\mathcal{X}(0,1) \to \mathcal{X}(0,0)]$. Since the category $\mathcal{A}''$ (resp. $\mathcal{B}''$) is equivalent to $\mathcal{X}(1,1)$ (resp. $\mathcal{X}(0,1)$) and $\mathcal{A}'$ (resp. $\mathcal{B}'$) is equivalent to $C_{p-1}[\mathcal{X}(1,1) \to \mathcal{X}(1,0)]$ (resp. $C_{p-1}[\mathcal{X}(0,1) \to \mathcal{X}(0,0)]$), Proposition 1.2 enables us to carry out an induction on p, hence the case $n = 2$ is done.

Now let $n \geq 3$, then we have

$$C_{p_2 \ldots p_n}\mathcal{X}(\varepsilon_1 = 1) = C_{p_2}[C_{p_3 \ldots p_n}\mathcal{X}(\varepsilon_1 = 1, \varepsilon_2 = 1) \to C_{p_3 \ldots p_n}\mathcal{X}(\varepsilon_1 = 1, \varepsilon_2 = 0)]$$
$$C_{p_2 \ldots p_n}\mathcal{X}(\varepsilon_1 = 0) = C_{p_2}[C_{p_3 \ldots p_n}\mathcal{X}(\varepsilon_1 = 0, \varepsilon_2 = 1) \to C_{p_3 \ldots p_n}\mathcal{X}(\varepsilon_1 = 0, \varepsilon_2 = 0)]$$

whence everything follows via the induction on n by virtue of the case $n = 2$ □

PROPOSITION 1.6 (fibration theorem for C-construction). — *Given a cube of exact categories $\mathcal{X}_{(n)}$, $n \geq 1$, strictly dominant in the first direction, the sequence*

$$C\mathcal{X} \to S.\mathcal{X}(\varepsilon_1 = 1) \to S.\mathcal{X}(\varepsilon_1 = 0)$$

is a fibration.

Proof. By Lemma 5.2 of [W1], it suffices to check that for every $p_2, \ldots, p_n \geq 0$ the sequence

$$C[C_{p_2 \ldots p_n}\mathcal{X}(\varepsilon_1 = 1) \to C_{p_2 \ldots p_n}\mathcal{X}(\varepsilon_1 = 0)] \to S.C_{p_2 \ldots p_n}\mathcal{X}(\varepsilon_1 = 1) \to$$
$$\to S.C_{p_2 \ldots p_n}\mathcal{X}(\varepsilon_1 = 0)$$

is a fibration up to homotopy. The functor $C_{p_2 \ldots p_n}\mathcal{X}(\varepsilon_1 = 1) \to C_{p_2 \ldots p_n}\mathcal{X}(\varepsilon_1 = 0)$ is strictly dominant by Proposition 1.5. Hence, Theorem 2.1 of [G1] gives the desired result. □

COROLLARY 1.7. — *Let $\mathcal{X} \to \mathcal{Y}$ be an exact functor of cubes, each of them being strictly dominant in the first direction. If the square*

$$\begin{array}{ccc} S.\mathcal{X}(\varepsilon_1 = 1) & \to & S.\mathcal{Y}(\varepsilon_1 = 1) \\ \downarrow & & \downarrow \\ S.\mathcal{X}(\varepsilon_1 = 0) & \to & S.\mathcal{Y}(\varepsilon_1 = 0) \end{array}$$

is homotopy cartesian, then the map $C\mathcal{X} \to C\mathcal{Y}$ is a homotopy equivalence □

The following assertion shows that it suffices to check the above condition on the vertices of given cubes.

COROLLARY 1.8. — *Let $\mathcal{X}_{(n)} \to \mathcal{Y}_{(n)}$ be an exact functor of cubes, each of the cubes being strictly dominant in the first direction. If for every $\varepsilon \in [1]^{n-1}$ the square*

$$\begin{array}{ccc} S.\mathcal{X}(1,\varepsilon) & \to & S.\mathcal{Y}(1,\varepsilon) \\ \downarrow & & \downarrow \\ S.\mathcal{X}(0,\varepsilon) & \to & S.\mathcal{Y}(0,\varepsilon) \end{array}$$

is homotopy cartesian, then the map $C\mathcal{X} \to C\mathcal{Y}$ is a homotopy equivalence.

Proof. We carry out the induction on n. If $n = 1$, then the assertion is equivalent to that of Corollary 1.7. If $n \geq 2$, then, according to Lemma 4.1 of [G3], there are fibration sequences

$$\begin{aligned}
&S.\mathcal{X}(\varepsilon_1 = 1, \varepsilon_2 = 1) \to S.\mathcal{X}(\varepsilon_1 = 1, \varepsilon_2 = 0) \to S.\mathcal{X}(\varepsilon_1 = 1) \\
&S.\mathcal{X}(\varepsilon_1 = 0, \varepsilon_2 = 1) \to S.\mathcal{X}(\varepsilon_1 = 0, \varepsilon_2 = 0) \to S.\mathcal{X}(\varepsilon_1 = 0) \\
&S.\mathcal{Y}(\varepsilon_1 = 1, \varepsilon_2 = 1) \to S.\mathcal{Y}(\varepsilon_1 = 1, \varepsilon_2 = 0) \to S.\mathcal{Y}(\varepsilon_1 = 1) \\
&S.\mathcal{Y}(\varepsilon_1 = 0, \varepsilon_2 = 1) \to S.\mathcal{Y}(\varepsilon_1 = 0, \varepsilon_2 = 0) \to S.\mathcal{Y}(\varepsilon_1 = 0).
\end{aligned}$$

The first and the second terms of these sequences form homotopy cartesian squares by the inductional hypothesis for the maps $\mathcal{X}(\varepsilon_2 = 1) \to \mathcal{Y}(\varepsilon_2 = 1)$ and $\mathcal{X}(\varepsilon_2 = 0) \to \mathcal{Y}(\varepsilon_2 = 0)$. Hence, so do the third terms, and we are done □

PROPOSITION 1.9. (additivity theorem for multidimension $S.$ construction). — *Let $\mathcal{X}_{(n)}$, $\mathcal{Y}_{(n)}$, $\mathcal{Z}_{(n)}$ be some cubes of exact categories, and let $i : \mathcal{Y} \to \mathcal{X}$, $j : \mathcal{Z} \to \mathcal{X}$ be exact inclusions. Suppose we are given exact functors $p : \mathcal{X} \to \mathcal{Z}$ and $q : \mathcal{X} \to \mathcal{Y}$, and an exact sequence of endofunctors of the cube $\mathcal{X}$: $0 \to jp \to 1_{\mathcal{X}} \to iq \to 0$. Then the map*

$$q \times p : S.\mathcal{X} \to S.\mathcal{Y} \times S.\mathcal{Z}$$

is a homotopy equivalence.

Proof. By Lemma 5.1 of [W1], it suffices to check that for every $p_1, \dots, p_n \geq 0$ the map $q \times p : S.C_{p_1 \dots p_n}\mathcal{X} \to S.C_{p_1 \dots p_n}\mathcal{Y} \times S.C_{p_1 \dots p_n}\mathcal{Z}$ is a homotopy equivalence. We can reduce this to the additivity theorem of Waldhausen (*cf.* Proposition 1.3.2(4) of [W2]) by the same argument as in the proof of Proposition 1.5. □

By virtue of Proposition 1.6, we obtain

COROLLARY 1.10 (additivity theorem for C-construction). — *Under the assumptions of Proposition 1.9 with $n \geq 1$, suppose that the cubes $\mathcal{X}_{(n)}$, $\mathcal{Y}_{(n)}$, $\mathcal{Z}_{(n)}$ are strictly dominant in the first direction. Then the map*

$$q \times p : C\mathcal{X} \to C\mathcal{Y} \times C\mathcal{Z}$$

is a homotopy equivalence.

In conclusion, we provide a sufficient condition for a functor to be strictly dominant.

PROPOSITION 1.11. — *Let $F_i : \mathcal{A} \to \mathcal{A}_i$, with $1 \leq i \leq m$, be a collection of strictly dominant functors. Then the functor $F = (F_1, \dots, F_m) : \mathcal{A} \to \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ is strictly dominant.*

Proof. It suffices to show the case $m = 2$. Given an admissible monomorphism $(f : A_1 \rightarrowtail B_1, g : A_2 \to B_2)$ in $\mathcal{A}_1 \times \mathcal{A}_2$, we decompose it trivially in $(f : A_1 \rightarrowtail B_1, 1 : A_2 \to A_2)$ and $(1 : B_1 \rightarrowtail B_1, g : A_2 \to B_2)$. Hence, by Lemma 1.4, we are reduced to show that an arrow of the form $(f : A_1 \to B_1, 1 : A_2 \to A_2)$ in $\mathcal{A}_1 \times \mathcal{A}_2$ admits a filtration of the required type. Since F_1 is strictly dominant, there exists an object B_1' in $\mathcal{A}_1$ and a filtration of F_1-monos $A_1 \rightarrowtail D_1 \rightarrowtail D_2 \rightarrowtail \cdots \rightarrowtail D_n = B_1 \oplus B_1'$ with the composite map equal to $f \oplus 0$. Let $D_1/A \cong F_1(C_1)$ and $D_i/D_{i-1} \cong F_1(C_i)$ for $2 \leq i \leq n$, where $C_1, \dots, C_n$ are some objects of $\mathcal{A}$. We put $E_i = F_2(C_i)$, $1 \leq i \leq n$. Then the filtration

$(A_1, A_2) \rightarrowtail (D_1, A_2 \oplus E_1) \rightarrowtail (D_2, A_2 \oplus E_1 \oplus E_2) \rightarrowtail \cdots \rightarrowtail (D_n, A_2 \oplus E_1 \oplus \cdots \oplus E_n)$ of F-monos is the one required $\square$

§2. The Multisimplicial Set $A(k; \mathcal{M})$

For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$, with $\varepsilon_i \in \{0,1\}$, let $\mathfrak{A}(k;\varepsilon)$ denote the set of all words $W = X_1 \ldots X_k$ of length k in the letters E, T, S, Q (the first letters of the words "exterior", "tensor", "symmetric" (products), and "quotient") satisfying the conditions

$$\text{(2.1)} \qquad \begin{array}{ll} \text{(i)} & \text{if } \mathrm{X_i = S, \text{ with } i \leq k-1, \text{ then } X_{i+1} = \cdots = X_k = S} \ ; \\ \text{(ii)} & \text{if } \mathrm{X_i = S, \text{ then } i \geq 2 \text{ and } X_{i-1} \neq E}; \\ \text{(iii)} & \mathrm{X_i = Q} \text{ if and only if } \varepsilon_\mathrm{i} = 1. \end{array}$$

Let $\alpha(W)$ be equal to 0 if $X_k = E$, to 1 if $X_k = T$, and to $n+1$ if $W = X_1 \ldots X_{k-n} S \ldots S$ with $X_{k-n} \neq S$; $\alpha(W)$ is not defined if $X_k = Q$.

We regard $\mathfrak{A}(k;\varepsilon)$ as a category with the minimal stock of morphisms including all arrows of the type

$$\text{(2.2)} \qquad \begin{array}{ll} \text{(i)} & X_1 \ldots X_n T \underbrace{S \ldots S}_{j} \leftarrow X_1 \ldots X_n \underbrace{S \ldots S}_{j+1} \text{ with } \mathrm{j = k - n - 1 \geq 0}; \\ \text{(ii)} & X_1 \ldots X_n \underbrace{E \ldots E}_{i} T \underbrace{S \ldots S}_{j-1} \leftarrow X_1 \ldots X_n \underbrace{E \ldots E}_{i-1} T \underbrace{S \ldots S}_{j} \text{ with} \\ & \qquad\qquad\qquad\qquad i + j = k - n, \ i, j \geq 1 \ ; \\ \text{(iii)} & X_1 \ldots X_n \underbrace{E \ldots E}_{i+1} \leftarrow X_1 \ldots X_n \underbrace{E \ldots E}_{i} T \text{ with} \\ & \qquad\qquad\qquad\qquad i = k - n - 1 \geq 0, \end{array}$$

where $n \geq 1$ and $X_n \in \{T, Q\}$. This set of morphisms consists of the identity isomorphisms and all formal compositions of arrows of the type (2.2), which we shall refer to as the *elementary arrows* in $\mathfrak{A}(k;\varepsilon)$. Notice that if $\varepsilon_k = 1$, then $\mathfrak{A}(k;\varepsilon)$ has no morphisms except the identity isomorphisms.

If $W \leftarrow W'$ is an elementary arrow in $\mathfrak{A}(k;\varepsilon)$, then both $\alpha(W)$ and $\alpha(W')$ are defined and we have $\alpha(W) = \alpha(W') - 1$. Consider a nonoriented graph with vertices corresponding to all words in $\mathfrak{A}(k;\varepsilon)$ and edges corresponding to all elementary arrows. This graph is a tree (suppose that, for the contrary, there exists a cycle, and take a word W with the minimal value $\alpha(W)$ in this cycle; this leads to a contradiction, since any word in $\mathfrak{A}(k;\varepsilon)$ is the target of at most one elementary arrow). Thus, the category $\mathfrak{A}(k;\varepsilon)$ is actually a partially ordered set.

Let $\mathfrak{A}(k) = \mathfrak{A}(k;(0,\dots,0))$ be the category of words in the letters E, T, S satisfying (2.1)(i)-(ii). We display the categories $\mathfrak{A}(2)$ and $\mathfrak{A}(3)$,

$$
\begin{array}{llr}
(2.3) & \begin{array}{l} TE \leftarrow TT \leftarrow TS \\ EE \quad ET \end{array} & \mathfrak{A}(2)
\end{array}
$$

. . . .

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & TTE & TTT \arrow[l] & TTS \arrow[l] \arrow[d, equal] & \\
 & & TEE & TET \arrow[l] & TTS \arrow[l] & TSS \arrow[l] \\
ETE & ETT \arrow[l] & ETS \arrow[l] & & & \\
EEE & EET & & & &
\end{tikzcd}
\qquad \mathfrak{A}(3)
$$

Exercise for the reader : the number of words in $\mathfrak{A}(k)$ is equal to $3.2^{k-1} - 1$.

Let $\mathfrak{G}(k;\varepsilon)$ be the set of words of length k in the letters E, T, Q satisfying (2.1) (iii); the number of such words is equal to 2^r, where r is the number of zeroes among ε_i. We regard $\mathfrak{G}(k;\varepsilon)$ as a category with identity isomorphisms only.

There is an obvious inclusion functor $\mathfrak{G}(k;\varepsilon) \to \mathfrak{A}(k;\varepsilon)$ (but $\mathfrak{G}(k;\varepsilon)$ is not a full subcategory in $\mathfrak{A}(k;\varepsilon)$, since, for example, the words TE and TT of $\mathfrak{G}(2;(0,0))$ are linked by an arrow of the type (2.2) (iii) in $\mathfrak{A}(2)$).

For $W = X_1 \dots X_k \in \mathfrak{A}(k;\varepsilon)$ (resp. for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in \{0,1\}^k$) and n, with $0 \le n \le k$, let $W_{>n}$, $W_{\ge n}$, $W_{<n}$, $W_{\le n}$ (resp. $\varepsilon_{>n}$, etc) denote the words $X_{n+1} \dots X_k, X_n \dots X_k, X_1 \dots X_{n-1}, X_1 \dots X_n$ (resp. the tuples $(\varepsilon_{n+1}, \dots, \varepsilon_k)$, etc.).

Suppose that $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n, 1, 0, \dots, 0)$ with $0 \le n \le k-1$. By definition, one checks that

(i) $W \in \mathfrak{A}(k;\varepsilon)$ if and only if $W_{\le n} \in \mathfrak{G}(n;\varepsilon_{\le n})$ and $W_{>n} \in \mathfrak{A}(k-n;\varepsilon_{>n})$;

(ii) $W \leftarrow W'$ is an elementary arrow in $\mathfrak{A}(k;\varepsilon)$ if and only if $W_{>n} \leftarrow W'_{>n}$ is an elementary arrow in $\mathfrak{A}(k-n;\varepsilon_{>n})$.

Thus, we have an isomorphism of categories (throughout the paper, we will speak of an isomorphism of categories instead of equivalence if some categories are actually isomorphic) :

$$
(2.4) \qquad \mathfrak{A}(k;\varepsilon) \cong \mathfrak{G}(n;\varepsilon_{\le n}) \times \mathfrak{A}(k-n;\varepsilon_{>n}) \text{ with n} \le \text{k} - 1,
$$

$$\text{and } \varepsilon_{>\mathrm{n}} = (1, 0, \ldots, 0).$$

Given a word W of length less or equal to k, we let $\mathfrak{A}_W(k;\varepsilon)$ denote the full subcategory of $\mathfrak{A}(k;\varepsilon)$ of the words with the beginning equal to W (this set may be empty). According to (2.2), if $W \leftarrow W'$ is an elementary arrow in $\mathfrak{A}(k)$, then W and W' begin with the equal number of E (which may be equal to zero). Thus we obtain

$$\mathfrak{A}(k) \cong \mathfrak{A}_T(k) \coprod \mathfrak{A}_E(k) \quad \text{(disjoint union of categories)}, \tag{2.5}$$

and obviously

$$\mathfrak{A}_E(k) \cong \mathfrak{A}(k-1) \quad , \quad EW \leftrightarrow W \ ; \tag{2.6}$$

$$\mathfrak{A}_T(k) \cong \mathfrak{A}(k;(1,0,\ldots,0)) \quad , \quad TW \leftrightarrow QW. \tag{2.7}$$

With the isomorphisms (2.4)-(2.7), one can reduce certain questions for the category $\mathfrak{A}(k;\varepsilon)$ with an arbitrary ε to those for $\mathfrak{A}_T(k)$.

We let $\overline{\mathfrak{A}}(k;\varepsilon)$ denote the category $\mathfrak{A}(k;\varepsilon)$ with a unique zero object in addition. Say that a sequence in $\overline{\mathfrak{A}}(k;\varepsilon)$ is "long exact" (formal Schur complex) if it has the form

$$\begin{aligned} 0 \leftarrow WE\ldots E \leftarrow WE\ldots ET \leftarrow WE\ldots ETS \leftarrow \ldots \\ \ldots \leftarrow WETS\ldots S \leftarrow WTS\ldots S \leftarrow WS\ldots S \leftarrow 0 \end{aligned} \tag{2.8)(i}$$

where $W = X_1\ldots X_n$, with $1 \le n \le k-1$, and $X_n \in \{T,Q\}$. In particular, "short exact sequences" are those of the type

$$0 \leftarrow WE \leftarrow WT \leftarrow WS \leftarrow 0. \tag{ii}$$

Thus, all horizontal sequences in the picture (2.3) are "exact".

If $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_k)$, $\varepsilon' = (\varepsilon'_1,\ldots,\varepsilon'_k)$ and $\varepsilon \le \varepsilon'$, i.e. $\varepsilon_i \le \varepsilon'_i$ for each $i = 1,\ldots,k$, we define a functor $\overline{\mathfrak{A}}(k;\varepsilon) \to \overline{\mathfrak{A}}(k;\varepsilon')$ which takes zero object to zero object, a word $W = X_1\ldots X_k$ to zero object if, for some i, $\varepsilon'_i = 1$ and $X_i = S$ (for this we need to add zero objects), otherwise $W \mapsto W' = X'_1\ldots X'_k$ where $X'_i = X_i$ if $\varepsilon'_i = 0$ and $X'_i = Q$ if $\varepsilon'_i = 1$. It is easy to check that this functor is "exact", i.e., it takes "long exact sequences" in $\overline{\mathfrak{A}}(k;\varepsilon)$ to "long exact sequences" in $\overline{\mathfrak{A}}(k;\varepsilon')$. For example, the functor $\overline{\mathfrak{A}}(4;(0,0,0,0)) \to \overline{\mathfrak{A}}(4;(0,0,1,0))$ takes the "exact" sequences

$$0 \leftarrow TEEE \leftarrow TEET \leftarrow TETS \leftarrow TTSS \leftarrow TSSS \leftarrow 0$$

of $\overline{\mathfrak{A}}(4;(0,0,0,0))$ to the "exact" sequence

$$0 \leftarrow TEQE \leftarrow TEQT \leftarrow TEQS \leftarrow 0 \leftarrow 0 \leftarrow 0$$

in $\overline{\mathfrak{A}}(4;(0,0,1,0))$. We also define the map $\mathfrak{G}(k;\varepsilon) \to \mathfrak{G}(k;\varepsilon')$ by the same rule.

The categories $\overline{\mathfrak{A}}(k;\varepsilon)$ (resp. $\mathfrak{G}(k;\varepsilon)$) together with the functors defined above form a covariant k-dimensional cube which we denote by $\overline{\mathfrak{A}}_{(k)}$ (resp. $\mathfrak{G}_{(k)}$). Since the inclusion functors $\mathfrak{G}(k;\varepsilon) \to \mathfrak{A}(k;\varepsilon)$ commute with the functors $\overline{\mathfrak{A}}(k;\varepsilon) \to \overline{\mathfrak{A}}(k;\varepsilon')$ and $\mathfrak{G}(k;\varepsilon) \to \mathfrak{G}(k;\varepsilon')$, we have an inclusion functor of cubes $\mathfrak{G}_{(k)} \to \overline{\mathfrak{A}}_{(k)}$.

Let $\mathcal{M}$ be an exact category with a distinguished zero object $*$.

Let $\mathcal{A}(k;\varepsilon) = \text{Exact}(\overline{\mathfrak{A}}(k;\varepsilon),\mathcal{M})$ denote the category of "exact" functors and their natural transformations, i.e. those functors that take the zero object of $\overline{\mathfrak{A}}(k;\varepsilon)$ to $*$ and take "long exact sequences" in $\overline{\mathfrak{A}}(k;\varepsilon)$ to long exact sequences in $\mathcal{M}$. Since the category $\mathfrak{A}(k;\varepsilon)$ is a final partially ordered set, $\mathcal{A}(k;\varepsilon)$ is actually a category of diagrams of a certain type in $\mathcal{M}$ (with certain exactness conditions) and morphisms of such diagrams.

Put $\mathcal{A}(k) = \mathcal{A}(k;(0,\dots,0))$. For example, $\mathcal{A}(2)$ and $\mathcal{A}(3)$ are the categories of diagrams of the type depicted in (2.3) with objects of $\mathcal{M}$ instead of words, where all horizontal sequences are exact including exactness at the ends.

It can be checked that each $\mathcal{A}(k;\varepsilon)$ is an exact category in the obvious sense. The "exact" functors $\overline{\mathfrak{A}}(k;\varepsilon) \to \overline{\mathfrak{A}}(k;\varepsilon')$ with $\varepsilon \le \varepsilon'$ induce the exact functors $\mathcal{A}(k;\varepsilon) \leftarrow \mathcal{A}(k;\varepsilon')$. Thus, the categories $\mathcal{A}(k;\varepsilon)$ form a k-dimensional cube of exact categories which we denote by $\mathcal{A}_{(k)}$.

It is convenient to regard the objects of $\mathcal{A}(k;\varepsilon)$ as "exact" functors $F : \overline{\mathfrak{A}}(k) \to \mathcal{M}$ (via $\overline{\mathfrak{A}}(k) \to \overline{\mathfrak{A}}(k;\varepsilon)$) with the properties :

(2.9) (i) if $W = X_1 \dots X_k \in \mathfrak{A}(k)$ and, for some i, $\varepsilon_i = 1$ and $X_i = S$, then $F(W) = *$;

(ii) if $W = X_1 \dots X_k$ and $W' = X'_1 \dots X'_k \in \mathfrak{A}(k)$ do not satisfy (i) and for each i the condition $\varepsilon_i = 0$ implies $X_i = X'_i$, then $F(W) = F(W')$.

The morphisms are natural transformations of such functors which respect the condition (2.9) (ii).

In other words, the objects of $\mathcal{A}(k;\varepsilon)$ can be thought of as diagrams in $\mathcal{M}$ with the disposition of objects and arrows given by the category $\mathfrak{A}(k)$ (with the exactness condition for the corresponding sequences), where some objects are linked by the equality sign and some objects are equal to $*$ accordingly to (2.9); morphisms of such diagrams are defined naturally. Thus, $\mathcal{A}(k;\varepsilon)$ can be regarded as an exact subcategory (not full) in $\mathcal{A}(k)$; more generally, $\mathcal{A}(k;\varepsilon')$ is a subcategory in $\mathcal{A}(k;\varepsilon)$ if $\varepsilon \leq \varepsilon'$.

We also consider the exact categories $\mathcal{G}(k;\varepsilon) = \mathrm{Funct}(\mathfrak{G}(k;\varepsilon),\mathcal{M})$; $\mathcal{G}(k;\varepsilon) \cong \mathcal{M} \times \cdots \times \mathcal{M}$ where the number of copies is equal to 2^r, and r is the number of zeroes among ε_i. The categories $\mathcal{G}(k;\varepsilon)$ form a k-dimensional cube of exact categories $\mathcal{G}_{(k)}$, with the functors $\mathcal{G}(k;\varepsilon) \leftarrow \mathcal{G}(k;\varepsilon')$ for $\varepsilon \leq \varepsilon'$ induced by the functors $\mathfrak{G}(k;\varepsilon) \to \mathfrak{G}(k;\varepsilon')$ defined above.

It is convenient to think of an object of $\mathcal{G}(k;\varepsilon)$ as a 2^k-tuple of objects of the category $\mathcal{M}$ positioned at the vertices of a k-dimensional cube, with any two objects linked by an edge in the i-th direction with $\varepsilon_i = 1$ being equal (and morphisms of such tuples respect those equalities).

We have the exact functor of cubes $\mathcal{A}_{(k)} \to \mathcal{G}_{(g)}$ induced by $\mathfrak{G}_{(k)} \to \overline{\mathfrak{A}}_{(k)}$.

The map $\mathcal{A}_{(2)} \to \mathcal{G}_{(2)}$ can be depicted in the following fashion,

$$\mathcal{G}_{(2)} = \left[\begin{array}{ccc} \bullet\ \bullet & & \bullet{=}\bullet \\ \|\ \| & \leftarrow & \|\ \| \\ \bullet\ \bullet & & \bullet{=}\bullet \\ \downarrow & & \downarrow \\ \bullet\ \bullet & & \bullet{=}\bullet \\ & \leftarrow & \\ \bullet\ \bullet & & \bullet{=}\bullet \end{array}\right] \leftarrow \mathcal{A}_{(2)} = \left[\begin{array}{ccccccccccc} \bullet & \twoheadleftarrow & \bullet & \leftarrowtail & \bullet & & \bullet & = & \bullet & \leftarrow & * \\ \| & & \| & & & \leftarrow & \| & & \| & & \\ \bullet & & \bullet & & & & \bullet & = & \bullet & & \\ & \downarrow & & & & & & \downarrow & & & \\ \bullet & \twoheadleftarrow & \bullet & \leftarrowtail & \bullet & & \bullet & = & \bullet & \leftarrow & * \\ & & & & & \leftarrow & & & & & \\ \bullet & & \bullet & & & & \bullet & = & \bullet & & \end{array}\right]$$

where, for example,

$$\begin{array}{ccccc} \bullet & \twoheadleftarrow & \bullet & \leftarrowtail & \bullet \\ \| & & \| & & \\ \bullet & & \bullet & & \end{array}$$

means the category $\mathcal{A}(2;(1,0))$ whose objects, by definition, are all diagrams in $\mathcal{M}$ of the type

$$\begin{array}{ccccc} X_{QE} & \twoheadleftarrow & X_{QT} & \leftarrowtail & X_{QS} \\ \| & & \| & & \\ X_{QE} & & X_{QT} & & \end{array}$$

with X_{QE} (resp. X_{QT}) written twice at the positions of the words EE and TE of $\mathfrak{A}(2)$ (resp. ET and TT), and the map

$$\begin{array}{ccccc} \bullet & \twoheadleftarrow & \bullet & \leftarrowtail & \bullet \\ \| & & \| & & \\ \bullet & & \bullet & & \end{array} \quad \rightarrow \quad \begin{array}{ccccc} \bullet & \twoheadleftarrow & \bullet & \leftarrowtail & \bullet \\ & & & & \\ \bullet & & \bullet & & \end{array}$$

is the equalities forgetful functor

$$\mathcal{A}(2;(1,0)) \to \mathcal{A}(2;(0,0)).$$

Let $A(k;\mathcal{M})$ denote the multisimplicial set $C\mathcal{A}_{(k)}$ (*cf.* §1). One checks that $C\mathcal{G}_{(k)}$ is naturally isomorphic to the iterated G-construction $G^{(k)}\mathcal{M}$ (see [GG], sect. 6, or [G2], sect. 3). Hence, the map of cubes $\mathcal{A}_{(k)} \to \mathcal{G}_{(k)}$ induces the map of k-fold multisimplicial sets $A(k;\mathcal{M}) \to G^{(k)}\mathcal{M}$, and we formulate the main result of this section.

PROPOSITION 2.1. — *The map $A(k;\mathcal{M}) \to G^{(k)}\mathcal{M}$ is a homotopy equivalence.*

Proof. All functors in the cube $\mathcal{G}_{(k)}$ are easily seen to be cofinal, hence $\mathcal{G}_{(k)}$ is strictly dominant in each direction (*cf.* §1). We shall check in §3 that the cube $\mathcal{A}_{(k)}$ (and also some its generalizations $\mathcal{A}_{(k,\ell)}$) is strictly dominant in the first direction (see Proposition 3.2). Hence, by Corollary 1.8, it suffices to check the following

LEMMA 2.2. — *For every $\varepsilon \in \{0,1\}^k$, the map $S.\mathcal{A}(k;\varepsilon) \to S.\mathcal{G}(k;\varepsilon)$ is a homotopy equivalence.*

Proof. Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n, 1, 0, \dots, 0)$ with $0 \le n \le k-1$. In view of (2.4), we obtain

$$\mathcal{A}(k;\varepsilon) \cong \mathrm{Exact}(\mathfrak{G}(n;\varepsilon_{\le n}) \times \mathfrak{A}(k-n;\varepsilon_{>n}), \mathcal{M}) \cong$$
$$\mathrm{Funct}(\mathfrak{G}(n;\varepsilon_{\le n}), \mathrm{Exact}(\mathfrak{A}(k-n;\varepsilon_{>n}), \mathcal{M})) \cong \mathrm{Funct}(\mathfrak{G}(n;\varepsilon_{\le n}), \mathcal{A}(k-n;\varepsilon_{>n})),$$

since $\mathfrak{G}(n; \varepsilon_{\leq n})$ is a trivial category, and obviously

$$\mathcal{G}(k;\varepsilon) \cong \mathrm{Funct}(\mathfrak{G}(n;\varepsilon_{\leq n}), \mathcal{G}(k-n;\varepsilon_{>n})) \text{ with } \varepsilon_{>n} = (1,0,\dots,0).$$

It results that the case $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n, 1, 0, \dots, 0)$ is reduced to that of $\varepsilon_{>n} = (1,0,\dots,0)$. In view of (2.7), it is equivalent to the fact that $S.\mathcal{A}_T(n-k) \to S.\mathcal{G}_T(n-k)$ is a homotopy equivalence, where we put

$$\mathcal{A}_T(m) : \mathrm{Exact}(\mathfrak{A}_T(m), \mathcal{M}) \ , \ \mathcal{G}_T(m) = \mathrm{Funct}(\mathfrak{G}_T(m), \mathcal{M}).$$

If $\varepsilon = (0, \dots, 0)$, accordingly to (2.5), (2.6) we have

$$\begin{gathered}\mathcal{A}(k) = \mathcal{A}(k;\varepsilon) = \mathrm{Exact}(\mathfrak{A}_T(k) \coprod \mathfrak{A}_E(k), \mathcal{M}) \cong \\ \mathcal{A}_T(k) \times \mathcal{A}_E(k) \cong \mathcal{A}_T(k) \times \mathcal{A}(k-1),\end{gathered}$$

and also

$$\mathcal{G}(k) \cong \mathcal{G}_T(k) \times \mathcal{G}_E(k) \cong \mathcal{G}_T(k) \times \mathcal{G}(k-1).$$

Hence everything in this case is reduced as well to the case of $\mathcal{A}_T(k) \to \mathcal{G}_T(k)$ by induction.

Suppose that $k \geq 3$. We let $\mathfrak{A}'_T(k)$ denote the category obtained from $\mathfrak{A}_T(k)$ by replacing the word $TS\dots S$ by the word $TES\dots S$ ($TES\dots S \notin \mathfrak{A}_T(k)$ in view of (2.1) (ii)) together with the new elementary arrow $TETS\dots S \leftarrow TES\dots S$, the new "exact sequence" $TE\dots E \leftarrow \cdots \leftarrow TETS\dots S \leftarrow TES\dots S$ in exchange for $TE\dots E \leftarrow \cdots \leftarrow TETS\dots S \leftarrow TTS\dots S \leftarrow TS\dots S$, and the "admissible epimorphism" $TTS\dots S \to TES\dots S$. We denote by $\mathfrak{A}''_T(k)$ the same category without the "admissible epimorphism".

We put $\mathcal{A}'_T(k) = \mathrm{Exact}(\mathfrak{A}'_T(k), \mathcal{M})$, where exact functors are supposed to take the "epimorphism" $TTS\dots S \to TES\dots S$ to admissible epimorphisms in $\mathcal{M}$; $\mathcal{A}''_T(k) = \mathrm{Exact}(\mathfrak{A}''_T(k), \mathcal{M})$.

LEMMA 2.3. — *The category $\mathcal{A}'_T(k)$ is naturally equivalent to $\mathcal{A}_T(k)$.*

Proof. If $F \in \mathcal{A}_T(k)$, we define the functor $G : \mathfrak{A}'_T(k) \to \mathcal{M}$ by letting $G(W) = F(W)$ for $W \neq TES\dots S$ and
$G(TES\dots S) = \mathrm{coker}(F(TS\dots S) \rightarrowtail F(TTS\dots S))$. Then the epimorphism $G(TTS\dots S) = F(TTS\dots S) \twoheadrightarrow G(TES\dots S)$ is naturally defined and the sequence $0 \leftarrow G(TE\dots E) \leftarrow \cdots \leftarrow G(TETS\dots S) \leftarrow G(TES\dots S) \leftarrow 0$ is exact, hence $G \in \mathcal{A}'_T(k)$.

On the other hand, if $G \in \mathcal{A}'_T(k)$, we define the functor $F : \mathfrak{A}_T(k) \to \mathcal{M}$ by letting $F(W) = G(W)$ for $W \neq TS\ldots S$ and $F(TS\ldots S) = \ker(G(TTS\ldots S) \twoheadrightarrow G(TES\ldots S))$, under some previous choice of kernels and cokernels in $\mathcal{M}$. Then the monomorphism $F(TS\ldots S) \rightarrowtail F(TTS\ldots S) = G(TTS\ldots S)$ is naturally defined and the sequence

$$0 \leftarrow F(TE\ldots E) \leftarrow \cdots \leftarrow F(TETS\ldots S) \leftarrow$$
$$\leftarrow F(TTS\ldots S) \leftarrow F(TS\ldots S) \leftarrow 0$$

is exact, hence $F \in \mathcal{A}_T(k)$.

The correspondence $F \leftrightarrow G$ provides the desired exact equivalence □

The inclusion functor $\mathfrak{A}''_T(k) \to \mathfrak{A}'_T(k)$ induces the epimorphism forgetfull exact functor $\mathcal{A}'_T(k) \to \mathcal{A}''_T(k)$.

LEMMA 2.4. — *The map $S.\mathcal{A}'_T(k) \to S.\mathcal{A}''_T(k)$ is a homotopy equivalence.*

Proof. First we show the case $k = 3$. The category $\mathcal{A}'_T(3)$ (resp. $\mathcal{A}''_T(3)$) consists of all diagrams in $\mathcal{M}$ of the type

$$\begin{array}{ccccc} X(TTE) & \twoheadleftarrow & X(TTT) & \leftarrowtail & X(TTS) \\ & & & & \downarrow \\ X(TEE) & \twoheadleftarrow & X(TET) & \leftarrowtail & X(TES) \end{array}$$

with exact rows (resp. without the vertical epimorphism). By the additivity theorem, $S.\mathcal{A}'_T(3)$ (resp. $S.\mathcal{A}''_T(3)$) is homotopy equivalent to the direct product of $S.$ constructions of the subcategories of diagrams of the type

$$\begin{array}{ccc} X(TTE) = X(TTT) & \leftarrow & * \\ & & \downarrow \\ X(TEE) = X(TET) & \leftarrow & * \end{array} \qquad \left(\text{resp. } \begin{array}{ccc} X(TTE) = X(TTT) & \leftarrow & * \\ X(TEE) = X(TET) & \leftarrow & * \end{array}\right),$$

and

$$\begin{array}{c} * \twoheadleftarrow X(TTT) = X(TTS) \\ \downarrow \\ * \twoheadleftarrow X(TET) = X(TES) \end{array} \qquad \left(\text{resp. } \begin{array}{ccc} * & \twoheadleftarrow & X(TTT) = X(TTS) \\ * & \twoheadleftarrow & X(TET) = X(TES) \end{array}\right).$$

Hence, in order to check that the map $S.\mathcal{A}'_T(3) \to S.\mathcal{A}''_T(3)$ is a homotopy equivalence, it suffices to show the following

SUBLEMMA 2.5. — *Let* Epi$\mathcal{M}$ *denote the category whose objects are all admissible epimorphisms in* $\mathcal{M}$ *and morphisms are transformations of arrows, with the natural structure of exact category which admits an exact equivalence* Epi$\mathcal{M} \leftrightarrow \mathcal{E}(\mathcal{M})$, *where* $\mathcal{E}(\mathcal{M})$ *is the exact category of short exact sequences in* $\mathcal{M}$. *Then the map* $S.\mathrm{Epi}\mathcal{M} \to S.(\mathcal{M} \times \mathcal{M})$ *given by* $(B \twoheadrightarrow C) \mapsto (B, C)$ *is a homotopy equivalence.*

Proof of the sublemma. It is equivalent to say that the functor f : $\mathcal{E}(\mathcal{M}) \to \mathcal{M} \times \mathcal{M}$, $(A \rightarrowtail B \twoheadrightarrow C) \mapsto (B, C)$, induces a homotopy equivalence on the $S.$ constructions. By the additivity theorem (see the proof of Lemma 1.4.3 in [W2]), the functor $g : \mathcal{M} \times \mathcal{M} \to \mathcal{E}(\mathcal{M})$, $(B, C) \mapsto (B \rightarrowtail B \oplus C \twoheadrightarrow C)$, is a homotopy equivalence. Hence, it suffices to prove that $fg : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$, $(B, C) \mapsto (B \oplus C, C)$, is a homotopy equivalence. Let $t : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ denote the exact functor $(B, C) \mapsto (C, 0)$. The map $|S.t|$ has a homotopy inverse $-|S.t|$ in the sense of the H-space structure on $|S.(\mathcal{M} \times \mathcal{M})|$ (*cf.* Lemma 3.2 of [GG]). Then $|S.fg| = |S.(1 \oplus t)| \sim 1 + |S.t|$, and $1 - |S.t|$ is its homotopy inverse, since $t^2 = 0$ □

We shall do something similar for an arbitrary $k \geq 4$, and the reader is invited to depict a diagram which represents an object of $\mathcal{A}'_T(4)$.

Let $\mathcal{A}'_T(k)_{\leq 1}$ (resp. $\mathcal{A}''_T(k)_{\leq 1}$) be the full subcategory of $\mathcal{A}'_T(k)$ (resp. $\mathcal{A}''_T(k)$) defined by the condition : $X(W) = *$, if $\alpha(W) \geq 2$. The categories $\mathcal{A}'_T(k)_{\leq 1}$ and $\mathcal{A}''_T(k)_{\leq 1}$ are obviously isomorphic.

Let $\mathcal{A}'_T(k)_{\geq 1}$ (resp. $\mathcal{A}''_T(k)_{\geq 1}$) denote the full subcategory in $\mathcal{A}'_T(k)$ (resp. $\mathcal{A}''_T(k)$) defined by the conditions

(i) $X(W) = *$ if $\alpha(W) = 0$;

(ii) $X(WTT) = X(WTS)$ if WTT, $WTS \in \mathfrak{A}_T(k)$.

By the additivity theorem, $S.\mathcal{A}'_T(k) \sim S.\mathcal{A}'_T(k)_{\leq 1} \times S.\mathcal{A}'_T(k)_{\geq 1}$ and $S.\mathcal{A}''_T(k) \sim S.\mathcal{A}''_T(k)_{\leq 1} \times S.\mathcal{A}''_T(k)_{\geq 1}$ (we cut all exact sequences in the corresponding diagrams by the conditions $\alpha(W) \leq 1$ and $\alpha(W) \geq 1$).

One checks that the rules

$$X(W) = \begin{cases} Y(W_{\leq k-1}) & \text{if } \alpha(W) \geq 1 \\ * & \text{if } \alpha(W) = 0 \end{cases} \text{ and } Y(W) = \begin{cases} X(WT) & \text{if } \alpha(W) = 0 \\ X(WS) & \text{if } \alpha(W) \geq 1 \end{cases}$$

define an isomorphism of exact categories

$$\mathcal{A}'_T(k)_{\geq 1} \cong \mathcal{A}'_T(k-1) \text{ (resp. } \mathcal{A}''_T(k)_{\geq 1} \cong \mathcal{A}''_T(k-1)),\ X \leftrightarrow Y,$$

whence Lemma 2.4 follows by induction □

We continue to prove Lemma 2.2. It now suffices to show that the map $S.\mathcal{A}''_T(k) \to S.\mathcal{G}_T(k)$ is a homotopy equivalence. Any diagram representing an object of $\mathcal{A}''_T(k)$ consists of two unlinked diagrams with the objects of the type $X(TTW)$ and $X(TEW)$, respectively, and each of the two diagrams represents some object of $\mathcal{A}_T(k-1)$. Formally speaking, we have

$$\mathcal{A}''_T(k) \cong \mathcal{A}''_{TT}(k) \times \mathcal{A}''_{TE}(k) \cong \mathcal{A}_T(k-1) \times \mathcal{A}_T(k-1)$$

where the isomorphism $\mathcal{A}''_{TT}(k) \cong \mathcal{A}_T(k-1)$ (resp. $\mathcal{A}''_{TE}(k) \cong \mathcal{A}_T(k-1)$) is given by the rule $X(TTW) = Y(TW)$ (resp. $X(TEW) = Y(TW)$). Hence the map $\mathcal{A}''_T(k) \to \mathcal{G}_T(k)$ is isomorphic to the direct product map

$$\mathcal{A}''_{TT}(k) \times \mathcal{A}''_{TE}(k) \to \mathcal{G}_{TT}(k) \times \mathcal{G}_{TE}(k)$$

which, in its turn, is isomorphic to the map

$$\mathcal{A}_T(k-1) \times \mathcal{A}_T(k-1) \to \mathcal{G}_T(k-1) \times \mathcal{G}_T(k-1).$$

We complete the proof of Lemma 2.2 and Proposition 2.1 by the induction on k. It remains to check the base $k = 2$, i.e., that the functor $\mathcal{A}_T(2) \to \mathcal{G}_T(2)$,

$$(X(TE) \twoheadleftarrow X(TT) \leftarrowtail X(TS)) \mapsto (X(TE), X(TT))$$

is a homotopy equivalence on $S.$ constructions, which is true by Sublemma 2.5. □□

§3. The Chain of Maps Connecting $A(k;\mathcal{M})$ to $G(k;\mathcal{M})$.

3.1. **The Multisimplicial Sets** $A(k,\ell;\mathcal{M})$. — We give briefly some definitions similar to those of §2 and refer there for more detailed account.

For $\ell = 1, \dots, k$ and $\varepsilon \in \{0,1\}^k$, we let $\mathfrak{A}(k,\ell;\varepsilon)$ denote the set of words of length k in the letters E, T, S, Q satisfying (2.1)(i)-(iii) and the condition

(3.1) if $W = X_1 \dots X_k$ and $X_i = T$ fore some i, with $\ell \le i < k$, then $X_{i+1} = S$ (and, consequently, accordingly to (2.1)(i),

$$X_{i+2} = \cdots = X_k = S.$$

We shall regard $\mathfrak{A}(k,\ell;\varepsilon)$ as a full subcategory in $\mathfrak{A}(k;\varepsilon)$ (see §2); we call a sequence in $\mathfrak{A}(k,\ell;\varepsilon)$ "exact" if and only if it is "exact" in $\mathfrak{A}(k;\varepsilon)$.

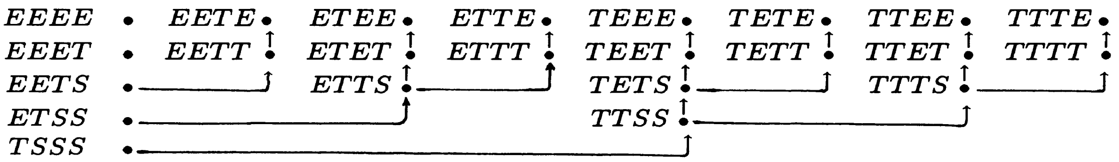
Obviously, $\mathfrak{A}(k,k;\varepsilon) = \mathfrak{A}(k;\varepsilon)$ and $\mathfrak{A}(k,\ell;\varepsilon) \subset \mathfrak{A}(k,\ell';\varepsilon)$ if $\ell \leq \ell'$. The "exact" functor $\overline{\mathfrak{A}}(k;\varepsilon) \to \overline{\mathfrak{A}}(k;\varepsilon')$, with $\varepsilon \leq \varepsilon'$, induces the "exact" functor $\overline{\mathfrak{A}}(k,\ell;\varepsilon) \to \overline{\mathfrak{A}}(k,\ell;\varepsilon')$, where we denote by a bar the same category with a unique zero object in addition.

We put $\mathcal{A}(k,\ell;\varepsilon) = \mathrm{Exact}(\mathfrak{A}(\mathrm{k},\ell;\varepsilon),\mathcal{M})$. Let $\mathcal{A}_{(k,\ell)}$ denote the k-dimensional cube formed by the exact categories $\mathcal{A}(k,\ell;\varepsilon)$ with $\varepsilon \in [1]^k$. The inclusions $\mathfrak{A}(k,\ell-1;\varepsilon) \subset \mathfrak{A}(k,\ell;\varepsilon)$ induce the exact functor of cubes p_ℓ : $\mathcal{A}_{(k,\ell)} \to \mathcal{A}_{(k,\ell-1)}$, $\ell = 2,\ldots,k$.

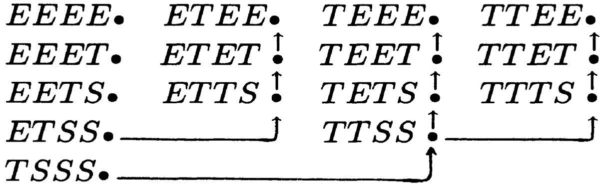
We put $A(k,\ell;\mathcal{M}) = C\mathcal{A}_{(k,\ell)}$ (see §1). Thus, we obtain a sequence of k-fold multisimplicial sets
$A(k;\mathcal{M}) = A(k,k;\mathcal{M}) \to A(k,k-1;\mathcal{M}) \to \cdots \to A(k,1;\mathcal{M})$, which is the main subject for study in the rest of section 3.1.

We depict the passage from $\mathfrak{A}(4,4)$ to $\mathfrak{A}(4,1)$, where $\mathfrak{A}(k,\ell)$ stands for $\mathfrak{A}(k,\ell;(0,\ldots,0))$. It looks like one cuts off branches of a tree beginning with the upper ones. The reader is invited to depict $\mathfrak{A}(4,\ell;\varepsilon)$, with $\ell = 4,3,2,1$ and some $\varepsilon \neq (0,0,0,0)$.

(3.1)



$\mathfrak{A}(4) = \mathfrak{A}(4,4)$



$\mathfrak{A}(4,3)$

$$\begin{array}{llcccc}
EEEE\bullet & TEEE\bullet & \qquad & EEEE\bullet & \qquad & \alpha(W)=0 \\
EEET\bullet & TEET\bullet & & EEET\bullet & & 1 \\
EETS\bullet & TETS\bullet & & EETS\bullet & & 2 \\
ETSS\bullet & TTSS\bullet & & ETSS\bullet & & 3 \\
TSSS\bullet & & & TSSS\bullet & & 4 \\
 & & & & & \\
\mathfrak{A}(4,2) & & & \mathfrak{A}(4,1) & &
\end{array}$$

All branches of these trees are "exact sequences", *cf.* (2.8).

Let $\ell \geq 2$, and let $\mathcal{B}(k,\ell;\varepsilon)$ denote the full subcategory of $\mathcal{A}(k,\ell;\varepsilon)$ whose objects are those $F \in \mathcal{A}(k,\ell;\varepsilon)$ with the property : if $W \in \mathfrak{A}(k,\ell-1;\varepsilon)$ then $F(W) = *$. We denote by i_ℓ the inclusion functor $\mathcal{B}(k,\ell;\varepsilon) \to \mathcal{A}(k,\ell;\varepsilon)$ and define an exact functor $q_\ell : \mathcal{A}(k,\ell;\varepsilon) \to \mathcal{B}(k,\ell;\varepsilon)$ by the rule : for $W \in \mathfrak{A}(k,\ell;\varepsilon)$ and $F \in \mathcal{A}(k,\ell;\varepsilon)$

$$q_\ell F(W) = \begin{cases} *, & \text{if } W \in \mathfrak{A}(k,\ell-1;\varepsilon) \\ \operatorname{coker}(F(W') \to F(W)), & \text{if } W \notin \mathfrak{A}(k,\ell-1;\varepsilon) \\ & \text{and the origin } W' \text{ of the unique elementary} \\ & \text{arrow } W' \to W \text{ belongs to } \mathfrak{A}(k,\ell-1;\varepsilon), \\ & \text{i.e. } \alpha(W) = k-\ell+1 \\ F(W) & \text{otherwise (i.e., if } \alpha(W) \leq k-\ell \text{ and} \\ & W \notin \mathfrak{A}(k,\ell-1;\varepsilon)). \end{cases}$$

We also define an exact functor $j_\ell : \mathcal{A}(k,\ell-1;\varepsilon) \to \mathcal{A}(k,\ell;\varepsilon)$ by the rule : for $F \in \mathcal{A}(k,\ell-1;\varepsilon)$ and $W \in \mathfrak{A}(k,\ell;\varepsilon)$

$$j_\ell F(W) = \begin{cases} F(W), & \text{if } W \in \mathfrak{A}(k,\ell-1;\varepsilon) \\ F(W'), & \text{if } W \notin \mathfrak{A}(k,\ell-1;\varepsilon) \text{ and the origin } W' \\ & \text{of the unique elementary arrow } W' \to W \\ & \text{belongs to } \mathfrak{A}(k,\ell-1;\varepsilon), \text{ i.e.} \\ & \alpha(W) = k-\ell+1 \\ *, & \text{otherwise (i.e. } \alpha(W) \leq k-\ell \text{ and} \\ & W \notin \mathfrak{A}(k,\ell-1;\varepsilon)). \end{cases}$$

Notice that the case $\varepsilon_k = 1$ is exceptional. In this case $\alpha(W)$ is not defined, there are no elementary arrows, the categories $\mathcal{A}(k,\ell;\varepsilon)$ and $\mathcal{B}(k,\ell;\varepsilon)$ are the direct products of some copies of $\mathcal{M}$, and we have obviously

$$\mathcal{A}(k,\ell;\varepsilon) \cong \mathcal{A}(k,\ell-1;\varepsilon) \times \mathcal{B}(k,\ell;\varepsilon).$$

The exact functors i_ℓ, j_ℓ, p_ℓ, q_ℓ give rise to the maps of cubes

$$\mathcal{B}_{(k,\ell)} \rightleftarrows \mathcal{A}_{(k,\ell)} \rightleftarrows \mathcal{A}_{(k,\ell-1)}$$

which we denote by the same letters.

PROPOSITION 3.1. — *a) For every k and ℓ, with $2 \leq \ell \leq k$, we have a short exact sequence of endofunctors of the cube $\mathcal{A}_{(k,\ell)}$,*

$$0 \longrightarrow j_\ell p_\ell \longrightarrow 1 \longrightarrow i_\ell q_\ell \longrightarrow 0;$$

b) If $\varepsilon, \varepsilon' \in \{0,1\}^k$ are such that $\varepsilon_\ell = 0$, $\varepsilon'_\ell = 1$, and $\varepsilon_i = \varepsilon'_i$ for $i \neq \ell$, then the map $\mathcal{A}(k,\ell;\varepsilon') \longrightarrow \mathcal{A}(k,\ell;\varepsilon)$ in the cube $\mathcal{A}_{(k,\ell)}$ induces an exact equivalence of categories $\mathcal{B}(k,\ell;\varepsilon') \longrightarrow \mathcal{B}(k,\ell;\varepsilon)$.

Proof a) One checks by definition that for every $\varepsilon \in \{0,1\}^k$ the same sequence of endofunctors of $\mathcal{A}(k,\ell;\varepsilon)$ is exact.

b) We consider the following cases.

1. If $\varepsilon_{\ell-1} = \varepsilon'_{\ell-1} = 1$, then $\mathfrak{A}(k,\ell;\varepsilon) = \mathfrak{A}(k,\ell-1;\varepsilon)$ and $\mathfrak{A}(k,\ell;\varepsilon') = \mathfrak{A}(k,\ell-1;\varepsilon')$, hence both $\mathcal{B}(k,\ell;\varepsilon)$ and $\mathcal{B}(k,\ell;\varepsilon')$ are trivial.

2. If $\ell \leq k-1$, $\varepsilon_{\ell-1} = \varepsilon'_{\ell-1} = 0$, and $\varepsilon_i = \varepsilon'_i = 0$ for every $i > \ell$, then we have a one-to-one correspondence

$$\begin{gathered}
\mathfrak{A}(k,\ell;\varepsilon) - \mathfrak{A}(k,\ell-1;\varepsilon) \xrightarrow{\sim} \mathfrak{A}(k,\ell;\varepsilon') - \mathfrak{A}(k,\ell-1;\varepsilon'), \\
X_1 \dots X_{\ell-2} TTS \dots S \leftrightarrow X_1 \dots X_{\ell-2} TQS \dots S \\
X_1 \dots X_{\ell-2} TEX_{\ell+1} \dots X_k \leftrightarrow X_1 \dots X_{\ell-2} TQX_{\ell+1} \dots X_k,
\end{gathered}$$

which gives rise to an exact isomorphism of categories $\mathcal{B}(k,\ell;\varepsilon') \longrightarrow \mathcal{B}(k,\ell;\varepsilon)$.

3. If $\ell \leq k-1$, $\varepsilon_{\ell-1} = \varepsilon'_{\ell-1} = 0$, and $\varepsilon_i = \varepsilon'_i = 1$ for some $i > \ell$, then the same argument with the bijection $\mathfrak{A}(k,\ell;\varepsilon) - \mathfrak{A}(k,\ell-1;\varepsilon) \xrightarrow{\sim} \mathfrak{A}(k,\ell;\varepsilon') - \mathfrak{A}(k,\ell-1;\varepsilon')$ given by

$$X_1 \dots X_{\ell-2} TEX_{\ell+1} \dots X_k \leftrightarrow X_1 \dots X_{\ell-2} TQX_{\ell+1} \dots X_k$$

gives the desired.

4. If $\ell = k$ and $\varepsilon_{\ell-1} = \varepsilon'_{\ell-1} = 0$, then $\mathcal{B}(k,\ell;\varepsilon)$ is a direct product of the categories of diagrams of the form $F(X_1 \dots TT) \xrightarrow{\sim} F(X_1 \dots TE)$

(two-term exact sequences), where the categories correspond bijectively to all elementary arrows $X_1 \dots TT \to X_1 \dots TE$ in $\mathfrak{A}(k,k;\varepsilon) - \mathfrak{A}(k,k-1;\varepsilon)$. On the other hand, $\mathcal{B}(k,\ell;\varepsilon')$ is a direct product of copies of $\mathcal{M}$ which are in one-to-one correspondence with all words $X_1 \dots TQ$ in $\mathfrak{A}(k,k;\varepsilon') - \mathfrak{A}(k,k-1;\varepsilon')$. The functor $\mathcal{B}(k,\ell;\varepsilon') \to \mathcal{B}(k,\ell;\varepsilon)$ takes an object $F \in \mathcal{B}(k,\ell;\varepsilon')$ to $G \in \mathcal{B}(k,\ell;\varepsilon)$ defined by the rule : the two-term exact sequence $G(X_1 \dots TT) \xrightarrow{\sim} G(X_1 \dots TE)$ coincides with $F(X_1 \dots TQ) \xrightarrow{1} F(X_1 \dots TQ)$. Hence this functor is an equivalence □

A good exercise illustrating Proposition 3.1 is to depict the cubes $\mathcal{A}_{(3,3)} \to \mathcal{A}_{(3,2)} \to \mathcal{A}_{(3,1)}$ and compute explicitly the cubes $\mathcal{B}_{(3,3)}$ and $\mathcal{B}_{(3,2)}$.

We notice that, for example, the composite map $\mathcal{A}(3,3) \to \mathcal{A}(3,1)$ does not admit a "kernel" category with such properties as those of $\mathcal{B}(k,\ell)$ with respect to the map $p_\ell : \mathcal{A}(k,\ell) \to \mathcal{A}(k,\ell-1)$. That is why we cannot prove directly, with the technique of §1, that the map $A(k,k;\mathcal{M}) \to A(k,1;\mathcal{M})$ is a homotopy equivalence and need the filtration $\{A(k,\ell;\mathcal{M})\}$.

PROPOSITION 3.2. — *For every k and ℓ, with $1 \le \ell \le k$, the cube $\mathcal{A}_{(k,\ell)}$ is strictly dominant in the first direction.*

Proof. We have to show that for every $\varepsilon \in \{0,1\}^{k-1}$, the functor $\mathcal{A}(k,\ell;(1,\varepsilon)) \to \mathcal{A}(k,\ell;(0,\varepsilon))$ is strictly dominant. We consider the following cases.

1. If $\ell = 1$ and $\varepsilon \neq (0,\dots,0)$, then we have a natural bijection $\mathfrak{A}(k,1;(0,\varepsilon)) \xrightarrow{\sim} \mathfrak{A}(k,1;(1,\varepsilon))$, $EW \leftrightarrow QW$. Therefore, the functor $\mathcal{A}(k,1;(1,\varepsilon)) \to \mathcal{A}(k,1;(0,\varepsilon))$ is an isomorphism of categories.

2. If $2 \le \ell \le k$ and $\varepsilon \neq (0,\dots,0)$, then $\mathfrak{A}(k,\ell;(0,\varepsilon))$ is the disjoint union of the subcategories $\mathfrak{A}_E(k,\ell;(0,\varepsilon))$ and $\mathfrak{A}_T(k,\ell;(0,\varepsilon))$, which consist of words beginning with E (resp. T). Each of the two subcategories admits an isomorphism to $\mathfrak{A}(k,\ell;(1,\varepsilon))$ given by $EW \leftrightarrow QW$ (resp. $TW \leftrightarrow QW$). Thus, the functor $\mathcal{A}(k,\ell;(1,\varepsilon)) \to \mathcal{A}(k,\ell;(0,\varepsilon)) \cong \mathcal{A}_E(k,\ell;(0,\varepsilon)) \times \mathcal{A}_T(k,\ell;(0,\varepsilon))$ is of the type of a diagonal functor $\mathcal{A} \to \mathcal{A} \times \mathcal{A}$, which is strictly dominant by Proposition 1.11.

3. If $\varepsilon = (0,\dots,0)$, then for every $\ell = 1,\dots,k$ the category $\mathfrak{A}(k,\ell) = \mathfrak{A}(k,\ell;(0,\varepsilon))$ is the disjoint union of the categories $\mathfrak{A}_i(k,\ell)$, where for $i = 0,1,\dots,k$, we denote by $\mathfrak{A}_i(k,\ell)$ the full subcategory of $\mathfrak{A}(k,\ell)$ which consists of all words of the form $E \dots E X_{i+1} \dots X_k$ with $X_{i+1} \neq E$ (some of $\mathfrak{A}_i(k,\ell)$ consist of one word only). Thus, $\mathcal{A}(k,\ell)$ is a direct product of the correspondent categories $\mathcal{A}_i(k,\ell)$. By virtue of Proposition 1.11, it suffices to show that

for each $i = 0, 1, \ldots, k$, the functor $\mathcal{A}(k,\ell;(1,\varepsilon)) \to \mathcal{A}_i(k,\ell)$ induced by the inclusion $\mathfrak{A}_i(k,\ell) \to \mathfrak{A}(k,\ell;(1,\varepsilon))$ given by

$$\begin{aligned} TX_2 \ldots X_k &\mapsto QX_2 \ldots X_k \ , \ \text{if i} = 0 \\ EX_2 \ldots X_k &\mapsto QX_2 \ldots X_k \ , \ \text{if } 1 \leq \text{i} \leq \text{k}, \end{aligned}$$

is strictly dominant. We claim that, in fact, all those functors are surjective. This is equivalent to the fact that given a part of the diagram corresponding to $\mathfrak{A}_i(k,\ell)$, we can expand it to the entire diagram corresponding to $\mathfrak{A}(k,\ell;(1,\varepsilon))$ in such a manner that all exactness conditions hold, i.e., to expand the diagram from a branch to the whole tree (*cf.* picture (3.1)). The last assertion is obvious, which completes the proof □.

PROPOSITION 3.3. — *For every k, ℓ, with $2 \leq \ell \leq k$, the map $A(k,\ell;\mathcal{M}) \to A(k,\ell-1;\mathcal{M})$ is a homotopy equivalence.*

Proof. By virtue of Corollary 1.7, it suffices to show that the maps $S.\mathcal{A}_{(k,\ell)}(\varepsilon_1 = 1) \to S.\mathcal{A}_{(k,\ell-1)}(\varepsilon_1 = 1)$ and $S.\mathcal{A}_{(k,\ell)}(\varepsilon_1 = 0) \to S.\mathcal{A}_{(k,\ell-1)}(\varepsilon_1 = 0)$ are homotopy equivalences. We treat the case $\varepsilon_1 = 1$, the case $\varepsilon_1 = 0$ is similar.

Consider the diagram

(3.2)

$$\begin{array}{ccccc} S.\mathcal{A}_{(k,\ell)}(\varepsilon_1 = 1, \varepsilon_\ell = 1) & \to & S.\mathcal{A}_{(k,\ell)}(\varepsilon_1 = 1, \varepsilon_\ell = 0) & \to & S.\mathcal{A}_{(k,\ell)}(\varepsilon_1 = 1) \\ \downarrow & & \downarrow & & \downarrow \\ S.\mathcal{A}_{(k,\ell-1)}(\varepsilon_1 = 1, \varepsilon_\ell = 1) & \to & S.\mathcal{A}_{(k,\ell-1)}(\varepsilon_1 = 1, \varepsilon_\ell = 0) & \to & S.\mathcal{A}_{(k,\ell-1)}(\varepsilon_1 = 1) \end{array}$$

where horizontal sequences are fibrations by Lemma 4.1 of [G3].

By virtue of the additivity theorem (Proposition 1.9) and Proposition 3.1 a), we have the homotopy equivalences

$$\begin{aligned} S.\mathcal{A}_{(k,\ell)}(\varepsilon_1 = 1, \varepsilon_\ell = 1) &\sim S.\mathcal{A}_{(k,\ell-1)}(\varepsilon_1 = 1, \varepsilon_\ell = 1) \times S.\mathcal{B}_{(k,\ell)}(\varepsilon_1 = 1, \varepsilon_\ell = 1), \\ S.\mathcal{A}_{(k,\ell)}(\varepsilon_1 = 1, \varepsilon_\ell = 0) &\sim S.\mathcal{A}_{(k,\ell-1)}(\varepsilon_1 = 1, \varepsilon_\ell = 0) \times S.\mathcal{B}_{(k,\ell)}(\varepsilon_1 = 1, \varepsilon_\ell = 0). \end{aligned}$$

The map of cubes $\mathcal{B}_{(k,\ell)}(\varepsilon_\ell = 1) \to \mathcal{B}_{(k,\ell)}(\varepsilon_\ell = 0)$ is an exact equivalence by Proposition 3.1 b), hence the map

$$S.\mathcal{B}_{(k,\ell)}(\varepsilon_1 = 1, \varepsilon_\ell = 1) \to S.\mathcal{B}_{(k,\ell)}(\varepsilon_1 = 1, \varepsilon_\ell = 0)$$

is a homotopy equivalence. Therefore, the diagram (3.2) results with the desired homotopy equivalence $S.\mathcal{A}_{(k,\ell)}(\varepsilon_1 = 1) \to S.\mathcal{A}_{(k,\ell-1)}(\varepsilon_1 = 1)$ □

We notice that, in general, the cube $\mathcal{B}_{(k,\ell)}$ is not strictly dominant in the first direction. Hence, we cannot use the additivity theorem for C-construction in the above proof.

3.2. *The Map* $\operatorname{Diag} A(k,1;\mathcal{M}) \to G(k;\mathcal{M})$. — Let $\mathcal{E}(k)$ denote the exact category of exact sequences in $\mathcal{M}$ of length k, i.e. those of the form $0 \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_k \leftarrow 0$, and let $F_k : \mathcal{E}(k) \to \mathcal{M}^{k+1}$ be the arrows forgetful functor. We denote by $\mathcal{D}_n(k)$ (resp. $\mathcal{E}_n(k)$) the n-dimensional cube with the category $\mathcal{M}^{k+1}$ at the vertex $(0,\dots,0)$ and $\mathcal{E}(k)$ at all other vertices (resp. $\mathcal{E}(k)$ at all vertices). Its structural functors coincide with F_k and the identity functor of $\mathcal{E}(k)$ (resp. the identity functor of $\mathcal{E}(k)$). We set

$$D_n(k) = C\mathcal{D}_n(k)\ ,\ E_n(k) = C\mathcal{E}_n(k).$$

We shall define simplicial maps

$$\operatorname{Diag} D_n(k) \to D_1(k) \text{ and } \operatorname{Diag} \mathrm{E_n(k)} \to \mathrm{E_1(k)}$$

which coincide on vertices with the identity maps $\mathcal{M}^{k+1} \to \mathcal{M}^{k+1}$ and $\mathcal{E}(k) \to \mathcal{E}(k)$, respectively, where Diag stands for the total diagonal functor. In order to do this, we should be given a choice of cokernels for all admissible monomorphisms in $\mathcal{M}$.

First we define the maps $\operatorname{Diag} D_2(k) \to D_1(k)$ and $\operatorname{Diag} E_2(k) \to E_1(k)$ by an argument similar to that in sect. 4.2 of [N].

A (p,p)-simplex in $D_2(k)$ is determined by the data : some objects $X_{i,j}$ of $\mathcal{M}^{k+1}$ with $0 \le i,j \le p$; admissible monomorphisms equipped with a choice of cokernels, $X_{i,j} \rightarrowtail X_{i,j'} \twoheadrightarrow X_{i,j'/j}$ and $X_{i,j} \rightarrowtail X_{i',j} \twoheadrightarrow X_{i'/i,j}$ for every $i < i', j < j'$; the induced monomorphisms of cokernels with a choice of their cokernels $X_{i'/i,j} \rightarrowtail X_{i'/i,j'} \twoheadrightarrow X_{i'/i,j'/j}$ and $X_{i,j'/j} \rightarrowtail X_{i',j'/j} \twoheadrightarrow X_{i'/i,j'/j}$, where all X with at least one fractional index are actually objects of $\mathcal{E}(k)$.

We take the diagonal objects $X_{i,i}$, $0 \le i \le p$, from the above data, and for any $i < j$, we associate with the admissible monomorphism $X_{i,i} \rightarrowtail X_{j,j}$ its cokernel $X_{j,j} \twoheadrightarrow X_{(j,j)/(i,i)}$ in $\mathcal{M}^{k+1}$ according to the given choice of cokernels in $\mathcal{M}$. This cokernel stands in the following natural diagram with exact rows and columns

$$\begin{array}{ccccc}
X_{i,j/i} & \rightarrowtail & X_{j,j/i} & \twoheadrightarrow & X_{j/i,j/i} \\
\| & & \uparrow & & \uparrow \\
X_{i,j/i} & \rightarrowtail & X_{(j,j)/(i,i)} & \twoheadrightarrow & X_{j/i,j} \\
 & & \uparrow & & \uparrow \\
 & & X_{j/i,i} & = & X_{j/i,i}
\end{array}$$

which is obviously a pull-back. Since all terms except the central one are objects of $\mathcal{E}(k)$, we obtain easily that $X_{(j,j)/(i,i)}$ can be regarded as an object of $\mathcal{E}(k)$ in a natural way. Thus, we have associated with an arbitrary p-simplex in $\operatorname{Diag} D_2(k)$ a p-simplex in $D_1(k)$ determined by the data $(X_{ii} \rightarrowtail X_{jj} \twoheadrightarrow X_{(j,j)/(i,i)})$, with $0 \leq i < j \leq p$. This correspondence is evidently compatible with face and degeneracy maps, hence we obtain a simplicial map $\operatorname{Diag} D_2(k) \to D_1(k)$. The map $\operatorname{Diag} E_2(k) \to E_1(k)$ is defined similarly but easier.

Now suppose that $n \geq 3$ and let $\mathcal{X}$ denote either $\mathcal{D}_n(k)$ or $\mathcal{E}_n(k)$. For $\varepsilon \in \{0,1\}^{n-2}$ we denote by $\mathcal{X}(\varepsilon)$ the two-dimensional face

$$\begin{bmatrix} \mathcal{X}(\varepsilon,0,1) & \leftarrow & \mathcal{X}(\varepsilon,1,1) \\ \downarrow & & \downarrow \\ \mathcal{X}(\varepsilon,0,0) & \leftarrow & \mathcal{X}(\varepsilon,1,0) \end{bmatrix}$$

of the cube $\mathcal{X}$. Then for every $r_1, \ldots, r_{n-2} \geq 0$ and $p \geq 0$, we have

$$\begin{aligned}
C_{r_1 \ldots r_{n-2} pp} \mathcal{X} \quad = \quad & C_{r_1 \ldots r_{n-2}}([C_{pp} \mathcal{X}(\varepsilon)]_{\varepsilon \in \{0,1\}^{n-2}}) \\
& \downarrow \\
& C_{r_1 \ldots r_{n-2}}([C_p \operatorname{diag} \mathcal{X}(\varepsilon)]_{\varepsilon \in \{0,1\}^{n-2}}) = \\
& = C_{r_1 \ldots r_{n-2} p}(\operatorname{diag}_{n-1,n} \mathcal{X})
\end{aligned}$$

where $[C_{pp}\mathcal{X}(\varepsilon)]_{\varepsilon\in\{0,1\}^{n-2}}$ and $[C_p \operatorname{diag} \mathcal{X}(\varepsilon)]_{\varepsilon\in\{0,1\}^{n-2}}$ are the $(n-2)$-dimensional cubes formed by the categories $C_{pp}\mathcal{X}(\varepsilon)$ (resp. $C_p \operatorname{diag} \mathcal{X}(\varepsilon)$) with the obvious functors, the vertical map is induced by the maps $C_{pp}\mathcal{X}(\varepsilon) \to C_p \operatorname{diag} \mathcal{X}(\varepsilon)$ defined above, and $\operatorname{diag}_{n-1,n} \mathcal{X}$ denotes the $(n-1)$-dimensional subcube of $\mathcal{X}$ given by $\varepsilon_{n-1} = \varepsilon_n$. Thus, we obtain a simplicial map

$$\operatorname{diag}_{n-1,n} C\mathcal{X} \to C \operatorname{diag}_{n-1,n} \mathcal{X}.$$

This enables us to define a map $\operatorname{Diag} C\mathcal{X} \to C \operatorname{Diag} \mathcal{X}$ by the induction on dimension. Since $\mathcal{X} = \mathcal{D}_n(k)$ or $\mathcal{E}_n(k)$, we obtain the maps $\operatorname{Diag} D_n(k) \to D_1(k)$ and $\operatorname{Diag} E_n(k) \to E_1(k)$.

Now let consider the cube $\mathcal{A}_{(k,1)}$. The categories $\mathcal{A}(k,1;\varepsilon)$ can be computed easily.

If $\varepsilon = (0,\dots,0)$, then $\mathfrak{A}(k,1;\varepsilon)$ consists of the words $EE\dots E, E\dots ET$, $E\dots TS,\dots, ETS\dots S, TS\dots S$; no two of these words are connected by an elementary arrow. Thus $\mathcal{A}(k,1) = \mathcal{A}(k,1;(0,\dots,0)) \cong \mathcal{M}^{k+1}$.

If $\varepsilon \neq (0,\dots,0)$, we let $r(\varepsilon)$ denote the maximal index i with $\varepsilon_i = 1$. Then all words in $\mathfrak{A}(k,1;\varepsilon)$ begin with the same word W of length $r(\varepsilon)$ in the letters E and Q accordingly to $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for $i = 1,\dots,r(\varepsilon)$. Those words form a "long exact sequence" $0 \leftarrow WE\dots E \leftarrow WE\dots ET \leftarrow \cdots \leftarrow WTS\dots S \leftarrow WS\dots S \leftarrow 0$ of length $k - r(\varepsilon) + 1$ (if $r(\varepsilon) = k$, then $\mathfrak{A}(k,1;\varepsilon)$ consists of a unique word which we regard as an exact sequence of length 1 with the identity isomorphism). Hence, we have $\mathcal{A}(k,1;\varepsilon) \cong \mathcal{E}(k - r(\varepsilon) + 1)$. In particular, $\mathcal{A}(k,1;(1,0,\dots,0)) \cong \mathcal{E}(k)$.

Thus, we can regard the cube $\mathcal{A}_{(k,1)}$ as a subcube of $\mathcal{D}_k(k)$ via the inclusions $\mathcal{A}(k,1;\varepsilon) \hookrightarrow \mathcal{E}(k)$ given for $\varepsilon \neq (0,\dots,0)$ by

$$F \mapsto (0 \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_{k-r(\varepsilon)+1} \leftarrow \underbrace{0 \leftarrow \cdots \leftarrow 0}_{r(\varepsilon)-1} \leftarrow 0)$$

where $X_i = F(W_i)$, and W_i is the unique word in $\mathfrak{A}(k,1;\varepsilon)$ with $\alpha(W_i) = i$, and via the isomorphism $\mathcal{A}(k,1) \xrightarrow{\sim} \mathcal{M}^{k+1}$. Hence we obtain the composite map $\operatorname{Diag} A(k,1;\mathcal{M}) \to \operatorname{Diag} D_k(k) \to D_1(k)$ with $D_1(k) \cong G(k;\mathcal{M})$, since $D_1(k) = [\mathcal{E}(k) \to \mathcal{M}^{k+1}]$, (*cf.* [N], sect. 2.2); the same simplicial set was called $0|SF_n$, with $n = k+1$, in [G1], sect. 8.

PROPOSITION 3.4. — *The map* $\operatorname{Diag} A(k,1;\mathcal{M}) \to G(k;\mathcal{M})$ *is a homotopy equivalence.*

In order to prove the proposition, we construct a section up to homotopy for the map in question and show that this section is a homotopy equivalence.

Let $\overline{G}(k;\mathcal{M})$ denote $G(k;\mathcal{M})$ regarded as a k-fold multisimplicial set, with all but the first directions being trivial. We denote by $\overline{\mathcal{A}}_{(k,1)}$ the k-dimensional cube with the functor $\mathcal{E}(k) \to \mathcal{M}^{k+1}$ at the edge $(1,0,\dots,0) \to (0,\dots,0)$ and the trivial category $*$ at the other vertices, and put $\overline{A}(k,1;\mathcal{M}) = C\overline{\mathcal{A}}_{(k,1)}$. Then we have an obvious inclusion $\overline{G}(k;\mathcal{M}) \to \overline{A}(k,1;\mathcal{M})$. Its image consists of the simplices given by those diagrams in which all isomorphisms in the directions other than the first are actually identities. One checks that the map $\overline{G}(k;\mathcal{M}) \to \overline{A}(k,1;\mathcal{M})$ is a homotopy equivalence by the argument similar to Corollary (2) of Lemma 1.4.1 in [W2].

Further, we have a natural inclusion $\overline{\mathcal{A}}_{(k,1)} \to \mathcal{A}_{(k,1)}$ which gives rise to the homotopy equivalence $\overline{A}(k,1;\mathcal{M}) \to A(k,1;\mathcal{M})$ by virtue of Corollary 1.8. One checks by definition that the composite map $|G(k;\mathcal{M})| = |\overline{G}(k;\mathcal{M})| \to |\overline{A}(k,1;\mathcal{M})| \to |A(k,1;\mathcal{M})| = |\operatorname{Diag} A(k,1;\mathcal{M})| \to |G(k;\mathcal{M})|$ is homotopic to the identity map (notice that the composite map is not equal to the identity, since the choice of cokernels in a given simplex of $G(k;\mathcal{M})$ does not coincide in general with the distinguished choice of cokernels in $\mathcal{M}$). Hence the proposition is proved. □

Propositions 3.3 and 3.4 result with a homotopy equivalence $\operatorname{Diag} A(k;\mathcal{M}) \to G(k;\mathcal{M})$ which depends up to simplicial homotopy on the choice of cokernels in the category $\mathcal{M}$.

§4. The Map $\Lambda^k : \mathrm{Sub}_k\, G\mathcal{M} \to A(k;\mathcal{M})$

We give a brief account of operations on an exact category $\mathcal{M}$ and refer to [G2], sect. 7, [G3], sect. 2, or [N], §3, for more details.

We suppose that for every $X_1,\dots,X_n \in \mathcal{M}$, a tensor product $X_1 \otimes \cdots \otimes X_n$ is determined in $\mathcal{M}$. For any admissible filtration $X_1 \rightarrowtail \cdots \rightarrowtail X_n$, we let an exterior product $X_1 \wedge \ldots \wedge X_n$ be determined. We prefer cosymmetric powers to symmetric ones and suppose that for every $X \in \mathcal{M}$ and $n \geq 1$, a cosymmetric power $X \circ \cdots \circ X$ (n copies) is determined in $\mathcal{M}$. In case of modules a cosymmetric (=divided) power is nothing but the invariants of symmetric group action on the corresponding tensor power. We notice that all the same can be done by means of symmetric powers and Schur complexes instead of cosymmetric powers and coSchur complexes.

Besides certain properties of naturality, we assume that the following three properties of these operations hold

(4.1) (i) tensor product is exact in each variable;

(ii) the multilinearity property of exterior products, i.e., given an admissible filtration $U \rightarrowtail \cdots \rightarrowtail V \rightarrowtail W' \rightarrowtail W \rightarrowtail X \rightarrowtail \cdots \rightarrowtail Y$, we have the exact sequence

$$0 \to U \wedge \ldots \wedge V \wedge W' \wedge X \wedge \ldots \wedge Y \to U \wedge \cdots \wedge V \wedge W \wedge X \wedge \ldots \wedge Y \to$$
$$\to U \wedge \ldots \wedge V \otimes W/W' \wedge X/W' \wedge \ldots \wedge Y/W' \to 0;$$

(iii) given an admissible filtration $X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n$, we have the exact sequence (coSchur complex)

$$0 \to X_0 \circ \cdots \circ X_0 \to X_1 \circ \cdots \circ X_1 \to (X_1/X_0) \otimes X_2 \circ \cdots \circ X_2$$
$$\to (X_1/X_0) \wedge (X_2/X_0) \otimes X_3 \circ \cdots \circ X_3 \to \cdots \to (X_1/X_0) \wedge \cdots \wedge (X_{n-1}/X_0) \otimes X_n$$
$$\to (X_1/X_0) \wedge \cdots \wedge (X_n/X_0) \to 0$$

where all terms contain n factors.

We suppose that for any admissible monomorphism $X \rightarrowtail Y$, a cokernel $Y \twoheadrightarrow Y/X$ is chosen in the category $\mathcal{M}$.

We recall that given a simplicial set X, its k-fold subdivision $\mathrm{Sub}_k X$ is a k-fold multisimplicial set with

$$(\mathrm{Sub}_k X)[P_1, \ldots, P_k] = X[P_1 \ldots P_k] \text{ for every } P_1, \ldots, P_k \in \Delta,$$

where $P_1 \ldots P_k$ is the concatenation of finite totally ordered sets, i.e. the disjoint union with each element of P_i declared to be less than each element of P_j if $i < j$ (*cf.* [G2], sect. 4). There is a natural homeomorphism of geometric realizations $|\mathrm{Sub}_k X| \cong |X|$.

We want to define a map $\Lambda^k : \mathrm{Sub}_k G\mathcal{M} \to A(k; \mathcal{M})$. Suppose we are given a simplex $x \in (\mathrm{Sub}_k G\mathcal{M})[P_1, \ldots, P_k] = G\mathcal{M}[P_1 \ldots P_k] = \mathrm{Exact}([\mathrm{Ar}(P_1 \ldots P_k) \to \mathrm{Ar}(BP_1 \ldots P_k)], [\mathcal{M} \xrightarrow{\mathrm{diag}} \mathcal{M} \times \mathcal{M}])$, *cf.* §1. Then we have to define its image $\Lambda^k x$ in $A(k; \mathcal{M})[P_1, \ldots, P_k] = C\mathcal{A}_{(k)}[P_1, \ldots, P_k] = \mathrm{Exact}(\Gamma(P_1, \ldots, P_k), \mathcal{A}_{(k)})$ (*cf.* §1), i.e., for every $\varepsilon \in \{0,1\}^k$ we have to define an exact functor

$$(\Lambda^k x)(\varepsilon) : \Gamma(P_1, \ldots, P_k; \varepsilon) \to \mathcal{A}(k, \varepsilon)$$

and those functors with different ε should be compatible.

Recall that $\mathcal{A}(k; \varepsilon) = \mathrm{Exact}(\mathfrak{A}(k; \varepsilon), \mathcal{M})$. Thus, given a collection $(i_1/j_1, \ldots, i_k/j_k) \in \Gamma(P_1, \ldots, P_k; \varepsilon)$ and a word $W \in \mathfrak{A}(k; \varepsilon)$, we have to define an object of $\mathcal{M}$

$$(4.2) \qquad (\Lambda^k x)(\varepsilon)(i_1/j_1, \ldots, i_k/j_k)(W)$$

and check the exactness condition in each i_m/j_m and in W. We define the required object by the following procedure.

Let $W = X_1 \dots X_k$ and $\{n_1 < \dots < n_r\} = \{n | X_n \in \{T, Q\}\}$. We divide the word W into the parts $W = W_0 W_1 \dots W_r$, where $W_0 = E \dots E$, $W_t = X_{n_t} E \dots E$ if $1 \le t \le r-1$, and W_r is equal to $X_{n_r} E \dots E$ or to $X_{n_r} S \dots S$. The number of E or S in each part may be equal to zero. Consider the intervals of the natural series $I_0 = (1, n_1 - 1), I_1 = (n_1, n_2 - 1), \dots, I_r = (n_r, k)$ which correspond to the distribution of indices between the words $W_0, \dots, W_r$; notice that $I_0 = \emptyset$ if $n_1 = 1$. We subdivide each of I_t, $0 \le t \le r$, into subintervals $I_{t,0}, I_{t,1}, \dots, I_{t,s(t)}$ according to the rule : let $\{n_{t,1} < \dots < n_{t,s(t)}\} = \{n \in I_t \mid j_n \neq B\}$ and $I_{t,0} = (n_t, n_{t,1} - 1), I_{t,1} = (n_{t,1}, n_{t,2} - 1), \dots, I_{t,s(t)} = (n_{t,s(t)}, n_{t+1} - 1)$ where we set $n_0 = 1$, $n_{r+1} = k+1$, and $n_{t,s(t)+1} = n_{t+1}$. Notice that $I_{t,0}$ is empty if $n_{t,1} = n_t$, and it is the case if $X_{n_t} = Q$.

We now associate an object $M_x(I_{t,p})$ of $\mathcal{M}$ with each nonempty interval $I_{t,p}$, with $0 \le t \le r$, $0 \le p \le s(t)$, according to the rule :

$$M_x(I_{0,0}) = \mathrm{pr}_1 x(i_1/B) \wedge \dots \wedge \mathrm{pr}_1 x(i_{n_{0,1}-1}/B) \text{ if } n_{0,1} > 1.$$

We use pr_1 and pr_2 to denote the projections $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and write simply $x(i/j)$ instead of $\mathrm{pr}_1 x(i/j) = \mathrm{pr}_2 x(i/j)$ if $j \neq B$. Notice that $j_n = B$ for $n \in I_{0,0}$;

$$M_x(I_{0,p}) = x(i_{n_{0,p}}/j_{n_{0,p}}) \wedge x(i_{n_{0,p}+1}/j_{n_{0,p}}) \wedge \cdots \wedge x(i_{n_{0,p+1}-1}/j_{n_{0,p}})$$

if $1 \le p \le s(0)$. We have $j_{n_{0,p}} \neq B$ by definition ;

$M_x(I_{t,0}) = \mathrm{pr}_2 x(i_{n_t}/B) \wedge \dots \wedge \mathrm{pr}_2 x(i_{n_{t,1}-1}/B)$ if $1 \le t \le r-1$ and $n_{t,1} > n_t$. In this case $j_n = B$ for each $n \in I_{t,0}$;

$$M_x(I_{t,p}) = x(i_{n_{t,p}}/j_{n_{t,p}}) \wedge x(i_{n_{t,p}+1}/j_{n_{t,p}}) \wedge \cdots \wedge x(i_{n_{t,p+1}-1}/j_{n_{t,p}})$$

if $1 \le t \le r-1$ and $1 \le p \le s(t)$;

if $W_r = X_{n_r} E \dots E$ then $M_x(I_{r,0}), M_x(I_{r,p})$ are defined according to the same rule as $M_x(I_{t,0})$, $M_x(I_{t,p})$ with $1 \le t \le r-1$;

suppose $W_r = X_{n_r} S \dots S$ and consider the subcases :

if $j_n \neq B$ for some $n > n_r$, then we associate the distinguished zero object $*$ with entire interval I_r ;

if $j_n = B$ for every n , with $n_r \le n \le k$, then $I_{r,0} = I_r$ and we put $M_x(I_r) = \mathrm{pr}_2 x(i_{n_r}/B) \circ \cdots \circ \mathrm{pr}_2 x(i_{n_r}/B)$ $(k - n_r + 1$ copies$)$;

if $j_{n_r} \neq B$ and $j_n = B$ for $n > n_r$, then $I_{r,0} = \emptyset$, $I_{r,1} = I_r$, and we put

$$M_x(I_r) = \frac{\mathrm{pr}_2 x(i_{n_r}/B) \circ \cdots \circ \mathrm{pr}_2 x(i_{n_r}/B)}{\mathrm{pr}_2 x(j_{n_r}/B) \circ \cdots \circ \mathrm{pr}_2 x(j_{n_r}/B)} \ (k - n_r + 1 \text{ copies})$$

accordingly to the given choice of cokernels in $\mathcal{M}$.

We now define the object (4.2) to be equal to the tensor product of all $M_x(I_{t,p})$ written in the natural order. In the case $M_x(I_r) = *$ we assume that the tensor product is also equal to $*$.

We check the exactness in W. Given a tuple $(i_1/j_1, \ldots, i_k/j_k) \in \Gamma(P_1, \ldots, P_k; \varepsilon)$ and a "long exact sequence"

$$0 \to W'S \ldots S \to W'TS \ldots S \to \cdots \to W'E \ldots ET \to W'E \ldots E \to 0$$

in $\overline{\mathfrak{A}}(k; \varepsilon)$ with $W' = X_1 \ldots X_m$, $X_m \in \{T, Q\}$ (*cf.* (2.8)), we have to show that the corresponding objects in $\mathcal{M}$ form an exact sequence. We consider the following cases.

1. If $j_m \neq B$ and $j_n = B$ for every $n > m$, then we obtain the sequence

$$\begin{aligned}
&0 \to M \otimes \frac{\mathrm{pr}_2 x(i_m/B) \circ \cdots \circ \mathrm{pr}_2 x(i_m/B)}{\mathrm{pr}_2 x(j_m/B) \circ \cdots \circ \mathrm{pr}_2 x(j_m/B)} \to \\
&\to M \otimes x(i_m/j_m) \otimes \mathrm{pr}_2 x(i_{m+1}/B) \circ \cdots \circ \mathrm{pr}_2 x(i_{m+1}/B) \to \\
&\to M \otimes x(i_m/j_m) \wedge x(i_{m+1}/j_m) \otimes \mathrm{pr}_2 x(i_{m+2}/B) \circ \cdots \circ \mathrm{pr}_2 x(i_{m+2}/B) \to \cdots \\
&\cdots \to M \otimes x(i_m/j_m) \wedge \cdots \wedge x(i_{k-1}/j_m) \otimes \mathrm{pr}_2 x(i_k/B) \to \\
&\to M \otimes x(i_m/j_m) \wedge \cdots \wedge x(i_k/j_m) \to 0,
\end{aligned}$$

where M is the tensor product corresponding to W'. In view of (4.1) (i), its exactness is equivalent to (4.1) (iii) for the filtration

$$\mathrm{pr}_2 x(j_m/B) \rightarrowtail \mathrm{pr}_2 x(i_m/B) \rightarrowtail \mathrm{pr}_2 x(i_{m+1}/B) \rightarrowtail \cdots \rightarrowtail \mathrm{pr}_2 x(i_k/B).$$

2. If $j_n = B$ for every $n \geq m$, then the corresponding sequence is exact by virtue of (4.1) (iii) for the filtration

$$0 \rightarrowtail \mathrm{pr}_2 x(i_m/B) \rightarrowtail \mathrm{pr}_2 x(i_{m+1}/B) \rightarrowtail \cdots \rightarrowtail \mathrm{pr}_2 x(i_k/B).$$

3. Let $\max\{t | j_t \neq B\} = n > m$. Then the first $(n - m)$ terms in the sequence are equal to $*$ and the rest are equal to those corresponding to the

sequence of words
$0 \to W''S\ldots S \to W''TS\ldots S \to \cdots \to W''E\ldots ET \to W''E\ldots E \to 0$, where $W'' = W'E\ldots ET$ is a word of length n. Hence we are reduced to the case 1.

Using the multilinearity property (4.1) (ii), we check immediately the exactness of (4.2) in each i_m/j_m, $1 \le m \le k$, as well as the compatibility of $(\Lambda^k x)(\varepsilon)$ with different ε. Thus, we have defined the map

$$\Lambda^k : \operatorname{Sub}_k G\mathcal{M} \to A(k;\mathcal{M}). \tag{4.3}$$

Composing it with the map $A(k;\mathcal{M}) \to G^{(k)}\mathcal{M}$ (*cf.* §2), we obtain the map Λ^k of Grayson. In order to prove this, it suffices to check that if $W \in \mathfrak{G}(k;\varepsilon)$, i.e., W does not contain S, then the definition of the object (4.2) is equivalent to that from [G2, sect. 7]. The verification is trivial.

We claim further that the composition of the total diagonal of the map (4.3) with the map $\operatorname{Diag} A(k;\mathcal{M}) \to G(k;\mathcal{M})$ defined in §3 is nothing but the map $\Lambda^k : \operatorname{Diag}\operatorname{Sub}_k G\mathcal{M} \to G(k;\mathcal{M})$ constructed in [N]. Notice that the two maps under comparison are defined up to a choice of cokernels of admissible monomorphisms in $\mathcal{M}$. It suffices to check coincidence of the two maps on vertices. For given a simplex of high dimension in the domain, its image under each of the maps is given by a certain diagram in which all arrows are uniquely determined by those in the given simplex according to the naturality properties of the operations, and by a given choice of cokernels in $\mathcal{M}$.

One checks by the construction of the map $\operatorname{Diag} A(k;\mathcal{M}) \to G(k;\mathcal{M})$ (*cf.* §3) that it takes a vertex $x \in \operatorname{Diag} A(k;\mathcal{M})[0] = \mathcal{A}(k)$ to the collection $(x(E\ldots E), x(E\ldots ET), x(E\ldots ETS), \ldots, x(ETS\ldots S), x(TS\ldots S)) \in G(k;\mathcal{M})[0]$. Given a vertex $y \in (\operatorname{Sub}_k G\mathcal{M})[0,\ldots,0] = G\mathcal{M}[k-1]$, we obtain under the map (4.3) a vertex x in $A(k;\mathcal{M})$ such that

$$\begin{aligned}
&x(E\ldots E) = \mathrm{pr}_1 y(0/B) \wedge \mathrm{pr}_1 y(1/B) \wedge \ldots \wedge \mathrm{pr}_1 y((k-1)/B)\\
&x(E\ldots ET) = \mathrm{pr}_1 y(0/B) \wedge \ldots \wedge \mathrm{pr}_1 y((k-2)/B) \otimes \mathrm{pr}_2 y((k-1)/B)\\
&x(E\ldots ETS) = \mathrm{pr}_1 y(0/B) \wedge \ldots \wedge \mathrm{pr}_1 y((k-3)/B) \otimes \mathrm{pr}_2 y((k-2)/B) \circ \mathrm{pr}_2 y((k-2)/B)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&x(ETS\ldots S) = \mathrm{pr}_1 y)(0/B) \otimes \mathrm{pr}_2 y(1/B) \circ \cdots \circ \mathrm{pr}_2 y(1/B)\\
&x(TS\ldots S) = \mathrm{pr}_2 y(0/B) \circ \cdots \circ \mathrm{pr}_2 y(0/B)
\end{aligned}$$

which coincides with the definition of the map Λ^k in [N], sect. 4.1. This completes the proof of the Main Theorem.

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