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HODGE-COMPONENTS OF CYCLIC HOMOLOGY FOR AFFINE QUASI-HOMOGENEOUS HYPERSURFACES

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0. Introduction

Let K be a field of characteristic zero, $R = K[X_1, \ldots, X_N]$ and the hypersurface $A = R/(F(X_1, \ldots, X_N))$ for a reduced polynomial $F \in R$. In [5] M. Gerstenhaber and S. Schack obtained a Hodge-decomposition of the Hochschild homology of commutative K-algebras A, where K is a field of characteristic zero. Using recent results by J. Majadas and A. Rodicio [12] and A. Lago and A. Rodicio [10], we are able to express the Hodge-components $HH_n^{(n-i)}(A, A)$ in terms of torsion submodules of exterior powers of $\Omega^1_{A/K}$, the module of Kaehler differentials. We find using the convention that $n + k = 2i$:

$$HH_n^{(i)}(A, A) \simeq \begin{cases} T(\Omega^k_{A/K}) & \text{for } i \neq n \ ; \\ \Omega^n_{A/K} & \text{for } i = n \ . \end{cases}$$

Moreover by Lemma 1, a straightforward generalization of theorem 4 in [14], we see $T(\Omega^k_{A/K}) = 0$ unless $codim\ Sing(A) < i < N + 1$. In [11] J.L.

Loday obtained a Hodge-decomposition of cyclic homology. We give an explicit formula for the Hodge-components of cyclic homology for reduced affine hypersurfaces defined by a quasi-homogeneous polynomial with an isolated singularity at the origin. Let $n + k = 2i$. Then in Theorems 2 and 3 we compute the Hodge-components of cyclic homology of a reduced hypersurface defined by a quasi-homogeneous polynomial. We get:

$$HC_n^{(i)} = \begin{cases} 0 & \text{for : } k \neq N-1 \ ; \\ T(\Omega^{N-1}_{A/K}) \simeq \Omega^N_{A/K} & \text{for } k = N-1 \ . \end{cases}$$

Thus I extend results of S. Geller, L. Reid and C. Weibel for curves [3] and of M.Vigué-Poirrier [15] for curves and surfaces defined by quasi-homogeneous polynomials with an isolated singularity at the origin. In [15] the author proves that for the reduced cyclic homology $\widetilde{HC_n} = \Omega^N_{A/K}$ or 0. In the last section we apply our results to $A-D-E$ singularities (c.f. [7] for a definition) and compute the dimension of the non-zero torsion submodules of some exterior power of the module of Kaehler differentials.

At this point the author would like to thank the organizers of the conference for providing me with an opportunity to present my work that is part of my Ph.D thesis at UC Berkeley under the supervision of Prof. M. Wodzicki. I also would like to thank Prof. C. Weibel and the referee for many helpful suggestions. After giving my talk I received a preprint by S. Geller and C. Weibel [4] that also computes the dimension of the Hodge-components of cyclic homology for hypersurfaces defined by a homogeneous polynomial.

There also is a paper by the Buenos-Aires group BACH in Advances of Mathematics [1] that contains some of my results using different methods.

1. Hodge-components of Hochschild homology

In [5] Corollary to Theorem 3.1 M. Gerstenhaber and S. Schack prove the following decomposition of $HH_n(A, A) := HH_n$ into A-modules for arbitrary commutative K-algebras A:

$$HH_n \simeq HH_n^{(1)} \oplus HH_n^{(2)} \oplus \ldots \oplus HH_n^{(n)}, \qquad with \quad HH_n^{(n)} = \Omega^n_{A/K}.$$

Theorem 1: *Let $n + k = 2i$. The Hodge-components of the Hochschild-homology modules of a reduced hypersurface A are given by:*

$$HH_n^{(i)}(A, A) \simeq \begin{cases} T(\Omega^k_{A/K}) & for\ i \neq n\ ; \\ \Omega^n_{A/K} & for\ i = n\ . \end{cases}$$

Proof: By M. Gerstenhaber and S.D Schack [5] p.231 we know that

$$HH_n^{(n-j)}(A, A) \simeq H_j(\bigwedge^{n-j} L_{A/K}),$$

where $L_{A/K}$ is the cotangent complex as defined in L. Illusie's book [8] II.1.2.3.1 p.123. By [8] Proposition III. 3.3.6. we have the following isomorphism in the derived category:

$$L_{A/K} \simeq (0 \to (F)/(F^2) \to \Omega^1_{R/K} \otimes_R A \to 0),$$

where $\Omega^1_{R/K} \otimes_R A$ sits in degree 0. For a definition of the terminology we refer to [8]. By [9] p.278 Corollaire 2.1.2.2 we have a canonical isomorphism in the derived category:

$$\bigwedge L_{A/K}[-*] = Kos^*(F/F^2 \xrightarrow{\phi} A \otimes \Omega^1_{R/K})$$

where $Kos^*(F/F^2 \xrightarrow{\phi} A \otimes \Omega^1_{R/K})$ denotes the Koszul complex of the A-module morphism $\phi : F/F^2 \to A \otimes \Omega^1_{R/K}$, given by: $\phi(F + (F^2)) = 1 \otimes_A dF$. By taking into account the shifting of degrees, we get:

$$H_j(\bigwedge^{n-j} L_{A/K}) \simeq H^{n-2j}(\bigwedge^{n-j} L_{A/K})[-n+j].$$

Moreover by a standard result of L. Illusie [9] p.278 corollaire 2.1.2.2., we have an isomorphism in the derived category

$$(\bigwedge^{n-j} L_{A/K})[-n+j] \sim Kos^*(\phi))_{n-j}.$$

So we get:

$$H^{n-2j}(\bigwedge^{n-j} L_{A/K})[-n+j] = H^{n-2j}(Kos^*(\phi)_{n-j}) =$$

$$H^{n-2j}((F^j)/(F^{j+1}) \otimes_A \Omega^{n-2j}_{R/K} \to (F^{j-1})/(F^j) \otimes_A \Omega^{n-2j+1}_{R/K}).$$

To compute

$$H^{n-2j}((F^j)/(F^{j+1}) \otimes_A \Omega^{n-2j}_{R/K} \to (F^{j-1})/(F^j) \otimes_A \Omega^{n-2j+1}_{R/K})$$

one considers the following filtration of the complex $\Omega^*_{R/K}$:

$$F^s\Omega^m_{R/K} = \begin{cases} I^{s-m}\Omega^m_{R/K} & \text{if } s > m \text{ ;} \\ \Omega^m_{R/K} & \text{otherwise .} \end{cases}$$

where I denotes the ideal (F) generated by our polynomial $F \in R$. The differential of the complex $\frac{F^{n-j}\Omega^*_{R/K}}{F^{n-j+1}\Omega^*_{R/K}}$ is given by:

$$\delta_{n-j} : \frac{I^j}{I^{j+1}} \otimes_R \Omega^{n-2j}_{R/K} \to \frac{I^{j-1}}{I^j} \otimes_R \Omega^{n-2j+1}_{R/K}$$

where:

$$\bar{F}^j \otimes dX_{i_1} \wedge \cdots \wedge dX_{i_{n-2j}} \to$$

$$\sum_{k=n-2j+1}^{N} (-1)^{n-2j} \overline{(\frac{\partial F}{\partial X_{i_k}})} \bar{F}^{j-1} \otimes dX_{i_1} \wedge \cdots \wedge dX_{i_{n-2j}} \wedge dX_{i_k}.$$

Here $i_j \in 1, \ldots, N$, and $\overline{(\frac{\partial F}{\partial X_{i_k}})}$ is the reduction of $\frac{\partial F}{\partial X_{i_k}} mod(F)$. Using that notation we have seen:

$$H^{n-2j}(Kos^*(\phi)_{n-j}) \simeq H^{n-2j}(\frac{F^{n-j}\Omega^*_{R/K}}{F^{n-j+1}\Omega^*_{R/K}}).$$

Note that

$$A \otimes_R \Omega^{n-j}_{R/K} \simeq \frac{F^{n-j}\Omega^*_{R/K}}{F^{n-j+1}\Omega^*_{R/K}}.$$

We always have an exact sequence:

$$\Omega^{n-2j-1}_{A/K} \to^{\theta_{n-2j-1}} A \otimes \Omega^{n-2j}_{R/K} \to^{\pi} \Omega^{n-2j}_{A/K} \to 0,$$

where (c.f. [12] p.375) $\theta_n : \Omega^n_{A/K} \to A \otimes \Omega^{n+1}_{R/K}$ is defined by:

$$\theta_n(da_1 \wedge \ldots \wedge da_n) = (1/n!) 1 \otimes dr_1 \wedge \ldots \wedge dr_n \wedge dF,$$

and the elements $r_i \in R$ are chosen such that $a_i = r_i + (F)$ for all $i = 1, \dots N$. Splicing two of these sequences together and using Theorem 4.11 of [12] we get:

$$H^{n-2j}\left(\frac{F^{n-j}\Omega^*_{R/K}}{F^{n-j+1}\Omega^*_{R/K}}\right) \simeq \frac{Ker\delta_{n-j+1}}{Im\delta_{n-j}} \simeq Ker\theta_{n-2j} \simeq T(\Omega^{n-2j}_{A/K}).$$

This completes the proof of Theorem 1.

Using a standard localization argument we generalize Theorem 4 of U. Vetter [14] as follows:

Lemma 1: *Let A be a reduced K-algebra and let $Sing(A)$ denote the singular locus of A, then*

$$T(\Omega^k_{A/K}) = 0 \quad if \quad codimSing(A) > inf(k, dimA).$$

Applying the above Lemma to a reduced hypersurface A with an isolated singularity at the origin:

$$T(\Omega^i_{A/K}) = 0, \qquad \text{for} \quad k \neq \quad N-1, N.$$

2. Hodge-components of cyclic homology

In [11] J.L. Loday obtained the following decomposition of cyclic homology: The cyclic homology decomposes as follows:

$$HC_n = HC_n^{(1)} \oplus HC_n^{(2)} \oplus \ldots \oplus HC_n^{(n)}$$

and there is a long exact $S - B - I$-sequence:

$$(1): \quad \cdots \to HH_n^{(i)} \to HC_n^{(i)} \to HC_{n-2}^{(i-1)} \to HH_{n-1}^{(i)} \to \cdots .$$

This result was proved by J.L. Loday in [11] using a double complex, which we henceforth denote by $\bar{B}(C)^{(i)}$. For a definition we refer to [11]. If we discard the first i rows of zeroes and filter $\bar{B}C^{(i)}$ by its columns, we obtain a spectral sequence $\{{}^{(i)}E^r\}_r$ such that the E^2 - part computes the iterated homology corresponding to taking homology first with respect to b, and then with respect to the induced B and its abutment is $HC_n^{(i)} = H_{n-2i}(Tot\bar{B}(C)^{(i)})$. The E^1 -components are given by: $E^1_{p,q} = HH^{(i-p)}_{i-p+q}$ and the E^2 -components are given by:

$$E^2_{p,q} = \begin{cases} \dfrac{HH^{(i)}_{i+q}}{B_{i+q-1}(HH^{(i-1)}_{i+q-1})} & p = 0\,; \\[2ex] \dfrac{Ker(B_{i-p+q}: HH^{(i-p)}_{i-p+q} \to HH^{(i-p+1)}_{i-p+q+1})}{B_{i-p+q-1}(HH^{(i-p-1)}_{i-p+q-1})} & 0 < p\,. \end{cases}$$

Theorem 2: *For $n \leq N = codim($ singular locus$)$, we compute*

$HC_n(A)$, for arbitrary reduced hypersurfaces A with an isolated singularity at the origin: For $n \le N$ and $n + k = 2i$ we have:

$$HC_n^{(i)} \simeq \begin{cases} \frac{\Omega^n_{A/K}}{d\Omega^{n-1}_{A/K}} & \text{if } i = n \ ; \\ H^k_{dR}(A) & \text{for: } n/2 \le i < n \ ; \\ 0 & \text{otherwise} \ . \end{cases}$$

Proof: This follows from Theorem 1 and Lemma 1 using the spectral sequence ${}^{(i)}E^r$. At the E^1 level everything is zero except the lines $p + q = 2i - N$ and $p + q = 2i - N + 1$ and the row $q = i$.

The following result is well-known (see e.g. [13]):

Lemma 2: *As always K is a field of characteristic zero. Let A be a finitely generated quasi-homogeneous K-algebra, then:*

$$\mathbf{H}^i_{dR}(A) = 0 \qquad for \quad i > 0 \quad and \quad \mathbf{H}^0_{dR}(A) = K.$$

3. Cyclic homology of quasi-homogeneous hypersurfaces

From now on, we assume that our reduced, affine hypersurface with an isolated singularity at the origin is defined by a quasi-homogeneous polynomial. Recall that if F is a nonzero polynomial then (c.f. E. Brieskorn and H. Knoerrer in [2]): *F is quasi-homogeneous with weights λ_i if it satisfies* *the generalized Euler equation:*

$$\sum_{i=1}^{N} \lambda_i X_i \frac{\partial F}{\partial X_i} = nF.$$

We note that such hypersurfaces A are augmented, graded algebras. By T.Goodwillie [6] we have:

$$HC_{2l+1}(A) = \widetilde{HC}_{2l+1}(A) \quad for \quad l \geq 1.$$

$$HC_{2l}(A) \simeq \widetilde{HC}_{2l}(A) \oplus HC_{2l}(K) \simeq \widetilde{HC}_{2l}(A) \oplus K,$$

where $\widetilde{HC}$ denotes reduced cyclic homology as defined in [11]. The Loday decomposition of HC and the splitting of the $S - B - I$ sequence pass to $\widetilde{HC}$, so we get:

$$(*) \ : \ 0 \to HC_{2l-1}^{(i-1)} \to^{B} HH_{2l}^{(i)} \to^{I} \widetilde{HC}_{2l}^{(i)} \to 0, \quad and$$

$$(**) \ : \ 0 \to \widetilde{HC}_{2l}^{(i-1)} \to^{B} HH_{2l+1}^{(i)} \to^{I} HC_{2l+1}^{(i)} \to 0.$$

Theorem 3: *Again let us assume that $n > N$ and $n + k = 2i$ then:*

$$\widetilde{HC}_n^{(i)} = \begin{cases} T(\Omega_{A/K}^{N-1}) & \text{for } k = N-1 \ ; \\ 0 & \text{otherwise} \ . \end{cases}$$

Cyclic homology is then obtained by summing up the components.

Proof:

Case I : N is odd: Then the only possibly non-zero components of cyclic homology are: $HC_{2l}^{(l)} = K, HC_{2l}^{(l+(N-1)/2)}$, and $HC_{2l+1}^{(l+(N+1)/2)}$. Using the decomposed $S - B - I$ sequence we obtain:

$$B(HC_{2l+1}^{(l+(N+1)/2)}) \subseteq HH_{2l+2}^{(l+2+(N-1)/2)} = 0, \quad and$$

$$S(HC_{2l+1}) = 0 = Ker(B : HC_{2l+1} \to HH_{2l+2}).$$

Hence:

$$HC_{2l+1} = 0, \quad HH_{2l}^{(l+(n-1)/2)} \simeq^I HC_{2l}^{(l+(N-1)/2)},$$

$$and \quad HC_{2l}^{(l+(N-1)/2)} \simeq^B HH_{2l+1}.$$

Case II : N is even: Then the only possibly non-zero components of cyclic homology are: $HC_{2l}^{(l)} = K, HC_{2l}^{(l+N/2)}$, and $HC_{2l+1}^{(l+(N-1)/2)}$. (Same reason).

$$B(HC_{2l}^{(l+N/2)}) \subseteq HH_{2l+1}^{(l+1+N/2)} = 0.$$

$$HC_{2l}^{(l+N/2)} = 0, \quad HC_{2l-1}^{(l-1+N/2)} \simeq^B HH_{2l}^{(l+N/2)},$$

$$and \quad HH_{2l+1} \simeq^I HC_{2l+1}.$$

In both cases we have seen: $T(\Omega_{A/K}^{N-1}) \simeq \Omega_{A/K}^{N}$.

Remark: The above spectral sequence degenerates at the E^2-level for reduced affine quasi-homogeneous hypersurfaces with an isolated singularity at the origin.

So we get for reduced, affine, quasi-homogeneous hypersurfaces with an isolated singularity at the origin: $E^{\infty}_{n,0} = 0$, if $n \neq i$. $E^{\infty}_{i,0} = K$. Here again E^{∞} denotes the limit term of our spectral sequence obtained from the bicomplex for $HC^{(i)}_n$.

4. Results for $A - D - E$ singularities

An important group of examples of quasi-homogeneous hypersurfaces with an isolated singularity at the origin are the so called $A-D-E$ singularities. It is known that all $A - D - E$ singularities over a field of characteristic 0 are singularities defined by one of the following quasi-homogeneous polynomials:

$$A_k : F(A,k) = X_1^{k+1} + X_2^2 + X_3^2 + \cdots + X_N^2$$

$$D_k : F(D,k) = X_1(X_1^{k-2} + X_2^2) + X_3^2 + \cdots + X_N^2$$

$$E_6 : F(E,6) = X_1^4 + X_2^3 + X_3^2 + \cdots + X_N^2$$

$$E_7 : F(E,7) = X_2(X_1^3 + X_2^2) + X_3^2 + \cdots + X_N^2$$

$$E_8 : F(E,8) = X_1^5 + X_2^3 + X_3^2 + \cdots + X_N^2$$

We get that for a singularity of type A_k, D_k, E_k:

$$dim_K T(\Omega^{N-1}_{A/K}) = dim_K\Big(\frac{K[X_1, X_2, \ldots, X_N]}{(\frac{\partial F}{\partial X_1}, \ldots \frac{\partial F}{\partial X_N})}\Big) = k.$$

We see that $dim_K T(\Omega^{N-1}_{A/K})$ is independent of N. If our ground field K is the complex numbers, this dimension is known as the Milnor number. So reduced cyclic homology computes a topological invariant of singularities defined by a quasi-homogeneous polynomial.

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