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Algebraic K-Theory of operator ideals
(after Mariusz Wodzicki)

by

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This is a report on some of Wodzicki's results on the algebraic K-theory of ideals in the ring of bounded operators on a Hilbert space. Unless indicated otherwise an ideal is always a two-sided ideal. We use the following notations. Let H denote a countably infinite-dimensional Hilbert space, and let $\mathcal{B}(H)$ denote the algebra of bounded operators on H. It is known that the ordered set of proper ideals in $\mathcal{B}(H)$ has a maximal element, namely $\mathcal{K} = \mathcal{K}(H)$ of compact operators on H, and a minimal element, namely $\mathcal{F} = \mathcal{F}(H)$ of finite rank operators on H. These assertions are proved in Calkin [1941].

In Suslin-Wodzicki [1990] we find an isomorphism

$$K_*(B \otimes \mathcal{K}) \longrightarrow K_*^{\text{top}}(B \otimes \mathcal{K}) = K_*^{\text{top}}(B)$$

for every C^*-algebra B where $\otimes$ denotes a C^*-completion of the algebraic tensor product. In LN 725, Karoubi conjectured that

$$K_*(B \widehat{\otimes}_\pi \mathcal{K}) \longrightarrow K_*^{\text{top}}(B \widehat{\otimes}_\pi \mathcal{K})$$

is an isomorphism where B is any Banach algebra with unit and $\widehat{\otimes}_\pi$ is Grothendieck's projective tensor product. The tensor product $B\widehat{\otimes}_\pi\mathcal{K}$ is not a C^*-algebra here even if B is a C^*-algebra. The C^*-analogue of Karoubi's conjecture, that has been circulated among people working on C^*-algebras under the name of the Karoubi's conjecture, was proved in Suslin-Wodzicki [1990]. The original Karoubi conjecture has been proved by Wodzicki [unpublished]. When we put B equal to the complex numbers $\mathbb{C}$, we have the isomorphism

$$K_m(\mathcal{K}) \longrightarrow K_m^{\text{top}}(\mathbb{C}) = \begin{cases} \mathbb{Z} & \text{for even } m \\ 0 & \text{for odd } m. \end{cases}$$

The comparison map $K_m(\mathcal{K}) \longrightarrow K_m^{\mathrm{top}}(\mathbb{C})$ factors through

$$K_m(\mathcal{B}(H), \mathcal{K}) \longrightarrow K_m^{\mathrm{top}}(\mathbb{C})$$

where the relative K-groups $K_i(\mathcal{B}(H), \mathcal{K})$ for $i > 0$ are defined in the first section. Suslin and Wodzicki prove that the excision morphism $K_*(\mathcal{K}) \longrightarrow K_*(\mathcal{B}(H), \mathcal{K})$ is an isomorphism. In particular, they get the index isomorphism

$$\mathrm{Ind} : K_{2i}(\mathcal{B}(H), \mathcal{K}) \longrightarrow \mathbb{Z} \quad \text{and} \quad K_{2i-1}(\mathcal{B}(H), \mathcal{K}) = 0$$

for all i. The isomorphism Ind is the classical index coming from the index of Fredholm operators.

In recent work M. Wodzicki studies other (two-sided) ideals $J \subset \mathcal{B}(H)$ and their algebraic K-theory $K_*(\mathcal{B}(H), J)$. He introduces a class of ideals J in $\mathcal{B}(H)$ which, for the purpose of this paper, we call B-ideals, and he proves the following theorem which analyses index homomorphisms with values in cyclic homology for B-ideals.

(2.8) *Main exact index sequence.* There is an exact sequence functorial under inclusion $J \subset J'$ for B-ideals of the form

$$\begin{array}{ccccccc} 0 & \rightarrow & HC_{2j-1}(\mathcal{B}(H), J) & \rightarrow & K_{2j}(\mathcal{B}(H), J) & & \\ & & & & \downarrow & & \\ & & & & \mathbb{Z} & & \\ & & & & \downarrow & & \\ & & & & HC_{2j-2}(\mathcal{B}(H), J) & \rightarrow & K_{2j-1}(\mathcal{B}(H), J) \rightarrow 0. \end{array}$$

Every Schatten ideal $\mathcal{C}_p$ is a B-ideal. For the definition of B-ideal, see (2.6).

In order to describe which ideals $J \subset \mathcal{B}(H)$ are B-ideals, we recall the definition of the power J^r of J to a strictly positive real number r which generalizes the positive integer power J^n of J. The real power J^r of J is the ideal generated by all $|A|^r$ for any $A \in J$ where $|A| = (AA^*)^{1/2}$. Observe that if $r \leq r'$, then we have the inclusion $J^r \supset J^{r'}$. We denote by $J_\infty = \bigcup_{r>0} J^r$, and one sees that J_∞ is an ideal which contains J. We call the ideal J_∞

the root completion of J. A B-ideal J will be defined in terms of J_∞ having certain properties. For this, see definition (2.6) and for the derivation of the five term sequence, see (2.7).

This text is part of a lecture given at the Strasbourg K-theory Conference, 1992. The author profitted from many discussions with Mariusz Wodzicki while we were both guests of the Forschergruppe, Topology and Noncommutative geometry, Heidelberg, May and June 1992 and during the preparation period of this text.

§1. Definition of relative K_* and HC_* groups

In this section we review the definition of the relative K_* and cyclic homology groups in order to fix notations and list the exact triangles that we will be using.

(1.1) *Definition.* Let R be a $\mathbb{Z}$-algebra (with 1). The algebraic K-groups are $K_i(R) = \pi_i(BGL(R)^+)$, $i \geq 1$. Let I be an ideal in R, let $R \to R/I$ denote the quotient morphism, and let $F(R, I)$ denote the homotopy fibre of the induced mapping on the plus constructions

$$BGL(R)^+ \longrightarrow B\overline{GL}(R/I)^+$$

where $\overline{GL}(R/I) = \mathrm{im}(GL(R) \to GL(R/I))$. The relative K-groups of the pair (R, I) are the homotopy groups $K_i(R, I) = \pi_i(F(R, I))$, $i \geq 1$.

(1.2) *Definition.* We can extend the K-groups to negative degrees with the following recursive formulas

$$K_{-i}(R) = \mathrm{coker}\Big(K_{1-i}(R[t]) \oplus K_{1-i}(R[t^{-1}]) \to K_{1-i}(R[t, t^{-1}])\Big)$$

and

$$K_{-i}(I) = \ker(K_{-i}(\mathbb{Z} \ltimes I) \to K_{-i}(\mathbb{Z})) \text{ for } i > 0.$$

The relative $K_i(R, I)$ for $i \leq 0$ are defined either by the double ring construction as in Milnor's book or by the suspension functor as in Bass's book.

(1.3) *Remark.* From the fibre space exact triangle of homotopy groups we have the following exact triangle of K-groups

$$\begin{array}{ccc} K_*(R,I) & \longrightarrow & K_*(R) \\ \uparrow & {}_{-1}\nwarrow \quad \swarrow & \\ & K_*(R/I) & \\ K_*(\mathbb{Z}\ltimes I, I) = K_*(I). & & \end{array}$$

The last terms of the exact sequence resulting from the fibre space homotopy sequence are the following

$$K_1(R) \to \overline{K}_1(R/I) = \pi_1(B\overline{GL}(R/I)^+) = \mathrm{im}(K_1(R) \to K_1(R/I)) \to 0.$$

The negative degree terms are exact from the nature of the definition of the relative K-groups in negative degrees.

The vertical morphism in this diagram leads to the following definition.

(1.4) *Definition.* An ideal I satisfies K_*-excision provided the above vertical arrow $K_*(I) \to K_*(R,I)$ is an isomorphism for all rings R containing I up to isomorphism.

We know that $K_i(I) \to K_i(R,I)$ is always an isomorphism for $i \leq 0$.

(1.5) *Remark.* Let I be an ideal satisfying K_*-excision. Then we have the following $\mathbb{Z}$-graded exact triangle

$$\begin{array}{cccc} K_*(I) \cong & K_*(R,I) & \longrightarrow & K_*(R) \\ & & {}_{-1}\nwarrow \quad \swarrow & \\ & & K_*(R/I). & \end{array}$$

Now we consider the relative cyclic homology groups. Here we use the conventions of Wodzicki compatible with the usual conventions in K-theory, but which differ by a shift of one in degree from Goodwillie. Let k denote a subring of the complex numbers containing the rational numbers, i.e. $\mathbb{Q} \subset k \subset \mathbb{C}$.

(1.6) *Notation.* For an algebra R over k let $CC_*(R)$ denote the associated single complex of the Tsygan double complex $CC_{**}(R)$ defining cyclic homology. The reader should note that $CC_*(R)$ is sometimes used for the quotient of the standard Hochschild complex $C_*(R)$ under the cyclic group action of $\mathbb{Z}/(n+1)\mathbb{Z}$ in degree n.

(1.7) *Definition.* Let R be an algebra over k with an ideal I. The relative cyclic homology $HC_*(R,I)$ of the pair R and I is given by $H_*(\ker(C_*(R) \to C_*(R/I)))$.

Observe that under the assumption that both R and I are flat k-modules

$$HC_*(R,I) = H_* \left[\frac{\operatorname{Tot} CC_{**} \overset{(0)}{(R} \leftarrow \overset{(1)}{I)}}{CC_{**}(R)} \right] [+1].$$

Here $\operatorname{Tot} CC_{**}(R \leftarrow I)$ denotes the associated single complex of the triple complex which arises from putting the differential algebra $R \leftarrow I$ into the Tsygan cyclic homology double complex functor. Here R has degree 0 and I degree 1.

(1.8) *Remark.* There is a short exact sequence of complexes

$$0 \to CC_{**}(R) \to \operatorname{Tot} CC_{**}(R \leftarrow I) \to \frac{\operatorname{Tot} CC_{**} \overset{(0)}{(R} \leftarrow \overset{(1)}{I)}}{CC_{**}(R)} [+1] \to 0,$$

and the associated homology sequence is the following

$$\begin{array}{ccccccc} HC_i(R,I) & \to & HC_i(R) & \to & HC_i(R/I) & \to & HC_i(R,I)[+1] \\ \uparrow & & & & & & \| \\ HC_i(I) = HC_i(k \ltimes I, I) & & & & & & HC_{i-1}(R,I) \to HC_{i-1}(R) \to \cdots. \end{array}$$

This is an exact triangle of the form

$$\begin{array}{ccc} HC_*(R,I) & \longrightarrow & HC_*(R) \\ {}_{-1}\nwarrow & & \swarrow \\ & HC_*(R/I). & \end{array}$$

Now we bring up the concept of excision, due to Wodzicki, for cyclic homology.

(1.9) *Definition.* An exact sequence of k-modules

$$0 \to K \to L \to M \to 0$$

is called pure provided

$$0 \to K \otimes_k V \to L \otimes_k V \to M \otimes_k V \to 0$$

is exact for every k-module V.

This concept is due to P. M. Cohn. A result given in Bourbaki, [Alg. X, ex.1 for §5, p. 187] is that an exact sequence is pure if and only if it is a filtered inductive limit of split exact sequences. In fact Bourbaki states this for pure acyclic complexes.

(1.10) *Definition.* A sequence $0 \to I \to R \to R/I \to 0$ is called k-pure provided it is k-pure as an extension of k-modules.

(1.11) *Definition.* A k-algebra I (possibly without unit) satisfies HC_*-excision provided for all k-pure extensions

$$0 \to I \to R \to R/I \to 0$$

the induced morphism $HC_*(I) \to HC_*(R,I)$ is an isomorphism.

When I satisfies excision and $0 \to I \to R \to R/I \to 0$ is pure, the exact cyclic homology triangle takes the following form

$$\begin{array}{ccc} HC_*(I) & \longrightarrow & HC_*(R) \\ {}_{-1}\nwarrow & & \swarrow \\ & HC_*(R/I). & \end{array}$$

The main reference for these questions of excision is Wodzicki [1989].

§2. Basic diagrams for the root completion and the extended Goodwillie isomorphism

We come back to the concept of associating to each ideal J in $\mathcal{B}(H)$ a larger ideal J_∞ which was considered in the introduction.

(2.1) *Definition.* The root completion J_∞ of an ideal $J \subset \mathcal{B}(H)$ is the union $J_\infty = \bigcup_{r>0} J^r$. The root completion sequence associated with J is the following kernel sequence for algebras

$$0 \to J_\infty/J \to \mathcal{B}(H)/J \to \mathcal{B}(H)/J_\infty \to 0.$$

Observe that if $0 < r \leq r'$, then we have $J^r \supset J^{r'}$, and in particular, the root completion of J can be defined by using just n-th roots, that is, $J_\infty = \bigcup_{n\geq 1} J^{1/n}$. Of course we have $J \subset J_\infty$.

In the next two sections we consider the K-theory and the HC_*-theory sequences associated with the root completion sequence associated with an ideal $J \subset \mathcal{B} = \mathcal{B}(H)$ where H is always a separable Hilbert space.

(2.2) *K-theory diagram for the root completion sequence.*

$$\begin{array}{ccccccc}
 & & \vdots & & & & \\
 & & \downarrow & & & & \\
 & & K_i(\mathcal{B}/J, J_\infty/J) & & & & \\
 & & \downarrow & & & & \\
 & & K_i(\mathcal{B}/J) & \xrightarrow[\sim]{\partial} & K_{i-1}(\mathcal{B}, J) & & \\
 & \nearrow & & & & \searrow & \\
0 = K_i(\mathcal{B}) & & \Big\downarrow & & & & K_{i-1}(\mathcal{B}) = 0 \\
 & \searrow & & & & \nearrow & \\
 & & K_i(\mathcal{B}/J_\infty) & \xrightarrow[\sim]{\partial} & K_{i-1}(\mathcal{B}, J_\infty) & & \\
 & & \downarrow & & & & \\
 & & \vdots & & & &
\end{array}$$

Recall that $\mathcal{B}$ is a flabby ring, and hence $K_*(\mathcal{B}) = 0$, that is the algebraic K-theory is zero. A reference for the basic definitions and this result is Wagoner [1972].

(2.3) *HC_*-theory diagram for the root completion sequence.*

$$
\begin{array}{ccccccc}
 & & \vdots & & & & \\
 & & \downarrow & & & & \\
 & & HC_i(\mathcal{B}/J, J_\infty/J) & & & & \\
 & & \downarrow & & & & \\
 & & HC_i(\mathcal{B}/J) & \xrightarrow[\sim]{\partial} & HC_{i-1}(\mathcal{B}, J) & & \\
 & \nearrow & & & & \searrow & \\
0 = HC_i(\mathcal{B}) & & \Big\downarrow & & & & HC_{i-1}(\mathcal{B}) = 0 \\
 & \searrow & & & & \nearrow & \\
 & & HC_i(\mathcal{B}/J_\infty) & \xrightarrow[\sim]{\partial} & HC_{i-1}(\mathcal{B}, J_\infty) & & \\
 & & \downarrow & & & & \\
 & & \vdots & & & &
\end{array}
$$

For the proof of $HC_*(\mathcal{B}) = 0$, see Wodzicki [1989]. In all cases the horizontal connecting morphisms ∂ are isomorphisms.

(2.4) *Remark.* In Goodwillie [1988] we find a morphism from rational algebraic K-theory to cyclic homology over $\mathbb{Q}$ defined using the Dennis trace map which is functorial for pairs of an algebra over $\mathbb{Q}$ together with an ideal. In our case we consider this morphism in the following case

$$\gamma : K_i(\mathcal{B}/J, J_\infty/J)_{\mathbb{Q}} \to HC_{i-1}(\mathcal{B}/J, J_\infty/J).$$

Goodwillie shows that if the ideal J_∞/J were nilpotent, then γ would be an isomorphism.

An extension of Goodwillie's methods gives the following for the locally nilpotent ideal J_∞/J. This is a result of Wodzicki which we state without proof.

(2.5) THEOREM. — *For the root completion of an ideal J in $\mathcal{B}$ the Goodwillie morphism*

$$\gamma : K_i(\mathcal{B}/J, J_\infty/J) \to HC_{i-1}(\mathcal{B}/J, J_\infty/J)$$

is an isomorphism. The group $K_i(\mathcal{B}/J, J_\infty/J)$ is uniquely divisible.

The second assertion of the theorem is the result of a general assertion of Wodzicki to the effect that $K_*(R, I)$ is uniquely divisible where I is an ideal with the property that any finite set $x_1, \ldots, x_n \in I$ generates in I a nilpotent $\mathbb{Q}$-algebra. Using a different method, Weibel [1982] proved a similar result.

Now we list two properties of an ideal J in $\mathcal{B}(H)$ such that these two properties used in the previous two diagrams (2.2) and (2.3) along with the extended Goodwillie isomorphism will yield the five term index sequence of the introduction.

(2.6) *Definition.* An ideal J in $\mathcal{B}(H)$ is a B-ideal provided it satisfies the following two conditions.

(1) $HC_*(\mathcal{B}, J_\infty) = 0$,

(2) $K_i(\mathcal{B}, J_\infty) \to K_i(\mathcal{B}, \mathcal{K}) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$

is an isomorphism where $\mathcal{K}$ is the ideal of compact operators in $\mathcal{B} = \mathcal{B}(H)$.

(2.7) *Remark.* For a B-ideal $J \subset \mathcal{B}$ we can connect the diagrams (2.2) and (2.3) using the isomorphism (2.5) to obtain the following commutative diagram with indicated isomorphisms

$$\begin{array}{ccccc}
 & & K_i(\mathcal{B}/J, J_\infty/J) & \xrightarrow{\sim} & HC_{i-1}(\mathcal{B}/J, J_\infty/J) \\
 & & \downarrow & \nwarrow \sim & \downarrow \sim \\
 & & K_{i-1}(\mathcal{B}, J) & \longrightarrow & HC_{i-2}(\mathcal{B}, J) \\
 & & \downarrow & & \downarrow \\
K_{i-1}(\mathcal{B}, \mathcal{K}) & \xleftarrow{\sim} & K_{i-1}(\mathcal{B}, J_\infty) & & HC_{i-2}(\mathcal{B}, J_\infty) = 0 \\
 & & \downarrow & & \downarrow \\
 & & K_{i-1}(\mathcal{B}/J, J_\infty/J) & \xrightarrow{\sim} & HC_{i-2}(\mathcal{B}/J, J_\infty/J).
\end{array}$$

In this diagram the key facts are the isomorphisms

$$K_{i-1}(\mathcal{B},\mathcal{K}) \xleftarrow{\sim} K_{i-1}(\mathcal{B},J_\infty) \quad \text{and} \quad HC_{i-2}(\mathcal{B},J_\infty)=0.$$

For $i-1=2j$ even, we obtain the index sequence in the next theorem.

(2.8) THEOREM. — *For B-ideals $J \subset \mathcal{B}$ we obtain the following exact sequence which is functorial under inclusions $J \subset J'$*

$$\begin{array}{ccc} 0 \to HC_{2j-1}(\mathcal{B}(H),J) \to K_{2j}(\mathcal{B}(H),J) & & \\ \downarrow & & \\ \mathbb{Z} & & \\ \downarrow & & \\ HC_{2j-2}(\mathcal{B}(H),J) \to K_{2j-1}(\mathcal{B}(H),J) \to 0. & & \end{array}$$

Now we establish properties of ideals J in $\mathcal{B}(H)$ which will assure that the ideals are B-ideals and look for examples of B-ideals.

§3. Generalities on ideals in $\mathcal{B}(H)$

We will concentrate on the first property, Condition (1) of Definition (2.6). Condition (2) for an ideal to be a B-ideal was introduced by Wodzicki in 1992.

(3.1) *Examples.* Recall from the introduction that an ideal J in $\mathcal{B}=\mathcal{B}(H)$ satisfies the inclusions $\mathcal{F} \subset J \subset \mathcal{K}$, and this means each $T \in J$ is compact and thus TT^* is compact, selfadjoint, and positive. We denote by $\mu_n(T)$ the nth eigenvalue of $|T|=(TT^*)^{1/2}$ repeated with multiplicity and arranged in sequence such that $\mu_n(T) \geq \mu_{n+1}(T)$. A basic example is the Schatten ideal $\mathcal{C}_p=\mathcal{C}_p(H)$ consisting of all $T \in \mathcal{K}$ such that

$$\mathrm{Tr}(|T|^p)=\sum_{n\geq 1}\mu_n(T)^p < +\infty.$$

Note that $p \leq p'$ implies that $\mathcal{C}_p \subset \mathcal{C}_{p'}$, and for real powers we have the relation $\mathcal{C}_p=(\mathcal{C}_{p'})^{p'/p}$ for any p, $p'>0$. In particular $(\mathcal{C}_p)^p=\mathcal{C}_1$, the ideal of trace class operators, and $\mathcal{C}_2$ is the ideal of Hilbert Schmidt operators.

(3.2) *Remark.* The operator norm closure $\overline{\mathcal{F}}$ is $\mathcal{K}$, the ideal in $\mathcal{B}$ of compact operators so that all proper ideals J in $\mathcal{B}$ have closure $\mathcal{K}$. The ideal of compact operators $\mathcal{K}$ is the only proper (two-sided) closed ideal in $\mathcal{B}$.

(3.3) *Definition.* The functoriality of ideals related to two Hilbert spaces H and H' is defined by choosing any isomorphism $\phi : H \to H'$ and forming $\mathrm{Ad}(\phi) : \mathcal{B}(H) \to \mathcal{B}(H')$ defined by

$$\mathrm{Ad}(\phi)(T) = \phi T \phi^{-1}.$$

For an ideal $J(H) \subset \mathcal{B}(H)$ we define $J(H') = \mathrm{Ad}(\phi)(J(H)) \subset \mathcal{B}(H')$.

(3.4) *Remarks.* For given $J(H)$ the definition of $J(H')$ is independent of the choice of the isomorphism ϕ since two ϕ's differ by an invertible element of $\mathcal{B}(H)$ and since $J(H)$ is an ideal. The examples

$$\mathcal{F}(H) \subset \mathcal{C}_p(H) \subset \mathcal{K}(H)$$

are related to the respective ideals for the Hilbert space H'.

Now we come to a basic construction used to study the ideals $J = J(H)$ in order to determine when they are B-ideals. For two Hilbert spaces H and H' let $H \tilde{\otimes} H'$ denote the Hilbert space tensor product, that is, the ℓ_2-completion of the algebraic tensor product $H \otimes H'$. Observe that the Hilbert spaces H and $H \tilde{\otimes} H$ are isomorphic.

(3.5) *Definition.* For an ideal $J = J(H)$ the tensor multiplier ideal $TM(J)$ of J is the ideal of all $T \in \mathcal{B}(H)$ such that

$$T \otimes J(H) \subset J(H \tilde{\otimes} H).$$

Sometimes we use the notation $TM(J(H))$ for the tensor multiplier ideal $TM(J)$ to indicate the functorial dependence on H.

(3.6) *Example.* For both the ideal of compact operators and the Schatten ideals we have $TM(\mathcal{K}) = \mathcal{K}$ and $TM(\mathcal{C}_p) = \mathcal{C}_p$.

(3.7) *Definition.* An ideal $I = I(H)$ of $\mathcal{B}$(H) is very large provided for the group of units $G = \mathcal{B}(H)^*$ in $\mathcal{B}(H)$, the induced map from the ideal of finite rank operators $\mathcal{F} \to I$ on the G-coinvariants

$$\mathcal{F}_G \to I_G$$

is zero. It is large provided that some root $I^{1/n}$ is very large.

(3.8) *Example.* The Schatten ideal $\mathcal{C}_p$ is very large for $p > 1$ and is not very large for $p \leq 1$. If $J \subset J'$ and J is very large, then J' is very large. If J contains an element T with $\mu_n(T) = 1/n$, then J is very large.

(3.9) *Definition.* An ideal J is ample provided $TM(J)$ is large, and J is very ample provided $TM(J)$ is very large.

(3.10) *Example.* If J is ample, then J_∞ is very ample. The ideal $\mathcal{F}$ is not ample.

The above definitions are background for the determination of which ideals J satisfy condition (1) (2.6), that is, $HC_*(\mathcal{B}, J) = 0$ for J to be a B-ideal.

§4. Vanishing of Hochschild homology $H_*(\mathcal{B}(H); J)$ with coefficients in J.

The main result of this section is the following vanishing theorem for Hochschild homology with values in an ideal. The proof is based on an action of the tensor multiplier ideal of J on the Hochschild homology $H_*(\mathcal{B}(H); J)$ with coefficients in the ideal J.

(4.1) *Theorem.* For an ideal J in $\mathcal{B}(H)$ such that $TM(J)$ is very large, we have $H_*(\mathcal{B}(H); J) = 0$.

Proof. Step 1. We define the operation θ_T for each $T \in TM(J)$ on the Hochschild homology module $H_*(\mathcal{B}(H); J)$. It is the composite of the morphisms on Hochschild homology where $H^{2\otimes}$ denotes $H\widetilde{\otimes}H$

$$\phi_T : H_*(\mathcal{B}(H); J(H)) \longrightarrow H_*(\mathcal{B}(H^{2\otimes}); J(H^{2\otimes}))$$

induced by the pair of morphisms where $A \in \mathcal{B}(H)$ maps to $\mathrm{id}_H \otimes A \in \mathcal{B}(H\widetilde{\otimes}H)$ and $X \in J(H)$ maps to $T \otimes X \in J(H^{2\otimes})$ and the morphism induced by any isomorphism $H \longrightarrow H^{2\otimes}$

$$H_*(\mathcal{B}(H^{2\otimes}); J(H^{2\otimes})) \longrightarrow H_*(\mathcal{B}(H); J(H)).$$

Morita invariance of Hochschild homology allows us to show that the second morphism in the composite is independent of the isomorphism $H \longrightarrow H^{2\otimes}$ used to define it. This defines an additive morphism

$$TM(J) \xrightarrow{\theta} \mathrm{End}_{\mathbf{C}} H_*(\mathcal{B}, J).$$

For $G = \mathcal{B}(H)^*$, the group of invertible elements in $\mathcal{B}(H)$, we have for $S \in G$ the relation $\theta_{STS^{-1}} = \theta_T$, and hence, the additive morphism θ factors through the G-coinvariants

$$TM(J)_G \xrightarrow{\theta} \mathrm{End}_{\mathbf{C}} H_*(\mathcal{B}, J).$$

Step 2. For $T \in \mathcal{F}(H)$ with $T = T^*$ we have the trace relation

$$\theta_T(\alpha) = (tr(T))\alpha.$$

To establish the trace relation, we have only to prove it in the special case where $T^2 = T = T^*$ and $\dim T(H) = 1$ by a suitable reduction as in step 1 with Morita equivalence. Such a T gives an orthogonal splitting $H = \mathbf{C} \oplus H$, and hence also, an isomorphism $H \tilde{\otimes} H \to (\mathbf{C} \oplus H) \tilde{\otimes} H = H \oplus (H \tilde{\otimes} H)$ which in turn is isomorphic to $H \oplus H$. Under these identifications the morphism ϕ_T is induced by the morphisms $A \in \mathcal{B}(H)$ maps to $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in M_2(\mathcal{B}(H))$ and $X \in J(H)$ maps to $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{B}(H))$. Thus the morphism ϕ_T is defined

$$\phi_T : H_*(\mathcal{B}(H); J(H)) \to H_*\left(\begin{pmatrix} \mathcal{B}(H) & 0 \\ 0 & \mathcal{B}(H) \end{pmatrix}; \begin{pmatrix} J(H) & 0 \\ 0 & 0 \end{pmatrix}\right)$$

which composed with the natural

$$H^*\left(\begin{pmatrix} \mathcal{B}(H) & 0 \\ 0 & \mathcal{B}(H) \end{pmatrix}; \begin{pmatrix} J(H) & 0 \\ 0 & 0 \end{pmatrix}\right) \longrightarrow H_*(\mathcal{B}(H); J(H))$$

gives the identity on $H_*(\mathcal{B}(H); J(H))$. This proves the trace relation.

Step 3. For an orthogonal projection P of rank 1, we have $\theta_P = 0$ since $TM(J)$ is very large, and $\theta_P(\alpha) = \alpha$ by step 2. Hence we have the vanishing result $H_*(\mathcal{B}(H), J) = 0$. This proves the theorem.

§5. Vanishing properties of relative cyclic homology

In the previous section we studied the vanishing of Hochschild homology with coefficients in an ideal. Now this is used to establish the vanishing of relative cyclic homology under an amplitude hypothesis of the tensor multiplier ideals.

(5.1) *Remark.* Let J be a (two-sided) ideal in a von Neumann algebra N. Then J is both left and right flat, and therefore, the natural morphism $J \otimes_N \overset{(n)}{\cdots} \otimes_N J = J^{n\otimes} \longrightarrow J^n$ is an isomorphism. Moreover, we have $(J^{1/n})^n = J$, and for $J_\infty = \bigcup_{r>0} J^r = \bigcup_{n\geq 1} J^{1/n}$, if J is ample, then each $TM(J_\infty^m)$ is very large. Observe that $J_\infty^m = J_\infty$.

(5.2) THEOREM. — *If J is an ideal in $\mathcal{B}(H)$ such that $TM(J^{n\otimes})$ is ampe for all $n \geq 1$, then we have the following vanishing of the relative cyclic homology* $HC_*(\mathcal{B}(H), J) = 0$.

Proof. For the relative cyclic homology of an algebra A relative to an ideal J which is right or left A-flat there is a spectral sequence in Quillen [1989], Theorem 4.3 with

$$E_{**}^r \Rightarrow HC_*(A, J)$$

and E^1-term given by the following

$$E_{i,j}^1 = \begin{cases} H_{j-i}(A; J^{i+1})_{cyl_{i+1}} & \text{for } 0 \leq i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

This form of the E^1-term comes from an equivariant version of proposition 2.16 stated near the bottom of p. 218 of Quillen [1989] for ideals J which are either left or right flat A-modules. The E^1-term is the coinvariants under an induced action of the cyclic group of order $i + 1$ as explained in proposition 2.16. We can display the E^1-term in a wedge as follows where the degrees are given in the left hand column and lower row.

...	...	...	...	...	...
3	$H_3(A; J)$	$H_2(A; J^2)_{cyl_2}$	$H_1(A; J^3)_{cyl_3}$	$H_0(A; J^4)_{cyl_4}$	
2	$H_2(A; J)$	$H_1(A; J^2)_{cyl_2}$	$H_0(A; J^3)_{cyl_3}$	0	
1	$H_1(A; J)$	$H_0(A; J^2)_{cyl_2}$	0	0	
0	$H_0(A; J)$	0	0	0	
deg	0	1	2	3	...

Now we apply theorem (4.1) to prove the theorem.

Combining the previous remark with the previous theorem, we have the following theorem immediately as a corollary.

(5.3) THEOREM. — *If J is an ample ideal in the von Neumann algebra $\mathcal{B}(H)$, then the following relative cyclic homology vanishes*

$$HC_*(\mathcal{B}(H), J_\infty) = HC_*(J_\infty) = 0.$$

Proof. The vanishing of the relative cyclic homology follows from the above discussion while the vanishing $HC_*(J_\infty) = 0$ follows from the first assertions using excision. This proves the theorem.

(5.4) *Remark.* The result of the previous theorem says that an ideal J in $\mathcal{B}(H)$ such that $TM(J)$ is ample satisfies the first condition for J to be an B-ideal given in definition (2.6).

(5.5) *Remark.* The last assertion of the previous theorem applies to $J = \mathcal{K}$ and $J = \mathcal{C}_\infty$, the union of all Schatten ideals.

(5.6) *Example.* Using the above theorem (5.3) and the results from Suslin, Wodzicki [1990], we deduce that all Schatten ideals are index ideals. Wodzicki recently proved that all Banach ideals $J \subset \mathcal{B}$ are B-ideals.

§6. Edge calculations of $HC_i(\mathcal{B}(H), J)$

Now we consider the implications of a situation where not all tensor multiplier ideals are very ample.

(6.1) *Basic hypothesis on J.* We assume that the ideals $J, \ldots, J^m$ are very ample, but J^{m+1} is not necessarily very ample. Recall that the following natural morphism is an isomorphism

$$J^{q+1}/[J, J^q] \to H_0(\mathcal{B}(H); J^{q+1})_{cyl_{q+1}} = (J^{q+1}/[\mathcal{B}(H), J^q])_{cyl_{q+1}}$$

for all q. This isomorphism is, for example, in Quillen [1989].

(6.2) PROPOSITION. — *Let J be an ideal in $\mathcal{B}(H)$ satisfing the above hypothesis. Then we have*

(1) $HC_i(\mathcal{B}(H), J) = 0$ *for* $i < 2m$, *and*

(2) $HC_{2m}(\mathcal{B}(H), J) = H_0(\mathcal{B}(H); J^{m+1})_{cyl_{m+1}} = J^{m+1}/[J, J^m]$.

(3) *There is an exact sequence*

$$H_1(\mathcal{B}(H); J^{m+2})_{cyl_{m+2}} \xrightarrow{d_1} H_2(\mathcal{B}(H); J^{m+1})_{cyl_{m+1}} \longrightarrow HC_{2m+2}(\mathcal{B}(H), J) \longrightarrow$$
$$\longrightarrow J^{m+2}/[J, J^{m+1}] \xrightarrow{d_1} H_1(\mathcal{B}(H); J^{m+1})_{cyl_{m+1}} \longrightarrow HC_{2m+1}(\mathcal{B}(H), J) \longrightarrow 0.$$

Proof. This follows by studying the spectral sequence used in the proof of (5.1). For this spectral sequence we have

$$E^1_{i,j} = \begin{cases} H_{j-i}(A; J^{(i+1)})_{cyl_{i+1}} & \text{for } 0 \leq i \leq j \\ 0 & \text{otherwise} \end{cases}$$

and therefore, by (4.1) we have $E^1_{i,j} = 0$ for $i < m$ or $j < m$, and hence the E^1-term is nonzero only in a wedge at $E^1_{m,m} = H_0(\mathcal{B}(H), J^{m+1})_{cyl_m}$ above the main diagonal. This proves the first two assertions, and for the exact sequence, we observe that it is just the exact sequence from the spectral sequence where $E^2_{i,j} = E^\infty_{i,j}$. This proves the proposition.

(6.3) *Example.* For $J = \mathcal{C}_p$ we have the following

$$\frac{J^{m+1}}{[J, J^m]} = \begin{cases} \mathbf{C} & \text{for } m < p < m+1 \\ \text{an uncountable infinite} & \\ \text{dimensional vector space}/\mathbf{C} & \text{for } m+1 = p \\ = \mathcal{C}_1/[\mathcal{B}(H), \mathcal{C}_1]. & \end{cases}$$

References

[1] Bourbaki, N. *Algèbre homologique*, Masson, 1980.

[2] Calkin, J.W. *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Annals of Mathematics, 42 (1941), pp. 839–873.

[3] Dixmier J. *Remarques sur les applications*, Arch. der Math., 3 (1952), pp. 290–297.

[4] Dixmier, J. *Formes linéaires sur un anneau d'opérateurs*, Bulletin Math. de France, (1953), pp. 9–39.

[5] Goodwillie, T. *Relative algebraic K-theory and cyclic homology*, Annals of Math., 124 (1986), pp. 347–402.

[6] Quillen, D. *Cyclic Cohomology and Algebra Extensions*, K-theory, 3 (1989), pp. 205–246.

[7] Suslin, A.A. and Wodzicki, M., *Excision in algebraic K-theory and Karoubi's conjecture.* Proc. Natl. Acad. Sci. USA, 87 (1990), pp. 9582–9584.

[8] Suslin, A.A. and Wodzicki, M., *Excision in algebraic K-theory*, Annals of Math., 136 (1992), pp. 51–122.

[9] WAGONER, J., *Delooping Classifiying Spaces in Algebraic K-theory*, Topology, 11 (1972), pp. 349–370.

[10] WEIBEL, C.A., *Mayer-Vietoris sequences and* mod p *K-theory*, Springer Lecture Notes in Math. 966 (1982), pp. 390–407.

[11] WODZICKI, M., *Excision in algebraic K-theory and in rational algebraic K-theory*, Annals of Math., 129 (1989), pp. 591–639.

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