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Exposé V

SEMI–STABLE REDUCTION AND CRYSTALLINE COHOMOLOGY WITH LOGARITHMIC POLES

by **Osamu Hyodo** and **Kazuya Kato**

Introduction

The results of this paper were obtained by the collaboration with J.–M. Fontaine and L. Illusie.

We say a scheme X over a discrete valuation ring A is with *semi–stable reduction* if etale locally on X, there is a smooth morphism $X \to \operatorname{Spec}(A[T_1, \ldots, T_r]/(T_1 \cdots T_r - \pi)$ for some $r \geq 0$, where π is a uniformizing parameter. This condition is equivalent to the condition that X is regular, the generic fiber of X is smooth, and the closed fiber of X is a reduced divisor with normal crossings on X.

Let A be a complete discrete valuation ring with field of fractions K and with residue field k such that $\operatorname{char}(K) = 0$, $\operatorname{char}(k) = p > 0$, and k is perfect, and let K_0 be the field of fractions of the ring $W = W(k)$ of Witt vectors. Let X be a proper scheme over A with semi–stable reduction, and let $Y = X \otimes_A k$. Then, the crystalline cohomology group $H^m_{crys}(Y/W) \otimes_W K_0$ $(m \in \mathbb{Z})$ is not a "good cohomology" when Y is singular. However U. Jannsen conjectured in [J] that there is a "new crystalline cohomology group" D, which is a finite dimensional K_0–vector space endowed with

– a bijective frobenius–linear operator $\varphi : D \to D$ called the frobenius,

– a nilpotent operator $\mathcal{N} : D \to D$ called the monodromy operator, satisfying $\mathcal{N}\varphi = p\varphi\mathcal{N}$,

– a K–isomorphism with the de Rham cohomology

$$\rho : D \otimes_{K_0} K \xrightarrow{\sim} H^m_{DR}(X_K/K) \qquad (X_K = X \otimes_A K).$$

This space D is a mixed characteristic analogue of the limit Hodge structure [S].

The triple $(D, \varphi, \mathcal{N})$ is constructed in Hyodo [H2] by using some de Rham–Witt complex with logarithmic poles. In this paper, we give another construction of $(D, \varphi, \mathcal{N})$ using the crystalline cohomology theory with logarithmic poles and give the isomorphism ρ. The 4–ple $(D, \varphi, \mathcal{N}, \rho)$ has the following further properties.

– $(D, \varphi, \mathcal{N})$ depends only on the scheme $X \otimes_A A/m_A^2$ over A/m_A^2 where m_A denotes the maximal ideal of A (cf. (1.7)).

– The isomorphism ρ depends on a choice of a prime element π of A. If we indicate the choice of π as ρ_π, we have

$$\rho_{\pi u} = \rho_\pi \circ \exp(\log(u)\mathcal{N})$$

for $u \in A^\times$, where we denote the K–linear operator on $D \otimes_{K_0} K$ induced by $\mathcal{N}$ by the same letter $\mathcal{N}$. The K–linear operator $\rho_\pi \circ \mathcal{N} \circ \rho_\pi^{-1}$ on $H^m_{DR}(X_K/K)$ is independent of the choice of π (cf. Thm. 5.1)).

– As is shown in [H2], the triple $(D, \varphi, \mathcal{N})$ is $\otimes_W K_0$ of a triple $(H, \varphi, \mathcal{N})$ with H a canonically defined $W(k)$–module of finite type. L. Illusie has proposed a method to show that the operator $\mathcal{N} : H \to H$ is already nilpotent before $\otimes_W K_0$. This has been carried out by A. Mokrane, see [M].

The theory of crystalline cohomology with logarithmic poles used in this paper is based on the theory of "logarithmic structures" of Fontaine–Illusie reported in [K1] (cf. § 2 for a summary of this theory). In fact, by using this theory of logarithmic structures, we construct $(D, \varphi, \mathcal{N}, \rho)$ in this paper not only for X as above, but also for a scheme over A with a "smooth logarithmic structure whose reduction is of Cartier type" (for example, a product of schemes with semi–stable reduction is such a scheme). We give also the detailed study of the de Rham–Witt complexes with logarithmic poles associated to such general situation (§ 4).

In [J], Jannsen presented a conjecture on the relation between the 4–ple $(D, \varphi, \mathcal{N}, \rho)$ and the p–adic etale cohomology $H^m_{et}(X \otimes_A \overline{K}, \mathbf{Q}_p)$, which was formulated in a more precise form and proved in the case of abelian variety by Fontaine [Fo1]. We discuss this conjecture in another paper [K2].

The subject of this paper is studied independently by Faltings [Fa] § 4, and a different formulation of logarithmic structure is given in [Fa] § 2. The triple $(D, \varphi, \mathcal{N})$ is obtained also from his theory. The study of the de Rham–Witt complex of this paper is not contained in [Fa].

A different approach to this subject using syntomic sheaves is given in [Fo2]. The authors heard that P. Deligne considered a mixed characteristic analogue of the limit Hodge structure in rather old days (unpublished). Some related topics are discussed in [Il2], [Il3], [Il4].

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1. — A fast construction of $(D, \varphi, \mathcal{N})$

Before we start the use of the crystalline cohomology theory with logarithmic poles, we remark in this section that it is possible to construct $(D, \varphi, \mathcal{N})$ in the semi–stable reduction case without using such theory, but using only the classical theory of the de Rham–Witt complexes. The proofs of some statements are not given in this section. However proofs using the theory of log structures are given in later sections for generalized versions of the statements. Proofs without using the theory of log structures exist, but we do not discuss them.

In this section, let A be a discrete valuation ring with field of fractions K and with residue field k, and assume that k is a perfect field of characteristic $p > 0$. Let X be a scheme over A with semi–stable reduction, and let $Y = X \otimes_A k$.

All sheaves considered in this section are those on small etale sites.

(1.1). — We define a complex $W_n\omega^{\cdot}_Y$ on Y_{et} as follows. This complex is nothing but the de Rham–Witt complex in [H2], but the construction here is different and more elementary. Though this complex in fact depends in general on the scheme $X \otimes_A A/m_A^2$ over A/m_A^2, not only on Y, we use the notation $W_n\omega_Y^{\cdot}$ for simplicity.

Take a dense open subscheme U of Y which is smooth over k, and let $u : U \to Y$ be the inclusion map. We define $W_n\omega_Y^{\cdot}$ as a subcomplex of $u_*W_n\Omega_U^{\cdot}$ where $W_n\Omega_U^{\cdot}$ is the usual de Rham–Witt complex ([Il1]) of U. Let

$$Y \xrightarrow{i} X \xleftarrow{j} X_K$$

be the inclusion maps. Then $i^{-1}(\mathcal{O}_X^\times) \to i^{-1}j_*(\mathcal{O}_{X_K}^\times)$ is injective and the restriction of $i^{-1}j_*(\mathcal{O}_{X_K}^\times)/i^{-1}(\mathcal{O}_X^\times)$ to U is isomorphic to the constant sheaf $K^\times/A^\times$. From this we see that there exists a unique homomorphism

$$d\log : i^{-1}j_*(\mathcal{O}_{X_K}^\times) \longrightarrow u_*W_n\Omega_U^1$$

which induces on $u^{-1}i^{-1}(\mathcal{O}_X^\times)$ the composite map

$$u^{-1}i^{-1}(\mathcal{O}_X^\times) \longrightarrow \mathcal{O}_U^\times \xrightarrow{d\log} W_n\Omega_U^1$$

and induces the zero map on $K^\times$. Define $W_n\omega_Y^{\cdot}$ to be the $W_n(\mathcal{O}_Y)$–subalgebra of $u_*W_n\Omega_U^{\cdot}$ generated by $dW_n(\mathcal{O}_Y)$ and $d\log(i^{-1}j_*(\mathcal{O}_{X_K}^\times))$. Then $W_n\omega_Y^{\cdot}$ becomes a subcomplex of $u_*W_n\Omega_U^{\cdot}$. As it is easily seen, $W_n\omega_Y^{\cdot}$ is independent of the choice of U.

(1.2). — One can check the following facts easily. The operators

$$F : u_*W_{n+1}\Omega_U^q \longrightarrow u_*W_n\Omega_U^q\,, \qquad V : u_*W_n\Omega_U^q \longrightarrow u_*W_{n+1}\Omega_U^q$$

induce

$$F : W_{n+1}\omega_Y^q \longrightarrow W_n\omega_Y^q\,, \qquad V : W_n\omega_Y^q \longrightarrow W_{n+1}\omega_Y^q$$

respectively, satisfying $FV = p$, $VF = p$, $dF = pFd$, $Vd = pdV$, $FdV = d$. The absolute frobenius of Y induces an endomorphism of the differential algebra

$$\varphi : W_n\omega_Y^{\cdot} \longrightarrow W_n\omega_Y^{\cdot}$$

which induces p^qF on $W_n\omega_Y^q$.

(1.3). — Now fix $m \in \mathbb{Z}$ and let

$$D_n = H^m(Y, W_n\omega^{\cdot}_Y)\,, \quad D_\infty = \varprojlim H^m(Y, W_n\omega^{\cdot}_Y)\,, \quad D = D_\infty \otimes_W K_0\,.$$

If Y is proper over k, it can be shown that each D_n is a $W_n(k)$–module of finite length and D_∞ is a finitely generated $W(k)$–module (cf. (3.2)). The frobenius φ on $W_n\omega^{\cdot}_Y$ (1.2) induces $D_n \to D_n$, $D_\infty \to D_\infty$ and $D \to D$, which we denote also by φ. The map $\varphi : D \to D$ is bijective as is seen from the existence of the endomorphism of complexes $g = (p^{r-q}V : W_n\omega^q_Y \to W_n\omega^q_Y)_{q\in\mathbb{Z}}$ for r bigger than the dimension of Y which satisfies $\varphi g = g\varphi = p^{r+1}$.

(1.4). — To obtain the monodromy operator $\mathcal{N}$, we define a complex $W_n\widetilde{\omega}^{\cdot}_Y$ on Y (which also depends on $X \otimes_A A/m_A^2$). Let $u : U \to Y$ be as in (1.2), and consider the graded differential algebra

$$\mathcal{A}^{\cdot} = u_*(W_n\Omega^{\cdot}_U)[\theta]/(\theta^2)$$

where θ is an indeterminate in degree one satisfying

$$\theta a = (-1)^q a\theta \quad (a \in u_*W_n\Omega^q_U)\,, \quad d\theta = 0\,.$$

We are going to define $W_n\widetilde{\omega}^{\cdot}_Y$ to be a $W_n(\mathcal{O}_Y)$–subalgebra of $\mathcal{A}^{\cdot}$. Let

$$\mathrm{d}\log : i^{-1}j_*(\mathcal{O}^\times_{X_K}) \longrightarrow \mathcal{A}^1$$

be the unique homomorphism which induces on $u^{-1}i^{-1}(\mathcal{O}^\times_X)$ the composite map

$$u^{-1}i^{-1}(\mathcal{O}^\times_X) \longrightarrow \mathcal{O}^\times_U \xrightarrow{\mathrm{d}\log} W_n\Omega^1_U$$

and induces on $K^\times$ the map $a \to \mathrm{ord}_K(a)\theta$ (here ord_K is the normalized additive discrete valuation of K). We define $W_n\widetilde{\omega}^{\cdot}_Y$ to be the $W_n(\mathcal{O}_Y)$–subalgebra of $\mathcal{A}^{\cdot}$ generated by $dW_n(\mathcal{O}_Y)$ and the image of d log. Then $W_n\widetilde{\omega}^{\cdot}_Y$ becomes a subcomplex of $\mathcal{A}^{\cdot}$, and is independent of the choice of U.

Proposition (1.5). — *The sequence*

$$0 \longrightarrow W_n\omega^{\cdot}_Y[-1] \longrightarrow W_n\widetilde{\omega}^{\cdot}_Y \longrightarrow W_n\omega^{\cdot}_Y \longrightarrow 0$$
$$a \longrightarrow a\theta\,, \quad \theta \longrightarrow 0$$

is exact.

This will be proven in (4.20).

(1.6). — Define $\mathcal{N} : D_n \to D_n$ as the connecting homomorphism of the exact sequence (1.5). The commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_n\omega_Y^{\cdot}[-1] & \longrightarrow & W_n\widetilde{\omega}_Y^{\cdot} & \longrightarrow & W_n\omega_Y^{\cdot} & \longrightarrow & 0 \\ & & {\scriptstyle p\varphi}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow & & \\ 0 & \longrightarrow & W_n\omega_Y^{\cdot}[-1] & \longrightarrow & W_n\widetilde{\omega}_Y^{\cdot} & \longrightarrow & W_n\omega_Y^{\cdot} & \longrightarrow & 0 \end{array}$$

where the middle φ is induced from the ring homomorphism $\mathcal{A}^{\cdot} \to \mathcal{A}^{\cdot}$ which extends φ of $u_*W_n\Omega_U^{\cdot}$ by $\theta \longmapsto p\theta$, proves $\mathcal{N}\varphi = p\varphi\mathcal{N}$. If Y is proper over k, this equation and the bijectivity of φ show that $\mathcal{N} : D \to D$ is nilpotent.

(1.7). — One can check that the complexes $W_n\omega_Y^{\cdot}$ and $W_n\widetilde{\omega}_Y^{\cdot}$ depend only on the scheme $X \otimes_A A/m_A^2$ over A/m_A^2, and hence $(D_n, \varphi, \mathcal{N})$ $(n \geq 1)$ depends only on $X \otimes_A A/m_A^2$ over A/m_A^2. This last fact can also be seen, by using the theory of log structures, from the fact that $(D_n, \varphi, \mathcal{N})$ is determined by certain log structures on Y and on $\mathrm{Spec}(k)$ (cf. § 3) which depend only on the scheme $X \otimes_A A/m_A^2$ over A/m_A^2.

2. — Crystalline cohomology with logarithmic poles

In this section, we give a summary of the paper [K1] on the logarithmic structures of Fontaine–Illusie, and add a logarithmic version (2.24) of a result [BO2] (1.6) of Berthelot–Ogus.

In this section, monoids are assumed to be commutative and have a unit element, and homomorphisms of monoids are assumed to preserve the unit elements. For a monoid P, let P^{gp} be the associated commutative group $\{ab^{-1}; a, b \in P\}$. We call a monoid *integral* if $P \to P^{gp}$ is injective (i.e. if "$ab = ac \Longrightarrow b = c$" holds).

(2.1). — For a scheme X, a *pre-logarithmic structure* on X is a sheaf of monoids M on the etale site X_{et}, endowed with a homomorphism $\alpha : M \to \mathcal{O}_X$

with respect to the multiplicative law on $\mathcal{O}_X$. A pre–logarithmic structure is called a *logarithmic structure* (or a *log structure*) if

$$\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times \quad \text{via } \alpha .$$

For example, the sheaf $M = \mathcal{O}_X^\times$ with the inclusion map $\mathcal{O}_X^\times \to \mathcal{O}_X$ is a log structure which we call the *trivial* log structure. If M is a log structure, we identify $\mathcal{O}_X^\times$ with the subsheaf $\alpha^{-1}(\mathcal{O}_X^\times)$ of M via α.

A morphism between schemes with log structures is defined in the evident way.

(2.2). — For a pre–log structure M on X, the log structure M^a on X associated to M is defined as the push out of $\mathcal{O}_X^\times \leftarrow \alpha^{-1}(\mathcal{O}_X^\times) \to M$ in the category of sheaves of monoids. That is,

$$M^a = (\mathcal{O}_X^\times \oplus M)/\sim$$

where $\sim$ is the equivalent relation

$$(u,a) \sim (v,b) \iff \text{there exist (locally) } c,\ d \in \alpha^{-1}(\mathcal{O}_X^\times) \text{ such that}$$
$$\alpha(c)u = \alpha(d)v \quad \text{and} \quad ad = bc$$

(the map $M^a \to \mathcal{O}_X$ is the sum of $M \to \mathcal{O}_X$ and the inclusion map $\mathcal{O}_X^\times \to \mathcal{O}_X$). The natural morphism $M \to M^a$ is universal among morphisms from M to log structures on X.

(2.3). — For a morphism of schemes $f : X \to Y$ and a log structure M on Y, the *inverse image* f^*M of M is defined to be the log structure on X associated to the pre–log structure $f^{-1}(M)$ (endowed with $f^{-1}(M) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$).

(2.4). — The category of schemes with log structures has finite inverse limits. For a finite inverse system $(X_\lambda, M_\lambda)_\lambda$, its inverse limit (X, M) is described as follows. The scheme X is the inverse limit of the inverse system $(X_\lambda)_\lambda$. If $p_\lambda : X \to X_\lambda$ denote the projections and M' denotes the inductive limit of the system $(p_\lambda^{-1}(M_\lambda))_\lambda$ in the category of sheaves of monoids on X_{et}, M is the log structure associated to the pre–log structure M'.

(2.5). — For a morphism of schemes with log structures $f:(X,M)\to(Y,N)$, we define an $\mathcal{O}_X$–module $\omega^1_{(X,M)/(Y,N)}$, which is called the *sheaf of differential forms with logarithmic poles relative to* f and is often denoted simply as $\omega^1_{X/Y}$, to be the quotient of $\Omega^1_{X/Y}\oplus(\mathcal{O}_X\otimes_{\mathbf{Z}}M^{gp})$ divided by the $\mathcal{O}_X$–submodule generated by local sections of the forms $(d\alpha(a),0)-(0,\alpha(a)\otimes a)$ $(a\in M)$ and $(0,1\otimes a)$ $(a\in f^{-1}(N))$. The class of $(0,1\otimes a)$ $(a\in M)$ in $\omega^1_{X/Y}$ is denoted by $\mathrm{d}\log(a)$. Define $\omega^q_{X/Y}=\wedge^q_{\mathcal{O}_X}\omega^1_{X/Y}$ for $q\in\mathbf{Z}$. Then with the map $d:\omega^q_{X/Y}\to\omega^{q+1}_{X/Y}$; $d(a\,\mathrm{d}\log(b_1)\wedge\cdots\wedge\mathrm{d}\log(b_q))=da\wedge\mathrm{d}\log(b_1)\wedge\cdots\mathrm{d}\log(b_q)$ $(a\in\mathcal{O}_X,b_1,\dots,b_q\in M)$, $(\omega^{\cdot}_{X/Y},d)$ becomes a complex.

(2.6). — We say a log structure M on a scheme X is *fine* if etale locally on X, there is a finitely generated integral monoid P and a homorphism $h:P_X\xrightarrow{\alpha}\mathcal{O}_X$ where P_X denotes the constant sheaf defined by P, such that M is isomorphic to the log structure associated to the pre–log structure (P_X,α).

A standard example of a fine log structure is the following. Let X be a regular scheme and D a reduced divisor with normal crossings on X. Let

$$M=\mathcal{O}_X\cap j_*\mathcal{O}_U^\times\qquad(j:U=X-D\hookrightarrow X)$$

with the inclusion map $\alpha:M\to\mathcal{O}_X$. Then M is a fine log structure which is associated etale locally to

$$\mathbf{N}^r_X\longrightarrow\mathcal{O}_X:(m_i)_{1\le i\le r}\longmapsto\prod_i\pi_i^{m_i}$$

where $\pi_i\in\mathcal{O}_X$ define regular subschemes of X whose union is D. The reason why we work with the etale topology in the theory of log structures is that the definition of "normal crossing" is etale local.

(2.7). — For a morphism $f:(X,M)\to(Y,N)$ between schemes with fine log structures, a *chart* of f is a system $(P_X\xrightarrow{s}M,\ Q_Y\xrightarrow{t}N,\ Q\xrightarrow{h}P)$ where P and Q are finitely generated integral monoids and s, t, h are homomorphisms satisfying the following conditions : s and t induce isomorphisms $(P_X)^a\xrightarrow{\sim}M$ and $(Q_Y)^a\xrightarrow{\sim}N$, respectively (here P_X is regarded as a pre–log structure via $P_X \xrightarrow{s} M \longrightarrow \mathcal{O}_X$, and Q_Y similarly), and (f,h) commutes with (s,t) in the evident sense. A chart of f exists etale locally.

In the following (2.8)–(2.12), let $f : (X,M) \to (Y,N)$ be a morphism of schemes with fine log structures. We give definitions of several types of morphisms (cf. [K1] § 3 and § 4).

(2.8). — We say f is a *closed immersion* (resp. an *exact closed immersion*) if the underlying morphism $f : X \to Y$ is a closed immersion and the map $f^*N \to M$ is surjective (resp. bijective).

(2.9). — It can be proven that the following two conditions (i) and (ii) (resp. (i)' and (ii)') are equivalent. We say f is *smooth* (resp. *etale*) if the equivalent conditions (i) and (ii) (resp. (i)' and (ii)') are satisfied.

(i) (resp. (i)'). The underlying morphism $X \to Y$ is locally of finite presentation, and for any commutative diagram of schemes with fine log structures of the form

$$\begin{array}{ccc} (T',L') & \xrightarrow{s} & (X,M) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ (T,L) & \xrightarrow{t} & (Y,N) \end{array}$$

where i is an exact closed immersion such that the ideal of T' in T is nilpotent, there exists etale locally on T a morphism (resp. there exists a unique morphism) $g : (T,L) \to (X,M)$ such that $gi = s$ and $fg = t$.

(ii) (resp. (ii)'). Etale locally on X and on Y, there exists a chart $(P_X \to M,\ Q_Y \to N,\ Q \xrightarrow{h} P)$ of f such that the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of the induced homomorphism $h^{gp} : Q^{gp} \to P^{gp}$ are finite groups whose orders are invertible on X, and such that the induced map $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ is etale in the usual sense.

We have :

(2.9.1). — If f is smooth, the $\mathcal{O}_X$-module $\omega^q_{X/Y}$ is locally free of finite type for any q.

(2.9.2). — If X is locally of finite type over Y, there exists etale locally on X a factorization $(X,M) \xrightarrow{i} (Z,L) \xrightarrow{g} (Y,N)$ with L fine, i a closed immersion and g smooth. This follows from the existence of local charts of f [K1, 2.9 (2)][(1)]

(2.9.3). — If f is a closed immersion, there exists etale locally on X a factorization $(X,M) \xrightarrow{i} (Z,L) \xrightarrow{g} (Y,N)$ with L fine, i an exact closed immersion and g etale [K1, 4.10].

(2.10). — It can be proved that the following conditions (i) and (ii) are equivalent. We say f is *integral* if there equivalent conditions are satisfied.

(1) By [K1, 2.9 (2)] we may assume X and Y are affine and we have a global chart of $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ given by ($P \to A$, $Q \to B$, $u : Q \to P$). We thus have a factorization $(X,M) \xrightarrow{i_1} (X_1,L_1) \to (Y,N)$ where $X_1 = Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ and L_1 is the log structure associated to $P \to \mathbb{Z}[P] \to B_1 = A \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$. Moreover, $M = i_1^* L_1$. Now, choose a surjective map $v : \mathbb{N}^r \to P$, and consider the factorization of u given by $Q \to Q \oplus \mathbb{N}^r \to P$ where the first map sends a to $(a,0)$ and the second one (a,b) to $u(a)+v(b)$. Taking the pull–back by $Y \to \mathrm{Spec}(\mathbb{Z}[Q]$ of $\mathrm{Spec}(\mathbb{Z}[P]) \to \mathrm{Spec}(\mathbb{Z}[Q \oplus \mathbb{N}^r]) \to \mathrm{Spec}(\mathbb{Z}[P])$, we get a factorization $(X_1,L_1) \xrightarrow{i_2} (X_2,L_2) \to (Y,N)$ where i_2 is a closed immersion and $(X_2,L_2) \to (Y,N)$ is smooth. Here $X_2 = \mathrm{Spec}(B_2)$, with $B_2 = A \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[Q \oplus \mathbb{N}^r]$. Finally, choose a surjective map of B_1–algebras $B_1[t_1,\ldots,t_n] \to B$ and endow $Z_1 = \mathrm{Spec}(B_1[t_1,\ldots,t_n])$ (resp. $Z_2 = \mathrm{Spec}(B_2[t_1,\ldots,t_n])$) with the inverse image log structure of X_1 (resp. X_2). We thus get a factorization $(X,M) \to (Z_1,M_1) \to (Z_2,M_2) \to (Y,N)$ where $(X,M) \to (Z_2,M_2)$ is a closed immersion and $(Z_2,M_2) \to (Y,N)$ is smooth, as desired.

(i) For any scheme Y' with a fine log structure N' and for any morphism $(Y', N') \to (Y, N)$, the log structure of the fiber product $(X, M) \times_{(Y,N)} (Y', N')$ is fine.

(ii) Etale locally on X and on Y, there exists a chart $(P_X \to M,\ Q_Y \to N,\ Q \xrightarrow{h} P)$ of f such that the ring homomorphism $\mathbf{Z}[Q] \to \mathbf{Z}[P]$ induced by h is flat.

We have (cf. [K1] § 4) :

(2.10.1). — The morphism f is integral if $f^*N \xrightarrow{\sim} M$ or if $N_{\overline{y}}/\mathcal{O}^{\times}_{Y,\overline{y}}$ is generated by one element for any $y \in Y$ (($)_{\overline{y}}$ denotes the stalk at a geometric point dominating y).

(2.10.2). — If f is smooth and integral, the underlying morphism $X \to Y$ is flat.

(2.11). — We say f is *exact* if the diagram

$$\begin{array}{ccc} f^{-1}(N) & \longrightarrow & M \\ \downarrow & & \downarrow \\ f^{-1}(N)^{gp} & \longrightarrow & M^{gp} \end{array}$$

is cartesian (then a closed immersion (2.7) is exact if and only if it is an exact closed immersion in the sense of (2.7)).

(2.12). — For a prime number p and a scheme S over $\mathbf{F}_p$ with a fine log. str. L, the *absolute frobenius* $F_{(S,L)} : (S, L) \to (S, L)$ is defined to be the pair of the absolute frobenius $F_S : S \to S$ and the p-th power map $F_S^{-1}(L) \cong L \xrightarrow{p} L$ where we used the natural isomorphism $F_S^{-1}(\mathcal{F}) \cong \mathcal{F}$ of any sheaf $\mathcal{F}$ on S_{et}. In the case X and Y are schemes over $\mathbf{F}_p$, we say f is of *Cartier type* if f is integral and the morphism g in the following commutative diagram with a cartesian square is exact.

$$
\begin{tikzcd}
(X,M) \arrow[r, "g"] \arrow[rr, bend left, "F_{(X,M)}"] & (X',M') \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & (X,M) \arrow[d, "f"] \\
& (Y,N) \arrow[r, "F_{Y,N)}"] & (Y,N)
\end{tikzcd}
\tag{2.12.1}
$$

If f is smooth and of Cartier type, we have a Cartier isomorphism

$$C^{-1} : \omega^q_{X'/Y} \xrightarrow{\sim} \mathcal{H}^q(\omega^{\cdot}_{X/Y}) \qquad (q \in \mathbb{Z}) \tag{2.12.2}$$

characterized by

$$C^{-1}(a\mathrm{d}\log(h^*(b_1)) \wedge \cdots \wedge \mathrm{d}\log(h^*(b_q))) = g^*(a)\mathrm{d}\log(b_1) \wedge \cdots \wedge \mathrm{d}\log(b_q)$$

$(a \in \mathcal{O}_X, \ldots, b_1, \ldots, b_q \in M)$.

(2.13). — We give remarks on smooth morphisms and morphisms of Cartier type.

(2.13.1). — Let A be a discrete valuation ring. Then, we call the log structure on Spec(A) corresponding to the closed point (regarded as a reduced divisor with normal crossings) in the sense of (2.6) the *canonical log structure* of Spec(A). If A' is a discrete valuation ring which is finite over A and N, N' denote the canonical log structures on Spec(A) and Spec(A'), respectively, the following three conditions on $f : (\mathrm{Spec}(A'), N') \to (\mathrm{Spec}(A), N)$ are equivalent. (i) f is etale. (ii) f is smooth. (iii) A' is tamely ramified over A.

(2.13.2). — Let A be as in (2.13.1) and X be a scheme over A with semi-stable reduction. Let $Y = X \otimes_A k$ where k is the residue field of A, let

M be the log structure on X corresponding to Y (regarded as a divisor with normal crossings on X), and let N be the canonical log structure on $\mathrm{Spec}(A)$. Then, the morphism $(X, M) \to (\mathrm{Spec}(A), N)$ is smooth (and it is integral by (2.10.1)). If k is of positive characteristic, the morphism $(Y, \overline{M}) \to (\mathrm{Spec}(k), \overline{N})$, where $\overline{M}$ and $\overline{N}$ denote the inverse images of M and N, respectively, is smooth of Cartier type.

(2.13.3). — Let k be a field and $j : U \hookrightarrow X$ be a toroidal embedding of a smooth k–variety U into a normal k–variety X. Then if M denotes the log structure $\mathcal{O}_X \cap j_*\mathcal{O}_U^\times$, (X, M) is smooth over $\mathrm{Spec}(k)$ where $\mathrm{Spec}(k)$ is endowed with the trivial log structure (2.1). The equivalence between (2.9) (i) and (2.9) (ii) in the case $Y = \mathrm{Spec}(k)$ and N is the trivial log structure says that the notion of toroidal embeddings over k is essentially equivalent to the notion of a scheme with a fine log structure which is smooth over $\mathrm{Spec}(k)$.

(2.13.4). — Smooth integral morphisms are stable under base changes, compositions, and under taking fiber products. The same is true for smooth morphisms of Cartier type in characteristic $p > 0$. For example, for X and A as in (2.13.2) with Y singular and for a discrete valuation ring A' which is finite with ramification index > 1 over A, $X' \overset{\mathrm{def}}{=} X \otimes_A A'$ is not regular. However from the view point of log structures, X' is not so ugly : with the log structure M' as the fiber product, (X', M') is smooth over $(\mathrm{Spec}(A'), N')$ with N' the canonical log structure on $\mathrm{Spec}(A')$.

(2.14). — The theory of crystalline cohomology is generalized to schemes with fine log structures as follows. As a base, we take a 4–ple (S, L, I, γ) where S is a scheme such that $\mathcal{O}_S$ is killed by a non–zero integer, L is a fine log structure on S, I is a quasi–coherent ideal on S, and γ is a PD (= divided power) structure on I. Let (X, M) be a scheme with fine log structure over (S, L) such that γ extends to X. We keep these notations in (2.15)–(2.17) and in (2.19)–(2.22).

(2.15). — We define the *crystalline site* $((X, M)/(S, L, I, \gamma))_{crys}$ (which we abbreviate as $((X, M)/(S, L))_{crys}$ or as $(X/S)^{\log}_{crys}$) as follows. An object is a 5–ple (U, T, M_T, i, δ) where U is an etale scheme over X, (T, M_T) is a scheme with a fine log structure over (S,L), i is an exact closed immersion $(U,M_{|U}) \to (T,M_T)$ over (S,L), and δ is a PD-structure on the ideal of U in T which is compatible with γ. Morphisms are defined in the evident way. A family of morphisms

$$g_\lambda : (U_\lambda, T_\lambda, M_{T_\lambda}, i_\lambda, \delta_\lambda) \longrightarrow (U, T, M_T, i, \delta)$$

is a covering if the morphisms of schemes $g_\lambda : T_\lambda \to T$ are etale and form a covering for the etale topology, and $U_\lambda \simeq T_\lambda \times_T U$ for all λ.

The *structure sheaf* $\mathcal{O}_{X/S}$ of $(X/S)^{\log}_{crys}$ is defined by

$$\mathcal{O}_{X/S}(U,T,M_T,i,\delta) = \Gamma(T, \mathcal{O}_T).$$

(2.16). — Let $i : (X,M) \to (X',M')$ be a closed immersion over (S,L) with M' fine. Then, the *PD–envelope* (D,M_D) of (X,M) in (X',M') is defined having the following characterization. Etale locally on X, i factors as $(X,M) \xrightarrow{i'} (X'',M'') \xrightarrow{g} (X',M')$ with M'' fine, i' an exact closed immersion and g etale, and D is the usual PD-envelope of X in X'' with the inverse image M_D of M'. This (D,M_D) has the desired universality as in the classical case. If i is an exact closed immersion, then D is the usual PD-envelope of X in X' and M_D is the inverse image of M'.

For example, let $X = \mathrm{Spec}(k[t])$ with k a field and t an indeterminate, and let M be the log structure on X associated to the divisor "$t = 0$". Let $(X',M') \stackrel{\text{def}}{=} (X,M) \times_{\mathrm{Spec}(k)} (X,M)$ where $\mathrm{Spec}(k)$ is endowed with the trivial log structure (2.1) and let $i : (X,M) \to (X',M')$ be the diagonal morphism. Then, $X' = \mathrm{Spec}(k[t_1,t_2])$, M' is the log structure corresponding to the divisor "$t_1 = 0$" $\cup$ "$t_2 = 0$", i is a closed immersion but not exact. As (X'',M''), we can take $X'' = \mathrm{Spec}(k[t_1,t_2,t_1t_2^{-1},t_1^{-1}t_2])$ with the log structure M'' corresponding to the divisor "$t_1 = 0$" (= "$t_2 = 0$)". Hence the PD-envelope of (X,M) in (X',M') is $\mathrm{Spec}(k[t_1] < v >)$ where $v = t_1t_2^{-1} - 1$ regarded as an indeterminate endowed with the log structure associated to $N \to k[t_1] < v >;\ 1 \to t_1$. ($< >$ means the PD-polynomial ring.)

(2.17). — The theory of crystals is generalized to schemes with fine log structures as follows. A sheaf of $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ on $(X/S)^{\log}_{crys}$ is called a *crystal* if the map

$$\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F}_T \longrightarrow \mathcal{F}_T \,,$$

is an isomorphism for any morphism $T' \to T$ in $(X/S)^{\log}_{crys}$, where $\mathcal{F}_T$ and $\mathcal{F}_{T'}$ denote the sheaves on T_{et} and T'_{et} induced by $\mathcal{F}$, respectively. If $(X, M) \xrightarrow{i} (Z, N)$ is a closed immersion over (S, L) with N fine and (Z, N) smooth over (S, L), the following two categories (a) (b) are equivalent. Let (D, M_D) be the PD-envelope (2.16) of (X, M) in (Z, N).

(a) The category of crystals on $(X/S)^{\log}_{crys}$.

(b) The category of $\mathcal{O}_D$-modules $\mathcal{K}$ on $D_{et} = X_{et}$ endowed with

$$\nabla : \mathcal{K} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}_Z} \omega^1_{Z/S}$$

satisfying the following conditions (i)–(iii).

(i) ∇ is additive and

$$\nabla(am) = a\nabla(m) + m \otimes da \quad (a \in \mathcal{O}_D\,,\ m \in \mathcal{K})\,.$$

(ii) The composite

$$\mathcal{K} \xrightarrow{\nabla} \mathcal{K} \otimes_{\mathcal{O}_Z} \omega^1_{Z/S} \xrightarrow{\nabla} \mathcal{K} \otimes_{\mathcal{O}_Z} \omega^2_{Z/S}$$

is zero where we extend ∇ to

$$\mathcal{K} \otimes_{\mathcal{O}_Z} \omega^q_{Z/S} \xrightarrow{\nabla} \mathcal{K} \otimes_{\mathcal{O}_Z} \omega^{q+1}_{Z/S}\,; \quad \nabla(m \otimes \omega) = \nabla(m) \wedge \omega + m \otimes d\omega\,.$$

(iii) If $x \in D$ and y denotes the image of x in Z, and if $t_1, \dots, t_r$ are elements of $N^{gp}_{\overline{y}}$ such that $(\mathrm{d}\log(t_i))_i$ is a basis of the free $\mathcal{O}_{Z,\overline{y}}$-module $\omega^1_{Z/S,\overline{y}}$, then for any $m \in \mathcal{K}_{\overline{x}}$ we have

$$\left(\prod_{1 \le i \le r} \prod_{1 \le j \le c_i} (\nabla^{\log}_{t_i} - n_j) \right) (m) = 0$$

for some $c_i \geq 0$, $n_j \in \mathbb{Z}$. Here $\nabla^{\log}_{t_i}$ is defined by

$$\nabla(m) = \sum_{1 \le i \le r} \nabla^{\log}_{t_i}(m) \otimes \mathrm{d}\log(t_i) \qquad (m \in \mathcal{K}_{\overline{x}})\,.$$

The definition of the functor giving the equivalence of categories follows faithfully the classical case. In particular, $\mathcal{K} = \mathcal{F}_D$ as an $\mathcal{O}_D$-module.

Remark (2.17.1). — Under the conditions (i) and (ii), $(\mathcal{K} \otimes_{\mathcal{O}_Z} \omega^{\cdot}_{Z/S}, \nabla)$ becomes a complex.

Remark (2.17.2). — Under the conditions (i) and (ii), if (iii) is satisfied for one choice of $(t_i)_i$, then it is satisfied for any choice of $(t_i)_i$.

(2.17.3). — Let (Z', N') be the fiber product of two copies of (Z, N) over (S, L) in the category of schemes with fine log structures (2.13.5), let $(D', M_{D'})$ be the PD-envelope of (X, M) in (Z', N'), and let p_1, $p_2 : D' \to D$ be the two projections. Assume $t_1, \ldots, t_r$ as above are given globally. Then, $\mathcal{O}_{D'}$ is isomorphic via p_2 to the PD-polynomial ring $\mathcal{O}_D < s_1, \ldots, s_r >$ with $s_1, \ldots, s_r$ indeterminates, and the isomorphism is given by $s_i \longmapsto p_1^*(t_i)p_2^*(t_i)^{-1} - 1$. The composite

$$(*) \qquad \mathcal{K}_D \longrightarrow p_1^*\mathcal{K}_D \cong \mathcal{K}_{D'} \cong p_2^*\mathcal{K}_D \cong \bigoplus_{n \in \mathbb{N}^r} \prod_i s_i^{[n_i]} \otimes \mathcal{K}_D$$

is given by

$$(**) \qquad m \longmapsto \Big(\prod_{1 \le i \le r} s_i^{[n_i]} \cdot \prod_{1 \le i \le r} \prod_{1 \le j < n_i} (\nabla_{t_i}^{\log} - j)(m) \Big)_{n \in \mathbb{N}^r}.$$

A similar fact holds for crystals in the sense of derived categories ([B] V 3.6.1) : if $\mathcal{K}$ is a crystal in the derived category and $\nabla_{t_i}^{\log}$ denotes the $s_i^{[1]}$-component of $\mathcal{K}_D \to \mathcal{K}_{D'}$ then (*) is given by (**).

(2.18). — To give an explicit description of the crystalline cohomology of crystals (2.20), we give here a preliminary definition. For a morphism of schemes with fine log structures $f : (X, M) \to (S, L)$ (at this point we don't need any PD-structure) such that the underlying morphism $X \to S$ is locally of finite type, an *embedding system* for f is a pair of simplicial objects $(X^{\cdot}, M^{\cdot})$ and $(Z^{\cdot}, N^{\cdot})$ in the category of schemes with fine log structures endowed with morphisms

$$(X^{\cdot}, M^{\cdot}) \to (X, M), \quad (X^{\cdot}, M^{\cdot}) \longrightarrow (Z^{\cdot}, N^{\cdot}), \quad (Z^{\cdot}, N^{\cdot}) \longrightarrow (S, L)$$

(here (X, M) and (S, L) are regarded as constant simplicial objects) satisfying the following conditions (i)–(iv).

(i) The diagram

$$\begin{array}{ccc} (X^{\cdot}, M^{\cdot}) & \longrightarrow & (Z^{\cdot}, N^{\cdot}) \\ \downarrow & & \downarrow \\ (X, M) & \longrightarrow & (S, L) \end{array}$$

is commutative.

(ii) The morphism $X^{\cdot} \to X$ is a hyper–covering for the etale topology, and M^i $(i \geq 0)$ is the inverse image of M on X^i for each i.

(iii) Each $(Z^i, N^i) \to (S, L)$ is smooth.

(iv) Each $(X^i, M^i) \to (Z^i, N^i)$ is a closed immersion.

It is easily seen that embedding systems for f exist.

Let $(X^{\cdot})^{\sim}_{et}$ be the topos whose object is a system which associates to each $i \geq 0$ a sheaf $\mathcal{F}^i$ on X^i_{et}, and to each increasing map $s : \{0, \ldots, i\} \to \{0, \ldots, j\}$ a morphism $\rho_s : \underline{s}^{-1}(\mathcal{F}^i) \to \mathcal{F}^j$ where $\underline{s}$ denotes the morphism $X^j \to X^i$ corresponding to s, satisfying $\rho_{id.} = id.$ and $\rho_{st} = \rho_s \cdot \underline{s}^{-1}(\rho_t)$.

The obvious morphism of topoi $\theta : (X^{\cdot})^{\sim}_{et} \to X^{\sim}_{et}$ ($X^{\sim}_{et}$ denotes the topos of sheaves on X_{et}) satisfies

$$\mathcal{F} \xrightarrow{\sim} R\theta_* \theta^{-1}(\mathcal{F})$$

for any abelian sheaf $\mathcal{F}$ on X_{et} ([SD]). For a complex $\mathcal{F}^{\cdot}$ in $(X^{\cdot})^{\sim}_{et}$ bounded below, $R\theta_*(\mathcal{F}^{\cdot})$ is computed as follows. By replacing $\mathcal{F}^{\cdot}$ with a complex which is quasi–isomorphic to $\mathcal{F}^{\cdot}$, we assume $R^q\theta_{i*}(\mathcal{F}^{ij}) = 0$ for any $q > 0$ and any i, j, where θ_i denotes $X^i \to X$ and $\mathcal{F}^{ij}$ denotes the degree j part of the complex on X^i defined by $\mathcal{F}^{\cdot}$. Then $R\theta_*(\mathcal{F}^{\cdot})$ is represented by the double complex $(\theta_{i*}(\mathcal{F}^{ij}))_{ij}$ ([SD]).

Definition (2.19). — *With notations as in* (2.14), *assume X is locally of finite type over S. Fix an embedding system $((X^{\cdot}, M^{\cdot}), (Z^{\cdot}, N^{\cdot}))$ for $(X, M) \to (S, L)$, and let (D^i, M_{D^i}) be the PD–envelope of (X^i, M^i) in*

(Z^i, N^i). *For a crystal* $\mathcal{F}$ *on* $(X/S)^{\log}_{crys}$, *define the complex* $C_{X/S,\mathcal{F}}$ *in* $(X^{\cdot})^{\sim}_{et}$ *by*

$$C_{X/S,\mathcal{F}} = (\mathcal{F}_{D^{\cdot}} \xrightarrow{\nabla} \mathcal{F}_{D^{\cdot}} \otimes_{\mathcal{O}_{Z^{\cdot}}} \omega^1_{Z^{\cdot}/S} \xrightarrow{\nabla} \mathcal{F}_{D^{\cdot}} \otimes_{\mathcal{O}_{Z^{\cdot}}} \omega^2_{Z^{\cdot}/S} \xrightarrow{\nabla} \cdots),$$

and call it the crystalline complex of $\mathcal{F}$. *If* $\mathcal{F} = \mathcal{O}_{X/S}$ *we denote* $C_{X/S,\mathcal{F}}$ *simply by* $C_{X/S}$.

PROPOSITION (2.20). — *Let the situation be as in* (2.19). *Let* $u^{\log}_{X/S}$ *be the canonical morphism from* $(X/S)^{\log}_{crys}$ *to* X_{et}. *Then there exists a canonical isomorphism*

$$Ru^{\log}_{X/S*}(\mathcal{F}) \cong R\theta_*(C_{X/S,\mathcal{F}}).$$

(2.21). — In (2.19), crystalline complexes associated to two different embedding systems E, E' are related to each other as follows. There is an embedding system E'' having morphisms $E'' \to E$ and $E'' \to E'$ of embedding systems. Denote the crystalline complex associated to E (resp. E', resp. E'') by $C_{X/S,\mathcal{F}}$ (resp. $C'_{X/S,\mathcal{F}}$, resp. $C''_{X/S,\mathcal{F}}$). Then the canonical morphisms

$$R\theta_*(C_{X/S,\mathcal{F}}) \longrightarrow R\theta''_*(C''_{X/S,\mathcal{F}}) \longleftarrow R\theta'_*(C'_{X/S,\mathcal{F}})$$

with θ, θ', θ'' the evident morphisms of topoi, are isomorphisms, and compatible with the isomorphism of (2.20).

The following lemma on crystalline complexes is used frequently in this paper.

LEMMA (2.22). — *In* (2.19), *if* $f : (X, M) \to (S, L)$ *factors as* $(X, M) \xrightarrow{\overline{f}} (\overline{S}, \overline{L}) \xrightarrow{i} (S, L)$ *with* $\overline{f}$ *smooth integral and* i *an exact closed immersion such that the ideal of* $\overline{S}$ *in* S *is a sub-PD-ideal of* I, *then* $C_{X/S}$ *is flat over* $\theta^{-1}f^{-1}(\mathcal{O}_S)$ *for any choice of an embedding system.*

PROOF. It is enough to show that D^i is flat over S. First, we show that it is sufficient to prove (2.22) for one choice of an embedding system. If $((X^{\cdot}, M^{\cdot}), (Z^{\cdot}, N^{\cdot}))$ is an embedding system, $(Z^{\cdot}, N^{\cdot})$ is integral over $(S^{\cdot}, L^{\cdot})$

on a neighbourhood of X in Z. So we may assume $(Z^\cdot, N^\cdot)$ is integral over (S, L). Assume we have two embedding systems $((X^\cdot, M^\cdot), (Z^\cdot, N^\cdot))$, $((X^\cdot)', (M^\cdot)')$, $((Z^\cdot)', (N^\cdot)'))$ with $(Z^\cdot, N^\cdot)$ and $((Z^\cdot)', N^\cdot)')$ integral over (S, L), and let $(((X^\cdot)'', (M^\cdot)''), ((Z^\cdot)'', (N^\cdot)'')$ be the third embedding system defined by $(X^i)'' = X^i \times_X (X^i)'$ and $((Z^i)'', (N^i)'') = (Z^i, N^i) \times_{(S,L)} ((Z^i)', (N^i)')$. If $(D^i, M_D i)$ (resp. $((D^i)', M_{(D^i)'})$) denotes the PD–envelope of (X^i, M^i) in (Z^i, N^i) (resp. $((X^i)', (M^i)')$ in $((Z^i)', (N^i)')$), and similarly $((D^i)'', M_{(D^i)''})$ denotes the PD–envolope of (X^i, M^i) in $(Z^i)'', (N^i)'')$ then etale locally

$$(D^i)'' \cong \operatorname{Spec}(\mathcal{O}_{D^i} < t_1, \ldots, t_r >)$$

$$(D^i)'' \cong \operatorname{Spec}(\mathcal{O}_{(D^i)'} < t'_1, \ldots, t'_{r'} >)$$

where $t_1, \ldots, t_r, t'_1, \ldots, t'_{r'}$ are indeterminates and $< >$ means the PD polynomial ring (same proof as in [K1] (6.5)). Hence the flatness of D^i over S is equivalent to that of $(D^i)'$. We may work locally on X, so we can choose an embedding system such that $X^i = X$ for any i, (Z^i, N^i) is a constant simplicial object (Z, N), $X = Z \times_S \overline{S}$ and $(Z, N) \to (S, L)$ is smooth and integral. Then, $D^i = Z$ for any i and Z is flat over S by (2.10.2).

The base change theorem for crystalline cohomology ([B] V 3.5) is generalized to log structures (cf. [K1] (6.10)). We shall use the following special case of the generalization.

Proposition (2.23). — *Let*

$$\begin{array}{ccc}
(X', M') & \xrightarrow{g} & (X, M) \\
\downarrow & \square & \downarrow \\
(\overline{S}', \overline{L}') & \longrightarrow & (\overline{S}, \overline{L}) \\
\downarrow & & \downarrow \\
(S', L', I', \gamma') & \longrightarrow & (S, L, I, \gamma)
\end{array}$$

be a commutative diagram of schemes with fine log structures such that : $f : (X, M) \to (\overline{S}, \overline{L})$ is smooth integral and the upper square is cartesian, the *lower two vertical arrows are exact closed immersions,* $\overline{S}$ (resp. $\overline{S}'$) *is defines in* S (resp. S') *by a PD–subideal of* I (resp. I'), *and* $\overline{S}' \to \overline{S}$ *is radicial. Then for a flat crystal* $\mathcal{F}$ *on* $(X/S)^{\log}_{crys}$, *we have*

$$\mathcal{O}_{S'|X'} \otimes^{L}_{\mathcal{O}_{S|X'}} Ru^{\log}_{X/S^*}(\mathcal{F})_{|X'} \xrightarrow{\sim} Ru^{\log}_{X'/S'}(g^*_{crys}(\mathcal{F}))\,.$$

Here for a sheaf $\mathcal{G}$ on X, S or S', $\mathcal{G}_{|X'}$ denotes the inverse image of $\mathcal{G}$ on X'. This is proved by using

$$\mathcal{O}_{S'|X'} \otimes_{\mathcal{O}_{S|X'}} (C_{X/S,\mathcal{F}|X'}) \xrightarrow{\sim} C_{X'/S',g^*_{crys}(\mathcal{F})}$$

and the flatness of $C_{X/S,\mathcal{F}}$ over $\mathcal{O}_S$ (2.22).

The following result (2.24) is the log structure version of the result of Berthelot–Ogus [BO2] (1.4) on the bijectivity of the relative frobenius map $\otimes\mathbb{Q}$ on the relative crystalline cohomology.

Proposition (2.24). — *Let* p *be a prime number and let* $f : (X, M) \to (S, L)$ *be a smooth morphism of Cartier type* (2.12) *between schemes over* $\mathbb{F}_p$ *with fine log structures. Assume we are given schemes with fine log structures* (T_n, L_n), $n \geq 1$ *with exact closed immersions*

$$(S, L) \hookrightarrow (T_1, L_1) \hookrightarrow (T_2, L_2) \hookrightarrow \cdots$$

and a PD–structure on the ideal of S *in* T_n *for each* n, *and assume that the following* (i)–(iv) *are satisfied.*

(i) *Each* $T_n \to T_{n+1}$ *is a PD–morphism.*

(ii) T_n *is a flat scheme over* $\mathbb{Z}/p^n\mathbb{Z}$ *and* $T_n \xrightarrow{\sim} T_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z}$.

(iii) $\{\mathrm{rank}_x(\omega^1_{X/S})\}_{x\in X}$ *is bounded. (Remark : the condition (iii) is satisfied if* X *is quasi–compact).*

Consider the diagram (2.12.1) and let

$$\varphi : Ru^{\log}_{X'/T_n^*}(\mathcal{O}_{X'/S}) \longrightarrow Ru^{\log}_{X/T_n^*}(\mathcal{O}_{X/T_n})$$

be the morphism of projective systems induced by g *in (2.12.1), where we identify* X_{et} *and* X'_{et} *via the canonical equivalence. Then, if* r $=$

$\max\{\mathrm{rank}_x(\omega^1_{X/S})\,;\ x \in X\}$, *there exists a homomorphism of projective systems* $\psi : Ru^{\log}_{X/T^*_n}(\mathcal{O}_{X/T_n}) \to Ru^{\log}_{X'/T^*_n}(\mathcal{O}_{X'/T_n})$ *satisfying* $\varphi\psi = p^r$ *and* $\psi\varphi = p^r$.

PROOF. We follow faithfully the method in [BO2].

There exist an etale covering $U \to X$, schemes with fine log structures (Z_n, N_n) and (Z'_n, N'_n) over $\mathbb{Z}/p^n\mathbb{Z}$ ($n \geq 1$), smooth integral morphisms $(Z_n, N_n) \to (T_n, L_n)$ and $(Z'_n, N'_n) \to (T_n, L_n)$, exact closed immersions $(Z_n, N_n) \to (Z_{n+1}, N_{n+1})$ and $(Z'_n, N'_n) \to (Z_{n+1}, N_{n+1})$ over (T_{n+1}, L_{n+1}) which induce $Z_n \xrightarrow{\sim} Z_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z}$ and $Z'_n \xrightarrow{\sim} Z'_{n+1} \otimes \mathbb{Z}/p^n\mathbb{Z}$, respectively, morphisms $(Z_n, N_n) \to Z'_n, N'_n)$ over (T_n, L_n) which are compatible with the above closed immersions, and exact closed immersions $(U, M_{|U}) \to (Z_1, N_1)$ and $(U', M'_{|U'}) \to (Z'_1, N'_1)$ where $U' = U \times_X X'$ such that the two squares in the diagram

$$\begin{array}{ccccc}
(U, M_{|U}) & \longrightarrow & (U', M'_{|U'}) & \longrightarrow & (S, L) \\
\downarrow & & \downarrow & & \downarrow \\
(Z_1, N_1) & \longrightarrow & (Z'_1, N'_1) & \longrightarrow & (T_1, L_1)
\end{array}$$

are commutative and cartesian. We identify $(X^{\cdot})^{\sim}_{et}$ and $(X'^{\cdot})^{\sim}_{et}$. We consider the crystalline complex C_{X/T_n} (resp. C_{X'/T_n}) defined with respect to the embedding system $((X^i, M^i), (Z^i_n, N^i_n))$ (resp. $((X'_n)^i, (M'_n)^i), ((Z'_n)^i, (N'_n)^i)))$ where X^i (resp. $(X')^i$ is the fiber product of $i+1$ copies of U (resp. U') over X (resp. X') and (Z^i_n, N^i_n) (resp. $((Z')^i_n, (N')^i_n)$ is the fiber product of $i+1$ copies of (Z_n, N_n) (resp. (Z'_n, N'_n)) over (T_n, L_n). Note that C_{X/T_n} is flat over $\mathbb{Z}/p^n\mathbb{Z}$ (2.22) and $C_{X/T_{n+1}} \otimes \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} C_{X/T_n}$, and the same things hold for C_{X'/T_n}. We define the complex $E^{\cdot}_n$ on $(X^{\cdot})_{et}$ as follows. Let

$$\widetilde{E}^q_n = \{a \in p^q C^q_{Y/T_n};\ da \in p^{q+1} C_{Y/T_n}\} \subset C^q_{Y/T_n}\,,$$

$$E^q_n = \widetilde{E}^q_m / p^n \widetilde{E}^q_m \quad \text{for } m > n+q\,.$$

Then E^q_n is independent of the choice of $m > n+q$, and with $d : E^q_n \to E^{q+1}_n$ induced by $d : C^q_m \to C^{q+1}_m$ for $m > n+q+1$, $(E^{\cdot}_n, d)$ becomes a complex. As is seen easily, the image of $\varphi : C^q_{X'/T_n} \to C^q_{X/T_n}$ is contained in $\widetilde{E}^q_n$, and thus $\varphi : C^q_{X'/T_m} \to C^q_{X/T_m}$ for $m > n+q$ defines $\overline{\varphi} : C^q_{X'/T_n} \to E^q_n$.

LEMMA (2.25). — *$\overline{\varphi}$ defines a quasi-isomorphism of complexes* $(C^{\cdot}_{X'/T_n}, d) \to (E^{\cdot}_n, d)$.

The proof of (2.25) is identical with the classical case given in [BO1] § 8, [BO2] § 1 and we omit it.

Now we finish the proof of (2.24). For any complex C and for $i \in \mathbf{Z}$, let $\tau_{\leq i}C$ be the subcomplex of C whose degree q part is C^q (resp. 0, resp. $\mathrm{Ker}(d : C^q \to C^{q+1})$) for $q < i$ (resp. $q > i$, resp. $q = i$). Let $r = \max\{\mathrm{rank}_x(\omega^1_{X/S})\}$. Then, the canonical morphism $\tau_{\leq r}C \to C$ is a quasi-isomorphism if $C = C_{X/T_n}, C_{X'/T_n}$ or $E^{\cdot}_n$ since $\mathcal{H}^q(C) = 0$ if $q > r$ for these C. Let $\psi' : \tau_{\leq r}C_{X/T_n} \to \tau_{\leq r}E^{\cdot}_n$ be the map induced from $\tau_{\leq r}C_{X/T_m} \to \tau_{\leq r}E^{\cdot}_m$, $a \longmapsto p^r a$ with $m > n+r$, and define ψ to be the composite map in the derived category

$$Ru^{\log}_{X/T_n^*}(\mathcal{O}_{X/T_n}) \xrightarrow{\text{by } \psi'} R\theta_* E^{\cdot}_n \xleftarrow[\cong]{\text{by } \overline{\varphi}} Ru^{\log}_{X'/T_n^*}(\mathcal{O}_{X'/T_n}) .$$

It is easy to see that $\varphi\psi = p^r$ and $\psi\varphi = p^r$.

3. — Crystalline construction of $(D, \varphi, \mathcal{N})$

We construct $(D, \varphi, \mathcal{N})$ using the theory of crystalline cohomology with logarithmic poles in a more general situation than § 1.

DEFINITION (3.1). — *Let p be a prime number and let S be a scheme over $\mathbf{F}_p$ with a log structure M. Let $n \geq 1$, and let $W_n(S) = \mathrm{Spec}(W_n(\mathcal{O}_S))$. We define the log structure $W_n(M)$ on $W_n(S)$ called the canonical lifting of M to be $M \oplus \mathrm{Ker}(W_n(\mathcal{O}_S)^\times \to \mathcal{O}_S^\times)$ which is endowed with the homomorphism to $W_n(\mathcal{O}_S)$ induced by $M \to W_n(\mathcal{O}_S)$; $a \longmapsto (\alpha(a), 0, \ldots, 0)$. The morphism $(W_n(S), W_n(M)) \to (W_n(S), W_n(M))$ defined by the usual frobenius F : $W_n(S) \to W_n(S)$ and by $F^{-1}(W_n(M)) \cong W_n(M) \to W_n(M)$; (p on M) $\oplus$ (F^* on $\mathrm{Ker}(W_n(\mathcal{O}_S)^\times \to \mathcal{O}_S^\times)$ is called the frobenius of $(W_n(S), W_n(M))$.*

(3.2). — Let k be a perfect field of characteristic $p > 0$ and fix a fine log structure L on $\mathrm{Spec}(k)$. Let $W_n = \mathrm{Spec}(W_n(k))$. Then we have the log structure $W_n(L)$ on W_n. Let Y be a scheme with a fine log structure M and with a smooth morphism $f:(Y,M)\to(\mathrm{Spec}(k),L)$. Take $m\in\mathbb{Z}$ and let

$$D_n = H^m(((Y,M)/(W_n,W_n(L)))_{crys},\ \mathcal{O}_{Y/W_n})$$

be the m–th crystalline cohomology group of (Y,M) over $(W_n,W_n(L))$, where W_n is endowed with the usual PD–structure on pW_n. In particular, it follows from (2.20) that

$$D_1 = H^m(Y,\omega_Y^{\cdot}) \quad \text{where} \quad \omega_Y^{\cdot} = \omega^{\cdot}_{(Y,M)/(\mathrm{Spec}(k),L)}\,. \tag{3.2.1}$$

The absolute frobenius of (Y,M) and that of $(W_n,W_n(L))$ induce

$$\varphi : D_n \longrightarrow D_n\,.$$

Let $D_\infty=\varprojlim D_n$, $D=D_\infty\otimes\mathbb{Q}$.

If f is smooth and integral and if Y is proper over k, (3.2.1) and the exact sequence of crystalline complexes

$$0\longrightarrow C_{Y/(W_m,W_m(L))}\xrightarrow{p^n}C_{Y/(W_{m+n},W_{m+n}(L))}\longrightarrow C_{Y/(W_n,W_n(L))}\longrightarrow 0$$

(2.22) (here Y is endowed with M; we do not abbreviate $W.(L)$ since sometimes we shall consider also the trivial log structure on W) show that D_n is of finite length over $W_n(k)$ and D_∞ is finitely generated over $W(k)$. If f is smooth and of Cartier type, $\varphi: D\to D$ is bijective by (2.23) and (2.24). (Here, (2.23) is applied by taking the frobenius $(W_n,W_n(L))\to(W_n,W_n(L))$ as $(S',L')\to(S,L)$ of (2.23), and the frobenius $(\mathrm{Spec}(k),L)\to \mathrm{Spec}(k),L)$ as $(\overline{S}',\overline{L}')\to(\overline{S},\overline{L})$. We then obtain

$$W_n(k)\underset{\varphi}{\searrow}\otimes_{W_n(k)}Ru^{\log}_{Y/(W_n,W_n(L))*}(\mathcal{O}_{Y/W_n})\cong Ru^{\log}_{Y'/(W_n,W_n(L))*}(\mathcal{O}_{Y'/W_n})\,.)$$

The bijectivity of $\varphi: D\to D$ is proved also using the de Rham–Witt complex of § 4, by the same argument as in (1.3).

Without the Cartier type assumption, the bijectivity of φ need not hold as in the simple example (3.3) below.

The monodromy operator will be discussed in (3.4)–(3.6). In the situation of semi–stable reduction, we will see in (4.20) that this $(D_n, \varphi, \mathcal{N})$ coincides with that of § 1.

(3.3). — Consider the case $Y = \mathrm{Spec}(k[t]/(t^r))$ with t and indeterminate, $(p, r) = 1, L$ (resp. M) is the log structure on $\mathrm{Spec}(k)$ (resp. Y) associated to $\mathbb{N} \to \mathcal{O}_{\mathrm{Spec}(k)}$; $1 \longmapsto 0$, (resp. $\mathbb{N} \to \mathcal{O}_Y$; $1 \longmapsto t$), and $f : (Y, M) \to (\mathrm{Spec}(k), L)$ is induced from $\mathbb{N} \to \mathbb{N}$; $1 \longmapsto r$. Such $(Y, M) \to (\mathrm{Spec}(k), L)$ appears as the reduction of a tamely ramified extension of a discrete valuation ring (2.13.1). Then, f is smooth and integral, but if $r > 1$, it is not of Cartier type. The crystalline cohomology of degree m of (Y, M) over $(W_n, W_n(L))$ vanishes for $m \neq 0$, and for $m = 0$ we have $D_n = W_n[t]/(t^r)$ with the frobenius φ which extends the usual frobenius of $W_n(k)$ by $\varphi(t) = t^p$. Hence $\varphi : D \to D$ is not bijective if $r > 1$.

(3.4). — Now we define the monodromy operator.

Let $f : (Y, M) \to (\mathrm{Spec}(k), L)$ and (by fixing m) D_n be as in (3.2), and assume that f is smooth and L is the log structure associated to $\mathbb{N} \to k$; $1 \longmapsto 0$. We define the *monodromy operator* $\mathcal{N} : D_n \to D_n$ in two ways (3.5) (3.6).

(3.5). — Let (D, L_D) be the PD–envelope of $(\mathrm{Spec}(k), L)$ in the fiber product of two copies of $(W_n, W_n(L))$ over W_n where the last W_n is endowed with the trivial log structure, and let $p_i : (D, L_D) \to (W_n, W_n(L))$ be the two projections. Let e be any section of L whose image in $L/\mathbb{G}_m \cong \mathbb{N}$ is $1 \in \mathbb{N}$, and regard it as a section of $W_n(L)$ via the embedding $L \subset W_n(L)$. Then, $D \cong \mathrm{Spec}(W_n < u - 1 >)$ where u is the image of $p_1^*(e)p_2^*(e)^{-1}$, which is independent of the choice of e and which is regarded as an indeterminate in this isomorphism. Let

$$\mathcal{K} = R\Gamma(((Y, M)/(W_n, W_n(L)))_{crys}, \mathcal{O}_{Y/W_n})$$
$$\mathcal{K}' = R\Gamma(((Y, M)/(D, L_D))_{crys}, \mathcal{O}_{Y/D}) .$$

Then we have a morphism

$$\mathcal{K} \longrightarrow Lp_1^*(\mathcal{K}) \cong \mathcal{K}' \cong Lp_2^*(\mathcal{K}) \cong \bigoplus_{i \in N} (u-1)^{[i]} \otimes \mathcal{K} \tag{3.5.1}$$

where the first and the second isomorphisms are by the base change theorem (2.23). We define the endomorphism $\mathcal{N} : D_n = H^m(\mathcal{K}) \to D_n$ to be the map induced from the $(u-1)^{[1]}$–component $\mathcal{K} \to \mathcal{K}$ of the morphism (3.5.1) (then, (3.5.1) is given by

$$\sum_{i \geq 0} \big((u-1)^{[i]} \otimes \prod_{0 \leq j < i} (\mathcal{N} - j) \big) \,).$$

The property $\mathcal{N}\varphi = p\varphi\mathcal{N}$ is easily verified.

(3.6). — Another construction of $\mathcal{N}$ is as follows. Consider the exact closed immersion $(W_n, W_n(L)) \to (\mathrm{Spec}(W_n[t]), \mathcal{L})$ where $\mathcal{L}$ is the log structure associated to $\mathbb{N} \to W_n[t]$; $1 \longmapsto t$ (here t is an indeterminate) defined by $W_n[t] \to W_n$; $t \longmapsto 0$ and $\mathcal{L} \to W_n(L)$; $1 \in \mathbb{N} \longmapsto 1 \in \mathbb{N}$. Take an embedding system $((Y^{\cdot}, M^{\cdot}), (Z^{\cdot}, N^{\cdot}))$ of $(Y, M) \to (\mathrm{Spec}(W_n[t]), \mathcal{L})$. Let C_{Y/W_n}, where W_n is endowed with the trivial log structure (resp. $C_{Y/\mathrm{Spec}(W_n<t>)}$, where $W_n < t >$ is the PD polynomial ring over W_n in one variable t and $\mathrm{Spec}(W_n < t >)$ is endowed with the inverse image of $\mathcal{L}$), be the crystalline complex associated to the embedding system $((Y^{\cdot}, M^{\cdot}), (Z^{\cdot}, N^{\cdot}))$ (resp. $((Y^{\cdot}, M^{\cdot}), (Z^{\cdot} \times_{\mathrm{Spec}(W_n[t])} \mathrm{Spec}(W_n < t >), (N^{\cdot})'))$ where $(N^{\cdot})'$ is the inverse image of $N^{\cdot}$). We obtain an exact sequence

$$0 \longrightarrow C_{Y/\mathrm{Spec}(W_n<t>)}[-1] \longrightarrow C_{Y/W_n} \longrightarrow C_{Y/\mathrm{Spec}(W_n<t>)} \longrightarrow 0$$

where the second arrow is $a \longmapsto a \wedge d\log(t)$. Since $W_n \otimes_{W_n<t>} C_{Y/\mathrm{Spec}(W_n<t>)}$ with respect to $W_n < t > \to W_n$; $t^{[i]} \to 0$ $(i \geq 1)$ is the crystalline complex $C_{Y/(W_n, W_n(L))}$ with respect to the embedding system $((Y^{\cdot}, M^{\cdot}), (Z^{\cdot} \times_{\mathrm{Spec}(W_n[t])} W_n, (N^{\cdot})''))$ where $(N^{\cdot})''$ is the inverse image of $N^{\cdot}$, we obtain an exact sequence

$$0 \longrightarrow C_{Y/(W_n, W_n(L))} \longrightarrow W_n \otimes_{W_n<t>} C_{Y/W_n} \longrightarrow C_{Y/(W_n, W_n(L))} \longrightarrow 0 .$$

We define $\mathcal{N}$ to be the connecting homomorphism of this exact sequence.

The coincidence of the two definitions of $\mathcal{N}$ given in (3.5) and (3.6) is proved by the method of [B] V 3.6.

4. — De Rham–Witt complexes

In this section, k denotes a perfect field of characteristic $p > 0$, and Y denotes a scheme with a fine log structure M and with a smooth morphism of Cartier type $f : (Y, M) \to (\mathrm{Spec}(k), L)$.

We consider in this section the de Rham–Witt complex of (Y, M) over $(\mathrm{Spec}(k), L)$ generalizing [H1] [H2] which treated the semi-stable reduction case. We give the definition (4.1), descriptions of the structure of the de Rham–Witt complex in (4.4)–(4.7), relation with the crystalline cohomology in (4.19), and the relation with § 1 in (4.20).

In this section, we shall consider the two log structures $W_n(L)$ and $\mathcal{O}_{W_n}^{\times}$ on W_n. We do not abbreviate $W_n(L)$ when W_n is endowed with $W_n(L)$, and abbreviate $\mathcal{O}_{W_n}^{\times}$ when W_n is endowed with $\mathcal{O}_{W_n}^{\times}$. For example, in the notation $Ru^{\log}_{Y/(W_n, W_n(L))}$ (resp. $Ru^{\log}_{Y/W_n}$), Y is endowed with M and W_n is endowed with $W_n(L)$ (resp. $\mathcal{O}_{W_n}^{\times}$).

(4.1). — We define the de Rham–Witt complex as follows. Let

$$W_n\omega_Y^q = R^q u^{\log}_{Y/(W_n, W_n(L))*}(\mathcal{O}_{Y/W_n}).$$

We define the operators

$$d : W_n\omega_Y^q \to W_n\omega_Y^{q+1}, \quad F : W_{n+1}\omega_Y^q \to W_n\omega_Y^q, \quad V : W_n\omega_Y^q \to W_{n+1}\omega_Y^q$$

satisfying

$$(4.1.1) \quad d^2 = 0, \quad FV = VF = p, \quad dF = pFd, \quad Vd = pdV, \quad FdV = d$$

as below, following the classical case [IR]. $W_n\omega_Y^{\cdot}$ becomes a complex with respect to the differential d.

In this section, we choose embedding systems $((Y^i, M^i), (Z_n^i, N_n^i))$ as in the proof of 2.24 (with X replaced by Y and (T_n, L_n) by $(W_n, W_n(L))$), and denote the corresponding crystalline complexes $C_{Y/(W_n,W_n(L))}$ simply by $C^{\cdot}_n$. Note $C^{\cdot}_n$ is flat over W_n and $C^{\cdot}_n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\sim} C^{\cdot}_m$ for $m \leq n$.

First, d is defined to be the connecting homomorphism in the exact sequence of cohomology sheaves associated to the exact sequence of crystalline complexes

$$0 \longrightarrow C^{\cdot}_n \xrightarrow{p^n} C^{\cdot}_{2n} \longrightarrow C^{\cdot}_n \longrightarrow 0\,.$$

Next, F (resp. V) is the map induced by $C^{\cdot}_{n+1} \to C^{\cdot}_n$ (resp. $p : C^{\cdot}_n \to C^{\cdot}_{n+1}$).

The relations (4.1.1) are proved easily. For $n = 1$, we have the Cartier isomorphism

$$C : W_1\omega^q_Y = \mathcal{H}^q(\omega^{\cdot}_Y) \xrightarrow{\sim} \omega^q_Y \quad \text{(cf. (3.2.1) and (2.12.2))}\,. \tag{4.1.2}$$

(4.2). — We define a canonical homomorphism

$$\pi_n : W_{n+1}\omega^{\cdot}_Y \longrightarrow W_n\omega^{\cdot}_Y$$

as follows. With the notations of the proof of 2.24, the map $p^q : C^q_{n+1} \to C^q_{n+q+1}$ sends $\mathrm{Ker}(C^q_{n+1} \to C^{q+1}_{n+1})$ into $\mathrm{Ker}(\widetilde{E}^q_{n+q+1} \xrightarrow{d} E^{q+1}_n)$ and induces $\pi_n : \mathcal{H}^q(C^{\cdot}_{n+1}) = W_{n+1}\omega^q_Y \to \mathcal{H}^q(E^{\cdot}_n) \cong W_n\omega^q_Y$ where the last isomorphism is by (2.25).

We call this map π_n and its composite $W_m\omega^q_Y \to W_n\omega^q_Y$ ($m \geq n$) the canonical projection.

Definition (4.3). — *Define a chain of subsheaves of* ω^q_Y

$$0 = B_0\omega^q_Y \subset B_1\omega^q_Y \subset B_2\omega^q_Y \subset \cdots \subset Z_2\omega^q_Y \subset Z_1\omega_Y \subset Z_0\omega^q_Y = \omega^q_Y$$

by the formulas

$$B_0\omega^q_Y = 0\,, \qquad Z_0\omega^q_Y = \omega^q_Y\,,$$

$$B_1\omega^q_Y = d\omega^{q-1}_Y\,, \qquad Z_1\omega^q_Y = \mathrm{Ker}(d : \omega^q_Y \longrightarrow \omega^{q-1}_Y)\,,$$

$$B_n\omega^q_Y \underset{\cong}{\xrightarrow{C^{-1}}} B_{n+1}\omega^q_Y/B_1\omega^q_Y$$

$$Z_n\omega^q_Y \underset{\cong}{\xrightarrow{C^{-1}}} Z_{n+1}\omega^q_Y/Z_1\omega^q_Y$$

and by induction on n (cf. Illusie [Il1]0.(2.2.2)).

We can generalize the structure theorem for the de Rham–Witt complexes [Il1] I §3B to the case with log structures.

THEOREM (4.4). — *The map $\pi_n : W_{n+1}\omega_Y^q \to W_n\omega_Y^q$ is surjective and the composite map*

$$\omega_Y^q \oplus \omega_Y^{q-1} \xrightarrow[\cong]{C^{-1}} W_1\omega_Y^q \oplus W_1\omega_Y^{q-1} \xrightarrow{(V^n, dV^n)} W_{n+1}\omega_Y^q$$

induces an isomorphism

$$(\omega_Y^q \oplus \omega_Y^{q-1})/R_n^q \xrightarrow{\sim} \mathrm{Ker}(\pi_n : W_{n+1}\omega_Y^q \longrightarrow W_n\omega_Y^q)$$

where R_n^q is defined by the exact sequence

$$0 \longrightarrow R_n^q \longrightarrow B_{n+1}\omega_Y^q \oplus Z_n\omega_Y^{q-1} \xrightarrow{(C^n, dC^n)} B_1\omega_Y^q \longrightarrow 0\,.$$

PROOF. The problem is local and hence we consider the crystalline complexes for embedding systems consisting of constant simplicial objects. The facts that $\mathrm{Image}(V^n, dV^n) \subset \mathrm{Ker}(\pi_n)$ and R_n^q dies in $\mathrm{Ker}(\pi_n)$ are easy and left to the reader. The surjectivity of π_n follows from the surjectivity of $p^q : C^q_{n+1,d=0} \to E^q_{n,d=0}$ which is checked easily. The surjectivity of $(V^n, dV^n) : W_1\omega_Y^q \oplus W_1\omega_Y^{q-1} \to \mathrm{Ker}(\pi_n)$ is proved also easily. Indeed, if $a \in C^q_{n+1,d=0}$ and the class of a in $W_{n+1}\omega_Y^q$ is annihilated by π_n, then $p^q a = d(p^{q-1}b) + p^n(p^q c)$ in C^q_{n+q+1} for some $b \in C^{q-1}_{n+q+1}$ and for some $c \in C^q_{n+q+1}$ such that $dc \in pC^{q+1}_{n+q+1}$. If $\bar{b}$ (resp. $\bar{c}$) denotes the class of b (resp. c) in $W_1\omega_Y^{q-1}$ (resp. $W_1\omega_Y^q$), we obtain $a = dV^n(\bar{b}) + V^n(\bar{c})$ in $W_{n+1}\omega_Y^q$.

Finally we prove that the kernel of the map in problem $s : \omega_Y^q \oplus \omega_Y^{q-1} \to$

$\mathrm{Ker}(\pi_n)$ coincides with R_n^q. The commutative diagram

$$\begin{array}{ccc}
W_1\omega_Y^q \oplus W_1\omega_Y^{q-1} & \xrightarrow{(V^n, dV^n)} & W_{n+1}\omega_Y^q \\
\Big\downarrow{\scriptstyle pr_2} & & \Big\downarrow{\scriptstyle F} \\
W_1\omega_Y^{q-1} & \xrightarrow{dV^{n-1}} & W_n\omega_Y^q
\end{array}$$

and induction on n show that if $(a,b) \in \mathrm{Ker}(s)$, then $b \in Z_n\omega_Y^{q-1}$. Hence $(a,b) \equiv (a',0) \bmod R_n^q$ for some $a' \in \omega_Y^q$. Since the kernel of $V : W_n\omega_Y^q \to W_{n+1}\omega_Y^q$ comes from the boundary map $W_1\omega_Y^{q-1} \to W_n\omega_Y^q$ which coincides with dV^{n-1} as is easily seen, we have $V^{n-1}C^{-1}(a') = dV^{n-1}C^{-1}(c)$ for some $c \in \omega_Y^{q-1}$. Therefore, by induction on n, a' belongs to $B_n\omega_Y^q$ ($= \mathrm{Ker}\, C^n : B_{n+1}\omega_Y^q \to B_1\omega_Y^q$) and hence $(a',0) \in R_n^q$. Q.E.D.

By the method as in the classical case [Il1], we can deduce the following facts from (4.4).

COROLLARY (4.5). — (1) *If $m \geq n$, $p^n : W_m\omega_Y^q \to W_m\omega_Y^q$ factors through the canonical projection $W_m\omega_Y^q \to W_{m-n}\omega_Y^q$. The induced map "$p^n$" : $W_{m-n}\omega_Y^{\cdot} \to W_m\omega_Y^{\cdot}$ is injective and $W_m\omega_Y^{\cdot}/$ "p^n"$(W_{m-n}\omega_Y^{\cdot}) \to W_n\omega_Y^{\cdot}$ is a quasi-isomorphism.*

(2) $\varprojlim_n W_n\omega_Y^q$ *is torsion free for any q.*

We give two presentations (4.6) (4.7) of $W_n\omega_Y^q$.

PROPOSITION (4.6). — *$W_n\omega_Y^q$ is canonically isomorphic to*

$$(4.6.1)\quad ((W_n\mathcal{O}_Y \otimes \overset{q}{\wedge}_{\mathbb{Z}}\ M^{gp}/f^{-1}(L^{gp})) \oplus (W_n\mathcal{O}_Y \otimes \overset{q-1}{\wedge}_{\mathbb{Z}}\ M^{gp}/f^{-1}(L^{gp})))/\mathcal{F}$$

where $\mathcal{F}$ is the subsheaf of the direct sum generated by local sections of the forms

$$(\varepsilon_i(\alpha(a_1)) \otimes (a_1 \wedge \cdots \wedge a_q), 0) - p^i(0, \varepsilon_i(\alpha(a_1)) \otimes (a_2 \wedge \cdots \wedge a_q))$$

$$(a_1, \ldots, a_q \in M, \quad 0 \leq i < n).$$

Here for $b \in \mathcal{O}_Y$, we denote $(\underbrace{0,\ldots,0}_{i \text{ times}}, b, 0, \ldots, 0) \in W_n(\mathcal{O}_Y)$ by $\varepsilon_i(b)$.

PROPOSITION (4.7). — (1) *There exists a canonical isomorphism of graded differential algebras*

$$\left(\bigoplus_{q\in\mathbb{Z}} \omega^q_{W_n(Y)/(W_n,W_n(L))}\right)/\mathcal{I} \cong \bigoplus_{q\in\mathbb{Z}} W_n\omega^q_Y$$

where $W_n(Y)$ is endowed with the canonical lifting $W_n(M)$ (3.1) of M and $\mathcal{I}$ is the graded subideal of the algebra generated locally by local sections of the forms $\eta_{i,j,a,b}$ and $d\eta_{i,j,a,b}$ $(0 \le j \le i < n,\ a \in \mathcal{O}_Y,\ b \in M)$ where

$$\eta_{i,j,a,b} = \varepsilon_i(a)d\varepsilon_j(\alpha(b)) - \varepsilon_i(a\alpha(b)^{p^{i-j}})\mathrm{d}\log(b)\,.$$

(2) *For each q and each $x \in Y$, $\mathcal{I}_{\overline{x}}$ coincides with the image of*

$$(\omega^{\cdot}_{\mathrm{Spec}(W(\mathcal{O}_{Y,\overline{x}}))/\,\mathrm{Spec}(W(\overline{k}))})_{\overline{x},tor} \longrightarrow \omega^{\cdot}_{W_n(Y)/(W_n,W_n(L)),\overline{x}}$$

where $\mathrm{Spec}(W_n(\mathcal{O}_{Y,\overline{x}}))$ (resp. $\mathrm{Spec}(W(\overline{k}))$) is endowed with the log structure associated to $M_{\overline{x}} \to W(\mathcal{O}_{Y,\overline{x}})$ (resp. $\Gamma(\mathrm{Spec}(\overline{k}), L) \to W(\overline{k}))$; $a \longmapsto (\alpha(a), 0, 0, \ldots)$, and tor denotes the torsion part.

PROPOSITION (4.8). — *Let T be an object of $((\mathrm{Spec}(k), L)/W_n)_{crys}$ (W_n is endowed here with the trivial log structure). Then, there exists a functorial homomorphism between graded $\mathcal{O}(T)$–algebras*

$$\bigoplus_{q\geq 0} \mathcal{O}(T) \otimes_{W_n(k)} W_n\omega^q_Y \longrightarrow \bigoplus_{q\geq 0} R^q u^{\log}_{Y/T*}(\mathcal{O}_{Y/T})\,, \tag{4.8.1}$$

which is an isomorphism if T is flat over W_n.

(4.9). — We prove (4.6)–(4.8) together. We may work etale locally, and hence we can take in (2.24) $Y^{\cdot} = Y$, $(Z^{\cdot}, N^{\cdot})$ to be a constant simplical object (Z, N). Consider the crystalline complex for this system. We define a ring homomorphism $\tau : W_n(\mathcal{O}_Y) \to \mathcal{H}^0(C^{\cdot}_{Y/T})$, an additive map $\delta : W_n(\mathcal{O}_Y) \to \mathcal{H}^1(C_{Y/T})$ which is a derivation with respect to τ and a homomorphism $\mathrm{d}\log : M^{gp} \to \mathcal{H}^1(C_{Y/T})$, by

$$\tau : (a_0, \ldots, a_{n-1}) \longmapsto \sum_{i=0}^{n-1} p^i \widetilde{a}_i^{p^{n-i}}$$

$$\delta : (a_0, \ldots, a_{n-1}) \longmapsto \sum_{i=0}^{n-1} \widetilde{a}_i^{p^{n-i}-1} d\widetilde{a}_i$$

$$\mathrm{d}\log : b \longmapsto \mathrm{d}\log(\widetilde{b}) ,$$

($a_i \in \mathcal{O}_Y$, $b \in M$) where $\widetilde{a}_i$ is any lifting of a_i to $\mathcal{O}_D$ and $\widetilde{b}$ is any lifting of b to the log structure N of Z. The map τ (resp. δ, resp. $\mathrm{d}\log$) is well defined by virtue of the following fact : for $a \in \mathcal{O}_D$ and $h \in \mathrm{Ker}(\mathcal{O}_D \to \mathcal{O}_Y)$,

$$p^i(a+h)^{p^{n-i}} = p^i a^{p^{n-i}} \quad \text{in} \quad \mathcal{O}_D$$

(resp. for $a \in \mathcal{O}_D$ and $h \in \mathrm{Ker}(\mathcal{O}_D \to \mathcal{O}_Y)$,

$$(a+h)^{p^{n-i}-1} d(a+h) - a^{p^{n-i}-1} da = d\left(\sum_{1 \le j \le p^{n-i}} c_j a^{p^{n-i}-j} h^{[j]} \right)$$

in $\mathcal{O}_D \otimes_{\mathcal{O}_Z} \omega^1_{Z/T}$ where $c_j = (p^{n-i})!((p^{n-i}-j)!)^{-1} p^{i-n} \in \mathbf{Z}_p$, resp. for $a \in N$ and $u \in \mathrm{Ker}(\mathcal{O}_Z^\times \to \mathcal{O}_Y^\times)$,

$$\mathrm{d}\log(au) - \mathrm{d}\log(a) \in \mathcal{O}_D \otimes_{\mathcal{O}_Z} \omega^1_{Z/T} \text{ is the image of } \log(u) \in \mathcal{O}_D$$

under $d : \mathcal{O}_D \to \mathcal{O}_D \otimes_{\mathcal{O}_Z} \omega^1_{Z/T}$).

In the case $T = W_n$, τ is a ring homomorphism $W_n(\mathcal{O}_Y) \to W_n\omega^0_Y$, and δ coincides with the composite $W_n(\mathcal{O}_Y) \xrightarrow{\tau} W_n\omega^0_Y \xrightarrow{d} W_n\omega^1_Y$. It is not difficult to see that there exists a $W_n(\mathcal{O}_Y)$-homomorphism $\omega^1_{W_n(Y)/(W_n, W_n(L))} \to \mathcal{H}^1(C_{Y/T})$ which sends da ($a \in W_n(\mathcal{O}_Y)$) to $\delta(a)$ and $\mathrm{d}\log(b)$ ($b \in M$)

to $d\log(b)$, that this map induces a homomorphism of $W_n(\mathcal{O}_Y)$–algebras $\psi : \bigoplus_{q\geq 0} \omega^q_{W_n(Y)/(W_n,W_n(L))} \to \bigoplus_{q\geq 0} \mathcal{H}^q(C_{Y/T})$ and that the last map annihilates $\mathcal{I}$. If $(W_n\omega^q_Y)'$ denotes the sheaf (4.6.1) and $\mathcal{I}_q$ denotes the degree q part of $\mathcal{I}$, we have a commutative diagram

$$\begin{tikzcd}[column sep=small] (W_n\omega^q_Y)' \arrow[rr, "s"] \arrow[dr, "t"'] & & \mathcal{H}^q(C_{Y/T}) \\ & (\omega^q_{W_n(Y)/(W_n,W_n(L))})/\mathcal{I}_q \arrow[ur, "\text{by } \psi"'] & \end{tikzcd} \tag{4.9.1}$$

where s (resp. t) sends $(w \otimes (b_1 \wedge \cdots \wedge b_q), 0)$ to $\tau(w)d\log(b_1)\cdots d\log(b_q)$ (resp. $w d\log(b_1) \wedge \cdots \wedge d\log(b_q)$) and $(0, w \otimes (b_1 \wedge \cdots \wedge b_{q-1}))$ to $\delta(w)d\log(b_1)\cdots d\log(b_{q-1})$ (resp. $dw \wedge d\log(b_1) \wedge \cdots \wedge d\log(b_{q-1})$).

We prove s is bijective in the case $T = W_n$.

Let $Fil^i((W_n\omega^q_Y)')$ be the image of

$$\left((V^i(W_n(\mathcal{O}_Y))\otimes \bigwedge\nolimits^q_{\mathbf{Z}} M^{gp}) \oplus (V^i(W_n(\mathcal{O}_Y))\otimes \bigwedge\nolimits^{q-1}_{\mathbf{Z}} M^{gp}\right) \longrightarrow (W_n\omega^q_Y)'.$$

Then, s sends Fil^i into the kernel of the canonical projection $W_n\omega^q_Y \to W_i\omega^q_Y$, and the isomorphism (4.4) $(V^i, dV^i)C^{-1} : \omega^q_Y \oplus \omega^{q-1}_Y \xrightarrow{\sim} \mathrm{Ker}(\pi_i)$ factors as

$$\omega^q_Y \oplus \omega^{q-1}_Y \longrightarrow Fil^i/Fil^{i+1} \xrightarrow{s} \mathrm{Ker}(\pi_i),$$

where the first map is a surjection defined by

$$(a d\log(b_1) \wedge \cdots \wedge d\log(b_q), 0) \longmapsto (\varepsilon_i(a) \otimes (b_1 \wedge \cdots \wedge b_q), 0)$$

$$(0, a d\log(b_1) \wedge \cdots \wedge d\log(b_{q-1}) \longmapsto (0, \varepsilon_i(a) \otimes (b_1 \wedge \cdots \wedge b_{q-1}))$$

which is well defined as is checked easily. This shows that s is an isomorphism and proves (4.6).

Next we show that t in (4.9.1) is surjective. This will prove (4.7)(1). We are reduced to the case $q = 1$. As a sheaf of abelian groups, $\omega^1_{W_n(Y)/(W_n,W_n(L))}$

is generated locally by $\varepsilon_i(a)d\varepsilon_j(b)$ $(a, b \in \mathcal{O}_Y^\times,\ 0 \le i,\ j < n)$ and $w d\log(b)$ $(w \in W_n(\mathcal{O}_Y),\ b \in M)$. If $i \ge j$, $\varepsilon_i(a)d\varepsilon_j(b)$ clearly belongs to $\mathcal{I}_1 + \mathrm{Image}(t)$. If $i \le j$, $\varepsilon_i(a)d\varepsilon_j(b) = d(\varepsilon_i(a)\varepsilon_j(b)) - \varepsilon_j(b)d\varepsilon_i(a) \in \mathcal{I}_1 + \mathrm{Image}(t)$.

We prove (4.7.) (2). Let $G^\cdot = \omega_{\mathrm{Spec}(W(\mathcal{O}_{Y,\overline{x}}))/\mathrm{Spec}(W(\overline{k}))_{\overline{x}}}$. First, $\mathrm{Image}(G^\cdot_{tor}) \subset \mathcal{I}_{\overline{x}}$ follows from the fact that $\varprojlim_n W_n\omega_Y^q$ is torsion free. Next we show that $\eta_{i,j,a,b}$ is, when regarded as an element of G^1, a torsion element. Let $\varphi : G^q \to G^q$ be the map induced by the frobenius of $W(\mathcal{O}_{Y,\overline{x}})$ and $W(\overline{k})$ and the p-th power maps on $M_{\overline{x}}$ and $\Gamma(\mathrm{Spec}(\overline{k}), L)$. Then, $\varphi : G^q \otimes \mathbb{Q} \to G^q \otimes \mathbb{Q}$ is bijective. Indeed, this is reduced to the case $q \le 1$. For $q = 0$, $\varphi : G^0 \otimes \mathbb{Q} \to G^0 \otimes \mathbb{Q}$ has the inverse map $(a_0, a_1, \ldots) \longmapsto p^{-1}(0, a_0, a_1, \ldots)$ and hence is bijective. For $q = 1$, the inverse map is given by

$$adb \longmapsto \varphi^{-1}(a)d\varphi^{-1}(b)\ (a, b \in W(\mathcal{O}_{Y,\overline{x}}) \otimes \mathbb{Q}),$$

$$a d\log(b) \longmapsto p^{-1}\varphi^{-1}(a)\mathrm{d}\log(b)\ (a \in W(\mathcal{O}_{Y,\overline{x}}) \otimes \mathbb{Q},\ b \in M_{\overline{x}})$$

(cf. [Il1] I (4.3)). On the other hand, the $2j$-th iteration of $\varphi : G^1 \to G^1$ annihilates $\eta_{i,j,a,b}$ since

$$\begin{aligned}\varphi^{2j}(\eta_{i,j,a,b}) &= \varepsilon_i(a^{p^{2j}})d\varepsilon_j(\alpha(b^{p^{2j}})) - \varepsilon_i(a^{p^{2j}}\alpha(b)^{p^{i+j}})\mathrm{d}\log(b^{p^j}) \\ &= p^j\varepsilon_i(a^{p^{2j}})d\varepsilon_0(\alpha(b^{p^j})) - p^j\varepsilon_i(a^{p^{2j}})\varepsilon_0(\alpha(b)^{p^j})\mathrm{d}\log(b^{p^j}) = 0.\end{aligned}$$

Finally we prove (4.8). We define the $\mathcal{O}(T)$-homomorphism in (4.8) to be the one induced from

$$W_n\omega_Y^q \xleftarrow[(4.6)]{\cong} (W_n\omega_Y^q)' \xrightarrow{s} \mathcal{H}^q(C_{Y/T})$$

where s is as in (4.9.1). The bijectivity statement in the case T is flat over $\mathbb{Z}/p^n\mathbb{Z}$ is reduced to the case $n = 1$ by using the long exact sequence of cohomology sheaves associated to the exact sequence

$$0 \longrightarrow C_{Y/(T\otimes\mathbb{Z}/p\mathbb{Z})} \xrightarrow{p^{n-1}} C_{Y/T} \longrightarrow C_{Y/(T\otimes\mathbb{Z}/p^{n-1}\mathbb{Z})} \longrightarrow 0.$$

In the case $n = 1$, by working locally, we may assume that $Y = Z \times_T \mathrm{Spec}(k)$. Then, $D = Z$ and our task is to prove

$$\mathcal{O}(T) \otimes_k \mathcal{H}^q(\omega^{\cdot}_Y) \xrightarrow{\sim} \mathcal{H}^q(\omega^{\cdot}_{Z/T}) .$$

By Cartier isomorphism (2.12), this map is rewritten as

$$\mathcal{O}(T) \underset{\varphi}{\nwarrow\!\searrow} \otimes_k \omega^q_Y \longrightarrow \mathcal{O}(T) \underset{\varphi}{\nwarrow\!\searrow} \otimes_{\mathcal{O}(T)} \omega^q_{Z/T}$$

where $\varphi : a \longmapsto a^p$. Since $\varphi : \mathcal{O}(T) \to \mathcal{O}(T)$ factors through the canonical surjection $\mathcal{O}(T) \to k$ and $k \otimes_{\mathcal{O}(T)} \omega^q_{Z/T} = \omega^q_Y$, we are done.

Remark (4.10). — The existence of the canonical homomorphism $W_n\omega^q_Y \to R^q u^{\log}_{Y/T^\bullet}(\mathcal{O}_{Y/T})$ in (4.8) is not an evident fact, for we do not have a morphism $(T, L_T) \to (W_n, W_n(L))$. The authors do not know if this homomorphism comes from a homomorphism in the derived category

$$(*) \qquad Ru^{\log}_{Y/(W_n, W_n(L))^\bullet}(\mathcal{O}_{Y/W_n}) \longrightarrow Ru^{\log}_{Y/T^\bullet}(\mathcal{O}_{Y/T}) .$$

The meaning of (4.8) is that $W_n\omega^q_Y$ grows "neglecting" log structures when we take PD–thickenings of Spec(k). To see how the problem is delicate, assume only that f is smooth integral but not that f is of Cartier type. Then we have the following counterexample of (4.8).

Let L be the log structure associated to $\mathbf{N} \to k$; $1 \longmapsto 0$, $Y = \mathrm{Spec}(k[t]/(t^r))$, $(p, r) = 1$, M is the log structure associated to $\mathbf{N} \to k[t]/(t^r)$; $1 \longmapsto t$, and $(Y, M) \to (\mathrm{Spec}(k), L)$ induced by $\mathbf{N} \to \mathbf{N}$; $1 \longmapsto r$. Then, $\Gamma(Y, W_n\omega^0_Y) = W_n(k)[t]/(t^r)$. If we take $T = \mathrm{Spec}(W_n(k) < s >)$ with s an indeterminate and endow T with the log structure associated to $\mathbf{N} \to W_n(k) < s >$; $1 \longmapsto s$ and with the usual PD–structure,

$$\Gamma(T, R^0 u^{\log}_{Y/T^\bullet}(\mathcal{O}_{Y/T})) = W_n(k) < s > [t]/(t^r - s) .$$

But $W_n(k) < s > \otimes_{W_n(k)} W_n(k)[t]/(t^r)$ and $W_n(k) < s > [t]/(t^r - s)$ are not isomorphic as $W_n(k) < s >$–algebras if $r \geq 2$.

The following (4.13) says that a good homomorphism (*) desired in (4.10) exists at least "modulo torsion which is bounded independently of n" under a certain assumption. This (4.13) will play a key role in § 5 in the definition of ρ_π.

DEFINITION (4.11). — *For a sequence of functors between categories*

$$\mathcal{C}_{n+1} \xrightarrow{\lambda_n} \mathcal{C}_n \xrightarrow{\lambda_{n-1}} \cdots \xrightarrow{\lambda_1} \mathcal{C}_1 ,$$

let $ps(\mathcal{C}.)$ be the category of systems $\{(A_n, s_n)\}_{n\geq 1}$ where A_n is an object of $\mathcal{C}_n$ and s_n is a morphism $\lambda_n(A_{n+1}) \to A_n$. We often abbreviate $\{(A_n, s_n)\}_{n\geq 1}$ as $\{A_n\}_n$.

DEFINITION (4.12). — *For an additive category $\mathcal{C}$, we denote by $\mathbf{Q}\otimes\mathcal{C}$ the category whose set of objects is the same as $\mathcal{C}$ but whose set of morphisms between objects A, B is $\mathbf{Q}\otimes \mathrm{Hom}_{\mathcal{C}}(A,B)$. An object A of $\mathcal{C}$ is denoted by $\mathbf{Q}\otimes A$ when it is regarded as an object of $\mathbf{Q}\otimes\mathcal{C}$.*

PROPOSITION (4.13). — *Assume we are given for each $n\geq 1$ an object T_n of $((W_n, W_n(L))/W_n)_{crys}$ with the log structure L_n, a morphism $F : (T_n, L_n) \to (T_n, L_n)$ and an exact closed immersion $(T_n, L_n) \to (T_{n+1}, L_{n+1})$ which are compatible with PD–structures and have the following properties* (i)–(iii).

(i) *With respect to the morphisms $(W., W.(L)) \to (T., L.) \to W.$ (the last $W.$ is endowed with the trivial log structure), F commutes with the frobenius of $(W_n, W_n(L))$ and that of W_n, and $(T_n, L_n) \to (T_{n+1}, L_{n+1})$ commutes with F, with $(W_n, W_n(L)) \to (W_{n+1}, W_{n+1}(L))$ and with $W_n \to W_{n+1}$.*

(ii) *T_n is flat over W_n and $T_n \xrightarrow{\sim} T_{n+1}\otimes \mathbf{Z}/p^n\mathbf{Z}$ for each n.*

(iii) *For each $n\geq 1$, the ideal $\mathcal{I}_n$ of W_n in T_n is generated etale locally by local sections of the form*

$$a^{[i]} \quad (i\geq 1) \quad \textit{with } a\in \mathrm{Image}(L_n \longrightarrow \mathcal{O}_{T_n}) \cap \mathcal{I}_n .$$

For $n\geq 1$, define

$$\mathcal{K}_n = Ru^{\log}_{Y/T_n^*}(\mathcal{O}_{Y/T_n}), \quad \mathcal{K}'_n = Ru^{\log}_{Y/(W_n,W_n(L))^*}(\mathcal{O}_{Y/W_n})$$

and let $\beta_n : \mathcal{K}_n \to \mathcal{K}'_n$ be the canonical morphism. Then, in the category $\mathbf{Q}\otimes ps(D(Y_{et}, \mathcal{O}(T.)))$, there exists a unique isomorphism

$$h : \mathbf{Q}\otimes\{\mathcal{O}(T_n)\otimes_{W_n}\mathcal{K}'_n\}_n \xrightarrow{\sim} \mathbf{Q}\otimes\{\mathcal{K}_n\}_n$$

satisfying the following (4.13.1) (4.13.2).

(4.13.1). — $(\beta_n)_n \circ h$ *coincides with the morphism induced by* $\mathcal{O}(T_n) \to W_n(k)$.

(4.13.2). — *If we denote the morphisms* $\mathcal{O}(T_n) \to \mathcal{O}(T_n)$, $\mathcal{K}_n \to \mathcal{K}_n$ *and* $\mathcal{K}'_n \to \mathcal{K}'_n$ *induced by the frobenius morphisms by the same letter* φ, *then* h *commutes with* $\varphi \otimes \varphi$ *on* $\mathbf{Q} \otimes \{\mathcal{O}(T_n) \otimes_{W_n} \mathcal{K}'_n\}_n$ *and* φ *on* $\mathbf{Q} \otimes \{\mathcal{K}_n\}_n$.

The rough idea of the proof of (4.13) is that the frobenius on $\mathcal{K}'_n$ is near to an isomorphism and the frobenius on $\mathcal{I}_n$ is near to zero, and this forces the morphism $\{\mathcal{K}_n\}_n \to \{\mathcal{K}'_n\}_n$ to split in $\mathbf{Q} \otimes ps(D((X_.)_{et}, W_.(k)))$.

To prove (4.13), we use the following lemma.

LEMMA (4.14). — *Let* $\mathcal{C}$ *be a triangulated category and* $\Phi : \mathcal{C} \to \mathcal{C}$ *be an exact functor. Assume we are given a distinguished triangle* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ *and morphisms* $\varphi_A : \Phi(A) \to A$, $\varphi_B : \Phi(B) \to B$, $\varphi_C : \Phi(C) \to C$, *in* $\mathcal{C}$ *(we denote all of them simply by* φ*) such that*

$$\alpha\varphi = \varphi\Phi(\alpha), \quad \beta\varphi = \varphi\Phi(\beta), \quad \gamma\varphi = \varphi[1]\Phi(\gamma).$$

Let p *be a prime number,* $s_0 = 1$, s_1, $s_2, \ldots$ *be integers such that* $s_i | s_{i+1}$ *for all* $i \geq 0$ *and* $\lim_{i\to\infty} \mathrm{ord}_p(s_{i+1}s_i^{-1}) = \infty$, *let* $r \geq 0$ *be an integer, and assume the following* (i)–(iii) *are satisfied.*

(i) *There exist morphisms* $\varphi_i : \Phi^i(A) \to A$ *for* $i \geq 0$ *such that* $\varphi_0 = Id_A$, $\varphi\Phi(\varphi)\cdots\Phi^{i-1}(\varphi) = s_i\varphi_i$ *for all* $i \geq 1$, *and* $(s_{i+1}s_i^{-1})\varphi_{i+1} = \varphi\Phi(\varphi_i)$ *for all* $i \geq 0$.

(ii) *There exists* $\psi : C \to \Phi(C)$ *such that* $\varphi\psi$ (resp. $\psi\varphi$) *is the multiplication by* p^r *on* C (resp. $\Phi(C)$).

(iii) *The set of homomorphisms from* C *to* A *is annihilated by a power of* p.

Take an integer $c \geq 0$ *such that* $p^{rc+1} | s_c$, *and an integer* $d \geq 0$ *such that* $p^{r(i+1)} | s_i p^d$ *for all* $i \geq 0$. *Then the kernel (resp.cokernel) of*

$$[C, B]_\varphi \longrightarrow [C, C]_\varphi : h \longmapsto \beta h$$

is annihilated by p^{2cr} (resp. p^{cr+d}), *where* $[C, B]_\varphi$ *denotes the set of morphisms* $h : C \to B$ *such that* $h\varphi = \varphi\Phi(h)$, *and* $[C, C]_\varphi$ *is defined similarly.*

PROOF. For $X = A$ or C, denote $\varphi\Phi(\varphi)\cdots\Phi^{i-1}(\varphi) : \Phi^i(X) \to X$ by $\varphi^{(i)}$ and denote $\Phi^{i-1}(\psi)\cdots\Phi(\psi)\psi : X \to \Phi^i(X)$ by $\psi^{(i)}$. We show first $[C,A]_\varphi$ is annihilated by p^{cr}. Indeed, for $h \in [C,A]_\varphi$, we have

$$p^{rc}h = h\varphi^{(c)}\psi^{(c)} = \varphi^{(c)}\Phi^c(h)\psi^{(c)} = s_c\varphi_c\Phi^c(h)\psi^{(c)} = p^{rc}a\varphi_c\Phi^c(h)\psi^{(c)}$$

for some $a \in p\mathbb{Z}$. Hence

$$p^{rc}h = p^{rc}a\varphi_c\Phi^c(h)\psi^{(c)} = p^{rc}a^2\varphi_c\Phi^c(\varphi_c)\Phi^{2c}(h)\psi^{(2c)} = \cdots = 0\,.$$

This proof shows also $[C,A[1]]_\varphi = 0$. In particular, we have $p^{cr}\gamma = 0$.

Next we show that the kernel of $[C,B]_\varphi \to [C,C]_\varphi$ is killed by p^{2cr}. Indeed, let h be an element of this kernel. Then, there exists $h' : C \to A$ such that $h = \alpha h'$. Since $\alpha(\varphi\Phi(h') - h'\varphi) = \varphi\Phi(h) - h\varphi = 0$, there exists $h'' : \Phi(C) \to C[-1]$ such that $\varphi\Phi(h') - h'\varphi = \gamma[-1]h''$. By $p^{cr}\gamma = 0$, we have $p^{cr}h' \in [C,A]_\varphi$ and hence $p^{2cr}h' = 0$ and $p^{2cr}h = 0$.

Finally we show the cokernel of $[C,B]_\varphi \to [C,C]_\varphi$ is killed by p^{cr+d}. Let $h \in [C,C]_\varphi$. Since $p^{cr}\gamma = 0$, we have $\gamma(p^{cr}h) = 0$ and hence there exists $h' : C \to B$ such that $\beta h' = p^{cr}h$. We have $\beta(\varphi\Phi(h') - h'\varphi) = 0$. Hence there exists $h'' : \Phi(C) \to A$ such that $\varphi\Phi(h') - h'\varphi = \alpha h''$. Define $t : C \to B$ by

$$t = p^d h' + \sum_{i\geq 0}(p^{d-r(i+1)}s_i)\alpha\varphi_i\Phi^i(h'')\psi^{(i+1)}\,.$$

Then $t \in [C,B]_\varphi$ and $\beta t = p^{cr+d}h$.

(4.15). — To prove (4.13), we apply (4.14) by taking

$$A = I_n \otimes^L_{\mathcal{O}(T_n)} \mathcal{K}_n \quad \text{where} \quad I_n = \mathrm{Ker}(\mathcal{O}(T_n) \longrightarrow W_n(k))\,,$$

$$B = \mathcal{K}_n\,, \quad C = \mathcal{K}'_n\,, \quad \beta = \beta_n\,, \quad \Phi = W_n(k) \underset{\varphi}{\nwarrow} \otimes^L_{W_n(k)}(?)\,, \quad s_i = (p^i)!\,,$$

and φ_i as follows. For $m \geq 1$, we have $\varphi^i(I_m) \subset p^i!I_m$. Indeed, if $a \in L_m$ and $\alpha(a) \in I_m$ (here $\alpha : L_m \to \mathcal{O}_{T_m}$ is the canonical map),

$$\varphi^i(\alpha(a)^{[j]}) \in (\alpha(a)^{p^i})^{[j]}\mathcal{O}_{T_m} = u_{i,j}\alpha(a)^{[p^i j]}\mathcal{O}_{T_m}$$

where $u_{i,j} = (p^i j)!(j!)^{-1}$, and $u_{i,j} \in p^i!\mathbf{Z}_p$ if $j \geq 1$. Hence we have a map of projective systems $(p^i!)^{-1}\varphi : I. \to I.$, which defines $\varphi_i : I_n \to I_n$. We define $\varphi_i : \Phi^i(A) \to A$ as $(\varphi_i \text{ on } I_i) \otimes (\varphi \text{ on } \mathcal{K}_n)$. Then, the assumptions (i) (iii) in (4.14) are satisfied clearly and (ii) is satisfied by (2.24). Note when n varies, we can take the same s_i, r, c and d independently of n. Hence we have the uniquements of h stated in (4.13). We show the existence of such h as follows. By (4.14), we find $h'_n \in [\mathcal{K}'_n, \mathcal{K}_n]_\varphi$ such that $\beta_n h'_n = p^{cr+d}$. Now when we vary n, the morphism $p^{2cr}h'_n$ coincides with the morphism induced by $p^{2cr}h'_{n+1}$. Thus $(p^{2cr}h'_n)_n$ is a morphism $\{\mathcal{K}'_n\}_n \to \{\mathcal{K}_n\}_n$ in $ps(D(Y_{et}, W.(k)))$. Define $h'' = p^{-3cr-d} \otimes (p^{2cr}h'_n) : \mathbf{Q} \otimes \{\mathcal{K}'_n\}_n \to \mathbf{Q} \otimes \{\mathcal{K}_n\}_n$ and let $h : \mathbf{Q} \otimes \{\mathcal{O}(T_n) \otimes_{W_n} \mathcal{K}'_n\}_n \to \mathbf{Q} \otimes \{\mathcal{K}_n\}_n$ be the morphism induced by h''. It remains to prove that h is an isomorphism. By lemma (4.17) below, it suffices to show that the morphism induced by h

$$(4.15.1) \qquad \mathbf{Q} \otimes \{\mathcal{O}(T_n) \otimes_{W_n} W_n\omega_Y^q\}_n \longrightarrow \mathbf{Q} \otimes \{R^q u^{\log}_{Y/T_n^*}(\mathcal{O}_{Y/T_n})\}_n$$

is an isomorphism in $\mathbf{Q} \otimes ps(\mathcal{O}(T.)$–modules)) for each q. But this follows from

LEMMA (4.16). — *The morphism* (4.15.1) *coincides with the one induced by the isomorphism in* (4.8).

PROOF. By a similar argument to that in (4.14) we can show that there exists a unique morphism whose composite with

$$\mathbf{Q} \otimes \{R^q u^{\log}_{Y/T_n^*}(\mathcal{O}_{Y/T_n})\} \longrightarrow \mathbf{Q} \otimes \{W_n\omega_Y^q\}_n$$

coincides with the morphism induced by $\mathcal{O}(T_n) \longrightarrow W_n(k)$ and which commutes with frobenius. The morphism induced by (4.8) also has these properties.

In (4.15) we have used

LEMMA (4.17). — *Let $\mathcal{C}_i$ $(i \geq 1)$ be abelian categories, $D(\mathcal{C}_i)$ their derived categories, and let*

$$D(\mathcal{C}_{n+1}) \xrightarrow{\lambda_n} D(\mathcal{C}_n) \longrightarrow \cdots \xrightarrow{\lambda_1} D(\mathcal{C}_1)$$

be exact functors. Let $\{A_n\}_n$ *and* $\{B_n\}_n$ *be objects of* $ps(D(\mathcal{C}.))$ *and let* $h : \{A_n\}_n \longrightarrow \{B_n\}_n$ *be a morphism. Assume there exists* $r \geq 0$ *such that*

$$\#\{q \in \mathbb{Z}\,;\; H^q(A_n) \neq 0 \quad \text{or} \quad H^q(B_n) \neq 0\} \leq r$$

for all n*. Then, the following two conditions are equivalent.*

(i) $\mathbb{Q} \otimes \{A_n\}_n \to \mathbb{Q} \otimes \{B_n\}_n$ *is an isomorphism in* $\mathbb{Q} \otimes ps(D(\mathcal{C}.))$.

(ii) *There exists a non-zero integer* m *such that the kernel and the cokernel of* $H^q(A_n) \to H^q(B_n)$ *are killed by* m *for any* q *and any* n*. Here* $H^q : D(\mathcal{C}_n) \to \mathcal{C}_n$ *is the canonical cohomology functor.*

Proof. The implication (i) $\Longrightarrow$ (ii) is easily seen. We prove (ii) $\Longrightarrow$ (i). For each $n \geq 1$, take any distinguished triangle $A_n \xrightarrow{h} B_n \to C_n \to$. If (ii) is satisfied, $H^q(C_n)$ is killed by m^2 for any q and for any n. By (4.18) below, this shows that C_n is killed by $M = m^{4r}$ for any n. By the exact sequence $\mathrm{Hom}(B_n, A_n) \to \mathrm{Hom}(B_n, B_n) \to \mathrm{Hom}(B_n, C_n)$, there exists $g_n : B_n \to A_n$ such that $h_n g_n = M$. We see easily that $(Mg_n)_{n \geq 1}$ is a morphism $\{B_n\}_n \to \{A_n\}_n$ in $ps(D(\mathcal{C}.))$ and satisfies

$$h_n(Mg_n) = M^2\,, \quad (Mg_n)h_n = M^2 \quad \text{for any } n\,.$$

Lemma (4.18). — *Let* $\mathcal{C}$ *be an abelian category,* A *an object of the derived category* $D(\mathcal{C})$*,* S *a finite subset of* $\mathbb{Z}$*,* m_q $(q \in S)$ *integers, and assume that* $H^q(A) = 0$ *for* $q \notin S$*, and that* $H^q(A)$ *is killed by* m_q *for* $q \in S$*. Then* A *is killed by* $\prod_{q \in S} m_q$.

Proof. The case $\#(S) \leq 1$ is clear. Assume $\#(S) \geq 2$, let $r = \max(S)$, and consider the distinguished triangle

$$\tau_{\leq r-1} A \longrightarrow A \longrightarrow \tau_{\geq r} A \longrightarrow\,.$$

By induction on $\#(S)$, $\tau_{\leq r-1} A$ is killed by $\prod_{\substack{q \in S \\ q < r}} m_q$ and $\tau_{\geq r} A$ is killed by m_r.

By the exact sequence

$$\mathrm{Hom}(A, \tau_{\leq r-1} A) \longrightarrow \mathrm{Hom}(A, A) \longrightarrow \mathrm{Hom}(A, \tau_{\geq r} A)\,,$$

we see that the identity morphism of A is killed by $\prod_{q \in S} m_q$.

THEOREM (4.19). — *(cf. [Il1] II §1 in the case without log structures). In the derived category, we have a canonical isomorphism*

$$W_n\omega_Y^{\cdot} \cong Ru^{\log}_{Y/(W_n,W_n(L))*}(\mathcal{O}_{Y/W_n})$$

compatible with frobenius and with the transition maps when n varies.

Here the transition map $W_{n+1}\omega_Y^{\cdot} \to W_n\omega_Y^{\cdot}$ means the canonical projection.

PROOF. The proof is the same as in the classical case [Il1] II §1 (cf. also [Bl] III §2). Take an embedding system $((Y^{\cdot},M^{\cdot}),\ (Z^{\cdot},N^{\cdot}))$ for $(Y,M) \to W_n, W_n(L))$ such that there exists a morphism $(W_n(Y^{\cdot}), W_n(M^{\cdot})) \to (Z^{\cdot},N^{\cdot})$ for which the diagram

$$\begin{tikzcd}
 & (W_n(Y^{\cdot}),\ W_n(M^{\cdot})) \arrow[dd] \arrow[dr] & \\
(Y^{\cdot},M^{\cdot}) \arrow[ur] \arrow[dr] & & (W_n, W_n(L)) \\
 & (Z^{\cdot},N^{\cdot}) \arrow[ur] &
\end{tikzcd}$$

is commutative, and consider the crystalline complex C_n with respect to this system. We define a homomorphism of complexes $C_n \to \theta^{-1}(W_n\omega_Y^{\cdot})$, where $\theta : (Y^{\cdot})^{\sim}_{et} \to Y^{\sim}_{et}$, as follows. Let $(D^{\cdot}, M_{D^{\cdot}})$ be the PD-envelope of $(Y^{\cdot},M^{\cdot})$ in $(Z^{\cdot},N^{\cdot})$. By the universal property of the PD-envelope and the usual PD-structure on the ideal of Y in $W_n(Y)$ (which is characterized by $\varepsilon_i(a)^{[p]} = (p-1)!^{-1}\varepsilon_{ip-1}(a^{p^{i(p-1)}})$ for $a \in \mathcal{O}_Y$, $i \geq 1$) we have $(W_n(Y^{\cdot}), W_n(M^{\cdot})) \to (D^{\cdot}, M_D^{\cdot})$, and this morphism gives a homomorphism of complexes $C_n \to \omega^{\cdot}_{W_n(Y^{\cdot})/(W_n,W_n(L),[])}$ where $\bigoplus_{q\geq 0} \omega^q_{W_n(Y^{\cdot})/(W_n,W_n(L)),[]}$ is the quotient of $\bigoplus_{q\geq 0} \omega^q_{W_n(Y^{\cdot})/(W_n,W_n(L))}$ by the ideal generated locally by local sections of the form

$$\text{(4.19.1)}\qquad d(a^{[i]}) - a^{[i-1]}da \quad (a \in \mathrm{Ker}(W_n(\mathcal{O}_{Y^{\cdot}}) \longrightarrow \mathcal{O}_{Y^{\cdot}}),\quad i \geq 1).$$

Since the image of (4.19.1) in $W_n\omega^1_Y$ is zero (this is seen for example, by the fact that $\varprojlim_n W_n\omega^1_Y$ is torsion free and

$$i!(d(a^{[i]}) - a^{[i-1]}da) = d(a^i) - ia^{i-1}da = 0)\,,$$

we have $\omega^{\cdot}_{W_n(Y^{\cdot})/(W_n(L)),[]} \to W_n\omega^{\cdot}_Y = \theta^{-1}(W_n\omega^{\cdot}_Y)$. Thus we obtain the desired map $C_n \to \theta^{-1}(W_n\omega^{\cdot}_Y)$. By applying $R\theta_*$, we have $Ru^{\log}_{Y/(W_n,W_n(L))^*}(\mathcal{O}_{Y/W_n}) \to W_n\omega^{\cdot}_Y$. The fact that this is an isomorphism is reduced to the case $n=1$ by (4.5)(1). For $n=1$, if we take $Y^{\cdot} = Z^{\cdot} = Y$, $\omega^q_Y = C^q_1 \to W_1\omega^q_Y$ coincides with the Cartier isomorphism C^{-1}.

(4.20). — We show that in the semi-stable reduction case treated in § 1, the de Rham–Witt complex in § 1 is canonically isomorphic to the de Rham–Witt complex of this section. Let the situation be as in § 1 and define the log structures on Y and on $\mathrm{Spec}(k)$ as in (2.13.2). Let $W_n\omega^{\cdot}_{Y,I}$ (resp. $W_n\omega^{\cdot}_{Y,II}$) be the de Rham-Witt complex of § 1 (resp. § 4). Let U be a dense open subscheme of Y which is smooth over k, and let $u : U \to Y$ be the inclusion map. Then, $W_n\omega^{\cdot}_{U,I} = W_n\Omega^{\cdot}_U = W_n\omega^{\cdot}_{U,II}$ by the reduction to the classical case [IR], and hence $W_n\omega^{\cdot}_{Y,I}$ and $W_n\omega^{\cdot}_{Y,II}$ are regarded as subcomplexes of the same complex $u_*W_n\omega^{\cdot}_{U,I} = u_*W_n\omega^{\cdot}_{U,II}$ (here use (4.4) to see the map $W_n\omega^{\cdot}_{Y,II} \to u_*W_n\omega^{\cdot}_{U,II}$ is injective). By the presentation (4.6) of $W_n\omega^{\cdot}_{Y,II}$, we see that these subcomplexes are the same. Q.E.D.

We give a proof of the exactness of (1.5). Let $(W_n\widetilde{\omega}^q_Y)'$ be the sheaf obtained by replacing $f^{-1}(L^{gp})$ in (4.6.1) by the trivial group sheaf, and let $(W_n\omega^q_Y)'$ be the sheaf (4.6.1). Then we have an evident surjection $(W_n\widetilde{\omega}^q_Y)' \to W_n\widetilde{\omega}^q_Y$ which sits in a commutative diagram

$$\begin{array}{ccccccccc}
 & & (W_n\omega^{q-1}_Y)' & \longrightarrow & (W_n\widetilde{\omega}^q_Y)' & \longrightarrow & (W_n\omega^q_Y)' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_n\omega^{q-1}_Y & \longrightarrow & W_n\widetilde{\omega}^q_Y & \longrightarrow & W_n\omega^q_Y & \longrightarrow & 0
\end{array}$$

Here the upper row is exact, the left and the right vertical arrows are isomorphisms by (4.6), and the lower row is exact except possibly at $W_n\widetilde{\omega}_Y^q$. This shows the exactness of the lower row.

In the above, we obtained $(W_n\widetilde{\omega}_Y^q)' \xrightarrow{\sim} W_n\widetilde{\omega}_Y^q$. From this we obtain also $\big(\bigoplus_{q\geq 0} \omega^q_{W_n(Y)//W_n}\big)/\widetilde{\mathcal{I}} \xrightarrow{\sim} \bigoplus_{q\geq 0} W_n\widetilde{\omega}_Y^q$ where $W_n(Y)$ is endowed with $W_n(M)$, W_n is endowed with the trivial log structure, and $\widetilde{\mathcal{I}}$ is the ideal generated locally by $\eta_{i,j,a,b}$ and $d\eta_{i,j,a,b}$ (4.7) regarded as local sections of $\omega^{\cdot}_{W_n(Y)/W_n}$.

Finally the coincidence of the monodromy operator of § 1 and that of § 3 follows from the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n[-1] & \longrightarrow & W_n \otimes_{W_n\langle t\rangle} C_{Y/W_n} & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & \cong\big\downarrow & & \big\downarrow & & \cong\big\downarrow & & \\
0 & \longrightarrow & \theta^{-1}(W_n\omega^{\cdot}_Y)[-1] & \longrightarrow & \theta^{-1}(W_n\widetilde{\omega}^{\cdot}_Y) & \longrightarrow & \theta^{-1}(W_n\omega^{\cdot}_Y) & \longrightarrow & 0
\end{array}$$

where the upper row is the exact sequence in (3.6), the left and the right vertical arrows are as in the proof of (4.19), and the midlle vertical arrow is defined in the same way as the left and the right ones.

5. — de Rham cohomology

The aim of this section is to prove

THEOREM (5.1). — *Let A be a complete discrete valuation ring with field of fractions K and with perfect residue field k such that $char(K) = 0$ and $char(k) = p > 0$, and let N be the canonical log structure on* Spec(A) (2.13). *Let X be a scheme with a fine log structure M and with a smooth morphism $f : (X, M) \to$* (Spec$(A), N$) *and let $Y = X \otimes_A k$. Denote the inverse image of M* (resp. N) *on Y* (resp. Spec(k)) *by $\overline{M}$* (resp. L). *Assume that X is proper over A and the morphism $(Y, \overline{M}) \to$* (Spec$(k), L$) *is of Cartier type. Fix $m \in \mathbb{Z}$, and let*

$$D = \mathbb{Q} \otimes \varprojlim_{n} H^m(((Y, \overline{M})/(W_n, W_n(L)))_{crys}\,,\ \mathcal{O}_{Y/W_n})\,.$$

Then, to each prime element π of A, we can associate a canonical K–isomorphism

$$\rho_\pi : K \otimes_{K_0} D \xrightarrow{\sim} H^m_{DR}(X_K/K)$$

(K_0 denotes the field of fractions of $W(k)$, $X_K = X \otimes_A K$ endowed with the log structure induced by M, and $H^m_{DR}(X_K/K) = H^m(X_K, \omega^{\cdot}_{X_K/K})$), satisfying

$$\rho_{\pi u} = \rho_\pi \exp(\log(u)\mathcal{N}) \quad \text{for } u \in A^\times .$$

In particular, the linear operator $\rho_\pi \circ \mathcal{N} \circ \rho_\pi^{-1}$ on $H^m_{DR}(X_K/K)$ is independent of π.

We shall use the following notations. Let $A_n = A \otimes \mathbf{Z}/p^n\mathbf{Z}$, $X_n = X \otimes \mathbf{Z}/p^n\mathbf{Z}$. We endow $\mathrm{Spec}(A_n)$ (resp. X_n) with the inverse image of N (resp. M). We denote $Ru^{\log}_{*/*'}$ simply by $[*/*']$.

LEMMA (5.2). — *Fix a prime element π of A, and let $\mathrm{Spec}(R_n)$ be the PD–envelope of $\mathrm{Spec}(A_n)$ in $\mathrm{Spec}(W_n[t])$ with respect to the closed immersion $t \longmapsto \pi$. Endow $\mathrm{Spec}(R_n)$ with the log structure associated to $\mathbf{N} \to R_n$; $1 \longmapsto t$. Then, we have a canonical isomorphism in $\mathbf{Q} \otimes ps(D((X.)_{et}, R.))$*

$$h_\pi : \mathbf{Q} \otimes \{R_n \otimes^L_{W_n} [Y/(W_n, W_n(L))]\}_{n\geq 1} \cong \mathbf{Q} \otimes \{[X_n/\mathrm{Spec}(R_n)]\}_{n\geq 1} .$$

PROOF. Note $[X_n/\mathrm{Spec}(R_n)] = [X_1/\mathrm{Spec}(R_n)]$. Take $r \geq 0$ such that $(m_A)^{p^r} \subset pA$. We define h_π to be the composite of

$$(5.2.1)\qquad \begin{aligned} & \mathbf{Q} \otimes \{R_n \otimes^L_{W_n} [Y/(W_n, W_n(L))]\}_n \\ & \xleftarrow[1\otimes\varphi^r]{\cong} \mathbf{Q} \otimes \{R_n \underset{\varphi^r}{\nwarrow} \otimes^L_{W_n} [Y/(W_n, W_n(L))]\}_n && (2.24) \\ & \cong \mathbf{Q} \otimes \{R_n \underset{g}{\nwarrow} \otimes_{W_n<t>} [Y/\mathrm{Spec}(W_n < t >)]\}_n && (4.13) \\ & \xrightarrow{(*)} \mathbf{Q} \otimes \{[X_1/\mathrm{Spec}(R_n)]\}_n = \mathbf{Q} \otimes \{X_n/\mathrm{Spec}(R_n)]\}_n \end{aligned}$$

where φ are the frobeniuses, $\mathrm{Spec}(W_n < t >)$ is endowed with the log structure associated to $\mathbf{N} \to W_n < t >$; $1 \longmapsto t$, $g : W_n < t > \to R_n$ is the homomorphism

$$t \longmapsto t^{p^r}, \quad a \longmapsto \varphi^r(a) \quad \text{for } a \in W_n ,$$

and the arrow (*) is induced by the left big square of the commutative diagram below. Though the log structures are abbreviated for simplicity, this diagram is a commutative diagram of schemes with log structures. In this diagram the composites of the horizontal arrows are the r–th iteration of the frobenius.

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(**)"] & X_1 \arrow[d] \\
\mathrm{Spec}(A_1) \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(A_1) \arrow[d] \\
\mathrm{Spec}(R_n) \arrow[r, "g"] & \mathrm{Spec}(W_n < t >) \arrow[r] & \mathrm{Spec}(R_n)
\end{tikzcd}$$

It is easily seen that h_π is R_n–linear and is independent of the choice of r. The following (5.3) shows that h_π is an isomorphism.

LEMMA (5.3). — *The arrow* $(*)$ *in* (5.2.1) *is an isomorphism.*

PROOF.

$$\mathbf{Q}\otimes\{R_n \underset{g}{\nwarrow} \otimes^L_{W_n<t>}[Y/\operatorname{Spec}(W_n <t>)]\}_n \cong \mathbf{Q}\otimes\{R_n \underset{\varphi^r}{\nwarrow} \otimes^L_{R_n}[X_1/\operatorname{Spec}(R_n)]\}_n$$

(by (2.23) since (**) is cartesian)

$$\xrightarrow[1\otimes\varphi^r]{\cong} \mathbf{Q}\{[X_1/\operatorname{Spec}(R_n)]\}_n$$

by (2.24).

(5.4). — The isomorphism h_π induces an isomorphism in

$ps(D((X.)_{et}, A.))$

$$\mathbf{Q}\otimes\{A_n \otimes^L_{W_n}[Y/(W_n, W_n(L))]\}_n \xrightarrow{\sim} \mathbf{Q}\otimes\{A_n \otimes^L_{R_n}[X_n/\operatorname{Spec}(R_n)]\}_n$$

and the last object is isomorphic to $\mathbf{Q}\otimes\{[X_n/\operatorname{Spec}(A_n)]\}_n$ by the base change theorem (2.23). Since

$$\mathbf{Q}\otimes \varprojlim_n H^m(X_n, [X_n/\operatorname{Spec}(A_n)]) \cong H^m_{DR}(X_K/K),$$

we obtain an isomorphism

$$\rho_\pi : K \otimes_W \varprojlim_n H^m(((Y,\overline{M})/(W_n, W_n(L)))_{crys}, \mathcal{O}_{Y/W_n}) \xrightarrow{\sim} H^m_{DR}(X_K/K)\,.$$

(5.5). — Finally we prove the relation between ρ_π and $\rho_{\pi u}$ stated in (5.1). Since $A^\times$ is generated by $1 + m_A$ and the image of the multiplicative representative $\lambda : k^\times \longrightarrow A^\times$, it is sufficient to consider the case $u \equiv 1 \bmod m_A$ and the case $u = \lambda(c)$ for $c \in k^\times$.

We consider first the case $u \equiv 1 \bmod m_A$. Let r be as in the proof of (5.2). Consider the morphism

$$\mathbb{Q} \otimes \{[Y/(W_n, W_n(L))]\}_n \xleftarrow[\cong]{\varphi^r} \mathbb{Q} \otimes \{[Y/(W_n, W_n(L))]\}_n$$

$$\xrightarrow{s} \mathbb{Q} \otimes \{[Y/W_n < t >]\} \xrightarrow{f_i} \mathbb{Q} \otimes \{[X_1/A_n]\}_n$$

($i = 1, 2$), where s is the morphism induced by h in (4.13) by taking $T_n = \mathrm{Spec}(W_n < t >)$ whose log structure is as in the proof of (5.2), and the arrow f_1 (resp. f_2) is induced by

$$g_1 : W_n < t > \longrightarrow A_n\,; \quad t \longmapsto \pi^{p^r} u^{p^r}$$

(resp. $g_2 : W_n < t > \longrightarrow A_n\,;\, t \longmapsto \pi^{p^r}$). Then the map $\rho_{\pi u}$ (resp. ρ_π) $: D \longrightarrow H^m(X_K/K)$ coincides with $H^m(f_1 \circ s \circ \varphi^{-r})$ (resp. $H^m(f_2 \circ s \circ \varphi^{-r})$). Let $(D'_n, M_{D'_n})$ be the PD–envelope of $\mathrm{Spec}(W_n < t >)$ in $\mathrm{Spec}(W_n < t_1, t_2 >)$, where $\mathrm{Spec}(W_n < t >)$ is endowed with the log structure associated to $\mathbb{N} \longrightarrow W_n < t >;\ 1 \longmapsto t$ and $\mathrm{Spec}(W_n < t_1, t_2 >)$ is endowed with the product log structure. Let $p_i : D' \longrightarrow \mathrm{Spec}(W_n < t >)$ ($i = 1, 2$) be the i–th projection. Since $g_1 \bmod p$ and $g_2 \bmod p$ coincide (as morphisms of log schemes), we have by (2.17.3)

$$p_1^* = \sum_{i \geq 0} (t_1 t_2^{-1} - 1)^{[i]} p_2^* \circ \prod_{0 \leq j < i} (\mathcal{N} - j)$$

as morphisms $[Y/W_n < t >] \longrightarrow [Y/D'_n]$, where $\mathcal{N} = \nabla_t^{\log}$ and p_i^* denotes the pull back by p_i. From this we obtain

$$\begin{aligned} f_1 &= \sum_{i\geq 0}(u^{p^r}-1)^{[i]} f_2 \circ \prod_{0\leq j<i}(\mathcal{N}-j) \\ &= \sum_{i\geq 0} \log(u^{p^r})^{[i]} f_2 \circ \mathcal{N}^i . \end{aligned}$$

Hence we have

$$\begin{aligned} \rho_{\pi u} &= H^m\Big(\sum_{i\geq 0} \log(u^{p^r})^{[i]} f_2 \circ \mathcal{N}^i \circ s \circ \varphi^{-r}\Big) \\ &= H^m\Big(\sum_{i\geq 0} \log(u^{p^r})^{[i]} f_2 \circ s \circ \varphi^{-r}\Big) \circ (p^{-r}\mathcal{N})^i \end{aligned}$$

(by $\mathcal{N} \circ s = s \circ \mathcal{N}$ which is easily seen, and by $\mathcal{N}\varphi = p\varphi\mathcal{N}$)

$$= \sum_{i\geq 0} (i!)^{-1} (\log(u))^i \rho_\pi \circ \mathcal{N}^i .$$

Next assume $u \in \lambda(k^\times)$. Then, the PD-morphism over W_n

$$f : W_n < t > \longrightarrow W_n < t > ; \quad t \longmapsto ut$$

preserves the frobenius, and this fact and the characterization of the isomorphism (4.13) show that the diagram

$$\begin{tikzcd}[column sep=small] & \mathbf{Q} \otimes \{[Y/(W_n, W_n(L))]\}_n \arrow[dl, "\text{by (4.13)}"'] \arrow[dr, "\text{by (4.13)}"] & \\ \mathbf{Q} \otimes \{[Y/\operatorname{Spec}(W_n < t >)]\}_n \arrow[rr, "\text{by } f"'] & & \mathbf{Q} \otimes \{[Y/\operatorname{Spec}(W_n < t >)]\}_n \end{tikzcd}$$

is commutative. The fact $\rho_\pi = \rho_{u\pi}$ follows from this easily.

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