Astérisque

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Astérisque, tome 222 (1994), p. 303-326
<http://www.numdam.org/item?id=AST_1994__222__303_0>

QUASI-REGULARITY PROPERTY FOR UNFOLDINGS OF HYPERBOLIC POLYCYCLES

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1. INTRODUCTION.

Let X be a real analytical vector field on $\mathbf{R}^2$. A polycycle Γ of X is an immersion of the circle, union of trajectories (Singular points and separatrices whose α and ω limits are contained in this set of singular points). Moreover one supposes that Γ is oriented by the flow of X and that a return map $P(x)$ along Γ is defined on some interval σ with one end point on Γ : σ is parametrized by analytical variable $x \in [0, x_1]$, $\{x = 0\} = \sigma \cap \Gamma = \{q\}$ and $P(x) : [0, x_0] \longrightarrow [0, x_1]$ for some $x_0 \in]0, x_1[$

We say that Γ is an hyperbolic polycycle if all the singular points in γ are hyperbolic saddle points. Let $\{p_1, \dots, p_k\}$ the set of these singular points listing in the way they are encountered when we describe Γ starting at q. We define the hyperbolicity ratio of p_i, $i = 1, \dots, k$ to be $r_i = \dfrac{\mu_i'}{\mu_i''}$ where $-\mu_i'$, μ_i'' are the eigenvalues at p_i ($\mu_i', \mu_i'' > 0$).

The Poincaré map $P(x)$ is analytic for $x > 0$, and extends continuously at 0 by $P(0) = 0$.

In 1985, Yu. Ilyashenko [I1] introduced a notion (the almost-regularity) similar to the following one up to a composition by the logarithm :

Definition. *Let $g(x) : [0, x_0] \longrightarrow \mathbf{R}$ a function, analytic for $x > 0$, and continuous at $x = 0$. One says that g is quasi-regular if:*

QR_1) *$g(x)$ has a formal expansion of Dulac type. This means that there exists a formal series:*

$$\hat{g}(x) = \sum_{i=0}^{\infty} x^{\lambda_i} P_i(lnx)$$

where λ_i is a strictly increasing sequence of positive real numbers $0 < \lambda_0 < \lambda_1 < \ldots$ tending to infinity and for each i, P_i is a polynomial, and $\hat{g}$ is a formal expansion of $g(x)$ in the following sense:

$$\forall n \geq 0 \qquad g(x) - \sum_{i=0}^{n} x^{\lambda_i} P_i(lnx) = o(x^{\lambda_n}).$$

$\mathbf{Q}R_2$*) Let $G(\xi) = g(e^{-\xi})$ for $\xi \in [\xi_0 = -logx_0, \infty[$.*
Then G has a bounded holomorphic extension in a domain $\Omega(C) \subset \mathbf{C}$ where $\Omega(C) = \{\zeta = \xi + i\eta \mid \xi^4 \geq C(1+\eta^2)\}$ for some $C > 0$.

In the same paper [I1], Ilyashenko proved that the shift map $\delta(x) = P(x) - x$ is quasi-regular. The property $\mathbf{Q}R_1$ was already established by Dulac in [D].

As a consequence of the Phragmen-Lindelöf theorem (see [C]) a flat quasi-regular function ($g(x) = o(x^n), \forall n$) is necessarily equal to zero, and it follows from this that Γ cannot be accumulated by limit cycles of X (a limit cycle of X is an isolated periodic orbit).

This result was a first step in the solution of the "Dulac problem", for which one needs to look not only at hyperbolic polycycles but more generally at all elementary polycycles. As it is well known, this general solution ([EMMR], [E1], [E2], [I2], [I3]) involved more elaborated technics, and we limit ourselves to the hyperbolic polycycles in this paper.

Here we want to consider the unfoldings (X_λ, Γ) of a hyperbolic polycycle Γ, germs of finite parameter family (X_λ), with $X_0 = X$ defined by a representative family on $V \times W$ where V is a neighborhood of Γ and W neighborhood of 0 in the parameter space.

As it was shown in [R], it is useful to obtain quasi-regularity property for 1-parameter unfoldings, in order to study finite cyclicity for general unfoldings of hyperbolic polycycles. In the present paper, we extend to any 1-parameter unfoldings a result of [R], proved there for hyperbolic loops (singular cycles with just 1 singular point):

Theorem 1. *Let (X_ϵ ,Γ) a 1-parameter analytic unfolding of an hyperbolic polycycle Γ for X_0 with k vertices. Let $P(x,\epsilon)$ the unfolding of the return map where x is some analytic parameter defined as above for X_0. Let $\delta(x,\epsilon) =$*

$P(x,\epsilon) - x$. Let $\widehat{\delta}(x,\epsilon) = \sum_{i=0}^{\infty} \delta_i(x)\epsilon^i$ (i.e: $\delta_i(x) = \frac{1}{i!}\frac{\partial^i \delta(x,0)}{\partial \epsilon^i}$) the formal expansion of δ in ϵ.

Then, there exists some $R > 0$ (depending on $r_1(0), \dots, r_k(0)$) such that for $\forall i \in \mathbf{N}$, $x^{iR}\delta_i(x)$ is quasi-regular.

Remarks.

1) Given an unfolding X_ϵ and a transversal $\sigma \simeq [0, x_1]$ chosen as above for X_0, the return map $P(x,\lambda)$ is defined in a domain $D = \cup_{\epsilon \in W}[\alpha(\epsilon), x_1]$ where $\alpha(\epsilon)$ is a continuous function, such that $\alpha(0) = 0$. So, given any $x \in]0, x_1]$, the return map $P(x,\lambda)$ is defined for x if $|\epsilon|$ is small enough. From this it follows that the functions $\delta_i(x)$ in the above theorem, are defined for $\forall x \in]0, x_1]$.

2) Theorem 1 extends Ilyashenko's one which corresponds to the quasi-regularity of $\delta_0(x)$.

The generalization brought by theorem 1 is useful to study unfolding of <u>identical</u> polycycles, i.e polycycles such that $\delta(x) = P(x) - x \equiv 0$. Suppose for instance that $\lambda = \epsilon \in \mathbf{R}$. Then, if (Γ, X_0) is an identical polycycle, one can write:

$$\delta(x,\epsilon) = \epsilon^n \bar{\delta}(x,\epsilon)$$

for some $n \geq 1$, with a function $\bar{\delta}(x,\epsilon)$ such that $\bar{\delta}(x,0) \not\equiv 0$. Then from theorem 1, we have that $\bar{\delta}(x,0)$ has a non-trivial Dulac expansion.

So, the equation for limit cycles $\{\delta(x,\epsilon) = 0\}$, which is equivalent to $\{\bar{\delta}(x,\epsilon) = 0\}$, has the same properties that in the non-identical case ($\delta(x,0) \not\equiv 0$).

This allows us to develop for some identical unfoldings a proof similar to the one for unfolding of non-identical polycycles. In [R] these ideas were applied to prove the finite cyclicity of any analytic unfolding of loops (Singular cycles with just one singular hyperbolic point). Here we extend it to some polycycles with 2 singular points:

Theorem 2. *Let (X_λ, Γ) an analytic unfolding of an hyperbolic 2-polycycle Γ (a polycycle with 2 singular points p_1, p_2). Let $r_1(\lambda), r_2(\lambda)$ the λ-depending hyperbolicity ratio at p_1, p_2. Suppose that:*

1) For all λ, $r_1(\lambda)r_2(\lambda) \equiv 1$
2) at least one of the two saddle connexions remains unbroken (for all λ).
Then (X_λ, Γ) has a finite cyclicity.

Remark.

A part the conditions 1,2, no other conditions are imposed on (X_λ, Γ) and the polycycle Γ may be identical. The non-identical case was already worked out in a previous paper [El.M]. Moreover, if $r_1(0) = r_2(0)^{-1} \notin \mathbf{Q}$, a result of finite cyclicity was obtained in [M], without the conditions 1,2.

The conditions 1,2 in the theorem 2 may seem very restrictive. Netherthe-less the theorem has the following natural application to polynomial vector fields. Let P_{2p} be the family of all polynomial vector fields of some even degree $2p$, $p \geq 1$. It is easy to extend P_{2p} in an analytic family of vector fields on the sphere (X_λ). This family (X_λ) is equivalent to P_{2p} on the interior of a 2-disk D^2, whose boundary ∂D^2 corresponds to the "circle at infinity γ_∞". Singular points of (X_λ) appears at infinity in pairs of opposite points (p, q) and a consequence of the even degree is that the tangential eigenvalues at p, q are opposite and the same for the two radial eigenvalues. It follows that the product of the ratios of hyperbolicity at p and q is one. Then if for some value λ_0 (that we can suppose equal to 0), $X_{\lambda_0} = X_0$ has just a pair of singular points p, q on γ_∞ and if there exists a connection Γ_1 of p and q in $\mathrm{int}(D^2)$, one can apply theorem 2 to the unfolding (X_λ, Γ) where Γ is one of the 2 polycycles containing Γ_1 and an arc Γ_2 of γ_∞ joigning p and q; we have $r_1(\lambda)r_2(\lambda) \equiv 1$ as noted above and the connection Γ_2 at infinity remains unbroken. This applies to the quadratic family $\mathcal{P}_2$ and allows to prove the finite cyclicity of some of the 121 possible cases of periodic limit sets listed for this family in [DRR] (cases labelled: H_1^1, H_2^1 in this article).

In the first paragraph, we prove the theorem 1. Of course, we hope that the quasi-regularity property proved here will have a more general application that the one given in theorem 2 and proved below in the second paragraph. In fact the proof uses the existence of a well ordered expansion for $\delta(x, \lambda)$ at any order of differentiability. This expansion was established for unfoldings like in theorem 2 in [El.M] and we recall it bellow. In this paper it was used to prove the finite cyclicity in the non-identical case. Here, we use it to reduce in some sense the general case to the non-identical case, by the method already described in the loop case in [R]. This is made in the second paragraph.

Firstly a natural ideal in the space of parameter functions germs, the coefficient Ideal $\mathcal{J}$ is associated to any unfolding of identical graphic . If $\{\phi_1, \dots, \phi_l\}$ is a system of generators for $\mathcal{J}$, and, in our case, when it exists a well- ordered expansion for $\delta(x, \lambda)$, one can divide $\delta(x, \lambda)$ in the ideal:

$$\delta(x, \lambda) = \sum \phi_i(\lambda)\delta_i(x, \lambda)$$

with functions $\delta_i(x, \lambda)$ having also a well-ordered expansion. The theorem 1 is then used to prove that "for most of indices i", $\delta_i(x, 0)$ is quasi-regular and then has a non-trivial Dulac expansion; and we can apply, as in [R] for the loop case, a derivation-division algorithm similar to the one used for the non-identical case. Here we will use the precise procedure developed in [El.M] for the non identical case.

2. QUASI-REGULARITY PROPERTY

2.1 Reduction to the quasi-regularity property for saddle transitions.— We recall here a definition used in [I1]:

DEFINITION 1 [I1].— *A domain L of* **C** *is said to be of class $\mathcal{I}$ if it contains a domain $\Omega(C)$ of the form*

$$\Omega(C) = \{\zeta = \xi + \eta; \xi^4 \geq C(1 + \eta^2)\}.$$

for some $C > 0$.

In the neighbourhood of each saddle point p_i, choose as in [I1] a chart analytical in (x_i, y_i, ϵ) in which the field X_ϵ takes the form

$$\begin{cases} \dot{x}_i & = x_i \\ \dot{y}_i & = -y_i[r_i(\epsilon) + x_i^{n_i} y_i f_i(x_i, y_i, \epsilon)] \end{cases}$$

where $n_i \in \mathbf{N}$ and $n_i \geq r_i(0)$ and the functions f_i are analytical on $\Delta_i = \{|x_i| \leq 1\} \times \{|y_i| \leq 1\} \times]-\epsilon_0, \epsilon_0[$ and satisfy

$$\sup_{\Delta_i} | f_i | \leq \inf_{|\epsilon|<|\epsilon_0|} (1, r_i(\epsilon)/2).$$

Denote by $\sigma_i = \{(x_i, y_i); y_i = 1\}$, $\sigma_i^+ = \sigma_i \cap [0, 1] \times \{1\}$, $\tau_i = \{(x_i, y_i); x_i = 1\}$, $\tau_i^+ = \tau_i \cap \{1\} \times [0, 1]$, $D_i(., \epsilon)$ the Dulac map which send σ_i^+ on τ_i^+

$$y_i = D_i(x_i, \epsilon) = D_{i,\epsilon}(x_i)$$

and $G_i(.,\epsilon)$ the analytical map which send τ_i on σ_{i+1} (with cyclic notation)

$$x_{i+1} = G_i(y_i, \epsilon) = G_{i,\epsilon}(y_i).$$

The return map on σ_1^+ is given by

$$P(x_1, \epsilon) = G_{k,\epsilon} \circ D_{k,\epsilon} \circ G_{k-1,\epsilon} \circ D_{k-1,\epsilon} \circ \cdots \circ G_{1,\epsilon} \circ D_{1,\epsilon}.$$

Put $H_1 = D_1$, $H_2 = G_1, \ldots$, $H_{2k-1} = D_k$, $H_{2k} = G_k$ and agree that the composition is made with respect to the first variable

$$P(x_1, \epsilon) = H_{2k} \circ H_{2k-1} \circ \cdots \circ H_1(x_1, \epsilon),$$

for all $n \in \mathbf{N}$, we can write

$$\begin{aligned}\frac{\partial^n P}{\partial \epsilon^n}(x_1, \epsilon) = & \sum_{p_{2k}+q_{2k}=l_{2k}=1}^{n} \sum_{p_{2k-1}+q_{2k-1}=l_{2k-1}=1}^{n} \cdots \sum_{l_1=1}^{n} A_{l_1,p_2,q_2,\ldots,p_{2k},q_{2k}} \times \\ & \times \left(\frac{\partial^{l_{2k}} H_{2k}}{\partial x_1^{p_{2k}} \epsilon^{q_{2k}}} \circ H_{2k-1} \circ \ldots \circ H_1(x_1, \epsilon)\right) \times \\ & \times \left(\frac{\partial^{l_{2k-1}} H_{2k-1}}{\partial x_1^{p_{2k-1}} \epsilon^{q_{2k-1}}} \circ H_{2k-2} \circ \ldots \circ H_1(x_1, \epsilon)\right) \times \ldots \times \left(\frac{\partial^{l_1} H_1}{\partial \epsilon^{l_1}}(x_1, \epsilon)\right)\end{aligned}$$

the coefficients $A_{l_1,p_2,q_2,\ldots,p_{2k},q_{2k}} \in \mathbf{Z}$.

Using [I1], we see that the maps $x_1 \mapsto H_i \circ H_{i-1} \circ \cdots \circ H_1(x_1, 0)$ are quasi-regular and their continuation to the complex plane, after conjugacy by the map $e^{-\xi}$, send a domain of class $\mathcal{I}$ on a domain of class $\mathcal{I}$. Furthermore, the analytical maps G_i can be naturally continued to complex disks in biholomorphic maps and their partial derivatives of all order $\partial^{l_i} G_i / \partial x_1^{p_i} \epsilon^{q_i}(x_1, 0)$ are quasi-regular. So the Theorem 1 is a consequence of the Lemma 1 below.

2.2 Quasi-regularity for unfolding of hyperbolic saddle transition.—

This section is devoted to the proof of the following Lemma

LEMMA 1. *Let P be a saddle point of an analytical planar field X_0 and X_λ an analytical unfolding of X_0 near P (the parameter λ belonging to some neighbourhood $\mathcal{V}^m$ of 0 in $\mathbf{R}^m$). Let $D(.,\lambda)$ the Dulac map defined as above and $r(\lambda)$ the hyperbolicity ratio of the saddle point $P(\lambda)$. Let $R = Max(1, r(0))$, then for all $n \in \mathrm{N}$ and $p + q_1 + q_2 + \cdots + q_m = n$, the map*

$$x \mapsto x^{nR} \frac{\partial^n D}{\partial x^p \lambda_1^{q_1} \cdots \lambda_m^{q_m}}(x, 0)$$

is quasi-regular.

Remark—.

The Lemma for $n = 0$ is proved in [I1]; the proof in the general case is based on this one. The multiplying function x^{nR} is not the best one for some values of $(p, q_1, \dots, q_m)$. But this choice allows an easy estimate of the constant R in Theorem 1: If we put $R_i = \mathrm{Max}(1, r_i(0))$ then in Theorem 1, take $R = R_1 \cdots R_k$. The result of this Lemma and the remark above show that Theorem 1 may be extended to any unfolding X_λ with $m > 1$: the maps $x_1 \mapsto x_1^{nR} \partial^n P/\partial\lambda_1^{q_1} \dots \lambda_m^{q_m}(x_1, 0)$ are quasi-regular.

Proof of Lemma 1.— We use the notations of [I1]. Choose an analytical chart (x, y, λ) so that the field X_λ takes the form

$$\begin{cases} \dot{x} &= x \\ \dot{y} &= -y[r(\lambda) + x^n y f(x, y, \lambda)] \end{cases} \tag{2.2.1}$$

with $n \geq r_0 = r(0)$ and f analytical on $\Delta = \{|x| \leq 1\} \times \{|y| \leq 1\} \times \mathcal{V}^m$ and satisfying

$$\sup_{\Delta} |f| \leq \inf_{\lambda \in \mathcal{V}} (1, r(\lambda)/2) \tag{2.2.2}$$

the family (X_λ) is induced by the family (X_μ) given by

$$\begin{cases} \dot{x} &= x \\ \dot{y} &= -y[r(\mu) + x^n y f(x, y, \lambda)] \end{cases} \tag{2.2.3}$$

where $\mu = (\mu_0, \lambda) \in \mathcal{V}^{m+1} \subset \mathbf{R}^{m+1}$ and $r(\mu) = r_0 + \mu_0$. Denote by $\sigma^+ = \{(x, y); y = 1 \text{ and } x \in [0, 1]\}$, $\tau^+ = \{(x, y); x = 1 \text{ and } y \in [0, 1]\}$ and $D(., \mu)$ the Dulac map which sends σ^+ on τ^+

$$y = D(x, \mu).$$

Extend the real field X_μ to a field $\widehat{X}_\mu$ defined on $\mathbf{C}^2$ with local variables (z, w) and complexify the parameter λ to $\widehat{\lambda} \in \mathcal{V}_{\mathbf{C}}^m \subset \mathbf{C}^2$

$$\begin{cases} \dot{z} &= z \\ \dot{w} &= -w[r + z^n w f(z, w, \widehat{\lambda})] \end{cases} \tag{2.2.4}$$

we keep the parameter μ_0 real for reason given below.

We will say that the complex family (2.2.4) as above, with $\mu_0 \in \mathbf{R}$ belongs to the Siegel domain because the ratio of hyperbolicity remains real. Denote by d_0 and d_1 the punctured disks of coordinate z

$$d_0 = \{(z,w); 0 < |z| \leq 1, w = 0\},$$

$$d_1 = \{(z,w); 0 < |z| \leq 1, w = 1\},$$

and by $\widehat{d}_0$, $\widehat{d}_1$ their universal covering with base point respectively on (1,0), (1,1) and with coordinate $\zeta = -\mathrm{Ln}z$. Denote also by d_2 the punctured disk of coordinate w

$$d_2 = \{(z,w); 0 < |w| \leq 1, z = 1\}$$

and by $\widehat{d}_2$ his universal covering with base point on (1,1) and coordinate $\nu = -\mathrm{Ln}w$. Let us show that for $\widehat{\lambda} \in \mathcal{V}^m_{\mathbf{C}}$, ϵ small enough and $C > 0$ big enough, there exists a map $\widehat{D}$ holomorphic in $(\zeta, \widehat{\lambda}) \in \Omega(C) \times \mathcal{V}^m_{\mathbf{C}}$, analytical in $\mu_0 \in]-\epsilon, \epsilon[$ and with values in $\widehat{d}_2$; furthermore, for $\widehat{\lambda} \in \mathcal{V}^m = \mathcal{V}^m_{\mathbf{C}} \cap \mathbf{R}^m$, the map $\widehat{D}$ is the complex continuation of the map D defined on $\sigma^+ \subset d_1$.

Let $\zeta = \xi + i\eta \in \Omega(C)$ and α^ζ the union of the two segments $[0,\xi]$, $[\xi,\zeta]$ parametrized by the arc-length s (see fig. 2a): $s(0) = 0$, $s(\xi) = \xi$, $s(\zeta) = \xi + |\eta| = S$. Let γ_0 and γ_1 the curves on fig. 2b defined by

$$\gamma_1 = \gamma_{\zeta,1} : \quad [0,S] \mapsto \mathbf{C} \times \{1\}, \quad s \to (\mathrm{Exp}(-\alpha^\zeta(s)), 1),$$

$$\gamma_0 = \gamma_{\zeta,0} : \quad [0,S] \mapsto \mathbf{C} \times \{0\}, \quad s \to (\mathrm{Exp}(\alpha^\zeta(s) - \zeta), 0),$$

and $u = r(\mu)\mathrm{Ln}z + \mathrm{Ln}w$ the first integral of the linear field associated to the field $\widehat{X}_\mu$. The formula

$$\frac{d}{dt}(|w|^2) = -2|w|^2[r(\mu) + \mathcal{R}e(z^n w f(z,w,\widehat{\lambda}))]$$

and the hypothesis (2.2.2) show that through each point $p = (z,w)$ with $z \neq 0$ and $|w| \leq 1$ passes a curve, solution of the system (2.2.4), which cover the segment $[z, z/|z|]$ of the curve γ_0 under the projection $\pi_z : (z,w) \to z$ and which is entirely contained in the polydisk $P = \{|z| \leq 1, |w| \leq 1\}$. Let us show now that if $p = (z,1) = (\mathrm{Exp}(-\zeta), 1)$, then this curve can be extented to a curve $\widehat{\gamma}_\zeta$ which cover the curve $\gamma_{\zeta,0}$ under the projection π_z and is contained in the intersection of P with the surface φ_p (the complex solution of the system (2.2.4) passing through p). To prove this point, an estimate of u along this curve is usefull. Parametrize the arc of $\widehat{\gamma}_\zeta$ defined above by $s \in [0,\xi]$; on the end of this arc, we have $|w| < 1$. Hence, suppose that the curve $\widehat{\gamma}_\zeta$ exists for $s \in [0, T]$ with $T > \xi$ and differentiate u along this curve

$$\dot{u} = z^n w f(z, w, \widehat{\lambda});$$

one can compute $|z^n w| \leq |z^{r_0} w| = |z^{-\mu_0}|.|z^r w| = |z^{-\mu_0}|.|e^u|$; along the curve $\widehat{\gamma}_\zeta$, we have $e^{-\xi} \leq |z| \leq 1$, and $|z^{-\mu_0}| \leq A = \mathrm{Sup}(e^{\mu_0 \zeta}, 1)$; therefore, we get $|\dot{u}| \leq A.|e^u|$. Put $v(s) = |u(\widehat{\gamma}_\zeta(s)) - u(p)|$; as along the curve $\widehat{\gamma}_\zeta$ we have $|dt/ds| = 1$, one can easily verify that

$$|dv/ds| \leq A.e^{-r\xi + v(s)} \leq e^{-\frac{r_0}{2}\xi + v(s)}.$$

From now on, the same arguments as in [I1] can be used; so we conclude that for $C > 0$ big enough, we get $v(s) \leq 1$ along the curve $\widehat{\gamma}_\zeta$ for all $\zeta \in \Omega(C)$ and for all $\mu = (\mu_0, \widehat{\lambda}) \in]-\epsilon, \epsilon[\times \mathcal{V}_{\mathbf{C}}^m$. The extension of th curve $\widehat{\gamma}_\zeta$ for $s \in [0, S]$ is done as in [I1] and we put

$$\widehat{D}(\zeta, \mu) = \nu(\widehat{\gamma}_\zeta(S)) = u(\widehat{\gamma}_\zeta(S)) = r(\mu)\zeta + h(\zeta, \mu)$$

with $|h(\zeta, \mu)| = v(S) \leq 1$; the same estimate as above shows that $h(\zeta, \mu) \to 0$ as $\zeta \to \infty$ and $\zeta \in \Omega(C)$ uniformly on $\mu \in]-\epsilon, \epsilon[\times \mathcal{V}_{\mathbf{C}}^m$.

Remark that for $\mu_0 \in \mathbf{C}$, the results above are false in domains of class $\mathcal{I}$, but still valid in domains of the form $\omega(C, C') = \{(\xi, \eta);\quad \xi \geq C(1 + C'\eta^2)^{1/2}\}$. Unfortunately, the Phragmen-Lindelöf theorem does not apply on such domains.

The extension map $\widehat{D}$ is holomorphic in $(\zeta, \widehat{\lambda}) \in \Omega(C) \times \mathcal{V}_{\mathbf{C}}^m$ and analytic in $\mu_0 \in]-\epsilon, \epsilon[$ and we have $D(x, \mu) = e^{-\widehat{D}(-\mathrm{Ln}x, \mu_0, \lambda)}$ for $x \in]0, x_0]$ and $\mu = (\mu_0, \lambda) \in \mathcal{V}_{\mathbf{R}}^{m+1}$. Denote by $F(\zeta, \mu_0, \widehat{\lambda}) = e^{-\widehat{D}(\zeta, \mu_0, \widehat{\lambda})}$; the analytic extension of the partial derivatives $\partial^n D / \partial x^p \mu_0^q \lambda_1^{q_1} \dots \lambda_m^{q_m}$ to the domain $\Omega(C) \times]-\epsilon, \epsilon[\times \mathcal{V}_{\mathbf{C}}^m$ is a function of the form

$$e^{p\zeta} \sum_{l=1}^{p} a_l \frac{\partial^{n-p+l} F}{\partial \zeta^l \mu_0^q \widehat{\lambda}_1^{q_1} \dots \widehat{\lambda}_m^{q_m}}$$

with $a_l \in \mathbf{Z}$. As the function F is bounded on $\Omega(C) \times]-\epsilon, \epsilon[\times \mathcal{V}_{\mathbf{C}}^m$ and holomorphic in $(\zeta, \widehat{\lambda})$, we begin by studying the functions $\partial^n F / \partial \mu_0^n$; but we have $\widehat{D}(\zeta, \mu) = -\mathrm{Ln} w(t(S), p, \mu)$. Then if we put $w_n(t(S), p, \mu) = (\partial^n w / \partial \mu_0^n)(t(S), p, \mu)$, we see that it suffices to study the functions $u_n = w_n / w$. Let us begin by the function u_1: using the fact that $r(\mu) = r_0 + \mu_0$ and the second line of the system (2.2.4), we get

$$\dot{w}_1 = -w_1[r + z^n w f_{11}] - w$$

and then $\dot{u}_1 = u_1 z^n w f_1 - 1$, where f_{11} and f_1 are holomorphic functions in $(z, w, \widehat{\lambda})$ bounded on $\mathcal{L} = \{|z| \leq 1\} \times \{|w| \leq 1\} \times \mathcal{V}_{\mathbf{C}}^m$. Put $v_1 = |u_1|$, then one can easily show that $|dv_1/ds| \leq A_{11} v_1 |e^u| + 1$ where $A_{11} = A.S_1$ and $S_1 = \mathrm{Sup}\{|f_1(z, w, \widehat{\lambda})|, \quad (z, w, \widehat{\lambda}) \in \mathcal{L}\}$; so we get

$$|\frac{dv_1}{ds}| \leq A_1 v_1 e^{-\frac{r_0}{2}\xi} + 1$$

for some constant A_1; this yield after integration between $s = 0$ and $s = \xi + |\eta|$

$$v_1 \leq A_1^{-1} e^{\frac{r_0}{2}\xi} (e^{A_1(\xi+|\eta|)e^{-\frac{r_0}{2}\xi}} - 1)$$

and this show that there exists $B_1 > 0$ such that $|e^{-r_0\zeta} u_1(\zeta, \mu)| \leq B_1$ for all $(\zeta, \mu) \in \Omega(C) \times]-\epsilon, \epsilon[\times \mathcal{V}_{\mathbf{C}}^m$.

Remark that the multiplying function can be replaced by the function ζ^{-1} and this is optimal for the linear part of the field. The same procedure and an induction on n permit us to show that there exist $B_n > 0$ such that $|e^{-nr_0\zeta} u_n(\zeta, \mu)| \leq B_n$ for all $(\zeta, \mu) \in \Omega(C) \times]-\epsilon, \epsilon[\times \mathcal{V}_{\mathbf{C}}^m$.

Now, let $\mathcal{V}_{\mathbf{C}}^m = d_1(0, a_1) \times \cdots \times d_m(0, a_m)$ where $a_i > 0$, $(C_l)_{l \in \mathbf{N}}$ some strictly increasing sequences with $C_0 = C$ and tending to some $C' < \infty$ and $(a_{i,l})_{l \in \mathbf{N}}$ some strictly decreasing sequences with $a_{i,0} = a_i$, tending to some $a'_i > 0$ for all $i = 1, \ldots, m$. The theorem of derivation under the integral sign and the Cauchy's integral formulas show that for all $n \in \mathbf{N}$ and $p + q + q_1 + \cdots + q_m = n$, there exist $B_{p,q,q_1,\ldots,q_m} > 0$ such that

$$|e^{-(p+qr_0)\zeta} \frac{\partial^n F}{\partial \zeta^p \mu_0^q \widehat{\lambda}_1^{q_1} \ldots \widehat{\lambda}_m^{q_m}} (\zeta, \mu)| < B_{p,q,q_1,\ldots,q_m}$$

for all $(\zeta, \mu) \in \Omega(C') \times]-\epsilon, \epsilon[\times d_1(0, a'_1) \times \cdots \times d_m(0, a'_m)$ and this finish the proof of Lemma 1.

3. FINITE CYCLICITY RESULT.

3. .1 The well ordered expansion for the shift map

We consider a real analytic family of vector fields X_λ on the plane. This family depends on a parameter $\lambda \in \mathbf{R}^\Lambda$, for some $\Lambda \in \mathbf{N}$. Suppose that for $\lambda = 0$, X_λ has an identical hyperbolic polycycle Γ_0 with two vertices P_1 and P_2. In order to study the cyclicity of Γ_0 in the family X_λ, we restrict ourselves to a fixed neighbourhood U of Γ_0 in the plane. We choose U as union of three sets A_1, A_2 and A_3 i.e $U = A_1 \cup A_2 \cup A_3$, and we denote W a neighbourhood of $\lambda = 0$ in $\mathbf{R}^\Lambda$. Now X_λ will be represented in $A_1 \times W, A_2 \times W$ and $A_3 \times W$ respectively by X^1_λ, X^2_λ and X^3_λ three analytic vector fields depending analytically in $(m, \lambda) \in \mathbf{R}^2 \times R^\Lambda$. The three charts verify the following properties :

i) In A_1 with local coordinates (x_1, y_1), $A_1 = \{(x_1, y_1); |x_1| \leq 2$ and $|y_1| \leq 2\}$ the family X^1_λ has a unique singular point $P_1(\lambda)$, which is an hyperbolic saddle point situated in the origin of A_1 i.e $P_1(\lambda) \equiv 0$. Also the stable separatrix and the unstable one are respectively the axis oy_1 and ox_1. Finally, the 1– jet of X^1_λ in 0 is equal to :

$$j^1X^1_\lambda(0) = x_1\frac{\partial}{\partial x_1} - r_1(\lambda)y_1\frac{\partial}{\partial y_1} \tag{3.1.1}$$

this formula defines on W an analytic function $r_1(\lambda)$: the hyperbolicity ratio of the saddle point P_1.

ii) In A_2 with local coordinates (x_2, y_2), $A_2 = \{(x_2, y_2); |x_2| \leq 2$ and $|y_2| \leq 2\}$, the family X^2_λ has a unique singular point $P_2(\lambda)$ which is an hyperbolic saddle point, situated in the origin i.e $P_2(\lambda) \equiv 0$.The 1– jet of X^2_λ in 0 is given by :

$$j^1(-X^2_\lambda)(0) = y_2\frac{\partial}{\partial y_2} - r_2(\lambda)x_2\frac{\partial}{\partial x_2} \tag{3.1.2}$$

the stable and unstable separatrices of $(-X^2_\lambda)$ at 0 are respectively the axis ox_2 and oy_2; and the hyperbolicity ratio of $(-X^2_\lambda)$ at P_2 is $r_2(\lambda)$.

iii) In A_3 the vector field X^3_λ has no singularities. Furthermore the points $Q_i(1,0)$, $s_i(0,1)$ in the two charts A_i $i = 1,2$ and the regular segments of Γ_0 joining them are contained in A_3 (figure 3).

The family X_λ verifies the two following conditions :
a) for all λ in W $\quad r_1(\lambda) = r_2(\lambda)$.

b) at least one of the saddle connections remains unbroken for all λ.

Remark.

The condition a) is equivalent to the first condition in Theorem 2.

Now let us define the maps that will permit us to study the cyclicity of Γ_0. Firstly consider:

$$\sigma_i = \{(x_i, y_i) \in A_i; y_i = 1\} \quad \text{and} \quad \tau_i = \{(x_i, y_i) \in A_i; x_i = 1\} \quad i = 1,2$$

$$\sigma_i^+ = \{(x_i, y_i) \in \sigma_i; x_i \geq 0\} \quad \text{and} \quad \tau_i^+ = \{(x_i, y_i) \in \tau_i; y_i \geq 0\} \quad i = 1, 2$$

the segments σ_i, τ_i are parametrized respectively by x_i, y_i $i = 1, 2$ and are transversal to the vector field X_λ for all λ in W.
The flow of $X_\lambda^3, \lambda \in W$ in A_3 defines two analytic diffeomorphisms, the regular transition maps, $R_{1,\lambda}$ and $R_{2,\lambda}$

$$R_{1,\lambda} : \tau_1^+ \longrightarrow \sigma_2, \quad R_{2,\lambda} : \tau_2^+ \longrightarrow \sigma_1$$

The flow of X_λ^1 in A_1 (resp of $(-X_\lambda^2)$ in A_2) defines the transition map $D_{1,\lambda}$ (resp $D_{2,\lambda}$) called the Dulac map.

$$D_{1,\lambda} : \sigma_1^+ \longrightarrow \tau_1, \quad D_{2,\lambda} : \tau_2^+ \longrightarrow \sigma_2$$

the map $D_{1,\lambda}$ (resp $D_{2,\lambda}$) is analytic for $x_1 > 0$ (resp $y_2 > 0$), but it's extended by continuity in 0: $D_{1,\lambda}(0) = 0$ and $D_{2,\lambda}(0) = 0$ for all λ in W.

Remark.

To define the above maps, we have perhaps to reduce the neighbourhood W to a some smaller one.
Finally the shift map will be defined by :

$$\delta(x, \lambda) = R_{1,\lambda} \circ D_{1,\lambda} \circ R_{2,\lambda}(x) - D_{2,\lambda}(x)$$

where $x = y_2$ is the parametrization of the transversal τ_2.

Proposition 1. *Given K arbitrary integer, there exists a neighbourhood $W_K \subset W$ of 0 in $\mathbf{R}^\Lambda$, analytic functions $\gamma_{ij}^K : W_K \longrightarrow \mathbf{R}$ such that on $[0, x_0] \times W_K$ the map $\delta(x, \lambda)$ has the form :*

$$1)\delta(x, \lambda) = \sum_{ir(0)+j \leq K+1} \gamma_{ij}^K x^{ir(\lambda)+j} + \phi_K(x, \lambda) \qquad \textit{if} \quad r(0) \notin \mathbf{Q}$$

$$2)\delta(x, \lambda) = \sum_{0 \leq j \leq i \leq K+1} \gamma_{ij}^K x^{ir(0)} \omega^j + \phi_K(x, \lambda) \qquad \textit{if} \quad r(0) = \frac{p}{q} \in \mathbf{Q}, p \wedge q = 1$$

where $r(\lambda)$ is the common hyperbolicity's ratio of X_λ^1 at P_1 and $(-X_\lambda^2)$ at P_2. The function ω is defined by: let $\alpha_1(\lambda) = r(0) - r(\lambda)$,

$$\omega(x, \lambda) = \begin{cases} \dfrac{x^{-\alpha_1(\lambda)} - 1}{\alpha_1(\lambda)}, & \textit{for } \alpha_1(\lambda) \neq 0 \\ -\ln x, & \textit{for } \alpha_1(\lambda) = 0. \end{cases}$$

$\phi_K(x,\lambda)$ is a C^k function, k-flat at $x=0$ for any λ.

In order to prove this proposition we have to extend the field X_λ in the complex domain; besides we'll restrict ourselves to the case $r(0)=1$ because the other cases are resolved in the same way.

3. .2 The complex continuation of δ

The family $X_\lambda, \lambda \in \mathbf{R}^\Lambda$ has a natural holomorphic extension $\mathbf{X}_{\widehat{\lambda}}$. This extension is obtained by extending the vector fields $X^i_\lambda \quad i=1,2,3$ to holomorphic ones in domains extending the different charts $A_i \times W$.

In the following we will denote this holomorphic extension by $\mathbf{X}_{\widehat{\lambda}}$, $\widehat{\lambda} \in \mathbf{C}^\Lambda$, we will work with the same notations as in the real domains with caps symbols to subline that we are in the complex ones.

We can suppose, up to a holomorphic conjugacy, that the vector fields $\mathbf{X}_{\widehat{\lambda}}{}^i$ are defined in the charts $\mathbf{A}_i$: polydisks $|\mathbf{x}_i|^2+|\mathbf{y}_i|^2 \le 2; (\mathbf{x}_i,\mathbf{y}_i) \in \mathbf{C}^2, i=1,2.$ The origin in each chart $\mathbf{A}_i$ is the only singular point with hyperbolic 1-jet:

$$j^1\mathbf{X}_{\widehat{\lambda}}{}^1(0) = \mathbf{x}_1\frac{\partial}{\partial \mathbf{x}_1} - (1-\widehat{\alpha}_1)\mathbf{y}_1\frac{\partial}{\partial \mathbf{y}_1}$$

$$j^1(-\mathbf{X}_\lambda{}^2)(0) = \mathbf{y}_2\frac{\partial}{\partial \mathbf{y}_2} - (1-\widehat{\alpha}_1)\mathbf{x}_2\frac{\partial}{\partial \mathbf{y}_2}$$

where $1-\widehat{\alpha}_1(\widehat{\lambda}) = \mathbf{r}(\widehat{\lambda})$ is the complex continuation of the hyperbolicity ratio.

We define :

$$\sigma_i = \{(\mathbf{x}_i,\mathbf{y}_i) \in \mathbf{A}_i; \mathbf{y}_i = 1\} \quad \text{and} \quad \tau_i = \{(\mathbf{x}_i,\mathbf{y}_i) \in \mathbf{A}_i; \mathbf{x}_i = 1\}$$

σ_i^+ is (resp τ_i^+) a sector in σ_i (resp in τ_i) defined by :

$$\sigma_i^+ = \{(\mathbf{x}_i,\mathbf{y}_i) \in \sigma_i; |Arg(\mathbf{x}_i)| \le \theta_0\} \qquad 0 < \theta_0 \le \frac{\pi}{2}$$

respectively

$$\tau_i^+ = \{(\mathbf{x}_i,\mathbf{y}_i) \in \tau_i; |Arg(\mathbf{y}_i)| \le \theta_0\}$$

the disks σ_i, τ_i are transversal to the local invariant manifolds of $\mathbf{X}^i_{\widehat{\lambda}}$.

The Dulac and the regular transition maps defined above have unique holomorphic extensions. So the shift map has a unique holomorphic extension noted by $\widehat{\delta}(x,\widehat{\lambda})$ in $(\tau_2^+ \setminus \{0\} \times \mathbf{W})$ and prolonged in 0 by 0.
Let the function $\widehat{\omega}(\mathbf{x},\widehat{\lambda})$ be the continuation of the real one defined above.

Theorem 3. *For any arbitrary integer K, there exists a neighbourhood $\mathbf{W}_K$ of 0 in $\mathbf{W} \subset \mathbf{C}^\Lambda$ and holomorphic functions $\widehat{\gamma}_{ij}^K : \mathbf{W}_K \longrightarrow \mathbf{C}$ continuation of the real functions γ_{ij}^K such that on $\tau_2^+ \times \mathbf{W}_K$ the function $\widehat{\delta}(\mathbf{x}, \widehat{\lambda})$ has the form :*

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \sum_{0 \leq j \leq i \leq K} \widehat{\gamma}_{ij}^K(\widehat{\lambda}) \mathbf{x}^i \widehat{\omega}^j + \psi^K(\mathbf{x}, \widehat{\lambda})$$

where ψ^K is a function of class C^K, in the real sense, K-flat in $\mathbf{x} = 0$.

Remark.

This development is what we call the well ordered expansion of order K. The monomials $\mathbf{x}^i \widehat{\omega}^j$ are totally ordered by the lexicographic order : $\mathbf{x}^i \widehat{\omega}^j \preceq \mathbf{x}^m \widehat{\omega}^n$ if and only if $i < m$ or $i = m$ and $j > n$.

The proposition 1 is an immediat consequence of theorem 3, it suffices to restrict all the different neighbourhoods, charts and functions in $\mathbf{C}^\Lambda$ and $\mathbf{C}^2$ respectively to $\mathbf{R}^\Lambda$ and $\mathbf{R}^2$.

Proof of the theorem 3.
Given an integer $K \neq 0$ we may apply the results of [R]. There exists a neighbourhood $\mathbf{W}_K$ of 0 in $\mathbf{C}^\Lambda$, some transversals depending on the parameter $\widehat{\lambda} \in \mathbf{C}^\Lambda$, $\widetilde{\sigma}_i$ (resp $\widetilde{\tau}_i$) tangent to $\sigma_i$in 0 (resp to σ_i) such that the Dulac maps : $\widetilde{\mathbf{D}}_{1,\widehat{\lambda}} : \widetilde{\sigma}_1^+ \longrightarrow \widetilde{\tau}_1$ and $\widetilde{\mathbf{D}}_{2,\widehat{\lambda}} : \widetilde{\tau}_2^+ \longrightarrow \widetilde{\sigma}_2$ are written under the form :

$$\widetilde{\mathbf{D}}_{1,\widehat{\lambda}}(\widetilde{\mathbf{x}}_1) = \widetilde{\mathbf{x}}_1 + \sum_{1 \leq j \leq i \leq K} \widetilde{\alpha}_{ij}(\widehat{\lambda}) \widetilde{\mathbf{x}}_1^i \widetilde{\omega}^j + \cdots + \widetilde{\alpha}_{K+1,1} \widetilde{\mathbf{x}}_1^{K+1} \widetilde{\omega} + \widetilde{\psi}_K^1(\widetilde{\mathbf{x}}_1, \widehat{\lambda})$$

$$\widetilde{\mathbf{D}}_{2,\widehat{\lambda}}(\widetilde{\mathbf{y}}_2) = \widetilde{\mathbf{y}}_2 + \sum_{1 \leq j \leq i \leq K+1} \widetilde{\beta}_{ij} \widetilde{\mathbf{y}}_2^i \widehat{\omega}^j + \cdots + \widetilde{\beta}_{K+1,1} \widetilde{\mathbf{y}}_2^{K+1} \widehat{\omega} + \widetilde{\psi}_K^2(\widetilde{\mathbf{y}}_2, \widehat{\lambda})$$

where $\widetilde{\alpha}_{ij}$,$\widetilde{\beta}_{ij}$ are holomorphic functions on $\mathbf{W}_K$ $\widetilde{\psi}_K^1, \widetilde{\psi}_K^2$ are C^K in the real sense, K-flat resp. to $\widetilde{\mathbf{x}}_1 = 0$, $\widetilde{\mathbf{y}}_2 = 0$.

The same arguments as in [R] work here because $\mathbf{X}_{\widehat{\lambda}}^1$ and $(-\mathbf{X}_{\widehat{\lambda}}^2)$ have respectively the same hyperbolicity ratio $\mathbf{r}(\widehat{\lambda})$ in P_1 and P_2.
Now there exist $\varphi_{i,\widehat{\lambda}}$ $i = 1, 2$ (resp $\phi_{i,\widehat{\lambda}}$ $i = 1, 2$) holomorphic diffeomorphisms defined by the flow of $\mathbf{X}_{\widehat{\lambda}}^i$ between σ_i and $\widetilde{\sigma}_i$ (resp τ_i and $\widetilde{\tau}_i$.)

$$\varphi_{i,\widehat{\lambda}} : \sigma_i \longrightarrow \widetilde{\sigma}_i \qquad \phi_{i,\widehat{\lambda}} : \tau_i \longrightarrow \widetilde{\tau}_i$$

with $\forall\widehat{\lambda} \in \mathbf{W}_K$, $\phi_{i,\widehat{\lambda}}(0) = \varphi_{i,\widehat{\lambda}}(0) = 0 \qquad i = 1, 2.$

So the Dulac maps :

$$\mathbf{D}_{1,\widehat{\lambda}} : \sigma_1^+ \longrightarrow \tau_1 \quad \text{and} \quad \mathbf{D}_{2,\widehat{\lambda}} : \tau_2^+ \longrightarrow \sigma_2$$

are written :

$$\mathbf{D}_{1,\widehat{\lambda}}(\mathbf{x}_1) = \left(\phi_{1,\widehat{\lambda}}^{-1} \circ \widetilde{\mathbf{D}}_{1,\widehat{\lambda}} \circ \varphi_{1,\widehat{\lambda}}\right)(\widetilde{\mathbf{x}}_1)$$
$$\mathbf{D}_{2,\widehat{\lambda}}(\mathbf{y}_2) = \left(\varphi_{2,\widehat{\lambda}}^{-1} \circ \widetilde{\mathbf{D}}_{2,\widehat{\lambda}} \circ \phi_{2,\widehat{\lambda}}\right)(\widetilde{\mathbf{y}}_2)$$

by using the lemma 2 below we obtain that the Dulac maps have the following development for a choosen K :

$$\mathbf{D}_1(\mathbf{x}_1) = \mathbf{x}_1 + \sum_{1 \leq j \leq i \leq K} \alpha_{ij}(\widehat{\lambda})\mathbf{x}_1^i\widehat{\omega}^j + \cdots + \alpha_{K+1,1}\mathbf{x}_1^{K+1}\widehat{\omega} + \psi_K^1(\mathbf{x}_1, \widehat{\lambda})$$

$$\mathbf{D}_2(\mathbf{y}_2) = \mathbf{y}_2 + \sum_{1 \leq j \leq i \leq K} \beta_{ij}(\widehat{\lambda})\mathbf{y}_2^i\widehat{\omega}^j + \cdots + \beta_{K+1,1}\mathbf{y}_2^{K+1}\widehat{\omega} + \psi_K^2(\mathbf{y}_2, \widehat{\lambda})$$

the functions $\alpha_{ij}, \beta_{ij}, \psi_K^l$ have the same properties as $\widetilde{\alpha}_{ij}, \widetilde{\beta}_{ij}, \widetilde{\psi}_K^l$.

Lemma 2. *Let* $\mathbf{f}$ *be a holomorphic function of* $\sigma^+ \times \mathbf{W}_K$ *where* σ^+ *is a sector as the ones defined below. If* $\mathbf{f}(0, \widehat{\lambda}) = 0$ *then there exists a holomorphic function* $\mathbf{g}$ *such that :*

$$\widehat{\omega}(\mathbf{x}(1+\mathbf{f}), \widehat{\lambda}) = (1 + \widehat{\alpha}_1(\widehat{\lambda})\mathbf{g})\widehat{\omega}(\mathbf{x}, \widehat{\lambda}) + g$$
$$\textit{if} \quad \mathbf{a} \neq 0 : \widehat{\omega}(\mathbf{a}\mathbf{x}, \widehat{\lambda}) = (1 + o(\widehat{\alpha}_1))\widehat{\omega}(\mathbf{x}, \widehat{\lambda}) - \ln \mathbf{a}(1 + o(\widehat{\alpha}_1)$$
$$\widehat{\omega}(\mathbf{a}\mathbf{x}(1+\mathbf{f}), \widehat{\lambda}) = (1 + o(\widehat{\alpha}_1))\widehat{\omega}(\mathbf{x}, \widehat{\lambda}) - \ln \mathbf{a}(1 + o(\widehat{\alpha}_1)) + \mathbf{g}$$

To finish the proof, we have to develop :

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \left(\mathbf{R}_{1,\widehat{\lambda}} \circ \mathbf{D}_{1,\widehat{\lambda}} \circ \mathbf{R}_{2,\widehat{\lambda}}\right)(\mathbf{x}) - \mathbf{D}_{2,\widehat{\lambda}}(\mathbf{x})$$

where $\mathbf{R}_{1,\widehat{\lambda}}$ and $\mathbf{R}_{2,\widehat{\lambda}}$ are the regular transition maps. We can write them as :

$$\mathbf{R}_{1,\widehat{\lambda}}(\mathbf{x}) = \mathbf{b}_0(\widehat{\lambda}) + \mathbf{b}_1(\widehat{\lambda})\mathbf{x} + \mathbf{b}_2(\widehat{\lambda})\mathbf{x}^2 + \cdots + \mathbf{b}_K(\widehat{\lambda})\mathbf{x}^K + o(\mathbf{x}^K)$$

$$\mathbf{R}_{2,\widehat{\lambda}}(\mathbf{x}) = \mathbf{a}_1(\widehat{\lambda})\mathbf{x} + \mathbf{a}_2(\widehat{\lambda})\mathbf{x}^2 + \mathbf{a}_3(\widehat{\lambda})\mathbf{x}^3 + \cdots + \mathbf{a}_K(\widehat{\lambda})\mathbf{x}^K + o(\mathbf{x}^K)$$

Using again the lemma 2, we find the expansion of $\widehat{\delta}$.

3. .3 Division in the ideal of coefficients

Let's recall some definitions and results from [R] :
For all $x_0 \in]0, \varepsilon_0]$, the domain of the function $\delta_\lambda(x) = \delta(x, \lambda)$ and for all $\lambda \in W$, δ is an analytic function in (x_0, λ). Then we can write it as follows :

$$\delta(x, \lambda) = \sum_{i=0}^{\infty} a_i(\lambda, x_0)(x - x_0)^i \tag{3.3.1}$$

for x close to x_0.

Consider the ideal J_{x_0} generated by the germs of the functions a_i in $\lambda = 0$. We will note them by $\widetilde{a}_i$. In [R], it is proved that the ideal J_{x_0} does not depend on the point $x_0 \neq 0$. J is called the ideal of coefficients associated to δ. $J \subset \mathcal{O}$ the ring of the germs of analytic functions in $\lambda = 0$.

In the following we will suppose that $J \neq \mathcal{O}$ ie $\delta(x, 0) \equiv 0$. This corresponds to the case: Γ_0 is identical. The other case $J = \mathcal{O}$, ie $\delta(x, 0) \not\equiv 0$, was studied in [El.M] and corresponds Γ_0 non identical. The definitions introduced here are available also in the complex domain.
So let $\mathbf{J}$ the complexified ideal of J. It's easy to see that $\mathbf{J} = \mathbf{J}_{\mathbf{x}_0}$ for any $\mathbf{x}_0 \in \widehat{\tau}_2^+$ where $\mathbf{J}_{\mathbf{x}_0}$ is the ideal of coefficients of $\widehat{\delta}(\mathbf{x}, \widehat{\lambda})$, extension of $\delta(x, \lambda)$ defined above.

Proposition 1. *Let $\widehat{\gamma}_{ij}^K, K \geq 2$ the coefficients of the expansion of $\widehat{\delta}$ to an order K. Given any k such that $1 \leq k < K$ then the germ $\widetilde{\widehat{\gamma}}_{ij}^K \in \mathbf{J}$ for $0 \leq j \leq i \leq k$.*

Proof.

We will apply the same algorithm as in [El.M]
Consider the well ordered development of $\widehat{\delta}$ up to order K :

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \sum_{0 \leq j \leq i \leq k} \widehat{\gamma}_{ij}^K \mathbf{x}^i \widehat{\omega}^j + \cdots + \widehat{\gamma}_{k+1.1}^K \mathbf{x}^{k+1} \widehat{\omega} + \psi^K(\mathbf{x}, \widehat{\lambda}) \tag{3.3.2}$$

For $\mathbf{x} \neq 0$, the germ in $\widehat{\lambda} = 0$ of the function $\widehat{\lambda} \longmapsto \widehat{\delta}_{\mathbf{x}}(\widehat{\lambda}) = \widehat{\delta}(\mathbf{x}, \widehat{\lambda})$ is in $\mathbf{J}$. Moreover each monomial in (3.3.2), apart the first one which is equal to 1, corresponds to a nonzero power of $\mathbf{x}$. It follows that :

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \widehat{\gamma}_{00} + \psi_0(\mathbf{x}, \widehat{\lambda}) \tag{3.3.3}$$

where $\psi_0(\mathbf{x}, \widehat{\lambda}) \longmapsto 0$ when $\mathbf{x} \longmapsto 0$, uniformly in $\widehat{\lambda}$. Because the ideal **J** is closed we conclude that $\widehat{\gamma}_{00}^K$ has its germ in **J**.

Suppose now that we have proved that the germs of $\widehat{\gamma}_{ij}^K$ are in **J** for all the monomials $\mathbf{x}^i\widehat{\omega}^j \preceq \mathbf{x}^m\widehat{\omega}^n$ where $\preceq$ is the order introduced above. We use also the lexicographic order between the couples $(i,j) < (m,n)$. Let :

$$\begin{aligned}\widehat{\delta}_{mn} &= \widehat{\delta} - \sum_{(i,j)<(m,n)} \widehat{\gamma}_{ij}^K \mathbf{x}^i\widehat{\omega}^j \\ &= \widehat{\gamma}_{mn}^K(\widehat{\lambda})\mathbf{x}^m\widehat{\omega}^n + \cdots + \psi^K(\mathbf{x}, \widehat{\lambda})\end{aligned} \tag{3.3.4}$$

For each $\mathbf{x} \neq 0$, the germ in $\widehat{\lambda} = 0$ of the function $\widehat{\lambda} \longmapsto \widehat{\delta}_{mn}(\mathbf{x}, \widehat{\lambda})$ is in **J**. But we have to remark that the sequence of monomials $\mathbf{x}^i\widehat{\omega}^j$ does not form a scale of infinitisimals in $\mathbf{x}$ (in uniform way in $\widehat{\lambda}$), because the ratio of $\mathbf{x}^i\widehat{\omega}^j$ and $\mathbf{x}^i\widehat{\omega}^l$, for $j < l$, is equal to $\widehat{\omega}^{-j+l}$ and does not tend to zero, uniformly in $\widehat{\lambda}$, if $\mathbf{x} \longmapsto 0$. So we cannot apply directly to $\widehat{\gamma}_{mn}^K$, the same argument we have applied to $\widehat{\gamma}_{00}^K$. We will apply it after a first step where we will transform $\widehat{\delta}_{mn}$ by division and derivation. This is based on the following observation : if a function $\varphi(\mathbf{x}, \widehat{\lambda})$ has a well ordered development up to some order K, like the function $\widehat{\delta}$, then for $\forall s, l \in \mathbf{R}$ and any order of derivation $r < K$, the function $\widehat{\lambda} \longmapsto \mathbf{x}^l\widehat{\omega}^s \dfrac{\partial^r\varphi}{\partial\mathbf{x}^r}(\mathbf{x}, \widehat{\lambda})$ has a germ in $\widehat{\lambda} = 0$ in **J**, for $\forall \mathbf{x} \neq 0$.

Starting with the monomials $\mathbf{x}^i\widehat{\omega}^j$, $\;i, j \in \mathbf{N}$, the derivation with respect to $\mathbf{x}$ produces more general monomials $\mathbf{x}^{i+s\widehat{\alpha}_1}\widehat{\omega}^j$. So, firstly we extend the total order introduced above in a *partial* one between these new monomials. For $i, j \in \mathbf{N}$ and $s, l \in \mathbf{Z}$ we take :

$$\mathbf{x}^{i+l\widehat{\alpha}_1}\widehat{\omega}^r \preceq \mathbf{x}^{j+k\widehat{\alpha}_1}\widehat{\omega}^s \Longleftrightarrow \begin{cases} i < j & \text{or} \\ i = j, l = k & \text{and} \quad r > s. \end{cases}$$

the notation "$f + \cdots$ " will be for a sum of f and a combination of monomials with larger order.

Now let us explain our first step. Starting with $\Delta_0 = \widehat{\delta}_{mn}$, we divide it by $\mathbf{x}^m$:

$$\Delta_1 = \mathbf{x}^{-m}\Delta_0 = \widehat{\gamma}_{mn}^K\widehat{\omega}^n + \widehat{\gamma}_{mn-1}^K\widehat{\omega}^{n-1} + \cdots + R_1 \tag{3.3.5}$$

with $R_1 = \psi^K\mathbf{x}^{-m}$ of order $(K - m)$, is more differentiable and flatter than the last term in $+\cdots$

If $n = 0$, our first step is achieved: $\Delta_1 = \widehat{\gamma}_{mn} + \varphi_1(\mathbf{x}, \widehat{\lambda})$ where $\varphi_1(\mathbf{x}, \widehat{\lambda}) \mapsto 0$ when $\mathbf{x} \longmapsto 0$, uniformly in $\widehat{\lambda}$, and we can repeat now, the argument used above for $\widehat{\gamma}_{00}^K$.

If $n > 0$, after noticing that $\dfrac{\partial \widehat{\omega}}{\partial \mathbf{x}} = -\mathbf{x}^{-1-\widehat{\alpha}_1}$, we have :

$$\Delta_2 = \mathbf{x}^{1+\widehat{\alpha}_1}\frac{\partial \Delta_1}{\partial \mathbf{x}} = n\widehat{\gamma}^K_{mn}\widehat{\omega}^{n-1} + \cdots + R_2 \tag{3.3.6}$$

where R_2 is a convenient remaining term as R_1 above. Repeating $n-1$ times again the same procedure, we obtain finally :

$$\begin{aligned}\Delta_{n+1} &= n!\widehat{\gamma}^K_{mn} + \cdots + R_{n+1} \\ &= n!\widehat{\gamma}^K_{mn} + \varphi_{n+1}(\mathbf{x}, \widehat{\lambda})\end{aligned} \tag{3.3.7}$$

with a convenient remaining term R_{n+1}. Now φ_{n+1} has an expansion whose first monomial has a positive power in $\mathbf{x}$, so that $\varphi_{n+1}(\mathbf{x}, \widehat{\lambda}) \longmapsto 0$ when $\mathbf{x} \longmapsto 0$, uniformly in $\widehat{\lambda}$. As above this implies that the germ of $\widehat{\gamma}^K_{mn}$ is in $\mathbf{J}$.

3. .4 The proof of theorem 2

As $\mathcal{O}$ is noetherian, $\mathbf{J}$ has a finite system of generators $\widetilde{\phi}_1, \widetilde{\phi}_2, \cdots, \widetilde{\phi}_l$; where $(\phi_1, \phi_2, \cdots, \phi_l)$ are holomorphic in $\mathbf{W}$.
Using the proposition 1 and the same arguments as in theorem 7 of $[R]$,we can write $\widehat{\delta}(\mathbf{x}, \widehat{\lambda})$ under the form :

$$\widehat{\delta}(\mathbf{x}, \widehat{\lambda}) = \sum_{i=1}^{l} \phi_i(\widehat{\lambda})\mathbf{h}^K_i(\mathbf{x}, \widehat{\lambda}) \tag{3.4.1}$$

where K is an arbitrary integer,the functions $\mathbf{h}^K_i(\mathbf{x}, \widehat{\lambda})$ are holomorphic for $\mathbf{x} \neq 0$ and have the well ordered expansions of order K. We deduce the following proposition in the real domain :

Proposition 2 $[R]$. *Let $(\phi_1, \phi_2, \cdots, \phi_l)$ analytic functions in W whose germs in $\lambda = 0$ generate the ideal of coefficients J. Let K an arbitrary integer, then there exists a neighbourhood $W_K \subset W$ in $\mathbf{R}^\Lambda$ and functions $h^K_i(x, \lambda)$, with $1 \le i \le l$ having well ordered expansions of order K :*

$$h^K_i(x, \lambda) = D^K h^K_i(x, \lambda) + \psi^K_i(x, \lambda) \qquad \text{in} \qquad [0, x_0] \times W_K$$

$$D^K h^K_i(x, \lambda) = \sum_{0 \le n \le m \le K} \gamma^{iK}_{mn} x^m \omega^n + \cdots + \gamma^{iK}_{K+1,1} x^{K+1}\omega$$

the h_i^K *are analytic for* $x \neq 0$. *They permit us to write* $\delta(x,\lambda)$ *as follows :*

$$\delta(x,\lambda) = \sum_{i=1}^{l} \phi_i(\lambda) h_i^K(x,\lambda) \tag{3.4.2}$$

we can choose a system of generators $(\phi_1, \phi_2, \cdots, \phi_l)$ verifying some properties as in [R] :

i) $(\phi_1, \phi_2, \cdots, \phi_l)$ is minimal in the sense that it is a basis of the vector space $J/\mathcal{M}J$ where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$.

ii) For $\lambda = 0$ the values of $h_i^K(x,0)$ of the expression (3.4.2) don't depend on K. So we can define the functions $h_i(x) = h_i^K(x,0)$ for any K. In the neighbourhood of $x = 0$, we can associate to them a formal power serie called the Dulac's development, we note by :

$$D^\infty h_i(x) = \sum_{0 \leq n \leq m}^{\infty} \gamma_{mn}^i(o) x^m (-\ln x)^n \tag{3.4.3}$$

where $\gamma_{mn}^i(0) = \gamma_{mn}^{iK}(0)$ for any $K \geq Sup\{m,n\}$. This development is unique. We obtain it from (3.4.2) by remarking that for $\lambda = 0$: $\quad x^m \omega^n = x^m(-\ln x)^n$. The functions h_i are analytic for $x \neq 0$ and $h_i \not\equiv 0$. But, we cannot assert that $D^\infty h_i \not\equiv 0$, this would be true if h_i was quasi–regular. If $D^\infty h_i \not\equiv 0$ then it will be equivalent to x^m or $x^m \ln x$. This equivalency allows us to define an order of flatness between the h_i such that $D^\infty h_i \not\equiv 0$ by : $order(h_i) < order(h_j)$ if and only if $h_j/h_i \longmapsto 0$ when $x \longmapsto 0$. We say that $order(h_i) = \infty$ if $D^\infty h_i \equiv 0$.

iii) There exists an index $s, 0 \leq s \leq l$ such that :

$$order(h_1) < order(h_2) < \cdots < order(h_s) < \infty$$

and

$$order(h_j) = \infty \qquad \text{for} \quad j \geq s+1$$

we say that h_i are ordered.

The properties **i,ii,iii** of the system $(\phi_1, \phi_2, \cdots, \phi_l)$ are not sufficient to conclude the finite cyclicity of Γ_0. That is why, we consider the map of desingularization of the set $\{\lambda \setminus \phi_1\phi_2\cdots\phi_l = 0\}$.

There exists $\varphi : \widetilde{W} \longmapsto W$ a proper analytic map of a compact domain $\widetilde{W}$ onto W neighbourhood of $0 \in R^\Lambda$. φ is the map of Hironaka's desingularization.

We consider the family $X_{\widehat{\lambda}} = X_{\varphi(\lambda)}$ for $\widehat{\lambda} \in \widetilde{W}$. Take $D = \widehat{\varphi}^{-1}\{0\}$ we associate to $X_{\widehat{\lambda}}$ at every point $\widehat{\lambda}_0 \in D$, an ideal of coefficients noted $\widetilde{J}_{\widehat{\lambda}} \subset \widehat{\mathcal{O}}_{\widehat{\lambda}}$, ring of analytic germs at $\widehat{\lambda} = 0$. As D is compact, the cyclicity of Γ_0 in X_λ will be finite if this is true for the $X_{\widehat{\lambda}}$-germ at every $\widehat{\lambda}_0 \in D$.

Let $\widehat{\delta}(x, \widehat{\lambda}) = \delta(x, \varphi(\widehat{\lambda}))$ the shift map of $X_{\widehat{\lambda}}$, $\widehat{\lambda}$ in a neighbourhood of $\widehat{\lambda}_0 \in D$. Then it's easy to see that $\widetilde{J}_{\widehat{\lambda}_0}$ will be generated by $\widehat{\phi}_i(\widehat{\lambda}) = \phi_i \circ \varphi(\widehat{\lambda})$. Furthermore, there exists $\widetilde{W}_{\widehat{\lambda}_0}$ a neighbourhood of $\widehat{\lambda}_0$ with coordinates $z_1, z_2, \cdots, z_\Lambda$(where $\widehat{\lambda}_0 = (0, 0, 0, \cdots, 0)$) such that :

$$\widehat{\phi}_i(\widehat{\lambda}) = u_i(\widehat{\lambda}) \prod_{i=1}^{\Lambda} z_j^{p_j^i} \tag{3.4.4}$$

the functions $u_i(\widehat{\lambda})$ are analytic and nonzero for all $\widehat{\lambda} \in \widetilde{W}_{\widehat{\lambda}_0}$, p_j^i are integers. Let's note $\prod_{j=1}^{i} z^{p_j^i} = \psi_i(\widehat{\lambda})$, then $\widetilde{\phi}_i(\widehat{\lambda}) = u_i(\widehat{\lambda})\psi_i(\widehat{\lambda})$.

Proposition 3 [*R*]. *From the system* $(\widehat{\phi}_1, \widehat{\phi}_2, \cdots, \widehat{\phi}_l)$ *we can extract a system* $(\widehat{\phi}_{i_1}, \widehat{\phi}_{i_2}, \cdots, \widehat{\phi}_{i_L})$ *possessing properties* **i,ii,iii** *as* $(\phi_1, \phi_2, \cdots, \phi_l)$. *i.e there exists* $s, 0 \leq s \leq L$ *such that if* $\widehat{\delta}(x, \widehat{\lambda}) = \sum_{j=1}^{L} \widehat{\phi}_{i_j} H_j^K(x, \widehat{\lambda})$ *then* $order(H_1) < order(H_2) < \cdots < order(H_s) < \infty$ *and* $order(H_j) = \infty$, *where* H_i *is defined as below.*

Remark.

The division of $\widehat{\delta}$ in $\widetilde{J}_{\widehat{\lambda}_0}$ is not degenerated in a sense we will explain below. Until the end we are going to work with the family $X_{\widehat{\lambda}}, \widehat{\delta}(x, \widehat{\lambda}), \widehat{\phi}_{i_j}, H_i^K, \ldots$ and we will show the finite cyclicity of Γ_0 for this family.

From now on, we discard the caps in the notation $\widehat{\lambda}, \widehat{\phi}, \ldots$ So that, we suppose we have a family X_λ with :

$$\delta(x, \lambda) = \sum_{i=1}^{L} \phi_i(\lambda) h_i^K(x, \lambda)$$

where

$$order(h_1) < order(h_2) < \cdots < order(h_s) < \infty$$

and

$$order(h_i) = \infty \quad i > s$$

and $\phi_i(\lambda) = u_i(\lambda)\psi_(\lambda)$ with $\psi_i(\lambda) = \prod_{j=1}^{\Lambda} z_j^{p_j^i}$ and $(z_1, z_2, \cdots, z_\Lambda)$ local coordinates in W_{λ_0}.

Lemma 3 [*R*]. *Consider $W_i = \{\lambda \in W_{\lambda_0} \setminus |\psi_i(\lambda)| \geq |\psi_j(\lambda)| \quad \text{for } j \neq i\}$. Let $I = \{i \setminus \lambda_0 \in W_i\}$. Then we have the following results :*
i) Let $i \in I$ then there exists an analytic arc $\lambda(\varepsilon) : [0, \varepsilon_0] \longmapsto W_{\lambda_0}$, with $\lambda(0) = \lambda_0$ such that :

$$order(\psi_i \circ \lambda)_{\varepsilon=0} < order(\psi_j \circ \lambda)_{\varepsilon=0} \qquad \text{for} \qquad j \neq i$$

ii) $\cup_{i \in I} W_i$ is a neighbourhood of λ_0.

Proposition 3. *If $i \in I$ then $D^\infty h_i \not\equiv 0$. This means that $I \subset \{1, 2, \cdots, s\}$.*

Proof.
Let $i \in I$ and $\lambda(\varepsilon)$ the analytic arc in W_{λ_0}. Let us consider the subfamily depending on 1–parameter :

$$X_\varepsilon = X_{\lambda(\varepsilon)} \qquad \varepsilon \in [0, \varepsilon_0]$$

let $\widetilde{\delta}(x, \varepsilon)$ the map δ associated to this family. obviously, $\widetilde{\delta}(x, \varepsilon) = \delta(x, \lambda(\varepsilon))$ where $\delta(x, \lambda)$ is the shift map of X_λ. So, for a given integer K, we can write :

$$\widetilde{\delta}(x, \varepsilon) = \sum_{j=1}^{L} \phi_j \circ \lambda(\varepsilon) h_j^K(x, \lambda(\varepsilon)) \tag{3.4.5}$$

for all indices j :

$$h_j^K(x, \lambda_\varepsilon)) = h_j(x) + O(\varepsilon)$$

and

$$\phi_j(\lambda(\varepsilon)) = a_j \varepsilon^{n_j} + O(\varepsilon^{n_j}) \qquad a_i \neq 0$$

We replace in equality (3.4.5) to obtain :

$$\widetilde{\delta}(x, \varepsilon) = a_i h_i(x) \varepsilon^{n_i} + O(\varepsilon^{n_i}) \tag{3.4.6}$$

the formula (3.4.6) indicates that up a multiplication by nonzero coefficient, $h_i(x)$ is the principal part of the development of $\widetilde{\delta}$ in serie of ε. Furthermore, by theorem 1 of this article, there exist quasi regular functions $I_j(x)$, up to some factor x^{jR}, such that :

$$\widetilde{\delta}(x,\varepsilon) = \varepsilon I_1(x) + \varepsilon^2 I_2(x) + \cdots + \varepsilon^j I_j(x) + O(\varepsilon^j) \qquad (3.4.7)$$

for any integer j.

If we equalize the two expressions (3.4.6) and (3.4.7), we find that :

$$a_i h_i(x) = I_{n_i}(x)$$

as $h_i(x)$ is not identically zero, its Dulac's development that coincides with the one of $I_i(x)$ is not identically null. (Here I_{n_i} is eventually quasi- regular because it is the first non zero term in the expansion (3.4.7).)

Remember that $\phi_i(x,\lambda) = u_i(\lambda)\psi_i(\lambda)$ with $u_i(\lambda) \neq 0$ for every $\lambda \in W_{\lambda_0}$, so we may find a real $r : 0 < r \leq 1$ such that : if $V_i^r = \{\lambda; |\phi_i(\lambda)| \geq r|\phi_j(\lambda)|$ for $i \neq j\}$ then $\cup_{i=1}^s V_i^r$ is a neighbourhood of λ_0.

To end the proof, we remark that we have the same situation as in paragraph 8 of [*R*], therefore we can conclude that for all $i : 1 \leq i \leq s$, there exists $N_i \in \mathbf{N}$, a neighbourhood W_i of λ_o and a real $x_i : 0 < x_i \leq x_0$ such that $\delta(x,\lambda)$ has less than N_i zeros in $[0, x_i]$ for all $\lambda \in V_i^r \cap W_i$.

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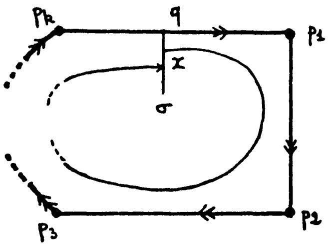

Figure 1.

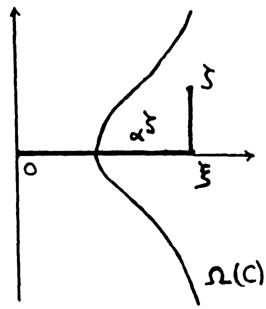

Figure 2.a

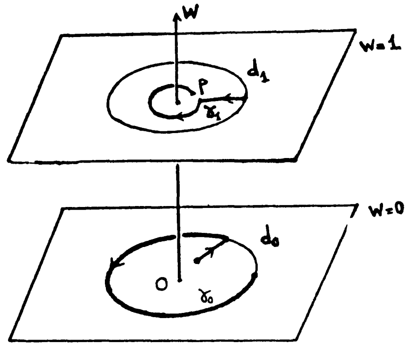

Figure 2.b

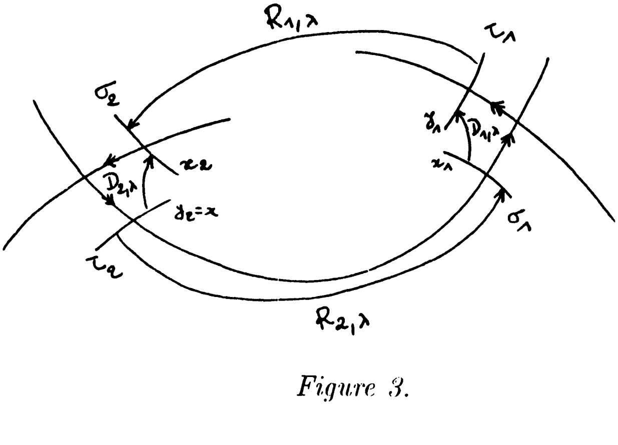

Figure 3.