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COMPLEX DYNAMICS IN HIGHER DIMENSION. I.

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1. Introduction

Given a polynomial equation $P(x) = a_n x^n + \cdots + a_0 = 0$, in one variable, x, one asks what are the solutions. The main advantage of the complex number system is that if x is allowed to be complex then the solutions always exist. However, to find the actual values of the solutions is impossible. One can only find approximate solutions.

A traditional method is Newton's method. One starts with a value x_0 and finds inductively a sequence $\{x_n\}$, $x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}$. If x_0 is near a simple root, this sequence converges to this root.

Shröder [Sc] was the first to study Newton's method for complex numbers. He was led to the study of iteration of the rational function $R(z) := z - \frac{P(z)}{P'(z)}$. Mainly he studied the local behavior of rational functions near attractive fixed points, $R(z_0) = z_0$, $| R'(z_0) |< 1$. He actually studied general rational functions rather than the special ones from Newton's method, because he discovered that Newton's could be replaced by infinitely many rational functions.

If instead one considers polynomial equation in two (or more) variables, $P(x,y) = Q(x,y) = 0$, where $P(x,y) = \sum a_{n,m} x^n y^n$, one is likewise led to study iteration of rational fonctions in two or more variables. In this case Newton's method takes the inductive form

$$(x_{n+1}, y_{n+1}) = R(x_n, y_n)$$

where the rational map R is given by

$$R(x,y) = (x,y) - \frac{1}{P_x Q_y - P_y Q_x}(PQ_y - QP_y, QP_x - PQ_x).$$

As in one variable there is an infinite family of other rational maps that could be used as well. The simplest one is $R(x,y) = (x,y) - A(P,Q)$ where A is a constant matrix equal to the inverse of the Jacobian matrix of $(P,Q)'$ at some point close to a fixed point.

More precisely consider the mapping in $\mathbf{C}^2$ given by

$$(P,Q) = \left(\frac{1}{2}x - (x-2y)^2, \frac{1}{2}y - x^2\right).$$

Obviously $(0,0)$ is a root of the system $P=0$, $Q=0$. If we apply the Schroder method to this system with $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ we get in homogeneous coordinate the mapping $f[x : y : t] = [2(z-2w)^2 : 2z^2 : t^2]$ which is a holomorphic map in $\mathbf{P}^2$. For any invertible matrix B the map $g = I - B(P,Q)$ in homogeneous coordinates is a holomorphic map of $\mathbf{P}^2$.

The analogue of Schröder's study indicated above is the local study of R around attractive fixed points. This was studied extensively in dimension 2, starting by Leau [Le] in the end of the last century and carried through by Lattes [La] and Fatou [Fa].

As far as the global study of iteration is concerned, that is, if we start with a value x_0, perhaps far from the roots of the polynomial, does Newton's method still converge? Schröder was able to decide this only for quadratic polynomials. In this case he found that there is a circle in the sphere, $\mathbf{C} \cup \{\infty\} = \mathbf{P}^1$, dividing it into two open sets. Each of these open sets contains one of the two roots and each starting point x_0 in these open sets give a sequence $\{x_n\}$ by Newton's method converging to the root in the same open set.

The global study of iteration in one variable only became possible in the second decade of this century after the introduction by Montel of normal families, in particular the normality of the family of holomorphic maps from the unit disc to the sphere $\mathbf{P}^1$ minus three points is crucial.

The analogue of this in higher dimensions was unavailable at Fatou's time, so essentially all the study of iteration of rational maps was local.

In this paper we will discuss mainly global questions of iteration of rational maps in higher dimension. The analogue of Montel's Theorem comes from the Kobayashi hyperbolicity of the complement of certain complex hypersurfaces in $\mathbf{P}^k$, the complex projective space of dimension k.

We start, here, with some basic facts on holomorphic endomorphisms of $\mathbf{P}^k$ (i.e. holomorphic maps). For simplicity we sometimes restrict our attention to $\mathbf{P}^2$. In a forthcoming paper we will study the structure of Julia's and Fatou components.

In section 2 we discuss some basic properties of holomorphic and meromorphic maps on $\mathbf{P}^k$.

Section 3 is an estimate of the number of periodic points, counted without multiplicity.

Then in section 4 we give a description of the family of exceptional maps. This family generalizes the map $z \to z^d$ on $\mathbf{P}^1$ which is characterized by the property that the points $\{0, \infty\}$ are totally invariant.

In section 5 we discuss the Kobayashi hyperbolicity of the complement of part of the critical orbit. We show that this holds for a Zariski dense set of maps. See Theorem 5.3 for a precise statement.

In section 6 we consider expansive properties of the map in the complement of the closure of the critical orbit under suitable hyperbolicity assumption and finally, in section 7, we classify critically finite maps in $\mathbf{P}^2$.

2. Holomorphic maps, Fatou and Julia sets.

We first describe the holomorphic maps from $\mathbf{P}^k$ to $\mathbf{P}^k$.

THEOREM 2.1. *Let f be a non constant holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. Then f is given in homogeneous coordinates by $[f_0 : f_1 : \cdots : f_k]$ where each f_j is a homogeneous polynomial of degree d and the f_j have no common zero except the origin.*

Proof. Let $[z_0 : z_1 : \cdots : z_k]$ be homogeneous coordinates in $\mathbf{P}^k$. We can assume that the image of f is not contained in any $(z_j = 0)$ (otherwise rotate coordinates). By the Weierstrass-Hurwitz Theorem [Gu] it follows that each of the meromorphic functions $\frac{z_i}{z_0} \circ f$ is a quotient of two homogeneous polynomials $\frac{F_j}{G_j}$ of the same degree.

Let $\tilde{F}$ denote the map $[\tilde{F}_0 : \cdots : \tilde{F}_k]$ where the $\tilde{F}_j$'s are homogeneous polynomials of the same degree obtained by dividing out common factors from the polynomials $\frac{F_j}{G_j} \cdot \Pi G_\ell$. We will show that $\tilde{F}$ is a lifting of f to $\mathbf{C}^{k+1}$. For this we only need to show that the $\tilde{F}_j$ have no common zeros except the origin. Suppose to the contrary that $p \in \mathbf{C}^{k+1} \mid (0)$ is a common zero. Choose a local lifting $\tilde{f} = [f_0 : \cdots : f_k]$ of f in a neighborhood of p. We may assume that one of the $f_j \equiv 1$. Say $f_0 \equiv 1$. Then it follows that $\tilde{F}_j = \tilde{F}_0 f_j$ and that $\tilde{F}_0(p) = 0$. But this implies that the common zero set of the $\tilde{F}_j$ is a complex hypersurface, which implies that they have a common factor, contradicting that we have already divided out all common factors.

Let $\mathcal{H}$ denote the space of non constant holomorphic maps on $\mathbf{P}^k$ and $\mathcal{H}_d$ the holomorphic maps given by homogeneous polynomials of degree d. Observe that $\mathcal{H}$ is stable under composition.

On the other hand there are the (not necessarily everywhere well defined) maps of degree d from $\mathbf{P}^k$ to $\mathbf{P}^k$, which are given in homogeneous coordinates by $[f_0 : f_1 : \cdots : f_k]$, but now the degree d homogeneous polynomials f_j are allowed to have common zeros. This later space is easily identified with $\mathbf{P}^N$ where $N = (k+1)\frac{(d+k)!}{d!k!} - 1$.

We will also consider the space $\mathcal{M}_d$ of meromorphic maps, consisting of those $[f_0 : \cdots : f_k]$ in P^N which have maximal rank on some nonempty open set.

It follows from Bezout's theorem that for f in $\mathcal{H}_d$ the number of points in $f^{-1}(a)$ is d^k counting multiplicity. Consequently f is of maximal rank and hence $\mathcal{H}_d \subset \mathcal{M}_d \subset \mathbf{P}^N$.

In analogy with one complex variable we define the Fatou set and Julia sets of a holomorphic map f in $\mathcal{H}_d$. More precisely we have the following definition.

DEFINITION 2.2. Given $f : \mathbf{P}^k \to \mathbf{P}^k$ in $\mathcal{H}_d$. $0 \leq \ell \leq k-1$, a point $p \in \mathbf{P}^k$ belongs to the Fatou set $\mathcal{F}_\ell$ if there exists a neighborhood $U(p)$ such that for every $q \in U(p)$ there exists a complex variety X_q through q of codimension ℓ and $\{f^n |_{X_q}\}$ is equicontinuous.

Observe that $\mathcal{F}_0$ is the largest open set where $\{f^n\}$ is equicontinuous. We call $\mathcal{F}_0$ the Fatou set. Also observe that each $\mathcal{F}_\ell$ is open and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{k-1}$.

Correspondingly, let $\mathcal{J}_\ell = \mathbf{P}^k \backslash \mathcal{F}_\ell$. We call $\mathcal{J}_0$ the Julia set.

THEOREM 2.3. *The Julia set of a holomorphic map in* $\mathcal{H}_d$, $d \geq 2$, *is always non empty.*

Proof. Assume $\mathcal{F}_0 = \mathbf{P}^k$. Let h be the limit of a subsequence $\{f^{n_k}\}$. Then h is a non constant holomorphic map of finite degree. As in one variable this contradicts that the degrees of f^{n_k} are unbounded, see [Mi].

THEOREM 2.4. *The sets* $\mathcal{H}_d$ *and* $\mathcal{M}_d$ *are Zariski open sets of* $\mathbf{P}^N$. *In particular* $\mathcal{H}_d$ *and* $\mathcal{M}_d$ *are connected. If* $f \in \mathcal{H}_d$, *then the critical set of* f *is an algebraic variety of degree* $(k+1)(d-1)$.

Proof. Consider $\sum$, the analytic set in $\mathbf{P}^N \times \mathbf{P}^k$ defined by the equation $f(z) = 0$. Let $\sum_d$ be the projection of $\sum$ in $\mathbf{P}^N$. Then $\sum_d$ is equal to $\mathbf{P}^N \backslash \mathcal{H}_d$. Since the projection is proper, by Tarski Theorem, we get that $\sum_d$ is an analytic set. The fact that $\mathcal{M}_d$ is Zariski open follows from the equation $\mathbf{P}^N \backslash \mathcal{M}_d = \bigcap_{z \in \mathbf{P}^k} \{f; J(f,z) = 0\}$ where $J(f,z)$ is the Jacobian of the lifted map on $\mathbf{C}^{k+1}$.

Let $f = [f_0 : f_1 : \cdots : f_k] \in \mathcal{H}_d$. Then the critical set of f is the projection under the canonical map $\Pi : \mathbf{C}^{k+1} \to \mathbf{P}^k$ of the critical set of $(f_0, f_1, \cdots, f_k)$. The degree of this set is clearly $\leq (k+1)(d-1)$ so the degree of the critical set of f in $\mathbf{P}^k$ is $\leq (k+1)(d-1)$. On the other hand for the map $[z_0^d : z_1^d : \cdots : z_k^d]$ the critical set has degree $(k+1)(d-1)$ and therefore since $\mathcal{H}_d$ is connected we get that for any $f \in \mathcal{H}_d$ the critical set has degree exactly $(k+1)(d-1)$.

3. Periodic points.

We show that the fixed point set of $f \in \mathcal{H}_d$ is discrete. More precisely we have :

THEOREM 3.1. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map of degree $d \geq 2$, and g be a meromorphic map of degree $d' < d$. Let Ω be the Zariski dense open subset where g is holomorphic. There can be no compact algebraic curve Z such that $f = g$ on $Z \cap \Omega$ and $Z \cap \Omega \neq \emptyset$. If g is holomorphic, the number of points where $f = g$ equals $(d^{k+1} - d'^{k+1})/(d-d')$ counted with multiplicity if g is holomorphic.*

The proof is to apply the Bezout theorem. We include it for the convenience of the reader.

Proof. Suppose that $\{f = g\}$ contains an open set of a compact complex subvariety Z of dimension one. We will arrive at a contradiction. First we write $f = [f_0 : f_1 : \cdots : f_k]$ and $g = [g_0 : g_1 : \cdots : g_k]$, where the f_j's are homogeneous holomorphic polynomials of degree $d > 1$ and the g_j's are homogeneous holomorphic polynomials of degree $d' \geq 1$, $d' < d$. Hence we can lift f, g to map on $\mathbf{C}^{k+1}$, $F = (f_0, f_1, \cdots, f_k)$, $G = (g_0, \cdots, g_k)$. Also the variety Z lifts to conic two dimensional surface X in $\mathbf{C}^{k+1}$. Introduce one more complex variable t and consider the $k+1$ equations $f_j - t^{d-d'} g_j = 0$. These are homogeneous equations of degree d in $\mathbf{C}^{k+2}$. Hence the common zero set is a conic complex variety Y. Consider at first the intersection with the hyperplane $t = 0$. Then the equations reduce to $f_0 = f_1 = \cdots = f_k = 0$. Since f is a well defined holomorphic map this zero set consists only of the origin. The natural projection of Y to P^{k+1} is therefore a compact complex space which does not intersect the hyperplane $t = 0$ at infinity. Hence the image is a compact subvariety in $\mathbf{C}^{k+1}$ and hence must be finite. This means that Y consists of a finite number of complex lines in $\mathbf{C}^{k+2}$ through the origin. Suppose next that p is in $Z \cap \Omega$, so $f(p) = g(p)$. Then there exists a complex value $t \neq 0$ and $(z_0, \cdots, z_k) \neq 0$ such that $p = [z_0 : \cdots : z_k]$ and $f_j(z_0, \cdots, z_k) = t^{d-d'} g_j(z_0, \cdots, z_k)$. Hence the point $(z_0, \cdots, z_k, t)$ belongs to Y. But this implies that Y is two dimensional, a contradiction. Hence we have shown that there is no such Z. In the case g is holomorphic this implies that $\{f = g\}$ is finite. Next we need to count the number of points. First we count the number of solutions using Bezout's theorem on the equations $f_j - t^{d-d'} g_j = 0$. There are d^{k+1} of these. However d'^{k+1} of these occur at the point $[0 : 0 : \cdots : 1]$, so this gives $d^{k+1} - d'^{k+1}$ solutions, but rotation of t by a $d - d'$ root of unity produces an equivalent solution, so the total number of solutions to $f = g$ is $(d^{k+1} - d'^{k+1})/(d - d')$. This complete the proof of the Theorem.

Using the above Theorem in the case $g = Id$ we obtain the number of periodic points.

COROLLARY 3.2. *Let* $f : \mathbf{P}^k \to \mathbf{P}^k$, $f \in \mathcal{H}_d$, $d \geq 2$. *The number of periodic points of order* n *counted with multiplicity is* $(d^{n(k+1)} - 1)/(d^n - 1)$.

We show that any holomorphic map of degree $d \geq 2$ has infinitely many disjoint periodic orbits.

THEOREM 3.3. *Let* $f : \mathbf{P}^2 \to \mathbf{P}^2$ *be a holomorphic map of degree* $d \geq 2$. *Then there exists infinitely many distinct periodic orbits.*

Proof. Recall that we have shown in Corollary 3.2 that the iterate f^n has $d^{2n} + d^n + 1$ fixed points counted with multiplicity. So to prove the theorem we only need to control the multiplicity. The control is trivial in dimension one but less obvious in higher dimension.

LEMMA 3.4. *Let 0 be a fixed point for a germ at zero of a local holomorphic map* $f : \mathbf{C}^2 \to \mathbf{C}^2$. *Assume that 0 is an isolated point of* $\{f^n(z,w) = (z,w)\}$ *for all integers* $n \geq 1$. *Then there exists an integer* N *such that for all iterates* f^n, $n \geq 1$, *the inequality* $\| f^n(z,w) - (z,w) \| \geq c_n \| (z,w) \|^N, c_n > 0$ *holds in some neighborhood (depending on* n*) of zero.*

We prove at first the lemma before we continue with the proof of the theorem.

Proof of the Lemma. Let the eigenvalues of f' be α, γ. Then we can assume that the map f has the form $f(z,w) = \alpha z + \beta w + P(z,w), \gamma w + Q(z,w)$ where P, Q vanish to at least second order and $\beta = 0$ if $\alpha \neq \gamma$. Then $f^n = (\alpha^n z + \beta_n w + P_n(z,w), \gamma^n w + Q_n(z,w)))$ for some constants β_n and functions P_n, Q_n vanishing to at least second order. It follows that the estimate holds immediately if $| \alpha |, | \gamma | \neq 1$ with $N = 1$. Next consider the situation where say $1\alpha | \neq 1$, $| \gamma | = 1$. Then $\beta = \beta_n = 0$ so the situation is equivalent to the case with α, γ reversed. We write $\gamma = e^{2\pi i\theta}$ where θ belongs to the unit interval. If θ is irrational it follows again that the estimate in the lemma holds for $N = 1$. Suppose then that $\theta = p/q$ for relatively prime integers p, q where $p = 0$, $q = 1$ or $q \geq 2$ and $p \in \{1, 2, \cdots, q-1\}$. In that case the estimate holds with $N = 1$ for all powers f^n as soon as n is not a multiple of q. To handle the missing case, we can replace f by the iterate f^q. This then is the case when $\gamma = 1$. So we can write $f = (\alpha z + P(z,w), w + Q(z,w))$.

We consider for this the more general case $\alpha \neq 1$ because we will anyhow need this below. The equation $\alpha z + P(z,w) = z$ can be solved implicitly in a small neighborhood of zero. We obtain $z = h(w)$ for some holomorphic function h vanishing at least to second order at the origin. We can now rewrite the map f in terms of holomorphic functions $P'(z,w), Q'(z,w)$ vanishing to at least first order at the origin and $Q''(w)$ vanishing to some finite order $k > 1$ at the origin as:

$$f(z,w) = (\alpha(z - h(w)) + h(w) + (z - h(w))P'(z,w),$$

$$w + Q''(w) + (z - h(w))(Q'(z,w)).$$

For any point (z_0, w_0) close to zero, let $(z_n, w_n) := f^n(z_0, w_0)$. We will write $(z_n, w_n) = (h(w_0) + \Delta_n, w_0 + \delta_n)$ and inductively estimate the error terms (Δ_n, δ_n). So at first, $(\Delta_0, \delta_0) = (z_0 - h(w_0), 0)$. We get $(z_{n+1}, w_{n+1}) = f(z_n, w_n) = (\alpha(h(w_0) + \Delta_n - h(w_0 + \delta_n)) + (h(w_0 + \delta_n) + (h(w_0) + \Delta_n - h(w_0 + \delta_n))P'(h(w_0) + \Delta_n, w_0 + \delta_n), w_0 + \delta_n + Q'(w_0 + \delta_n) + (h(w_0) + \Delta_n - h(w_0 + \delta_n))Q'(h(w_0) + \Delta_n, w_0 + \delta_n)) = o(\Delta_n + h(w_0), w_0 + \delta_n + Q'(w_0))$. Hence it follows that $(\Delta_{n+1}, \delta_{n+1}) = (\alpha\Delta_n + o(\mid \Delta_n \mid, \mid \delta_n \mid), \delta_n + Q'(w_0) + o(\mid \Delta_n \mid, \mid \delta_n \mid))$. From this we inductively prove the estimates $(\Delta_n, \delta_n) = (\alpha^n \Delta_0 + o(\mid \Delta_0, Q'(w_0) \mid))), nQ'(w_0) + o(\mid \Delta_0, Q'(w_0) \mid))$.

With these error estimates we estimate $f^n(z_0, w_0) - (z_0, w_0)$. We get $\mid (z_n - z_0, w_n - w_0) \mid = \mid (\Delta_n - \Delta_0, \delta_n) \mid \geq \mid ((\alpha^n - 1) \mid \Delta_0, nQ'(w_0)) \mid - o(\mid \Delta_0, Q'(w_0) \mid)$. Hence for all n such that $\alpha^n \neq 1$, we have that $\mid f^n(z,w) - (z,w) \mid \geq c_n(\mid z - h(w) \mid + \mid Q'(w) \mid)$ close enough to the origin for some $c_n > 0$. But if $\mid z \mid \leq \mid w \mid$ the second factor is of order of $\mid w \mid^k$ while if $\mid z \mid \geq \mid w \mid$ the first factor is at least $\mid z \mid /2$, so the estimate of the lemma follows with $N = k$ in all these cases. In particular we are done when $\mid \alpha \mid \neq 1$, $\mid \gamma \mid = 1$ (or vice versa).

We continue with the case when both α and γ have modulus one. Suppose at first that $\alpha = \gamma$ and they are irrational rotations. Then $f^n = (\alpha^n z + \beta_n w + P_n(z,w), \gamma^n w + Q_n(z,w))$ as above and clearly the estimate of the Lemma holds with $N = 1$. Suppose next that $\alpha = \gamma = e^{2\pi i p/q}$ where $(p,q) = (0,1)$

or $q \geq 2$ and $p = 1, \cdots, q-1$ relatively prime to q. Then the estimate of the Lemma holds for all iterates with n not a multiple of q and $N = 1$. It remains to consider iterates which are multiples of q. But this reduces to the case $\alpha = \gamma = 1$ and β arbitrary.

First consider the case when $\beta \neq 0$. We may then assume that $\beta = 1$. So the map has the form $f = (z + w + P(z,w), w + Q(z,w))$ where P, Q vanish to at least second order at the origin. There are several cases to consider. Assume at first that P, Q vanish on the z-axis. Then $f(z,0) = (z,0)$, contradicting that 0 is an isolated point of $\{f(z,w) = (z,w)\}$. Next assume that $P = wP'(z,w)$ while $Q(z,0) = az^\ell + \cdots$ for some integer $\ell \geq 2$ and $a \neq 0$. Inductively we see that $f^n = (z + nw(1 + O(|(z,w)|)) + O(|z^\ell|), w + N_n a z^\ell + o(|(w, z^\ell)|))$. If $|w| \leq |z|^{\ell-1/2}$ then the second component of $f^n - Id$ is at least of the order of $\|(z,w)\|^\ell$ while if $|w| \geq |z|^{\ell-1/2}$, the first component is at least of the order of $\|(z,w)\|^{\ell-1/2}$. Hence the estimate in the lemma follows. Now consider the case when $P(z,0) = az^k + \cdots$ for some integer $k \geq 2$ and some constant $a \neq 0$, while $Q(z,w) = wQ'(z,w)$. Then note that $w + P(z,w)$ vanishes on a complex manifold $w = h(z)$ for some holomorphic function h vanishing to finite order at least 2 at the origin.

Make the change of coordinates $w' = w - h(z)$, $z' = z$. In this coordinate system f has the same linear terms. But $f(z',0) = (z', h(z')Q'(z', h(z'))$ and the second component can only vanish to finite order at the origin. (Otherwise in the (z,w) coordinate system $f(z,h(z)) = (z,h(z))$ and there is a whole curve of fixed points.) But then we are back in the previously considered case.

The next case is when $P(z,0) = az^k + \cdots$, $Q(z,0) = bz^\ell + \cdots$ for integers $k, \ell \geq 2$ and numbers $a, b \neq 0$. In this case we can again use the same coordinate change as above, $w' = w - h(z)$, $z' = z$ with $h(z) + P(z, h(z)) = 0$ to reduce to the case $P(z,0) = 0$.

Next consider the case when $\beta = 0$. Then the map has the form $f(z,w) = (z + p(z,w), w + Q(z,w))$ where P, Q vanish to at least second order. Inductively we can then prove the estimate $f^n = (z + nP + o(\|(P,Q)\|)$, $w + nQ + o(\|(P,Q)\|))$. Hence $\|f^n - Id\| \geq \|(P,Q)\| \geq \|(z,w)\|^N$ for fixed N, close enough to the origin. The last inequality is just the Lojasiewicz inequality since (0,0) is an isolated fixed point of f.

Now we investigate the case when $\alpha \neq \gamma$ but they both have modulus one. In both and θ are irrational, then $f^n(z,w) = \alpha^n z + \cdots, \gamma^n w + \cdots$ and it is clear that $f - Id$ vanishes to first order for any n. Next assusme that φ is irrational and that $\gamma = p/q$ where $(p,q) = (0,1)$ or $q > 1$ and $0 < p < q$ is relatively prime to q. But now this case follows as above when $|\alpha| \neq 1$ and θ is rational. So assume that both φ and θ are rational. Then for some iterates, the map is of this form with both powers equal one. In that case the above applies. If one of the eigenvalues is one and the other is different from one, then this case is also handled above, and if both are different from one we are also done. Hence we have covered all cases.

Having thus finished the proof of the lemma we can continue with the proof of the theorem. Suppose that there are only finitely many periodic orbits. Then for some point p the multiplicity of $f^n - Id$ at p can be chosen arbitrarily large. Taking local coordinates we can assume $p = 0$. From the lemma we have $| f^n(z,w) - (z,w) | \geq c_n | (z,w) |^N$. Let P_N denote the Taylor polynomial of $f^n - Id$ of order N. Then for r sufficiently small $| f^n - Id - P_N | < | f^n - Id |$ on the sphere around zero of radius r. Hence by Rouché's Theorem [AY] the multiplicity of $f^n - Id$ at zero is at most N^2, a contradiction.

4. Exceptional varieties.

For a rational map on $\mathbf{P}^1$, a finite set E is exceptional if $f^{-1}(E) = E$. Similarly we introduce the following notion.

DEFINITION 4.1. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be in $\mathcal{H}_d$ and V a compact subvariety in $\mathbf{P}^k$. Then V is exceptional if $f^{-1}(V) = V$.*

If V is exceptional, necessarily $f(V) = V$ also. Let $f : \mathbf{P}^k \to \mathbf{P}^k$ denote a holomorphic map of degree $d \geq 2$. Assume f has an exceptional hypersurface V. Note that replacing f with an iterate we may assume that each irreducible branch of V is mapped to itself. Hence any collection of irreducible branches of V is also exceptional.

PROPOSITION 4.2. *Let $V_1, \cdots, V_\ell$ denote irreducible branches of the exceptional variety of f. Assume V_i is the zero set of the irreducible polynomial h_i. Then $\sum$ degree$(h_i) \leq k+1$. In particular there are at most $k+1$ irreducible branches of the exceptional set, and if there are $k+1$ they are all linear.*

Proof. Let $\pi : \mathbf{C}^{k+1} \to \mathbf{P}^k$ be the natural projection. We denote by $\tilde{V}_i$ the pull back of V_i to $\mathbf{C}^{k+1}$. Denote also by f the lifted map $f = (f_0, f_1, \cdots, f_k)$ to $\mathbf{C}^{k+1}$. Then $\tilde{V}_i$ is an irreducible homogeneous complex hypersurface, so we can write it as $\tilde{V}_i = \{h_i(z_0, z_1, \cdots, z_k) = 0\}$ for an irreducible homogeneous polynomial h_i.

LEMMA 4.3. *Let X be an exceptional hypersurface for $f \in \mathcal{H}_d$. Assume $X =$*

$\{h = 0\}$ whereh is a homogeneous polynomial. Then there exists a non zero constantc such thath, f satisfy the Böttcher functional equationh $\circ f = ch^d$.

Proof. Since X is totally invariant, the polynomial $h \circ f$ only vanishes on $\tilde{X}$ the pull back of X to $\mathbf{C}^{k+1}$. However the degree of this composition is $(\deg h) \cdot d$. Hence the equality follows.

We continue the proof of the proposition. An exceptional variety given by $h = 0$ as above is part of the critical set of f. Moreover the Jacobian determinant of f has $(\Pi h_i)^{d-1}$ as a factor. From Theorem 2.4 it follows that $\sum \deg h_i \leq k+1$.

PROPOSITION 4.4. *The set of holomorphic maps without exceptional hypersurfaces is a nonempty Zariski open set in$\mathcal{H}_d$.*

Proof. We identify homogeneous polynomials h of degree $\ell \leq k+1$ with their space of coefficients $\mathbf{P}^{N_\ell}$ and we define $\sum_\ell := \{(f, h) \in \mathcal{H}_d \times \mathbf{P}^{N_\ell}$; $h \circ f = ch^d$ for some constant $c\}$. If f has an exceptional variety then (f, h) is in $\sum_\ell$ for some h and some $\ell \leq k+1$. The projection of $\sum_\ell$ on $\mathcal{H}_\ell$ is again an analytic variety. Since there exists one map in each $\mathcal{H}_d$ which is not exceptional, the proposition follows.

We will next discuss the various possibilities in $\mathbf{P}^2$. At first suppose that there is an exceptional variety V_1 which is a line. By a linear change of coordinates we can suppose that this line is given by $\{[z : w : t]\}; t = 0$, i.e. V_1 is the hyperplane at infinity. Since V_1 is totally invariant this means that f is a polynomial map of degree d on $\mathbf{C}^2(z, w)$. Moreover the condition that the map is well defined on $\mathbf{P}^2$ simply means that the highest degree terms of the two components of the polynomial are of degree d and have no common zeros except at the origin. This means that if the hyperplane at infinity is exceptional, then the map has the form $[z : w : t] \to [f_0 : f_1 : t^d]$ where the functions $f_0(z, w, 0)$, $f_1(z, w, 0)$ have nondegenerate degree d-terms. Next assume that the exceptional variety contains two complex lines. We may then assume that they are given by $t = 0$, $w = 0$ respectively. It follows that f_1 has the form w^d. Hence in this case the map has the form $[z : w : t] \to [f_0 : w^d : t^d]$ where f_0 is a homogeneous polynomial of degree d with a nonzero coefficients in front of the z^d term.

Then consider the case when the exceptional variety contains three lines. We may assume that two of them are given by $t = 0$, $w = 0$ so that the map has the above mentioned form. We first consider the possibility that all three lines intersect at the same point, i.e. $[1 : 0 : 0]$. Hence the third line must have the form $w = \alpha$ in the $t = 1$ coordinate system. But this contradicts that the line is exceptional because all roots of $w^d = \alpha^d$ define lines in the preimage.

So the three lines only intersect in pairs. Hence we can assume that the lines are $z = 0$, $w = 0$, $t = 0$. But then we see that the map f_0 must be of the form $f_0 = z^d$.

In conclusion we have proved :

THEOREM 4.5. **1.** *If the map* $f \in \mathcal{H}_d$ *has an exceptional variety containing one complex line, then we can assume that* f *has the form* $[f_0([z:w:t]) : f_1([z:w:t]) : t^d]$ *where the functions* $f_0(z,w,0)$, $f_1(z,w,0)$ *have nondegenerate degreed-terms.*

2. *If the map* $f \in \mathcal{H}_d$ *has an exceptional variety containing two complex lines, then we can assume that* f *has the form* $[f_0([z:w:t]) : w^d : t^d]$ *where the function* $f_0(z,0,0) = z^d$.

3. *If the map* $f \in \mathcal{H}_d$ *has an exceptional variety containing three complex lines, then we can assume that* f *has the form* $[z^d : w^d : t^d]$.

Assume next that the exceptional variety $\{h = 0\}$ is cubic, but not a product of three lines. There are three possibilities, first $\{h = 0\}$ is the union of an irreducible quadric and a complex line, second $\{h = 0\}$ is a smooth torus and finally $\{h = 0\}$ could be a singular cubic variety.

We will show that none of these cases can occur.

If $\{h = 0\}$ is the union of an irreducible quadric Q, and a complex line L, notice that $f\mid_Q$ and $f\mid_L$ are both exceptional maps on $\mathbf{P}^2$. Hence they must have two critical points. Since these critical points must be intersection points of Q and L, it follows that $Q \cap L$ consists of two points. We can assume that $Q = (zw = t^2)$ and $L = (t = 0)$ after a linear change of coordinates.

By Theorem 4.5 we can also assume that f has the form $[f_0 : f_1 : t^d]$. Hence we have the identity

$$f_0 f_1 - t^{2d} = c(zw - t^2)^d$$

for some $c \neq 0$, see Lemma 4.3.

It follows that $f_0 f_1 = \prod_{i=1}^{d}(t^2 - c_i(zw - t^2))$ for distinct constants c_i. Hence some irreducible factor g_0 of f_0 will divide some term $t^2 - c_i(zw - t^2)$ while some irreducible factor g_1 of f_1 will divide another term $t^2 - c_j(zw - t^2)$, $c_i \neq c_j$. Let p be a common zero of g_0 and g_1. It follows that $f_0(p) = f_1(p) = t(p) = 0$ so f is undefined at p.

This controdiction proves that an exceptional variety cannot consist of an irreducible quadric and a line.

Our next case is when the exceptional set $\{h = 0\} := V$ is a smooth cubic. We are grateful to R. Narasimhan for proving that this is impossible. His argument goes as follows.

Let $\Omega := \mathbf{P}^2 \backslash V$. Then f is a covering map of Ω, $f : \Omega \to \Omega$. It is known that $\pi_1(\Omega)$ is finite ([D]), in fact $\pi_1(\Omega) = \mathbf{Z}/3\mathbf{Z}$. Moreover $f* : \pi_1(\Omega) \to \pi_1(\Omega)$

is injective and hence bijective. But this is impossible since f is nontrivial. Hence the exceptional set cannot be a torus.

We turn to the remaining cubic case, when $\{h = 0\}$ is irreducible and singular.

If $\{h = 0\}$ has a normal crossing singularity then $\pi_1(\Omega)$ is still $\mathbf{Z}/3\mathbf{Z}$ ([D]) so Narasimhan's proof above still applies. So suppose that $\{h = 0\}$ has a cusp singularity. Then the normalization of the exceptional set is a $\mathbf{P}^1$ and the map restricted to it has only one critical point. Since f is d to 1 on $\{h = 0\}$ this is impossible.

There is only one remaining possibility for the exceptional set, namely a nonsingular quadratic curve. In this case we can assume h has the form $zw - t^2$ and that $f_0 f_1 - f_2^2 = (zw - t^2)^d$.

Then d must be an odd integer : if $d = 2k$ is even, we can write

$$f_0 f_1 = \left(f_2 - (zw - t^2)^k\right)\left(f_2 + (zw - t^2)^k\right)$$

and we can show as above that f_0, f_1 and f_2 must have a common zero contradicting the assumption that f is well defined.

We discuss now finite exceptional sets.

THEOREM 4.6. *For fixed $d \geq 2$ the set $\tilde{\mathcal{H}}_d$ of holomorphic maps f from $\mathbf{P}^k \to \mathbf{P}^k$ that have no exceptional finite set is a nonempty Zariski open set of $\mathcal{H}_d$.*

Proof. Given $f \in \mathcal{H}_d$, and $a \in \mathbf{P}^k$ let $\Phi_i(a, f)$ denote the solutions of $f(z) = a$. If E is an exceptional finite set, then f induces a bijection of E. If a is in E then $f^{-1}(a)$ is one point, i.e. all the $\Phi_i(a, f)$ coincide. Hence $\tilde{\mathcal{H}}_d$ is a Zariski open set in $\mathcal{H}_d$. Since there are maps without exceptional points, $\tilde{\mathcal{H}}_{\mathbf{D}}$ is nonempty.

Next we give an example of a holomorphic map on $\mathbf{P}^2$ with an exceptional point belonging to the Julia set, contradicting the situation in $\mathbf{P}^1$ where all exceptional points are superattractive

$$f([z : w : t]) = [w^d + \lambda z t^{d-1} : z^d : t^d + P(z, w)]$$

where P is any homogeneous polynomial of degree d. Then $p = [0 : 0 : 1]$ is an exceptional point : $f^{-1}(p) = p$. If $|\lambda| > 1$ then the point is in the Julia set, since the two eigenvalues of $f'(p)$ are λ and 0. However, it follows from the stable manifold theorem, see [Ru] or [Sh], that $p \in \mathcal{F}_1$, since one of the eigenvalue of f at p is zero.

THEOREM 4.7. *There exist constantsc(d) so that for any$f : \mathbf{P}^2 \to \mathbf{P}^2$, $f \in \mathcal{H}_d$, any finite exceptional set has at mostc(d) points.*

Proof. We can assume that $d \geq 3$ since $c(2) \leq c(4)$. Observe that the degree of the map f is d^2, counting multiplicity. Notice that an exceptional point (wether it is fixed or on a periodic orbit) can have only one preimage. Hence, necessarily, all exceptional points lie in the critical set. Let p be an exceptional point. Assume at first that p is a regular point of the critical set C and $f(p)$ is a regular point for $f(C)$. Then we can choose local coordinates near p and $f(p)$ such that the map has the form $(z, w) \to (z, w^\ell)$ for some integer ℓ. So the map is locally ℓ to 1. But by Theorem 2.4, $\ell \leq 3(d-1)+1$, and this last number is $< d^2$ if $d \geq 3$. Hence p cannot be exceptional. It follows that p is a singular point of C or $f(p)$ is a singular point of $f(C)$. Since $f(C)$ has degree at most $3d(d-1)$ and the number of singular points of C or $f(C)$ is bounded by Bezout's Theorem, by a constant, the Theorem follows.

5. Generic hyperbolicity in $\mathbf{P}^2$.

One of the main tools in holomorphic dynamics in one variable is Montel's Theorem, more precisely, a family of holomorphic maps from the unit disc to $\mathbf{P}^1 \backslash \{0, 1, \infty\}$ is locally equicontinuous.

We want to prove here an analogue for maps on $\mathbf{P}^2$. We recall some properties of the Kobayashi-Royden infinitesimal distance, and we refer to Lang's book [La] for background. Let $\Delta(0, r)$ denote the disc of radius r centered at the origin in $\mathbf{C}$.

DEFINITION 5.1. Let M be a complex manifold and (p, ξ) be in the unit tangent bundle. Define

$$K_M(p, \xi) = \inf\{\frac{1}{r}; \exists \varphi : \Delta(0, r) \to M \text{ holomorphic}, \varphi(0) = p, \ \varphi'(0) = \xi\}.$$

In case M is the unit disc $\Delta(0, 1)$, K_M is the Poincaré metric, more precisely $K_M(z, \xi) = \frac{|\xi|}{1-|z|^2}$.

If $f : M \to N$ is a holomorphic map, then f is distance decreasing, i.e.

$$K_N(f(p), f'(p)\xi) \leq K_M(p, \xi).$$

If f is a covering map, then f is a local isometry, i.e.

$$K_N(f(p), f'(p)\xi) = K_N(p, \xi).$$

DEFINITION 5.2. The complex manifold M is Kobayashi hyperbolic if for every $p \in M$ there exists a neighborhood $U(p)$ and a constant $c > 0$ such that for all $q \in U$, $K_M(q, \xi) \geq c \mid \xi \mid$.

The integrated form of K_M is then a distance d_M, the Kobayashi metric, on M. If M is hyperbolic and complete with respect to d_M, we say that M is complete hyperbolic. If $M \subset N$ is an open subset of a complex manifold N with a hermitian metric ds, then M is said to be hyperbolically embedded, if M is Kobayashi hyperbolic and for every $p \in \partial M$ there is an open neighborhood $U(p)$ in N and a constant $c > 0$ such that $K_M(q, \xi) \geq c\, ds(q, \xi)$ for every $q \in U(p) \cap M$. This definition is independent of the choice of Hermitian metric on N.

THEOREM 5.3. *Fix an integerd* ≥ 2. *Then there exists a Zariski dense open set*$\mathcal{H}' \subset \mathcal{H}_d$ *with the following properties. If* $f \in \mathcal{H}'$ *and*C *denotes it's critical set. Then*
i) No point of$\mathbf{P}^2$ *lies in*$f^n(C)$ *for three differentn*, $0 \leq n \leq 4$.
ii)$\mathbf{P}^2 \setminus \left(\bigcup_{n=0}^4 f^n(C) \right)$ *is Kobayashi complete hyperbolic and hyperbolically embedded in*$\mathbf{P}^2$.

We prove first a lemma.

LEMMA 5.4. *Let*$f = [z^d : w^d : t^d]$. *There exists an arbitrarily small perturbationg of*f *such that the five (reducible) varieties*$g^n(C)$, $n = 0, \cdots, 4$ *have no triple intersections.*

Proof. We will find such a g by choosing a suitable $(3,3)$ matrix A arbitrarily close to the identity and letting $g = g_A := A(f)$. This ensures that the critical set of g is the same as the critical set of f. Let us denote by C_1, C_2, C_3 the components $(z = 0)$, $(w = 0)$, $(t = 0)$ respectively. For each choice of integers $0 \leq n_1 < n_2 < n_3 \leq 4$ and $1 \leq m_1, m_2, m_3 \leq 3$ let $A_{n,m}$ denote those matrices A for which $g^{n_i}(C_{m_i})$, $i = 1, 2, 3$ have a triple intersection. Then $A_{n,m}$ is necessarily a closed subvariety. Namely we consider the complex manifold $\mathbf{P}^2 \times G$ where G is the space of invertible matrices. In there consider the complex varieties $V_n := \{g_A^n(C).A\}$. By Tarski's proper mapping theorem these are varieties and the projection to G is proper with one dimensional compact fibers. Hence the projection of the intersection of any three of these is a complex subvariety of G, again by Tarski's proper mapping theorem.

Hence, if the lemma is false, then some $A_{n,m} = G$. Notice that if the m_s's are all distinct, then $A_{n,m}$ must be empty (near the Identity). Hence, by the symmetry in the situation we may assume that $m_1 = 1$, $1 \leq m_2$, $m_3 \leq 2$. Consider next the special case $g = [z^d + \epsilon t^d : w^d + \epsilon t^d : t^d]$. Observe that if $h(z) = z^d + \epsilon$ then $g^n(w = 0) = \{[z : h^n(0) : 1]\}$ and $g^n((z = 0)) = \{[h^n(0) : w : 1]\}$. It is clear that if some $m_i = 2$, there are no triple intersections if $\epsilon \neq 0$ is small enough. It remains to consider the case when all the $m_i = 1$, $i = 1, 2, 3$.

First let us prove that $n_1 > 0$. Consider at first the family $g_\epsilon = [z^d + \epsilon w^d : w^d : t^d]$. Then $g_\epsilon^n(C_1)$ are lines of the form $z = \eta_n w$ (in $(t = 1)$) where the η_n are distinct for suitable small ϵ while $\eta_0 = 0$. Next consider for fixed ϵ the maps $g_{\epsilon,\delta} = [z^d + \epsilon w^d + \delta t^d : w^d : t^d]$. The image of $g(C_1)$ is parametrized by $[\epsilon w^d + \delta : w^d : 1]$. In general

$$g^n(C_1) = [\eta_n w^{d^n} + O(\delta)(w, \cdots, w^{2^d - 1}) + \delta + O(\delta^2) : w^{2^d} : 1].$$

It follows that the intersection points with C_1 are estimated by

$$w = (-\delta/\eta_n)^{1/d^n}(1 + O(\delta^{1/\delta^n}).$$

Hence their w-coordinate is $-\delta/\eta_n \cdot (1 + O(\delta^{1/d^n}))$ which rules out triple intersections with C_1.

Hence we are reduced to the case $1 \leq n_1 < n_2 < n_3 \leq 4$. Observe that in the previous case we may assume that the intersections of the various images of C_1 with C_1 have different modulus, otherwise modify ϵ. This will enable us to deal with the next case when $n_1 = 1$. Assume that there always is a triple intersection when $n_1 = 1$ for some n_2, n_3 as above. Let p be such a triple intersection. Notice that there are at most d preimages of p, all of which are in C_1 and all with the same $| w |$ value. But necessarily also, one of these points must be in $g^{n_2 - 1}(C_1) \cap C_1$ and one of these points must be in $g^{n_3 - 1}(C_1) \cap C_1$, which contradicts that such intersection points have different modulus.

Hence we have only one more case to consider, $n_1 = 2$, $n_2 = 3$, $n_3 = 4$. For this consider a map far from the Identity composed with A, namely $g = [t^d : z^d : w^d]$. Then the orbit of C_1 is $(z = 0) \to (w = 0) \to (t = 0) \to (z = 0) \to (w = 0)$. In particular the images $g^2(C_1)$, $g^3(C_1)$, $g^4(C_1)$ have no point on common. Hence this is true as well for some small perturbations of f. This completes the proof of the lemma.

We will use the following two theorems by M. Greene.

THEOREM 5.5 ([Gr1]). *If V is a compact complex manifold and $D_1, \cdots, D_m$ are hypersurfaces (possibly singular), then $V \backslash D$ is complete hyperbolic and hyperbolically embedded provided*

1) There is no non constant holomorphic map $\mathbf{C} \to V \backslash D$.

2) There is no non constant holomorphic map $\mathbf{C} \to D_{i_1} \cap \cdots \cap D_{i_k} \backslash (D_{j_1} \cup \cdots \cup D_{j_\ell})$ *for any choice of indices* $\{i_1, \cdots, i_k, j_1, \cdots, j_\ell\} = \{1, \cdots, m\}$.

THEOREM 5.6 ([Gr2]). *Suppose* f *is a holomorphic map from* $\mathbf{C}$ *to* $\mathbf{P}^k$ *omitting* $k+2$ *distinct irreducible compact hypersurfaces. Then* $f(\mathbf{C})$ *is contained in a compact complex hypersurface.*

We now prove Theorem 5.3.

Proof. By the Lemma and Tarski's Theorem there exists a Zariski dense open set $\mathcal{H}' \subset \mathcal{H}_d$ satisfying condition 1 of Theorem 5.3. We prove 2. Let $f \in \mathcal{H}'$ we will show that $\Omega := \mathbf{P}^2 \backslash \bigcup_{n=0}^4 f^n(C)$ satisfies the conditions of Greene's Theorem. Let φ be a holomorphic map from $\mathbf{C}$ to Ω. By Theorem 5.6 $\varphi(\mathbf{C})$ is contained in an algebraic subvariety $X \backslash \bigcup_{n=0}^4 f^n(C)$. Then $\varphi(\mathbf{C})$ omits at least three points in X. Hence φ is constant. Similarly we check condition 2. This completes the proof of Theorem 5.3.

If three varieties have a common point, then their image under a map must also have a common point. Hence we get the following immediate corollary.

COROLLARY 5.7. *Fix an integer* $d \geq 2$. *Let* $f \in \mathcal{H}'$, *defined in Theorem 5.3, and let* C *denote it's critical set. Then*
1. *No point lies in* $f^n(C)$ *for three different* n, $k-4 \leq n \leq k$ *if* $k \leq 3$.
2. $\mathbf{P}^2 \backslash \left(\bigcup_{n=k-4}^{n=k} f^n(C) \right)$, $k \leq 0$ *is Kobayashi complete hyperbolic and hyperbolically embedded.*

We show next that generically components of $f^n(C)$ have genus larger then 1.

PROPOSITION 5.8. *For every* $d \geq 2$, $n \in \mathbf{Z}$, *there exists an open set* $\Omega \subset \mathcal{H}_d$ *such that for every* $g \in \mathcal{H}_d$ *there is an open neighborhood* $U(g)$ *of* g *such that* $U(g) \backslash \Omega$ *is a countable union of compact subsets of varieties. Moreover, for every* $f \in \Omega$, *all the irreducible components of* $f^n(C)$ *have genus at least one.*

Remark. We define the genus of a singular curve to be the genus of the normalization.

We first prove a lemma.

LEMMA 5.9. *For each* $n \in \mathbf{Z}$, *there exists a nonempty open subset* V *of* $\mathcal{H}_d$ *so that for each* $f \in V$, *with critical set* C, *each irreducible component of* $f^n(C)$ *is a compact curve of genus at least 1.*

Proof. We study the maps

$$f_r := [z^d + rwt^{d-1} + rtw^{d-1} : w^d + rzt^{d-1} + rtz^{d-1} : t^d + rzw^{d-1} + rwz^{d-1}]$$

for complex numbers r close to zero. Note that these maps are symmetric in the coordinates. Hence it suffices to study the map in the $(t = 1)$ coordinates in the set $|z|, |w| \leq 1$. There the map has the form

$$(z, w) \to \left(\frac{z^d + rw + rw^{d-1}}{1 + rzw^{d-1} + rwz^{d-1}}, \frac{w^d + rz + rz^{d-1}}{1 + rzw^{d-1} + rwz^{d-1}} \right).$$

We compute the Jacobian determinant J. We obtain

$$J = d^2(zw)^{d-1} - r^2(1 + (d-1)z^{d-2})(1 + (d-1)w^{d-2}) + rR(z, w) + r^2 Q(z, w)$$

where Q vanishes to order at least d and R vanishes to order at least $2d - 1$ and each term contains a factor $(zw)^{d-1}$. Consider the set $|z|, |w| \leq s_d$ for some small $s_d > 0$ independant of r. Suppose that $J(z, w) = 0$. It follows that $|zw| > k_d |r|^{2/(d-1)}$ for some fixed constant $k_d > 0$. This implies that the gradient of J is nonzero. It follows that inside this disc of radius s_d, the critical set consists of $d - 1$ branches each of which is a small perturbation of $zw = cr^{2/d-1}$. Since $\{J = 0\}$ is close to $\{zwt = 0\}$ and since there is a closed noncontractible curve in $\{zwt = 0\}$ joining $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$, it follows that each irreducible branch of C has genus at least one. For any fixed $n > 0$ and for r small enough the n^{th} forward image of the noncontractible curves in C are still noncontractible in $f_r^n(C)$. Since there is no nonconstant holomorphic map from $\mathbf{P}^1$ to a compact Riemann surface of genus larger or equal to one, it follows also that each irreducible branch of $f_r^{-n}(C)$ has genus ≥ 1. The same argument holds for an open neighborhood of f_r in $\mathcal{H}_d$.

Proof of the Proposition. There is a proper subvariety $\sum$ of $\mathcal{H}_d$ such that for $h \in \sum$ there exists a regular point in $h^n(C_h)$ of sheet number strictly less than the local maximum.

For each $g \in \mathcal{H}_d \backslash \sum$ there is an open neighborhood U_g of g and a proper subvariety X of U_g such that for $h \in X$ there exists a point $(z_1, z_2) \in S := h^n(C_h)$ and a polydisc $\Delta_1 \times \Delta_2$ around (z_1, z_2) such that S is a ramified cover over Δ_1 and either the number of irreducible branches of the germ $S_{(z_1,z_2)}$ is less than maximal or one of them has sheet number larger than minimal.

For maps $h \in U_g \backslash X$ the irreducible branches have constant topology.

The lemma now implies the proposition.

6. Expansion in the presence of hyperbolicity.

We show that periodic orbits of holomorphic self maps of P^k are non attractive in the complement of the critical orbits under the hypothesis of Kobayashi hyperbolicity.

THEOREM 6.1. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map with critical set C. Let $\mathcal{C}$ be the closure of $\bigcup_{j=0}^{\infty} f^j(C)$. Assume that $P^k \backslash \mathcal{C}$ is Kobayashi hyperbolic and hyperbolically embedded. If p is a periodic point for f, $f^\ell(p) = p$, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ and $p \notin C$, then $| \lambda_i | \geq 1$, $1 \leq i \leq k$. Also $| \lambda_1 \cdots \lambda_k | > 1$ or f is an automorphism of the component of $P^k \backslash \mathcal{C}$ containing p.*

Proof. We show at first that the eigenvalues of the derivative of the ℓ^{th} iterate at the periodic point are all at least one. Let $U := P^k \backslash C$ and let $U_1 := U \backslash f^{-1}(C)$. Observe that $U_1 \subset U$. As $f : U_1 \to U$ is a covering map we obtain that the infinitesimal Kobayashi metric at a point x and tangent vector ξ satisfy

$$K_U(f(x), f'(x)\xi) = K_{U_1}(x, \xi) \geq K_U(x, \xi).$$

So if $x \notin C$, and $f^\ell(x) = x$, then all eigenvalues λ_i of $(f^\ell)'(x)$ have modulus at least one.

Next we consider the Jacobian determinant of the ℓ^{th} iterate. First, let Ω be a component of U, $p \in \Omega$, and let $\Omega_\ell \subset \Omega$ be the connected component of $f^{-\ell}(\Omega)$ containing p. Let M be the universal covering of Ω_ℓ and $\pi : M \to \Omega_\ell$ the projection. Observe that M is hyperbolic and that for the Kobayashi metric biholomorphic mappings are isometries. Also observe that $(M, f^\ell \circ \pi =: \pi')$ is the universal cover of Ω. Pick any nonvanishing holomorphic k-form α at p. Fix a Hermitian metric on TM. Let $\| \ \|$ be a volume form on the space of $(0, k)$ forms, such that holomorphic automorphisms preserve the volume. Fix a point $q \in M$ with $\pi(q) = p$. Define

$$E_\Omega^M(p, q, \alpha) := \inf\{\| \gamma \|_q^2 ; g(q) = p, g_*(\gamma) = \alpha\}$$

where g runs through all holomorphic maps with non vanishing Jacobian from M to Ω with $g(q) = p$. Similarly define

$$E_{\Omega_1}^M(p, q, \alpha) := \inf\{\| \gamma \|_q^2 ; g(q) = p, \pi(q) = p, g_*(\gamma) = \alpha\}$$

where g runs through all holomorphic maps with non vanishing Jacobian and $\pi(q) = p$ from M to Ω_ℓ, $g(q) = p$. To proceed we need a lemma.

LEMMA 6.2. *The extremal maps exist and are surjective.*

We prove the lemma for E_Ω^M. The proof for $E_{\Omega_\ell}^M$ is the same.

Proof of lemma. Let g_n be a minimizing sequence. Consider the g_n's as maps from M to $\mathbf{P}^k \backslash \mathcal{C}$ which is hyperbolically embedded. Then g_n is equicontinuous with respect to a metric on $\mathbf{P}^k$. Hence by Ascoli theorem there exists a subsequence $g_{n_k} \to g$ and $g(p) = p$ and $\det g' \neq 0$ and hence g has values in Ω. Let $\tilde{g}$ be such that $\pi' \circ \tilde{g} = g$ and $\tilde{g}(q) = q$. If $| \det \tilde{g}'(q) | < 1$ then by the chain rule, this will contradict that g is extremal. Since M is hyperbolic we must have that $| \det \tilde{g}'(q) | \leq 1$, ([K, Thm 3. 3]). Hence, it follows that $\tilde{g}$ is an automorphism, ([K, Thm 3. 3]), and hence g is surjective.

We next continue with the proof of the last assertion of the theorem. Since f^ℓ is a covering map from Ω_ℓ to Ω, since $f^\ell(p) = p$

$$(*) \qquad E_{\Omega_\ell}^M(p, q, \alpha) = E_\Omega^M(f^\ell(p), q, (f^\ell)_*(p)(\alpha)).$$

If $\Omega_\ell = \Omega$, then (*) implies that $| \det(f^\ell)'(p) | = 1$. Hence f^ℓ is an automorphism of Ω ([Ko]).

If Ω_ℓ is a proper subset of Ω, then the Lemma implies that $E_{\Omega_\ell}^M(z, \alpha) > E_\Omega^M(z, \alpha)$. Hence $| \det(f^\ell)'(p) | > 1$.

Observe that $E_\Omega^M(p, q, \alpha) = \| \gamma \|_q$ where $\pi'_*(q)\gamma = \alpha$. Since $\| \; \|$ is invariant under biholomorphisms, this is independent of q. We will denote by $E_\Omega^M(p, \alpha)$ this volume form.

Remark 6.3. *If $k = 1$, then Theorem 6.1 says that attractive, rationally indifferent and Cremer points are in C.*

The following result generalizes a classical result of Fatou and Julia on rational maps on P^1. Let $f : \mathbf{P}^k \to \mathbf{P}^k$ and let $C = C_1 \cup \cdots \cup C_\ell$ denote the irreducible components of the critical set.

DEFINITION 6.4. *We say that a Fatou component $\Omega \subset P^k$ is a Siegel domain if there exists a subsequence f^{n_i} converging to the identity map on Ω.*

PROPOSITION 6.5. *Let C denote the critical set of a holomorphic map $f : \mathbf{P}^k \to \mathbf{P}^k$ of degree at least 2. Assume that the complement of the closure of $\bigcup_{n=0}^\infty f^{-n}(C)$ is hyperbolically embedded. Then*

$$J \subset \bigcup_{N>0} \overline{\cup_{n \geq N} f^{-n}(C)} =: J(C).$$

Hence all periodic points with one eigenvalue of modulus strictly larger than 1 are in $J(C)$.

Proof. Assume that $p \notin J(C)$. Then $\exists N$ such that $B(p,r)$ does not intersect $\overline{\bigcup_{n\geq N} f^{-n}(C)}$. This implies that for $n \geq N$, $f^n(B(p,r)) \cap \overline{\bigcup_{k=0}^{\infty} f^{-k}(C)} = \emptyset$. Hence f^n is a normal family on $B(p,r)$, which proves the statement.

THEOREM 6.6. *Under the assumptions of Theorem 3.3 we have : If there is a component* U *of the Fatou set of* f *such that* $f^n(U)$ *does not converge u. c.c to* $\mathcal{C}$, *then* U *is preperiodic to a Siegel domain* Ω *with* $\partial\Omega \subset \mathcal{C}$.

Note : This does not exclude that part of the Siegel domain is in $\mathcal{C}$.

Proof. Let U be a component of the Fatou set satisfying the hypothesis of the theorem. Assume that f^n does not converge to $\mathcal{C}$ on compact subsets of U. Then there exist a nonempty open subset $U' \subset U$ and a neighborhood N of $\mathcal{C}$ and a subsequence f^{n_i} such that $f^{n_i}(U') \cap N = \emptyset$. Moreover we may assume that $f^{n_i} \to h$ on U'. Since $K_{\mathbf{P}^k\setminus\mathcal{C}}(f(x), f'(x)(\xi)) \geq K_{\mathbf{P}^k\setminus\mathcal{C}}(x,\xi)$, and the Kobayashi metric is upper semicontinuous, it follows that h is nondegenerate. Let Ω be the Fatou component containing $h(U')$. On Ω there is a subsequence such that $f^{n_{k+1}-n_k} \to I$, and Ω is periodic under f^ℓ. Let E_X denote the Kobayashi volume form with respect to holomorphic maps of maximal rank at every point on the complex manifold X. Denote $E_{\mathbf{P}^k\setminus f^{-m\ell}(\mathcal{C})}$ by E_m. We have $E_0(x,\alpha) \leq E_m(x,\alpha) \leq E_{m'}(x,\alpha) = E_0(f^{m'\ell}(x), Df^{m'\ell}(x)(\alpha))$ for any $0 < m < m'$. Taking limits and using the upper semicontinuity of the Kobayashi volume form gives $E_{\mathbf{P}^k\setminus\mathcal{C}}(x,\alpha) = E_m(x,\alpha)$ for $x \in \Omega\setminus\mathcal{C}$. Let $\mathcal{C}' = \overline{\bigcup f^{-m}(C)}$. Passing to the limit we obtain that $E_{\mathbf{P}^k\setminus\mathcal{C}}(x,\alpha) = E_{\mathbf{P}^k\setminus\mathcal{C}'}(x,\alpha)$ for $x \in \Omega\setminus\mathcal{C}$ nonempty.

If a point of $\partial\Omega$ is not in $\mathcal{C}$, then $E_{\mathbf{P}^k\setminus\mathcal{C}'}(x,\alpha)$ must blow up, see ([FS], Theorem 8.4) which is a contradiction.

The proof of Theorem 1.8 shows that if U is a Fatou component which is not preperiodic to a Siegel domain, then all limit functions are in $\mathcal{C}$.

We are going to prove the dual statement of Theorem 1.7 for non attractive fixed points.

Remark 6.7. *The same statement holds if instead of* C *we consider an analytic variety* A *such that the complement of the closure of* $\bigcup_{n=0}^{\infty} f^n(A)$ *is hyperbolic, in which case* $J \subset \bigcap_{N>0} \overline{\bigcup_{n>N} f^{-n}(A)} := J(A)$. *In general we*

don't have $J = J(A)$:
Example : $[z : w : t] \to [(z-2w)^2 : z^2 : t^2]$ $(z = \alpha w) \to (Z = ((\alpha-2)/\alpha)^2 W)$ and preimages of lines are lines. The union of the closure of $\cup f^{-n}(C)$ equals $P^2 \cdot J \neq P^2$.

COROLLARY 6.8. *Assume that the complement of the closure of*

$$\bigcup_{n=0}^{\infty} f^{-n}(C)$$

is hyperbolic and hyperbolically embedded. Then any Siegel domain is a Fatou component and a domain of holomorphy which is hyperbolic.

Proof. Let Ω be a Siegel domain. It is clear that $\Omega \cap \bigcup_{n=0}^{\infty} f^{-n}(C) = \emptyset$. Hence Ω is a component of $P^k \backslash J(C)$ which is clearly locally Stein. The domain Ω is hyperbolic since $\Omega \subset P^k \backslash \overline{\bigcup_{n=0}^{\infty} f^{-n}(C)}$.

We describe more precisely the behavior of f at a fix point when some of the eigenvalues are of modulus 1.

PROPOSITION 6.7. *Let* $f : \mathbf{P}^k \to \mathbf{P}^k$ *be a holomorphic map in* $\mathcal{H}_d$, $d \geq 2$. *Assume* $\Omega := \mathbf{P}^k \backslash \mathcal{C}$ *is Kobayashi hyperbolic.*

Suppose p is a fix point for f, $f(p) = p$, and $p \in J \backslash \mathcal{C}$. Let $\lambda_1, \cdots, \lambda_k$ be the eigenvalues of $f'(p)$. Assume $\mid \lambda_i \mid = 1$, $1 \leq i \leq s$, $\mid \lambda_j \mid > 1$ for $j > s$. Then there exists a subvariety $\sum$ through p such that $f_{|\sum}$ is linearizable.

We first prove a Lemma.

LEMMA 6.8. *Let* g *be a holomorphic map from* $\mathbf{C}^s$ *to* $\mathbf{C}^s$. *Assume* $g(0) = 0$ *and* $g'(0) = A$ *is unitary. If* (g^n) *is a normal family in a neighborhood of 0, then* g *is linearizable.*

Proof. Observe first that (A^n) and (A^{-n}) are bounded. Define, as in one variable,

$$\varphi(z) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} A^{-j} \circ g^j.$$

The mapping φ is well defined in a neighborhood of 0 and $\varphi(g(z)) = A\varphi(z)$.

We proceed to prove the proposition.

Proof. We know that f is a local isometry from $\Omega_1 = \mathbf{P}^k \backslash f^{-1}(\mathcal{C})$ to $\Omega = \mathbf{P}^k \backslash \mathcal{C}$. Let d_Ω denote the Kobayashi distance of Ω, and let $B(p, r)$ be the ball of radius r and center p for d_Ω. Since $\Omega_1 \subset \Omega$, $d_\Omega(f(z), f(p)) \geq d_\Omega(z,p)$. The Kobayashi distance induces the topology of Ω, hence we can assume that f is invertible in a neighborhood of $\bar{B}(p,r)$ and let $g = f^{-1}$. We have $g(B(p,r)) \subset B(p,r)$, and the family g^n is normal. Let $h = \lim g^{n_i}$ and define $\sum' := h(B(p,r))$. Clearly we have $g(\sum) \subset \sum$. If we consider the sequence $g^{n_{i+1}-n_i}$, we can assume $h_{|\sum}$ is identity. It follows that $\sum$ is contained in the complex manifold $S := \{q \in B(p,r) \ \ h(q) = q\}$. Let $\sum$ be the connected component of S through p. $\sum$ is a complex manifold of dimension s, $g^n_{|\sum}$ is normal hence by the Lemma $g_{|\sum}$ is linearizable around p, and the same result holds for f.

Example. Let $f[z : w : t] = [\lambda zt + z^2 : w^2 + ct^2 : t^2]$ where $|\lambda| = 1$ is such that $\lambda z + z^2$ is linearizable in a neighborhood of 0 in $\mathbf{C}$. Let $p = [0 : w_0 : 1]$ where w_0 is such that $f(p) = p$. One easily check that if $|c| >> 1$ then $p \notin \mathcal{C}$ and that f is linearizable in a disc through p.

In one dimension, it is well known that if the Julia set J of $R : \mathbf{P}^1 \to \mathbf{P}^1$ is disjoint from the closure of the orbit of the critical set then J is hyperbolic in the dynamical sense, i.e. there exists an infinitesimal metric in a neighborhood of J such that R is expanding for that metric.

We prove here a similar result in higher dimension.

THEOREM 6.9. *Let $f \in \mathcal{H}_d$, $f : \mathbf{P}^k \to \mathbf{P}^k$, $d \geq 2$. Let $\mathcal{C} = \overline{\bigcup_{n\geq 0} f^n(C)}$. Assume $\mathbf{P}^k \backslash \mathcal{C}$ is Kobayashi hyperbolic and hyperbolically embedded. Let X be a compact set in J, the Julia set of f. Assume $f(X) \subset X$ and $X \cap \mathcal{C} = \emptyset$, then there exists a continuous volume form E in a neighborhood of X, and a constant $k > 1$ such that*

$$E(f^n(x), (f^n)'(x)\alpha) \geq k^n E(x, \alpha).$$

Here α denotes a nonvanishing holomorphic k-form.

Proof. Let (Ω_j), $1 \leq j \leq \ell$ denote the components of $U := \mathbf{P}^2 \backslash \mathcal{C}$ intersecting X. Define $\Omega = \bigcup_{j=1}^{\ell} \Omega_j$. Observe that for every j, the components of $f^{-1}(\Omega_j)$ are contained in one of the Ω_s, $1 \leq s \leq \ell$. We claim that there is an N such no component of $f^{-N}(\Omega)$ coincide with one of the Ω_j's. Otherwise, since the number of components is finite, there are two integers n_1, n_2 such that say a component of $f^{-n_1}(\Omega_1)$ is equal to Ω_2 and a component of $f^{-n_2}(\Omega_2)$ is equal to Ω_1 and hence $f^{n_1+n_2}$ is an isometry of the Kobayashi hyperbolic domain Ω_1. As a consequence Ω_1 is a Siegel domain, contradicting that X is in the Julia set and intersects Ω_1.

Let $\Omega' := f^{-N}(\Omega)$, f^N is a covering map from Ω' to Ω. Denote by $E(p,\alpha)$ the volume form $E_\Omega^M(p,\alpha)$ constructed in the proof of Theorem 6.1. Observe that $E(p,\alpha)$ is continuous and that for all p in Ω'

$$E_\Omega^M(f(p), f'(p)\alpha) = E_{\Omega'}^M(p,\alpha) > E_\Omega^M(p,\alpha)$$

the inequality holds since no component of Ω' coincide with a component of Ω. The continuity of E implies the theorem.

7. Classification of critically finite maps in $\mathbf{P}^2$.

In the study of iteration of rational maps $R : P^1 \to P^1$ the case where every critical point is preperiodic or periodic is quite central, see ([Th]). Here we consider the corresponding problem in higher dimension. The question was raised by McMullen [B].

It is quite difficult to construct non trivial examples of critically finite maps in dimension 2. Some examples where studied in [FS]. It is proved in [FS] that the map $g : \mathbf{P}^2 \to \mathbf{P}^2$ defined by

$$g[z : w : t] = [(z-2w)^2 : (z-2t)^2 : z^2]$$

has $\mathbf{P}^2$ as Julia's set. It is probably an interesting question to study maps in $\mathcal{H}_d$ close to g, as was done in one variable by M. Rees [Re].

DEFINITION. *Let$F : M^2 \to M^2$ be a finite holomorphic self mapping of a compact complex manifold of dimension 2. Denote by$C$ the critical set. LetC_i be the irreducible components of$C$. We assume that eachC_i is periodic or preperiodic,*

$$C_i \to F(C_i) \to \cdots \to F^{\ell_i}(C_i) \to \cdots \to F^{\ell_i+n_i}(C_i) = F^{\ell_i}(C_i)$$

where$\ell_i \geq 0$, $n_i \geq 1$ are (minimally chosen) integers. We then say that$F$ is critically finite. We say that$F$ is strictly critically finite if all the mapsF^{n_i} : $F^{\ell_i+j} \to F^{\ell_i+j}$ are critically finite self maps (on possibly singular Riemann surfaces).

Observe that F is critically finite if $V := \bigcup_{n=0}^{\infty} F^n(C)$ is a closed complex hypersurface of M, in this case we define $W = F^{-1}(V)$ which is a complex hypersurface containing V.

We recall the following result, see ([Su],[Th]).

THEOREM 7.2. *Let$R : \mathbf{P}^1 \to \mathbf{P}^1$ be a rational map of degree$d \geq 2$. Assume that$R$ is critically finite. Then the only Fatou components of R are preperiodic to superattractive components.*

Recall that Ω is a superattractive component if $\exists p \in \Omega$, $\exists \ell$, $R^\ell(p) = p$ and $(R^\ell)'(p) = 0$. The theorem shows in particular that if R is critically finite and if every critical point is preperiodic that the Julia set of R is $\mathbf{P}^1$.

We give here a proof of Theorem 7.2 that does not use Sullivan's non wandering theorem nor the construction of an expanding metric.

Proof. Let Ω be a Fatou component for R. It follows from Theorem 6.6 that Ω is not preperiodic to a Siegel domain and that all the limits are in $\bigcup_{j=0}^{\infty} R^j(C)$ which is finite. Replacing R by some iterate we can assume that all the possible limits $a_1, a_2, \cdots, a_r$ are fix points for R. Let $U_1, U_2, \cdots, U_r$ be neighborhoods of $a_1, a_2, \cdots, a_r$, 2 by 2 disjoint and such that $R(U_j)$ are also 2 by 2 disjoint. Let $\omega \subset\subset \Omega$. For $n \geq N_0$, $R^n(\omega) \subset \bigcup_{j=1}^{r} U_j$ but since $R(U_j)$ are disjoint there exists j_0 such that for $n \geq N_0$, $R^n(\omega) \subset U_{j_0}$. Hence $\{R^n\}$ converges to $a_{j_0} =: a$. Let $\lambda := f'(a)$, clearly $|\lambda| \leq 1$.

Assume that $|\lambda| = 1$, $\lambda = e^{2\pi i \alpha}$. The map R cannot be linearizable near a since the corresponding Fatou component requires an infinite orbit of critical points. Similarly if α is rational or a Cremer point, see ([Mi]). So $|\lambda| < 1$ but then if $\lambda \neq 0$ we still need an infinite orbit of critical points. So finally $\lambda = 0$ and Ω is preperiodic to a superattractive Fatou component.

We will need to apply the above theorem to Riemann surfaces with singularities whose normalizations are biholomorphic to $\mathbf{P}^1$. More precisely, let X be an irreducible analytic set of dimension 1, let $\hat{X}$ denote it's normalization with covering map $\pi : \hat{X} \to X$. If f is a holomorphic map from X to X, we will denote by $\hat{f}$ the lifted map from $\hat{X}$ to $\hat{X}$. We will say that a point p in X is critical for f if $\pi^{-1}(p)$ is critical for $\hat{f}$. It is straightforward to verify that p is critical if and only if f restricted to a neighborhood of p is not injective. The terminology of Julia sets and Fatou components extends to this context.

DEFINITION 7.3. *Let$X$ be a compact analytic set. If$f : X \to X$ is holomorphic, we will say that$f$ is critically finite if$\hat{X} = \mathbf{P}^1$ and the orbit of the critical set is finite.*

Observe that this implies that $\hat{f} : \mathbf{P}^1 \to \mathbf{P}^1$ is critically finite. If a is a fixed point in $\hat{X}$, we will say that it has eigenvalue λ if $\hat{f}'(\hat{a}) = \lambda$ where $\pi(\hat{a}) = a$.

The following version of the above theorem is then clear.

THEOREM 7.4. *Let $f : X \to X$ be a d to 1 holomorphic map from an irreducible compact analytic set to itself. Assume that $\hat{X} = \mathbf{P}^1$. If f is critically finite, then the only Fatou components for f are preperiodic to superattractive basins.*

Proof. We just apply the above theorem to $\hat{f} : \hat{X} \to \hat{X}$.

We will need the following result which shows that if an algebraic curve X is invariant under f then f is not injective on X.

PROPOSITION 7.5. *Let $X \subset \mathbf{P}^2$ be an algebraic curve of degree r. Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be a holomorphic map which is d^2 to 1. Assume that $f(X) \subset X$. If f is an ℓ to 1 map on X, then $\ell \geq d$.*

Proof. Let H be a generic complex plane such that the number of points in $H \cap X$ is r. Suppose that $H = \{z_2 = 0\}$. Let $f^n = [f_0^n : f_1^n : f_2^n]$ in homogeneous coordinates. The number of points of $X \cap \{f_2^n = 0\}$ is $r\ell^n$. So there exists a homogeneous polynomial h_n of degree $r\ell^n$ vanishing on $X \cap \{f_2^n = 0\}$ but not identically. Let $\pi : \mathbf{C}^3 \to \mathbf{P}^2$ be the canonical map, and denote $\tilde{X} = \pi^{-1}(X)$. Define $\Phi_n = (h_n f_0^n/f_2^n, h_n f_1^n/f_2^n, h_n)$ from $\tilde{X}$ with values in $\mathbf{C}^3$. The map is weakly holomorphic in $\tilde{X}$ so there exists p such that for every n, $z_0^p\Phi_n$ extend as a holomorphic map. Since $z_0^p\Phi_n$ is homogeneous of degree $p + r\ell^n$ we can assume that the extension is also homogeneous of degree $p + r\ell^n$. Let g_n be this extension, $g_n = f^n$ on X. Then by Theorem 1.5 we should have $p + r\ell^n \geq d^n$ for every n, which implies $\ell \geq d$.

COROLLARY 7.6. *If $f(X) = X$, and if f is holomorphic of degree $d \geq 2$ then f does not induce an automorphism on X.*

THEOREM 7.7. *Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be a holomorphic map with critical set C. Assume that f is strictly critically finite and that $\mathbf{P}^2 \backslash C$ is hyperbolic. Then the only Fatou components of f are (preperiodic or equal to) superattractive components. In particular if no critical point is periodic, then the Julia set of f is $\mathbf{P}^2$.*

Proof. Recall that $V := \bigcup_{j=0}^{\infty} f^j(C)$ is a closed hypersurface of $\mathbf{P}^2$ and that every irreducible component of V is preperiodic or periodic. Without lack of generality (replacing f by an iterate if necessary), we can assume that every component is preperiodic to some irreducible V_j which is invariant under f. Let $\hat{V}_j$ denote the normalization of V_j. Observe first that $\hat{V}_j$ cannot be a hyperbolic Riemann surface since one of the iterates of f will be the identity on V_j and we have seen in Theorem 3.1 that this is not possible. So V_j can be $\mathbf{P}^1$ or a torus T. Let A denote the singular set of V. Then A consists of intersections of different branches as well as self intersections and cusps. Note that A is a finite set, $\{a_1, a_2, \cdots, a_r\}$. We can assume all superattractive periodic points with respect to some V_j are fixed.

Fix a Fatou component Ω. Assume at first that the iterates $f^n \mid \Omega$ converges u. c. c. to A. The argument is the same as in Theorem 5.2. Replacing f by an iterate we may assume that $f^n \mid \Omega$ converges u. c. c. to a_1. We will show that this implies that a_1 must be superattractive, completing the proof in this first case. For this, let λ_1, λ_2 be the eigenvalues of f' at a_1. Since $\{f^n\}$ is normal on Ω it follows that necessarily $\mid \lambda_1 \mid, \mid \lambda_2 \mid \leq 1$. Suppose that $a_1 \in V_1$ say. Then we exclude at first the possibility that $\hat{V}_1$ is a torus. If $\hat{V}_1$ is a torus, then if $f \mid V_1$ is not an automorphism, then this contradicts that a_1 is a nonrepelling fixed point. So necessarily $f \mid V_1$ is an automorphism. This is impossible by Theorem 3.1. Hence we may assume that $\hat{V}_1 = \mathbf{P}^1$. Since $f \mid V_1$ is critically finite, the eigenvalue corresponding to $f \mid V_1$ at a_1 is zero.

Assume at first that V_1 has a cusp singularity at a_1 which is mapped to itself. In local coordinates we can parametrize this singularity as $t \to (t^p + t^q + \cdots)$ for some integers $1 < p < q$, where q is not a multiple of p. If f' has a nonzero eigenvalue at 0, then we may assume that $f(z, w) = (O(\mid z, w \mid^2), \alpha w + O(\mid z, w \mid^2))$ for some nonzero α. Clearly this contradicts that f maps this singularity to itself.

Hence a_1 can be assumed to be a reducible singular point of V and all branches of V there, which are mapped to themselves by some iterate are nonsingular. If these branches are not all tangent to each other then both eigenvalues of f' must be zero, so we are done. Assume next that there are two such irreducible branches that are tangent. We may assume these have the form $w = 0$ and $w = \alpha z^k + O(\mid z \mid^{k+1})$ for some integer $k > 1$. If there is a nonzero eigenvalue at a_1 we may again assume that f has the form: $(z, w) \to (O(\mid z, w \mid^2), \beta w + O(\mid z, w \mid^2))$ for some nonzero β. This is again impossible. It remains to consider the case when there is one nonsingular branch, $w = 0$ which is mapped to itself and at least one more branch which is mapped to $w = 0$. So again we may assume that the other eigenvalue is nonzero and then that the map has the form $f(z, w) = (O(w* \mid z, w \mid^2), \alpha w + w * O(\mid z, w \mid)$ for some nonzero α. Hence no other branches can be mapped into the z-axis.

The second case to consider is when for some nonempty open subset $\omega \subset\subset \Omega$ there exists some subsequence $\{f^{n_j}\}$ converging to h with values in $V_1 \setminus$ (neighborhood of order ϵ of $A =: A_\epsilon$). We first assume that Ω intersects some preimage of V. Then some forward orbit of Ω intersects some V_j. Clearly this intersection must be with the Fatou set of the restriction map to V_j. But since this restriction is critically finite this implies that any such Fatou component is superattractive as a Fatou component of V_j for a superattractive fixed point p. It remains to show that this is a superattractive fixed point for f on $\mathbf{P}^2$ as well. If $p \in A$ this is as in the first part of the proof. So we may assume that p is a nonsingular point of V. So we may assume that $V_j = (w = 0)$ near the origin and that f has the form $f(z,w) = (z^k + O(\mid z \mid^{k+1}, w* \mid z, w \mid)$, $\alpha w + w * O(\mid z, w \mid)$ for some $k > 1$. Computing the Jacobian we see hat this forces them to have a branch of the critical set through the axis, implying that $p \in A$, a contradiction. Note that if we replace f by a high iterate, the set A may increase, but the argument in the first part still applies. So if some forward orbit of Ω intersects V_j, then this is a superattractive basin, and we are done.

Hence we can assume that Ω does not intersect any preimage of V. Since the Kobayashi infinitesimal metric of $\mathbf{P}^2 \backslash V$ does not blow up in directions parallell to $V_1 \backslash A_\epsilon$, when we approach V_1, we find that h is nonconstant.

We can assume that $\hat{V}_1 = \mathbf{P}^1$ or a torus. Then the image of h cannot contain a repelling periodic point. Say if $f^\ell(p) = p$, $\mid f^{\ell'}(p) \mid > 1$. Then $\{f^{n_j + \ell r_j}\}$ will have some derivative blowing up in ω. So $\hat{V}_1 = \mathbf{P}^1$ and the image of h will be in a Fatou component of $f \mid V_1$. Hence Ω is preperiodic to a superattractive component. Indeed $f \mid V_1$ has a superattractive component since $f \mid V_1$ is critically finite, the same proof as above shows that the superattractive points with respect to $f \mid V_1$ is superattractive.

The only remaining case is when $f^n \mid \Omega$ does not converge to V. But this cannot happen, by Theorem 6.6, because $\mathbf{P}^2 \backslash V$ is hyperbolic.

We now consider some cases when we do not assume that for some N, $\mathbf{P}^2 \backslash \bigcup_{n=0}^N f^n(C)$ is hyperbolic.

THEOREM 7.8. *Assume that* $f : \mathbf{P}^2 \to \mathbf{P}^2$ *is strictly critically finite. Suppose that for some irreducible component* C_0 *of the critical set we have* $f(C_0) = C_0$. *Then the only Fatou components are preperiodic to superattractive basins.*

We start at first with two lemmas of independent interest.

LEMMA 7.9. *Suppose that* $g : \mathbf{P}^2 \to \mathbf{P}^2$ *is a holomorphic map of degree* d *and that* g *maps a compact complex hypersurface* Z *to itself and that* Z *is contained in the critical set of* g. *Then we have the estimate* $\text{dist}(f(z), Z) = o(\text{dist}(z, Z))$.

Proof of the Lemma. We reduce this to a local statement in $\mathbf{C}^2$. Let X, Y

be one dimensional closed analytic subsets of the unit ball B containing zero. Suppose that $f : B \to B$, $f(0) = 0$, is a holomorphic map with $f(X) \subset Y$ such that the Jacobian of f vanishes on X. Moreover assume that
Moreover assume that f is a finite map. Then for every constant $c > 0$ there exists a neighborhood V of 0 such that $\text{dist}(f(p), Y) \geq c\text{dist}(p, X)$ for all $p \in V$. Note that we may assume that X, Y are irreducible at zero. After changes of coordinates we may assume that X can be parametrized by $t \to (t^p, t^q(g(t))$ where $g(0) = 1$ and $1 \leq p < q$. Similarly Y can be parametrized by $\tau \to (\tau^{p'}, \tau^{q'}(g'(\tau))$ where $g'(0) = 1$ and $1 \leq p' < q'$. Next observe that in a small enough neighborhood of 0, we can measure the distance to X parallell to the w-axis, i.e. $d_w := \text{dist}((z_0, w_0), X \cap \{(z_0, w) \in X\} \geq \text{dist}((z_0, w_0), X) \geq \frac{1}{2} d_w$. To prove that, the left inequality is obvious. For the right inequality, observe at first that if we consider any smooth curve in X, then the total variation in w is less than ϵ times the total variation in z. It follows that the bidisc $\Delta((z_0, w_0); d_w/2)$ does not intersect X. Hence the right inequality follows. If the derivative $\frac{\partial f}{\partial w}$ vanishes, it follows from considering the image of lines parallell to the w-axis that $\text{dist}(f(p), Y) \leq c\,\text{dist}(p, X)$ for all p close enough to zero. Hence we are left with the case that $\frac{\partial f}{\partial w} \neq 0$ at the origin. The image of lines parallell with the origin are therefore nonsingular curves (near 0). Note that since the Jacobian vanishes on X, necessarily these lines must hit Y tangentially except possibly at the origin. However this implies by continuity that $\frac{\partial f_1}{\partial w} \neq 0$ and $\frac{\partial f_2}{\partial w} = 0$ at the origin, where we have written $f = (f_1, f_2)$. Next consider a line L parallell to the w-axis through (z_0, w_0). Inside L consider the straight line γ of lenght r from (z_0, w_0) to the nearest point q in $\{(z_0, w) \cap X\}$. Consider the image $f(\gamma)$. Then the horizontal length of this image is of order of magnitude r, while the vertical lenght is $o(r)$ where the little $o(r)$ refers to an expression bounded by an arbitrarily small multiple of r as the neighborhood shrinks. Also there must be a point on Y with the same z-coordinate as $f(z_0, w_0)$ with a w-coordinate differing by $o(r)$. But this shows that $\text{dist}(f(z_0, w_0), Y) \leq c\,\text{dist}((z_0, w_0), X)$ as desired.

LEMMA 7.10. *Let $f \in \mathcal{H}$, $f : \mathbf{P}^2 \to \mathbf{P}^2$. If C_0 is a critical invariant component, then except for Siegel domains all limit functions for $\{f^n\}$ in a Fatou component are in the closure of $\bigcup_{n=0}^{\infty} f^n(C)$.*

Proof of the Lemma. By the previous lemma there exists a neighborhood U of C_0 such that $f(U) \subset\subset U$. It is easy to construct enough algebraic hypersurfaces contained in U and to show that $\mathbf{P}^2 \setminus \bar{U}$ is hyperbolic.

Let Ω be a Fatou component for f, which is not a Siegel domain. If $f^{n_0}(\Omega)$ intersects U for some n_0, then all possible limits are in C_0. Otherwise $f^n(\Omega)$ stays in the hyperbolic set $\mathbf{P}^2 \setminus \bar{U}$ and the argument about Siegel domains of Theorem 1.8 applies to show that Ω is preperiodic to a Siegel domain, a contradiction.

The proof of the theorem then follows the same lines as the previous theorem.

We give here an example of a critically finite map on $\mathbf{P}^2$ where the Julia set has nonempty interior, but the Julia set is not all of $\mathbf{P}^2$. This contrasts with the one dimensional case where the Julia set is either everything or has empty interior.

Define $f : \mathbf{P}^2 \to \mathbf{P}^2$, $[z : w : t] \to [(z-2w)^2 : z^2 : t^2]$. Then the critical orbits are

$$(z = 2w) \to (z = 0) \to (w = 0) \to (z = w) \leftarrow$$

$$(t = 0) \leftarrow .$$

First, observe that f is strictly critically finite.

Note that inside $(t = 0)$ the map is of the form $z \to \frac{(z-2)^2}{z^2}$ which is strictly preperiodic. Hence the Julia set contains all of $(t = 0)$. Since $(t = 0)$ is critical, there is, (Lemma 7.9) an open neighborhood U of $(t = 0)$ such that $f^n(U) \to (t = 0)$. Hence if we restrict to any nonempty open subset of U, the iterates cannot be a normal family there, since there is no superattractive point on $(t = 0)$. However the point $[0 : 0 : 1]$ is an attractive fixed point, so has a nonempty open basin of attraction. Hence the Fatou set is nonempty as well.

One can also give a direct proof. Note that f maps the space of lines to itself. $(w = \alpha z) \to (w = (1-2\alpha)^{-2} z)$ and high iterates of an open set of lines cover the space of all lines. One shows that the Fatou set is just the basin of attraction of $[0 : 0 : 1]$. It is Kobayashi hyperbolic since if $\| (z, w) \|$ is large enough, then $f^k[z : w : 1] \to (t = 0)$. And on the dense set of periodic lines, the map is exceptional so has only two Fatou components. As in the end of the proof of Theorem 5.7 the derivatives of the iterates of f must blow up on any open set disjoint from the basin of attraction of $[0 : 0 : 1]$. Hence the Julia set has nonempty interior. However int J is not Kobayashi hyperbolic since it contains $(t = 0)$.

Note that if $[P(z, w) : Q(z, w)]$ is any rational map on $\mathbf{P}^1$ which is strictly preperiodic, critically finite, the same argument works to show that the map $[P(z, w) : Q(z, w) : t^d]$ has nonempty Fatou set and Julia set with interior, $d = \deg P, Q$. This is a class of examples where the complement of the closure of $\bigcup_{n=0}^{\infty} f^n(C)$ is not Kobayashi hyperbolic for any rational map $[P : Q]$. Indeed $[0 : 0 : 1]$ is superattractive. The action of f on $(t = 0)$ has Julia set $\mathbf{P}^1$ and f maps lines to lines. Hence the argument is as above.

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