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Contents

0 Introduction

Suppose X is a smooth and projective scheme over a perfect field k with Witt ring W. Mazur's fundamental theorem [23] establishes a striking link between the action of Frobenius and the Hodge filtration on the crystalline cohomology of X/W and suggests a close connection and analogy between F-crystals and Hodge structures. Applications of Mazur's theorem and its concomitant philosophy include Katz's conjecture on Newton polygons [*op. cit.*], a crystalline Torelli theorem for certain K3 surfaces [26], and a simple proof of the degeneration of the Hodge spectral sequence [7]. The theorem also underlies the deeper manifestations of the theory of p-adic periods, developed by Fontaine-Messing [12], Faltings [9], and Wintenberger [29].

Our main goal in this monograph is to formulate and prove a version of Mazur's theorem with coefficients in an F-crystal. In order to do this it is necessary to define and describe a "Hodge filtration" on an F-crystal and on its cohomology. This suggests our second goal, the development of a crystalline version of the notion of a complex variation of Hodge structure, which we call a "T-crystal" (the "T" is for transverse). These objects make sense on any formal scheme of finite type over W, especially for schemes smooth over W or over $W_\mu =: W/p^\mu W$ for $\mu \in \mathbf{Z}^+$. Putting together the "F" and the "T," we obtain the notion of "F-T-crystal," which we hope is a reasonable analog of a variation of Hodge structure, on any smooth complete scheme over W_μ. In particular, F-T-crystals of level one should correspond to p-divisible groups. [1] We show how to attach to a T-crystal (E', A) to

[1]This seems to follow easily from a recent result of Kato, which is essentially the case

a suitable F-crystal (E, Φ), and our formulation of Mazur's theorem relates the action of Frobenius on the cohomology of the F-crystal (E, Φ) to the corresponding Hodge filtration on the cohomology of (E', A).

This paper can be seen as a continuation of our previous study [24, 25] of Griffiths transversality and crystalline cohomology, here with emphasis on global, rather than local, results. An important feature of our approach is that we work systematically with the Hodge filtration on crystalline cohomology over W, not just its image in the De Rham cohomology over k.

Before describing the plan of our paper, it is helpful to begin by briefly reviewing the statement of Mazur's theorem. Let k be a perfect field of characteristic p and let W be its Witt ring, with F denoting the Frobenius automorphisms of k and of W. Then a nondegenerate F-crystal on k/W is a finitely generated free W-module E, together with an F-semi-linear injective endomorphism Φ. Even if k is algebraically closed, the classification of such objects is quite complicated, as is the case for Hodge structures. Now a Hodge structure can be greatly simplified by forgetting its integral lattice—one then just obtains a filtered vector space (H, Fil) determined up to isomorphism by the Hodge numbers $h^i =: \dim \operatorname{Gr}^i_{Fil} H$. Mazur's crystalline analog of this simplification is the passage from an F-crystal Φ to the associated F-span $\Phi\colon E' \to E$, in which one simply forgets that the source and target of Φ are one and the same W-module. It is easy to classify F-spans up to isomorphism. Namely, still following Mazur, we define a filtration on E by taking $M^iE' =: \Phi^{-1}(p^iE)$; it is then easy to to see that our span is determined up to isomorphism by the Hodge numbers of the filtered k-vector space $(E' \otimes k, M)$. Actually it turns out to be more convenient to work with a slightly different filtration A, given by $A^iE' =: \sum_j p^{[i-j]} M^j E'$, which in fact induces the same filtration as does M on $E' \otimes k$. This construction defines a functor $\alpha_{k/W}$ from the category of F-crystals on k/W to the category of filtered W-modules. We can now state Mazur's fundamental result [4, 8.41] in the following way: if we apply $\alpha_{k/W}$ to the canonical F-crystal structure on the crystalline cohomology of a suitable X/k, the resulting filtration A is just the Hodge filtration:

$$A^iE = H^n(X/W, J^{[i]}_{X/W}).$$

Now suppose that we have a *family* of F-crystals (E, Φ) on a smooth X/k, *i.e.*, an injective map of locally free crystals $\Phi\colon F^*_{X/W}E \to E$. Such an object is usually just called an "F-crystal on X/W," and we view it

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$\mu = 1$.

as an analog of a variation of Hodge structure. Similarly, one can view an F-span $\Phi\colon F^*_{X/W}E' \to E$ as an analog of a complex variation of Hodge structure. For each point x of X, one can perform Mazur's construction, and obtain a filtered $W(k(x))$-module $(E'(x), A(x))$. It turns out that these filtrations vary nicely in a family: they fit together to form a filtration A of the crystal E'. As in the complex case, A is not a filtration by subcrystals, but rather by sheaves in the crystalline topos, satisfying a crystalline version of Griffiths transversality. For example the filtration associated to the constant F-crystal is the filtration by the divided powers of the ideal $J_{X/W}$. We call the data (E', A) a "T-crystal," and thus we obtain a functor $\alpha_{X/W}$ from the category of F-spans on X/W to the category of T-crystals. It turns out that this functor is even an equivalence for crystals of level less than p. Now our generalization (*c.f.* (7.4.3) and (7.5.3)) of Mazur's theorem says that the functor α commutes with the formation of higher direct images, under suitable conditions.

The use of logarithmic structures in crystalline cohomology greatly increases its range of applicability, so we begin in Section 1 by reviewing and extending the theory of logarithmic crystals, due originally to Faltings [10], Fontaine and Illusie, Hyodo, and Kato [15], [20]. The main new features of our presentation are the systematic study of logarithmic differential operators and the theory of p-curvature and Cartier descent in a logarithmic context. This section may be of some foundational interest independent of the rest of the article. On the other hand, those readers who want to avoid the technical difficulties of logarithmic structures can omit it and just work with the trivial logarithmic structures throughout the rest of the paper.

The first real task in our program is the systematic study of Griffiths transversality in the crystalline setting. The advantage of this viewpoint, aside from its aesthetic appeal, is that it allows us to work in arithmetic and geometric directions simultaneously. The basic idea is the following: If (E, A) is a filtered $\mathcal{O}$-module over a ring $\mathcal{O}$ and J is an $\mathcal{O}$-ideal, we say that (E, A) is "G-transversal to J" if $JE \cap A^iE = JA^{i-1}E$ for every i. If (J, γ) is a divided power ideal ("PD-ideal"), this notion must be modified to read:

$$JE \cap A^iE = JA^{i-1}E + J^{[2]}A^{i-2}E + \cdots ;$$

we then say that (E, A) is "G-transversal to (J, γ)" or just "PD-transversal to J." Section 2 discusses this notion in detail, investigates its behavior under pullback, and establishes the technical and geometric underpinnings of our work.

We begin the study of crystals and Griffiths transversality *per se* in Section 3. Recall that if Y/W is smooth, there is an equivalence between the category of crystals of $\mathcal{O}_{Y/W}$-modules and the category $MIC(Y/W)$ of pairs (E_Y, ∇) consisting of a quasi-coherent sheaf E_Y of $\mathcal{O}_Y$-modules endowed with an integrable and p-adically nilpotent connection ∇. Consider the category of triples (E_Y, ∇, A_Y), where (E_Y, ∇) is an object of $MIC(Y/W)$ and A_Y is a filtration on E_Y which is Griffiths transversal to ∇. We shall see that this category is equivalent to the category of pairs (E, A), where E is as before a crystal of $\mathcal{O}_{Y/W}$-modules and A is a filtration of E by subsheaves in the crystalline site which is PD-transversal to the PD-ideal $J_{Y/W}$ of $\mathcal{O}_{Y/W}$. It is then easy to give a natural generalization of this condition for arbitrary schemes X/W (for example, for smooth schemes over k); we thus construct the category of T-crystals on X/W.

Section 4 develops the language and techniques that we shall use to interpolate the various filtrations that arise in our work on Mazur's theorem. For example, if (K, A, B) is a bifiltered object then for any subset σ of $\mathbf{Z} \times \mathbf{Z}$, we obtain a subobject $K_\sigma =: \sum\{A^i \cap B^j : (i,j) \in \sigma\}$. This defines a filtration of K indexed by the lattice of subsets of $\mathbf{Z} \times \mathbf{Z}$, and the correspondence $\sigma \mapsto K_\sigma$ is compatible with the lattice structure—a fact which plays a key technical role in our proofs. There is also a close connection between this lattice and the lattice of gauges ("1-gauges" in our terminology) considered by Mazur in [23]. After slightly modifying his notion of a "tame gauge structure," we discover a close connection between such structures and G-transversality. The section ends with a discussion of the cohomology of tame gauge structures, generalizing and simplifying the results of §2 and §3 of [23] and §8 of [4].

Section 5 prepares the way for our formulation of the generalization of Mazur's theorem. Suppose for simplicity that X is smooth over a perfect field k (and we are working with trivial log structures.) Instead of studying F-crystals, it is more natural and general to work with F-spans, *i.e.* p-isogenies $\Phi: F_X^* E' \to E$ in the category of crystals on X/W (*c.f.* (5.2.1)). We find a close connection between F-spans and T-crystals. Namely, we construct a functor $\alpha_{X/W}$ from the category of F-spans to the category of T-crystals, interpolating Mazur's construction of the filtration M on E' when X is a point. For spans of small level (or "width," *c.f.* (5.1.1)), this functor even turns out to be an equivalence of categories. We then introduce the notion of an "F-T-crystal" on a smooth lifting Y of X to W_μ; this is an F-crystal on X/W together with a lifting of its associated T-crystal to Y/W. The section ends with a discussion of the relationship between such F-T-crystals and the category MF^∇ of Fontaine-modules, including a simple proof of Faltings' structure theorem for Fontaine-modules.

Section 6 discusses the cohomology of T-crystals. It includes a filtered Poincaré lemma for T-crystals and some technical preparations that allow us to study bifiltered complexes and the associated hypercohomology spectral sequences. In particular, we show that T-crystals can often be "pushed forward." Thus if $f: X \to Y$ is a smooth proper morphism of smooth W_μ-schemes and if (E', A) is a T-crystal on X/W, the crystalline cohomology sheaves $R^q f_* E'$ inherit a T-crystal structure, under suitable hypotheses, *c.f.* (6.3.2). We take care to describe as carefully as possible the behavior of the Hodge filtration even when the dimension is large compared to p.

Section 7 is devoted to the formulation and proof of our analog of Mazur's theorem. The main formulation (7.3.1) of this theorem takes place on the level of complexes. We prove it by an unscrewing procedure based on the interpolation techniques of Section 4 until we are essentially reduced to a filtered version of the Cartier isomorphism. On the level of cohomology, our theorem asserts (7.4.3) that, with suitable hypotheses, the functor $\alpha_{X/W}$ commutes with higher direct images. This result allows us to show in (7.5.3) that (with suitable hypothesis), the higher direct images of F-T-crystals again form F-T-crystals. As this manuscript was nearing completion, I learned with great interest that Kazuya Kato [18] is working on a theory (cohomology of F-gauges), which is closely related to our treatment of Mazur's theorem, but uses a different point of view. (The original definition of F-gauges is due to Ekedahl, [8], and is inspired by work of Fontaine, Lafaille, Nygaard, and of course Mazur.)

Section 8 contains examples and applications of our theory. It begins with a very cursory discussion of liftings of T-crystals in mixed characteristic, leading to generalizations of the decomposition theorems of Deligne and Illusie [7] as well as vanishing theorems of Kodaira-type, all with coefficients in the Hodge complexes associated to an F-T-crystal. We also give a slight refinement (8.2.2) of a result of Faltings [9, IVb], which shows that the Hodge spectral sequence and torsion in crystalline cohomology are well-behaved, provided that the prime p is large compared to the dimension of the space and the width of the crystal. Next we discuss Hodge and Newton polygons associated to F-spans and F-crystals, and in particular establish a form of Katz's conjecture with coefficients in an F-crystal (fulfilling, at least partially, a hope expressed in [1]). One application of our use of logarithmic structures is the link we find between the mixed Hodge structure of a smooth variety in characteristic zero and the Newton polygon of its reduction modulo a suitable prime p (8.3.7). Finally we work out what our theory says about the cohomology of symmetric powers of F-T crystals on curves, with an eye toward the theory of modular forms (compare work of Ulmer [28]).

I wish at this point to express my gratitude to Luc Illusie for sending me an early version of his manuscript [17], and to Pierre Deligne for a letter [5] about the degeneration of the Hodge spectral sequence (with constant coefficients) and its application to the lifted form of Katz's conjecture. I also want to thank the N.S.A., the C.N.R.S., and the Universities of Rennes and Orsay for their support and hospitality. My conversations with the équipes of both universities were a source of many ideas and great pleasure. Finally, special thanks are due to the referee, who did a truly heroic job on both the local and global levels.

1 Logarithmic structures and crystals

In this section we discuss some of the foundational aspects of logarithmic crystals and crystalline cohomology. Although we cannot give a complete treatment of the foundations here, we shall review the basic notions for the convenience of the reader. If $(X, \mathcal{O}_X)$ is a scheme, a "prelogarithmic structure" on X is a pair (M_X, α_X), where M_X is a sheaf of commutative monoids on X and α_X is a morphism from M_X into the multiplicative monoid of $\mathcal{O}_X$. If t is a local section of $\mathcal{O}_X$, then $\alpha_X^{-1}(t)$ can be thought of as the set (possibly empty) of local logarithms of t defined by the prelogarithmic structure. Therefore we shall write the monoid law of M_X additively. From now on, "monoid" will mean "commutative monoid." We write $M \to M^{gp}$ for the universal map from a monoid M into a group, and recall that M is called "integral" if this map is injective. We often write M^* for the group of invertible elements of M.

If (M_X, α_X) is a prelogarithmic structure on a scheme $(X, \mathcal{O}_X)$, we say that $(X, \mathcal{O}_X, M_X, \alpha_X)$ is a "prelogarithmic scheme," and sometimes write $X^\times$, or even just X, for this entire set of data. A morphism of prelogarithmic schemes $f^\times : X^\times \to Y^\times$ is a morphism f of the underlying schemes, together with a morphism $f_M^{\times *} : f^{-1}M_Y \to M_X$ such that the obvious square commutes. We will usually just write f^* instead of $f_M^{\times *}$.

Kato has observed that it is desirable to declare that each unit should have a unique logarithm. Thus, he defines a logarithmic structure as a prelogarithmic structure for which the map

$$\alpha_{X|\alpha^{-1}(\mathcal{O}_X^*)} \colon \quad \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$$

is an isomorphism. Kato shows that any prelogarithmic structure α maps to a corresponding universal logarithmic structure, called the logarithmic structure associated to α. For example, if A is a ring and $\alpha: M \to A$ is a map from M to the multiplicative monoid of A, then there is an associated logarithmic scheme $\mathrm{Spec}(A, M, \alpha)$. If in addition M is finitely generated and integral, $\mathrm{Spec}(A, M, \alpha)$ is called "fine and affine," and a fine logarithmic scheme is a scheme with an open cover (étale or Zariski, depending on one's taste) each member of which is isomorphic to a fine and affine logarithmic scheme. Without explicit mention to the contrary, all logarithmic schemes considered here will be fine. Fiber products exist in the category of (fine) logarithmic schemes (but the constructions may be different in the different categories). For a fundamental example, view the polynomial algebra $\mathbf{Z}[t_1, \ldots t_n]$ as the monoid algebra associated to the monoid $\mathbf{N}^n$. The natural map $\mathbf{N}^n \to \mathbf{Z}[t_1, \ldots t_n]$ sending $(i_1, \ldots i_n)$ to $t_1^{i_1} \cdots t_n^{i_n}$ is a prelogarithmic structure on $\mathbf{Z}[t_1, \ldots t_n]$, and the associated logarithmic scheme is called "logarithmic affine n-space $\mathbf{A}^{n\times}$."

If (X, M_X) is a logarithmic scheme, the natural map $M_X^* \to \mathcal{O}_X^*$ is an isomorphism, and hence there is an injective morphism of monoids $\lambda: \mathcal{O}_X^* \to M_X$. Let $\overline{M}_X$ be the sheaf of orbits of the action of $\mathcal{O}_X^*$ on M_X, together with its induced monoid structure. Then we have a canonical "exact sequence":

$$0 \longrightarrow \mathcal{O}_X^* \xrightarrow{\lambda} M_X \longrightarrow \overline{M}_X \longrightarrow 0 \tag{1.0.0.1}$$

If $\alpha: M \to \mathcal{O}_X$ is a logarithmic structure and m is a section of M, then one sees immediately that m is invertible if and only if $\alpha(m)$ is. Furthermore, if $f^\times: X^\times \to S^\times$ is a morphism of logarithmic schemes sending a point x of X to $s \in S$, then a section m of $M_{S,s}$ is invertible if and only if $f^*(m)$ is invertible in $M_{X,x}$. (This follows from the preceding statement and the fact the the map $f^*: \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is a local homomorphism.) We have to refer to Kato's paper [20] for discussions of the more subtle notions of exact and integral morphisms.

If there is more than one prelogarithmic structure we want to consider, we may write X^* for another prelogarithmic scheme with the same underlying scheme, and then we may find it necessary to write $M_{X^\times}$ or M_{X^*}. We hope the context will make our notational abuses acceptably clear.

1.1 Logarithmic crystals and differential operators

We shall begin by reviewing and expanding on Kato's discussion of logarithmic calculus. In this section we work with schemes annihilated by a power of

p. Alternatively one could use p-adic formal schemes; in this case all tensor products are taken to be completed tensor products.

A morphism of logarithmic schemes $i^\times: X^\times \to T^\times$ is said to be a "closed immersion" if the underlying morphisms of schemes is a closed immersion and the map $i^*: i^{-1}M_T \to M_X$ is surjective, and a closed immersion is "exact" if the induced map $i^{-1}\overline{M}_T \to \overline{M}_X$ is an isomorphism. If this is the case and if J is the kernel of $i^*: \mathcal{O}_T \to i_*\mathcal{O}_X$, then we have an exact sequence of sheaves of monoids:

$$0 \longrightarrow i^{-1}(1+J) \xrightarrow{\lambda} i^{-1}M_T \longrightarrow M_X \longrightarrow 0 \qquad (1.1.0.1)$$

where $1+J$ is the kernel of the map $\mathcal{O}_T{}^* \to \mathcal{O}_X{}^*$.

Now suppose that $i^\times: X^\times \to T^\times$ is an exact closed immersion over $S^\times$ and suppose we are given two $S^\times$-morphisms $g_1^\times$ and $g_2^\times : T^\times \to Y^\times$ such that $g_1^\times \circ i^\times = g_2^\times \circ i^\times = f^\times$. Then for each section a of $\mathcal{O}_Y$, we let $\tilde{D}(a) =: i^{-1}g_2^* a - i^{-1}g_1^* a \in i^{-1}J$, and we have the familiar equations:

$$\tilde{D}(ab) = g_1^*(a)\tilde{D}(b) + g_1^*(b)\tilde{D}(a) + \tilde{D}(a)\tilde{D}(b).$$

Thus, projecting to J/J^2 we obtain a derivation $D: \mathcal{O}_Y \to g_*J/J^2$. Similarly, for each section m of M_Y, we can consider the two sections $g_i^*(m)$ of M_T. As these have the same image in M_X, we see from the exactness of (1.1.0.1) that there is a unique section $\mu(m)$ of $i^{-1}(1+J)$ such that

$$i^{-1}g_2^*(m) = i^{-1}g_1^*(m) + \lambda(\mu(m)).$$

We shall sometimes write $g_2^*(m) - g_1^*(m)$ for $\lambda(\mu(m))$. Note that μ is a homomorphism of monoids $M_Y \to f_* i^{-1}(1+J)$. Let $\tilde{\delta}(m) =: \mu(m) - 1$, a section of J. If m' is another section of M_T, we have the equation

$$\tilde{\delta}(m+m') = \tilde{\delta}(m) + \tilde{\delta}(m') + \tilde{\delta}(m)\tilde{\delta}(m'),$$

and projecting to J/J^2 we find an additive map $\delta: M_Y \to g_*J/J^2$. One sees immediately that

$$\tilde{D}(\alpha(m)) = g_1^*(\alpha(m))\tilde{\delta}(m), \qquad \text{hence} \quad D(\alpha(m)) = \alpha(m)\delta(m),$$

which is compatible with our view of m as a logarithm of $\alpha(m)$. For future reference, we summarize these definitions and formulas as follows:

1.1.1 Formulas: *If g_1 and g_2 are two log morphisms $T^\times \to Y^\times$ which agree modulo the ideal J of an exact closed immersion $i: X^\times \to Y^\times$, then there are* *monoid morphisms*

$$\begin{aligned} \tilde{D}: f^{-1}\mathcal{O}_Y &\longrightarrow i^{-1}J \\ \mu: f^{-1}M_Y &\longrightarrow 1+i^{-1}J \subseteq i^{-1}\mathcal{O}_T^* \end{aligned}$$

If we let $\tilde{\delta}(m) =: \mu(m) - 1 \in i^{-1}J$, *then for* $a \in \mathcal{O}_Y$ *and* $m \in M_Y$,

$$\begin{aligned} i^{-1}g_2^*a &= i^{-1}g_1^*a + \tilde{D}(a) \\ \tilde{D}(aa') &= g_1^*(a)\tilde{D}(a') + g_1^*(a')\tilde{D}(a) + \tilde{D}(a)\tilde{D}(a') \\ g_2^*(m) &= g_1^*(m) + \lambda(\mu(m)) \\ \tilde{\delta}(m) &= \mu(m) - 1 \\ \tilde{\delta}(m+m') &= \tilde{\delta}(m) + \tilde{\delta}(m') + \tilde{\delta}(m)\tilde{\delta}(m') \\ \tilde{D}(\alpha(m)) &= g_1^*(\alpha(m))\tilde{\delta}(m) \end{aligned}$$

These formulas justify the following definition:

1.1.2 Definition: *Suppose* (X, M_X, α_X) *is a prelogarithmic scheme and* E *is a sheaf of* $\mathcal{O}_X$*-modules. Then a "logarithmic derivation of* $(\mathcal{O}_X, M_X)$ *with values in* E*" is a pair* $\partial =: (D, \delta)$, *where* $D: \mathcal{O}_X \to E$ *is a derivation and* $\delta: M_X \to E$ *is a monoid homomorphism, such that* $D(\alpha(m)) = \alpha(m)\delta(m)$ *for each* $m \in M_X$. *If* $(X, M_X, \alpha_X) \to (S, M_S, \alpha_S)$ *is a morphism, then* ∂ *is a "derivation relative to* (S, M_S, α_S)*" if* $Df^*(b) = 0 = \delta f^*(n)$ *for every section* b *of* $\mathcal{O}_S$ *and* n *of* M_S.

If $\partial = (D, \delta)$ is a logarithmic derivation, we usually just write ∂a and ∂m for Da and δm, respectively. It is clear that there is a universal logarithmic derivation $d: (\mathcal{O}_X, M_X) \to \Omega^1_{X^\times/S^\times}$ relative to $S^\times$; $\Omega^1_{X^\times/S^\times}$ is the sheaf of "relative logarithmic Kahler differentials". If u is a local section of $\mathcal{O}_X^*$, then it follows from our definitions that $d\lambda(u) = u^{-1}du$ *i.e.* $\operatorname{dlog} u$. A logarithmic derivation (D, δ) on a *logarithmic* scheme is determined uniquely by δ, but not by D in general.

Logarithmic smoothness and étaleness are defined using Grothendieck's infinitesimal lifting properties for nilpotent exact closed immersions. We warn the reader that such morphisms can be quite complicated, and refer to Kato's paper for some important results about them. In particular, if $X^\times/S^\times$ is smooth $\Omega^1_{X^\times/S^\times}$ is locally free. We say that a sequence $(m_1, \ldots m_n)$ of sections of M_X is a "system of logarithmic coordinates" if and only if $(dm_1, \ldots dm_n)$ form a basis for $\Omega^1_{X^\times/S^\times}$. Since $\Omega^1_{X^\times/S^\times}$ is locally generated by the image of M, it is clear that such systems always exist locally on X, if $X^\times/S^\times$ is smooth. A sequence $(m_1, \ldots m_n)$ of sections of M_X determines an $S^\times$-morphism from $X^\times$ to logarithmic affine n-space $\mathbf{A}^{n\times}_{S^\times}$ over $S^\times$, and it follows from [20, 3.12] that this morphism is logarithmically étale if and only if $(m_1, \ldots m_n)$ is a system of logarithmic coordinates.

If $i^\times: X^\times \to T^\times$ is a locally closed immersion of schemes with fine log structure, Kato has shown in [20, 4.10] that, locally on X, there is a factorization $i^\times = u^\times \circ i'^\times$, where $u^\times$ is log étale and $i'^\times: X^\times \to T'^\times$ is an exact closed immersion. This process usually involves some blowing up in T. The formal completion of $X^\times$ in along $i'^\times$ is independent of the choices and exists globally, and is called the "exact formal completion" of $T^\times$ along $X^\times$. The divided power envelope of the ideal of $i'^\times$ in $T'^\times$ along $i'^\times$ also exists globally, and is just called the divided power envelope of $X^\times$ in $T^\times$ [20, 5.4]. In particular, if J is the ideal of the exact formal diagonal, then one has a canonical isomorphism $J/J^2 \cong \Omega^1_{X^\times/S^\times}$.

Let $\Omega^i_{X^\times/S^\times}$ be the i^{th} exterior power of $\Omega^1_{X^\times/S^\times}$. Then there is a unique way to define operators

$$d^i: \Omega^i_{X^\times/S^\times} \longrightarrow \Omega^{i+1}_{X^\times/S^\times}$$

such that $d^2 = 0$ (in particular $d(dm) = 0$ if $m \in M$) and $d(\alpha \wedge \beta) = (d\alpha)\beta + (-1)^{\deg \alpha}\alpha(d\beta)$ for all α and β. The resulting complex is called the (logarithmic) De Rham complex of $X^\times/S^\times$. There is a also the notion of a logarithmic connection $\nabla: E \to E \otimes \Omega^1_{X^\times/S^\times}$, which is defined just as in the classical case. However, in order to really understand the relationship of connections to crystals, it is best to study differential operators of higher order. We give only a sketch, following the methods of [4].

Suppose that $g: Y^\times \to S^\times$ is a morphism of logarithmic schemes, with $i: X^\times \to Y^\times$ a closed immersion. Let $D^\times_{Y/S}(n)$ be the logarithmic divided power envelope of $X^\times$ embedded in $(Y^\times)^{n+1}$ via the diagonal map. Let $D^\times =: D^\times_{Y/S}(0)$, and for any quasi-coherent sheaf E on Y, we write $D^\times(1) \otimes E$ for $p_{1*}p_2^*E$.

1.1.3 Definition: *If E and F are quasi-coherent sheaves of $\mathcal{O}^\times_D$-modules, an "HPD-differential operator from E to F" is an $\mathcal{O}_{D^\times}$-linear map*

$$\phi: D^\times(1) \otimes E \to F;$$

ϕ has order less than or equal to n if it annihilates $J^{[n+1]}$, where J is the divided power ideal of X in $D^\times_{Y/S}(n)$. For any such operator ϕ, $\phi^\flat: M_Y \times E \to F$

is the map defined by

$$\begin{aligned} \phi^\flat(m,e) &=: \phi(\mu(m)(p_2^*(e)), \quad \text{and} \\ \phi^\flat(e) &=: \phi^\flat(0,e) = \phi(p_2^* e), \end{aligned}$$

where $\mu(m)$ *is given as in (1.1.1).*

If ψ is an HPD differential operator $\mathcal{O}_X \to \mathcal{O}_X$, we shall write $\psi(m)$ for $\psi(m,1)$. Since the map $\phi \mapsto \phi^\flat$ is not injective, composition of HPD differential operators has to be defined "formally," *i.e.* geometrically, as described in [4, 4.6]. We have a natural isomorphism:

$$D^\times(2) \cong D^\times(1) \times_{D^\times} D^\times(1),$$

and we let Δ^* be the composition:

$$\Delta^*\colon \mathcal{O}_{D^\times(1)} \xrightarrow{p_{13}^*} \mathcal{O}_{D^\times(2)} \cong \mathcal{O}_{D^\times(1)} \otimes_{\mathcal{O}_{D^\times}} \mathcal{O}_{D^\times(1)}$$

Then if ψ is an HPD-differential operator from F to G, we define

$$\psi \circ \phi =: \psi \circ (id_{D^\times(1)} \otimes \phi) \circ (\Delta^* \otimes id_E).$$

1.1.4 Lemma: *If* ϕ *and* ψ *are HPD differential operators and if* m *is a local section of* M_Y *and* e *is a local section of* E*,*

$$\phi^\flat(m', \alpha(m)e) = \alpha(m)\phi^\flat(m+m', e), \quad \text{and}$$

$$(\psi\phi)^\flat(m,e) = \psi^\flat(m, \phi^\flat(m,e)).$$

Proof: For the first formula, we just calculate:

$$\begin{aligned} \alpha(m)\phi^\flat(m+m',e) &= \alpha(m)\phi\big[\mu(m+m')p_2^*(e)\big] \\ &= \alpha(m)\phi\big[\mu(m)\mu(m')p_2^*(e)\big] \\ &= \phi\big[p_1^*[\alpha(m)]\mu(m)\mu(m')p_2^*(e)\big] \\ &= \phi\big[p_2^*[\alpha(m)]\mu(m')p_2^*(e)\big] \\ &= \phi\big[\mu(m')p_2^*(\alpha(m)e)\big] \\ &= \phi^\flat(m', \alpha(m)e) \end{aligned}$$

To compute the formula for $\psi^\flat \circ \phi^\flat$, we first need to check that for any section m of M_Y,

$$\Delta^*(\mu(m)) = (\mu(m) \otimes \mathbf{1})(\mathbf{1} \otimes \mu(m)) = \mu(m) \otimes \mu(m) \qquad (1.1.4.2)$$

To prove this cocycle condition, observe that the three projection maps $p_i: D^\times(2) \to D^\times$ all agree on the strict diagonal, which is an exact closed subscheme of $D^\times(2)$. Thus for each $m \in M_Y$ and each pair (i, j), there is a unique $\mu_{ij}(m) \in \mathcal{O}^*_{D^\times(2)}$ such that $\lambda(\mu_{ij}(m)) + p_i^* m = p_j^* m$. Consequently

$$\lambda(\mu_{ij}(m)) + \lambda(\mu_{jk}(m)) = \lambda(\mu_{ik}(m)) \qquad \text{and}$$

$$\mu_{ij}(m)\mu_{jk}(m) = \mu_{ik}(m)$$

Furthermore, $p_{ij}^*\mu(m) = \mu_{ij}(m)$, since

$$\lambda(p_{ij}^*\mu(m))p_i^*(m) = p_{ij}^*(\lambda(\mu(m))p_1^*(m)) = p_{ij}^*(p_2^*(m)) = p_j^*(m).$$

It follows that $p_{13}^*(\mu(m)) = p_{12}^*(\mu(m))p_{23}^*(\mu(m))$, and (1.1.4.2) follows immediately.

Let us write $\phi_m^\flat(e)$ for $\phi^\flat(m, e)$, μ_m for $\mu(m)$, and $p_{2,m}(e)$ for $\mu_m p_2^*(e)$. Then there is a diagram:

$$\begin{array}{ccccccc}
E & \xrightarrow{p_{2,m}^*} & D^\times(1) \otimes E & \xrightarrow{\phi} & F & & \\
\big\downarrow{\scriptstyle p_{2m}^*} & & \big\downarrow{\scriptstyle (\cdot 1 \otimes \mu_m) \otimes p_2^*} & & \big\downarrow{\scriptstyle p_{2,m}^*} & \searrow{\scriptstyle \psi_m^\flat} & \\
D^\times(1) \otimes E & \xrightarrow{\Delta^* \otimes id_E} & D^\times(1) \otimes D^\times(1) \otimes E & \xrightarrow{id_{D^\times(1)} \otimes \phi} & D^\times(1) \otimes F & \xrightarrow{\psi} & G
\end{array}$$

The square on the left commutes because of (1.1.4.2). The counterclockwise circuit from E to G is $(\psi\phi)_m$, and the clockwise circuit is $\psi_m \phi_m$. ∎

When $Y^\times/S^\times$ is (logarithmically) smooth we can calculate explicitly using Kato's local description [20, 6.5] of $D^\times(n)$, which we now recall. Locally on Y we can choose a set of logarithmic coordinates $(m_1, \ldots m_n)$ for $g^\times$. Let $t_i =: \alpha(m_i)$, $u_i =: \mu_(m_i)$, and $\eta_i =: u_i - 1$. Then η_i belongs to the the ideal of the diagonal and in fact $(\eta_1, \ldots \eta_n)$ forms a set of PD-generators for this ideal. We have the equation

$$p_2^*(t_i) = u_i p_1^*(t_i) = p_1^*(t_i)(\eta_i + 1). \tag{1.1.4.3}$$

Now using p_1^* as structure morphism, one finds an isomorphism of PD-algebras over $\mathcal{O}_{D^\times}$

$$\mathcal{O}_{D^\times(1)} \cong \mathcal{O}_{D^\times}\langle \eta_1, \ldots \; \eta_n \rangle.$$

Thus the set of PD-monomials $\eta^{[I]}$ is a topological basis for $\mathcal{O}_{D^\times_{Y/S}(1)}$ as $\mathcal{O}_{D^\times}$-module (via p_1^*). Let ∂_I be the $\mathcal{O}_{D^\times}$-linear map such that $\partial_I(\eta^{[J]}) = \delta_{IJ}$, a PD-differential operator of order $|I|$. We shall see that ∂_I acts like the differential operator $t_I\partial^I/\partial t^I$. If $N = (n_1, \ldots n_n)$ and $J = (j_1, \ldots j_n)$ are multi-indices, we write $J!$ for $\prod_i (j_i!)$ and $\binom{n}{j}$ for $\prod_i \binom{n_i}{j_i}$, where as usual

$$\binom{n_i}{j_i} =: \frac{n_i(n_i-1)\cdots(n_i-j_i+1)}{j_i(j_i-1)\cdots 1}.$$

We also say $J \leq N$ if and only if $j_i \leq n_i$ for all i.

1.1.5 Lemma: *With the notations above, we have the following formulas:*

$$\begin{aligned}
\partial^\flat_J(\sum_j n_j m_j, 1) &= J!\binom{N}{J} \\
\partial^\flat_J(t^N) &= J!\binom{N}{J} t^N \\
\partial_J &= \prod_{0 \leq J' < J} (\partial_i - j'_i)
\end{aligned}$$

Proof: We calculate

$$\begin{aligned}
\partial^\flat_J(\sum_i n_i m_i, 1) &= \partial_J(\sum_i \mu(n_i m_i)) \\
&= \partial_J(\prod_i u_i^{n_i}) \\
&= \partial_J(\prod_i (1+\eta_i)^{n_i}) \\
&= \partial_J(\prod_i \sum_{r_i} r_i! \binom{n_i}{r_i} \eta_i^{[r_i]} \\
&= \partial_J \sum_{0 \leq R \leq N} R! \binom{N}{R} \eta_i^{[r_i]}
\end{aligned}$$

The first formula follows immediately, and the second is a consequence: apply (1.1.4.1) with $m = \sum_i n_i m_i$, $e = 1$, and $m' = 0$.

For the composition formula, recall that we can identify $D(2)^\times$ with the divided power envelope of X in $Y(1) \times_Y Y(1)$. We write $\mathbf{1}$ for the identity element of $\mathcal{O}_{D(1)^\times}$. Since $u_i = \mu_(m_i)$, it follows from (1.1.4.2) that $\Delta^*(u_i) = u_i \otimes u_i$. Since $\eta_i = u_i - 1$, we find

$$\begin{aligned}
\Delta^*(\eta_i) = \Delta^*(u_i) - \mathbf{1} &= u_i \otimes u_i - \mathbf{1} \\
&= (\eta_i + \mathbf{1}) \otimes (\eta_i + \mathbf{1}) - \mathbf{1} \\
&= \eta_i \otimes \eta_i + \eta_i \otimes \mathbf{1} + \mathbf{1} \otimes \eta_i
\end{aligned}$$

Now we can compute the divided powers:

$$\begin{aligned}\Delta^*(\eta_i^{[n]}) &= \sum_{a+b+c=n} (\eta_i \otimes \eta_i)^{[c]}(\eta_i \otimes \mathbf{1})^{[a]}(\mathbf{1} \otimes \eta_i)^{[b]} \\ &= \sum_{a+b+c=n} \eta_i^{[a]}\eta_i^{[c]} \otimes \eta_i^{[b]}\eta_i^{[c]} \\ &= \sum_{a+b+c=n} \binom{a+c}{c}\binom{b+c}{c}\eta_i^{[a+c]} \otimes \eta_i^{[b+c]}\end{aligned}$$

The same equation holds with multi-indices. Let ϵ_i be the multi-index with a 1 in the i^{th} place and zeroes elsewhere, so that $\partial_{\epsilon_i} = \partial_i$. Then we find

$$(id \otimes \partial_{\epsilon_i})\Delta^*(\eta^N) = \eta^{[N-\epsilon_i]} + n_i\eta^{[N]}$$

It follows from this that

$$\partial_J\partial_{\epsilon_i} = \partial_{J+\epsilon_i} + j_i\partial_J$$

for any multi-index J. Then (1.1.5.3) follows easily by induction. ∎

Here is the relationship between logarithmic derivations and differential operators.

1.1.6 Lemma: *To each logarithmic derivation $\partial\colon \mathcal{O}_X \to E$ there is a unique logarithmic differential operator $\partial^\sharp$ of order ≤ 1 such that $\partial^{\sharp\flat}(m) = \partial(m)$ for all $m \in M_X$. Furthermore,*

1. *For any sections m of M_X and a of $\mathcal{O}_X$ we have*

$$\partial^{\sharp\flat}(m,a) = \partial(a) + a\partial(m).$$

2. *If $\psi\colon F \to \mathcal{O}_X$ is an HPD differential operator and ∂ is a logarithmic derivation, then for $x \in F$ and $m \in M_X$,*

$$(\partial^\sharp\psi)^\flat(m,x) = \partial(\psi^\flat(m,x)) + \partial(m)\psi^\flat(m,x).$$

Proof: Let $i\colon X \to D^\times(1)$ be the inclusion (via the diagonal) and let J be the corresponding ideal of $D^\times(1)$. For any $a \in D^\times(1)$, let $\tilde{\phi}(a) =: a - p_1^*i^*(a)$. Then $\tilde{\phi}\colon D^\times(1) \to J$ is an HPD differential operator, and its composition ϕ with the projection to $J/J^{[2]}$ has order less than or equal to 1. It is immediate that $\phi^\flat(b) = Db$ and $\phi^\flat(m,1) = \delta(m)$ for $b \in \mathcal{O}_X$ and $m \in M_X$, in the notation of (1.1.1). Thus $\phi^\flat$ is a logarithmic derivation $\mathcal{O}_X \to J/J^{[2]}$, and it is clear that it is the universal one. This gives us the identification $J/J^{[2]} \cong \Omega^1_{X^\times/S^\times}$ and allows us to identify d with $\phi^\flat$. Now any logarithmic derivation $\partial =: (D,\delta)$ is the composition of d with a linear map $\tilde{\partial}\colon \Omega^1_{X^\times/S^\times} \to E$. Then $\partial^\sharp =: \tilde{\partial}\circ\phi$ is evidently the unique differential operator $\partial^\sharp$ of order less than or equal to 1 such that $\partial^{\sharp\flat}(m,1) = \partial(m)$; furthermore $\partial(a) = \partial^{\sharp\flat}(a)$ for $a \in \mathcal{O}_X, m \in M_X$. Moreover, because $\partial^\sharp$ has order less than or equal to 1 it annihilates $J^{[2]}$, and hence $\partial^\sharp\tilde{\delta}(m)p_2^*(a) = \partial^\sharp\tilde{\delta}(m)p_1^*(a)$. It follows that

$$\begin{aligned}
\partial^{\sharp\flat}(m,a) &= \partial^\sharp(p_2^*a + \tilde{\delta}(m)p_2^*a)\\
&= \partial^{\sharp\flat}(a) + \partial^\sharp(\tilde{\delta}(m)p_1^*a)\\
&= \partial^{\sharp\flat}(a) + a\partial^{\sharp\flat}(\tilde{\delta}(m),1)\\
&= \partial(a) + a\partial(m)
\end{aligned}$$

■

This proves the first of the formulas above; the second follows by substituting $a = \psi^\flat(m,x)$ and using (1.1.4). ■

1.1.7 Proposition: *The sheaf $T_{X^\times/S^\times}$ of logarithmic derivations with values in $\mathcal{O}_X$ relative to $S^\times$ becomes a Lie algebra over $f^{-1}\mathcal{O}_S$ with bracket operation defined by*

$$[(D_1,\delta_1),(D_2,\delta_2)] =: ([D_1,D_2], D_1\delta_2 - D_2\delta_1). \tag{1.1.7.4}$$

If ∂_1 and ∂_2 are two logarithmic derivations and if $\partial_1^\sharp$ and $\partial_2^\sharp$ are the corresponding differential operators (1.1.6), then $[\partial_1,\partial_2]^\sharp = [\partial_1^\sharp,\partial_2^\sharp]$, where the latter is computed in the associative algebra of HPD differential operators.

Proof: It is not difficult to check directly that (1.1.7.4) defines a logarithmic derivation. Furthermore, if we let $\partial_i = (D_i,\delta_i)$, then we can compute from (1.1.6.2) that for any $m \in M_X$

$$(\partial_1^\sharp\partial_2^\sharp - \partial_2^\sharp\partial_1^\sharp)^\flat(m) = D_1\delta_2(m) - D_2\delta_1(m),$$

and for any section a of $\mathcal{O}_X$,

$$(\partial_1^\sharp\partial_2^\sharp - \partial_2^\sharp\partial_1^\sharp)^\flat(a) = D_1D_2(a) - D_2D_1(a).$$

This proves that the formula for the Lie bracket is compatible with the forming commutators in the algebra of HPD differential operators. ■

If $\omega \in \Omega^1_{X^\times/S^\times}$ and ∂_1,∂_2 are logarithmic derivations, it is easy to verify that

$$\langle d\omega, \partial_1\wedge\partial_2\rangle = \partial_1\langle\omega,\partial_2\rangle - \partial_2\langle\omega,\partial_1\rangle - \langle\omega,[\partial_1,\partial_2]\rangle. \tag{1.1.7.5}$$

The proof is the same as in the classical case: although neither the right nor the left side of the purported equality is a linear function of ω, one computes easily that the difference between them is. As $\Omega^1_{X^\times/S^\times}$ is generated by elements of the form dm with $m \in M_X$, it therefore suffices to check that the formula is true when $\omega = dm$. The left side is zero for such dm, and the right side is

$$\begin{aligned}\partial_1\langle dm, \partial_2\rangle - \partial_2\langle dm, \partial_1\rangle - \langle dm, [\partial_1, \partial_2]\rangle &= \\ D_1(\delta_2(m)) - D_2(\delta_1(m)) - [\delta_1, \delta_2](m),\end{aligned}$$

which vanishes by (1.1.7.4).

A logarithmic connection $\nabla\colon E \to E \otimes \Omega^1_{Y^\times/S^\times}$ prolongs uniquely to maps

$$\nabla^i\colon E \otimes \Omega^i_{Y^\times/S^\times} \to E \otimes \Omega^{i+1}_{Y^\times/S^\times}$$

such that $\nabla^i(e \otimes \omega) = \nabla(e) \wedge \omega + e \otimes d\omega$; the connection is said to be "integrable" if $\nabla^2 \circ \nabla^1 = 0$. Using (1.1.7.5) one sees easily that a connection ∇ is integrable if and only if ∇ is compatible with bracket, *i.e.* if and only if $\nabla_{[\partial_1,\partial_2]} = [\nabla_{\partial_1}, \nabla_{\partial_2}]$ for any two logarithmic derivations ∂_1 and ∂_2.

An integrable connection on a p-adic formal scheme is said to be "locally quasi-nilpotent" if for every local system of logarithmic coordinates and every local section e of E, $\nabla(\partial_J)e$ tends to zero as $|J|$ tends to infinity. (Later we shall see that this condition is in fact independent of the coordinate system.) One can define the notion of a PD-stratification and an HPD stratification just as in the classical case [4, 4.3,4.3H]

Theorem 1.1.8 (Kato) *If E is a quasi-coherent sheaf on $D^\times_X(Y/S)$ the following sets of data are equivalent:*

1. *An integrable (resp. integrable and quasi-nilpotent) logarithmic connection*

$$\nabla\colon E \to \Omega^1_{Y^\times/S^\times} \otimes E$$

   such that that $\nabla(fe) = df \otimes e + f\nabla e$ for local sections f of $\mathcal{O}_D$ and e of E.

2. *A PD stratification (resp. HPD stratification) $\epsilon\colon p_2^*E \to p_1^*E$.*

If $(m_1, \ldots m_n)$ is a logarithmic system of coordinates and $\eta_i =: \tilde{\delta}(m_i)$ (1.1.1), then ϵ is given explicitly by the formula

$$\epsilon(p_2^*e) = \sum_J \eta^{[J]} p_1^*(\nabla(\partial_J)e) =: \sum_J \eta^{[J]} p_1^*\Big(\prod_{J'<J}\big(\nabla(\partial_i) - j_i'\big)e\Big). \qquad (1.1.8.6)$$

Proof: The method is basically the same as in [4, 4.8,4.12]. One has to show that ∇ prolongs uniquely to a homomorphism ρ from the ring of PD-differential operators $D^\times \to D^\times$ to the ring of PD-differential operators $E \to E$. This can be checked locally, with the aid of the explicit formulas (1.1.5): $\{\partial_J\}$ is a basis for the algebra of of operators $D^\times \to D^\times$ as a left $D^\times$-module, and the only possibility is

$$\rho(\partial_J) =: \prod_{0 \leq J' < J} \left(\nabla(\partial_i) - j'_i\right).$$

One then must show that ρ extends to a homomorphism of operator algebras. There is one tricky point, which is to show that the differential operators $\rho(\partial_i)$ commute. The fact that ∇ is integrable just tells us that the corresponding endomorphisms $\rho(\partial_i)^\flat$ of E commute, and to conclude that the same is true in the ring of differential operators one must show that *a priori* the commutator $[\rho(\partial_i), \rho(\partial_j]$ has order less than or equal to one. This is the logarithmic version of [4, 4.9], and can be proved in much the same way. We know that the ideal J is generated as a PD ideal by the set of elements of the form $\delta(m)$, where m ranges over local sections of M_Y. Then it is easy to prove from (1.1.4.2) that the image of $J^{[2]}$ under the map

$$D^\times(1)/J^{[3]} \longrightarrow D^\times(1)/J^{[2]} \otimes D^\times(1)/J^{[2]}$$

induced by Δ^* is generated by the set of elements of the form $\delta(m) \otimes \delta(m)$. Then one can check just as in *op. cit.* that for any connection ∇ on E and any two differential operators ϕ and ψ: $\mathcal{O}_Y \to \mathcal{O}_Y$ of order less than or equal to one, $[\nabla(\phi), \nabla(\psi)]$ is again a differential operator of order less than or equal to one. ∎

If $S^\times$ is a fine logarithmic scheme endowed with a PD -ideal (I, γ) and if $X^\times$ is a fine logarithmic scheme over $S^\times$ to which γ extends, Kato explains in [20, 5.2], how to define the logarithmic crystalline site: one considers all exact closed PD-immersions $(U^\times, T^\times, \delta)$ of open subsets of $X^\times$ over $(S^\times, \gamma)$; the nilpotent crystalline site is defined similarly. Then quasi-coherent logarithmic crystals on the nilpotent site can be interpreted in terms of modules with logarithmic connection on logarithmic PD-envelopes as above, and crystals on the full site correspond to modules with nilpotent connection. Cohomology in this topos can be calculated with the aid of a filtered Poincaré lemma, just as in the classical case.

1.2 p-Curvature and the Cartier operator

Our first task is to show that the module of tangent vector fields on a logarithmic scheme in characteristic p has a natural structure of a restricted Lie algebra. I am grateful to H. Lenstra, G. Hochschild, and G. Bergman for some enlightening discussions concerning the identities necessary to prove this fact.

1.2.1 Proposition: *Suppose that $f\colon X^\times/S^\times$ is a morphism of logarithmic schemes in characteristic p. Then the sheaf $T_{X^\times/S^\times}$ of logarithmic derivations with values in $\mathcal{O}_X$ relative to $S^\times$ becomes a restricted Lie algebra over $f^{-1}\mathcal{O}_S$, with a p-linear mapping $\partial \mapsto \partial^{(p)}$ defined by the formula*

$$(D,\delta)^{(p)} = (D^p, F_X^* \circ \delta + D^{p-1} \circ \delta) \tag{1.2.1.1}$$

Furthermore, if $a \in \mathcal{O}_X$ and $\partial \in T_{X^\times/S^\times}$, we have the formula [2]

$$(a\partial)^{(p)} = a^p \partial^{(p)} - a\partial^{p-1}(a^{p-1})\partial \tag{1.2.1.2}$$

Proof: It is well-known that the p^{th} iterate D^p of D is an (ordinary) derivation; we must show that there is a corresponding monoid morphism

$$\delta^{(p)}\colon M_X \to \mathcal{O}_X \quad \text{such that} \quad D^p(\alpha(m)) = \alpha(m)\delta^{(p)}(m)$$

for $m \in M_X$. To do this, consider the corresponding differential operator $\phi =: \partial^\sharp$. If $m \in M_X$, we have, by the first of the formulas in (1.1.4), $D^p(\alpha(m)) = \alpha(m)\phi^{p\flat}(m)$. Let $\delta^{(p)}(m) =: \phi^{p\flat}(m)$. If we can prove that $\delta^{(p)}$ is a monoid morphism, it will follow that $\partial^{(p)} =: (D^p, \delta^{(p)})$ is again a logarithmic derivation. Furthermore, the axioms for a restricted lie algebra, as well as (1.2.1.2), will hold, by the general formula of Hochschild [14, Lemma 1].

The additivity of $\delta^{(p)}$, as well as (1.2.1.1)), will follow from the explicit formula

$$(\phi^p)^\flat = F_X^* \circ \delta + D^{p-1} \circ \delta,$$

of which there are several proofs. The most elementary is due to H. Lenstra; this is the one we choose to present.

Let $\mathcal{E}_n$ denote the set of partitions of the set $\{1 \ldots n\}$. Then I claim that for each positive integer n we have

$$(\phi^n)^\flat(m) = \sum_{\epsilon \in \mathcal{E}_n} \prod_{c \in \epsilon} D^{\#c-1}\partial(m). \tag{1.2.1.3}$$

[2] This formula is attributed to Deligne in [21, 5.2.3], where it appears with an incorrect sign; I believe it is originally due to Hochschild [14, p. 481].

This formula is clear for $n = 1$, and we proceed by induction on n. First let us observe that if $\delta \in \mathcal{E}_{n+1}$, then there is a unique member δ^* of δ which contains the element $n+1$. Let $c' =: \delta^* \backslash \{n+1\}$ and $\delta' =: \delta \backslash \{c'\}$. Then define

$$\overline{\delta} =: \begin{cases} \delta', & \text{if } \#\delta^* = 1; \\ \delta' \cup \{c'\} & \text{otherwise.} \end{cases}$$

Thus $\delta \mapsto \overline{\delta}$ defines a surjection $\mathcal{E}_{n+1} \to \mathcal{E}_n$; the fiber of an element ϵ of $\mathcal{E}_n$ becomes identified with the set of elements of ϵ, together with one additional element ϵ^*.

We can now proceed with the induction step in the proof of (1.2.1.3). Let us write b for $\phi^\flat(m) = \partial(m)$. From (1.1.6) and the induction hypothesis, we obtain

$$\begin{aligned}
(\phi^{n+1})^\flat(m) &= D(\phi^n)^\flat(m) + b(\phi^n)^\flat(m) \\
&= D \sum_{\epsilon \in \mathcal{E}_n} \prod_{c \in \epsilon} D^{\#c-1} b + b \sum_{\epsilon \in \mathcal{E}_n} \prod_{c \in \epsilon} D^{\#c-1} b \\
&= \sum_{\epsilon \in \mathcal{E}_n} \sum_{c \in \epsilon} D^{\#c} b \prod_{c' \in \epsilon \backslash \{c\}} D^{\#c'-1} b + b \sum_{\epsilon \in \mathcal{E}_n} \prod_{c \in \epsilon} D^{\#c-1} b \\
&= \sum_{\epsilon \in \mathcal{E}_n} \sum_{\delta : \overline{\delta} = \epsilon} \prod_{c \in \delta} D^{\#c-1} b \\
&= \sum_{\delta \in \mathcal{E}_{n+1}} \prod_{c \in \delta} D^{\#c-1} b
\end{aligned}$$

Now if $\epsilon \in \mathcal{E}_n$ has r elements $\{c_1 \ldots c_r\}$, with $\#c_1 \geq \#c_2 \ldots$, let $I(\epsilon) =: (\#c_1 - 1, \#c_2 - 1, \ldots)$. For each multi-index I, we let $c(I) =: \#\{\epsilon \in \mathcal{E}_n : I(\epsilon) = I\}$. Then the formula above can be rewritten

$$(\phi^n)^\flat(m) = \sum_I c(I) \prod_j D^{I_j} b.$$

The symmetric group S_n operators on $\{1, \ldots n\}$ and hence on $\mathcal{E}_n$; the orbits of this action are precisely the fibers of the function c. Let the cyclic group $\mathbf{Z}/n\mathbf{Z}$ act by sending its generator to the cycle $(1, \ldots n)$; it is clear that the only elements of $\mathcal{E}_n$ fixed under this action are the two trivial partitions, with n elements and with 1 element, respectively. In particular, if $n = p$ is prime, all the other orbits have cardinality divisible by p. Thus modulo p our formula reduces to

$$\delta^{(p)}(m) =: (\phi^p)^\flat(m) = b^p + D^{p-1}(b) = \delta(m)^p + D^{p-1}\delta(m),$$

as desired.

1.2.2 Remark: If $X^\times/S^\times$ is smooth and $(m_1,\ldots m_n)$ is a system of logarithmic coordinates, let $(\partial_1,\ldots\partial_n)$ denote the corresponding basis for $T_{X^\times/S^\times}$. Then I claim that that $\partial_i^{(p)} = \partial_i$ for all i. Let ∂_{ϵ_i} also denote the differential operator $\partial_i^\sharp$, as in (1.1.6). (Here ϵ_i is the multi-index with 1 in the i^{th} place and zeroes elsewhere, consistent with the notation we used in (1.1.5).) Since the ring of HPD differential operators is killed by p, $\partial_{\epsilon_i}^p - \partial_{\epsilon_i} = \prod_{j=0}^{p-1}(\partial_{\epsilon_i} - j)$. By (1.1.5), this is $\partial_{p\epsilon_i}$. It is clear from (1.1.5.2) that $\partial_{p\epsilon_i}^\flat(m_j) = 0$ for all i, and it follows that $\partial_i^{(p)}(m_j) - \partial_i(m_j) = 0$ for all j; proving the claim.

Now we can define the p-curvature of a sheaf (E,∇) with integrable logarithmic connection, following the usual method. Namely for each logarithmic derivation ∂ of $\mathcal{O}_X$ into $\mathcal{O}_X$, we consider

$$\psi(\partial) =: \nabla(\partial)^p - \nabla(\partial^{(p)}).$$

It follows from the Jacobson identity and the formula (1.2.1.2) for $(f\partial)^{(p)}$ that ψ is a p-linear map from the set of logarithmic derivations of $X^\times$ into the set of horizontal endomorphisms of (E,∇).

Suppose that $Y^\times/S^\times$ is smooth, with S a p-adic formal scheme, and let (E,∇) be a quasi-coherent $\mathcal{O}_Y$-module with integrable logarithmic connection. It is now easy to see that (E,∇) is quasi-nilpotent (1.1.8) if and only if the p-curvature ψ of its reduction mod p is quasi-nilpotent, in the following sense: for every local logarithmic derivation ∂ and every local section e of E/pE on a quasi-compact set, $\psi(\partial)^n e = 0$ for n large enough.

1.2.3 Remark: A logarithmic scheme $X^\times$ in characteristic p has a canonical Frobenius endomorphism: the underlying map of schemes is just the usual absolute Frobenius, and the morphism on the sheaf of monoids is multiplication by p. Kato has given a delicate analysis of the relative Frobenius map of a morphism $X^\times/S^\times$ of fine log schemes in characteristic p, necessary for understanding the Cartier isomorphism. He begins by forming the usual fiber product X^1 of $X^\times$ with $X^\times$ over $S^\times$, in the category of log schemes. In general, this log scheme may not be integral, and the fiber product $X^\sharp$ computed in the category of fine log schemes is a closed subscheme of X^1. (If the morphism $X^\times \to S^\times$ is integral [20, 4.3], however, then $X^\sharp \cong X^1$.) It is still true, however, that the pullback of $\Omega^1_{X^\times/S^\times}$ to $X^\sharp$ is $\Omega^1_{X^\sharp/S^\times}$. Furthermore [20, 4.10], the map $X^\times \to X^\sharp$ induced by Frobenius can be factored canonically $X^\times \to X' \to X^\sharp$, where $X^\times \to X'$ is exact [20, 4.6] and weakly purely inseparable [20, 4.9] and $X' \to X^\sharp$ is log étale; we shall call the map

$$F_{X^\times/S^\times}: X^\times \to X' \qquad (1.2.3.4)$$

the "exact relative Frobenius morphism" and shall denote the map $X' \to X$ by $\pi_{X^\times/S^\times}$. In particular, the differentials of the complex $\Omega^{\cdot}_{X^\times/S^\times}$ are $\mathcal{O}_{X'}$-linear, and it is still true that $\Omega^i_{X'/S^\times} \cong \pi^*\Omega^i_{X^\times/S^\times}$. Notice that if $X^\times \to S^\times$ is integral, then the map $X' \to S^\times$ is logarithmically smooth.

If the map $g: X' \to X^\sharp$ is an isomorphism and if $X^\times/S^\times$ is an integral morphism of log schemes [20, 4.3], then the map $X^\times \to S^\times$ is said to be a morphism of "Cartier type." We shall be primarily interested in morphisms which are of Cartier type and log smooth; we call such morphisms "perfectly smooth" for the sake of brevity. For example, suppose that $f: X \to S_0$ is a smooth morphism of schemes with trivial log structure and $Z \subset X$ is a relative smooth divisor with normal crossings. Then the monoid of sections of $\mathcal{O}_X$ which are invertible outside Z defines a fine logarithmic structure on X, and the associated logarithmic map to S is perfectly smooth. If S_0 is locally factorial and $D \subseteq S_0$ is a divisor, then the same construction for $D \subseteq S_0$ defines a fine log structure on S_0; if also X is locally factorial, we also get a fine log structure associated to $Z \cup f^{-1}(D) \subseteq X$, and a perfectly smooth morphism of fine log schemes $X^\times \to S^\times$. In fact X/S_0 need not be smooth: it is enough for it to have semi-stable reduction along D.

Warning: even for perfectly smooth morphisms, the relative Frobenius map need not be flat. For example, if P is any integral saturated monoid, the spectrum of the monoid algebra $k[P]$, endowed with the log structure associated to the inclusion $P \to k[P]$ is perfectly smooth over $\operatorname{Spec} k$ with the trivial structure. The Frobenius morphism is not flat unless $k[P]$ is regular.

Now we can copy the proof of [24, 2.11] to obtain:

1.2.4 Proposition: *Let (E, ∇) be a sheaf of $\mathcal{O}_X$-modules with integrable connection on a fine log scheme $X^\times/S^\times$ in characteristic p, and let $F: X^\times \to X'$ be the exact relative Frobenius morphism. Then there is a unique $\mathcal{O}_{X'}$-linear map (the Cartier operator):*

$$C: F_*\underline{Z}^1(E, \nabla) \longrightarrow \Omega^1_{X'/S^\times} \otimes F_*E$$

such that for any logarithmic derivation ∂ on $X^\times$ and any section ω of $\underline{Z}^1(E, \nabla)$,

$$\langle C(\omega), \pi^*\partial \rangle = \langle \omega, \partial^{(p)} \rangle - \nabla_\partial^{p-1} \langle \omega, \partial \rangle. \qquad (1.2.4.5)$$

Furthermore, if ∂' is a logarithmic derivation on X', and if e is any local section of E, we have

$$\nabla_\partial \langle C(\omega), \partial' \rangle = -\psi_{\partial'} \langle \partial, \omega \rangle \qquad (1.2.4.6)$$

$$\langle C(\nabla e), \partial' \rangle = -\psi_{\partial'}(e) \tag{1.2.4.7}$$

∎

If the p-curvature ψ of (E, ∇) vanishes, then the last two equations of (1.2.4) imply that we have a commutative diagram

$$\begin{array}{ccc} F_*\underline{Z}(E,\nabla) & \xrightarrow{C} & F_*E \otimes \Omega^1_{X'/S^\times} \\ \downarrow & & \uparrow \\ F_*\underline{H}^1(E,\nabla) & \xrightarrow{\overline{C}} & F_*\underline{H}^0(E,\nabla) \otimes \Omega^1_{X'/S^\times} \end{array}$$

In the case of the constant connection $(\mathcal{O}_X, d)$, Kato has defined the inverse Cartier isomorphism [20, 4.12]. This is a canonical isomorphism

$$C^{-1}: \Omega^q_{X'/S^\times} \to \underline{H}^q(\Omega^{\cdot}_{X^\times/S^\times}), \tag{1.2.4.8}$$

characterized by the fact that

$$C^{-1}(d\pi^* m_1 \wedge \cdots d\pi^* m_q) = [m_1 \wedge \cdots m_q] \tag{1.2.4.9}$$

for any sequence of sections $(m_1, \ldots m_q)$ of M_X.

To justify our notation, we should verify that Kato's C^{-1} is indeed the inverse of our mapping $\overline{C}$ in the diagram above. It will suffice to check that $\overline{C}C^{-1}(d\pi^* m) = \pi^* dm$ for every local section m of M_X, and for this it suffices to verify that they have the same contraction with $\pi^*\partial$ for every logarithmic derivation $\partial = (D, \delta)$ on $X^\times/S^\times$. Using (1.2.4.9) and (1.2.1.1) we compute:

$$\begin{aligned} F^*\langle \overline{C}C^{-1} d\pi^* m, \pi^*\partial \rangle &= \langle C dm, \pi^*\partial \rangle \\ &= \langle dm, \partial^{(p)} \rangle - \partial^{p-1}\langle dm, \partial \rangle \\ &= \langle dm, F_X^* \circ \delta + D^{p-1} \circ \delta \rangle - \partial^{p-1}\langle dm, \partial \rangle \\ &= (\delta(m))^p + D^{p-1}(\delta(m) - D^{p-1}(\delta(m)) \\ &= (\delta(m))^p \\ &= F^*\langle \pi^* dm, \pi^*\partial \rangle \end{aligned}$$

This proves the formula. ∎

We shall need a slightly more precise form of Kato's theorem.

1.2.5 Theorem: *Suppose that $X^\times \to S^\times$ is a smooth morphism of logarithmic schemes in characteristic p, with exact relative Frobenius morphism $F =: F_{X^\times/S^\times}: X^\times \to X'$, and suppose that E' is a quasi-coherent sheaf of $\mathcal{O}_{X'}$-modules.*

1. *The sheaf $E'' =: F^*(E')$ has a canonical integrable connection ∇'', whose p-curvature vanishes. Furthermore, one has a canonical Cartier isomorphism*
$$E' \otimes \Omega^i_{X'/S^\times} \cong F_*\underline{H}^i(E'', \nabla'').$$

2. *Suppose that A is a quasi-coherent filtration on E' and let M denote the filtration of E'' induced by A. Then we have canonical isomorphisms:*

$$\begin{aligned} C^{-1}\colon A^iE' \otimes \Omega^q_{X'/S} &\cong F_*\underline{H}^q(M^iE'', \nabla'') \\ C^{-1}\colon \mathrm{Gr}^i_A E' \otimes \Omega^q_{X'/S} &\cong F_*\underline{H}^q(\mathrm{Gr}^i_M E'', \nabla'') \end{aligned}$$

Proof: The differentials of the complex $F_*\Omega^{\cdot}_{X^\times/S^\times}$ are $\mathcal{O}_{X'}$-linear, and we can form the complex of $\mathcal{O}'_X$-modules: $E' \otimes F_*\Omega^{\cdot}_{X^\times/S^\times}$. But

$$E' \otimes F_*\Omega^q_{X^\times/S^\times} \cong F_*E'' \otimes \Omega^q_{X^\times/S^\times},$$

and we can identify this complex with the De Rham complex of an integrable connection $\nabla'' =: id_{E'} \otimes d$ on E''. It is clear that the p-curvature ∇'' vanishes. To prove the second statement, it will be enough to prove that we have canonical isomorphisms

$$E' \otimes F_*\underline{H}^q(\Omega^{\cdot}_{X^\times/S^\times}) \cong \underline{H}^q(E' \otimes F_*\Omega^{\cdot}_{X^\times/S^\times})$$

for all E', compatibly with filtrations. Some care is required because the map F is not flat, and the maps $F^*A^iE' \to M^iE''$ are not injective, in general. It is clear that we may work locally on $X^\times$—*e.g.* étale locally (in the classical sense).

Let $\oplus_q\Omega^q_{X'/S}[-q]$ be the complex consisting of $\Omega^q_{X'/S}$ in degree q but with zeroes as boundary maps. In fact, Kato's proof shows that, locally in the classical étale topology, C^{-1} comes from a homotopy equivalence of complexes of $\mathcal{O}_{X'}$-modules

$$\oplus_q\Omega^q_{X'/S}[-q] \longrightarrow F_*\Omega^{\cdot}_{X^\times/S^\times}$$

This immediately implies that formation of its cohomology is compatible with any base change, and with filtrations in the sense of the proposition. ∎

We shall also need to use Mazur's formula for the Cartier isomorphism, which relates C^{-1} to local liftings of F^*. We shall be using these local liftings extensively, so it is worthwhile to formalize the definition.

1.2.6 Definition: *Let $S^\times$ be a formal W-scheme with the p-adic topology, equipped with a fine logarithmic structure. A "lifted situation over $S^\times$" is a a logarithmically smooth and integral $X^\times/S_0$, together with a lifting*

$$F_{Y^\times/S^\times}: Y^\times \to Y'$$

of the exact relative Frobenius morphism of $X^\times/S_0^\times$. The lifted situation is "parallelizable" if there exist systems of logarithmic coordinates $(m_1, \dots m_n)$ for $Y^\times/S^\times$ and $(m'_1, \dots m'_n)$ for $Y'^\times/S^\times$ such that $F^*_{Y^\times/S^\times}(m'_i) = pm_i$ for all i.

Notice that the relative Frobenius morphisms considered above are $\mathcal{O}_S$-linear, and in particular we do not assume that the absolute Frobenius endomorphism of S_0 lifts. Furthermore, because $X^\times/S^\times$ is integral, Y/S and Y'/S are flat [20, 4.5].

Let us check that such liftings exist locally. Let Y' and $Y^\times$ be liftings of X' and $X^\times$, respectively, and let $(m_1, \dots m_n)$ be a system of logarithmic coordinates for $Y^\times/S^\times$. These induce logarithmic coordinates $(\overline{m}_1, \dots \overline{m}_n)$ for $X^\times/S^\times$ and $(\overline{m}'_1, \dots \overline{m}'_n)$ for $X'/S^\times$, and $F^*_X(m'_i) = pm_i$. Lift $(\overline{m}'_1, \dots \overline{m}'_n)$ to a system of logarithmic coordinates $(m'_1, \dots m'_n)$ for $Y'/S^\times$. We get a diagram,

$$\begin{array}{ccc} Y^\times & & Y' \\ \downarrow{\scriptstyle h^\times} & & \downarrow{\scriptstyle h'} \\ \mathbf{A}^{n\times}_{S^\times} & \xrightarrow{g} & \mathbf{A}^{n\times}_{S^\times} \end{array}$$

where $h^\times$ and h' are the étale maps defined by the coordinate systems and g is the map sending m'_i to pm_i. Modulo p this square is filled in by the Frobenius map $X^\times \to X'$, and by the infinitesimal lifting property for étale maps, we can fill in the square above as well.

1.2.7 Lemma: *Let $\mathcal{Y}^\times =: (Y^\times/S^\times, F)$ be a lifted situation. Then*

$$dF^*: \Omega^1_{Y'/S^\times} \to \Omega^1_{Y^\times/S^\times}$$

is divisible by p, and $p^{-1}dF^*$ when reduced modulo p defines an injective map

$$\eta^1_X: \Omega^1_{X'/S^\times} \to F_* Z^1_{X^\times/S^\times}.$$

The composite of this map with the natural projection is the Cartier isomorphism (1.2.4.8).

Proof: It follows from the exactness of $F_{X^\times/S^\times}$ that $M^{gp}_{X'}/M^{gp}_{S_0} \cong M^{gp}_{X}/M^{gp}_{S_0}$, and with this identification the Frobenius map $M^{gp}_{X'}/M^{gp}_{S_0} \cong M^{gp}_{X}/M^{gp}_{S_0}$ becomes multiplication by p. In particular, for any local section m' of $M^{gp}_{Y'}/M^{gp}_{S}$, F^*m' can be written as $pm + a$, where $a \in \operatorname{Ker}\left(M^{gp}_{Y} \to M^{gp}_{X}\right)$. Since $X \to Y$ is an exact closed immersion, this kernel is just the set of sections of $\mathcal{O}^*_Y$ which are congruent to one modulo p, so we can write $a = \lambda(1+pb)$. Then

$$c =: (b - pb^2/2 + p^2b^3/3 - p^3b^4/4 + \cdots) \in \mathcal{O}_Y,$$

and in fact $da = d\log(1+pb) = pdc$. Hence

$$dF^*(dm') = d(F^*m') = d(pm) + da = pdm + pdc$$

This shows that the dF^* is divisible by p and that $p^{-1}dF^*$ takes dm' to $dm + dc$, which maps to the class of $dm = C^{-1}(dm')$ in cohomology, by (1.2.4.9). The injectivity of our map is a consequence of the fact that C^{-1} is an isomorphism. ∎

1.3 Residues and Cartier descent

If X/S is a morphism of smooth schemes with trivial log structure in characteristic p, then the "classical" theory of Cartier descent gives an equivalence between the category of quasi-coherent sheaves of $\mathcal{O}_{X'}$-modules and the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules equipped with an integrable and p-integrable connection [21]. However in the logarithmic case, a module E with integrable and p-integrable connection ∇ does not descend, in general, to X': the canonical map $F^*\underline{H}^0(E,\nabla) \to E$ need be neither injective nor surjective. (For example, let X be the logarithmic affine line. Then the ideal $(t) \subseteq k[t]$ is stable under the standard logarithmic connection ∇ and has zero p-curvature. However, the map $F^*\underline{H}^0((t),\nabla) \to (t)$ is not surjective, and the map $F^*\underline{H}^0(k[t]/(t),\nabla) \to k[t]/(t)$ is not injective.) Similar phenomena occur over a power series ring R in characteristic zero: the map $R \otimes H^0(E,\nabla) \to E$ need be neither injective nor surjective, in general. In order for the descent to work one needs in addition conditions on residues.

If $f^\times: X^\times \to S^\times$ is a morphism of logarithmic schemes, we let X^* denote X with the logarithmic structure induced from that of S. Note that we have $\Omega^1_{X/S} \cong \Omega^1_{X^*/S^\times}$. Then f factors as $f^\times = f^* \circ u$, where $u: X^\times \to X^*$ and $f^*: X^* \to S^\times$. Thus, we have an exact sequence

$$\Omega^1_{X^*/S^\times} \to \Omega^1_{X^\times/S^\times} \to R_{X^\times/S^\times} \to 0, \tag{1.3.0.1}$$

where $R_{X^\times/S^\times} =: \Omega^1_{X^\times/X^*}$. The map $d\colon \mathcal{O}_X \to \Omega^1_{X^\times/S^\times}$ factors through $\Omega^1_{X^*/S^\times}$, and hence the composite map $\mathcal{O}_X \to R_{X^\times/S^\times}$ is zero. If x is a point of X and m is a section of $M_{X,x}$, then dm lies in $\Omega^1_{X^\times/S^\times,x}$, and we can look at its image in the $k(x)$-vector space $\Omega^1_{X^\times/S^\times}(x)$. Because the latter is a group, our map extends uniquely to $M^{gp}_{X,x}$, and because it is a $k(x)$ vector space, it extends further to a map

$$k(x) \otimes M^{gp}_{X,x} \to \Omega^1_{X^\times/S^\times}(x).$$

Similarly, we find that $d\log$ induces a map

$$k(x) \otimes \mathcal{O}^*_{X,x} \to \Omega^1_{X/S}(x).$$

These maps are compatible and fit into the commutative diagram in the next lemma.

1.3.1 Lemma: *If x is a geometric point of X and $s = f(x)$, there is a commutative diagram with surjective columns and exact rows:*

$$\begin{array}{ccccccc} k(x)\otimes \mathcal{O}^*_X & \to & k(x)\otimes M^{gp}_{X^\times,x}/M^{gp}_{S^\times,s} & \to & k(x)\otimes \overline{M}^{gp}_{X^\times,x}/\overline{M}^{gp}_{S^\times,s} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \overline{d}(x)} & & \\ \Omega^1_{X/S}(x) & \to & \Omega^1_{X^\times/S^\times}(x) & \to & R_{X^\times/S^\times}(x) & \to & 0. \end{array}$$

Furthermore, $\overline{d}(x)$ is an isomorphism.

Proof: Only the last statement requires proof. Let

$$\pi\colon M^{gp}_{X^\times,x} \to k(x) \otimes \overline{M}^{gp}_{X^\times,x}$$

be the map sending m to $1 \otimes \overline{m}$, where $\overline{m}$ is the image of m in $\overline{M}^{gp}_{X^\times,x}$. Then $\alpha(m)\pi(m) = 0$ for all m. Hence the pair $(0, \pi)$ is a logarithmic derivation and so factors through $d : M^{gp}_{X^\times,x} \to \Omega^1_{X^\times/S^\times,x}$. We get an induced map

$$R_{X^\times/S^\times}(x) \to k(x) \otimes \overline{M}^{gp}_{X^\times,x}/\overline{M}^{gp}_{S^\times,s}$$

which is inverse to the map $\overline{d}(x)$. ∎

1.3.2 Definition: *A sequence $(m_1, \ldots m_n)$ of logarithmic coordinates for $X^\times/S^\times$ is a called "strict" at x if and only if there exists an r such that $\{dm_1(x) \ldots dm_r(x)\}$ form a basis for the image of*

$$\Omega^1_{X/S}(x) \to \Omega^1_{X^\times/S^\times}(x).$$

We shall say that $f\colon X^\times \to S^\times$ is "strictly residual" at a point x if the map $\Omega^1_{X^\times/S^\times}(x) \to R_{X^\times/S^\times}(x)$ is an isomorphism.

It is clear that if $X^\times/S^\times$ is smooth and x is a point of X then there exists, in a neighborhood of x, a strict system of logarithmic coordinates for $X^\times/S^\times$. Note that any étale map is strictly residual. We have the following useful lemma.

1.3.3 Lemma: *Suppose that $f^\times: X^\times \to S^\times$ is smooth and x is a point of X. Then in a neighborhood of x there exists a factorization $f^\times = g^\times \circ h^\times$, where $h^\times: X^\times \to Z^\times$ is strictly residual and smooth at x, $Z^\times \to S^\times$ is smooth, and $Z^\times$ has the logarithmic structure induced from $S^\times$. If $f^\times$ is perfectly smooth, so are $g^\times$ and $h^\times$.*

Proof: If x is a point of X, we can find a strict system of logarithmic coordinates $(m_1, \ldots m_n)$ at x, with $(m_1, \ldots m_r)$ a basis for the image of $\Omega^1_{X/S}(x)$ in $\Omega^1_{X^\times/S^\times}(x)$. Let $Z^\times$ be affine r-space over $S^\times$, with the logarithmic structure induced from that of $S^\times$, and let $h^\times: X^\times \to Z^\times$ be the map induced by $(\alpha(m_1), \ldots \alpha(m_r))$. Then we have an exact sequence:

$$h^*\Omega^1_{Z^\times/S^\times} \to \Omega^1_{X^\times/S^\times} \to \Omega^1_{X^\times/Z^\times} \to 0,$$

and it is clear from our construction that the first map is injective and that the sequence is split in a neighborhood of x. Hence by [20, 3.12], h is smooth; of course the projection map $g: Z^\times \to S^\times$ is also smooth. Since the logarithmic structures on $Z^\times$ and on $S^\times$ are essentially the same, the statements about perfect smoothness are clear. ∎

Suppose that (E, ∇) is a sheaf of $\mathcal{O}_X$-modules with integrable logarithmic connection. Then the map ∇ induces a map

$$\rho: E \to E \otimes R_{X^\times/S^\times};$$

this map is $\mathcal{O}_X$-linear because $d: \mathcal{O}_X \to R_{X^\times/S^\times}$ vanishes; it is called the "residue map of ∇."

Now we can state the logarithmic analog of Cartier descent.

1.3.4 Theorem: *Suppose that $X^\times/S^\times$ is a smooth morphism of logarithmic schemes in characteristic p, and let $F: X^\times \to X'$ be the exact relative Frobenius morphism of $X^\times/S^\times$. If (E, ∇) is any coherent sheaf with integrable connection, then $E' =: E^\nabla$ is a coherent sheaf of $\mathcal{O}_{X'}$-modules and there is a canonical horizontal map $(F^*E', \nabla'') \to (E, \nabla)$. If the residue ρ and p-curvature of ∇ vanish, this map is surjective, and if in addition $\mathrm{Tor}_1^{\mathcal{O}_X}(E, R_{X^\times/S^\times}) = 0$, then it is bijective.*

Proof: We may argue locally in a neighborhood of a point x of X. Suppose that $(m_1, \dots m_n)$ is a set of logarithmic coordinates for $X^\times/S^\times$, with corresponding basis $(\partial_1, \dots \partial_n)$ for $T_{X^\times/S^\times}$. Let h be the HPD differential operator defined by

$$h =: \sum_I (-1)^{|I|} \frac{\partial_I}{I!},$$

where ∂_I is the differential operator defined in (1.1.5), where I ranges over all multi-indices $(I_1, \dots I_n)$ with each $0 \le I_i < p$, and where $I!$ means $\prod_i I_i!$. Let $\nabla(h)$ denote the corresponding endomorphism of E. Then it is clear that $\nabla(h)e = e$ if $\nabla e = 0$. Moreover, if the p-curvature vanishes, then I claim that $\nabla \circ \nabla(h) = 0$. To see this, recall from (1.1.5) that

$$\partial_{j\epsilon_i} = \prod_{j'=0}^{j-1} (\partial_{\epsilon_i} - j'),$$

and define, for each integer $k \in [0, p)$,

$$h_{i,k} =: \sum_{j=0}^{k} (-1)^j \frac{\partial_{j\epsilon_i}}{j!}.$$

An easy induction on k (valid in any characteristic) shows that

$$h_{i,k} = \frac{(-1)^k (\partial_{\epsilon_i} - 1)(\partial_{\epsilon_i} - 2) \cdots (\partial_{\epsilon_i} - k)}{k!},$$

and hence that

$$\partial_{\epsilon_i} h_{i,k} = \frac{(-1)^k (\partial_{\epsilon_i})(\partial_{\epsilon_i} - 1) \cdots (\partial_{\epsilon_i} - k)}{k!} = \frac{(-1)^k \partial_{(k+1)\epsilon_i}}{k!}$$

Letting $h_i =: h_{i,p-1}$, we find (in characteristic p) that

$$\partial_{\epsilon_i} h_i = \frac{\partial_{p\epsilon_i}}{(p-1)!} = -\partial_{p\epsilon_i} = \partial_{\epsilon_i} - \partial^p_{\epsilon_i}.$$

Recall from (1.2.2) that $\partial_i^{(p)} = \partial_i$. Hence if ∇ is any integrable connection, we find

$$\begin{aligned}
\nabla(\partial_i)\nabla(h_i) &= \nabla(\partial_{\epsilon_i})\nabla(h_i) \\
&= \nabla(\partial_{\epsilon_i} h_i) \\
&= \nabla(\partial_{\epsilon_i}) - \nabla(\partial^p_{\epsilon_i}) \\
&= \nabla(\partial_i) - \nabla(\partial_i)^p \\
&= \nabla(\partial_i^{(p)}) - \nabla(\partial_i)^p \\
&= -\psi(\partial_i)
\end{aligned}$$

We have

$$\begin{aligned} h &=: \sum_I \frac{(-1)^{|I|}}{I!}\partial_I \\ &= \sum_I \frac{(-1)^{|I|}}{I!}\prod_{i=1}^n \frac{(-1)^{I_i}}{I_i!}\partial_{I_i\epsilon_i} \\ &= \prod_{i=1}^n \sum_{I_i=0}^{p-1} \frac{(-1)^{I_i}}{I_i!}\partial_{I_i\epsilon_i} \\ &= \prod_{i=1}^n h_i \end{aligned}$$

Thus we see that, if the p-curvature vanishes, h is a projection operator onto the horizontal subsheaf E^∇ of E. Note also for further reference the simple formula

$$h = \prod_{i=1}^n \prod_{j_i=1}^{p-1} \frac{(\partial_i - j_i)}{(-j_i)} \tag{1.3.4.2}$$

As $h = \prod_i h_i$ and h_i commutes with ∂_j, our claim follows.

Suppose now that (E, ∇) has vanishing residue map and p-curvature. Consider first the case in which $f^\times$ is strictly residual at x, so that

$$\Omega^1_{X^\times/S^\times}(x) \cong R_{X^\times/S^\times}(x).$$

Since the residue map is zero at x, it follows that each $\nabla(\partial_i)$ maps E into the maximal ideal m_x times E. This implies that h is congruent to the identity map modulo m_x, and hence that $E^\nabla \to E(x)$ is surjective. It follows from Nakayama's lemma that the map $F^*E^\nabla \to E$ is surjective in a neighborhood of x.

Let K denote the kernel of $F^*E^\nabla \to E$, so that we have an exact sequence: $0 \to K \to F^*E^\nabla \to E \to 0$. Applying (1.2.5.1), we see that the map $(F^*E^\nabla)^\nabla \to E^\nabla$ is an isomorphism, and hence $K^\nabla = 0$. It is clear that K inherits an integrable connection from that of F^*E^∇, and its p-curvature must vanish because the p-curvature of F^*E^∇ does. If in addition $\mathrm{Tor}_1(E, R_{X^\times/S^\times}) = 0$, then it also follows that the residue map of K vanishes. But the result of the previous paragraph now applies to K, telling us that the map $F^*K^\nabla \to K$ is surjective. Then $K = 0$ and the proof is complete in this case.

To deduce the general case, we use a factorization $f^\times = g^\times \circ h^\times$ as in Lemma (1.3.3). The exact relative Frobenius map $F\colon X^\times \to X'$ for $X^\times/S^\times$

factors as $F = F' \circ F_{X^\times/Z^\times}$, where $F_{X^\times/Z^\times}: X^\times \to X'_{Z^\times}$ is the exact relative Frobenius map for $X^\times/Z^\times$ and F' is the exact relative Frobenius map for $Z^\times/S^\times$ pulled back to $X^\times_{S^\times}$. We have an exact sequence

$$0 \to h^*\Omega^1_{Z^\times/S^\times} \to \Omega^1_{X^\times/S^\times} \to \Omega^1_{X^\times/Z^\times} \to 0,$$

and we let $\nabla_{X/Z}: E \to E \otimes \Omega^1_{X^\times/Z^\times}$ be the composite of ∇ with the map $E \otimes \Omega^1_{X^\times/S^\times} \to E \otimes \Omega^1_{X^\times/Z^\times}$. Let E'' denote the kernel of $\nabla_{X/Z}$, which we view as a sheaf on $X^\times_{Z^\times}$. It is clear that the p-curvature and residues of $\nabla_{X/Z}$ vanish if those of ∇ do. Then by the strictly residual case discussed above, the map $F^*_{X^\times/Z^\times} E'' \to E$ is surjective, and bijective if the Tor vanishes. We may assume that X/Z is affine, and then we view E'' as a quasi-coherent sheaf on Z. It is clear that ∇ induces a map $\nabla_{Z/S}: E'' \to E'' \otimes \Omega^1_{Z/S}$, and this $\nabla_{Z/S}$ is an integrable connection whose p-curvature vanishes if that of ∇ does. Furthermore the kernel of $\nabla_{Z/S}$ is precisely the kernel of ∇. Hence by the standard version of Cartier descent, *c.f.* [21, 5.1], we find an isomorphism $F'^* E^\nabla \cong E''$, and hence $F^*_{X^\times/Z^\times} F'^* E^\nabla \cong F^*_{X^\times/Z^\times} E''$. Our result is now clear. ∎

1.3.5 Corollary: *Suppose that $\Sigma =: 0 \to E_1 \to E_2 \to E_3 \to 0$ is an exact sequence of quasi-coherent sheaves with integrable connection on $X^\times/S^\times$ and that the p-curvature, residue map, and $\mathrm{Tor}_1(\ , R_{X^\times/S^\times})$ of E_2 all vanish. Then if Σ is split as a sequence of $\mathcal{O}_X$-modules, it is also split as a sequence of sheaves with connection. In particular, the sequence*

$$0 \to E_1^\nabla \to E_2^\nabla \to E_3^\nabla \to 0$$

is also exact.

Proof: We begin with the last statement. Let Q denote the cokernel of the map $E_2^\nabla \to E_3^\nabla$. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} F^*E_2^\nabla & \longrightarrow & F^*E_3^\nabla & \longrightarrow & F^*Q & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow & & \\ E_2 & \longrightarrow & E_3 & \longrightarrow & 0 & & \end{array}$$

The vanishing of the residue map and p-curvature of E_2 imply their vanishing for E_3, and because Σ is split, $\mathrm{Tor}_1(R_{X^\times/S^\times}, E_3)$ vanishes also. It follows that the arrows α_2 and α_3 are both isomorphisms, and hence that $F^*Q = 0$. But the morphism F is surjective, and hence $Q = 0$.

The above argument also applies to the sequence

$$0 \longrightarrow \mathrm{Hom}(E_3, E_1) \longrightarrow \mathrm{Hom}(E_3, E_2) \longrightarrow \mathrm{Hom}(E_3, E_3) \longrightarrow 0,$$

so that the map $\mathrm{Hom}^{\nabla}(E_3, E_2) \to \mathrm{Hom}^{\nabla}(E_3, E_3)$ is surjective. Thus, there is a horizontal splitting $E_3 \to E_2$. ∎

1.3.6 Corollary: *Suppose that (E'', ∇'') is a quasi-coherent sheaf of $\mathcal{O}_X$ modules with an integrable logarithmic connection and endowed with a horizontal and finite filtration M by quasi-coherent subsheaves. Suppose that the p-curvature and residue map of ∇'' vanish, and that $\mathrm{Tor}_1(\mathrm{Gr}_M E'', R_{X^\times/S^\times}) = 0$. Then if $E' =: E''^{\nabla}$ and $A^iE' =: E' \cap M^iE''$, the natural map $F^*_{X/S}A^iE' \to M^iE''$ is an isomorphism for all i. In other words, (E'', M) descends canonically to a quasi-coherent sheaf with quasi-coherent filtration (E', A).*

Proof: By induction on i, $\mathrm{Tor}_1(E''_X/M^iE''_X, R_{X/S})$ and $\mathrm{Tor}_1(M^iE''_X, R_{X/S})$ vanish for all i. Hence by (1.3.4), E''_X is spanned by its sheaf of horizontal sections E'_X, and in fact the map $\zeta_X\colon F^*_X E'_X \to E''_X$ is an isomorphism. The subsheaf $M^kE''_X$ of E''_X is stable under ∇'' and hence its p-curvature is also zero. Because the map

$$M^kE''_X \otimes R_{X/S} \to E''_X \otimes R_{X/S}$$

is injective, we can conclude that the residue map $\rho\colon M^kE''_X \to M^kE''_X \otimes R_{X/S}$ also vanishes. Thus by Theorem (1.3.4), the map $F^*_X[(M^kE''_X)^{\nabla}] \to M^kE''_X$ is also an isomorphism. By definition, $A^kE'_{X'} = (M^kE''_X)^{\nabla} = E' \cap (M^kE'')$. ∎

The following useful corollary is of course trivial when $F_{X^\times/S^\times}$ is flat.

1.3.7 Corollary: *If Q is a quasi-coherent sheaf on X' such that $F^*_{X^\times/S^\times}Q$ is locally free of finite rank, then Q is also locally free of finite rank.*

Proof: The statement is local, so we may and shall assume that there exists an exact sequence $K \to E \to Q \to 0$, where E is locally free on X'. Let M be the image of $F^*_{X^\times/S^\times}(K)$ in $F^*_{X^\times/S^\times}E$, so that we have an exact sequence

$$0 \to M \to F^*_{X^\times/S^\times}(E) \to F^*_{X^\times/S^\times}(Q) \to 0$$

of modules with integrable connection. Because the last term is locally free, the sequence is locally split as a sequence of $\mathcal{O}_X$-modules. It follows from Corollary (1.3.5) that it is also locally split as a sequence of modules with connection. This implies that the map $E \to Q$ is also locally split, and hence that Q is locally free. The finiteness result is easy. ∎

Let us change our notation slightly, letting $S^\times$ denote a flat formal W-scheme with ideal of definition S_0 defined by p, endowed with a fine logarithmic structure, and let $X^\times/S_0^\times$ be a logarithmically smooth and integral morphism, where $X^\times$ is a fine logarithmic scheme. The exact relative Frobenius morphism $F_{X/S_0}\colon X \to X'$ is S_0-linear and induces a morphism of crystalline topoi:

$$F_{X/S_0 cris}\colon (X/S)_{cris} \to (X'/S)_{cris}.$$

At the risk of some confusion, we shall reduce the number of subscripts by just writing $F_{X/S}$ instead of $F_{X/S_0 cris}$. Notice that $F_{X/S}$ is $\mathcal{O}_S$-linear, and no lifting of the absolute Frobenius endomorphism of S_0 enters.

1.3.8 Corollary: *Suppose that E' is a crystal of p-torsion free $\mathcal{O}_{X'/S^\times}$-modules on $X'^\times/S$ and let $E'' =: F^*_{X^\times/S^\times}E'$. Then if E' is p-torsion free, the natural map $H^0(X'/S^\times, E') \to H^0(X^\times/S^\times, E'')$ is an isomorphism.*

Proof: The assertion is local in the Zariski topology of X and therefore may be proved locally, in a lifted situation (1.2.6). Let us just write E' and E'' for the values of E' and E'' on Y' and Y, respectively, and let ∇' and ∇'' be the corresponding connections. It follows from (1.2.5) that the natural map $E'_{X'} \to F_{X/S*}F^*_{X/S}E''_X$ is injective, and it is easy to see that this implies that the map $H^0(X'/S^\times, E') \to H^0(X^\times/S^\times, E'')$ is also injective.

To prove the surjectivity we shall need the following lemma.

1.3.9 Lemma: *Suppose that $\alpha \in E'' \otimes \Omega^{i-1}_{Y^\times/S^\times}$ and $\beta \in E' \otimes \Omega^i_{Y'^\times/S^\times}$ are such that $p^r d''\alpha \equiv F^*\beta \pmod{p^{r+1}}$. Then $d''\alpha \in pE'' \otimes \Omega^i_{Y^\times/S^\times}$.*

Proof: We have a commutative diagram

$$\begin{array}{ccc} E' \otimes \Omega^{i-1}_{Y'^\times/S^\times} & \xrightarrow{d'} & E' \otimes \Omega^i_{Y'^\times/S^\times} \\ \downarrow{\scriptstyle \eta^{i-1}_Y} & & \downarrow{\scriptstyle p\eta^i_Y} \\ E'' \otimes F_*\Omega^{i-1}_{Y^\times/S^\times} & \xrightarrow{d''} & E'' \otimes F_*\Omega^i_{Y^\times/S^\times} \end{array}$$

in which η^i_Y is the map induced by $p^{-i}F^*$. Our assumption says that there is a $\gamma \in \Omega^i_{Y^\times/S^\times} \otimes E''$ such that

$$p^r d''\alpha = p^i\eta^i_Y\beta + p^{r+1}\gamma$$

If $\beta = 0$ our assertion is trivial. Otherwise we may write $\beta = p^j\beta'$, where $\beta' \in \Omega^i_{Y'^\times/S^\times}$ and p does not divide β'. If $i + j > r$ our assertion is again trivial, and otherwise we may divide by p^{i+j} to obtain

$$p^{r-i-j}d''\alpha = \eta^i_Y\beta' + p^{r-i-j+1}\gamma.$$

Reducing modulo p and recalling that d'' modulo p is just $d \otimes \mathrm{id}_{E'}$, we see that C^{-1} applied to the reduction of β' modulo p is zero. This implies that β' is divisible by p, which is a contradiction. ∎

To prove (1.3.8), suppose that e'' is a horizontal section of E'', and that we have found $e'_n \in E'$ and $e''_n \in E''$ such that $e'' = \eta^0_Y(e'_n) + p^n e''_n$. Then

$$0 = F^*(\nabla'(e'_n)) + p^n\nabla''(e''_n),$$

and by Lemma (1.3.9), it follows that $\nabla''(e''_n)$ is divisible by p. Then by (1.2.5.1), we can write $e''_n = \eta^0_Y(\delta'_n) + p\delta''_{n+1}$. Then

$$e'' = \eta^0_Y(e'_n + p^n\delta'_n) + p^{n+1}e''_{n+1}.$$

It is clear that the limit e' of the Cauchy sequence (e'_n) satisfies $e'' = \eta^0_Y(e')$ and that e' is horizontal. ∎

We include the following result for the sake of completeness.

1.3.10 Proposition: *Suppose that E is a crystal of finite type locally free sheaves of $\mathcal{O}_{X^\times/S^\times}$-modules on $X^\times/S^\times$. Then there is a crystal E' of locally free $\mathcal{O}_{X'/S^\times}$-modules on the nilpotent crystalline site of $X'/S^\times$ such that $E \cong F^*_{X^\times/S^\times}E'$ if and only if, in characteristic p, the p-curvature and residue map of (E_X, ∇) vanish.*

Proof: The necessity of our condition is clear. It follows from Corollary (1.3.8) (which also works for the nilpotent site) that the functor $F^*_{X^\times/S}$ is fully faithful, so we may prove the sufficiency locally, in a lifted situation. We write E for the value of E on Y, and let ∇ be its connection. We may and shall assume that E is free, and choose a basis (e_i). Let θ be the connection matrix *i.e.*, the matrix of one-forms such that $\nabla(e_i) = \sum_j e_j \otimes \theta_{ji}$. We have to prove that we can choose the basis such that $\theta = F^*(\theta')$ for some matrix of one-forms on Y'. Theorem (1.3.4) tells us that this is true modulo p, so that we may assume that we have chosen (e_i) such that

$$\theta = F^*(\phi) + p^n\omega$$

with $n = 1$. Let us show how to modify the choice of basis so that the same equation holds with $n+1$ in place of n.

Because the connection is integrable, we have the matrix equation $d\theta = \theta \wedge \theta$, and it follows that

$$p^n d\omega \equiv F^*(\phi \wedge \phi - d\phi) \pmod{p^{n+1}}.$$

Then Lemma (1.3.9) tells us that ω is exact modulo p. Thus the reduction of ω modulo p is homologous to an element in the image of the inverse Cartier isomorphism, and so we can find matrices of forms δ on Y'/S and u and ϵ on Y/S such that $\omega = p^{-1}F^*(\delta) + du + p\epsilon$. Then we have

$$\theta = F^*(\phi + p^{n-1}\delta) + p^n du + p^{n+1}\epsilon.$$

Now let $e'_i =: e_i - p^n \sum_j e_j u_{ji}$, which is again a basis for E because $n > 0$. Because θ is divisible by p, we have

$$\nabla e'_i \equiv \sum_j e_j \otimes \theta_{ji} - p^n e_j \otimes du_{ji} \equiv \sum_j e_j \otimes \theta_{ji} - p^n e'_j \otimes du_{ji} \pmod{p^{n+1}}.$$

Thus there is a matrix ω'' such that the new connection matrix θ' is give by

$$\theta' = \theta - p^n du + p^{n+1}\omega'' = F^*(\phi + p^{n-1}\delta) + p^{n+1}(\omega'' + \epsilon).$$

It is clear that our procedure converges to a basis with desired property. Let us remark that the p-curvature of the resulting connection need not be nilpotent, and hence the corresponding crystal lives only on the nilpotent site, in general. ∎

2 Transversality and divided power ideals

2.1 First notions

In this section we study various notions of "transversality" of a filtration with respect to an ideal. We use these in the next section during our crystalline interpretation of Griffiths transversality. Usually our filtrations F will be decreasing and indexed by $\mathbf{Z}$, and we write F^∞ for $\cap_i F^i$ and $F^{-\infty}$ for $\cup_i F^i$; as usual $F[i]$ is the filtration defined by setting $F[i]^j =: F^{i+j}$. If $g\colon T' \to T$ is a morphism and (E, A) is a filtered $\mathcal{O}_T$-module, we let A_g denote the filtration on g^*E induced by A, *i.e.* $A_g^i =: \operatorname{Im}(g^*A^iE \to g^*E)$. If there is no danger of confusion we may sometimes write (g^*E, A) or even $(E_{T'}, A)$ instead of (g^*E, A_g)

2.1.1 Definition: *If (E, A) is a filtered sheaf of quasi-coherent $\mathcal{O}_T$-modules and $g\colon T' \to T$ is a morphism, we say that (E, A) is "normally transversal to g" if the maps $g^*A^kE \to g^*E$ are all injective.*

For example, if E is flat over $\mathcal{O}_T$, then (E, A) is normally transversal to g if and only if each $\operatorname{Tor}_1^{\mathcal{O}_T}(E/A^kE, \mathcal{O}_{T'})$ vanishes. It is easy to see in general that if (E, A) is normally transversal to g, then the natural map

$$g^* \operatorname{Gr}_A^k E \to \operatorname{Gr}_A^k g^*E$$

is an isomorphism. The converse is true if g^*E is separated with respect to the A_g topology—for example if $A^iE = 0$ for $i >> 0$. Of course, if the inclusions $A^kE \to E$ are, locally on T, filtering direct limits of direct factors, then (E, A) is normally transversal to every g (and conversely, by a theorem of Lazard).

To justify our terminology, we recall that two T-schemes X and Y are sometimes said to meet transversally over T if for all $i > 0$, $\mathrm{Tor}_i^{\mathcal{O}_T}(\mathcal{O}_X, \mathcal{O}_Y) = 0$. Then if Y is a subscheme of T defined by an ideal I, the I-adic filtration of $\mathcal{O}_T$ is normally transversal to $X \to T$ if X meets the normal cone to Y in T transversally over T. (We have so far had no need to consider the higher Tor's, so we have omitted them from the definition.)

If $j: T' \to T$ is a closed subscheme defined by an ideal J, we shall say that (E, A) is normally transversal to T' or to J instead of to j. The condition says in this case that $JE \cap A^k E = JA^k E$ for all k. We shall be interested in a weakened version of this condition. Namely, we shall say that A is "G′-transversal" to J if $JE \cap A^k E \subseteq JA^{k-1}E$ for every k. Note for example that the J-adic filtration on E is G′-transversal to J, but not normally transversal to J. In fact, if A is the J-adic filtration, $JE \cap A^k E = JA^{k-1}E$ for all k, in which case we say that (E, A) is "G-transversal to J."

In fact we shall need to generalize the condition of G-transversality to take into account divided powers. Suppose that T is a scheme and $\mathcal{J}$ is a filtration of $\mathcal{O}_T$ by quasi-coherent sheaves of ideals. We say that $\mathcal{J}$ is "multiplicative" if $\mathcal{O}_T = \mathcal{J}^0$ and $\mathcal{J}^i \mathcal{J}^j \subseteq \mathcal{J}^{i+j}$ for all i and j. For example if $J \subseteq \mathcal{O}_T$ is a sheaf of ideals we can consider the J-adic filtration, given by $\mathcal{J}^i =: J^i$, and if (J, γ) is a sheaf of ideals with divided power structure we can also consider the $J - PD$-adic filtration: $\mathcal{J}^i =: J^{[i]}$. In any case we just write J for $\mathcal{J}^1$.

2.1.2 Definition: *Let $\mathcal{J}$ be a quasi-coherent and multiplicative filtration of $\mathcal{O}_T$ and let (E, A) be a filtered $\mathcal{O}_T$-module. We say (E, A) is "G′-transversal to $\mathcal{J}$" if for all k*

$$JE \cap A^k E \subseteq \mathcal{J}^1 A^{k-1} E + \mathcal{J}^2 A^{k-2} E + \cdots,$$

is "saturated with respect to $\mathcal{J}$" or "$\mathcal{J}$-saturated" if $\mathcal{J}^i A^{j-i} E \subseteq A^j E$ for all i and j, and is "G-transversal to $\mathcal{J}$" if it is both G′-transversal to $\mathcal{J}$ and $\mathcal{J}$-saturated.

Thus, (E, A) is G-transversal to $\mathcal{J}$ if and only if for all k we have

$$JE \cap A^k E = \mathcal{J}^1 A^{k-1} E + \mathcal{J}^2 A^{k-2} E + \cdots. \qquad (2.1.2.1)$$

For example, it is clear that the filtered $\mathcal{O}_T$-module $(\mathcal{O}_T, \mathcal{J})$ is G-transversal to $(\mathcal{O}_T, \mathcal{J})$. If $\mathcal{J}$ is the J-adic filtration, $\mathcal{J}^i = \mathcal{J}\mathcal{J}^{i-1}$ if $i > 0$, so this definition is consistent with the simple one given above. Note that if (E, A) is $\mathcal{J}$-saturated, then $\mathrm{Gr}_A E$ is annihilated by J. Furthermore, if (E, A) is saturated with respect to $\mathcal{J}$ it is also saturated with respect to the J-adic filtration, as $J^i \subseteq \mathcal{J}^i$ for all i.

If J is the sheaf of ideals determined by a locally closed immersion of schemes $i: X \to T$, we shall also say that (E, A) is G-transversal to i or to X instead of to J when convenient.

2.1.3 Definition: *Suppose that A is a filtration on E which is G-transversal to $\mathcal{J}$, and that m and n are integers. We say that "the $\mathcal{J}$-level of (E, A) is within the interval $[m, n]$" if $A^{n+1}E \subseteq JE$ and $A^m E = E$, and we say that (E, A) has "width less than or equal to $n - m$."*

When there seems to be no danger of confusion as to which ideal $\mathcal{J}$ we are using, we just say "level" or "width" instead of "$\mathcal{J}$-level." or "$\mathcal{J}$-width."

2.1.4 Lemma: *If A is G-transversal to $\mathcal{J}$, then the following conditions are equivalent:*

1. *A has level within $(-\infty, n]$.*

2. *For all $j \geq 0$,*

$$A^{n+j}E = \mathcal{J}^j A^n E + \mathcal{J}^{j+1} A^{n-1} E + \cdots.$$

Proof: It is clear that the second condition implies the first. We check the reverse implication by induction on j, the case of $j = 0$ being trivial. Supposing that $A^{n+1}E \subseteq JE$ and that the second condition holds for j, we note that $A^{n+j+1}E \subseteq A^{n+1}E \subseteq JE$, and hence

$$A^{n+j+1}E \subseteq JE \cap A^{n+j+1}E = \mathcal{J}^1 A^{n+j}E + \mathcal{J}^2 A^{n+j-1}E + \cdots + \mathcal{J}^{j+1} A^n E + \cdots.$$

It now suffices to apply the induction hypothesis. ∎

2.1.5 Remark: When J is an invertible ideal and E is J-torsion free, the data of a filtration M on E which is G-transversal to J and of finite level has a very simple interpretation in terms of J-isogenies, see Lemma (5.1.2).

2.2 Transversality and pullbacks

Suppose that T' and T are schemes, endowed with quasi-coherent multiplicative filtrations $\mathcal{J}'$ and $\mathcal{J}$ on their respective structure sheaves. Then by a morphism $(T', \mathcal{J}') \to (T, \mathcal{J})$ we mean a morphism of schemes $g: T' \to T$ such that g^* maps $g^{-1}(\mathcal{J}^i)$ into $\mathcal{J}'^i$ for every i.

2.2.1 Lemma: *Suppose that $g: (T', \mathcal{J}') \to (T, \mathcal{J})$ is a morphism of schemes with multiplicative quasi-coherent filtrations. Let $i: X \to T$ and $i': X' \to T'$ be the inclusions defined by the ideals J and J', so that we have a commutative diagram:*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & T' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{i} & T \end{array}$$

Let (E, A) be a filtered sheaf of $\mathcal{O}_T$-modules, and suppose that (i^*E, A) is normally transversal to f.

1. *If (E, A) is normally transversal to i, then (g^*E, A_g) is normally transversal to i'.*

2. *If (E, A) is G'-transversal to $\mathcal{J}$, then (g^*E, A_g) is G'-transversal to $\mathcal{J}'$.*

Proof: Write $h =: i \circ f = g \circ i'$. Let $\mathrm{Ker}^k_A =: JE \cap A^kE/JA^kE$, so that we have an exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathrm{Ker}^k_A \to i^*A^kE \to A^ki^*E \to 0.$$

Define $\mathrm{Ker}^k_{A_g}$ in the same way, so as to obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & f^*\,\mathrm{Ker}^k_A & \longrightarrow & f^*i^*A^kE & \longrightarrow & f^*A^ki^*E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \mathrm{Ker}^k_{A_g} & \longrightarrow & i'^*A^k_gg^*E & \longrightarrow & A^kh^*E & \longrightarrow & 0 \end{array}$$

We have a surjection: $g^*A^kE \to A^kg^*E$ and hence β is also surjective. Since (i^*E, A) is normally transversal to f, the map

$$f^*A^ki^*E \to f^*i^*E \cong h^*E$$

is injective, and hence so is the map γ. It follows from the diagram that α is surjective.

It is now easy to prove our claims. It is clear that the normal transversality of (E, A) to J is equivalent to the vanishing of Ker_A^k for each k, and normal transversality of (g^*E, A_g) to i' is equivalent to the vanishing of $\mathrm{Ker}_{A_g}^k$. Since α is surjective, the former implies the latter. For the G-transversal version, let

$$Q^k(A, \mathcal{J}) =: A^kE/(\mathcal{J}^1A^{k-1}E + \mathcal{J}^2A^{k-2}E + \cdots) \cap A^kE.$$

As $Q^k(A, \mathcal{J})$ is killed by J we can regard it as as an $\mathcal{O}_X$-module. Furthermore, the projection $A^kE \to Q^k(A, \mathcal{J})$ factors through a map $\tilde{\theta}^k: i^*A^kE \to Q^k(A, \mathcal{J})$, and it is clear that the the condition of G$'$-transversality amounts to the vanishing of the restriction θ^k of $\tilde{\theta}^k$ to Ker_A^k for all k. We find a commutative diagram:

$$\begin{array}{cccc} f^*\theta^k: & f^*\mathrm{Ker}_A^k & \rightarrow & f^*Q^k(A, \mathcal{J}) \\ & \downarrow{\scriptstyle\alpha} & & \downarrow \\ \theta'^k: & \mathrm{Ker}_{A_g}^k & \rightarrow & Q^k(A_g, \mathcal{J}') \end{array}$$

Since α is surjective, the vanishing of $f^*\theta^k(A, \mathcal{J})$ implies that of $\theta'^k(A_g, \mathcal{J}')$. ∎

In particular, the lemma applies when f is flat or when (i^*E, A) is locally split. For example, if $\mathcal{J}'$ is coarser than $\mathcal{J}$ and if $J' = J$ or (i^*E, A) is locally split, then any filtration which is normally (resp. G$'$-transversal) to $\mathcal{J}$ is also normally (resp. G$'$-transversal) to $\mathcal{J}'$. Here is another example.

2.2.2 Corollary: *Suppose (E, A) is a filtered sheaf of $\mathcal{O}_X$-modules and that $i: X \to T$ admits a retraction $f: T \to X$. Then (f^*E, A_f) is normally transversal to i.*

Proof: We have a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{i} & T \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{id_X} & X \end{array}$$

It is trivial that (E, A) is normally transversal to the zero ideal, and as id_X is flat, the lemma implies that (f^*E, A_f) is normally transversal to i. ∎

2.2.3 Lemma: *Suppose that, in the situation of Lemma (2.2.1), g is faithfully flat and $\mathcal{J}' = f^*\mathcal{J}$. Then the converse statements also hold. That is, if (g^*E, A_g) is normally (respectively, G$'$or G) transversal to $\mathcal{J}'$, the same is true of (E, A) with respect to $\mathcal{J}$.*

Proof: The flatness of g implies that $g^*A^kE \cong A_g^k g^*E$ and that g^* commutes with intersections and sums. These statements make the proof straightforward. ∎

2.3 Saturation

2.3.1 Lemma: *Let (E, A) be a filtered sheaf of $\mathcal{O}_T$-modules, G'-transversal to $\mathcal{J}$, and let $(E, A)_{\mathcal{J}} =: (E, A_{\mathcal{J}})$ be the "saturation of A with respect to $\mathcal{J}$," defined by*

$$A^k_{\mathcal{J}}E =: A^kE + \mathcal{J}^1A^{k-1}E + \mathcal{J}^2A^{k-2}E + \cdots.$$

1. *$A_{\mathcal{J}}$ is G-transversal to $\mathcal{J}$ and is coarser than A. It is the finest filtration on E which is coarser than A and which is saturated with respect to $\mathcal{J}$.*
2. *$A_{\mathcal{J}}$ induces the same filtration on E/JE as does A.*

Suppose that also that $A^mE = E$ for $m << 0$. Then

3. *$A_{\mathcal{J}}$ is the coarsest filtration on E which is G'-transversal to $\mathcal{J}$, is coarser than A, and induces the same filtration on E/JE as A.*
4. *$A_{\mathcal{J}}$ is the unique filtration on E which is coarser than A, is G-transversal to $\mathcal{J}$, and induces the same filtration on E/JE as A.*

Proof: For any integers i and j,

$$\mathcal{J}^i\mathcal{J}^jA^{k-j}E \subseteq \mathcal{J}^{i+j}A^{k-j}E \subseteq A^{k+i}_{\mathcal{J}}E.$$

This implies that $\mathcal{J}^iA^k_{\mathcal{J}}E \subseteq A^{k+i}_{\mathcal{J}}E$, and hence that $A_{\mathcal{J}}$ is saturated with respect to $\mathcal{J}$. It is clearly coarser than A and is the finest $\mathcal{J}$-saturated filtration with this property. To see that it is G′-transversal to $\mathcal{J}$, suppose that $x \in JE \cap A^k_{\mathcal{J}}E$, and write $x = x' + x''$, where $x' \in A^kE$ and $x'' \in \mathcal{J}^1A^{k-1}E + \mathcal{J}^2A^{k-2}E + \cdots$. Since x and x'' belong to JE, x' belongs to

$$\begin{aligned} JE \cap A^kE &\subseteq \mathcal{J}^1A^{k-1}E + \mathcal{J}^2A^{k-2}E + \cdots \\ &\subseteq \mathcal{J}^1A^{k-1}_{\mathcal{J}}E + \mathcal{J}^2A^{k-2}_{\mathcal{J}}E + \cdots. \end{aligned}$$

Since x'' also belongs to this sum, so does x.

It is clear that $A^kE \subseteq A^k_{\mathcal{J}}E \subseteq A^kE + JE$, *i.e.* that $A_{\mathcal{J}}$ induces the same filtration on E/JE as does A. Suppose that B is another filtration with this property and which is coarser than A and G$'$-transversal to E; we shall check that $B^kE \subseteq A^k_{\mathcal{J}}E$ for all k. For $k << 0$ this is trivial, and we proceed by induction on k. Assuming the result for $k-1$, suppose that $b \in B^kE$. Since B and A induce the same filtration on E/JE, we can write $b = a + c$, where $a \in A^kE$ and $c \in JE$. Since B is coarser than A, $a \in B^kE$, and it follows that $c \in JE \cap B^kE$. Since B is G-transversal to $\mathcal{J}$, it follows that

$$c \in JE \cap B^kE \subseteq \mathcal{J}^1B^{k-1}E + \mathcal{J}^2B^{k-2}E + \cdots.$$

By the induction hypothesis we see that

$$c \in \mathcal{J}^1A^{k-1}_{\mathcal{J}}E + \mathcal{J}^2A^{k-2}_{\mathcal{J}}E \cdots \subseteq A^k_{\mathcal{J}}E.$$

Since $a \in A^kE \subseteq A^k_{\mathcal{J}}E$, we conclude that indeed $b \in A^k_{\mathcal{J}}E$. It is clear that the last statement of the lemma follows from the others. ∎

2.3.2 Corollary: *Suppose A and B are two filtrations on E, both of which are G-transversal to $\mathcal{J}$. Suppose that A and B induce the same filtration on E/JE and that A is coarser than B. Then in fact $A = B$.* ∎

2.3.3 Definition: *Suppose that (E, A) is a filtered sheaf of $\mathcal{O}_T$-modules which is G-transversal to $\mathcal{J}$, let $i: X \hookrightarrow T$ be the closed immersion defined by J, and let $f: X' \to X$ be a morphism. Then we say that "(E, A) is compatible with f" if and only if (i^*E, A) is normally transversal to f. In a situation as in Lemma (2.2.1), if (E, A) is compatible with f, it follows that the $\mathcal{J}'$-saturation of (g^*E, A) is G-transversal to $\mathcal{J}'$; we call it the "transverse pullback of (E, A)."*

If in the above situation f is the closed immersion associated to an ideal I, we may say that (E, A) is "compatible with I" instead of with f. In particular, we see from (2.2.1) and (2.3.3) that if $\mathcal{J}$ and $\mathcal{J}'$ are two multiplicative filtrations on $\mathcal{O}_T$ and if $\mathcal{J}'$ is coarser than $\mathcal{J}$ and (E, A) is G$'$-transversal to $\mathcal{J}$ and compatible with J', then $(E, A_{\mathcal{J}'})$ is G-transversal to $\mathcal{J}'$.

2.3.4 Remark: Suppose that $g': (T'', \mathcal{J}'') \to (T', \mathcal{J}')$ is another morphism, defining a commutative diagram:

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & T'' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} \\ X' & \xrightarrow{i'} & T' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{i} & T \end{array}$$

Suppose (E, A) is a filtered sheaf of $\mathcal{O}_T$-modules such that (i^*E, A) is normally transversal to f and (f^*i^*E, A) is normally transversal to f'. Then (i^*E, A) is normally transversal to $f \circ f'$. Furthermore, the $\mathcal{J}''$-transverse pull-back of (E, A) by g' is naturally isomorphic to the $\mathcal{J}''$- transverse pull back of $(f^*E, A_{\mathcal{J}'})$ by g'.

2.3.5 Remark: Suppose that m is an integer such that $\mathcal{J}^i = J^i$ for all $i < m$ and that (E, A) is G-transversal to $\mathcal{J}$, of level within $[a, \infty)$. Then for $i < m + a$ we have $A^{i-m+1}E = E$ and hence

$$\begin{aligned} JE \cap A^iE &= \mathcal{J}^1A^{i-1}E + \cdots \mathcal{J}^{m-1}A^{i-m+1}E + \mathcal{J}^mA^{i-m}E + \cdots \\ &= J^1A^{i-1}E + \cdots J^{m-1}E \end{aligned}$$

This shows that the filtration B defined by $B^iE =: A^iE$ if $i < m + a$ and $B^{m+a}E = 0$ is G$'$-transversal to the ideal J. It follows that the J-saturation B_J of B is G-transversal to J, and $B^i_JE = A^iE$ for $i < m + a$. In particular, we see that there is a simple equivalence between the data of a filtration which is G-transversal to $\mathcal{J}$ and of width less than m and that of a filtration which is G-transversal to J and of width less than m.

Suppose that $\mathcal{I}$ and $\mathcal{K}$ are two multiplicative filtrations on $\mathcal{O}_T$ and define

$$\mathcal{J}^j =: \sum\{\mathcal{I}^i\mathcal{K}^k : i + k = j\}.$$

One checks immediately that $\mathcal{J}$ is again multiplicative; we denote this filtration by $\mathcal{I}+\mathcal{K}$ and call it the "sum of $\mathcal{I}$ and $\mathcal{K}$." For example, if $J = I+K$, the J-adic filtration is the sum in this sense of the I-adic and K-adic filtrations, and similarly for PD-ideals.

2.3.6 Lemma: *Suppose that the $\mathcal{J}$ is the sum of two multiplicative filtrations $\mathcal{K}$ and $\mathcal{I}$, and suppose that A is a filtration on an $\mathcal{O}_T$-module E which is saturated with respect to $\mathcal{I}$ and G$'$-transversal to $\mathcal{K}$. Then $A_{\mathcal{J}} = A_{\mathcal{K}}$. If A is G$'$-transversal to $\mathcal{K}$, then $A_{\mathcal{J}}$ is G-transversal to $\mathcal{K}$.*

Proof: If A is G$'$-transversal to $\mathcal{K}$, $A_{\mathcal{K}}$ is G-transversal to $\mathcal{K}$, and hence the second statement follows from the former. We note that

$$\mathcal{J}^j = \mathcal{K}^j + \cdots \mathcal{K}^i\mathcal{I}^{j-i} + \cdots \mathcal{I}^j,$$

and since A is $\mathcal{I}$-saturated, $\mathcal{K}^i\mathcal{I}^{j-i}A^{k-j}E \subseteq \mathcal{K}^iA^{k-i}E$. Then

$$\mathcal{J}^jA^{k-j}E \subseteq \mathcal{K}^jA^{k-j}E + \cdots \mathcal{K}^iA^{k-i}E + \cdots A^kE \subseteq A^k_{\mathcal{K}}E.$$

We conclude that

$$A^k_{\mathcal{J}}E \quad =: \quad A^kE + \cdots \mathcal{J}^jA^{k-j}E + \cdots \subseteq A^k_{\mathcal{K}}E.$$

This shows that $A^k_{\mathcal{J}}E \subseteq A^k_{\mathcal{K}}E$; the reverse inclusion is trivial. ∎

Note that if $(E/KE, A)$ is locally split, it follows from (2.3.3) that $(E, A_{\mathcal{J}})$ is in fact G-transversal to $\mathcal{J}$ also. This shows that a filtration (E, A) can be G-transversal to more than one multiplicative filtration $\mathcal{J}$.

The following rather technical lemma will be used in our proof of the Griffiths transversality theorem (3.3.3).

2.3.7 Lemma: *Suppose that (E, B) is a filtered $\mathcal{O}_T$-module with $B^aE = E$, and that $\mathcal{I}$ and $\mathcal{K}$ are multiplicative filtrations on $\mathcal{O}_T$ such that*

1. *(E, B) is saturated with respect to $\mathcal{I}$.*
2. *$(\mathrm{Gr}_{\mathcal{K}} E, B)$ is G-transversal to $\mathcal{I}$.*
3. *$(E, \mathcal{K}E)$ is normally transversal to I and (E, B) is normally transversal to each $\mathcal{K}^i$.*

Then $(E, B_{\mathcal{K}}) = (E, B_{\mathcal{I}+\mathcal{K}})$ is G-transversal to $\mathcal{I}$. Suppose further that I' is an ideal containing I such that $(\mathrm{Gr}_{\mathcal{K}}(E/IE), B)$ is normally transversal to I'. Then $(E/IE, B_{\mathcal{K}})$ is also normally transversal to I'.

Proof: We see from Lemma (2.3.6) that $(E, B_{\mathcal{I}+\mathcal{K}}) = (E, B_{\mathcal{K}})$. Thus for the first statement it will suffice to prove that $(E, B_{\mathcal{K}})$ is G′-transversal to $\mathcal{I}$. Let $A =: B_{\mathcal{K}}$ and take $k \geq a$. We have by the third hypothesis

$$\begin{aligned} A^kE \quad &=: \quad B^kE + \mathcal{K}^1B^{k-1}E + \cdots \mathcal{K}^{k-a}B^aE \\ &= \quad B^kE + \mathcal{K}^1E \cap B^{k-1}E + \cdots \mathcal{K}^{k-a}E \cap B^{k-a}E \end{aligned}$$

It follows that $\mathcal{K}^jE \cap A^kE = \mathcal{K}^jE \cap B^{k-j}E = \mathcal{K}^jB^{k-j}E$ and that $A^k \mathrm{Gr}^j_{\mathcal{K}} E \cong B^{k-j} \mathrm{Gr}^j_{\mathcal{K}} E$. We shall prove by descending induction on $j \leq k - a$ that

$$IE \cap \mathcal{K}^jE \cap A^kE \subseteq \mathcal{I}^1A^{k-1}E + \mathcal{I}^2A^{k-2}E + \cdots. \qquad (2.3.7.1)$$

For $j = k - a$, we have on the left $IE \cap \mathcal{K}^kE$, which is the same as $I\mathcal{K}^kE$ by the last hypothesis. But $I\mathcal{K}^kE \subseteq IA^kE$ by definition, so (2.3.7.1) is certainly true in this case.

For the induction step, let x be a member of the left side of (2.3.7.1) and let $i =: k - j$. Then $x \in IE \cap \mathcal{K}^j E = I\mathcal{K}^j E$ by the third hypothesis again. Let x'' be the image of x in $\mathrm{Gr}^j_{\mathcal{K}} E$. It is clear that x'' belongs to

$$\begin{aligned} I \,\mathrm{Gr}^j_{\mathcal{K}} E \cap A^k \,\mathrm{Gr}^j_{\mathcal{K}} E &= I \,\mathrm{Gr}^j_{\mathcal{K}} E \cap B^i \,\mathrm{Gr}^j_{\mathcal{K}} E \\ &= \mathcal{I}^1 B^{i-1} \,\mathrm{Gr}^j_{\mathcal{K}} E + \cdots \mathcal{I}^a B^{i-a} \,\mathrm{Gr}^j_{\mathcal{K}} E, \end{aligned}$$

by hypothesis (2). Choose

$$z \in \mathcal{I}^1 \mathcal{K}^j B^{i-1} E + \cdots \mathcal{I}^a \mathcal{K}^j B^{i-a} E \subseteq IE \cap \mathcal{K}^j B^i E \subseteq IE \cap A^k E$$

mapping to x''. Then $x - z$ lies in $\mathcal{K}^{j+1} E \cap A^k E \cap IE$, and, we can apply the induction hypothesis to see that

$$x - z \in \mathcal{I}^1 A^{k-1} E + \cdots \mathcal{I}^{k-a} A^a E.$$

But $z \in \mathcal{I}^1 \mathcal{K}^j B^{i-1} E + \cdots + \mathcal{I}^a \mathcal{K}^j B^{i-a} E \subseteq \mathcal{I}^1 A^{k-1} E + \mathcal{I}^2 A^{k-2} E + \cdots$, so the proof is complete.

The second statement takes place entirely modulo I, so we may as well assume that $I = 0$. There is a commutative diagram with vertical isomorphisms:

$$\begin{array}{ccc} A^i \,\mathrm{Gr}^j_{\mathcal{K}} E \otimes \mathcal{O}_T/I' & \longrightarrow & \mathrm{Gr}^j_{\mathcal{K}} E/I'E \\ \downarrow & & \downarrow \\ B^{i-j} \,\mathrm{Gr}^j_{\mathcal{K}} E \otimes \mathcal{O}_T/I' & \longrightarrow & \mathrm{Gr}^j_{\mathcal{K}} E/I'E \end{array}$$

As the bottom arrow is an injective by assumption, so is the top one. It now follows by induction that each map

$$(A^i E / A^i E \cap \mathcal{K}^j E) \otimes \mathcal{O}_T/I' \to E/(\mathcal{K}^j E + I'E)$$

is injective. Hence any element of the kernel of $A^i E \otimes \mathcal{O}_T/I' \to E/I'E$ is the image of some $x \in (A^i E \cap \mathcal{K}^j E) \otimes \mathcal{O}_T/I'$; furthermore this element x maps to zero in $E/I'E$. Taking $j = i$ we have $\mathcal{K}^i E \subseteq A^i E$, and we see that x belongs to the kernel of the map $\mathcal{K}^i E \otimes \mathcal{O}_T/I' \to E/I'E$. Because $(E, \mathcal{K})$ is normally transversal to I', x vanishes, and this shows that $A^i E \otimes \mathcal{O}_T/I' \to E/I'E$ is injective—*i.e.* that (E, A) is normally transversal to I'. ∎

2.4 Uniform filtrations

2.4.1 Definition: *Suppose that (E, A) is a filtered sheaf of $\mathcal{O}_T$-modules, G-transversal to $\mathcal{J}$. Suppose that E is separated and complete for the J-adic topology. We say that (E, A) is "$\mathcal{J}$-uniform," or just "uniform" if there is no risk of confusion, if E is locally free of finite rank over $\mathcal{O}_T$ and $\mathrm{Gr}_A(E/JE)$ is locally free over $\mathcal{O}_X =: \mathcal{O}_T/J$.*

Note that if (E, A) is $\mathcal{J}$-uniform and $\mathcal{J}'$ is coarser than $\mathcal{J}$, then the $\mathcal{J}'$-saturation $(E, A'_{\mathcal{J}})$ of (E, A) to $\mathcal{J}'$ is G-transversal to $\mathcal{J}'$, by lemma (2.2.1). We shall also call $(E, A_{\mathcal{J}'})$ the "expansion of A to $\mathcal{J}'$," and we call A a "contraction of $A_{\mathcal{J}'}$ to $\mathcal{J}$." Note that we have an isomorphism of filtered objects:

$$(E/J'E, A_{\mathcal{J}'}) \cong (E/JE, A) \otimes \mathcal{O}_T/J' \tag{2.4.1.1}$$

It follows from this that $(E, A_{\mathcal{J}'})$ is $\mathcal{J}'$-uniform. Note also that the process of expansion is transitive in the obvious sense.

2.4.2 Proposition: *Let (E, A) be a filtered sheaf of $\mathcal{O}_T$-modules uniformly G-transversal to $\mathcal{J}$. Let $\mathcal{J}E$ or $\mathcal{J}_E$ denote the filtration on E defined by $\mathcal{J}_E^i E =: \mathcal{J}^i E$.*

1. *Locally on T, A can be contracted to any subfiltration of $\mathcal{J}$.*

2. *For any k and any $i \geq 0$,*

$$\mathcal{J}^i E \cap A^k E = \mathcal{J}^i A^{k-i} E + \mathcal{J}^{i+1} A^{k-i-1} E + \cdots.$$

3. *There are canonical isomorphisms:*

$$\operatorname{Gr}^i_{\mathcal{J}} \mathcal{O}_T \otimes (E/JE, A[-i]) \cong (\operatorname{Gr}^i_{\mathcal{J}_E} E, A)$$

$$\operatorname{Gr}^i_{\mathcal{J}} \mathcal{O}_T \otimes \operatorname{Gr}^{k-i}_A (E/JE) \cong \operatorname{Gr}^i_{\mathcal{J}_E} \operatorname{Gr}^k_A E$$

Proof: If T is affine and if $\operatorname{Gr}_A(E/JE)$ is free, we can clearly find a filtration B such that $\operatorname{Gr}_B E$ is free and such that B is finer than A and induces the same filtration on E/JE as A. Corollary (2.3.2) shows that $B_J = A$. Then B is uniformly G-transversal to the zero ideal, and we can expand it to any ideal contained in J.

The next statement is also local, and thus we may assume that there exists a contraction B of A to the zero ideal. I claim that for any $i \geq 0$,

$$\mathcal{J}^i E \cap A^k E = \mathcal{J}^i B^{k-i} E + \mathcal{J}^{i+1} B^{k-i-1} E + \cdots. \tag{2.4.2.2}$$

This clearly implies statement (2) above. We prove it by induction on i, the case of $i = 0$ being just the fact that $A = B_{\mathcal{J}}$. If it is true for i and if $x \in \mathcal{J}^{i+1} E \cap A^k E$, then by the induction hypothesis we can write $x = y + z$ where $y \in \mathcal{J}^i B^{k-i} E$ and $z \in \mathcal{J}^{i+1} B^{k-i-1} E + \cdots$. Then $y \in \mathcal{J}^{i+1} E \cap B^{k-i} E$, and since $E/B^{k-i}E$ is flat, $y \in \mathcal{J}^{i+1} B^{k-i} E \subseteq \mathcal{J}^{i+1} B^{k-i-1} E$. The result follows.

For the last statement, note that the image of the multiplication map $\mathcal{J}^i \otimes A^{k-i}E \to E$ is contained in $A^i E \cap \mathcal{J}^i E$, and hence we find a commutative diagram of filtered objects

$$\begin{array}{ccc} (\mathcal{J}^i \otimes (E, A[-i]) & \longrightarrow & (\mathcal{J}^i E, A) \\ \downarrow & & \downarrow \\ (\mathcal{J}^i/\mathcal{J}^{i+1} \otimes (E/JE, A[-i])) & \longrightarrow & (\mathrm{Gr}^i_{\mathcal{J}_E}, A) \end{array}$$

The bottom arrow is an isomorphism because E is locally free; to prove that it is strictly compatible with the filtrations we may work locally, choosing a contraction B of A to the zero ideal. The strictness is then clear from (2.4.2.2) ∎

2.4.3 Corollary: *A filtered sheaf of $\mathcal{O}_T$-modules (E, A) is uniformly G-transversal to $\mathcal{J}$ if and only if, locally on T, (E, A) is isomorphic to a finite direct sum of copies of filtered objects of the form $(\mathcal{O}_T, \mathcal{J}[d])$, for various d.*

Proof: If (E, A) is uniformly G-transversal to $\mathcal{J}$, Proposition (2.4.2.1) shows that, locally on T, we may choose a contraction B of A to the zero ideal. We may also choose a splitting C of the filtration B of E; then for all i

$$B^i = C^i \oplus C^{i+1} \oplus \cdots.$$

If d_j is the rank of C_j we find

$$A^i E \cong \bigoplus_{j=0}^{\infty} \mathcal{J}^{i-j} C^j = \bigoplus (\mathcal{O}_T, \mathcal{J}[-j])^{d_j}.$$

This proves the nontrivial direction of the corollary. ∎

2.4.4 Corollary: *Suppose that (E, A) and (E', A') are two filtered $\mathcal{O}_T$-modules, both uniformly G-transversal to $\mathcal{J}$. Then $(E \otimes E')$ and $\mathrm{Hom}(E, E')$, endowed with the usual filtrations, are also uniformly G-transversal to $\mathcal{J}$.*

Proof: Thanks to the previous corollary, the first statement follows from the fact that

$$(\mathcal{O}_T, \mathcal{J}[a]) \otimes (\mathcal{O}_T, \mathcal{J}[b]) \cong (\mathcal{O}_T, \mathcal{J}[a+b]),$$

and the second from

$$\mathrm{Hom}((\mathcal{O}_T, \mathcal{J}[a]), (\mathcal{O}_T \mathcal{J}[b])) \cong (\mathcal{O}_T, \mathcal{J}[b-a])$$

when each $\mathcal{J}^i$ is invertible. ∎

3 Griffiths transversality

Suppose that Y/S is a smooth scheme and (E, ∇) is a sheaf of $\mathcal{O}_Y$-modules with integrable connection. Recall that a filtration A of E by sub-$\mathcal{O}_Y$-modules is said to satisfy Griffiths transversality if and only if $\nabla A^k \subseteq A^{k-1} \otimes \Omega^1_{Y/S}$. We shall also say that A is "Griffiths transversal" or just "G-transversal" to ∇. Our goal in this section is to give a crystalline interpretation of this condition.

Throughout this section and most of the rest of this article we shall work with the following notation. Let S be a formal scheme for the p-adic topology, and suppose that (I, γ) is a divided power ideal of $\mathcal{O}_S$, compatible with the divided power structure of $(p) \subseteq \mathbf{Z}_p$ and defining a closed formal subscheme S' of S. We suppose that S is flat over $\mathbf{Z}_p$ and for each integer $\nu > 0$ we denote by S_ν the reduction of S modulo p^ν; we also let $S_0 =: S_1$ and $S_\infty =: S$. We shall allow S to carry a fine logarithmic structure, *e.g.* the trivial one, and just use the notation S to stand for the corresponding logarithmic formal scheme. Let us fix $\mu \in [0, \infty]$ such that $S' \subseteq S_\mu$ and let X/S'_μ be a fine log scheme of finite type to which the divided powers (I, γ) extend. Then $\mathrm{Cris}(X/S)$ is by definition the site whose objects are exact PD-thickenings T of open subsets of X such that $p^n\mathcal{O}_T = 0$ for some $n > 0$, and with the usual morphisms and covering families. (This site is denoted by $\mathrm{Cris}(X/\hat{S})$ in [4].) Let $(X/S)_{cris}$ be the corresponding topos with sheaf of rings and PD-ideals $(\mathcal{O}_{X/S}, J_{X/S}, \gamma)$. (Since the PD-structures on I and on $J_{X/S}$ are by definition compatible, we will not run into any serious difficulties by using the same letter γ for both of them.)

If $U \subseteq X$ can be embedded as a closed subscheme of a log smooth Y/S, then for each $n \geq \mu$ we can form the divided power envelope $D_X(Y_n)$ of X in Y_n, and (for $n' > n$) the reduction of $D_X(Y_{n'})$ modulo p^n is $D_X(Y_n)$. We denote by $D_U(Y)$ the inverse limit of the system $\{D_U(Y_n) : n \geq \mu\}$, a formal scheme for the p-adic topology. Such an object is called a "fundamental thickening of U relative to S." Although $D_U(Y)$ itself is not an object of $\mathrm{Cris}(X/S)$ as we have defined it, each $D_U(Y_n)$ is, and by passing to the limit we see that a sheaf E of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$ in fact defines a sheaf E_D of $\mathcal{O}_D$-modules on $D_U(Y)$.

The site formed in the obvious way by considering fundamental thickenings of open sets of X is called the "restricted crystalline site" and is denoted by $\mathrm{Rcris}(X/S)$. In practice, it will suffice (*e.g.* for cohomology calculations) to consider the value of sheaves on the fundamental thickenings. In particular, giving a crystal of $\mathcal{O}_{X/S}$-modules on Rcris is the same as giving a crystal of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$. In fact, if D is a fundamental thickening of X/S, then to give a crystal of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$ is the same as to give a compatible collection of crystals on each $\mathrm{Cris}(X/S_n)$, and is again the same as to give a sheaf of $\mathcal{O}_D$-modules with connection ∇ which is integrable, p-adically nilpotent, and compatible with the divided powers.

3.0.5 Remark: A sheaf E such that $E_D = 0$ for each fundamental thickening D is called "parasitic." For example, if X/S_μ is log smooth (or a local complete intersection in a log smooth scheme), then the kernel of multiplication by p on $\mathcal{O}_{X/S}$ is parasitic. For a thorough discussion of the restricted site and parasitic sheaves over bases on which p is nilpotent, we refer to [3, IV,2].

3.0.6 Remark: The category of crystal of $\mathcal{O}_{X/S}$-modules is abelian, with the evident construction of kernels and cokernels. However, the inclusion functor from the category of crystals of $\mathcal{O}_{X/S}$-modules to the category of all sheaves of of $\mathcal{O}_{X/S}$-modules is not left exact—for example, if X/S_μ is log smooth, then multiplication by p on $\mathcal{O}_{X/S}$ is injective in the category of crystals, but not in the category of sheaves. We will be dealing with crystals endowed with filtrations which consist sometimes but not always of subcrystals. For the sake of clarity, we should therefore explain how we associate to a filtered object (E, N) in the category of crystals a filtered object $(E, \tilde{N})$ in the category of sheaves of $\mathcal{O}_{X/S}$-modules. For each object T of $\mathrm{Cris}(X/S)$, we let $\tilde{N}^i E$ denote the sheaf-theoretic image of $N^i E \to E$ in the category of sheaves. For each fundamental thickening D of X relative to S, the natural map $N^i E_D \to \tilde{N}^i E_D$ is an isomorphism, and the kernel of the natural surjection $N^iE \to \tilde{N}^iE$ is parasitic. Because of this the distinction between $\tilde{N}$ and N is not very important, and we shall eventually drop the tildes from the notation.

3.1 Griffiths transversality and G-transversality

If (E, A) is a filtered sheaf of $\mathcal{O}_{X/S}$-modules in $(X/S)_{cris}$ such that (E_T, A_T) is G-transversal to J_T for each object T of Cris (X/S), then we shall say that (E, A) is "G-transversal to $(J_{X/S}, \gamma)$." For any morphism $f\colon (T', J_{T'}, \gamma) \to (T, J_T, \gamma)$ in Cris (X/S), Lemma (2.2.1) tells us that the filtration A_f on f^*E_T induced by A_T is G′-transversal to $(J_{T'}, \gamma)$, and hence by Lemma (2.3.1), its saturation $A_{f,J_{T'},\gamma}$ is G-transversal to $(J_{T'}, \gamma)$.

3.1.1 Lemma: *Let E be a crystal of $\mathcal{O}_{X/S}$-modules on* Cris $((X/S))$, *endowed with a filtration A which is G-transversal to $(J_{X/S}, \gamma)$. The following conditions are equivalent:*

1. *For any morphism $f\colon T' \to T$ in* Cris (X/S),

$$A^kE_{T'} = A_f^kE_{T'} + J_{T'}A_f^{k-1}E_{T'} + \cdots J_{T'}^{[i]}A^{k-i}E_{T'} + \cdots$$

2. *For any object (U, T, γ) of* Cris (X/S), *the filtration induced by A_T on E_U is A_U*

These conditions are automatically satisfied if X/S is log smooth.

Proof: Clearly (2) is a special case of (1), with f the inclusion morphism: $(U, U, 0) \to (U, T, \gamma)$. For the converse, note that since A^kE is a sheaf on Cris (X/S), we have a commutative diagram

$$\begin{array}{ccc} f^*A^kE_T & \longrightarrow & A^kE_{T'} \\ \downarrow & & \downarrow \\ f^*E_T & \longrightarrow & E_{T'}. \end{array}$$

It follows that the filtration $A_{T'}$ of $E_{T'}$ is coarser than the filtration A_f induced by the filtration A_T of E_T, and (2) implies that $A_{T'}$, A_T, and (hence) A_f all induce the same filtration on $(E_{U'})$. Lemma (2.2.1) implies that A_f is G′-transversal to $J_{T'}$, and it follows that $A_{f,J_{T'},\gamma}$ is G-transversal to $(J_{T'}, \gamma)$, finer than $A_{T'}$, and induces the same filtration on $E_{U'}$ as does $A_{T'}$. By Corollary (2.3.2), the two filtrations coincide. Finally, we verify that (2) is automatic if X/S is log smooth. This is local on T, so we may assume that T is affine and that there exists a retraction $f: T \to X$. Then as we have seen, the filtration A_f on E_T induced by the filtration A_X on E_X is finer than the the filtration A_T on E_T, and the filtration on E_X induced by A_T is finer than A_X. Pulling back the containment $A_f \leq A_T$, we obtain that $A_X \subseteq i^* A_T$. As also $i^* A_T \subseteq A_X$, the filtrations are equal. ∎

3.1.2 Theorem: *Suppose that Y/S is log smooth and E is a crystal of $\mathcal{O}_{Y/S}$-modules on* Cris (Y/S)*, corresponding to a module with integrable connection (E_Y, ∇) on Y/S. Let A be a filtration of E by subsheaves of $\mathcal{O}_{Y/S}$-modules and let A_Y be the corresponding filtration of E_Y. Then if A is G-transversal to $(J_{Y/S}, \gamma)$, the filtration A_Y is Griffiths transversal to ∇. Conversely, given a filtration A_Y on E_Y which is Griffiths transversal to ∇, there is a unique filtration A on the crystal E which is G-transversal to $(J_{Y/S}, \gamma)$ and whose value on Y is the given filtration A_Y.*

Proof: Given (E, A), consider the first infinitesimal neighborhood $P^1_{Y/S}$ of the diagonal of $Y \times_S Y$. The ideal of Y in $P^1_{Y/S}$ is $\Omega^1_{Y/S}$, and since it is a square zero ideal, we may endow it with the trivial divided power structure. Then $(P^1_{Y/S}, Y, 0)$ becomes an object T of the crystalline site of Y/S, and (E_T, A_T) is a filtered sheaf on $P^1_{Y/S}$. We have maps $p_i: T \to Y$ for $i = 1, 2$, and hence canonical isomorphisms: $\rho_i: p_i^* E_Y \to E_T$, reducing to the identity on Y. If e is a section of E_Y, then

$$\nabla e = \rho_2 p_2^*(e) - \rho_1 p_1^*(e) \in \Omega^1_{Y/S} E_Y.$$

Now if the filtration A_T is G-transversal to $\Omega^1_{Y/S}$ and $e \in A^k E_Y$, then $\rho_2 p_2^*(e)$ and $\rho_1 p_1^*(e)$ lie in $A^k E_T$ and

$$\nabla e \in \Omega^1_{Y/S} E_T \cap A^k E_T = \Omega^1_{Y/S} A^{k-1} E_T \cong A^{k-1} E_Y \otimes \Omega^1_{Y/S}.$$

This shows that (E_Y, A_Y) is Griffiths transversal to ∇.

Conversely suppose that (E_Y, ∇, A_Y) is a module with integrable connection and Griffiths transversal filtration. Let $(D(1), J, \gamma)$ denote the divided power envelope of the diagonal of $Y \times_S Y$, and let $\epsilon: p_2^* E_Y \to p_1^* E_Y$ be the isomorphism induced by ∇. By Lemma (2.2.1.2), the filtrations A_{p_i} on $p_i^* E_Y$ are G′-transversal to (J, γ), and hence by Lemma (2.3.1) the filtrations $A_i =: A_{p_i, J_Y}$ are G-transversal to (J, γ).

3.1.3 Lemma: *The isomorphism $\epsilon: p_2^* E_Y \to p_1^* E_Y$ induces an isomorphism:*

$$A_2^k p_2^* E_Y \to A_1^k p_1^* E_Y.$$

Proof: This is a local statement, and we may work with a system of logarithmic coordinates $(m_1, \ldots m_n)$ for Y/S. Let $\eta_i =: \tilde{\delta}(m_i) \in J$ and let $\{\partial_I\}$ denote the corresponding basis for the ring of PD-differential operators (1.1.5). Recall from (1.1.8.6) that if e is a local section of E_Y, $\epsilon p_2^*(e) = \sum_I \eta^{[I]} p_1^*(\nabla(\partial_I)e)$. Now if $e \in A^k E_Y$, the fact that A is G-transversal to ∇ implies that $\nabla(\partial_I)e \in A^{k-|I|}$, and hence $\eta^{[I]} p_1^* \partial_I(e) \in A_1^k p_1^* E_Y$. It follows immediately that ϵ maps $A_2^k p_2^* E_Y$ to $A_1^k p_1^* E_Y$. Since the formula for the inverse of ϵ is the same, with only the indices interchanged (a consequence of the cocycle condition), the same argument shows that the inverse of ϵ maps $A_1^k p_1^* E_Y$ to $A_2^k p_2^* E_Y$. This proves the lemma. ∎

It follows immediately from the lemma that if r_1 and r_2 are two retractions $(T, J_T, \gamma) \to (U, 0, 0)$ in Cris(Y/S), then the two filtrations A_{r_i, J_T} are equal. Since Y/S is log smooth, such retractions always exist locally, and we can use any retraction to define a filtration A_T on E_T which is G-transversal to (J_T, γ). It is now straightforward to verify that $A^k E$ forms a sheaf of $\mathcal{O}_{X/S}$-modules, and that the filtration (E, A) satisfies the conditions of the theorem. Finally, we observe that Corollary (2.3.2) implies that the filtration thus constructed is unique. ∎

3.2 T-crystals

I hope that the previous theorem provides justification for the following definition.

3.2.1 Definition: *A "proto-T-crystal" on X/S is a pair (E, A), where E is a crystal of $\mathcal{O}_{X/S}$-modules and A is a filtration on E which is G-transversal to $(J_{X/S}, \gamma)$, and satisfies the equivalent conditions of the Lemma (3.1.1). We say (E, A) is "uniform" if E is locally free of finite type over $\mathcal{O}_{X/S}$ and* $\mathrm{Gr}_A E_X$ *is locally free on X. A proto-T-crystal (E, A) is a called a "T-crystal" if (E, A) is compatible (2.3.3) with the closed subscheme of X defined by p^i for every $i > 0$.*

Note that the the condition that (E, A) be compatible with p^i just says that (E_X, A) is normally transversal to the ideal $p^i \mathcal{O}_X$; this is of course automatic if $p^i \mathcal{O}_X = 0$ or if (E, A) is uniform.

Now suppose that $i: X \to Y$ is a closed immersion, with Y/S log smooth. Temporarily we will denote by δ the PD-structure of $J_{Y/S}$. Recall that $D_X(Y) =: i_{cris*} \mathcal{O}_{X/S}$ is a crystal of $\mathcal{O}_{Y/S}$-algebras, endowed with a sheaf of PD-ideals $(i_{cris*} J_{X/S}, \gamma)$ [4, 6.2]. The next result can be thought of as a crystalline version of the Griffiths transversality theorem for the closed immersion i.

3.2.2 Lemma: *Suppose that (E, A) is a proto-T-crystal on* Cris (X/S), *and let $i_{cris*}(E, A)$ denote the filtered crystal of $\mathcal{O}_{Y/S}$-modules defined by*

$$A^k i_{cris*} E =: i_{cris*} A^k E.$$

Then $i_{cris*}(E, A)$ is G-transversal to $(J_{Y/S}, \delta)$ and to $(i_{cris*} J_{X/S}, \gamma)$.

Proof: Suppose that $T =: (U, T, \delta)$ is an object of Cris (Y/S), and J_T is the ideal of U in T. To simplify the notation, we may and shall assume that $U = Y$. Recall that $i^*_{cris} T$ is represented in Cris (X/S) by $D_X(T) =: D_{X,\delta}(T)$, the divided power envelope of X in T compatible with the divided power structure δ on the ideal J_T of Y in T (and of course with γ and the canonical divided power structure on (p).) Thus, $i_{cris*} A^k E_T$ is $\lambda_* A^k E_{D(T)}$, where $\lambda \colon D_{X,\delta}(T) \to T$ is the canonical projection. The fact that (E_T, A) is G-transversal to the ideal $(J_{D(T)}, \gamma)$ of X in $D_X(T)$ comes from the definitions, but to prove that it is G-transversal to the ideal (J_T, δ) of Y in T takes more work. Dropping the λ's from the notation, we can write what we have to prove as:

$$A^k E_{D(T)} \cap J_T E_{D(T)} = J_T A^{k-1} E_{D(T)} + \cdots + J_T^{[i]} A^{k-i} E_{D(T)} + \cdots \qquad (3.2.2.1)$$

This assertion is local on Y, so we may and shall assume that Y is affine. Since Y/S is log smooth, there exist a retraction $r \colon T \to Y$, which by functoriality induces a map $f \colon D_X(T) \to D_X(Y)$ of divided power envelopes. We find a commutative diagram:

$$\begin{array}{ccccc} X & \longrightarrow & D_X(Y) & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X_T & \longrightarrow & D_X(T) & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle r} \\ X & \longrightarrow & D_X(Y) & \longrightarrow & Y \end{array}$$

Here $X_{T'}$ is the inverse image of X in $D_X(T)$, so that the bottom square on the left is Cartesian. Recall that the very fact that $i_{cris*}\mathcal{O}_{X/S}$ is a crystal asserts that the bottom square on the right is also Cartesian; this is one of the key technicalities of the foundations of crystalline cohomology [4, 6.2]. It follows trivially that the squares on the top are Cartesian. Since the ideal

$J_{D(Y)}$ of X in $D_X(Y)$ is a PD-ideal and the map $(D_X(T), X) \to (D_X(Y), X)$ is a PD-morphism, it follows that the ideal $K = J_{D(Y)}\mathcal{O}_{D(T)}$ of X_T in $D_X(T)$ is a sub PD-ideal of the ideal $J_{D(T)}$ of X in $D_X(T)$. Similarly, the ideal I of $D_X(Y)$ in $D_X(T)$ is a sub PD-ideal of $J_{D(T)}$. In fact, it is clear from the fact that the top squares in the diagram are Cartesian that this ideal is just $J_T\mathcal{O}_{D(T)}$ and that $J_{D(T)} = I + K$. Thus $J_T^{[j]}A^iE_{D(T)} = I^{[j]}A^iE_{D(T)}$, are reduced to proving equation (3.2.2.1) with I in place of J_T. Write J for $J_{D(T)}$.

Let $A^k_f E_{D_X(T)}$ denote the image of the natural map $f^*A^kE_{D(Y)} \to E_{D(T)}$, so that $A^kE_{D(T)} = A^k_{f,J,\gamma}E_{D(T)}$. Since $(E_{D(Y)}, A)$ is saturated with respect to $(J_{D(Y)}, \gamma)$, $(E_{D(T)}, A_f)$ is saturated with respect to (K, γ). By Lemma (2.2.1.1), $(E_{D(Y)}, A_f)$ is normally transversal to I and hence G$'$-transversal to (I, δ). Lemma (2.3.6) now tells us that $(E_{D(T)}, A)$ is G-transversal to (I, δ). ∎

Recall that the crystal $i_{cris*}E$ is also a sheaf of $D_X(Y)$-modules. As we have seen $(i_{cris*}E, A)$ gives rise to a module with connection (E_Y, ∇), with a filtration A which is G-transversal to ∇. This sheaf has a structure of $D_X(Y)$-module; we refer to [4] for the various compatibilities these data satisfy. It is clear that we have proved:

3.2.3 Theorem: *The construction $(E, A) \mapsto (i_{cris*}E, A)$ described above defines an equivalence between the category of proto-T-crystals of $\mathcal{O}_{X/S}$-modules on X/S and the category of triples (E_Y, ∇, A_Y), where E_Y is a sheaf of $D_X(Y)$-modules on Y endowed with an integrable connection ∇ and A_Y is a filtration of E_Y which is G-transversal to ∇ and to the sheaf of PD-ideals (J_X, γ) of X in $D_X(Y)$. Under this equivalence, (E, A) is a T-crystal if and only if (E_X, A) is normally transversal to each $p^i\mathcal{O}_X$.* ∎

In particular, we see that if X/S' is log smooth and if Y/S is any lifting of X/S', the data of a T-crystal on X/S amounts to the data of a triple (E_Y, A_Y, ∇), where (E_Y, ∇) is a module with integrable connection and A_Y is a filtration on E_Y which is Griffiths transversal to ∇ and G-transversal to the ideal (I, γ) of S' in S (compatibly with each p^i).

3.2.4 Remark: If X/S is a formal scheme, (and $\mu = \infty$) we define a T-crystal on X/S to be a compatible collection of T-crystals on each X_n. If X/S is formally log smooth, a p-torsion free and coherent T-crystal on X amounts to a p-torsion free coherent sheaf of $\mathcal{O}_X$-modules E_X, together with an integrable connection ∇ relative to S and a filtration A which is Griffiths transversal to ∇ such that $\mathrm{Gr}_A\, E_X$ is p-torsion free. To see this, suppose that

(E, A) is a T-crystal on X; then for any n the underlying crystal on X_n/S gives us a module with connection (E_X, ∇) on X/S which is independent of n, as well as a filtration A_n which is Griffiths transversal to ∇ and to the PD-ideal (p^n, γ). Furthermore, for $n' > n$ the filtration $A_{n'}$ is compatible with the ideal p^n and its saturation (2.3.1) with respect to (p^n, γ) is A_n. This implies that the image of $A_{n'}$ in E_n is the same as the image of A_n, and we let $A^i E =: \varprojlim A^i_n E_n$. It is clear that (E, A) is Griffiths transversal to ∇ and it remains only to prove that $\mathrm{Gr}_A E_X$ is p-torsion free. The hypothesis that the filtration (E, A_n) be compatible with p says precisely that (E_n, A) is normally transversal to the ideal (p), and hence that $\mathrm{Tor}_1(E_n/A^i_n E_n, \mathcal{O}_X/p\mathcal{O}_X)$ vanishes. Then $\varprojlim \mathrm{Tor}_1(E_n/A^i_n E_n, \mathcal{O}_X/p\mathcal{O}_X)$ also vanishes, and it follows easily that each $E/A^i E$ and hence also $\mathrm{Gr}_A E_X$ is p-torsion free. As an exercise, the reader can also prove that in fact $A^i E_X = \cap_n A^i_n E_X$.

For example, take $X = S =: \mathrm{Spf}\, W$. Then a proto-T-crystal on W/W is just a W-module E endowed with a filtration A, since any such A is automatically G-transversal to the zero ideal. For (E, A) to be a T-crystal, we require (E, A) to be normally transversal to each p^i; it is clear that if E is finitely generated, this is equivalent to saying that the filtration A is a filtration by direct factors. If $X = \mathrm{Spec}\, W_n$, a coherent T-crystal on X/W is a finitely generated filtered W-module (E, A), where A is G-transversal to p^n and $(E \otimes W_n, A)$ is a filtration by direct factors.

3.2.5 Remark: Suppose that, in the situation of Theorem (3.2.3) the filtration A_Y is stable by ∇. Then in fact we can regard A as defining a filtration of E by subcrystals, and following the procedure described in (3.0.6), we obtain a filtration $\tilde{A}$ of the the sheaf E by sheaves of $\mathcal{O}_{X/S}$-modules. If (E, A) is the corresponding T-crystal on X/S, it is clear that the filtration A_T of E_T can be computed by taking the (J_T, γ)-expansion of the filtration $\tilde{A}_T$. We shall call T-crystals arising in this way "horizontal." The failure of a T-crystal to be horizontal is measured by the mapping

$$\xi_Y \colon \mathrm{Gr}_A E_Y \longrightarrow \mathrm{Gr}_A E_Y \otimes \Omega^1_{Y/S}$$

induced by ∇; this mapping has degree -1 and is in fact a linear map of sheaves of $\mathcal{O}_X$-modules. If X/S_μ is log smooth and Y/S is a lifting of X/S_μ, then $\mathrm{Gr}_A E \otimes \Omega^1_{Y/S} \cong \mathrm{Gr}_A E_Y \otimes \Omega^1_{X/S}$, and it is easy to check that ξ_Y is independent of the choice of Y/S. Note that the map ξ_Y is compatible with the filtration (p, γ) of $\mathrm{Gr}_A E_Y$; it is useful to think of ξ_Y as a filtered complex. For more on these Kodaira-Spencer maps, *c.f.* §6.2.

3.3 Functoriality

3.3.1 Remark: Suppose $f: X' \to X$ is a morphism of logarithmic S-schemes and (E, A) is a proto-T-crystal on X/S. If (E_X, A) is normally transversal to f then there is a natural way to define a proto-T-crystal $f^*(E, A)$ on X'/S. Namely, if T is an object of $\mathrm{Cris}(X/S)$ and T' an object of $\mathrm{Cris}(X'/S)$ and if $g: T' \to T$ is a PD-morphism covering the restriction of f to $X' \cap T$, it follows from (2.2.1) and (2.3.3) that the $(J_{T'}, \gamma)$-saturation of $(g^* E_T, A_g)$ is G-transversal to $(J_{T'}, \gamma)$. Moreover, this saturation is, up to canonical isomorphism, independent of the choice of g. To see this, suppose that g_1 and g_2 are two such choices, and let $h: T' \to T(1)$ be the corresponding map to the fiber product of T with itself in the category $\mathrm{Cris}(X/S)$. Let (E_i, A) be the transverse pullback of (E_T, A) to $T(1)$ via the canonical projection π_i; since (E, A) is a proto-T-crystal, we have canonical isomorphisms $(E_i, A) \cong (E_{T(1)}, A)$. On the other hand, it follows from (2.3.4) that the transverse pullback of (E_i, A) via h can be identified with the transverse pullback of (E_T, A) via g_i, and hence these coincide. If (E, A) is a T-crystal, one has to check that $(f^* E_X, A)$ is still compatible with each p^i; this is of course automatic if $p\mathcal{O}_{X'} = 0$ or if (E, A) is uniform.

3.3.2 Remark: In keeping with the usual notation for Hodge structures, we let $\mathcal{O}_{X/S}(m)$ denote the T-crystal obtained by endowing $\mathcal{O}_{X/S}$ with filtration $A^i =: J^{i+m}_{X/S}$. If (E, A) and (E', A) are two uniform T-crystals on X/S, we can use (2.4.4) to construct a T-crystal structure on $E \otimes E'$ and on $\mathrm{Hom}(E, E')$. Suppose that (E, A) is uniform. Then by a "principal polarization on (E, A) of weight m" we mean an isomorphism of T-crystals

$$(E, A) \to \mathrm{Hom}\big[(E, A), \mathcal{O}_{X/S}(-m)\big];$$

the associated bilinear map

$$\langle \ , \ \rangle: E \times E \to \mathcal{O}_{X/S}(-m)$$

is supposed to be alternating if m is odd and symmetric if m is even. The form $\langle \ , \rangle$ identifies $A^i E_X$ with the annihilator of $A^{m+1-i} E_X$, and it follows that the width of (E, A) is less than or equal to m.

The following result can perhaps be thought of as a version of the Griffiths transversality theorem for a log smooth morphism, on the level of the crystalline topos.

3.3.3 Proposition: *Suppose that $S' \subseteq S$ is defined by a PD-ideal (I, γ), that X/S' is log smooth and that (E, A) is a proto-T-crystal on X/S. Then if $D =: D_X(Y)$ is the PD-envelope of X in a log smooth Y/S, (E_D, A) is G-transversal to the PD-ideal $(I, \gamma)\mathcal{O}_D$. Furthermore, if (E, A) is a T-crystal, $(E_D/IE_D, A)$ is normally transversal to each p^i.*

Proof: Since (E_D, A) is G-transversal to (J, γ) and $I \subseteq J$, it follows that it is also saturated with respect to (I, γ). The G′-transversality will require more work. Suppose that the lower level of (E, A) is at least a.

The assertion is local on Y, so we may and shall assume that Y is affine and that X admits a smooth lifting Z/S embedded in Y. Since $X \subseteq Z$ is defined by the PD-ideal (I, γ), Z is in fact contained in $D_X(Y)$. The ideal K of Z in $D_X(Y)$ is a sub PD-ideal of the ideal J of X in $D_X(Y)$, and in fact $J = K + I$. Furthermore, since Z/S is log smooth, there exists a retraction $g: D_X(Y) \to Z$. Recall from (3.1.1) that

$$A^k E_D = A_g^k g^* E_Z + J A_g^{k-1} g^* E_Z + J^{[2]} A_g^{k-2} g^* E_Z + \cdots + J^{[k-a]} g^* A_g^a E_Z.$$

Since (E_Z, A) is saturated with respect to (I, γ) and since $J = K + I$, it follows that:

$$A^k E_D = A_g^k g^* E_Z + K A_g^{k-1} g^* E_Z + K^{[2]} A_g^{k-2} g^* E_Z + \cdots + K^{[k-a]} A_g^a g^* E_Z.$$

Thus, (E_D, A) is the saturation of of (E_D, A_g) with respect to (K, γ).

Note first of all from [4, 3.32] that locally on Z, $\mathcal{O}_D$ looks a divided power polynomial algebra, compatibly with the filtrations. Hence $\mathcal{O}_D$ and $\mathrm{Gr}_{K,\gamma} \mathcal{O}_D$ are flat over Z. It follows from this that (E_D, K_E, γ) is normally transversal to I and that (E_D, A_g) is normally transversal to each $K^{[i]}$. Furthermore, (E_Z, A) is G-transversal to (I, γ) and hence Lemma (2.2.1) tells us that (E_D, A_g) and $(\mathrm{Gr}_{K,\gamma} E_D, A_g)$ are G′-transversal to (I, γ); since the filtration remains saturated with respect to (I, γ), it is in fact G-transversal to (I, γ). Thus it follows from Lemma (2.3.7) that (E_D, A) is G-transversal to (I, γ).

It remains for us to prove that if (E, A) is a T-crystal on X/S', then $(E_D/IE_D, A)$ is normally transversal to p^i. Let $D' =: D \times_S S'$ and let D'' be the reduction of D' modulo p^i, with similar notation for Z and X; note that $Z' = X$. We want to prove that $A^i E_{D'} \otimes \mathcal{O}_{D''} \to E_{D''}$ is injective. By assumption, $A^i E_X \otimes \mathcal{O}_{X''} \to E_{X''}$ is injective, and because $\mathrm{Gr}_{K,\gamma} \mathcal{O}_{D'}$ is flat over X, the top map of the diagram

$$\begin{array}{ccc} A^i E_X \otimes \mathcal{O}_{X''} \otimes \mathrm{Gr}_{K,\gamma} \mathcal{O}_{D'} & \longrightarrow & E_{X''} \otimes \mathrm{Gr}_{K,\gamma} \mathcal{O}_{D'} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{K,\gamma} A_g^i E_{D'} \otimes \mathcal{O}_{D''} & \longrightarrow & \mathrm{Gr}_{K,\gamma} E_{D''} \end{array}$$

is injective. The vertical maps are isomorphisms, and hence $(E_{X'}, (K, \gamma))$ is normally transversal to p^i and the bottom map is injective. Thus $(\mathrm{Gr}_{\mathcal{K}}\, E_{D'}, A_g)$ is normally transversal to $(I + p^i)$. The second part of Lemma (2.3.7) now finishes the proof. ∎

3.3.4 Corollary: *Suppose $i: X \to X'$ is a closed immersion of log smooth S'-schemes. Then if (E, A) is a T-crystal on X/S, $i_{cris*}(E, A)$ defines a T-crystal on (X'/S).*

Proof: Locally on X' we may choose a smooth lifting Y'/S. Then if $D =: D_X(Y')$, we have seen that $(i_{cris*}E_D, A)$ is G-transversal to the ideal (I, γ) of X' in Y', compatibly with each p^i. Furthermore, it has a canonical integrable connection, hence defines a T-crystal on X'/S. It is clear that this construction is independent of the choice of the lifting. ∎

The above corollary applies in particular to the T-crystal $(\mathcal{O}_X, J_{X/S})$. The value of $i_{cris*}(\mathcal{O}_X, J_{X/S})$ on Y is $(\mathcal{O}_{D_X(Y)}, J_{D_X(Y)})$. Note that this T-crystal does not have bounded upper level.

4 Filtrations, bifiltrations, and gauges

4.1 Lattice filtrations and bifiltrations

Let $\mathcal{A}$ be an abelian category with direct and inverse limits and in which direct (*i.e.* filtering inductive) limits are exact. Let $\mathbf{L}$ be a a partially ordered set. By an "$\mathbf{L}$-filtration" on an object E of $\mathcal{A}$ we mean a morphism from $\mathbf{L}$ to the partially ordered set of subobjects of E. For example, if $\mathbf{L} = \mathbf{Z}$ this amounts to the usual notion of an (increasing) filtration indexed by $\mathbf{Z}$. If A is the filtration and $\lambda \in \mathbf{L}$, we denote by $A_\lambda E$ the corresponding subobject. If A is an $\mathbf{L}^{op}$-filtration of E, we write instead $A^\lambda E$ and refer to A as a "decreasing filtration." A filtration is said to be "exhaustive" if, for all $b \in \mathbf{L}$, $\cup_a\{A_a E : a \geq b\} = E$, and is called "separated" if, for all $b \in \mathbf{L}$, $\cap_a\{A_a E : a \leq b\} = 0$. Similarly, A is said to be "finite" if for all b there exist elements a and c of $\mathbf{L}$ such that $a \leq b \leq c$ and $A_a E = 0$ and $A_c E = E$. If A is increasing and $A_s E = \sum_{\sigma \in S} A_\sigma E$ whenever S is a totally ordered subset of $\mathbf{L}$ with supremum $s \in \mathbf{L}$, then we say that A is "continuous." If A is decreasing we say that A is continuous if the same equation holds with "supremum" replaced by "infimum".

For each $a < b$ in $\mathbf{L}$, we may consider the quotient A_b/A_a, which we could reasonably denote by $\mathrm{Gr}^A_{a,b} E$. We let $\mathrm{GR}^A E$ denote the direct sum of these, for all $a < b$ in $\mathbf{L}$.

4.1.1 Definition: *Suppose that* $\mathbf{L}$ *is a lattice,* i.e. *that any two elements* x *and* y *have an infimum* $x \wedge y$ *and a supremum* $x \vee y$. *We say that an* $\mathbf{L}$*-filtration* A *on* E *is a "lattice filtration" if the following two equivalent* *conditions are fulfilled, for any σ and $\tau \in \mathbf{L}$.*

1. *$A_\sigma E + A_\tau E = A_{\sigma\vee\tau}E$ and $A_\sigma E \cap A_\tau E = A_{\sigma\wedge\tau}E$.*

2. *The natural map $A_\sigma E/A_{\sigma\wedge\tau}E \to A_{\sigma\vee\tau}E/A_\tau E$ is an isomorphism.*

To see that the second condition implies the first one, contemplate the following diagram, in which the exactness of all the columns and rows except the central row implies the exactness of the central row as well.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & A_\tau E & \longrightarrow & A_\tau E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{\sigma\wedge\tau}E & \longrightarrow & A_\sigma E \oplus A_\tau E & \longrightarrow & A_{\sigma\vee\tau}E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{\sigma\wedge\tau}E & \longrightarrow & A_\sigma E & \longrightarrow & A_\sigma E/A_{\sigma\wedge\tau}E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

For example, we could take for $\mathbf{L}$ the set of integers with the usual ordering, or the set of subsets of some set S, or the set of closed subsets of a topological space.

If (E, A) is an $\mathbf{L}$-filtration and $E' \subseteq E$ is a subobject, then one can define in the usual way an induced filtration on E', and similarly for quotients. Note that lattice filtrations do not induce lattice filtrations, in general. If $f\colon (E, A) \to (E', A')$ is a morphism of $\mathbf{L}$-filtered objects, we say that f is "strict" if A and A' induce the same filtration on $\operatorname{Im} f$. For example, if f is injective and A is exhaustive, one sees immediately that f is strict if and only if $\mathrm{GR}\, f$ is injective. If A is exhaustive and f is any morphism, one sees that f is strict if and only if the sequence

$$0 \to \mathrm{GR}^A(\ker f) \to \mathrm{GR}^A E \to \mathrm{GR}^{A'} E \to \mathrm{GR}^{A'}(\operatorname{Cok} f) \to 0$$

is exact.

Suppose (E, A, B) is a bifiltered object in $\mathcal{A}$, where each of A and B is a decreasing filtration of E indexed by $\mathbf{Z}$. If $(i, j) \in \mathbf{Z} \times \mathbf{Z}$ we will often write $E_{i,j}$, A^iB^j, or $A^i \cap B^j$ for $A^iE \cap B^jE$. It is sometimes convenient to set $A^{-\infty} =: \cup A_i$, and $A^{\infty} =: \cap A_i$, and similarly for $B^{\pm\infty}$. If (i', j') is another element of $\mathbf{Z} \times \mathbf{Z}$, we write $(i', j') \geq (i, j)$ if $i' \geq i$ and $j' \geq j$. Let $\overline{\mathbf{Z}} =: \mathbf{Z} \cup \{-\infty, \infty\}$. For any subset σ of $\overline{\mathbf{Z}} \times \overline{\mathbf{Z}}$, let

$$E_\sigma =: (E, A, B)_\sigma =: \sum A^i \cap B^j : (i, j) \in \sigma.$$

Note that $E_\sigma = E_{\overline{\sigma}}$ where $\overline{\sigma} =: \{(a', b') : \exists (a, b) \in \sigma : (a', b') \geq (a, b)\}$. On the other hand, if $\overline{\tau}$ is a proper subset of $\overline{\sigma}$ for every proper subset τ of σ we will say that σ is "reduced." When no confusion seems possible we will also write (a, b) for the singleton set containing (a, b). Observe that $\sigma \subseteq \overline{\sigma}$, $\overline{\overline{\sigma}} = \overline{\sigma}$ and that $\overline{\sigma \cup \tau} = \overline{\sigma} \cup \overline{\tau}$. Thus, $\sigma \mapsto \overline{\sigma}$ defines a topological closure operator on $\mathbf{Z} \times \mathbf{Z}$. Actually it is even true that $\overline{\sigma} = \cup\{\overline{\tau} : \tau \in T\}$ whenever $\sigma = \cup\{\tau : \tau \in T\}$, so that any union of closed sets is again closed. We see that the operation $\sigma \mapsto E_\sigma$ defines a filtration of E indexed by the lattice of subsets of $\mathbf{Z} \times \mathbf{Z}$. Clearly we lose no information by restricting to the lattice $\mathbf{L}$ of nonempty closed subsets; this lattice $\mathbf{L}$ even has infinite infima and suprema. Note that the lattice $\mathbf{L}$ has an obvious involution $\sigma \mapsto \sigma'$, taking σ to its transpose (obtained by interchanging the two coordinates). For any $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ we also have the translation operation $T_{a,b}$, defined by $T_{a,b}(\sigma) =: \{(i + a, j + b) : (i, j) \in \sigma\}$. This is clearly also a lattice automorphism.

4.1.2 Lemma: *Let (E, A, B) be a bifiltered object of $\mathcal{A}$.*

1. *If T is any subset of $\mathbf{L}$ and $\sigma =: \bigcup_{\tau \in T} \tau$,*

$$(E, A, B)_\sigma = \sum_{\tau \in T} (E, A, B)_\tau$$

2. *If σ and τ are any two subsets of $\mathbf{Z} \times \mathbf{Z}$,*

$$(E, A, B)_\sigma \cap (E, A, B)_\tau = (E, A, B)_{\overline{\sigma} \cap \overline{\tau}}.$$

In other words, $\sigma \mapsto (E, A, B)_\sigma$ defines a continuous lattice filtration of E, indexed by $\mathbf{L}$.

Proof: The first statement is obvious, as is the inclusion "$\supseteq$" in the second statement. It suffices to prove the reverse inclusion for finite subsets τ, since direct limits commute with finite intersections. We do this by induction on the cardinality of τ. There is nothing to prove if τ is empty, so we consider first the case in which τ consists of the single element (m, n). Our statement is obvious if $\tau \subseteq \overline{\sigma}$, so let us assume that this is not the case. Let $\rho =: \{(a,b) \in \sigma : a < m\}$, and let $\sigma' =: \{(a,b) \in \sigma : a \geq m\}$, so that $E_\sigma = E_\rho + E_{\sigma'}$. Since $(m,n) \notin \overline{\sigma}$, $b > n$ for every $(a,b) \in \rho$. Suppose $x \in E_\sigma \cap E_{mn}$ and write $x = y + x'$ where $y \in E_\rho$ and $x' \in E_{\sigma'}$. I claim that $y \in E_{\overline{\sigma}\cap\overline{\tau}}$. If ρ is empty this is trivial; if not we let d be the smallest integer which occurs as the second coordinate of an element of ρ. Then $d > n$ and $(m,d) \in \overline{\sigma} \cap \overline{\tau}$. Now $x \in A^m E$ and $x' \in A^m E$ so $y \in A^m E$. As $E_\rho \subseteq B^d E$ we conclude that $y \in A^m E \cap B^d E \subseteq E_{\overline{\sigma}\cap\overline{\tau}}$. It follows that $x' \in E_{\sigma'} \cap E_\tau$, and it suffices to prove that $x' \in E_{\overline{\sigma}'\cap\overline{\tau}}$. Let $\sigma'' =: \{(a,b) \in \sigma' : b \geq n\}$; then a similar argument shows that $x' \in E_{\overline{\sigma}'\cap\overline{\tau}} + E_{\sigma''}$. As $\sigma'' \subseteq \overline{\tau}$, it is obvious that $E_{\sigma''} \subseteq E_{\overline{\sigma}\cap\overline{\tau}}$; this completes the proof when τ has cardinality one.

For the induction step, write $\tau = \tau' \cup \tau''$, where τ'' has cardinality one. Write an element x of $E_\sigma \cap E_\tau$ as $y' + y''$, with $y' \in E_{\tau'}$ and $y'' \in E_{\tau''}$. Then y'' belongs to

$$\begin{aligned} E_{\tau''} \cap (E_\sigma + E_{\tau'}) &= E_{\tau''} \cap E_{\sigma\cup\tau'} \\ &= E_{\overline{\tau}''\cap(\overline{\sigma}\cup\overline{\tau}')} \\ &= E_{(\overline{\tau}''\cap\overline{\sigma})\cup(\overline{\tau}''\cap\overline{\tau}')} \\ &= E_{\overline{\tau}''\cap\overline{\sigma}} + E_{\overline{\tau}''\cap\overline{\tau}'} \end{aligned}$$

Write $y'' = z'' + z'$, with $z'' \in E_{\overline{\tau}''\cap\overline{\sigma}}$ and $z' \in E_{\overline{\tau}''\cap\overline{\tau}'}$. Then $x = y' + z' + z''$, with $z'' \in E_{\overline{\tau}''\cap\overline{\sigma}} \subseteq E_{\overline{\tau}\cap\overline{\sigma}}$ and $y' + z' \in E_{\tau'}$. But then $x' =: x - z'' = y' + z' \in E_\sigma \cap E_{\tau'}$, so by the induction hypothesis $x' \in E_{\overline{\sigma}\cap\overline{\tau}'}$. As $z'' \in E_{\overline{\sigma}\cap\overline{\tau}}$, the proof is complete. ∎

4.1.3 Definition: *A morphism $\eta: (E, A, B) \to (E', A, B)$ of bifiltered objects is "bistrict" if it is strictly compatible with the filtrations A and B and if additionally the maps*

$$\mathrm{Gr}_A\, \eta: (\mathrm{Gr}_A\, E, B) \to (\mathrm{Gr}_A\, E', B)$$

$$\mathrm{Gr}_B\, \eta: (\mathrm{Gr}_B\, E, A) \to (\mathrm{Gr}_B\, E', A)$$

are strictly compatible with the filtrations.

4.1.4 Lemma: *Suppose $\eta: (E, A, B) \to (E', A, B)$ is an injective morphism of bifiltered objects in $\mathcal{A}$. Then η is bistrict if and only if for every $\sigma \subseteq \mathbf{Z} \times \mathbf{Z}$,*

$$\eta^{-1}(E', A, B)_\sigma = (E, A, B)_\sigma.$$

Proof: Suppose η is bistrict. For any integer k we have a commutative diagram

$$\begin{array}{ccccccc}
(\mathrm{Gr}_A^{m+k} E, B) & \hookrightarrow & (A^m E/A^{m+k+1}E, B) & \rightarrow & (A^m E/A^{m+k}E, B) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
(\mathrm{Gr}_A^{m+k} E', B) & \hookrightarrow & (A^m E'/A^{m+k+1}E', B) & \rightarrow & (A^m E/A^{m+k}E', B) & \rightarrow & 0
\end{array}$$

The two rows are exact sequences of filtered objects (the arrows are strictly compatible with the filtrations), and hence the associated graded sequences are also exact. Using the strictness of $\mathrm{Gr}_A \eta$, we can now easily prove by induction on k that the map

$$(A^m E/A^{m+k}E, B) \rightarrow (A^m E'/A^{m+k}E', B)$$

is strictly compatible with the filtrations. In other words, we have

$$\eta^{-1}(B^{n'}E' \cap A^m E' + A^{m'}E') = B^{n'}E \cap A^m E + A^{m'}E$$

whenever $m' \geq m$. As η is also strictly compatible with the filtration B, it follows also that if $n' \geq n$,

$$\eta^{-1}(B^{n'}E' \cap A^m E' + A^{m'}E' \cap B^n E') = B^{n'}E \cap A^m E + A^{m'}E \cap B^n E' \quad (4.1.4.1)$$

Because direct limits are exact, it suffices to prove the lemma for finite subsets σ. Note that if the cardinality of σ is one, the result is trivial, and we proceed by induction on the cardinality of σ. We may further assume that σ is reduced and that it has cardinality at least two. Suppose $(a, b) \in \sigma$ has the smallest possible first coordinate and let (c, d) be the element of σ such that $c > a$ and such that c is minimal with this property. Then necessarily $d < b$ and d is maximal with this property. Let τ be the set obtained from σ by removing (a, b) and (c, d) and inserting (a, d). By the induction hypothesis, the lemma is true for τ. We have $E_\tau = E_\sigma + E_{ad}$, and similarly for E'. By Lemma (4.1.2), $E_\sigma \cap E_{ad} = E_{ab} + E_{cd}$. We deduce the existence of the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \rightarrow & E_{ad}/(E_{ab} + E_{cd}) & \rightarrow & E/E_\sigma & \rightarrow & E/E_\tau & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & E'_{ad}/(E'_{ab} + E'_{cd}) & \rightarrow & E'/E'_\sigma & \rightarrow & E'/E'_\tau & \rightarrow & 0
\end{array}$$

The induction hypothesis tells us that the arrow on the right is injective and (4.1.4.1) says that the arrow on the left is injective; it follows that the middle arrow is injective, and the proof is complete. We leave the easy proof of the converse to the reader. ∎

4.1.5 Corollary: *Let* $\mathbf{L}$ *be the lattice of closed subsets of* $\mathbf{Z} \times \mathbf{Z}$*, for the closure operator defined above. Then there is an equivalence between the category of bifiltered objects of* $\mathcal{A}$ *and the category of objects of* $\mathcal{A}$ *endowed with a continuous lattice* $\mathbf{L}$*-filtration. In this equivalence, bistrict arrows correspond to strict arrows.*

Proof: Suppose we are given a lattice filtered object (E, C). For each integer i, let $\alpha_i =: [i, \infty) \times \mathbf{Z}$ and $\beta_i =: \mathbf{Z} \times [i, \infty)$. Define $A^i E =: C_{\alpha_i}$ and $B^j E =: C_{\beta_j}$. As $\alpha_i \cap \beta_j = [i, \infty) \times [j, \infty)$, it follows easily that $C_\sigma = (E, A, B)_\sigma$ for any $\sigma \in \mathbf{L}$. This gives a functor from the category of lattice $\mathbf{L}$-filtered objects to the category of bifiltered objects which is quasi-inverse to the functor constructed above. ∎

4.2 Gauges

Let us explain the relationship between the lattice $\mathbf{L}$ and Mazur's gauges. Let $\mathbf{G}$ denote the set of nonincreasing functions $\mathbf{Z} \to \mathbf{Z}$, endowed with the usual partial ordering and lattice structure (defined pointwise). Let n be a positive integer. We shall call an element ϵ of $\mathbf{G}$ an "n-gauge" if $\epsilon(j+1) \geq \epsilon(j) - n$ for all j, and we let $\mathbf{G}_n$ denote the set of all n-gauges. [3] If $e \in \overline{\mathbf{Z}}$, we write c_e for the constant function whose value is always e. We shall also consider the constant $c_{-\infty}$ to be an n-gauge. If $a \in \mathbf{N}$ and $j \in \mathbf{Z}$, we let a_j denote the a-gauge given by $a_j(i) = a$ if $i \leq j$ and $a_j = 0$ if $i > j$. One checks easily that the set $\mathbf{G}_n$ of n-gauges forms a sublattice of $\mathbf{G}$. If $\epsilon \in \mathbf{G}_n$ and $(a, b) \in \mathbf{Z} \times \mathbf{Z}$, let $T_{a,b}(\epsilon)(i) =: \epsilon(i-a) + b$. Then $T_{a,b}(\epsilon) \in \mathbf{G}_n$, and $T_{a,b}$ is a lattice automorphism $\mathbf{G} \to \mathbf{G}$. Finally, we define an involution $\epsilon \mapsto \epsilon'$ on $\mathbf{G}_n$ by setting $\epsilon'(i) =: \epsilon(-i) - i$.

4.2.1 Proposition: *For each* $\epsilon \in \mathbf{G}_1$*, let* $\sigma(\epsilon) =: \{(\epsilon(i) + i, \epsilon(i)) : i \in \mathbf{Z}\}$, $\sigma(c_{-\infty}) =: \mathbf{Z} \times \mathbf{Z}$*. Then the mapping*

$$\epsilon \mapsto \overline{\sigma}(\epsilon)$$

defines an anti-isomorphism of lattices $\mathbf{L} \to \mathbf{G}_1$ *and is compatible with the involutions* $'$ *and the translation operations* $T_{a,b}$.

Proof: Suppose that $\sigma \in \mathbf{L}$; we will find a 1-gauge ϵ such that $\overline{\sigma}(\epsilon) = \sigma$. We may suppose that σ is a proper subset of $\mathbf{Z} \times \mathbf{Z}$. Let A be the image

[3] Strictly speaking, our "1-gauges" correspond to Mazur's "gauge functions" [23]. We make no use of his "cutoffs," which seem to be partially responsible for his Lemma (5.2), which is not desirable in our situation.

of σ under the first projection map $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$. For each $a \in A$, let $B(a) =: \{b : (a, b) \in \sigma\}$, and let $b(a) \in \overline{\mathbf{Z}}$ be $\inf B(a)$. Notice that, since σ is closed, $a < a'$ implies that $b(a') \leq b(a)$. In particular, if $b(a) = -\infty$, then the same is true of $b(a')$ for all $a' \geq a$; we let a_∞ be the smallest a such that $b(a) = -\infty$ if such an a exists. (Note that the set of a's such that $b(a) = -\infty$ is bounded below, since σ is a proper subset of $\mathbf{Z} \times \mathbf{Z}$.) Let A' be the set of all $a \in A$ such that $b(a)$ is finite. For each $a \in A'$, let $i(a) =: a - b(a)$, and note that if a and a' are two members of A' with $a < a'$ we have

$$i(a') - i(a) = a' - b(a') - a + b(a) > b(a) - b(a') \geq 0.$$

Then $a' - b(a) = i(a') - (b(a) - b(a'))$ lies in the interval $[i(a), i(a')]$, and we can define a function ϵ on this interval by the rule

$$\epsilon(i) =: \begin{cases} b(a) & \text{if } i \leq a' - b(a) \\ a' - i & \text{if } i \geq a' - b(a) \end{cases}$$

Then $\epsilon(i(a)) = b(a)$, $\epsilon(i(a')) = b(a')$, and ϵ is the restriction of a 1-gauge to $[i(a), i(a')]$. If the set A' is bounded below and if a_0 is its greatest lower bound, define

$$\epsilon(i) =: a_0 - i \text{ for } i \leq i(a_0).$$

If A' is bounded above, let a'_∞ be its greatest upper bound, so that $a_\infty = a'_\infty + 1$. Set $i(a_\infty) =: a_\infty - b(a'_\infty)$, and define

$$\epsilon(i) =: \begin{cases} b(a'_\infty) & \text{if } i \in [i(a'_\infty), i(a_\infty)] \\ a_\infty - i & \text{if } i \geq i(a_\infty) \end{cases}$$

Then $\epsilon \in \mathbf{G}_1$, and I claim that $\overline{\sigma}(\epsilon) = \sigma$. Suppose first that $(a, b) \in \sigma$. If $B(a)$ is bounded below, $b \geq b(a)$. As $(\epsilon(i(a)) + i(a), \epsilon(i(a))) = (a, b(a))$, we see that $(a, b(a)) \in \sigma(\epsilon)$ and hence $(a, b) \in \overline{\sigma}(\epsilon)$. If $B(a)$ is not bounded below, $a \geq a_\infty$ and we let $b' =: b \wedge b(a'_\infty)$. Then setting $i =: a_\infty - b'$, we see that $i \geq i(a_\infty)$ and so $\epsilon(i) = a_\infty - i = b'$. Then $(\epsilon(i) + i, \epsilon(i)) = (a_\infty, b')$ belongs to $\sigma(\epsilon)$ and so $(a, b) \in \overline{\sigma}(\epsilon)$. For the converse it suffices to check that $\sigma(\epsilon) \subseteq \sigma$. First suppose that $i \in [i(a), i(a')]$, with a and a' in A'. If $i \leq a' - b(a)$, $(\epsilon(i) + i, \epsilon(i)) = (b(a) + i, b(a))$. The first coordinate of this pair is at least a, and as $(a, b(a))$ belongs to σ, so does the pair. If $i \geq a' - b(a)$, $(\epsilon(i) + i, \epsilon(i)) = (a', a' - i)$ and the second coordinate is greater than $b(a')$, so again the pair belongs to σ. Now if $i < i(a_0)$, $(\epsilon(i) + i, \epsilon(i)) = (a_0, a_0 - i)$, and $a_0 - i > b_0$ so again our pair belongs to σ. If $i \in [i(a'_\infty), i(a_\infty)]$, then $\epsilon(i) + i, \epsilon(i)) = ((b(a'_\infty) + i, (b(a'_\infty))$ whose first coordinate is at least a'_∞ and so belongs to σ. Finally, if $i > i(a_\infty)$, $(\epsilon(i)+i, \epsilon(i)) = (a_\infty, a_\infty - i)$, which trivially belongs to σ because $B(a_\infty)$ is not bounded below. This concludes the proof that $\overline{\sigma}(\epsilon) = \sigma$.

Suppose ϵ and δ are two 1-gauges and suppose that $\overline{\sigma}(\epsilon) \subseteq \overline{\sigma}(\delta)$. I claim that then $\epsilon \geq \delta$. Indeed, if $i \in \mathbf{Z}$,

$$(\epsilon(i)+i, \epsilon(i)) \in \overline{\sigma}(\epsilon) \subseteq \overline{\sigma}(\delta),$$

so there exists a j such that $(\epsilon(i)+i, \epsilon(i)) \geq (\delta(j)+j, \delta(j))$. If $i \leq j$, we use the fact that δ is a 1-gauge to see that

$$\epsilon(i)+i \geq \delta(j)+j \geq \delta(i)+i,$$

and hence $\epsilon(i) \geq \delta(i)$. If on the other hand $j \leq i$, we use the fact that δ is decreasing to see that $\epsilon(i) \geq \delta(j) \geq \delta(i)$.

It follows that $\epsilon = \delta$ if $\overline{\sigma}(\epsilon) = \overline{\sigma}(\delta)$, and hence that our mapping $\sigma \colon \mathbf{G} \to \mathbf{L}$ is bijective and order reversing, and hence defines an anti-isomorphism of lattices. The verification that our isomorphism is compatible with the involutions and translation operations is immediate. ∎

We next discuss a generalization of Mazur's notion of "tame" gauges.

4.2.2 Definition: *A "control function" is a nondecreasing $g \colon \mathbf{N} \to \mathbf{N}$ such that $g(0) = 0$ and $g(i)+g(j) \geq g(i+j)$ for $i, j \in \mathbf{N}$. For any control function g, set*

$$\mathbf{G}_g =: \{\epsilon \in \mathbf{G} : \epsilon(i) - \epsilon(j) \geq g(j-i) \text{ whenever } j \geq i\}.$$

Elements of $\mathbf{G}_g$ will be called "g-tame."

Notice that $\mathbf{G}_g \subseteq \mathbf{G}_{g(1)} = \mathbf{G}_{g(1)id}$. For example, the function

$$i \mapsto \langle i \rangle =: \inf\{\operatorname{ord}_p p^j/j! : j \geq i\}$$

is a control function. It is easy to see that if g is any control function, the subset $\mathbf{G}_g$ of $\mathbf{G}$ satisfies the following conditions:

1. $\mathbf{G}_g$ contains the constant functions.
2. $\mathbf{G}_g$ is stable under the operations $T_{a,b}$.
3. $\mathbf{G}_g$ is closed in $\mathbf{G}$ under infima and suprema.

4.2.3 Proposition: *For each $n \in \mathbf{N}$, the assignment $g \mapsto \mathbf{G}_g$ defines a bijection between the set of control functions g with $g(1) \leq n$ and the set of subsets of $\mathbf{G}_n$ satisfying the above conditions.*

Proof: Suppose that $\mathbf{G}_\gamma \subseteq \mathbf{G}_n$ satisfies the three conditions above. If $j \in \mathbf{Z}$, let $\mathbf{G}_{\gamma,j} =: \{\epsilon \in \mathbf{G}_\gamma : \epsilon(j) = 0\ \}$. Note that if $\epsilon \in \mathbf{G}_{\gamma,j}$, $\epsilon(i) \leq (j-i)n \vee 0$ for all i, and hence

$$\epsilon_j =: \sup \mathbf{G}_{\gamma,j} \tag{4.2.3.1}$$

exists and belongs to $\mathbf{G}_\gamma$. This function will be the maximal tame gauge which vanishes at j, and we will use it to construct our control function g. Note that $c_0 \in \mathbf{G}_{\gamma,j}$, and hence $\epsilon_j(i) = 0$ for $i \geq j$. Furthermore, if $\epsilon \in \mathbf{G}_\gamma$ $T_{0,-\epsilon(j)}(\epsilon) \in \mathbf{G}_{\gamma,j}$, so $T_{0,-\epsilon(j)}(\epsilon) \leq \epsilon_j$, that is,

$$\epsilon \leq \epsilon_j + \epsilon(j) \quad \text{for any } \epsilon \in \mathbf{G}_\gamma \text{ and } j \in \mathbf{Z} \tag{4.2.3.2}$$

Next, I claim that

$$T_{i,0}(\epsilon_j) = \epsilon_{i+j} \text{ for all } i, j. \tag{4.2.3.3}$$

Indeed, $T_{i,0}(\epsilon_j(i+j)) = \epsilon_j(j) = 0$, so $T_{i,0}(\epsilon_j) \leq \epsilon_{i+j}$ for all i and j. Apply this with i replaced by $-i$ and j by $i+j$ to conclude that $T_{-i,0}(\epsilon_{i+j}) \leq \epsilon_j$, hence $\epsilon_{i+j} \leq T_{i,0}(\epsilon_j)$, whence we have equality.

Let $g(j) =: \epsilon_j(0) = \epsilon_0(-j)$ for $j \geq 0$. Then $g(0) = 0$ and $g(j) \geq 0$ for all $j \geq 0$. Furthermore, note that

$$\epsilon_{i+j}(j) - \epsilon_i(0) = T_{j,0}(\epsilon_i)(j) - \epsilon_i(0) = \epsilon_i(0) - \epsilon_i(0) = 0.$$

Hence $\epsilon_{i+j} - \epsilon_i(0) \in \mathbf{G}_{\gamma,j}$, and thus

$$\epsilon_{i+j} - \epsilon_i(0) \leq \epsilon_j. \tag{4.2.3.4}$$

This implies that $\epsilon_{i+j}(0) \leq \epsilon_i(0) + \epsilon_j(0)$, so g is a control function.

I claim that $\mathbf{G}_\gamma = \mathbf{G}_g$. Suppose that $\epsilon \in \mathbf{G}_\gamma$ and $i \leq j$. Then by (4.2.3.2), $\epsilon(i) \leq \epsilon_j(i) + \epsilon(j)$, so

$$\epsilon(i) - \epsilon(j) \leq \epsilon_j(i) = \epsilon_{j-i}(0) = g(j-i),$$

and hence $\epsilon \in \mathbf{G}_g$. Conversely, if $\epsilon \in \mathbf{G}_g$ and if $j \geq i$, we have

$$\epsilon(i) - \epsilon(j) \leq g(j-i) = \epsilon_j(i).$$

On the other hand, if $j \leq i$ we have

$$\epsilon(i) \leq \epsilon(j) = \epsilon(j) + \epsilon_j(i),$$

so in any case we conclude that $\epsilon(i) \leq \epsilon_j(i) + \epsilon(j)$ for all i and j. Let $\delta_j =: \epsilon_j + \epsilon(j)$, and note that $\delta_j \in \mathbf{G}_\gamma$ and that $\epsilon \leq \delta_j$ for all j. It follows that $\delta =: \inf\{\delta_j : j \in \mathbf{Z}\} \in \mathbf{G}_\gamma$ and that $\epsilon \leq \delta$. On the other hand, we have $\epsilon(j) = \delta_j(j) \geq \delta(j)$ for all j, and hence $\epsilon = \delta$. This proves the proposition, except for the uniqueness of the control function g. For this, note that given a control function g, we can define $\alpha: \mathbf{Z} \to \mathbf{Z}$ by $\alpha(i) =: g(-i)$ if $i \leq 0$ and $\alpha(i) = 0$ if $i \geq 0$. Then using the fact that g is a control function we check that $\alpha \in \mathbf{G}_g$; it is in fact clear that $\alpha = \epsilon_0$ and hence that g is determined by $\mathbf{G}_g$. ∎

4.2.4 Remark: For any gauge ϵ, there is a maximal g-tame gauge ϵ_g which lies under ϵ. Specifically $\epsilon_g(j) = \inf\{\epsilon(k) + g(k-j) : k \geq j\}$, and is called the "$g$-tame closure of ϵ."

Warning: The operation of taking the g-tame closure of a gauge does not commute with the operation $\vee$. Notice also that if $\limsup i - g(i) = \infty$, $\epsilon_g = c_{-\infty}$ if ϵ is not bounded below. This is the case, for example, for the control function $\langle\ \rangle$.

When we are dealing with 1-gauges, it is also possible to describe the notion of tameness in terms of the lattice $\mathbf{L}$ of closed subsets of $\mathbf{Z} \times \mathbf{Z}$.

4.2.5 Lemma: *Let g be a control function with $g(1) = 1$ and say that a subset σ of $\mathbf{Z}\times\mathbf{Z}$ is g-tame if $(a, b) \in \sigma$ implies that $(a+g(k)-k, b+g(k)) \in \overline{\sigma}$ for all $k \geq 0$. Then a gauge ϵ is g-tame if and only if the corresponding σ is.*

Proof: First suppose that ϵ is g-tame. Suppose $(a, b) \in \sigma(\epsilon)$, and choose $j \in \mathbf{Z}$ such that $(a, b) = (\epsilon(j) + j, \epsilon(j))$. For $k \in \mathbf{N}$, let $i =: j - k$. Then by definition of $\sigma(\epsilon)$, $(\epsilon(i) + i, \epsilon(i)) \in \sigma(\epsilon)$. Since ϵ is g-tame, we have $\epsilon(i) \leq \epsilon(j) + g(k)$, and hence

$$(a + g(k) - k, b + g(k)) = (\epsilon(j) + g(k) + i, \epsilon(j) + g(k)) \in \overline{\sigma}(\epsilon).$$

Conversely, suppose that $\sigma(\epsilon)$ is g-tame and $i < j$. Set $k =: j - i$. We have $(\epsilon(j) + j, \epsilon(j)) \in \sigma(\epsilon)$, and since the latter is g-tame, it follows that

$$(\epsilon(j) + i + g(k), \epsilon(j) + g(k)) = (\epsilon(j) + j + g(k) - k, \epsilon(j) + g(k)) \in \overline{\sigma}(\epsilon).$$

Then there exists an integer m such that

$$(\epsilon(j) + i + g(k), \epsilon(j) + g(k)) \geq (\epsilon(m) + m, \epsilon(m)).$$

Now if $m \leq i$, $\epsilon(m) \geq \epsilon(i)$ and we conclude that $\epsilon(j)+g(k) \geq \epsilon(i)$, as required. On the other hand, if $i \leq m$, $\epsilon(m)+m \geq \epsilon(i)+i$ and so $\epsilon(j)+i+g(k) \geq \epsilon(i)+i$, which yields the same conclusion. ∎

Note that σ is g-tame if and only if $\overline{\sigma}$ is. We let $\mathbf{L}_g$ denote the lattice of g-tame elements of $\mathbf{L}$. It is clear that for any $\sigma \subseteq \mathbf{Z} \times \mathbf{Z}$, there is a unique minimal element in the set of all $\sigma' \in \mathbf{G}_g$ which contain σ; we denote this element by σ_g and call it the "g-tame closure" of σ.

4.3 Tame gauge structures and G-transversality

4.3.1 Definition: *Let t be a nonzero divisor of a ring R. A multiplicative filtration $\mathcal{J}$ of R is "t-principal" if each $\mathcal{J}^i$ is principal, generated by some power of t.*

If $\mathcal{J}$ is t-principal we can define $g: \mathbf{N} \to \mathbf{N}$ by the rule $\mathcal{J}^i = (t^{g(i)})$; then g is a control function. Conversely, if g is a control function, the same rule defines a filtration $\mathcal{J}$ such that $\mathcal{J}^i \mathcal{J}^j \subseteq \mathcal{J}^{i+j}$. To avoid overloading the notation we shall suppose in what follows that t is fixed in advance. In practice, t will always be our prime number p. Recall that $J =: \mathcal{J}^1$.

4.3.2 Definition: *Suppose that $\mathcal{J}$ is t-principal, given by a control function g, and that (E, A) is a filtered R-module. We say that (E, A) is "G-transversal to (g, t)" (or just to g) if and only if A is exhaustive, G-transversal to $\mathcal{J}$, and compatible with t.*

We shall see from (4.3.4) that in fact if (E, A) is G-transversal to g then $(E/JE, A)$ is necessarily normally transversal to t^i for every i. Note again that this condition is trivial if $g(1) = 1$.

4.3.3 Definition: *Let $\breve{E}$ be an R-module on which t acts bijectively. A "g-gauge structure," or "$\mathbf{G}_g$-structure," on $\breve{E}$ is a decreasing exhaustive $\mathbf{G}_g$-filtration A on $\breve{E}$ such that*

1. *$A^{\epsilon \vee \delta} \breve{E} = A^{\epsilon} \breve{E} \cap A^{\delta} \breve{E}$ for every ϵ and δ in $\mathbf{G}_g$.*
2. *$A^{\epsilon} \breve{E} = \sum_{\delta} A^{\delta} \breve{E}$ if $\epsilon = \inf\{\delta \in D\}$, whenever D is a subset of $\mathbf{G}_g$ which is bounded below.*
3. *$A^{\epsilon+1} \breve{E} = tA^{\epsilon} \breve{E}$ if $\epsilon \in \mathbf{G}_g$*

The first two conditions just say that A is a continuous lattice filtration on $\breve{E}$. We write E for $A^{c_0} \breve{E}$, and note that $\breve{E}$ is necessarily the localization of E by t. Indeed, it is clear that there is an injective map $E_t \to \breve{E}$. If $x \in \breve{E}$, $x \in A^{\epsilon}_{c_{-\infty}} \breve{E}$, since the filtration A is exhaustive. As $\epsilon_{c_{-\infty}} = \inf\{c_n : n \in \mathbf{Z}\}$, it follows from axiom 2 that for some $n << 0$, $x \in A^{c_n} \breve{E}$. Then by axiom 3, $A^{c_n} \breve{E} \subseteq t^n E$, so $x \in t^n E$ and our map is surjective. We shall often write $A^{\epsilon} E$ instead of $A^{\epsilon} \breve{E}$. Recall that for each $i \in \mathbf{Z}$, $\epsilon_i(j) =: g(i-j)$ if $i \geq j$ and $\epsilon_i(j) =: 0$ otherwise.

We say that $(\breve{E}, A)$ has "level within $[a, \infty)$" if whenever $\epsilon(a) = d$, $A^{\epsilon} \breve{E} \supset A^{c_d} \breve{E}$, or equivalently, $A^{\epsilon \vee c_d} \breve{E} = A^{c_d} \breve{E}$, or $A^{\epsilon \wedge c_d} \breve{E} = A^{\epsilon} \breve{E}$. We say that $(\breve{E}, A)$

has "g-level within $(\infty, b]$ if and only if whenever ϵ is a g-tame gauge with $\epsilon(b) = e$, $A^\epsilon \breve{E} \subseteq A^{c_e}\breve{E}$ (equivalently, $A^\epsilon \breve{E} = A^{\epsilon \vee c_e}\breve{E}$, or $A^{c_e}\breve{E} = A^{\epsilon\wedge c_e}\breve{E}$.)

4.3.4 Proposition: *Let $\mathcal{J}$ be a principal multiplicative filtration on R with associated control function g, let E be a t-torsion-free R-module with a filtration A which is G-transversal to g, and let $\breve{E} =: E_t$. For each $\epsilon \in \mathbf{G}_g$ define $A^\epsilon \breve{E}$ by*

$$A^\epsilon \breve{E} = \sum_i A^{\epsilon_i + \epsilon(i)}\breve{E} = \sum_i t^{\epsilon(i)} A^i E. \tag{4.3.4.1}$$

Then $\epsilon \mapsto A^\epsilon \breve{E}$ defines a $\mathbf{G}_g$-structure on $\breve{E}$. Furthermore, every $\mathbf{G}_g$-structure arises in this way, and in fact there is an equivalence between the category of R-modules (E, A) with filtration G-transversal to g and the category of $\mathbf{G}_g$-structures.

Proof: Suppose that A is a filtration on E which is G-transversal to g and define $A^\epsilon\breve{E}$ as above. Then I claim that this defines a $\mathbf{G}_g$-structure on $\breve{E}$. The only nontrivial property to check is (4.3.3.1). To verify it, suppose that ϵ and δ belong to $\mathbf{G}_g$. Suppose also for the moment that ϵ and δ are constant outside a finite interval.

We proceed by induction on the sum of the lengths of such an interval for ϵ and δ. Note that if $\epsilon \geq \delta$ or $\delta \geq \epsilon$, the result is trivial, and in particular it is trivial if ϵ and δ are both constant. Let n be the smallest extended integer such that $\epsilon(n') = \epsilon(n)$ for all $n' \geq n$, and let m be the analogue for δ.

Case 1. ϵ is not constant and $\epsilon(n) \geq \delta(n)$.
In this case, write $x \in A^\epsilon\breve{E} \cap A^\delta\breve{E}$ as $\sum_{i<n} t^{\epsilon(i)}x_i + t^{\epsilon(n)}x_n$ with $x_i \in A^iE$. Then $x'' =: t^{\epsilon(n)}x_n \in A^{\epsilon\vee\delta}\breve{E}$ and so $x' =: x - x'' \in A^\epsilon\breve{E} \cap A^\delta\breve{E}$. But in fact $x' \in A^{\epsilon'}\breve{E}$, where $\epsilon' = \epsilon \vee c_{\epsilon(n-1)}$. As ϵ' is constant outside a smaller interval than ϵ, we can conclude by induction that $x' \in A^{\epsilon'\vee\delta}\breve{E} \subseteq A^{\epsilon\vee\delta}\breve{E}$, and hence the same is true of x.

Case 2. δ is not constant and $\delta(m) \geq \epsilon(m)$.
This case is proved in the same way as Case 1.

Case 3. $\epsilon(n) < \delta(n)$ and $\delta(m) < \epsilon(m)$.
We may suppose without loss of generality that $m > n$; in this case ϵ is allowed to be constant but δ cannot be. We argue by induction on $\epsilon(m) - \delta(m)$. Dividing by $t^{\delta(m)}$, we may assume without loss of generality that $\delta(m) = 0$. Write $x = \sum_i t^{\delta(i)}x_i$, with $x_i \in A^iE$. As $\epsilon(n) = \epsilon(m) > 0$ and $x \in A^\epsilon\breve{E}$, $x \in tE$, and as $\delta(i) \geq 1$ for $i < m$, $x_m \in tE$. It follows that $x_m \in tE \cap A^mE$. Since $(E/JE, A)$ is normally transversal to t, we see that

x_m lies in $tA^mE + JE$. Write $x_m = y_m + z$ with $y_m \in tA^mE$ and $z \in JE$; then

$$\begin{aligned} z \in JE \cap A^mE &= t^{g(1)}A^{m-1}E + t^{g(2)}A^{m-2}E + \cdots \\ &= t^{\epsilon_m(m-1)}A^{m-1}E + t^{\epsilon_m(m-2)}A^{m-2}E + \cdots, \end{aligned}$$

since $g(i) = \epsilon_m(m-i)$. As $\delta(m) = 0$, $\delta \leq \epsilon_m$, and so we can write $x_m = ty_m + \sum_{i<m} t^{\delta(i)}y_i$ with $y_i \in A^iE$. Set $x'_i =: x_i + y_i$ for $i < m$ and observe that we have $x = ty_m + \sum_{i<m} t^{\delta(i)}x'_i$ with $x'_i \in A^iE$. This shows that in fact $x \in A^{\delta'}\breve{E}$, where $\delta' =: \delta \vee c_1$. The induction hypothesis then implies that $x \in A^{\delta' \vee \epsilon}\breve{E} \subseteq A^{\delta \vee \epsilon}\breve{E}$.

Notice that if ϵ is constant and δ is not and ϵ and δ cross, we are covered by Case 3. Thus the above cases cover all possibilities and our result is proved provided ϵ and δ are constant outside a finite interval.

Suppose next that ϵ and δ are eventually constant for large values of i and that (E, A) has level within $[a, \infty)$, where a is finite. Let $\overline{\epsilon} =: \epsilon \wedge c_{\epsilon(a)}$. Since (E, A) has level within $[a, \infty)$, we have

$$A^\epsilon \breve{E} = \sum_{i \geq a} t^{\epsilon(i)}A^iE = A^{\overline{\epsilon}}\breve{E},$$

and $\overline{\epsilon}$ is constant outside a finite interval. Using the same notation for δ, we find that

$$\begin{aligned} A^\epsilon \breve{E} \cap A^\delta \breve{E} &= A^{\overline{\epsilon}}\breve{E} \cap A^{\overline{\delta}}\breve{E} \\ &= A^{\overline{\epsilon} \vee \overline{\delta}}\breve{E} \\ &= A^{\overline{\epsilon \vee \delta}}\breve{E} \\ &= A^{\epsilon \vee \delta}\breve{E} \end{aligned}$$

Now suppose that (E, A) has finite level and that ϵ is arbitrary but that δ is eventually constant, and let $\rho_n(\epsilon) =: \epsilon \vee c_{\epsilon(n)}$. Note that $\rho_n(\epsilon)$ still belongs to $\mathbf{G}_g$ and is constant for large i. As $\epsilon = \inf\{\rho_n : n \in \mathbf{Z}\}$ and $\rho_{n+1} \leq \rho_n$, we have

$$\begin{aligned} A^\epsilon \breve{E} \cap A^\delta \breve{E} &= \Big(\sum_n A^{\rho_n(\epsilon)}\breve{E}\Big) \cap A^\delta \breve{E} \\ &= \Big(\bigcup_n A^{\rho_n(\epsilon)}\breve{E}\Big) \cap A^\delta \breve{E} \\ &= \bigcup_n \Big(A^{\rho_n(\epsilon}\breve{E} \cap A^\delta \breve{E}\Big) \\ &= \bigcup_n A^{\rho_n(\epsilon) \vee \delta}\breve{E} \end{aligned}$$

$$
\begin{aligned}
&= A^{\inf\{\rho_n(\epsilon)\vee\delta\}}\breve{E} \\
&= A^{\epsilon\vee\delta}\breve{E}
\end{aligned}
$$

Arguing similarly for δ, we easily deduce our formula for arbitrary ϵ and δ, provided that (E, A) has finite level.

To eliminate the hypothesis on the level, note first that for any a, the induced filtration (A^aE, A) is still G-transversal to g, and now has finite level. But we have $A^\epsilon E = \cup_a A^\epsilon A^a E$ for any ϵ, and so the general case follows by an argument similar to that of the previous paragraph.

Conversely, suppose that $(\breve{E}, A)$ is a $\mathbf{G}_g$-structure, and set $E =: A^{c_0}\breve{E}$. By property 3 of the definition of $\mathbf{G}_g$-structures, $t^jE = A^{c_j}\breve{E}$ for each $j \in \mathbf{Z}$. For each integer i, let $A^iE =: A^{\epsilon_i}\breve{E} \subseteq E$, where ϵ_i is defined in equation (4.2.3.1). We have already seen in the proof of (4.2.3) that if $\epsilon \in \mathbf{G}_g$, $\epsilon = \inf\{\epsilon_i + \epsilon(i)\}$, and hence

$$
A^\epsilon\breve{E} = \sum_i A^{\epsilon_i+\epsilon(i)}\breve{E} = \sum_i t^{\epsilon(i)}A^iE. \tag{4.3.4.2}
$$

I claim that the filtration (E, A) is G-transversal to g. Indeed, to see that it is $\mathcal{J}$-saturated, recall from (4.2.3.4) that $\epsilon_{i+j} \leq \epsilon_j + \epsilon_i(0)$, for all $i \geq 0$ and $j \in \mathbf{Z}$. Therefore we have

$$
\begin{aligned}
A^{\epsilon_j+\epsilon_i(0)}\breve{E} &\subseteq A^{\epsilon_{i+j}}\breve{E} \\
t^{\epsilon_i(0)}A^{\epsilon_j}\breve{E} &\subseteq A^{\epsilon_{i+j}}\breve{E} \\
\mathcal{J}^iA^jE &\subseteq A^{i+j}E
\end{aligned}
$$

Now suppose that $0 \leq e \leq g(1)$. Then

$$
\epsilon_j(i) \vee e =: \begin{cases} g(j-i) & \text{if } i < j \\ e & \text{if } i \geq j \end{cases}
$$

Hence

$$
\begin{aligned}
t^eE \cap A^jE = A^{c_e}\breve{E} \cap A^{\epsilon_j}\breve{E} &= A^{\epsilon_j \vee c_e}\breve{E} \\
&= \sum_{i \geq j} t^eA^iE + \sum_{i<j} t^{g(j-i)}A^iE \\
&= t^eA^jE + \mathcal{J}^1A^{j-1}E + \mathcal{J}^2A^{j-2}E + \cdots
\end{aligned}
$$

Taking $e = g(1)$ we see that (E, A) is G$'$-transversal to $\mathcal{J}$. Taking $e = 1$ and reducing modulo JE we see that $(E/JE, A)$ is normally transversal to t.

It is clear that our constructions define an equivalence of categories as described. We should also remark that they preserve level. ∎

For example, we see easily that to give a $\mathbf{G}_1$-structure on $\breve{E}$ is the same as giving a submodule $E \subseteq \breve{E}$ such that $E_t \cong \breve{E}$ and a filtration A of E which is G-transversal to (t). Let t_E be the t-adic filtration of $\breve{E}$ defined by $E \subset \breve{E}$. Then one verifies immediately that, in the notation of (4.2.1),

$$(\breve{E}, A, (t))_{\sigma(\epsilon)} = A^{\epsilon}\breve{E}. \tag{4.3.4.3}$$

Warning: More generally, any control function g with $g(1) = 1$ defines a lattice of tame gauges $\mathbf{G}_g$ and a corresponding lattice $\mathbf{L}_g \subseteq \mathbf{L}$. Then to give a $\mathbf{G}_g$-structure on $\breve{E}$ is the same as giving a submodule E as above and a filtration A which is G-transversal to (g, t). However, (4.3.4.3) is not valid for all g-tame gauges ϵ.

4.3.5 Remark: If $(\breve{E}, A)$ has g-level within $[a, b]$, then for any g-tame gauge ϵ, $A^{\epsilon}E = A^{\epsilon \wedge c_{\epsilon(a)} \vee c_{\epsilon(b)}}E = \sum_a^b p^{\epsilon(i)}A^iE$.

4.3.6 Remark: Let $\mathbf{G}_g^{\flat}$ denote the set of all elements of $\mathbf{G}_g$ which are bounded below. Because every member of $\mathbf{G}_g$ is the infimum of a subset of $\mathbf{G}_g^{\flat}$, it follows from axiom (2) of (4.3.3) that a $\mathbf{G}_g$-structure is determined by its restriction to $\mathbf{G}_g^{\flat}$. Furthermore, arguing as we did at the end of the proof of Proposition (4.3.4), we see that any $\mathbf{G}_g^{\flat}$-structure extends to a $\mathbf{G}_g$-structure, and the two notions are essentially equivalent.

4.3.7 Lemma: *Suppose that $\mathcal{J}$ is the filtration of R defined by a control function g and a nonzero divisor $t \in R$, and let (E, A) be a $\mathbf{Z}$-filtered R-module.*

1. *If $(E, A_{\mathcal{J}})$ is the $\mathcal{J}$-saturation of (E, A) and if $\epsilon \in \mathbf{G}_g$, then*

$$A^{\epsilon}_{\mathcal{J}}\breve{E} = A^{\epsilon}\breve{E}.$$

2. *If (E, A) is $\mathcal{J}$-saturated and if ϵ_g is the g-tame closure (4.2.4) of ϵ, then*

$$A^{\epsilon_g}\breve{E} = A^{\epsilon}\breve{E}.$$

Proof: For the first statement, we have:

$$t^{\epsilon(j)}A^j_{\mathcal{J}}E \quad = \quad t^{\epsilon(j)} \sum_{i=-\infty}^{j} t^{g(j-i)}A^iE$$

$$
\begin{aligned}
&\subseteq t^{\epsilon(j)} \sum_{i=-\infty}^{j} t^{\epsilon(i)-\epsilon(j)} A^i E \\
&\subseteq \sum_{i=-\infty} t^{\epsilon(i)} A^i E \\
&\subseteq A^\epsilon E
\end{aligned}
$$

This implies that $A^\epsilon_{\mathcal{J}} E \subseteq A^\epsilon E$; the reverse inclusion is obvious.

To prove the second statement, choose for each j a $k \geq j$ such that $\epsilon_g(j) = \epsilon(k) + g(k-j)$. Then

$$
\begin{aligned}
t^{\epsilon_g(j)} A^j E &= t^{\epsilon(k)+g(k-j)} A^j E \\
&\subseteq t^{\epsilon(k)} A^k E \\
&\subseteq A^\epsilon \breve{E}
\end{aligned}
$$

∎

For example, let $\epsilon_k(i) =: \langle k - i \rangle$ and $\sigma_k =: \sigma(\epsilon_k)$. Then if A is G-transversal to p, the saturation $A_{p,\gamma}$ of A with respect to (p, γ) is given by $A^k_{p,\gamma} E = A^{\epsilon_k} E = (E, A)_{\sigma_k}$.

4.4 Cohomology

We now discuss cohomology. As we have to deal with filtrations which are not finite, we cannot really rely on the standard references [16] and [2] for the filtered and bifiltered derived categories. This is not serious, since we won't need any of the deeper theorems from these sources. If $\mathbf{L}$ is a partially ordered set, we say a morphism $(K, A) \to (K', A')$ of filtered complexes in $\mathcal{A}$ is a "filtered quasi-isomorphism" if for each $\lambda \in \mathbf{L}$, the induced morphism $A_\lambda K \to A'_\lambda K'$ is a quasi-isomorphism. Of course, this implies that the induced map $\mathrm{GR}^A K \to \mathrm{GR}^{A'} K'$ is a quasi-isomorphism, but the converse is not true without further hypotheses. In order to save space we shall sometimes write A^i instead of $A^i K$.

4.4.1 Lemma: *Let (K, A, B) be a bifiltered complex in $\mathcal{A}$. Suppose that for each j, $A^k \mathrm{Gr}^j_B K$ is acyclic for $k >> 0$, and for each i, $B^k \mathrm{Gr}^i_A K$ is acyclic for $k >> 0$. Then the following are equivalent:*

1. *For each i and j, the intersection $A^i K \cap B^j K$ is an acyclic complex.*
2. *Whenever $i < i'$ and $j < j'$, the complex $(A^i \cap B^j)/(A^{i'} \cap B^j + A^i \cap B^{j'})$ is acyclic, and for some pair (m, n), the complex $A^m K \cap B^n K$ is acyclic.*

3. *For each i and j, the complex $\mathrm{Gr}_A^i \mathrm{Gr}_B^j K$ is acyclic, and for some pair (m,n), the complex $A^m K \cap B^n K$ is acyclic.*

4. *For every $\sigma \subset \mathbf{Z} \times \mathbf{Z}$, the complex $(K, A, B)_\sigma$ is acyclic.*

Proof: Suppose that $i' > i$ and $j' > j$. We have exact sequences of complexes:

$$0 \to A^{i'} \cap B^j \to A^i \cap B^j \to (A^i \cap B^j)/(A^{i'} \cap B^j) \to 0$$

$$0 \to A^i \cap B^{j'}/A^{i'} \cap B^{j'} \to (A^i \cap B^j)/(A^{i'} \cap B^j) \to (A^i \cap B^j)/(A^{i'} \cap B^j + A^i \cap B^{j'}) \to 0.$$

These sequences immediately show that condition (1) implies condition (2), and condition (3) is the special case of condition 2 obtained by setting $i' = i+1$ and $j' = j+1$. Suppose that condition 3 holds. If $k \geq j$ we have an exact sequence:

$$0 \to \mathrm{Gr}_B^k K \,\mathrm{Gr}_A^i K \to B^j K \,\mathrm{Gr}_A^i K / B^{k+1} K \,\mathrm{Gr}_A^i K \to B^j K \,\mathrm{Gr}_A^i K / B^k \,\mathrm{Gr}_A^i K \to 0$$

By induction on k we see that each $B^j \,\mathrm{Gr}_A^i K / B^k \,\mathrm{Gr}_A^i K$ is acyclic, and because $B^k \,\mathrm{Gr}_A^i K$ is acyclic for k large we conclude that each $B^j \,\mathrm{Gr}_A^i K$ is acyclic. A similar induction shows that the complexes $(A^i \cap B^j)/(A^{i'} \cap B^j)$ are acyclic, and, symmetrically, the complexes $(A^i \cap B^j)/(A^i \cap B^{j'})$ are acyclic. Applying the first of these with i or $i' = m$, we conclude that the complexes $(A^i \cap B^n)$ are acyclic for all i, and applying the second with j or $j' = n$ we see that condition (1) is also satisfied.

Supposing the first three conditions are satisfied, we prove condition (4) by induction on the cardinality of σ, the case of cardinality one being trivial. For the induction step, suppose $(m,n) \in \sigma$. We may assume without loss of generality that σ is reduced, for otherwise we would have $K_\sigma = K_\tau$ for some proper subset τ of σ, and we could apply the induction hypothesis. Now let $\sigma' =: \sigma \setminus \{(m,n)\}$, let m' be the smallest $i > m$ such that $(i,j) \in \sigma'$ for some j, and let n' be the smallest $j > n$ such that $(i,j) \in \sigma'$ for some i. Applying Lemma (4.1.2), we find:

$$A^m \cap B^n \cap K_{\sigma'} = A^{m'} \cap B^n + A^m \cap B^{n'}$$

As $K_\sigma = K_{\sigma'} + A^m \cap B^n$, the above equation implies that we have an exact sequence of complexes:

$$0 \to K_{\sigma'} \to K_\sigma \to A^m \cap B^n/(A^{m'} \cap B^n + A^m \cap B^{n'}) \to 0.$$

As we have seen, condition (1) of the lemma implies that the last complex is acyclic, and the induction hypothesis implies that the complex $K_{\sigma'}$ is acyclic.

Finally we remark that because filtering direct limits are exact in $\mathcal{A}$, we can deduce the truth of (4) for infinite subsets from its truth for all finite subsets. This completes the proof that (1) implies (4). On the other hand, the last exact sequence above also makes it clear that (4) implies (3). ∎

Applying the standard yoga of filtered mappings cones [16, V,1.2], we can conclude that formation of $(K, A, B)_\sigma$ is compatible with bifiltered quasi-isomorphisms, *i.e.*,

4.4.2 Corollary: *Suppose* $f\colon (K, A, B) \to (K', A', B')$ *is a bifiltered quasi-isomorphism and* $\sigma \subseteq \mathbf{Z} \times \mathbf{Z}$ *is a finite subset. Then* f *induces a quasi-isomorphism*

$$(K, A, B)_\sigma \to (K', A', B')_\sigma.$$

We can construct the filtered homotopy category $K(\mathcal{A}, \mathbf{L})$ by taking as objects $\mathbf{L}$-filtered complexes in $\mathcal{A}$, and as morphisms the filtered homotopy classes of maps. We then obtain the filtered derived category $D(\mathcal{A}, \mathbf{L})$ by localizing this category by the filtered quasi-isomorphisms. We have a similar notion for the bifiltered derived category. Suppose the abelian category $\mathcal{A}$ satisfies Grothendieck's axioms AB1-AB5 and admits a generator, so that every object of A can be embedded in an injective object $I(A)$. In fact Grothendieck shows in [13] that this can be done in a functorial way: there exist a functor $I\colon A \to A$ and a natural transformation $\iota\colon id_{\mathcal{A}} \to I$ such that each $I(A)$ is injective and each ι_A is a monomorphism. An investigation of his construction reveals more: if $A \subseteq B$, then $I(A) \subseteq I(B)$, and also $I(A)/A \subseteq I(B)/B$ and I applied to the zero morphism is zero. It is now clear that any object (K, A) of $D^+(A, \mathbf{L})$ is isomorphic to an object $\tilde{I}(K, A)$ each of whose terms is injective. For any additive functor $\Gamma\colon \mathcal{A} \to \mathcal{A}'$ we define $R\Gamma(K, A)$ to be the class of $\Gamma\tilde{I}(K, A)$. One can see that if (K', A) is any object of $D^+(\mathcal{A}, \mathbf{L})$ all of whose terms are acyclic for Γ, then any isomorphism $(K, A) \to (K', A)$ in $D^+(\mathcal{A}, \mathbf{L})$ induces a natural isomorphism $R\Gamma(K, A) \to \Gamma(K', A)$ in $D^+(\mathcal{A}', \mathbf{L})$. In the context of the bifiltered derived category it is worth noting that if (K', A, B) is a bifiltered object such that each $A^iK' \cap B^jK'$ is acyclic for Γ, then the same is true of each $(K, A, B)_\sigma$ for each finite subset σ of $\mathbf{Z} \times \mathbf{Z}$. The same is true for all subsets of σ provided that Γ preserves direct limits. Notice that if (K, A) is a continuous lattice filtration, then so is (K', A'), at least in the sense of the derived category. (That is, if $\sigma, \tau \in \mathbf{L}$, the natural maps $A'_\sigma K'/A'_{\sigma\wedge\tau}K' \to A'_{\sigma\vee\tau}K'/A'_\tau K'$ are quasi-isomorphisms, and similarly for direct limits.)

4.4.3 Proposition: *Let $(\breve{E}, A)$ be a $\mathbf{G}_g$-structure of level within $[a, \infty)$. Suppose that $\{(H^q, \delta^q) : q \in \mathbf{N}\}$ is a cohomological δ-functor, and suppose that for $q = n$ and $n+1$,*

1. *The A-module $H^q(E)$ is t-torsion free.*
2. *For all i the maps $H^q(A^iE/A^iE \cap tE) \to H^q(E/tE)$ are injective.*
3. *The functor H^q commute with direct limits.*

Then for all $\epsilon'' \geq \epsilon' \geq \epsilon \in \mathbf{G}_\gamma$, the sequences

$$0 \to H^n(A^{\epsilon'}\breve{E}) \to H^n(A^{\epsilon}\breve{E}) \to H^n(A^{\epsilon}\breve{E}/A^{\epsilon'}\breve{E}) \to 0$$

and

$$0 \to H^n(A^{\epsilon'}\breve{E}/A^{\epsilon''}\breve{E}) \to H^n(A^{\epsilon}\breve{E}/A^{\epsilon''}\breve{E}) \to H^n(A^{\epsilon}\breve{E}/A^{\epsilon'}\breve{E}) \to 0$$

are exact. Furthermore, if we let $A^\epsilon \breve{H}^n(E) =: H^n(A^\epsilon\breve{E}) \hookrightarrow \breve{H}^n(E)$ then $(H^n(E), A)$ defines a $\mathbf{G}_\gamma$-structure (4.3.3), and we have natural isomorphisms

$$A^\epsilon \breve{H}^n(E)/A^{\epsilon'}\breve{H}^n(E) \cong H^n(A^\epsilon\breve{E}/A^{\epsilon'}\breve{E})$$

Proof: The following argument, which works with just one value of q, replaces the method of comparing gauges by means of "simple augmentations," used in [23] and [4] and which does not apply in the current context.

4.4.4 Lemma: *With the notations of (4.4.3), suppose that the hypotheses (1)–(3) are satisfied for q. Then whenever $\epsilon' \geq \epsilon$, the map $H^q(A^{\epsilon'}\breve{E}) \to H^q(A^\epsilon\breve{E})$ is injective.*

Proof: If $\epsilon \in \mathbf{G}_g$, we write $H^q(\epsilon)$ for $H^q(A^\epsilon\breve{E})$, and if $\epsilon' > \epsilon$, we write $H^q(\epsilon/\epsilon')$ for $H^q(A^\epsilon\breve{E}/A^{\epsilon'}\breve{E})$, and $H^q_\infty(\epsilon/\epsilon')$ for the image of $H^q(\epsilon)$ in $H^q(\epsilon/\epsilon')$. We can and shall identify $H^q(\breve{E})$ with the localization of $H^q(E)$ by t. Notice that if $\epsilon'' > \epsilon' > \epsilon$, then we have a commutative diagram

$$\begin{array}{ccccc}
H^q(\epsilon') & \longrightarrow & H^q(\epsilon) & \xrightarrow{\cong} & H^q(\epsilon) \\
\downarrow & & \downarrow & & \downarrow \\
H^q(\epsilon'/\epsilon'') & \longrightarrow & H^q(\epsilon/\epsilon'') & \longrightarrow & H^q(\epsilon/\epsilon') \\
\downarrow & & \downarrow & & \downarrow \\
H^{q+1}(\epsilon'') & \xrightarrow{\cong} & H^{q+1}(\epsilon'') & \longrightarrow & H^{q+1}(\epsilon')
\end{array}$$

in which the columns and central row are exact. This diagram shows that the sequence

$$H^q_\infty(\epsilon'/\epsilon'') \longrightarrow H^q_\infty(\epsilon/\epsilon'') \longrightarrow H^q_\infty(\epsilon/\epsilon')$$

is exact.
Step 1: If $e' \geq e$, the maps

$$H^q(c_{e'}) \to H^q(c_e) \quad \text{and} \quad H^q_\infty(c_{e'}/c_{e'+1}) \to H^q_\infty(c_e/c_{e'+1})$$

are injective.
Proof: If $e' = e + d$, we have a commutative diagram

$$\begin{array}{ccc} H^q(E) & \xrightarrow{t^d} & H^q(E) \\ \downarrow{\scriptstyle f_{e'}} & & \downarrow{\scriptstyle f_e} \\ H^q(c_{e'}) & \longrightarrow & H^q(c_e) \end{array}$$

in which the map f_e (resp. $f_{e'}$) is induced by multiplication by t^e (resp. $t^{e'}$). These maps are isomorphisms, and because $H^q(E)$ is t-torsion free, it follows that $H^q(c_{e'}) \to H^q(c_e)$ is injective. Now consider the diagram

$$\begin{array}{ccccc} H^q(c_{e'+1}) & \longrightarrow & H^q(c_{e'}) & \xrightarrow{\pi} & H^q_\infty(c_{e'}/c_{e'+1}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ H^q(c_{e'+1}) & \longrightarrow & H^q(c_e) & \longrightarrow & H^q_\infty(c_e/c_{e'+1}) \end{array}$$

In this diagram α is an isomorphism, β is injective, and π is surjective. It follows that γ is also injective. This proves Step 1.

If $g(1) = 0$ there is nothing left to prove, so we assume that $g(1) > 0$. Let $1_i =: e_i \wedge c_1 \in \mathbf{G}_g$ for all $i \in \mathbf{Z}$.

Step 2: For any $i \in \mathbf{Z}$ and $e \in \mathbf{Z}$, the maps $H^q(1_i + c_e/c_{e+1}) \to H^q(c_e/c_{e+1})$ are injective.
Proof: First note that

$$A^i E/A^i E \cap tE \cong A^{\epsilon_i}\breve{E}/A^{\epsilon_i \vee c_1}\breve{E} \cong A^{\epsilon_i \wedge c_1}\breve{E}/A^{c_1}\breve{E} = A^{1_i}\breve{E}/A^{c_1}\breve{E}.$$

Thus the second hypothesis implies that the top horizontal arrow in the diagram below is is injective.

$$\begin{array}{ccc} H^q(1_i/c_1) & \longrightarrow & H^q(c_0/c_1) \\ \downarrow{\scriptstyle t^e} & & \downarrow{\scriptstyle t^e} \\ H^q(1_i + c_e/c_{e+1}) & \longrightarrow & H^q(c_e/c_{e+1}) \end{array}$$

As the vertical arrows are isomorphisms, it follows that the bottom arrow is also injective. This proves Step 2.

Step 3. For any $\epsilon \in \mathbf{G}_g$ and any $e \in \mathbf{Z}$, the map

$$H^q(\epsilon \vee c_e/\epsilon \vee c_{e+1}) \to H^q(c_e/c_{e+1})$$

is injective.

Proof: If $\epsilon \geq c_{e+1}$, $\epsilon \vee c_e = \epsilon \vee c_{e+1} = \epsilon$ and the injectivity is trivial. Similarly, if $\epsilon \leq c_e$, then $\epsilon \vee c_e = c_e$ and $\epsilon \vee c_{e+1} = c_{e+1}$ and the injectivity is again trivial. In the remaining case, there exists an $i \in \mathbf{Z}$ such that $\epsilon(i) \geq e+1$ and $\epsilon(i+1) \leq e$. Then if we set $\delta = \epsilon \vee c_e$ we find that $\epsilon \vee c_{e+1} = \delta \vee c_{e+1}$, so

$$\epsilon \vee c_e/\epsilon \vee c_{e+1} = \delta/\delta \vee c_{e+1} = \delta \wedge c_{e+1}/c_{e+1} = (1_i + c_e)/c_{e+1}.$$

Thus Step 3 follows from Step 2.

Step 4. For any $\epsilon \in \mathbf{G}_g$, $e \in \mathbf{Z}$ and $k \in \mathbf{N}$, the map

$$H^q_\infty(\epsilon \vee c_e/\epsilon \vee c_{e+k}) \to H^q_\infty(c_e/c_{e+k})$$

is injective.

Proof: We proceed by induction on k; the case $k = 0$ being trivial. We have a commutative diagram with exact rows:

$$\begin{array}{ccccc} H^q_\infty(\epsilon \vee c_{e+k}/\epsilon \vee c_{e+k+1}) & \xrightarrow{f} & H^q_\infty(\epsilon \vee c_e/\epsilon \vee c_{e+k+1}) & \xrightarrow{g} & H^q_\infty(\epsilon \vee c_e/\epsilon \vee c_{e+k}) \\ \downarrow a & & \downarrow b & & \downarrow c \\ H^q_\infty(c_{e+k}/c_{e+k+1}) & \xrightarrow{f'} & H^q_\infty(c_e/c_{e+k+1}) & \xrightarrow{g'} & H^q_\infty(c_e/c_{e+k}) \end{array}$$

In this diagram, a is injective by Step 3, c is injective by the induction assumption, and f' is injective by Step 1. It follows that b is also injective, proving Step 4.

Step 5: For any $\epsilon > \delta \in \mathbf{G}_g$, the map $H^q(\epsilon) \to H^q(\delta)$ is injective.

Proof: We may assume without loss of generality that $\delta = c_{-\infty}$, and that $\epsilon \neq c_\infty$. Fix integers d and e with $d \geq \epsilon(a) \vee e$. Then $(\epsilon \vee c_d - c_d)(a) = 0$, so that $(\epsilon \vee c_d - c_d) \leq \epsilon_a$ and $A^{\epsilon \vee c_d - c_d}\breve{E} \supseteq A^{\epsilon_a}\breve{E}$. Since (E, A) has level within $[a, \infty)$, $A^{\epsilon_a}\breve{E} = E$, and we conclude that $A^{\epsilon \vee c_d}\breve{E} \supseteq A^{c_d}\breve{E}$. Then in fact $A^{\epsilon \vee c_d}\breve{E} = A^{c_d}\breve{E}$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccc} H^q(\epsilon \vee c_d) & \xrightarrow{f} & H^q(\epsilon \vee c_e) & \xrightarrow{g} & H^q_\infty(\epsilon \vee c_e/\epsilon \vee c_d) \\ \downarrow a & & \downarrow b & & \downarrow c \\ H^q(c_d) & \xrightarrow{f'} & H^q(c_e) & \xrightarrow{g'} & H^q_\infty(c_e/c_d) \end{array}$$

In this diagram a is an isomorphism, c is injective by Step 4, and f' is injective by Step 1; it follows that b is injective. This is true for any integer e, and in particular if we take $e =: \epsilon(n)$ for n large. As $\epsilon = \inf\{\epsilon \vee c_{\epsilon(n)} : n \in \mathbf{N}\}$, we find that $H^q(\epsilon) = \varinjlim H^q(\epsilon \vee c_{\epsilon(n)})$, and Step 5 follows. This completes the proof of the first statement of the proposition.

Step 6. For any $\epsilon \in \mathbf{G}_g$, $H^q(\epsilon + 1) = tH^q(\epsilon)$.
Proof: This is is clear: we see from Step 5 that each $H^q(\epsilon)$ is t-torsion free, and we have by definition that $A^{\epsilon+1}\breve{E} = tA^\epsilon\breve{E}$. ∎

4.4.5 Remark: As a matter of fact, in the proof we used only a hypothesis which is slightly weaker than (2) above; it is enough to require that the maps

$$H^q_\infty(A^iE/A^iE \cap tE) \to H^q_\infty(E/tE)$$

be injective.

Now suppose that H^{q+1} also satisfies the above hypotheses. In this case, we conclude that the maps $H^{q+1}(\epsilon') \to H^{q+1}(\epsilon)$ are all injective. This implies:

4.4.6 Claim: *For any ϵ and δ in $\mathbf{G}_g$,*

$$H^q(\epsilon \wedge \delta) \cong H^q(\epsilon) \cap H^q(\delta) \text{ and } H^q(\epsilon \vee \delta) \cong H^q(\epsilon) + H^q(\delta).$$

Proof: We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\epsilon\breve{E} \vee A^\delta\breve{E} & \longrightarrow & A^\epsilon\breve{E} \oplus A^\delta\breve{E} & \longrightarrow & A^\epsilon\breve{E} \wedge A^\delta\breve{E} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \breve{E} & \longrightarrow & \breve{E} \oplus \breve{E} & \longrightarrow & \breve{E} & \longrightarrow & 0 \end{array}$$

Applying the cohomology functor $\{H^q : q \in \mathbf{N}\}$ we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^q(\epsilon \vee \delta) & \xrightarrow{a^q} & H^q(\epsilon) \oplus H^q(\delta) & \xrightarrow{b^q} & H^q(\epsilon \wedge \delta) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^q(\breve{E}) & \longrightarrow & H^q(\breve{E}) \oplus H^q(\breve{E}) & \longrightarrow & H^q(\breve{E}) & \longrightarrow & 0 \end{array}$$

It is obvious that the bottom row is exact, and we know by Step 5 that a^q is injective. Since a^{q+1} is also injective, it follows from the long exact sequence of cohomology that b^q is surjective, so that the top row is short exact. The claim follows easily.

Using the fact that H^q is compatible with direct limits we see that H^q is also compatible with infinite infima, and hence that $\epsilon \mapsto H^q(\epsilon)$ defines a $\mathbf{G}_g$-structure on $H^q(E)$ which evidently has level within $[a, \infty)$. Thus our proposition follows from (4.3.4). ∎

4.4.7 Corollary: *With the notations and hypotheses of (4.4.3), let P denote the filtration of $\mathrm{Gr}_A E$ induced by the t-adic filtration on E. Then for all i and j, the sequence*

$$0 \to H^n(P^i \operatorname{Gr}_A^j E) \to H^n(\operatorname{Gr}_A^j E) \to H^n(\operatorname{Gr}_A^j / P^i \operatorname{Gr}_A^j E) \to 0$$

is exact.

Proof: By the calculus of gauges, we have

$$P^i \operatorname{Gr}_A^j E = t^i E \cap A^j E / t^i E \cap A^{j+1} E \cong A^{c_i \vee \epsilon_j} \breve{E} / A^{c_i \vee \epsilon_{j+1}} \breve{E}, \qquad \text{and}$$

$$\operatorname{Gr}_A^j E / P^i \operatorname{Gr}_A^j E \cong A^{\epsilon_j} / A^{\epsilon_{j+1} \wedge (\epsilon_j \vee c_i)}$$

Of course, the same equations hold for the gauge structure $(H^q(E), A)$. Thus the previous corollary tells us that we have a commutative diagram

$$\begin{array}{ccccc}
H^q(P^i \operatorname{Gr}_A^j E) & \to & H^q(\operatorname{Gr}_A^j E) & \to & H^q(\operatorname{Gr}_A^j E / P^i \operatorname{Gr}_A^j E) \\
\downarrow & & \downarrow & & \downarrow \\
P^i \operatorname{Gr}_A^j H^q(E) & \to & \operatorname{Gr}_A^j H^q(E) & \to & \operatorname{Gr}_A^j H^q(E) / P^i \operatorname{Gr}_A^j H^q(E)
\end{array}$$

As the vertical arrows are isomorphisms and the bottom sequence is short exact, so is the top. ∎

4.4.8 Remark: Suppose that the base ring A in (4.4.3) is a discrete valuation ring with uniformizing parameter t and that all the cohomology groups $H^q(A^\epsilon \breve{E})$ and $H^q(A^\epsilon \breve{E}/A^{\epsilon'} \breve{E})$ are finitely generated A-modules. Then we can define the q^{th} Betti number $h^q(E)$ of E to be the rank of the (free part of) the finitely generated A-module $H^q(E)$. Because $E_0 =: E/tE$ is killed by t, $H^q(E_0)$ is a vector space over the residue field k of A; we let $h^q(E_0)$ denote its dimension (necessarily finite). Then one sees immediately from the long exact sequence associated to $0 \to E \to E \to E_0$ that $h^q(E_0) \geq h^q(E)$, with equality if and only if both $H^q(E)$ and $H^{q+1}(E)$ are t-torsion free. Similarly it follows from the exact sequences

$$H^q(A^i E_0) \longrightarrow H^q(E_0) \longrightarrow H^q(E_0/A^i E_0) \longrightarrow H^{q+1}(A^i E_0) \longrightarrow H^{q+1}(E_0)$$

that hypothesis (4.4.3.2) is true for q and for $q+1$ if and only if for all i, the sequence

$$0 \longrightarrow H^q(A^i E_0) \longrightarrow H^q(E_0) \longrightarrow H^q(E_0/A^i E_0) \longrightarrow 0$$

is exact. In fact, each $H^q(\mathrm{Gr}^i_A E_0)$ is a finite-dimensional vector space over k, and if we let $h^{i,q-i}(E_0, A)$ denote its dimension, it is easy to deduce the following statement:

$$h^q(E) \leq h^q(E_0) \leq \sum_i h^{i,q-i}(E_0, A),$$

and equality holds if and only if H^q *and* H^{q+1} *satisfy the hypotheses (4.4.3.1) and (4.4.3.2)*

4.4.9 Proposition: *Let* (K, A, B) *be a bifiltered complex of abelian sheaves on a noetherian topological space* X. *Let* $E(X, K, A)$ *denote the spectral sequence of hypercohomology [6, §3]:*

$$E(X, K, A) \Rightarrow H(X, K).$$

Suppose that the following hypotheses are satisfied:

1. *The complex* K *is bounded below.*

2. *The filtrations* A *and* B *are exhaustive, and the maps*

   $$H^q(B^i) \to \varprojlim H^q(B^i/B^{i+m})$$

   are isomorphisms for all q.

3. *For each* j, *the complex* $A^k \mathrm{Gr}^j_B K$ *is acyclic for* $k >> 0$, *and for each* i *the complex* $B^k \mathrm{Gr}^i_A K$ *is acyclic for all* $i' < i$ *and* $k >> 0$.

4. *The spectral sequences of the two filtered complexes* $E(X, K, B)$, *and* $E(X, \mathrm{Gr}_B K, A)$ *degenerate at* E_1 $(E_1 = E_\infty)$.

Then for all q, i, j *we have*

1. $H^q(X, A^i K) \subseteq H^q(X, K), \quad H^q(X, B^j K) \subseteq H^q(X, K)$

2. $H^q(X, A^i K \cap B^j K) \cong H^q(X, A^i K) \cap H^q(X, B^j K.)$

3. *The spectral sequence of each of the two filtered complexes* $E(X, K, A)$ *and* $E(X, \mathrm{Gr}_A K, B)$ *degenerates at* E_1 $(E_1 = E_\infty)$.

4. *If* σ *is a subset of* $\mathbf{Z} \times \mathbf{Z}$, *there is a natural isomorphism:*

   $$H^q(X, (K, A, B)_\sigma) \cong (H^q(X, K), A, B)_\sigma.$$

4.4.10 Remark: Instead of assuming that the arrows

$$H^q(X, B^i) \to \varprojlim H^q(X, B^i/B^{i+m})$$

are isomorphisms, we could instead assume that the maps $H^q(X, B^{i+1}) \to H^q(X, B^i)$ are injective for all i.

4.5 Gauges and the derived category

We shall also find it useful to discuss a construction which I learned from a letter from Deligne to Illusie, dated December, 1988. The origin of this technique is hard to trace; it goes back to ideas of Fontaine-Lafaille [11], Kato [19], and Fontaine-Messing [12]. Essentially this construction can be viewed as a derived category version of the gauge-theoretic operation $(E, C) \mapsto C^\epsilon E$.

Let R be a noetherian ring, separated and complete for the p-adic topology. Suppose that $C =: \{C^i, j_i \colon C^i \to C^{i-1} : i \in \mathbf{Z}\}$ is an inverse system of R-modules. If $\epsilon \colon \mathbf{Z} \to \mathbf{Z}$ is a nonincreasing function, let $\Delta(i) =: \epsilon(i) - \epsilon(i-1)$ and consider the chain complex given by:

$$T^\epsilon(C) =: \bigoplus_i C^i \xrightarrow{\partial^\epsilon} \bigoplus_i C^i, \tag{4.5.0.1}$$

where ∂^ϵ sends c_i to $(-p^{\Delta(i)}c_i, j_i(c_i))$ in $C^i \oplus C^{i-1}$. We let $L_0^\epsilon(C) =: H_0(T^\epsilon(C))$ and $L_1^\epsilon(C) =: H_1(T^\epsilon(C))$. Of course, a filtered module (E, C) can be viewed as an inverse system, and in this case we write $T^\epsilon(E, C)$ for the corresponding construction. It is clear that $L_1^\epsilon(E, C) = 0$ for any filtered module, since in that case the maps j_i are injective.

It is worth remarking that if $\epsilon' \geq \epsilon$, then there is a morphism of complexes

$$\beta^{\epsilon',\epsilon} \colon T^{\epsilon'}(C) \to T^\epsilon(C)$$

defined by

$$\begin{array}{rcll} \beta_1(c_i) & = & p^{\epsilon'(i-1)-\epsilon(i-1)}c_i & \text{for } c_i \in C^i \subseteq T_1^{\epsilon'}(C) \\ \beta_0(c_i) & = & p^{\epsilon'(i)-\epsilon(i)}c_i & \text{for } c_i \in C^i \subseteq T_0^{\epsilon'}(C) \end{array}$$

If $\epsilon'' \geq \epsilon' \geq \epsilon$ the corresponding morphisms are of course compatible. Note that if we identify $T^{\epsilon+1}(C)$ with $T^\epsilon(C)$, the arrow $\beta^{\epsilon+1,\epsilon}$ becomes identified with multiplication by p.

If (E, C) is a filtered module there is a natural map

$$\alpha^\epsilon \colon T_0^\epsilon(E, C) \to C^\epsilon E, \qquad \text{sending} \quad \oplus c_i \mapsto \sum_i p^{\epsilon(i)} c_i \tag{4.5.0.2}$$

It is clear that α^ϵ determines a surjection $L_0^\epsilon(E,C) \to C^\epsilon E$. If $\operatorname{Gr}_C E$ is p-torsion free, it is easy to see that this surjection is an isomorphism.

The functor T^ϵ passes over to inverse systems of complexes in the obvious way: if C is an inverse system of complexes then we let $T^\epsilon(C)$ denote the total complex formed from the double complex constructed by functoriality. It is obvious from the construction that the functor T^ϵ thus formed is exact, *i.e.* that it takes quasi-isomorphisms to quasi-isomorphisms.

4.5.1 Remark: (The following remark is not used in the sequel; it justifies our statement that T^ϵ can be viewed as a derived category version of $(E,C) \mapsto C^\epsilon E$.) If E is any R-module, let $F(E)$ denote the free R-module spanned by the nonzero elements of E. Then $F(E)$ becomes a functor of E in an obvious way; it is not additive but takes the zero morphism to zero. Furthermore we have a natural surjection $\sigma\colon F(E) \to E$. Suppose that R is p-torsion free and that (E,C) is a filtered complex of R-modules such that each C^iE^q is separated and complete for the C-adic topology and such that $C^iE^q = E^q$ for $i \leq a$. Let $K^{0q} =: \prod_{j\geq a} F(C^jE^q)$, and let $C^iK^{0q} =: \prod_{j\geq i} F(C^jE^q) \subseteq K^{0q}$. The boundary maps of E induce maps $K^{0q} \to K^{0q+1}$, and in fact (K^0, C) becomes a filtered complex. We can define a map $(K^{0q}, C) \to (E^q, C)$ by sending $(x_i : i \geq a)$ to $\sum_i \sigma(x_i)$; this map is strictly compatible with the filtrations, surjective, and compatible with the boundary maps. Let K^{-1} be its kernel, with the induced structure of a filtered complex. We have a strict exact sequence $0 \to (K^{-1}, C) \to (K^0, C) \to (E,C) \to 0$. As $\operatorname{Gr}_C K^{-1}$ is contained in $\operatorname{Gr}_C K^0 \cong F(C^i)$, both these are p-torsion free. Taking the total filtered complex (K,C) associated to the double complex $K^{-1} \to K^0$ we see that (K,C) is quasi-isomorphic to (E,C) and has a p-torsion free Gr. Hence we have quasi-isomorphisms:

$$T^\epsilon(K,C) \cong T^\epsilon(E,C) \text{ and } T^\epsilon(K,C) \cong C^\epsilon K.$$

This shows that $T^\epsilon(K,C)$ can be viewed as the derived functor of the operation $(K,C) \mapsto C^\epsilon K$.

For example, suppose X/W is formally smooth and D is a fundamental thickening of X/W. Then the filtered object $(\mathcal{O}_D, J_D)$ has a p-torsion free associated graded, and hence $T^\epsilon(\mathcal{O}_{X/W}, J_D) \cong J_D^\epsilon$. On the other hand, if ϵ is (p^μ, γ)-tame, then in fact we have a quasi-isomorphism $J_D^\epsilon \cong (i_{cris*} J^\epsilon_{X_\mu/W})_D$. This shows that the sheaves J_D^ϵ we have been considering can be regarded as the left derived functor $L^\epsilon(\mathcal{O}_{X/W}, J_D)$ for any lifting X/W of X_μ/W, and that this derived functor is independent of the choice of lifting, when ϵ is (p^μ, γ)-tame.

If (K, C) is a filtered complex, we denote by $H^q(C)$ the inverse system obtained by taking the q^{th} cohomology of the inverse system $\{C^i K\}$. Since formation of T^ϵ commutes with taking cohomology, we have a short exact sequence of complexes:

$$0 \to T_0^\epsilon(K, C) \to T^\epsilon(K, C) \to T_1^\epsilon(K, C)[1] \to 0$$

and an associated long exact sequence

$$T_1^\epsilon(H^q(C)) \to T_0^\epsilon(H^q(C)) \to H^q(T^\epsilon(K, C)) \to T_1^\epsilon(H^{q+1}(C)) \to T_0^\epsilon(H^{q+1}(C)),$$

We deduce the following short exact sequence, which we note for further reference:

$$0 \to L_0^\epsilon H^q(C) \to H^q(T^\epsilon(K, C)) \to L_1^\epsilon H^{q+1}(C) \to 0 \qquad (4.5.1.3)$$

We say that an inverse system C has "level within the interval $[a, b]$" if $C^i = 0$ for $i > b$ and j_i is an isomorphism for $i \leq a$. If there exists such an interval we say that C is "essentially finite."

Proposition 4.5.2 (Deligne) *Let C be an essentially finite inverse system.*

1. *If C has level within $[a, b]$, then the inclusion map from the subcomplex of T^ϵ given by*
$$\bigoplus_{a+1}^{b} C^i \to \bigoplus_{a}^{b} C^i$$
into T^ϵ is a quasi-isomorphism.

2. *If each C^i has finite length, then $\lg L_0^\epsilon(C) = \lg L_1^\epsilon(C) + \lg C^{-\infty}$*

3. *If each j_i is injective, then $L_1^\epsilon(C) = 0$, and the converse holds if ϵ is strictly decreasing and multiplication by p is nilpotent on each C^i.*

4. *Suppose that ϵ is strictly decreasing. If each $j_i \colon C^i \to C^{i-1}$ is a split monomorphism, $L_0^\epsilon(C) \cong C^{-\infty}$, and the converse is true if each C^i is a noetherian R-module.*

Proof: To prove the first statement it is sufficient to prove that the quotient complex is acyclic. This quotient Q looks like:

$$\bigoplus_{-\infty}^{a} C^i \to \bigoplus_{-\infty}^{a-1} C^i,$$

and in this case each j_i is an isomorphism. We can define an inverse to the boundary map by mapping an element d of Q_0 to the element c of Q_1 given by

$$c_i = d_{i-1} + p^{\Delta(i-1)} d_{i-2} + p^{\Delta(i-2)} d_{i-3} + \cdots .$$

This proves statement (1), and statement (2) follows immediately from this and the consequent exact sequence:

$$0 \to L_1^\epsilon(C) \to \bigoplus_{a+1}^{b} C^i \to \bigoplus_{a}^{b} C^i \to L_0^\epsilon(C) \to 0.$$

If each j_i is injective it is clear that ∂^ϵ is injective. For the converse, suppose $j_i(x) = 0$. Using induction and the nilpotence of multiplication by p, we may replace x by $p^k x$ for some k so that $px = 0$; then it suffices to prove that this $x = 0$. But then $\partial^\epsilon(x) = (j_i(x), -px) = 0$, so $x \in L_1^\epsilon = 0$. For the last statement, suppose that $s_i: C^i \to C^{i+1}$ is a splitting of j_i. Define $\gamma: C^a \to T_1^\epsilon$ by $\gamma(c) = (c, s_0(c), (s_1(s_0(c)), \ldots)$, and define $\beta: L_0^\epsilon(C) \to \bigoplus_{a+1}^b C^i$ by $\beta(c_0, c_1, \ldots) = (s_0(x_0) - x_1, s_1(x_1) - x_2, \ldots)$. It is clear that β is the quotient of T_0^ϵ by γ, and that the induced map from $T_1^\epsilon(C)$ to this quotient sends, in degree i, c to $c - p^{\Delta(i)}c$. Since $\Delta(i) > 0$, this is an isomorphism, and the proof is complete. To prove the converse, notice that an isomorphism $L_0(C) \cong C^{-\infty}$ induces an isomorphism $L_0^\epsilon(C \otimes R') \cong C^{-\infty} \otimes R'$ for every Artinian R-algebra R'. Hence by statement (2), $\lg L_1^\epsilon(C \otimes R') = 0$ for every such R' Then by the third statement, each $C^i \otimes R' \to C^{i-1} \otimes R'$ is injective. This implies that j_i remains injective when tensored with any A-module, and hence is a split monomorphism. ∎

Our application of these techniques will rest on the following result.

4.5.3 Proposition: *Suppose that h and g are control functions and δ is a positive integer such that $h(i) \geq g(i) + \delta$ for all $i \geq 1$. If (E, C) is a p-torsion free filtered module which is G-transversal to h and of level within $[a, \infty)$, and if ϵ is a g-tame gauge, then there is a natural quasi-isomorphism:*

$$T^\epsilon(E/p^\delta E, C) \cong C^\epsilon E / p^\delta C^\epsilon E$$

Proof: Let $E_\delta =: E/p^\delta E$. In the following diagram, α is the map α^ϵ defined in (4.5.0.2), and K_δ is the kernel of π^δ.

$$\begin{array}{ccccc} & & K_\delta & & p^\delta C^\epsilon E \\ & & \downarrow & & \downarrow \\ T_1^\epsilon(E,C) & \xrightarrow{\partial} & T_0^\epsilon(E,C) & \xrightarrow{\alpha} & C^\epsilon E \\ \downarrow & & \downarrow{\scriptstyle \pi^\delta} & & \downarrow \\ T_1^\epsilon(E_\delta,C) & \xrightarrow{\partial_\delta} & T_0^\epsilon(E_\delta,C) & & C^\epsilon E/p^\delta C^\epsilon E \end{array}$$

The two columns on the right in the diagram form short exact sequences, but the middle row is not exact, in general. Since by construction (E_δ, C) is a filtered object, the map ∂_δ is injective, and we know that α is surjective. Thus it is clear that our proposition will follow from the following fact.

4.5.4 Lemma: *With the notation above,* $\alpha^{-1}(p^\delta C^\epsilon E) = K_\delta + \operatorname{Im}\partial$.

Proof: We begin by showing that α maps K_δ to $p^\delta C^\epsilon E$. Suppose that $x \in K_\delta$; without loss of generality we may assume that x is homogeneous, say $x = (0, \ldots, x_j, \ldots)$ lies in degree j. Then $x_j \in p^\delta E \cap C^j E$, and as

$$\begin{aligned} p^\delta E \cap C^j E &= p^\delta C^j E + p^{h(1)} C^{j-1} E + p^{h(2)} C^{j-2} E \cdots \\ &\subseteq p^\delta (C^j E + p^{g(1)} C^{j-1} E + p^{g(2)} C^{j-2} E \cdots), \end{aligned}$$

we have

$$\alpha(x) = p^{\epsilon(j)} x_j \in p^\delta (p^{\epsilon(j)} C^j E + p^{\epsilon(j)+g(1)} C^{j-1} E + \cdots.$$

As ϵ is g-tame, we have $\epsilon(j) + g(j-i) \geq \epsilon(i)$ for all i and j, and hence we see that $\alpha(x) \in p^\delta C^\epsilon E$, as required. Since $\operatorname{Im}\partial \subseteq \operatorname{Ker}\alpha$, it is clear that the right side of our purported inequality is contained in the left.

For the reverse inclusion, suppose that $c \in T_0^\epsilon(E,C)$ is such that $\alpha(c) \in p^\delta C^\epsilon E$; say $\alpha(c) = p^\delta z$, with $z \in C^\epsilon E$. Since the map α is surjective, we can choose y such that $\alpha(y) = z$. Then $\alpha(c - p^\delta y) = 0$, and since $p^\delta y \in K_\delta$, it will suffice to prove that $c - p^\delta y \in K_\delta + \operatorname{Im}\partial$. Thus we may as well assume that $c \in \operatorname{Ker}\alpha$. We choose j such that $c_m = 0$ for $m > j$ and argue by induction on j. Write

$$c = (c_j, c_{j-1}, \ldots) \qquad \text{with } c_j \in C^j E.$$

Since $\alpha(c) = 0$, we have

$$p^{\epsilon(j)} c_j + p^{\epsilon(j-1)} c_{j-1} + \cdots p^{\epsilon(a)} c_a = 0.$$

Dividing by $p^{\epsilon(j)}$, we find that

$$c_j \in p^{\Delta(j)}E \cap C^jE \subseteq p^{\Delta(j)}C^jE + p^{h(1)}C^{j-1}E + \cdots p^{h(j-i)}C^iE + \cdots.$$

Write

$$c_j = p^{\Delta(j)}y_j + p^{h(1)}y_{j-1} + \cdots p^{h(j-i)}y_i + \cdots,$$

where $y_i \in C^iE$. For $i < j$ we have

$$\epsilon(j) + h(j-i) \geq \epsilon(j) + g(j-i) + \delta \geq \epsilon(i) + \delta,$$

so that

$$z_i =: p^{\epsilon(j)+h(j-i)-\delta-\epsilon(i)}y_i \in C^iE.$$

Let

$$c' =: (0, c_{j-1} + p^\delta z_{j-1} + y_j, c_{j-2} + p^\delta z_{j-2}, \ldots)$$

Then we find that

$$\begin{aligned}
\alpha(c') &= p^{\epsilon(j-1)}(c_{j-1} + p^\delta z_{j-1} + x_j) + \cdots p^{\epsilon(i)}(c_i + p^\delta z_i) + \cdots \\
&= p^{\epsilon(j-1)}y_j + p^{\epsilon(j-1)+\delta}z_{j-1} + \cdots p^{\epsilon(i)+\delta}z_i + \cdots + p^{\epsilon(j-1)}c_{j-1} + \cdots \\
&= p^{\epsilon(j)}(p^{\Delta(j)}y_j + p^{h(1)}y_{j-1} + \cdots) + p^{\epsilon(j-1)}c_{j-1} + \cdots \\
&= p^{\epsilon(j)}c_j + p^{\epsilon(j-1)}c_{j-1} + \cdots \\
&= 0.
\end{aligned}$$

By the induction assumption, $c' \in K_\delta + \operatorname{Im}\partial$. As each $p^\delta z_i$, viewed in degree i, belongs to K_δ, it follows that

$$c'' =: (0, c_{j-1} + y_j, c_{j-2}, \ldots) \in K_\delta + \operatorname{Im}\partial.$$

Now we have

$$c = c'' - \partial(y_j, 0, 0 \ldots) + (p^{h(1)}y_{j-1} + \cdots p^{h(j-i)}y_i + \cdots, 0, \ldots)$$

As each $p^{h(j-i)}y_i$, viewed in degree j, lies in $C^jE \cap p^\delta E \subseteq K_\delta$, the proof is complete. ∎

4.5.5 Remark: For example, suppose that g is the control function associated to the PD-ideal (p^a, γ) and that h is the control function associated to $(p^{a+\delta}, \gamma)$. Because $\gamma_i(p^{a+\delta}) = p^{i\delta}\gamma_i(p^a)$, it is clear that $h(i) \geq g(i) + \delta$ whenever $i \geq 1$. We shall apply this especially when $a = 1$.

5 T-crystals and F-spans

Throughout this section S will denote a logarithmic formal W-scheme with a subscheme of definition $S_0 \subseteq S$ defined by the PD-ideal (p). We let X/S_0 denote an integral [20, 4.1] and logarithmically smooth morphism of fine logarithmic schemes. Let $F_{X/S}: X \to X'$ denote the exact relative Frobenius morphism (1.2.3), which fits in the familiar diagram

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/S}} & X' & \xrightarrow{\pi_{X/S}} & X \\ & \searrow & \downarrow & & \downarrow \\ & & S_0 & \xrightarrow{F_{S_0}} & S_0 \end{array} \tag{5.0.5.4}$$

In this diagram, $\pi_{X/S} \circ F_{X/S}$ is the absolute Frobenius endomorphism of X. If X/S is perfectly smooth (1.2.3), the square is Cartesian. The integrality of X/S_0 assures that smooth liftings of X and X' will be flat over S, *c.f.* [20, 4.3]. Note again that $F_{X/S}$ is $\mathcal{O}_S$-linear and that we do not need a lifting of F_{S_0} to S for now.

Our main goal in this section is to describe the close relationship between T-crystals and F-spans (whose definition we recall below). In particular, we show in (5.2.13) how to associate a T-crystal to an "admissible" F-span in a natural way. Our construction interpolates to families the construction of the abstract Hodge filtration attached to an F-span over a point in [23] and enriches the mod p construction in [24]. For admissible F-spans of "width" less than p (5.1.1), we even obtain an equivalence of categories.

5.1 G-transversality and isogenies

Our constructions depend on the simple observation that when E is a p-torsion free object in an abelian category, the data of a filtration on E which is G-transversal to (p) and of finite level amounts to nothing more than a p-isogeny. We shall need to understand this correspondence rather thoroughly, so our first task is to explain it in some detail.

We say that an object in an abelian category is "p-torsion free" if the endomorphism of E induced by multiplication by p is injective, and we define the p-adic filtration P of E by letting P^iE be the image of multiplication by p^i for $i \geq 0$. Assuming henceforth that E is p-torsion free, we note that P^iE is isomorphic to E, and in fact we can define a system $\breve{E}$ of objects indexed by $\mathbf{Z}$ by setting $P^i\breve{E} =: E$ for all i, and for $k \leq i$ $P^i\breve{E} \rightarrow P^k\breve{E}$ the map defined by multiplication by p^{i-k}. Actually we are interested in $i << 0$, so we should regard this as a direct system and work with the ind-object "$\varinjlim P^iE$." In practice it will suffice for us to work with $P^a\breve{E}$ for some a sufficiently negative, so we need not insist on this point. Notice that for any other p-torsion free object E'' and any j,

$$\operatorname{Hom}(E'', \breve{E}) =: \varinjlim \operatorname{Hom}(E'', P^a\breve{E}) \cong \operatorname{Hom}(P^j\breve{E}'', \breve{E}) \cong \operatorname{Hom}(\breve{E}'', \breve{E}).$$

5.1.1 Definition: *A "p-isogeny" $E'' \rightarrow E$ is an isomorphism of ind-objects $\breve{\Phi}: \breve{E}'' \rightarrow \breve{E}$. We say that $\breve{\Phi}$ has "level within $[a, b]$" and "width less than or equal to $b-a$" if $\breve{\Phi}$ lies in* $\operatorname{Hom}(E'', P^a\breve{E})$ *and its inverse lies in* $\operatorname{Hom}(E, P^{-b}\breve{E}'')$.

The terminology and notation we are using are somewhat abusive because $\breve{\Phi}$ does not come from a map $E'' \rightarrow E$ unless $a \geq 0$, in which case we say that $\breve{\Phi}$ is "effective." Of course, one can always reduce to this case by a "twist." An effective p-isogeny of level within $[0, b]$ amounts to a map $\Phi: E'' \rightarrow E$ such that there exists a map $V: E \rightarrow E''$ with ΦV and $V\Phi$ both equal to multiplication by p^b.

If $\breve{\Phi}: E'' \rightarrow E$ is a p-isogeny we define filtrations M on $\breve{E}''$ and N on $\breve{E}$ by the following formulas:

$$M^j\breve{E}'' =: \breve{\Phi}^{-1}(P^j\breve{E})$$

$$N^j\breve{E} =: \breve{\Phi}(P^j\breve{E}'') \qquad (5.1.1.1)$$

The filtrations P, N, and M are G-transversal to (p), and Φ induces a bifiltered isomorphism

$$\breve{\Phi}: (\breve{E}'', M, P) \longrightarrow (\breve{E}, P, N). \qquad (5.1.1.2)$$

In particular, if σ is a finite subset of $\mathbf{Z} \times \mathbf{Z}$ and if $\epsilon \in \mathbf{G}_1^\flat$, we have isomorphisms:

$$\Phi_\sigma\colon (\breve{E}'', M, P)_\sigma \to (\breve{E}, N, P)_{\sigma'}\ ,\ \Phi^\epsilon\colon M^\epsilon E'' \to N^{\epsilon'} E, \tag{5.1.1.3}$$

where ϵ' and σ' are as described in (4.2). (These equations hold for all σ and ϵ if our category has exact direct limits.) Notice that (E, N) has level within $[-a, -b]$.

5.1.2 Lemma: *Let E'' be a p-torsion free object in an abelian category. The correspondence described above gives an equivalence of between the following three sets of data:*

1. *A p-isogeny $E'' \to E$ of level within $[a, b]$*
2. *A filtration M on $\breve{E}''$ which is G-transversal to (p) such that*
$$M^b \breve{E}'' \subseteq P^0 \breve{E}'' \subseteq M^a \breve{E}''$$
3. *A filtration M on E'' which is G-transversal to (p) and of level within $[a, b]$*

Similarly, if E is a p-torsion free object, to give a p-isogeny $E'' \to E$ of level within $[a, b]$ is the same as giving a filtration N on E which is G-transversal to (p) and of level within $[-b, -a]$.

Proof: We give only a sketch. We have already described above how to pass from a p-isogeny to a filtration M as in (2). Starting with the data of (2), we simply restrict it to $E'' =: P^0 \breve{E}$ to obtain the data of (3). On the other hand, if we start with (3), then we observe that for any $j \geq 0$, $P^j E'' \cap M^i E'' = p^j M^{i-j} E''$, because M is G-transversal to (p). It is clear that we can use the right side of this equation to define the left side for $j \leq 0$, and in this way we obtain a filtration M on all of $\breve{E}''$ which is still G-transversal to p. Since (E'', M) has level within $(\infty, b]$, $M^{b+1} E'' \subseteq pE''$, and it follows that $M^b \breve{E}'' \subseteq P^0 \breve{E}''$. Since (E, M) has level within $[a, \infty)$, $P^0 \breve{E}'' =: E'' \subseteq M^a E'' \subseteq M^a \breve{E}''$, and we have the data of (2). Starting from (2), we let $E =: M^0 \breve{E}''$, and let $\breve{\Phi}$ be the map coming from the inclusion $E'' = P^0 \breve{E}'' \to M^a \breve{E}'' = P^a \breve{E}$. Because $M^b \breve{E}'' \subseteq P^0 \breve{E}''$ we have also $M^0 \breve{E}'' \subseteq P^{-b} \breve{E}''$, hence a map $E \to P^{-b} \breve{E}''$ which is the inverse to $\breve{\Phi}$. ∎

There is an obvious notion of a morphism of p-isogenies, and the correspondence above is functorial.

5.2 The functors μ and α

Because X/S_0 is logarithmically smooth and integral, the structure sheaf of a fundamental thickening of any open subset of X/S is p-torsion free. We can then apply the above constructions in the abelian category of crystals of $\mathcal{O}_{X/S}$-modules. In particular, any locally free crystal will be p-torsion free.

5.2.1 Definition: *An "F-span on X/S" is a triple (E', E, Φ), where E' is a crystal of p-torsion free $\mathcal{O}_{X'/S}$-modules on X'/S, E is a crystal of p-torsion free $\mathcal{O}_{X/S}$-modules on X/S, and $\Phi\colon F^*_{X/S}E' \to E$ is a p-isogeny. A morphism of F-spans is a pair of morphisms of crystals, compatible with the given p-isogenies. The F-span $p^m\colon \mathcal{O}_{X'/S} \to \mathcal{O}_{X/S}$ is denoted $\mathcal{O}_{X/S}(-m)$.*

For example, if $X = S_0$, a crystal on X/S just amounts to a quasi-coherent sheaf of $\mathcal{O}_S$-modules, and $F_{X/S}$ is the identity map. Thus an F-span on S_0/S is just a p-isogeny of p-torsion free quasi-coherent sheaves of $\mathcal{O}_S$-modules. We shall often denote the category of F-spans on X/S by $FS(X/S)$. If $\Phi\colon F^*_{X/S}E' \to E$ is an F-span, we can apply (5.1.1.1) to obtain bifiltered ind-objects $(F^*_{X/S}\breve{E}', M, P)$ and $(\breve{E}, P, N)$ in the category of crystals of $\mathcal{O}_{X/S}$-modules, and a bifiltered isomorphism

$$\breve{\Phi}\colon (F^*_{X/S}\breve{E}', M, P) \to (\breve{E}, P, N). \tag{5.2.1.1}$$

5.2.2 Remark: An F-span is said to be "effective" if it has level within $[0, \infty)$, and is said to be "of finite type" if E' and E are of finite type. We shall say that an F-span is "uniform" if it has finite width and if $\mathrm{Gr}^i_M E''_0 \cong \mathrm{Gr}^{-i}_N E_0$ is a locally free $\mathcal{O}_{X/S_0}$-module of finite type for all i. If this is true, it follows that E and E'' are locally free $\mathcal{O}_{X/S}$-modules of finite type also. In fact, if Y/S is a lifting of X/S, it follows easily from (2.4.2) that, locally on Y, there exist bases (e''_i) of E''_Y and (e_i) of E_Y such that $\Phi(e''_i) = p^{d_i}e_i$ for some sequence of integers (d_i). It is now easy to prove that formation of the filtrations M and N is compatible with base change, tensor product, and internal Hom, for uniform F-spans. A principal polarization on an F-span is defined in the obvious way, *c.f.*(3.3.2).

We can describe our constructions explicitly in a lifted situation (1.2.6). In terms of the lifted situation $\mathcal{Y} =: (Y, F_{Y/S})$, the category $FS(X/S)$ can be described as the category of horizontal p-isogenies of the form:

$$F^*_{Y/S}(E'_Y, \nabla') \to (E_Y, \nabla).$$

Similarly, a T-crystal on X'/S can be described as a triple (E', ∇', A), where E' is a quasicoherent sheaf of $\mathcal{O}_{Y'}$-modules, ∇' is an integrable and quasi-nilpotent logarithmic connection on E', and A is a filtration on E' which is G-transversal to (p, γ) and Griffiths transversal to ∇'.

Note that in a lifted situation, the Frobenius mapping is divisible by p^i on i-forms, and we let

$$\eta^i_Y : \Omega^i_{Y'/S} \to F_{Y/S*}F^*_{Y/S}\Omega^i_{Y/S}$$

denote $p^{-i}F^*_{Y/S}$. If E' is any quasicoherent sheaf of $\mathcal{O}_Y$-modules we find a natural map

$$\eta^i_{E'} : \Omega^i_{Y'/S} \otimes E' \xrightarrow{\eta^i \otimes id} F_{Y/S*}(\Omega^i_{Y/S}) \otimes E' \cong F_{Y/S*}(\Omega^i_{Y/S} \otimes F^*_{Y/S}E'). \quad (5.2.2.2)$$

5.2.3 Definition: *An "F_γ-span on X/S" is a pair (E', M_γ), where E' is a p-torsion free crystal of $\mathcal{O}_{X'/S}$-modules on X'/S and M_γ is a filtration of $F^*_{X/S}E'$ by subcrystals which is G-transversal to the PD-ideal (p, γ).*

If $\Phi: F^*_{X/S}E' \to E$ is an F-span on X/S, we let Φ_γ denote the F_γ-span on X/S obtained by taking the saturation (2.3.1) of the filtration M with respect to (p, γ). Thanks to (5.1.2) and (2.3.5) we see that, for F-spans of width less than p, the functor $\Phi \mapsto \Phi_\gamma$ is an equivalence of categories.

5.2.4 Definition: *A T-crystal (E', A) on X'/S is "admissible" if E' is a crystal of finite type p-torsion free $\mathcal{O}_{X'/S}$-modules and the filtration A is compatible with $F_{X/S}$ (2.3.3),* i.e., *$(E'_{X'}, A)$ is normally transversal to $F_{X/S}$ (2.1.1).*

(This terminology is abusive, because admissibility of (E', A) depends not only on X'/S but also on X/S.)

5.2.5 Proposition: *Suppose that (E', A) is an admissible T-crystal on X'/S. Then the T-crystal $F^*_{X/S}(E', A)$ is in fact horizontal (3.2.5) and defines a finite type F_γ-span on X/S. Letting $\mu_{X/S}(E', A)$ denote this F_γ-span, we obtain a fully faithful functor from the category of admissible T-crystals to the category of F_γ-spans. For uniform T-crystals, the functor μ commutes with base change, tensor product, and internal Hom. Finally, if E' is of finite type, $\mu(E', A)$ is uniform if and only if (E', A) is uniform.*

Proof: The fact that the T-crystal $F^*_{X/S}(E', A)$ is defined is an immediate consequence of our assumption that the filtration A of E' is compatible with $F_{X/S}$. (3.3.1). To compute it, let us begin by working in a lifted situation $\mathcal{Y}$. Let M_γ denote the filtration of $E''_Y =: F^*_{Y/S}(E'_{Y'})$ induced by $A_{Y'}$. We know by (2.2.1) that M_γ is G-transversal to (p, γ). Because the ideal of X in Y is just (p) and the filtration M_γ is saturated with respect to (p, γ), M_γ is in fact the filtration of E''_Y giving the value of the T-crystal $F^*_{X/S}(E', A)$ on Y. In particular, it is independent of the choice of $F_{Y/S}$.

To prove that M_γ is horizontal, it suffices to check that each $M^k_\gamma E''_Y$ is locally generated by sections x such that $\nabla'' x \in M^k_\gamma E'' \otimes \Omega^1_{Y/S}$. If x is a local section of $A^k E'$,

$$\begin{aligned} \nabla''\eta^0(x) = p\eta^1(\nabla' x) &\in pF_{Y/S*}(\Omega^1_{Y/S} \otimes F^*_{Y/S} A^{k-1}_Y E') \\ &\subseteq F_{Y/S*}(\Omega^1_{Y/S} \otimes M^k_\gamma E'') \end{aligned}$$

as required.

We know from (1.3.8) that the functor $F^*_{X/S}$ from the category of finite type p-torsion free crystals of $\mathcal{O}_{X'/S}$ modules to the category of crystals of $\mathcal{O}_{X/S}$-modules is fully faithful. Thus the full faithfulness of μ will follow from the following lemma.

5.2.6 Lemma: *Suppose that (E', A) is an admissible T-crystal on X'/S. Then for any (p, γ) tame-gauge ϵ, the natural map*

$$F^*_{Y/S} A^\epsilon E'_{Y'} \to M^\epsilon_\gamma E''_Y$$

is an isomorphism. Furthermore, the map

$$\eta\colon (E'_{Y'}, A) \to F_{Y/S*}(E''_Y, M_\gamma)$$

is injective and strictly compatible with the filtrations.

Proof: Let us consider the cohomological δ functor

$$H^q(M) =: \mathrm{Tor}^{\mathcal{O}_{Y'}}_{-q}(M, F_{Y/S*}\mathcal{O}_Y).$$

We can apply Proposition (4.4.3) with $q = 0$ to conclude that each

$$F^*_{Y/S} A^\epsilon E'_{Y'} \to E''_Y$$

is injective, and that we get in this way a $\mathbf{G}_\gamma$-structure on E''_Y. It follows that the image of $F^*_{Y/S} A^\epsilon E'_{Y'}$ in $F^*_{Y/S} E'_{Y'}$ must be $M^\epsilon_\gamma E''_Y$.

For each i we can form the following commutative diagram, in which we have just written F for $F_{Y/S}$.

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathrm{Gr}_A^i E'_{Y'} & \rightarrow & E'_{Y'}/A^{i+1}E'_{Y'} & \rightarrow & E'_{Y'}/A^i E'_{Y'} \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \rightarrow & F_*F^*\mathrm{Gr}_A^i E'_{Y'} & \rightarrow & F_*F^*E'_{Y'}/A^{i+1}E'_{Y'} & \rightarrow & F_*F^*E'_{Y'}/A^i E'_{Y'} \end{array}$$

We know by Proposition (4.4.3) that the sequence along the bottom is exact. Furthermore, if Q is any quasi-coherent sheaf of $\mathcal{O}_{X'}$-modules, it follows from (1.2.5) that the map

$$\eta_0 : Q \rightarrow F_{X/S*}F^*_{X/S}Q$$

is injective. This implies that the arrow f in the diagram above is injective. It is now clear by induction on i that the arrows g and h are injective also. The strictness of η follows. ∎

To finish the proof of (5.2.5), we recall that a p-torsion free T-crystal of finite type (E', A) is uniform if and only if $\mathrm{Gr}_A E'_{X'}$ is locally free, and the corresponding F-span is uniform if and only if $\mathrm{Gr}_M E''_X \cong F^*_{X/S}\mathrm{Gr}_A E'_{X'}$ is uniform. The equivalence of these two follows from Corollary (1.3.7). ∎

5.2.7 Remark: Let $T_G(Y'/S)$ be the category whose objects are triples (E'_Y, ∇', A_Y), where $(E'_{Y'}, \nabla')$ is a module with integrable connection on Y'/S and A_Y is a filtration which is G-transversal to (p) and Griffiths transversal to ∇'. This category depends on Y'/S and not just on X'/S. If (E'_X, A) is normally transversal to $F_{X/S}$, then the filtration M induced by A on $E'' =: F^*_{Y/S}E'$ is G-transversal to (p), and the proof of (5.2.5) shows that it is also horizontal. In this way we obtain a functor from the category of objects of $T_G(Y'/S)$ which are compatible with $F_{X/S}$ to the category of F-spans on X/S. However, for objects of width at least p, the functor depends on the choice of $F_{Y/S}$.

5.2.8 Definition: *An F-span $\Phi: F^*_{Y/S}E' \rightarrow E$ on X'/S is "admissible" if and only if there exist an admissible T-crystal (E', A) on X'/S and an isomorphism of F_γ-spans $\mu(E', A) \cong (E', M_\gamma)$. We shall denote the T-crystal (E', A) associated to an admissible F-span Φ by $\alpha_{X/S}(\Phi)$.*

We should remark that the T-crystal associated to an admissible F-span Φ is unique, and even functorial in Φ, because μ is fully faithful. Consequently the question of the admissibility of an F-span is a local on the Zariski topology of X. Notice that we have an equivalence between the category of admissible F-spans of width less than p and the category of admissible T-crystals of width less than p.

The following result gives more substance to our constructions.

5.2.9 Theorem: *An F-span $\Phi: F^*_{X/S}E' \to E$ of finite type is admissible if and only if the filtration M induces on $F^*_{X/S}E'_X$ descends to E'_X. This is automatically the case if $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathrm{Gr}_N E, R_{X/S})$ vanishes (e.g. if Φ is uniform in a neighborhood of the support of $R_{X/S}$).*

Proof: The "only if" assertion is trivial, and the second assertion follows from (1.3.6) and the isomorphism $\mathrm{Gr}_N E_X \cong \mathrm{Gr}_M F^*_{X/S}E'_X$. It remains to prove the "if" part of the first assertion. In other words we have to prove that Φ is admissible, provided that the natural maps

$$F^*_{X/S}(M^i E''^{\nabla}_X) \to M^i E''_X$$

are isomorphisms for all i.

This is a local question, and so we may work locally on X/S, with the aid of a parallelizable lifted situation $\mathcal{Y}$. Let $\Phi: F^*_{Y/S}(E', \nabla') \to (E, \nabla)$ be the realization of Φ on $\mathcal{Y}$. If $X_0 = \mathrm{Spec}\, k$, $Y = SpfW$ and $F_{Y/S}$ is an isomorphism, so the filtration M automatically descends to a filtration on E' and there is nothing more to prove. (This is essentially the construction of Mazur in [23].) It will take us more work to carry out the construction in a family. Let $(E'', \nabla'') =: F^*_{Y/S}(E', \nabla')$, let $\eta_E: E' \to F_{Y/S*}E''$ be the natural map, and define

$$A^k_{\mathcal{Y}}E' =: \eta_E^{-1}(M^k E''). \tag{5.2.9.3}$$

The main difficulty is contained in the following lemma, which is the logarithmic version of [24, 2.2.1].

5.2.10 Lemma: *For all $q \geq 0$, the maps (5.2.2.2)*

$$\eta^q_X: (E'_X, A_{\mathcal{Y}}) \otimes \Omega^q_{X'/S} \to F_{X/S*}(E''_X, M) \otimes \Omega^q_{X/S}$$

are injective and strictly compatible with the filtrations.

Proof: If $(m_1, \cdots m_n)$ is a system of parallelizable logarithmic coordinates for $\mathcal{Y}$, we see that $\eta^1_Y(dm'_i) = dm_i$, so that η^i locally looks just like a direct sum of copies of η^0_Y. Thus it suffices to prove the lemma for $\eta =: \eta^0_X$.

With respect to our chosen set of coordinates we have $F_{\mathcal{Y}}^*(m_i') = pm_i$, and hence $F_{\mathcal{Y}}^* dm_i' = pdm_i$. Let $(\partial_1, \ldots, \partial_n)$ denote the basis for $T_{Y/S}$ dual to $(dm_1, \ldots, dm_n)$, and $(\partial_1', \ldots, \partial_n')$ the basis for $T_{Y/S}$ dual to $(dm_1', \ldots dm_n')$. Then $\partial_i(F_{\mathcal{Y}}^*\omega') = F_{\mathcal{Y}}^*(p\partial_i'\omega')$ for any section ω' of $\Omega^1_{Y'/S}$, and hence

$$\nabla''(\partial_i) \circ \eta = \eta \circ \nabla'(p\partial_i').$$

Define an endomorphism h'' of E'' as in the proof of (1.3.4); using formula (1.3.4.2) we can write

$$h'' =: \prod_{i=1}^{n} \prod_{j_i=1}^{p-1} \frac{\nabla''(\partial_i - j_i)}{(-j_i)}$$

Since the filtration M on E'' is horizontal, it is stable by h''. Furthermore, we know from (1.3.4) that for any section e'' of E'', $h''(e'')$ is horizontal mod p, and hence lies in the image of η modulo p. Now define an endomorphism h' of E' by the analogous formula:

$$h' =: \prod_{i=1}^{n} \prod_{j_i=1}^{p-1} \frac{\nabla'(p\partial_i' - j_i)}{(-j_i)}$$

Then h' is congruent to the identity modulo p, and $\eta h' = h''\eta$.

We can now prove that η is strictly compatible with the filtrations. Namely, suppose e' is a section of $E'_{X'}$ such that $\eta(e') \in M^i E''_X$. Then if $x' \in E'_{Y'}$ is any lifting of e', there exist $y \in M^i E''_Y$ and $z \in E''_Y$ such that $\eta(x') = y + pz$. We proceed to show that one can in fact choose x', y, and z such that $\eta(x') = y + p^j z$ for every $j > 0$, by induction on j. If this is true for j with $j > 0$ we get:

$$\begin{aligned} \eta(x') &= y + p^j z \\ h''\eta(x') &= h''(y) + p^j h''(z) \\ \eta h'(x') &= h''(y) + p^j\big(\eta(z') + pz''\big) \\ \eta\big(h'(x') - p^j z'\big) &= h''(y) + p^{j+1} z'' \end{aligned}$$

Since h' is congruent to the identity modulo p and $j > 0$, $h'(x') - p^j z'$ is still a lifting of e', and our claim is proved. But for $j \geq i$, $p^j z \in M^i E''$, and it follows that $x' \in A^i E'$, as required. ∎

5.2.11 Lemma: *The filtration $A_{\mathcal{Y}}$ is G-transversal to (p) and compatible with $F_{X/S}$, and the natural map $F_{Y/S}^* A_{\mathcal{Y}}^k E' \to M^k E''$ is an isomorphism.*

Proof: It follows immediately from the definitions that the filtration $A_{\mathcal{Y}}$ is G-transversal to (p). Indeed, if $a \in A^k_{\mathcal{Y}}E' \cap pE'$, then

$$\eta_E(a) \in M^kE'' \cap pE'' = pM^{k-1}E'',$$

and hence $a \in pA^{k-1}_{\mathcal{Y}}E'$. The reverse inclusion is even more obvious. Now we proceed to show that the natural map $F^*_{Y/S}A^k_{\mathcal{Y}}E' \to M^kE''$ is an isomorphism by induction on k. We have a commutative diagram

$$\begin{array}{ccccccc}
F^*_{Y/S}A^{k-1}_{\mathcal{Y}}E' & \xrightarrow{p} & F^*_{Y/S}A^k_{\mathcal{Y}}E' & \longrightarrow & F^*_{Y/S}A^k_{\mathcal{Y}}E'_X & \longrightarrow & 0 \\
\downarrow{\scriptstyle\zeta^{k-1}} & & \downarrow{\scriptstyle\zeta^k} & & \downarrow{\scriptstyle\zeta^k_X} & & \\
0\longrightarrow M^{k-1}E'' & \xrightarrow{p} & M^kE'' & \longrightarrow & M^kE''_X & \longrightarrow & 0.
\end{array}$$

The bottom row is exact because the filtration M is G-transversal to (p), and the top row is exact because A is G-transversal to (p). The map ζ^{k-1} is an isomorphism by the induction hypothesis, and (5.2.10) tells us that $M^kE''^{\nabla}_X$ is precisely $A^k_{\mathcal{Y}}E'_X$. Because we are assuming that the map

$$F^*_{X/S}(M^kE''^{\nabla}_X) \to M^kE''_X$$

is an isomorphism, we see that ζ^k_X is an isomorphism. It follows that ζ^k is an isomorphism, as required. Finally we note that the injectivity of ζ^k_X says that $A_{\mathcal{Y}}$ is compatible with $F_{X/S}$. ∎

In order to prove that $A_{\mathcal{Y}}$ is Griffiths transversal to ∇' we shall use the following converse to the argument of (5.2.5).

5.2.12 Lemma: *Let (E', ∇') be a quasi-coherent sheaf with integrable connection on Y'/S, let $A_{\mathcal{Y}}$ be a filtration on E' which is G-transversal to (p) and compatible with $F_{X/S}$. Let $(E'', \nabla'') =: F^*_{Y/S}(E', \nabla')$, with the filtration M induced by $A_{\mathcal{Y}}$. Then $A_{\mathcal{Y}}$ is Griffiths transversal to ∇' if and only if M is stable under ∇''.*

Proof: Note first that by (2.2.1), the filtration (E'', M) is G$'$-transversal to (p). If x is a local section of $A^k_{\mathcal{Y}}E'$ and M^kE'' is stable under ∇'', we have

$$p\eta^1(\nabla'x) = \nabla''\eta^0(x) \in F_{Y/S*}(\Omega^1_{Y/S} \otimes M^kE'').$$

As (E'', M) is G$'$-transversal to (p), it follows that

$$\eta^1(\nabla'x) \in F_{Y/S*}(\Omega^1_{Y/S} \otimes M^{k-1}E').$$

Since η^1 is injective and strictly compatible with the filtrations, $\nabla'(x) \in \Omega^1_{Y'/S} \otimes A^{k-1}_{\mathcal{Y}} E'$. ∎

To finish the proof of Theorem (5.2.9), we argue as in the proof of (5.2.5), using Proposition (4.4.3) and the compatibility of A with $F_{X/S}$, that for any one-gauge ϵ, the natural map

$$F^*_{Y/S} A^{\epsilon}_{\mathcal{Y}} E' \to M^{\epsilon} E''$$

is an isomorphism. Let $A_{\mathcal{Y},\gamma}$ denote the saturation of $A_{\mathcal{Y}}$ with respect to (p,γ). it is easy to check that $A_{\mathcal{Y},\gamma}$ is still Griffiths transversal to E', and hence defines a T-crystal (E', A) on X'/S. It is clear that this T-crystal is compatible with $F_{X/S}$, and that $\mu(E', A) \cong (E'', M_\gamma)$. This completes the proof that Φ is admissible. ∎

The proof of (5.2.9) also gives us a method for computing the T-crystal associated to an admissible F-span. For the sake of our later applications, it is useful to summarize our constructions in the following way.

5.2.13 Theorem: *There is a functor $\alpha_{X/S}$ from the category of admissible F-spans on X/S to the category of admissible T-crystals on X'/S which are compatible with $F_{X/S}$. This functor is uniquely characterized by the fact that, in a parallelizable lifted situation $\mathcal{Y}$, it takes $(\Phi\colon F^*_{Y/S} E' \to E)$ to $(E'_{Y'}, A_{\mathcal{Y},\gamma})$ (5.2.9.3). It induces an equivalence when restricted to objects of width less than p, and takes the constant span $id\colon \mathcal{O}_{X/S} \to \mathcal{O}_{X/S}$ to the constant T-crystal $(\mathcal{O}_{X'/S}, (J_{X'/S}, \gamma))$.*

∎

5.2.14 Remark: Suppose that $f\colon X \to Y$ is a log smooth morphism of log smooth and integral S_0-schemes and $\mathcal{Y}/S$ is a log smooth lifting of Y/S_0 We can form a relative Frobenius diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{F_{X/Y}} & \tilde{X} & \xrightarrow{\pi_{X'/Y'/S}} & X' & \xrightarrow{\pi_{X/Y/S}} & X \\
 & \searrow f & \downarrow \tilde{f} & & \downarrow f' & & \downarrow f \\
 & & Y & \xrightarrow{F_{Y/S}} & Y' & \xrightarrow{\pi_{Y/S}} & Y \\
 & & & \searrow & \downarrow & & \downarrow \\
 & & & & S_0 & \xrightarrow{F_{S_0}} & S_0
\end{array}$$

in which $F_{X/Y}$ is formed by taking the exact version of the induced map from X to the fiber product $X' \times_{Y'} Y$. We have

$$F_{X/S} = \pi_{X'/Y'/S} \circ F_{X/Y} \quad \text{and} \quad \pi_{X/Y} = \pi_{X/Y/S} \circ \pi_{X'/Y'/S}.$$

Now if (E', A) is an admissible T-crystal on X'/S which is compatible with $\pi_{X'/Y'/S}$, then it pulls back to a T-crystal $(\tilde{E}, A)$ on $\tilde{X}$ which is compatible with $F_{X/Y}$, giving an admissible T-crystal on $\tilde{X}/Y$. It is clear that $\mu_{X/\mathcal{Y}}(\tilde{E}, A) \cong \mu_{X/S}(E', A)$. Now if $\Phi_{X/S}: F^*_{X/S}E' \to E$ is an F-span on X/S, we can let $\tilde{E} =: \pi^*_{X'/Y'/S}E'$ and view $\Phi_{X/S}$ as a map $\Phi_{X/Y}: F^*_{X/Y}\tilde{E} \to E$ of crystals on X/S. Forgetting some structure, we can regard this as a map of crystals on $X/\mathcal{Y}$, and we find that we have an F-span on $X/\mathcal{Y}$. It is clear that this procedure defines a functor from the category of F-spans on X/S to the category of F-spans on $X/\mathcal{Y}$. We see that if $\Phi_{X/S}$ is admissible and if $\alpha_{X/S}(\Phi_{X/S})$ is compatible with $F_{X/Y}$, then $\Phi_{X/\mathcal{Y}}$ is admissible, and in fact

$$\alpha_{X/\mathcal{Y}}(\Phi_{X/Y}) \cong \pi^*_{X'/Y'/S}\alpha_{X/S}(\Phi_{X/S}).$$

Finally let us note from the exact sequence

$$f^*R_{Y/S} \to R_{X/S} \to R_{X/Y} \to 0$$

that the support of $R_{X/Y}$ is contained in the support of $R_{X/S}$, so that if $\Phi_{X/S}$ is uniform in a neighborhood of the support of $R_{X/S}$, then $\Phi_{X/Y}$ is uniform in a neighborhood of the support of $R_{Y/S}$, and all our compatibility and admissibility conditions are fulfilled.

5.3 F-crystals, T-crystals, and Fontaine modules

In this section we introduce the notion of F-T-crystals, which we propose as a p-adic analog of a variation of Hodge structure. We use the same notation as before, but we assume in addition that the logarithmic formal scheme S is provided with a lifting F_S of its absolute Frobenius morphism. If S is log smooth over $\operatorname{Spf} W$ (where W has the trivial log structure), such liftings always exists locally. More generally, W could be endowed with a log structure attached to a prelog structure sending every nonunit to zero (I propose calling log structures of this form "hollow".) On the other hand, the log structure on W associated to the prelog structure $\mathbf{N} \to W$ sending 1 to p does not allow such liftings of Frobenius.

If now X/S_0 is log smooth, we find from the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{F_S} & S \end{array}$$

a morphism of topoi $F_{X_{cris}}: X/S_{cris} \to X/S_{cris}$; frequently we just write F_X instead of $F_{X_{cris}}$. Recall (1.2.3) that X/S_0 is said to be perfectly smooth if it is log smooth and integral and the relative Frobenius morphism is exact. We say that a formal log scheme Y/S is perfectly smooth if it is logarithmically smooth and integral and its reduction modulo p is perfectly smooth.

Classically, an F-crystal is a p-torsion free crystal of $\mathcal{O}_{X/S}$-modules E together with an isogeny $\Phi: F_X^* E \to E$. We shall enrich this notion as follows:

5.3.1 Definition: *Suppose that Y/S_μ is a perfectly smooth morphism of fine log schemes, and let X/S_0 be its reduction modulo p. An "F-T crystal on Y/S" is a triple (E, Φ, B), where E is a p-torsion free crystal of X/S-modules, $\Phi: F_X^* E \to E$ is a p-isogeny, and (E, B) is a p-torsion free finite type T-crystal on Y/S which is compatible with F_X and with $\pi_{X/S}$, such that $\pi_{X/S}^*((E, B)|_{X/S}) = \alpha_{X/S}(\Phi)$.*

Of course, we should really have replaced the equal sign in the above definition with a specified isomorphism. Notice that admissibility of the F-span and T-crystal to which an F-T-crystal gives rise is automatic from the definition. When $S = \operatorname{Spf} W$ with the trivial log structure and X/k is perfectly smooth, the map $\pi_{X/W}: X' \to X$ is an isomorphism, so that the filtration $A_{X/W}$ of $E' = \pi_{X/W}^* E$ descends uniquely to E. Thus in this case an F-crystal is the same thing as an F-T-crystal, and we can view $\alpha_{X/W}(\Phi)$ as a T-crystal on X/W.

5.3.2 Remark: The hypothesis that the absolute Frobenius endomorphism of S_0 lift to S is a nuisance, especially in the logarithmic context. At the price of a more cumbersome definition, this hypothesis can be relaxed somewhat. Suppose that Y/S_μ is perfectly smooth, and suppose that we are given a lifting F_{S_μ} of F_{S_0} to S_μ. Then a "μ-F-T crystal on Y/S" consists of: an admissible F-span $\Phi: F_{Y/S}E' \to E$ on Y/S, an admissible T-crystal (E', B') on Y'/S lifting $\alpha_{Y_0/S}(\Phi)$, and an admissible T-crystal structure B on E_μ with an isomorphism $\pi_{Y/S_\mu}^*(E_\mu, B) \cong (E'_\mu, B')$. It is clear that, if F_{S_μ} lifts all the way to S, an F-T-crystal gives us a μ-F-T-crystal, and it turns out that the structure of a μ-F-T-crystal is enough for most of our applications.

The category of F-T-crystals is closely related to the category MF^∇ [1] defined in [9], at least in a local situation. We shall begin by reviewing this category and the main facts about it. Let Y/S_μ be flat (*e.g.* smooth), suppose

[1] I propose calling objects in the category $MF(\mathcal{Y})$ "modules de Fontaine" in French, or "Fontaine modules" in English.

$F_Y: Y \to Y$ is a lifting of the absolute Frobenius endomorphism of $X =: Y_1$ covering F_S and let $\mathcal{Y} =: (Y, F_Y)$. Consider the category $MF_{big}(\mathcal{Y})$ whose objects are pairs (M, ϕ), where $M =: \{M^i, j_i, i \in \mathbf{N}\}$ is an inverse system of $\mathcal{O}_Y$-modules and where $\phi =: \{\phi_i: F_Y^* M^i \to M^0\}$ is a collection of linear maps such that $\phi_{i-1} \circ j_i = p \circ \phi_i$ for all $i \geq 1$. To give such a system ϕ is the same as giving a map $\phi: F_Y^* L_0^\iota(M) \to M^0$, where $\iota(i) =: -i$ and L_0^ι is the cokernel of the map ∂^ι defined in (4.5.0.1). Morphisms in the category $MF_{big}(\mathcal{Y})$ are defined in the obvious way, and one obtains an abelian category, with the standard construction of kernels and cokernels. For any $a, b \in \mathbf{N}$ $MF_{[a,b]}(\mathcal{Y})$ is the full subcategory of $MF_{big}(\mathcal{Y})$ such that each M^i vanishes if $i > b$, each j_i with $i \leq a$ is an isomorphism, each M^i is coherent, and such that ϕ is an isomorphism. We let $MF_n(\mathcal{Y})$ denote the union of all $MF_{[a,b]}(\mathcal{Y})$ such that $b - a \leq n$ and $MF(\mathcal{Y})$ the union of all $MF_n(\mathcal{Y})$. If $\mathcal{Y}' \to \mathcal{Y}$ is a morphism (compatible with the lifts of Frobenius), we have an evident functor from $MF_{big}(\mathcal{Y})$ to $MF_{big}\mathcal{Y}'$. Since the functor L_0^ι commutes with base change, this functor sends $MF(\mathcal{Y})$ to $MF(\mathcal{Y}')$. When Y/S_μ is log smooth, one can also consider the category $MF^\nabla(\mathcal{Y}/S)$, in which there are also given maps $\nabla_i: M^i \to \Omega^1_{Y/S} \otimes M^{i-1}$ satisfying the evident connection rule. (The point is that the category $MF^\nabla_{p-1}(\mathcal{Y})$ is independent of the choice of F_Y, as explained by Faltings [9].) For the reader's convenience we state and give a proof of the following theorem of Faltings [9, 2.1], in a slightly more general form.

Theorem 5.3.3 (Faltings) *Suppose (M, Φ) is an object of $MF(\mathcal{Y})$.*

1. *Each M^i is locally a direct sum of sheaves of the form $\mathcal{O}_Y/p^e\mathcal{O}_Y$, and each $j_i: M^i \to M^{i-1}$ is injective and locally split (so the inverse system M can be regarded as a filtered object).*

2. *$MF(\mathcal{Y})$ is an abelian subcategory of $MF_{big}(\mathcal{Y})$, and any morphism in $MF(\mathcal{Y})$ is strictly compatible with the filtrations.*

Proof: We first discuss (5.3.3.1) in the case in which $\mu = 1$, so $Y = X$. It is then clear that the complex $T^\iota(B)$ (4.5.0.1) can be identified with the complex

$$\bigoplus_{i>0} M^i \xrightarrow{\oplus j_i} \bigoplus_{i>0} M^{i-1}.$$

Thus,

$$L_0^\iota(M) \cong G_0(M) =: \oplus \operatorname{Cok}(j_i), \text{ and}$$
$$L_1^\iota(M) \cong G_1(M) =: \oplus \operatorname{Ker}(j_i).$$

The following result, based on the technique of Deligne (4.5.2), replaces Faltings' argument using Fitting ideals.

5.3.4 Lemma: *Suppose that R is an Artinian local ring with residue field k and let $M =: \{M^i, j_i\}$ be an inverse system of finite type R-modules indexed by $\mathbf{N}$, such that $M^i = 0$ for $i >> 0$. Then*

$$\lg G_0(M \otimes k) \lg R \geq \lg M^0, \tag{5.3.4.1}$$

and equality holds if and only if each j_i is injective and each M^i and $G_0(M)$ are free.

Proof: There is an exact sequence:

$$0 \to G_1(M) \longrightarrow \bigoplus_{i>0} M^i \xrightarrow{\oplus j_i} \bigoplus_{i\geq 0} M^i \to G_0(M) \to 0$$

As all the terms have finite length, we see that

$$\lg G_0(M) = \lg M^0 + \lg G_1(M) \tag{5.3.4.2}$$

Let r be the length of R. It is clear from Nakayama's lemma that for any finite type R-module N, $r \lg(N \otimes k) \geq \lg N$, with equality if and only if N is free. Apply this to $G_0(M)$ and use the fact that the functor G_0 commutes with base change to conclude that $r \lg G_0(M \otimes k) \geq \lg G_0(M)$, with equality if and only if $G_0(M)$ is free. Using equation (5.3.4.2) we find

$$r \lg G_0(M \otimes k) \geq \lg G_0(M) = \lg M^0 + \lg G_1(M).$$

The claim is now obvious: equality in the equation (5.3.4.1) implies that $G_1(M) = 0$ and that $G_0(M)$ is free so that M is a finitely filtered object whose graded object is free. ∎

We now can easily prove (5.3.3.1) when $\mu = 1$. Without loss of generality we may assume that X is local, and even that $X = \operatorname{Spec} R$, where R is an Artinian local ring. We argue by induction on its length r. If $r > 1$ write $F_R = f \circ g$, where g is a map to a proper quotient R' of R. Then the induction hypothesis applies to g^*M and hence $g^*G_0(M)$ is free. It follows that the same is true of $F_R^* G_0(M)$. Of course, this is automatically true if $r = 1$. Thus we may proceed under this assumption. Then as $F_R^* G_0(M) \cong M$ we have

$$\begin{aligned} \lg M^0 &= \lg F_R^* G_0(M) = r \lg F_R^* G_0(M) \otimes k \\ &= r \lg(F_k^* G_0(M \otimes k)) = r \lg G_0(M \otimes k). \end{aligned}$$

Then (5.3.4) finishes the proof.

It is clear that it will suffice to prove the theorem when μ is finite, which we assume from now on.

5.3.5 Claim: *Suppose that* $f\colon (M,\Phi) \to (M'',\Phi'')$ *is a surjection in* $MF(\mathcal{Y})$ *and that* (M'',Φ'') *satisfies (5.3.3.1). Then* $(M',\Phi') =: \operatorname{Ker} f$ *lies in* $MF(\mathcal{Y})$.

Proof: For any integer $\mu' \le \mu$, let $Y_{\mu'}$ denote the reduction of Y_μ modulo $p^{\mu'}$. Because Y/S_μ is flat, we have an exact sequence

$$0 \to \mathcal{O}_{Y_{\mu''}} \to \mathcal{O}_Y \to \mathcal{O}_{Y_{\mu'}} \to 0,$$

where $\mu'' =: \mu - \mu'$. This sequence remains unchanged when pulled back by F_Y, and in particular remains exact on the left. This implies that

$$\operatorname{Tor}_1^{\mathcal{O}_Y}(F_{Y*}\mathcal{O}_Y, \mathcal{O}_{Y_{\mu'}}) = 0,$$

and then by induction that

$$\operatorname{Tor}_i^{\mathcal{O}_Y}(F_{Y*}\mathcal{O}_Y, \mathcal{O}_{Y_{\mu'}}) = 0 \text{ for all } i > 0. \tag{5.3.5.3}$$

Since the maps j_i'' are locally split monomorphisms, (4.5.2.4) implies that, locally on Y, $L_0^\iota(M'') \cong M''^0$, and the latter is locally a direct sum of modules of the form $\mathcal{O}_{Y_{\mu'}}$. We conclude that $\operatorname{Tor}_1^{\mathcal{O}_Y}(F_{Y*}\mathcal{O}_Y, L_0^\iota(M'')) = 0$. (Of course, all this is obvious when F_Y is flat.) Since the maps j_i'' of M'' are injective, $L_1^\iota(M'') = 0$, and it follows easily that we have an exact sequence

$$0 \to L_0^\iota(M') \to L_0^\iota(M) \to L_0^\iota(M'') \to 0.$$

This sequence remains exact when pulled back by F_Y^*, and we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & F_Y^*L_0^\iota(M') & \to & F_Y^*L_0^\iota(M) & \to & F_Y^*L_0^\iota(M'') & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M'^0 & \to & M^0 & \to & M''^0 & \to & 0 \end{array}$$

It follows that the vertical arrow on the left is also an isomorphism. ∎

We now proceed with the proof of (5.3.3) by induction on μ, closely following Faltings. Suppose $\mu > 1$ and the result proved for $\mu - 1$, and let (M,Φ) be an object of $MF(\mathcal{Y})$. Let $M'' =: M/pM$ and $M' =: pM$, so that we have an exact sequence:

$$0 \to M' \to M \to M'' \to 0.$$

As the functors L_0^ι and F_R^* commute with base change, the map

$$F_Y^*L_0^\iota(M'') \to M''^0$$

is an isomorphism, hence $(M'', \Phi'') \in MF(\mathcal{Y})$. As M'' is killed by p, (5.3.3.1) is true for it, and the claim then implies that $(M', \Phi') \in MF(\mathcal{Y})$ also. By the induction assumption, it also satisfies (5.3.3.1). It is now immediate that the maps $j_i \colon M^i \to M^{i-1}$ are injective. Furthermore, the injectivity of the maps j_i'' implies that $pM^0 \cap M^i = pM^i$. It is then easy to check that (5.3.3.1) is true for M. ∎

To prove (5.3.3.2), suppose that f is a morphism in $MF(\mathcal{Y})$. Since L_0^ι and F_R^* are right exact, it is clear that $\operatorname{Cok} f$ is in $MF(\mathcal{Y})$. Then by (5.3.5) and (5.3.3.1), $\operatorname{Im} f$ is in $MF(\mathcal{Y})$, and hence by the same reasoning $\operatorname{Ker} F$ is in $MF(\mathcal{Y})$. Finally, notice that the injectivity of transition maps of $\operatorname{Cok} f$ immediately implies that f is strictly compatible with the filtrations. ∎

Later we shall also find a derived category version of Theorem (5.3.3) useful. For complexes annihilated by p, see also (8.2.2). We begin with a definition.

5.3.6 Definition: *A "Fontaine-complex" on $\mathcal{Y}$ is a filtered complex (K, B) on Y together with an isomorphism in the derived category*

$$\psi \colon LF_Y^* T^\iota(K, B) \to K.$$

5.3.7 Corollary: *Suppose that $((K, B), \psi)$ is a Fontaine-complex on $\mathcal{Y}$ which is bounded above and such that $H(\operatorname{Gr}_F K)$ is coherent. Then*

1. *There are natural isomorphisms*

$$L_0^\iota(H^q(B)) \to H^q(T^\iota(K, B)) \quad \text{and} \quad F_Y^* L_0^\iota(H^q(B)) \to H^q(K).$$

   In particular,each $(H^q(B), \psi)$ becomes an object of $MF(\mathcal{Y})$.

2. *Each $H^q(B^i K) \to H^q(K)$ is injective and a local direct factor.*

3. *Each $H^q(B^i K)$ is, locally on Y, a direct sum of sheaves of the form $\mathcal{O}_Y / p^e \mathcal{O}_Y$ (for various e).*

Proof: Our assertion is local so we may and shall assume that Y is affine. We argue by descending induction on q, assuming that $H^{q'}(K, B)$ satisfies all three statements for all $q' > q$.

Begin by replacing (K, B) by a complex such that each $\operatorname{Gr}_K B$ is flat and of finite type over $\mathcal{O}_Y$. Note that the fact that the maps

$$H^{q'}(j_i) \colon H^{q'}(B^i K) \to H^{q'}(B^{i-1} K)$$

are injective implies by (4.5.2.3) that $L_1^\iota H^{q'}(B) = 0$ for $q' > q$, and hence from the exact sequence (4.5.1.3) that the map $L_0^\iota H^{q'}(B) \to H^{q'}(T^\iota(K,B))$ is an isomorphism for $q' \geq q$. Furthermore, since $H^{q'}(j_i)$ is split for $q' > q$, we see from (4.5.2.4) that $L_0^\iota H^{q'}(B) \cong H^{q'}(K)$ for $q' > q$, and hence by (5.3.5.3) and (5.3.7.3),

$$\mathrm{Tor}_i(F_{Y*}\mathcal{O}_Y, H^{q'}(T^\iota(K,B)) = 0$$

in this range. Now applying the universal coefficient theorem and using the quasi-isomorphism ψ, we find an isomorphism

$$F_Y^* L_0^\iota H^q(B) \cong F_Y^* H^q(T^\iota(K,B)) \cong H^q(F_Y^* T^\iota(K,B)) \cong H^q(K).$$

Thus we can apply (5.3.3) to the inverse system $H^q(B)$, so it too satisfies the conclusions of our proposition. ∎

5.3.8 Proposition: *Suppose that Y/S is flat and that (M,ϕ) is an object of $MF(\mathcal{Y})$. For each i, let M_t^i be the p^∞-torsion submodule of M^i and let M_f^i be the quotient of M^i by M_t^i, so that we have an exact sequence in $MF_{big}(\mathcal{Y})$*

$$0 \to (M_t, \phi_t) \to (M,\phi) \to (M_f, \phi_f) \to 0.$$

Then in fact, all the terms in this sequence lie in $MF(\mathcal{Y})$.

Proof: We may argue locally. Each $j_i \colon M^i \to M^{i-1}$ is locally split, and hence so is each $j_{i,t}$ and $j_{i,f}$ Thus by (4.5.2), there is an isomorphism $L_0^\iota(M_f) \cong M_f^0$, and hence $L_0^\iota(M_f)$ has no p-torsion; furthermore $L_1^\iota(M_f) = 0$. It follows that the sequence

$$0 \to L_0^\iota(M_t) \to L_0^\iota(M) \to L_0^\iota(M_f) \to 0$$

is exact, and that $L_0^\iota(M_t) \cong L_0^\iota(M)_t$ and similarly $L_0^\iota(M_f) \cong L_0^\iota(M)_f$. It is then clear that (M_t, ϕ_t) and (M_f, ϕ_f) lie in $MF(\mathcal{Y})$. ∎

Now we can explain the relationship between Fontaine-modules and F-T-crystals.

5.3.9 Proposition: *Suppose $\mathcal{Y} = (Y/S_\nu, F_Y)$ is a local situation, with Y/S_ν perfectly smooth and $\nu \geq 2$, and let $\mu =: \nu - 1$. Then there is a functor from the category of F-T-crystals on Y/S of width less than p to the category $MF_{p-1}^\nabla(Y_\mu/S_\mu)$, taking an object (E, Φ, B) to the inverse system $\{B^i E_\mu : i \in \mathbf{Z}\}$. When $\nu = \infty$, this functor induces an equivalence between the category of F-T-crystals on Y/S and the category of p-torsion free objects of $MF_{p-1}^\nabla(Y/S)$.*

5.3.10 Remark: In fact, to construct a Fontaine-module on a perfectly smooth Y/S_μ (with liftings of Frobenius to Y and to S_μ), we only need a μ-F-T-crystal (E, B, B', Φ) on Y/S, plus liftings of Y'/S and (E', B') to $S_{\mu+1}$.

Proof: It is perhaps worthwhile to begin with the following result, which holds without any restriction on the level.

5.3.11 Lemma: *Suppose that Y/S_μ is perfectly smooth, with a lifting of Frobenius F_Y, and that (E, Φ, B) is an F-T-crystal on Y/S. Then for every lifting of Y and (E, Φ, B) to $S_{\mu+1}$ and every p-tame gauge ϵ, there is a natural isomorphism*

$$F_Y^* L_0^\epsilon(E \otimes W_\mu, B) \cong (N^{\epsilon'} E) \otimes W_\mu.$$

More generally, we obtain an isomorphism as above from a μ-F-T-crystal (E, B, B', Φ) on Y/S plus liftings of Y' and of (E', B') to $S_{\mu+1}$.

Proof: Let g be the control function (4.2.2) associated with (p, γ) and h the control function associated with $(p^{\mu+1}, \gamma)$. Let $(E', B') =: \pi_{Y/S}^*(E, B)$, or, in the case of a μ-F-T-crystal, just the given T-crystal on Y'/S, and let C' be the lifting of B' to a lifting of Y' to $S_{\mu+1}$. Then $h(i) \geq g(i) + \mu$ when $i > 0$. Hence by (4.5.3), (4.5.5), and the fact that L_0^ϵ commutes with base change, we have isomorphisms

$$\pi_{Y/S_\mu}^* L_0^\epsilon(E_\mu, B) \cong L_0^\epsilon(E'_\mu, B') \cong L_0^\epsilon(E'_\mu, C') \cong C'^\epsilon E' \otimes W_\mu.$$

Since ϵ is p-tame and C' lifts $A =: \alpha_{X/W}(\Phi)$, we have $C'^\epsilon E' = A^\epsilon E'$ by (4.3.7). By (5.2.6) and equation (5.1.1.3),

$$F_{Y/S}^* A^\epsilon E' \cong M_\gamma^\epsilon F_{Y/S}^* E' = M^\epsilon F_{Y/S}^* E' \cong N^{\epsilon'} E.$$

Reducing modulo p^μ we find

$$\begin{aligned} F_Y^* L_0^\epsilon(E_\mu, B) &\cong F_{Y/S}^* \pi_{Y/S_\mu}^* L_0^\epsilon(E_\mu, B) \\ &\cong F_{Y/S}^*(A^\epsilon E') \otimes W_\mu \\ &\cong (N^{\epsilon'} E) \otimes W_\mu \end{aligned}$$

∎

In particular, suppose (E, Φ, B) has width less than p. We may and shall suppose, without loss of generality, that its level is within $[0, p-1]$. Let $\iota(i) =: -i$ for all $i \in \mathbf{Z}$. Then we have isomorphisms:

$$F_Y^* L_0^\iota(E \otimes W_\mu, B) \to E \otimes W_\mu \qquad (5.3.11.4)$$

Indeed, for any $m \geq 0$ let ϵ_m be the maximal p-tame gauge which vanishes at 0, and let ϵ'_m be its "transpose" (4.2.1). If $m < p$ it is easy to check, using the fact that $N^i E = p^i N^0 E$ for $i \geq 0$, that $N^{\epsilon'_m} E = p^m N^{-m} E$. If m is greater than or equal to the upper level of Φ, $N^{-m} E = E$, and we obtain that $N^{\epsilon'_m} E = p^m E$. Again from the facts that the level of (E, B) is within $[0, m]$ and that $m < p$ we see that the natural map

$$T^{\iota+m}(E \otimes W_\mu, B) \to T^{\epsilon_m}(E \otimes W_\mu, B)$$

is an isomorphism. Identifying $T^{\iota+m}$ with T^ι and $p^m E$ with E and applying the previous lemma, we find the desired isomorphism (5.3.11.4). This completes the construction of the functor. Thanks to Faltings' structure theorem, it is clear that any torsion free object of $MF^{\nabla}_{p-1}(Y/W)$ is given by a filtration B on $E =: M^0$ by local direct factors, and that Φ is divisible by p^i on $B^i E$. Finally, I claim that the saturation B_p of B with respect to p is precisely the filtration $A_{\mathcal{Y}}$ attached to Φ. Indeed, if $x \in F_Y^* A^i_{\mathcal{Y}} E$, $\Phi(x) = p^i z$ for some z. Since ϕ is an isomorphism, we can write $z = \phi(y)$, where $y \in F_Y^* L^\iota_0(B)$ is the class of $y_0 \oplus y_1 \cdots$. Then $p^i \phi(y) = \Phi(p^i y_0 + p^{i-1} y_1 + \cdots)$, and it follows that $x = p^i y_0 + p^{i-1} y_1 + \cdots \in B^i_p E$. This shows that $A_{\mathcal{Y}}$ is finer than B_p, and the reserve inclusion is obvious. ∎

5.3.12 Corollary: *If (E, B, Φ) is a μ-F-T-crystal on Y/S such that Y'_0 and (E', A) lift to S_2, then (E, B) is uniform. In particular, if $\mu > 1$, any μ-F-T-crystal on Y/S of width less than p is uniform.*

Proof: Our statement is local on Y, so we may assume that there exists a local lifting of Frobenius. Then by (5.3.9) we see that (E_X, B) underlies a Fontaine-module, and hence by (5.3.3) $\operatorname{Gr}_B E_X$ is locally free. In fact, we see even that $\operatorname{Gr}_B E_{\mu-1}$ is locally free, but we need to prove that $\operatorname{Gr}_B E_Y$ is locally free. . This follows from the following lemma.

5.3.13 Lemma: *Suppose that (E, B) is a T-crystal on Y/S_μ whose restriction to $X =: Y_0$ is uniform. Then (E, B) is uniform.*

Proof: We know that $\operatorname{Gr}_B E_X$ is locally free over $\mathcal{O}_X$. and we have to prove that $\operatorname{Gr}_B E_Y$ is locally free over $\mathcal{O}_Y$. By dévissage we see that E_X is locally free, and since E is p-torsion free, the local criterion for flatness implies that E_T is locally free for every local lifting T of Y. Thus E_Y is locally free. To prove that (E, B) is uniform, we still have to show that each $\operatorname{Gr}^j_B E_Y$ is locally free on Y. By the local criterion for flatness, it will suffice to prove that $\mathrm{Tor}_1(\mathrm{Gr}^j_B E_Y, \mathcal{O}_X) = 0$. which we do by induction on j. Assuming the result true for all $i < j$, we observe from the exact sequences

$$0 \to B^{i+1}E_Y \to B^i E_Y \to \mathrm{Gr}^i_B E_Y \to 0$$

and induction on i that $\mathrm{Tor}_1(B^j E_Y, \mathcal{O}_X) = 0$, and so again by the local criterion for flatness, $B^j E_Y$ is locally free over $\mathcal{O}_Y$. It follows that we have an exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_Y}(\mathrm{Gr}^j_B E_Y, \mathcal{O}_X) \to B^{j+1}E_Y \otimes \mathcal{O}_X \to B^j E_Y \otimes \mathcal{O}_X.$$

But our definition of T-crystals requires that the filtration (E_Y, B) be normally transversal to p, so the arrow on the right is injective and the Tor vanishes. ∎

5.3.14 Corollary: *If Y/S is a perfectly smooth logarithmic formal scheme, the operation of killing torsion defines a functor from the category $MF^{\nabla}(Y/S)$ to the category of uniform F-T-crystals on Y/S. Under this functor, the filtration M on M^0 corresponds to the filtration B.*

6 Cohomology of T-crystals

Let S be a fine logarithmic formal W-scheme, flat over W, which we view as a PD-scheme by using the unique PD-structure on the ideal (p). Let $S_\mu \subseteq S$ be the subscheme defined by $I =: (p^\mu)$, a sub-PD-ideal of (p). Let X/S_μ be integral and logarithmically smooth and suppose (E, A) is a T-crystal on X/S. We will be interested in studying the cohomology of (E, A) as a filtered object in several senses. For example, we shall want to study the bifiltered object $(E, A, (I, \gamma))$, as well as the bifiltered object (E, A, P), where P is the p-adic filtration. Furthermore, we shall want to study the gauge structure on E defined by A using formula (4.3.4.2) with respect to the element p. Let $\mathcal{J}^i_\mu \subseteq \mathcal{O}_S$ denote the i^{th} divided power of the ideal (p^μ). Then $\mathcal{J}_\mu$ is a multiplicative p-principal filtration and is determined by a control function $g = g_\mu$ with $g_\mu(1) = \mu$. Because the filtration A is saturated with respect to $(p^\mu, \gamma) = \mathcal{J}_\mu$, $A^{\epsilon_g}\check{E} = A^\epsilon \check{E}$ for any gauge $\epsilon \in \check{E}$, by (4.3.7.2). Thus we may as well restrict our attention to g_μ-tame gauges. Let us note also that if X can be embedded in a logarithmically smooth Y/S and $D =: D_X(Y)$, then by (3.3.3), (E_D, A_D) is G-transversal to (p^μ, γ). Hence by (4.3.4) we have a $\mathbf{G}_{(p^\mu,\gamma)} =: \mathbf{G}_{\mathcal{J}_\mu}$-structure on $\check{E}_D$ which we denote by $A_{\mathbf{G}}$.

6.1 The Poincaré Lemma

We begin with the analogue of the filtered Poincaré lemma for T-crystals. If $i\colon X/S \to Y/S$ is a closed immersion, with Y/S logarithmically smooth, the value E_D of E on the PD-envelope $D =: D_X(Y)$ of X in Y has a natural integrable connection ∇ and filtration A_D which is G-transversal to ∇. Let

$$A^i_D(\Omega^q_Y \otimes E_D) =: \Omega^q_Y \otimes A^{i-q}E_D;$$

one obtains in this way a filtered complex which we denote by $(E^{\cdot}_Y, A)$ or $(K_{Y/S}, A)$ and which we call the "De Rham complex of the T-crystal (E, A) with respect to Y/S."

6.1.1 Theorem: *Suppose $i: X/S \to Y/S$ is a closed immersion, with Y/S logarithmically smooth. Let (E, A) be a T-crystal on X/S, and let $\mathcal{J}$ be a filtration of $\mathcal{O}_{X/S}$ by sheaves of ideals. Then there is a bifiltered quasi-isomorphism:*

$$Ru_{X/S*}(E, A, \mathcal{J}) \cong (E^{\cdot}_Y, A, \mathcal{J}).$$

Furthermore, this quasi-isomorphism induces a filtered quasi-isomorphism of **G***-filtered objects:*

$$Ru_{X/S*}(E, A_{\mathbf{G}}) \cong (E^{\cdot}_Y, A_{\mathbf{G}}).$$

Proof: We use Grothendieck's technique of linearization of differential operators, discussed for example in §6 of [4]. We give only a sketch. Recall that for any $\mathcal{O}_Y$-module with integrable connection (E_Y, ∇) there is a complex $E^{\cdot}_Y$ of $\mathcal{O}_Y$-modules and differential operators and then a complex $L(E^{\cdot}_Y)$ of crystals of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}\,(X/S)$, with $\mathcal{O}_{X/S}$-linear boundary maps. Moreover, there is a natural isomorphism of complexes $L(\Omega^{\cdot}_{Y/S}) \otimes_{\mathcal{O}_{X/S}} E_Y \cong L(E^{\cdot}_Y)$ [4, 6.15]. The complex $L(\Omega^{\cdot}_{Y/S})$ comes equipped with a canonical filtration Fil_X (whose definition we recall below). If E is a crystal of $\mathcal{O}_{X/S}$-modules, the value E_D of E on the divided power envelope $D_X(Y)$ of X in Y has a natural integrable connection, and there is a canonical isomorphism $L(\Omega^{\cdot}_{Y/S}) \otimes E \cong L(E^{\cdot}_D)$. If (E, A) is a T-crystal, we shall want to consider the tensor product $(L(\Omega^{\cdot}_{Y/S}), Fil_X) \otimes (E, A)$, endowed with the tensor product filtration, which we denote by $(L(E^{\cdot}_Y), A)$.

6.1.2 Proposition: *There is a natural map of bifiltered complexes of $\mathcal{O}_{X/S}$-modules*

$$(E, A, \mathcal{J}) \to (L(E^{\cdot}), A, \mathcal{J}).$$

Locally on $(X/S)_{cris}$, this map is a bifiltered homotopy equivalence. In particular it is a bifiltered quasi-isomorphism, and it induces a filtered quasi-isomorphism of **G***-filtered objects*

$$(E, A_{\mathbf{G}}) \to (L(E^{\cdot}_Y), A_{\mathbf{G}}).$$

Proof: The proof of [4, 6.13] shows that the map

$$(\mathcal{O}_{X/S}, J_{X/S}, \gamma) \to (L(\Omega^{\cdot}), Fil_X)$$

is, locally on $(X/S)_{cris}$, a filtered homotopy equivalence. Furthermore, since the filtration A on E is saturated with respect to $(J_{X/S}, \gamma)$, the tensor product filtration of $\mathcal{O}_{X/S} \otimes E$ induced by A and $(J_{X/S}, \gamma)$ is just A. It follows that the map

$$(E, A) \to (L(E^{\cdot}), A) \cong (L(\Omega^{\cdot}_{Y/S}) \otimes (E, A))$$

is locally a filtered homotopy equivalence. The proposition now follows from the following lemma.

6.1.3 Lemma: *Let $f: (K, A) \to (K', A')$ be a morphism of filtered complexes of $\mathcal{O}_T$-modules. Then f induces a morphism of bifiltered complexes*

$$(K, A, \mathcal{J}) \to (K', A', \mathcal{J}).$$

Suppose, moreover, that f is locally a filtered homotopy equivalence. Then f also induces bifiltered homotopy equivalences, and in particular bifiltered quasi-isomorphisms:

$$(K, A, I) \to (K', A', I) \quad \text{and} \quad (K, A_{\mathbf{G}}) \to (K', A'_{\mathbf{G}})$$

Proof: Suppose $g: (K, A) \to (K', A')$ is a morphism of filtered sheaves of $\mathcal{O}_X$-modules. Then g automatically maps $I^n K$ to $I^n K'$ for all n, and hence also $I^n K \cap A^m K$ to $I^n K' \cap A'^m K'$ for all n and m. Thus, g is compatible with the bifiltrations. Applying this remark to a morphism of filtered complexes f as in the lemma, we find the first statement of the lemma. If f is a homotopy equivalence on some open subset U of X, then on U we have a retraction $s: (K', A') \to (K, A)$ and homotopy operators $h_1: (K, A) \to (K, A)$ and $h_2: (K', A') \to (K', A')$ to which we can also apply the remark. ∎

Warning: Caution is required because in general the maps

$$(K, A, \mathcal{J}) \to (K', A', \mathcal{J}) \quad \text{and} \quad (K, A_{\mathbf{G}}) \to (K', A'_{\mathbf{G}})$$

induced by a filtered quasi-isomorphism $(K, A) \to (K', A')$ are not (bi)filtered quasi-isomorphisms.

Let $\tilde{Y} =: i^*(Y)$ in $(X/S)_{cris}$, which we identify with the object $D_X(Y)$, which prorepresents it. Recall from [4, 5.26] that there is a map

$$\phi: (X/S)_{cris}|_{\tilde{Y}} \longrightarrow D_X(Y)_{zar} = X_{zar}.$$

Let us write $\Omega^q_{\tilde{Y}/S}$ for $\mathcal{O}_D \otimes \Omega^q_{Y/S}$.

6.1.4 Lemma: *There are natural isomorphisms:*

$$A^iE \cap P^jL(\Omega^q_{\tilde{Y}/S} \otimes E_{\tilde{Y}}) \cong j_{\tilde{Y}*}\phi^*(\Omega^q_{\tilde{Y}/S} \otimes A^iE_{\tilde{Y}} \cap P^jE_{\tilde{Y}})$$

$$A^\epsilon L(\Omega^q_{\tilde{Y}/S} \otimes E_{\tilde{Y}}) \cong j_{\tilde{Y}*}\phi^*(\Omega^q_{\tilde{Y}/S} \otimes A^\epsilon[-q]E_{\tilde{Y}})$$

Proof: An isomorphism $L(\Omega^q_{\tilde{Y}/S} \otimes E_{\tilde{Y}}) \cong j_{\tilde{Y}*}\phi^*(\Omega^q_{\tilde{Y}/S} \otimes E_{\tilde{Y}})$ was constructed in [4, 6.10]; we need only check that it strictly preserves the filtrations. This is clear for the p-adic filtration, and we check it for the A-adic filtration locally, on an object T of Cris(X/S) which we may assume admits a map h to $\tilde{Y}$. To simplify the notation, we just check the case $q = 0$, and let $D =: D_X(T \times_S Y)$, with $\pi_T : D \to T$ the natural projection. Recall that

$$j_{\tilde{Y}*}\phi^*(A^iE_{\tilde{Y}}) \cong \pi_{T*}A^kE_D.$$

Moreover, $L(\mathcal{O}_X) \cong \mathcal{O}_T \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_{Y/S}(1)}$, where $D_{Y/S}(1)$ is the PD envelope of X in $Y \times_S Y$, and $Fil^k_X \subseteq L(\mathcal{O}_{X/S})$ is the k^{th} divided power of the PD-ideal K generated by the ideals of X in T and in $D_{Y/S}(1)$. It is shown in [4, 6.10] that the natural map

$$g : \mathcal{O}_T \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_{Y/S}(1)} \to D_X(T \times_S Y)$$

is a PD isomorphism taking K to the ideal J of X. Thus, g identifies $A^iL(E)$ with the tensor product filtration of $(\pi_T^*E_T, A_T)$ and (J, γ). This is just the expansion of the filtration A_T with respect to (J, γ), which by Lemma (3.1.1) is the same as the filtration A of E_D, as desired. The assertion for gauges follows. (For more details on the gauge construction, *c.f.* pp. 8-15 and 8-16 of [4].) ∎

6.1.5 Lemma: *All the terms of the filtered complexes* $(L(E^\cdot), A, \mathcal{I})$ *and* $(L(E^\cdot), A_{\mathbf{G}})$ *are acyclic for* $u_{X/S*}$.

Proof: This lemma is essentially proved in [4, §7], but is not stated explicitly there. For the sake of clarity, we briefly sketch the argument, which uses the intermediate site Cris$(X/S.)$ defined on page 7-22 of [4]. We shall also use the notation of the diagram on the top of page 7-24 of *op. cit.* Suppose that F is one of the terms of one of the above complexes. For each $T \in$ Cris(X/S), F_T is quasi-coherent, so by [4, 7.22.2],

$$Ru_{X/S*}F \cong R\varprojlim Ru_{X/S_n*}(i_n^*F) =: R\pi_{X*}Rv_{X/S.*}j^*F.$$

Now by (6.1.4), [4, 6.10], and [4, 5.27], each sheaf $j_{n*}j^*F$ is acyclic for u_{X/S_n*}, and since these sheaves also satisfy the hypothesis of [4, 7.20], $Rv_{X/S.*}j^*F \cong v_{X/S.*}j^*F$ is acyclic for π_{X*} also. ∎

6.1.6 Lemma: *If $\epsilon \in \mathbf{G}_g$, the natural map $Ru_{X/S*}A^{\epsilon}E \to A^{\epsilon}E^{\cdot}_Y$ is a quasi-isomorphism.*

Proof: We have a filtered quasi-isomorphism $(E, A_{\mathbf{G}}) \to (L(E^{\cdot}_Y), A_{\mathbf{G}})$, and the terms of the latter are acyclic for $u_{X/S*}$, so we obtain an isomorphism in the filtered derived category:

$$Ru_{X/S*}(E, A_{\mathbf{G}}) \cong u_{X/S*}(L(E^{\cdot}_Y), A_{\mathbf{G}}).$$

For any $\epsilon \in \mathbf{G}$ we have $u_{X/S*}A^{\epsilon}L(E^{\cdot}_Y) = \varprojlim A^{\epsilon}E^{\cdot}_{D_n}$. Thus, the filtered complex $u_{X/S*}(L(E^{\cdot}_Y), A_{\mathbf{G}})$ is the p-adic completion of the filtered complex $(E^{\cdot}_Y, A_{\mathbf{G}})$ and we denote it by $(E^{\cdot}_Y, \hat{A}_{\mathbf{G}})$. We have a natural map of filtered complexes

$$(E^{\cdot}_Y, A_{\mathbf{G}}) \to (E^{\cdot}_Y, \hat{A}_{\mathbf{G}}),$$

and it remains only for us to show that it is a filtered quasi-isomorphism. In other words, we have to show that for any $\epsilon \in \mathbf{G}$, the induced map

$$\kappa_{\epsilon}: A^{\epsilon}E^{\cdot}_Y \to \hat{A}^{\epsilon}E^{\cdot}_Y$$

is a quasi-isomorphism. Because the filtration A on E_Y has level within $[a, \infty)$, we have $A^{\epsilon}E_Y = \sum_{i \geq a} p^{\epsilon(i)}A^i E_Y$. Moreover, if ϵ is bounded below, a finite sum suffices, and hence the map κ_{ϵ} is an isomorphism of complexes. (This is proved in [4, pp.8-15–16] in the special case of the constant T-crystal, but the general case is essentially the same.)

To deal with unbounded gauges ϵ (which are not very important), we claim first that for $j << 0$ the map

$$Ru_{X/S*}A^{\epsilon \vee c_j}E \to Ru_{X/S*}A^{\epsilon}E$$

is an isomorphism. We may check this fact locally, and hence we may choose a lifting Z of X. In this case, A is just the (p^{μ}, γ)-adic filtration of E_Z, and since (E_Z, A) has level within (∞, b), $A^{\epsilon \vee c_j}E_Z = A^{\epsilon}E_Z$ for $j << 0$. The same holds for the De Rham complexes, which calculate the derived functors in question, so the claim holds.

Returning the proof of the lemma, we observe that for $j << 0$, the maps

$$A^{\epsilon \vee c_j}E^{\cdot}_Y \to A^{\epsilon \vee c_{j-1}}E^{\cdot}_Y \to \hat{A}^{\epsilon}E^{\cdot}_Y$$

are quasi-isomorphisms. Taking the direct limit as j tends to $-\infty$, we see that the map $A^{\epsilon}E^{\cdot}_Y \to \hat{A}^{\epsilon}E^{\cdot}_Y$ is also a quasi-isomorphism, as required. ∎

Theorem (6.1.1) is an immediate consequence of the above lemmas. ∎

6.1.7 Corollary: *Suppose that $i: X \to Y$ and $i': X \to Y'$ are two embeddings of X, with Y/S and Y'/S formally log smooth. Suppose that $f: Y' \to Y$ is a morphism such that $f \circ i' = i$. Then f induces (bi)filtered quasi-isomorphisms:*

$$f^*: (E^{\cdot}_Y, A, \mathcal{J}) \to (E^{\cdot}_{Y'}, A, \mathcal{J})$$

$$f^*: (E^{\cdot}_Y, A_{\mathbf{G}}) \to (E^{\cdot}_{Y'}, A_{\mathbf{G}})$$

∎

6.1.8 Remark: If X/S cannot be embedded in a smooth scheme over S, we can use simplicial techniques to calculate $Ru_{X/S*}$. Let $\{X_i : i = 1, \dots r\}$ be an open cover of X such that there exist closed immersions $X_i \to Y_i$, with Y_i/S smooth. For each multi-index $I =: (I_0, \dots I_n)$ let $X_I =: \cap_j X_{I_j}$ and $Y_I =: \times_S Y_{I_j}$. Then we have a locally closed immersion $X_I \to Y_I$ and we can form the divided power envelope D_I of this closed immersion. For each n let $X_n =: \coprod\{X_I : |I| = n\}$ and similarly for Y_n and D_n; then the standard projection maps define simplicial schemes $X_\bullet$, $Y_\bullet$, and $D_\bullet$. We have an evident morphism $\epsilon^+: X_\bullet \to X$, and it is standard that for any abelian sheaf E on $X_\bullet$, the natural map $E \to R\epsilon^+_* \epsilon^{+*} E$ is an isomorphism. A similar statement holds for the bifiltered derived category. In our situation we have a commutative diagram of toposes (*c.f.* [4, pp. 7.13–7.14])

$$\begin{array}{ccc} (X_\bullet/S)_{cris} & \xrightarrow{u_{X_\bullet/S}} & X_\bullet \\ \downarrow \epsilon^+_{cris} & & \downarrow \epsilon^+ \\ (X/S)_{cris} & \xrightarrow{u_{X/S}} & X \end{array}$$

If (E, A) is a T-crystal on X/S we denote its inverse image on $X_\bullet/S$ by the same letter. Carrying out the construction of Theorem (6.1.1) with respect to the embedding $X_\bullet \to Y_\bullet$ we obtain an isomorphism $Ru_{X_\bullet/S*}\epsilon^+_{cris}(E, A, \mathcal{I}) \cong (E^{\cdot}_{Y_\bullet}, A, \mathcal{I})$, and hence we find an isomorphism

$$Ru_{X/S*}(E, A, \mathcal{I}) \cong R\epsilon^+_* Ru_{X_\bullet/S*}(E, A, \mathcal{I}) \cong R\epsilon^+_*(E^{\cdot}_{Y_\bullet}, A, \mathcal{I}).$$

Now if X/S is separated, $R\epsilon^+_*(E^{\cdot}_{Y_\bullet}, A, \mathcal{I})$ is represented by the total complex $(K_{Y_\bullet/S}, A, \mathcal{I})$ associated to the filtered double complex which in degree p, q is $\oplus_{|I|=p}(E_{D_I} \otimes \Omega^q_{Y_I/S}, A, \mathcal{I})$. Thus, we find a simplicial version of Theorem (6.1.1):

$$Ru_{X/S*}(E, A, \mathcal{I}) \cong (K_{Y_\bullet/S}, A, \mathcal{I}).$$

6.2 Kodaira-Spencer sheaves and complexes

In order to study the cohomology of a T-crystal (E, A) on X/S, it is useful to consider also the sheaves $\mathrm{Gr}_A E$ on $\mathrm{Cris}(X/S)$ and $\mathrm{Gr}_A E_\mu$, where $E_\mu =: E/p^\mu E$. (Note that if $i\colon \mathrm{Cris}(X/S_\mu) \to \mathrm{Cris}(X/S)$ is the inclusion, we have a natural isomorphism $Ru_{X/S*}(E_\mu, A) \cong Ru_{X/S_\mu}(i^*E, A)$.) It would be natural to call these either the "Kodaira-Spencer sheaves" or the "Hodge sheaves" attached to (E, A). For technical reasons it is often convenient to restrict our attention to the realization of these sheaves on the divided power envelopes of X in smooth schemes Y over S—*i.e.* to sheaves on the restricted crystalline site. Their values their can be endowed with several natural filtrations. For example, it is useful to consider the filtrations induced by p-adic filtration P, and the divided power I-adic filtration $I_\gamma =: (I_{\gamma,E})$ on E.

6.2.1 Proposition: *Suppose $X \to Y$ is a closed immersion, where Y/S is smooth. Then there are canonical filtered quasi-isomorphisms:*

$$\begin{aligned} Ru_{X/S*}\,\mathrm{Gr}_A E &\cong \mathrm{Gr}_A K_{Y/S} \\ Ru_{X/S*}\,\mathrm{Gr}_A(E, P) &\cong (\mathrm{Gr}_A K_{Y/S}, P) \\ Ru_{X/S*}\,\mathrm{Gr}_A(E, I_\gamma) &\cong (\mathrm{Gr}_A K_{Y/S}, I_\gamma) \\ Ru_{X/S_\mu *}\,\mathrm{Gr}_A E_\mu &\cong \mathrm{Gr}_A K_{Y_\mu/S} \end{aligned}$$

Furthermore, the complexes listed on the right are complexes of $\mathcal{O}_X$-modules with $\mathcal{O}_X$-linear boundary maps.

Proof: The existence of the quasi-isomorphisms follows from (6.1.1). Since $\mathrm{Gr}_A E_Y$ is annihilated by the ideal of X in $D_X(Y)$, the complexes on the right are in fact complexes of $\mathcal{O}_X$-modules. The fact that the boundary maps are linear follows as usual from the Griffiths transversality of A to ∇ and the Leibniz rule. ∎

For example, the proposition shows that we have natural isomorphisms:

$$Ru_{X/S_\mu *}\,\mathrm{Gr}_A E_\mu \cong \mathrm{Gr}_A K_{X/S_\mu}.$$

Recall that if we apply this to the constant T-crystal $(\mathcal{O}_{X/S}, J_{X/S}, \gamma)$ we get

$$Ru_{X/S_\mu *}\,\mathrm{Gr}^i_J \mathcal{O}_{S_\mu} \cong \Omega^i_{X/S_\mu}[-i].$$

6.2.2 Remark: Associated to the filtered sheaf (E_μ, A) on $(Cris X/S_\mu)$ and the functor $f_{X/S_\mu *}$ is the usual spectral sequence

$$E_1^{i,j}(E_\mu, A) \cong R^{i+j} f_{X/S_\mu *}\,\mathrm{Gr}^i_A E_\mu \Rightarrow R^{i+j} f_{X/S_\mu *} E_\mu.$$

Using the isomorphisms of (6.2.1), we can rewrite the E_1-term in terms of the cohomology of a complex of sheaves and $\mathcal{O}_X$-linear maps on X with its Zariski topology:

$$R^{i+j} f_{X/S_{\mu *}} \operatorname{Gr}^i_A E_\mu \cong R^{i+j} f_* \operatorname{Gr}^i_A K_{X/S},$$

where $\operatorname{Gr}^i_A K_{X/S}$ is the complex

$$\operatorname{Gr}^i_A E_X \to \operatorname{Gr}^{i-1}_A E_X \otimes \Omega^1_{X/S} \to \cdots \operatorname{Gr}^{i-q}_A E_X \otimes \Omega^q_{X/S} \to \cdots,$$

in which the term $\operatorname{Gr}^{i-q}_A E_X \otimes \Omega^q_{X/S}$ appears in degree q. In order to make the indices compatible with the usual Hodge spectral sequence in the case of the constant T-crystal, we introduce the following notation:

$$\Omega^{i\cdot}_{(E,A)/S} =: \operatorname{Gr}^i_A K_{X/S}[i] \cong Ru_{X/S*} \operatorname{Gr}^i_A E_\mu[i] \tag{6.2.2.1}$$

If the filtration A is understood, we will write $\Omega^{i\cdot}_{E/S}$ instead of $\Omega^{i\cdot}_{(E,A)/S}$. Then our spectral sequence can be written:

$$E^{i,j}_1(E_\mu, A) \cong R^{i+j} f_{X/S*}(\operatorname{Gr}^i_A E_\mu) \cong R^j f_*(X, \Omega^{i\cdot}_{E/S}) \Rightarrow R^{i+j} f_{X/S*}(E_\mu).$$

We shall call this spectral sequence the "Hodge spectral sequence" associated to a T-crystal (E, A).

The Kodaira-Spencer sheaves $\operatorname{Gr}_A E$ can be analyzed in terms of successive extension of quotients of the sheaves $\operatorname{Gr}_A E_\mu$, as the following result shows.

6.2.3 Proposition: *Let I_γ denote the filtration of $\operatorname{Gr}_A E$ induced by the (I,γ)-filtration on E, and let I denote the filtration of $\operatorname{Gr}_A E$ induced by the usual I-adic filtration. Then there are natural isomorphisms of sheaves on the restricted crystalline site:*

$$\operatorname{Gr}^j_{I,\gamma} \mathcal{O}_S \otimes_{\mathcal{O}_S} \operatorname{Gr}^{n-j}_A E_\mu \cong \operatorname{Gr}^j_{I,\gamma} \operatorname{Gr}^n_A E.$$

If (E, A) has level in $[a, \infty)$ and $n < p+a$, we also have natural isomorphisms:

$$A^n E_\mu \cong \operatorname{Gr}^0_I A^n E \quad \text{and}$$

$$\operatorname{Gr}^j_I \mathcal{O}_S \otimes_{\mathcal{O}_S} \operatorname{Gr}^{n-j}_A E_\mu \cong \operatorname{Gr}^j_I(A^n E / I A^n E) \quad \text{if } j > 0.$$

Proof: If D is a fundamental thickening of an open subset of X, then by Proposition (3.3.3) we see that (E_D, A) is G-transversal to (I,γ) and compatible with each p^i. It follows from Proposition (4.3.4) that (E_D, A) is given by a $\mathbf{G}_{\mathcal{I}_\mu}$-structure. In particular, if g is the control function corresponding to (I,γ), and if ϵ_n is the maximal g-tame gauge which vanishes at n, then $A^n E_D = A^{\epsilon_n} E_D$. Recall that $\epsilon_n(i) = g(n-i)$ for $i \leq n$, and $I^{[i]} = (p^{g(i)})$. Let c denote the constant gauge whose value is $g(j)$, by property (1) of (4.3.3), we see that $A^c E_D \cap A^{\epsilon_n} E_D = A^{c \vee \epsilon_n} E_D$. As $(c \vee \epsilon_n)(n-i) = g(i)$ if $i \geq j$ and $= g(j)$ if $i \leq j$, it follows that

$$I^{[j]} E_D \cap A^n E_D = I^{[j]} A^{n-j} E_D + I^{[j+1]} A^{n-j-1} E_D + \cdots \qquad (6.2.3.2)$$

This implies that the natural map

$$I^{[j]} \mathcal{O}_Y \otimes A^{n-j} E_D \longrightarrow I^{[j]} E_D \cap A^n E_D$$

is surjective modulo $I^{[j+1]} E_D$. Let $I_j \subseteq \mathcal{O}_S$ denote the annihilator of $\mathrm{Gr}^j_{I,\gamma} \mathcal{O}_S$, *i.e.* $(p^{g(j+1)-g(j)})$. We see that multiplication by $p^{g(j)}$ induces an isomorphism

$$A^{n-j} E_D / (I_j E_D \cap A^{n-j} E_D) \cong \mathrm{Gr}^j_{I_\gamma, E} A^n E_D.$$

Now $I \subseteq I_j$ and the filtration (E_{D_μ}, A) is normally transversal to I_j. It follows that multiplication by $p^{g(j)}$ also induces an isomorphism:

$$A^{n-j} E_D / (I_j E_D \cap A^{n-j} E_D) \cong \mathrm{Gr}^j_{I,\gamma} \mathcal{O}_S \otimes A^{n-j} E_{D_\mu}.$$

We thus obtain an isomorphism

$$\mathrm{Gr}^j_{I,\gamma} \mathcal{O}_S \otimes A^{n-j} E_{D_\mu} \cong \mathrm{Gr}^j_{I_\gamma, E} A^n E_D \qquad (6.2.3.3)$$

It is clear that this implies the first statement of the proposition.

For the second statement, first note from the standard formula [4, 3.3] for the ordinal of $p^n/n!$ that $I^j = I^{[j]}$ if $j < p$. Now if $0 \leq j \leq n-a < p$, formula (6.2.3.2) reads:

$$\begin{aligned} I^{[j]} E \cap A^n E &= I^{[j]} A^{n-j} E + I^{[j+1]} A^{n-j-1} E + \cdots + I^{[n-a]} A^a E \\ &= I^j A^{n-j} E + I^{j+1} A^{n-j-1} E + \cdots I^{n-a} A^a E \\ &= I^j A^{n-j} E \end{aligned}$$

Thus if $0 \leq j \leq n-a$ we have

$$\begin{aligned} I^j E \cap A^n E &= I^j A^{n-j} E \quad \text{and} \\ I^{j+1} E \cap A^n E &= I^j E \cap A^n E \cap I^{j+1} E \\ &= (I^j A^{n-j} E) \cap I^{j+1} E \\ &= I^j (A^{n-j} E \cap I E) \end{aligned}$$

Thus we have an isomorphism:

$$I^j/I^{j+1} \otimes A^{n-j} E_\mu \to \mathrm{Gr}_I^j A^n E. \tag{6.2.3.4}$$

If also $j > 0$, $I^j E \cap I A^n E = I^j A^{n-j+1} E$, so (6.2.3.4) also induces an isomorphism

$$I^j/I^{j+1} \otimes \mathrm{Gr}_A^{n-j} E_\mu \to \mathrm{Gr}_I^j (A^n E / I A^n E).$$

This proves the second statement if $j \leq n - a$. If $j > n - a$,

$$I^j E \subseteq I I^{[n-a]} E \subseteq I A^n E$$

so $\mathrm{Gr}_I^j(A^n E / I A^n E)$ vanishes, as predicted by the formula. ∎

6.3 Higher direct images of T-crystals

Throughout this section we suppose that X/S_μ is proper and logarithmically smooth and integral; here $\mu \in \mathbf{N}$ or $\mu = \infty$.

6.3.1 Proposition: *Let (E, A) be a T-crystal on X/S of level within $[a, b]$, and let d be the relative dimension of X/S. Suppose that n is an integer such that $R^n f_{X/S*}E$ and $R^{n+1} f_{X/S*}E$ are p-torsion free over $\mathcal{O}_S$ and that the Hodge spectral sequence (6.2.2) of the restriction of (E, A) to X_0 degenerates at E_1 in degree n, i.e. $E_1^{i,j} = E_\infty^{ij}$ whenever $i + j = n$. Then the maps*

$$A^k R^n f_{X/S*} E =: R^n f_{X/S*}(A^k E) \to R^n f_{X/S*} E$$

are injective for all k and define a filtration on $R^n f_{X/S*}E$ which is G-transversal to (g, p), of level within $[a, b + d]$. Furthermore, if ϵ is any p^μ-tame gauge, there is a natural isomorphism:

$$A^\epsilon R^n f_{X/S*} E \cong R^n f_{X/S*}(A^\epsilon E).$$

Proof: We describe the proof when X can be embedded in a logarithmically smooth and proper scheme Y/S, and let D be the divided power envelope of X in Y; the general case follows by means of the usual simplicial methods (6.1.8). According to Theorem (6.1.1), the cohomology of $(E, A_{\mathbf{G}})$ is calculated by the Zariski cohomology of the $\mathbf{G}$-filtered complex $(K_{Y/S}, A_{\mathbf{G}})$, where $K_{Y/S}$ is the De Rham complex of E_D. As we have observed, Proposition (3.3.3) implies that the terms in the complex $(K_{Y/S}, A)$ are G-transversal to (g, p), and it follows that we have a $\mathbf{G}_{\mathcal{J}}$-structure in the category of complexes of sheaves of modules on X_{zar}.

As X/S_μ is smooth, it follows from (6.1.1) that the natural map

$$(E_{Y/S}/p^\mu E_{Y/S}, A) \to (E_{X/S_\mu}, A)$$

is a quasi-isomorphism. Our hypothesis on the the hypercohomology spectral sequence of the filtered complex $(\mathrm{Gr}^0_p K_{Y/S}, A) \cong (K_{X/S}, A)$ implies that each

$$H^q(A^i \, \mathrm{Gr}^0_P \, K_{Y/S}) \to H^q(\mathrm{Gr}^0_P \, K_{Y/S})$$

is injective for $q = n$ and $n+1$. Thus we can apply (4.4.3) to conclude that the map $R^n f_{X/S*} A^k E \to R^n f_{X/S*} E$ is injective and defines a filtration which is G-transversal to (g, p), and that formation of $R^n f_{X/S*}$ is compatible with the construction A^ϵ.

It is clear that $A^a R^n f_{X/S*} E = R^n f_{X/S*} E$. On the other hand, if $k > b$, $A^k E_X = 0$ and hence if $k > b + d$, the complex $A^k \; K_{X/S_\mu} = 0$. It follows that $A^k R^n f_{X/S*} E_\mu$ vanishes if $k > \; b + d$. This shows that the level is as described. ∎

6.3.2 Theorem: *Let $f: X \to Y$ be a logarithmically smooth, proper, and integral morphism of logarithmic schemes which are logarithmically smooth over S_μ. Let (E, A) be a T-crystal on X/S and suppose that $R^n f_{cris*} E$ and $R^{n+1} f_{cris*} E$ are locally free over $\mathcal{O}_{Y/S}$, and that the Hodge spectral sequence relative to f_{X_0/Y_0} of the restriction of (E, A) to X_0 degenerates at E_1 in degree n. Then the filtration*

$$A^k R^n f_{X/Y*} E =: R^n f_{X/Y*}(A^k E) \subseteq R^n f_{X/Y*} E$$

defines a T-crystal on Y/S. Moreover, for any $\mu' < \mu$, the restriction of this T-crystal to $(Y_{\mu'}/W)$ is the q^{th} direct image of the restriction of (E, A) to $(X_{\mu'}/S)$.

Proof: We may work locally on Y, with the aid of a smooth lifting $\mathcal{Y}$ of Y to S, and use the description (3.2.3) of the category of T-crystals on Y_μ/S. It follows from the previous result that $(R^n f_{X/Y*} E, A)$ is G-transversal to the divided power ideal (p^μ, γ) of Y in $\mathcal{Y}$, compatibly with all p^i. Because f is logarithmically smooth and integral, it is flat [20, 4.5], and it follows that base changing for crystals works the same way as in the classical case [20, 6.12]. In particular, since $R^{n+1} f_{X/\mathcal{Y}*} E$ is locally free, the formation of $R^n f_{X/\mathcal{Y}*} E$ commutes with base change, and $R^n f_{X/\mathcal{Y}*} E$ forms a crystal on Y/S whose crystal structure is determined by the Gauss-Manin connection on $E_\mathcal{Y}$. The fact that the filtration induced by A is Griffiths transversal to ∇ is proved in the same way as the classical case. Finally, note that the restriction of $(R^n f_{X/S*}E, A)$ to $Y_{\mu'}$ is defined by taking the saturation $A_{\mu'}$ of A with respect to $(p^{\mu'}, \gamma)$. Thus, for each i, $A^i_{\mu'}R^n f_{X/Y*}E = A^{\epsilon_i}R^n f_{X/Y*}E$, where ϵ_i is the maximal $p^{\mu'}$-tame gauge which vanishes at i. Since ϵ_i is also p^μ-tame, we see that

$$A^i_{\mu'}R^n f_{X/Y*}E \cong R^n f_{X/Y*}A^{\epsilon_i}E = R^n f_{X/Y*}A^i_{\mu'}E.$$

This last is precisely the i^{th} level of the direct image of restriction of (E, A) to $X_{\mu'}/S$. ∎

6.3.3 Proposition: *With the notation of (6.3.2), suppose that Y_0 is reduced, that (E, A) is a uniform T-crystal on X/S, and that the cohomology sheaves $R^n f_{cris*}E$ are locally free $\mathcal{O}_{Y/S}$-modules for all n. Suppose also that for each closed point y of Y, the Hodge spectral sequence of the restriction of (E, A) to the fiber $X(y)/k(y)$ of f over y degenerates at E_1. Then for all n, (E, A) satisfies the satisfies the hypotheses and conclusion of Theorem (6.3.2), and the T-crystal $(R^n f_{cris*}E, A)$ is uniform on (Y_μ/S). If $\tilde{Y}$ is another smooth logarithmic S_μ-scheme, then formation of the T-crystal $(R^n f_{cris*}E, A)$ commutes with base change $\tilde{Y} \to Y$.*

Proof: Because X/Y is flat and the cohomology sheaves are locally free, their formation commutes with all base change. By hypothesis, the hypercohomology spectral sequence of $(\mathrm{Gr}^0_P E(y), A)$ degenerates at E_1, and therefore the rank of $H^n(X(y)/k(y), \mathrm{Gr}_A \mathrm{Gr}^0_P E(y))$ over $k(y)$ equals that of $H^n(X(y)/k(y), \mathrm{Gr}^0_P E(y))$, hence is locally constant. It follows that the coherent sheaf $R^n f_{X/S*} \mathrm{Gr}_A \mathrm{Gr}^0_P E$ is locally free on Y_0 and that its formation commutes with base change.

Choose a local lifting $\mathcal{Y}$ of Y as above and let $E =: R^n f_{X/S*}E$ with its induced filtration A, and let $E_i =: E/p^iE$. We know that (E, A) is G-transversal to g, and in particular that the induced filtration on E_μ is normally transversal to p^i, *i.e.* that $A^iE_\mu \otimes \mathcal{O}_{Y_1} \to E_1$ is injective. As E_μ is flat over $\mathcal{Y}_\mu$, this implies that $\mathrm{Tor}_1(E_\mu/A^iE_\mu, \mathcal{O}_{Y_1}) = 0$, where the Tor is computed on $Y = \mathcal{Y}_\mu$. As (E_1/A^iE) is flat over Y_1, the local criterion for flatness implies that (E_μ/A^iE_μ) is also flat over Y_μ. This proves that (E, A) is uniform as a T-crystal on Y_μ. The fact that its formation commutes with base change follows easily from the fact that this is true modulo p^μ and Lemmas (2.3.2) and (2.2.1). ∎

6.3.4 Remark: Instead of assuming that Y_0 is reduced and that the Hodge spectral sequence degenerates fiber by fiber, we could instead have assumed that the sheaves $R^n f_{X/Y*}(\mathrm{Gr}_A E_X)$ are locally free and that the Hodge spectral sequence for $f_{X/Y}$ degenerates at E_1. Let us also remark that if X and Y have trivial log structures and are smooth over W_μ, and if E comes from a crystal on X/W, then we know from [27, (1.16,3.7)] that the rank of $H^n(X(y)/W(y), E(y))$ is a locally constant function of y. If we assume that for every y these modules are torsion free, then they they are free, and it follows from the universal coefficient theorem for crystalline cohomology that the rank of

$$H^n(X(y)/W(y), E(y)) \otimes k \cong H^n(X(y)/k(y), \mathrm{Gr}_P^0 E(y))$$

is locally constant. Then standard techniques show that each of the sheaves $R^n f_{X/Y_1*}(\mathrm{Gr}_P^0 E)$ and $R^n f_{X/Y*}(E)$ is locally free and that its formation commutes with base change.

7 Cohomology of F-spans—Mazur's Theorem

This section is devoted to the formulation and proof of a generalization of Mazur's fundamental theorem [23] about Frobenius and the Hodge filtration to the case of cohomology with coefficients in an F-span. In essence, our theorem will assert the compatibility of the functor $\alpha_{X/W}$ with formation of higher direct images.

Throughout this section we let S denote a p-adic formal scheme, flat over W and endowed with a fine logarithmic structure. We let X/S_μ be a smooth and integral morphism of fine log schemes; μ will be zero (and hence $X = X_0$) unless explicitly stated otherwise. Eventually we will also require that X/S_0 be perfectly smooth (1.2.3); in any case we let $F_{X/S_0}: X_0 \to X_0'$ or just $F_{X/S}$ denote the exact relative Frobenius morphism (1.2.3.4). We do not need a lifting of the absolute Frobenius endomorphism F_{S_0} until later, when we discuss F-T- crystals.

7.1 Cohomology of the conjugate filtration

If $\Phi: F^*_{X/S}E' \to E$ is an F-span on X/S, we let $E'' =: F^*_{X/S}E'$. As described in (5.1.2), we have filtrations M and P on $\breve{E}'' =: E'' \otimes \mathbf{Q}$, and P and N on $\breve{E} =: E \otimes \mathbf{Q}$, as well as a bifiltered isomorphism

$$\breve{\Phi}: (\breve{E}'', M, P) \to (\breve{E}, P, N).$$

In particular, for each integer j we can apply the functor $Ru_{X/S*}$ to obtain an isomorphism in the bifiltered derived category of complexes of Zariski sheaves on X_{zar}:

$$(Ru_{X/S*}P^j\breve{E}'', M, P) \to (Ru_{X/S*}N^j\breve{E}, P, N).$$

These fit together for varying j's to form an inductive system of isomorphisms, which we denote abusively by

$$(Ru_{X/S*}\breve{E}'', M, P) \to (Ru_{X/S*}\breve{E}, P, N).$$

(Note that the systems $N^j\breve{E}$ and $P^j\breve{E}$ are cofinal, so "$\varprojlim$" $N^j\breve{E} \cong$ "$\varprojlim$" $P^j\breve{E}$.) The existence and functoriality of $(Ru_{X/S*}\breve{E}'', M, P)$ and $(Ru_{X/S*}\breve{E}, P, N)$ follow from the functoriality of the crystalline topos. The following proposition describes how they may be computed explicitly in terms of filtered De Rham complexes.

7.1.1 Proposition: *Suppose that X can be embedded as a locally closed subscheme of a logarithmically smooth S-scheme Y, let D be the (p-adically complete) PD-envelope of X in Y, and let $K_{Y/S}$ denote the De Rham complex of E on D, with $\breve{K}_{Y/S} =:$ "$\varprojlim$"$P^jK_{Y/S}$. Then there is a commutative diagram of isomorphisms in the bifiltered derived category:*

$$\begin{array}{ccc} (Ru_{X/S*}\breve{E}'', M, P) & \longrightarrow & (\breve{K}''_{Y/S}, M, P) \\ \downarrow \breve{\Phi} & & \downarrow \breve{\Phi}_Y \\ (Ru_{X/S*}\breve{E}, P, N) & \longrightarrow & (\breve{K}_{Y/S}, P, N) \end{array}$$

∎

Proof: This would be standard, except that we must compare the filtrations by subsheaves $\tilde{M}$ and $\tilde{P}$ (respectively $\tilde{P}$ and $\tilde{N}$) of E'' (resp. E), and the filtrations by subcrystals (*c.f.* Remark (3.0.6)). The difference between these is parasitic (3.0.5), and we have some technical work to do to show that it causes no essential problems when we pass to cohomology.

If $\theta: E' \to E$ is a morphism of crystals of $\mathcal{O}_{X/S}$-modules we let $K(\theta)$ denote the kernel of θ computed in the category of all sheaves of $\mathcal{O}_{X/S}$-modules. If θ is a monomorphism in the category of crystals, $K(\theta)$ will be parasitic [3, V,2], but need not vanish, in general. When working over a base scheme on which p is nilpotent, $Ru_{X/S*}$ annihilates all parasitic sheaves ([3, V,1.3.3]), but I do not know if this is true in our context, when S is a formal scheme. We shall have to make do with the following result.

7.1.2 Lemma: *Suppose that $\theta: E' \to E$ is a monomorphism in the category of crystals of $\mathcal{O}_{X/S}$-modules whose cokernel has bounded p^∞-torsion. Then $Ru_{X/S*}K(\theta) = 0$.*

Proof: This is a local question, and we may assume that X admits a smooth lifting Y/S. For $n \geq \mu$, we can restrict θ to the site $\text{Cris}\,(X/S_n)$ to obtain a map of crystals θ_n on X/S_n. Let K_n denote the kernel of θ_n, computed in the category of crystals of $\mathcal{O}_{X/S_n}$-modules. Because $\text{Cok}(\theta)$ has bounded torsion, the inverse system $\{\text{Tor}_1(\text{Cok}(\theta), \mathcal{O}_{X/S_n}) : n \geq \mu\}$ is essentially zero, and hence the same is true of its quotient system $\{K_n\}$. Because K_n is a crystal, $Ru_{X/S_n*}K_n$ is represented by the De Rham complex $K_n^{\cdot}$ with respect to Y/S, and in fact the inverse system of these complexes is essentially zero. There is an evident map of sheaves of $\mathcal{O}_{X/S_n}$-modules $\iota_n: K_n \to K(\theta_n)$, whose kernel K'_n cokernel K''_n are parasitic on $\text{Cris}(X/S_n)$.

In order to pass from the collection of sites $\text{Cris}(X/S_n)$ to $\text{Cris}(X/S)$, we use the intermediate site $\text{Cris}(X/S.)$ and the diagram on the top of page 7-28 of [4]. It is clear that we have an exact sequence of quasi-coherent sheaves

$$0 \to K'. \to K. \to K(\theta.) \to K''. \to 0$$

on $\text{Cris}(X/S.)$; furthermore $K(\theta.)$ is just the restriction $j^*K(\theta)$ of $K(\theta)$ to $\text{Cris}(X/S.)$. Since K'_n and K''_n are parasitic, $Ru_{X/S_n*}K'_n = Ru_{X/S_n*}K''_n = 0$, and hence by [4, 7.22.1] $Rv_{X/S.*}K'. = Rv_{X/S.*}K''. = 0$. It follows that $Rv_{X/S.*}K. \cong Rv_{X/S.*}(K(\theta.))$, and applying [4, 7.19] and [4, 7.22.2], we find

$$Ru_{X/S*}K(\theta) \cong Ru_{X/S*}Rj_*j^*K(\theta) \cong R\pi_{X*}Rv_{X/S.*}K(\theta.) \cong R\pi_{X*}Rv_{X/S.*}K..$$

Now $Rv_{X/S.*}K.$ is represented by the inverse system of De Rham complexes $\{K_n^{\cdot}\}$, which we have seen is essentially zero, and hence is annihilated by $R\pi_{X*}$. ∎

If D is a fundamental thickening of X relative to S and E is a sheaf of $\mathcal{O}_D$-modules, we let $L(E)$ be the corresponding crystal of $\mathcal{O}_{X/S}$-modules [4, 6.10.1].

7.1.3 Lemma: *Suppose that $\theta: E' \to E$ is an injective map of $\mathcal{O}_D$ modules whose cokernel has bounded p^∞-torsion. Then the image $\tilde{L}(\theta)$ of $L(\theta)$, computed in the category of sheaves of $\mathcal{O}_{X/S}$-modules, is acyclic for $u_{X/S*}$*

Proof: Observe that $L(\theta)$ is a map of crystals of $\mathcal{O}_{X/S}$-modules and that its cokernel has bounded p^∞-torsion. We have an exact sequence

$$0 \to K(\mathrm{L}(\theta)) \to L(E') \to \tilde{L}(\theta) \to 0$$

of sheaves of $\mathcal{O}_{X/S}$-modules. By Lemma (7.1.2), $Ru_{X/S*}K(L(\theta)) = 0$, and by (6.1.5), the sheaf $L(E')$ is acyclic for $u_{X/S*}$. It follows that the same is true for $\tilde{L}(\theta)$. ∎

If E is a crystal of $\mathcal{O}_{X/S}$-modules we let $E^{\cdot}_D$ denote the De Rham complex of the module with connection E_D, and we recall that there is a natural quasi-isomorphism of sheaves on Cris(X/S): $E \cong L(E^{\cdot}_D)$. Suppose that N is a filtration of N by subcrystals; then for each i we have a map of complexes of $\mathcal{O}_{X/S}$-modules $L(N^iE^{\cdot}_D) \to L(E_D)$, and we let $\tilde{N}^iL(E^{\cdot}_D)$ denote its image.

7.1.4 Lemma: *There is a natural filtered quasi-isomorphism of complexes of $\mathcal{O}_{X/S}$-modules $(E, \tilde{N}) \to (L(E^{\cdot}_D), \tilde{N})$.*

Proof: For each i we have a commutative diagram of complexes of sheaves of $\mathcal{O}_{X/S}$-modules

$$\begin{array}{ccccccc} N^iE & \longrightarrow & E & \longrightarrow & E/N^iE & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ L(N^iE^{\cdot}_D) & \longrightarrow & L(E^{\cdot}_D) & \longrightarrow & L(E/N^iE^{\cdot}_D) & \longrightarrow & 0 \end{array}$$

Notice that we have zeroes on the right because the inclusion functor from the category of crystals to the category of sheaves is right exact. We know that the vertical arrows are quasi-isomorphisms, and it follows that the natural map $\tilde{N}^iE \to \tilde{N}^iL(E^{\cdot}_D)$ is also a quasi-isomorphism. ∎

It is now easy to prove Proposition (7.1.1). It is clear that the previous lemma holds also for bifiltered complexes. Furthermore, by (7.1.3), the terms of the bifiltered complex $(L(E^{\cdot}_D), \tilde{P}, \tilde{N})$ are acyclic for $Ru_{X/S*}$, and hence we obtain an isomorphism in the bifiltered derived category:

$$Ru_{X/S*}(E_D, \tilde{P}, \tilde{N}) \cong u_{X/S*}(L(E^{\cdot}_D), \tilde{P}, \tilde{N}) \cong (E^{\cdot}_D, P, N).$$

These arguments also apply to the bifiltered complexes $(P^jE'', \tilde{P}, \tilde{M})$ and $(N^jE, \tilde{N}, \tilde{P})$. Thus, for any integer a, we have a commutative diagram:

$$\begin{array}{ccc} (Ru_{X/S*}P^jE'', \tilde{M}, \tilde{P}) & \longrightarrow & (P^jK''_{Y/S}, \tilde{M}, \tilde{P}) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi_Y} \\ (Ru_{X/S*}N^jE, \tilde{P}, \tilde{N}) & \longrightarrow & (N^jK_{Y/S}, \tilde{P}, \tilde{N}) \end{array}$$

All the arrows are isomorphisms in the derived category, and in fact the vertical arrow on the right is an isomorphism of filtered complexes. Fitting these together as a varies and noting that "$\varprojlim$" $P^j K \cong$ "$\varprojlim$" $N^j K$, we obtain the statement of the proposition. ∎

In fact, it is not the bifiltered object $(Ru_{X/S*}\breve{E}'', M, P)$ in which our primary interest lies, but rather the object obtained from it by applying Deligne's "décalé" operation [6] (also used by Illusie in [17] in a context very similar to ours). Recall from [6, 1.3] that if (K, P) is a filtered complex, we have

$$\begin{aligned}(\operatorname{Dec} P)^i K^q &=: \{x \in P^{i+q}K^q : dx \in P^{i+q+1}K^{q+1}\}\\ (\operatorname{Dec}^* P)^i K^q &=: \{P^{i+q}K^q + dP^{i+q-1}K^{q-1}\}\end{aligned}$$

These operations are not compatible with taking mapping cones but are compatible with filtered quasi-isomorphisms: a filtered quasi-isomorphism $h\colon (A, P) \to (B, P)$ induces filtered quasi-isomorphisms:

$$(A, \operatorname{Dec} P) \to (B, \operatorname{Dec} P) \qquad \text{and} \qquad (A, \operatorname{Dec}^* P) \to (B, \operatorname{Dec}^* P).$$

Furthermore, if A' is a subcomplex of A of A and if P' is the filtration on A' induced by P, then by its very definition $\operatorname{Dec} P'$ is the filtration of A' induced by the filtration $\operatorname{Dec} P$. Dually, the same is true of quotient complexes, with Dec replaced by Dec*. It is however not true that Dec is compatible with quotients or Dec* with subobjects. Nevertheless, it follows from the five lemma that if the filtered quasi-isomorphism h induces a filtered quasi-isomorphism $(A', P) \to (B', P)$, then it also induces filtered quasi-isomorphisms

$$(A', \operatorname{Dec} P) \to (B', \operatorname{Dec} P) \qquad \text{and} \qquad (A/A', \operatorname{Dec} P) \to (B/B', \operatorname{Dec} P),$$

and similarly for Dec*. We conclude that if $u\colon (A, F, P) \to (B, F, P)$ is a bifiltered quasi-isomorphism, then the induced maps

$$(A, F, \operatorname{Dec} P) \to (B, F, \operatorname{Dec} P) \qquad \text{and} \qquad (A, F, \operatorname{Dec}^* P) \to (B, F, \operatorname{Dec}^* P)$$

are also bifiltered quasi-isomorphisms. Thus the décalage operations pass to the bifiltered derived category.

Warning: Formation of the décalé does not preserve acyclicity with respect to $u_{X/S*}$ of the terms of a complex, and hence does not commute with higher direct images, in general. In particular, $(Ru_{X/S*}\breve{E}'', M, \operatorname{Dec} P)$ is not the same as $Ru_{X/S*}(\breve{E}'', M, \operatorname{Dec} P)$.

Note that the filtration N of $\breve{K}$ is G-transversal to (p); it follows immediately that $\operatorname{Dec} N$ is also. (This is not so for $\operatorname{Dec}^* N$, however.) Thus, we can also can view our bifiltered objects as objects endowed with a lattice filtration (4.1.1) indexed by the lattice **L**, or as endowed with a gauge structure indexed by the lattice of 1-gauges $\mathbf{G}_1$ (4.3.3). Furthermore, the dictionary (4.3.4.3) allows us to pass from one point of view to the other.

7.1.5 Remark: If X cannot be embedded in a smooth log scheme over S, we may compute $Ru_{X/S*}(E, \operatorname{Dec} N, P)$ by simplicial methods, as in Remark (6.1.8). Using the notation introduced there, note that if (K, A, B) is a bifiltered complex on X, we also have an isomorphism in the bifiltered derived category:

$$(K, A, \operatorname{Dec} B) \cong R\epsilon_*^+ \epsilon^{+*}(K, A, \operatorname{Dec} B).$$

Since the operation of décalage commutes with restricting to open subsets, $\epsilon^{+*}(K, A, \operatorname{Dec} B)$ can also be viewed as the décalage of the filtration B of $\epsilon^{+*}(K, A, B)$. In particular, it follows that the formation of the décalage of B in $\epsilon^{*+}(K, A, B)$ commutes with $R\epsilon_*^+$.

7.2 Local version of the main theorem

It seems desirable to begin our discussion of Mazur's theorem in the case of a parallelizable lifted situation $\mathcal{Y} =: (Y, F_{Y/S})$ (1.2.6). Let $\Phi: F_{Y/S}^* E' \to E$ be an admissible F-span (5.2.8) of level within $[a, b]$. As we have seen in theorem (5.2.13), there is a corresponding filtration $A_{\mathcal{Y}}$ on E' which is G-transversal to ∇' and G-transversal to (p); recall that

$$A_{\mathcal{Y}}^k E' =: (\Phi \circ \eta_E)^{-1}(p^k E).$$

We shall find it convenient to work with the extension of $A_{\mathcal{Y}}$ to $\check{E}'_{Y'}$ given by (5.1.2); this extension is also Griffiths transversal to ∇' and G-transversal to (p). As before, we let $\check{K}'_{Y'/S}$ denote the De Rham complex of $(\check{E}'_{Y'}, \nabla'_Y)$, with its filtrations P and $A_{\mathcal{Y}}$.

7.2.1 Theorem: *With the notation of the previous paragraph, $F_{Y/S}^*$ induces a bifiltered quasi-isomorphism:*

$$\Psi: (\check{K}'_{Y'/S}, A_{\mathcal{Y}}, P) \to F_{Y/S*}(\check{K}''_{Y/S}, M, \operatorname{Dec} P) \cong F_{Y/S*}(\check{K}_{Y/S}, P, \operatorname{Dec} N).$$

Proof: We begin with a preliminary remark in characteristic p, in which the lifting plays no role and need not exist. Recall that the Frobenius pull-back of any quasi-coherent sheaf inherits a canonical integrable and p-integrable connection.

7.2.2 Lemma: *Let (E_0', A) be a quasi-coherent sheaf of $\mathcal{O}_{X'}$-modules with a finite filtration by quasi-coherent subsheaves, and let (E_0'', ∇'', M) be the corresponding quasi-coherent sheaf of $\mathcal{O}_X$-modules, with its induced integrable connection ∇'' and horizontal filtration M. Let (K_0'', M) denote the De Rham complex of (E_0'', ∇''), with the filtration induced by M. Then:*

1. *There are natural isomorphisms of filtered sheaves:*

$$C^{-1}: (\Omega^n_{X'/S_0} \otimes E_0', A) \to (\underline{H}^n(K_0''), M)$$

2. *The spectral sequence of a filtered complex:*

$$E_1^{i,j} \cong \underline{H}^{i+j}(\operatorname{Gr}^i_M K_0'')) \Rightarrow \underline{H}^{i+j}(E_0'', \nabla)$$

   degenerates at E_1

3. *The boundary maps of the complex K_0'' are strictly compatible with the filtration M.*

Proof: Statement (1) is repeated from (1.2.5). The remaining two conditions, which are in fact equivalent [6, 1.3.2], are an easy consequence. In fact, we can use the Cartier isomorphism (7.2.2.1) to identify $\underline{H}^n(K_0'')$ with $\Omega^n_{X'/S} \otimes E_0'$, compatibly with the filtrations. As the maps

$$\Omega^n_{X'/S} \otimes A^i E_0' \to \Omega^n_{X'/S} \otimes E_0'$$

are injective, so are the maps

$$\underline{H}^n(M^i K_0'') \to \underline{H}^n(K_0'').$$

This proves the degeneration of the spectral sequence. ∎

Let us return to our study of an F-span in a lifted situation, keeping the notation and hypotheses of (7.2.1). We consider the spectral sequence $E_r(K, W)$ of a filtered complex (K, W). Recall from [6, 1.3.3.2] that there are natural maps:

$$\begin{aligned} u: E_0^{j,n-j}(K, \operatorname{Dec} W) \cong \operatorname{Gr}^j_{\operatorname{Dec} W} K^n &\to E_1^{j+n,-j}(K, W) \\ u^*: E_1^{j+n,-j}(K, W) &\to \operatorname{Gr}^j_{\operatorname{Dec}^* W} K^n \cong E_0^{j,n-j}(K, \operatorname{Dec}^* W) \end{aligned}$$

which induce quasi-isomorphisms (after a suitable renumbering):

$$\begin{aligned} u: \operatorname{Gr}_{\operatorname{Dec} W} K &\to E_1(K, W) \\ u^*: E_1(K, W) &\to \operatorname{Gr}_{\operatorname{Dec}^* W} K \end{aligned}$$

7.2.3 Proposition: *With the notation and hypotheses of (7.2.1)*

1. *The maps u and u^* induce filtered quasi-isomorphisms*

$$\begin{aligned} u\colon (\mathrm{Gr}_{\mathrm{Dec}\,P}\,\breve{K}''_{Y/S}, M) &\to (E_1(\breve{K}''_{Y/S}, P), M) \\ u^*\colon (E_1(\breve{K}''_{Y/S}, P), M) &\to (\mathrm{Gr}_{\mathrm{Dec}^*\,P}\,\breve{K}''_{Y/S}, M) \\ u\colon (\mathrm{Gr}_{\mathrm{Dec}\,N}\,\breve{K}_{Y/S}, P) &\to (E_1(\breve{K}_{Y/S}, N), P) \\ u^*\colon (E_1(\breve{K}_{Y/S}, N), P) &\to (\mathrm{Gr}_{\mathrm{Dec}^*\,N}\,\breve{K}_{Y/S}, P) \end{aligned}$$

2. *The natural maps*

$$\begin{aligned} (\breve{K}''_{Y/S}, M, \mathrm{Dec}\,P) &\to (\breve{K}''_{Y/S}, M, \mathrm{Dec}^*\,P) \\ (\breve{K}_{Y/S}, P, \mathrm{Dec}\,N) &\to (\breve{K}_{Y/S}, P, \mathrm{Dec}^*\,N) \end{aligned}$$

   are bifiltered quasi-isomorphisms.

Proof: Because it is natural, Deligne's morphism u is automatically compatible with the filtrations induced by the filtration M, and gives us a map of filtered complexes

$$u\colon (\mathrm{Gr}_{\mathrm{Dec}\,P}\,K'', M) \to (E_1(K'', P), M).$$

It is not automatic, however, that this arrow is a filtered quasi-isomorphism, and the proof of this fact will take further work. Because (E''_Y, M) is G-transversal to p, we see that multiplication by p^j induces natural filtered isomorphisms:

$$\mathrm{Gr}^0_P(\breve{E}''_Y, M[-j]) \to \mathrm{Gr}^j_P(\breve{E}''_Y, M).$$

As these isomorphisms are evidently compatible with the connections, they induce isomorphisms of filtered complexes:

$$\mathrm{Gr}^0_P(K'', M[-j]) = \mathrm{Gr}^0_P(\breve{K}'', M[-j]) \to \mathrm{Gr}^j_P(\breve{K}'', M).$$

It thus follows from Lemma (7.2.2) that the boundary maps of the complex $\mathrm{Gr}^j_P(\breve{K}''_{Y/S})$ are strictly compatible with the filtration M, and [6, 1.3.15] implies that the maps u and u^* are filtered quasi-isomorphisms. This proves (7.2.3.1) for $(\breve{E}'', M, P)$. It is easy to deduce (7.2.3.2). In fact, it follows that

$$u^* \circ u\colon \mathrm{Gr}_{\mathrm{Dec}\,P}\,Ru_{X/S*}(\breve{E}'', M) \to \mathrm{Gr}_{\mathrm{Dec}^*\,P}\,Ru_{X/S*}(\breve{E}'', M)$$

is a filtered quasi-isomorphism, so by (4.4.1) it will suffice to check that whenever $i >> j$ or $j >> i$, the induced map

$$u^* \circ u \colon M^i \cap (\operatorname{Dec} P^j)\breve{K}''_{Y/S} \to M^i \cap (\operatorname{Dec}^* P^j)K''_{Y/S}$$

is a quasi-isomorphism. Recall that

$$M^{b+i}\breve{E}'' \subseteq P^i\breve{E}''_Y \subseteq M^{a+i}\breve{E}''_Y$$

and it follows that if d is the relative dimension of X/S_0,

$$M^{b+d+i}\breve{K}''_{Y/S} \subseteq \operatorname{Dec} P^i\breve{K}''_{Y/S} \subseteq M^{a+i}\breve{K}''_{Y/S} \qquad (7.2.3.1)$$

and similarly for $\operatorname{Dec}^* P$. Thus for example if we take $j \geq i+b+d$ or $i \geq j+a$ our result is obvious. This proves (7.2.3) for (E'', M, P), and of course the analogs for $(\breve{E}, P, N)$ follow. ∎

We are now ready for the proof of Proposition (7.2.1). We check first that $F^*_{Y/S}$ induces a bifiltered morphism

$$\Psi \colon (K', A_{\mathcal{Y}}, P) \to F_{Y/S*}(K'', M, \operatorname{Dec} P).$$

Suppose x is a local section of

$$P^j K'^q \cap A^k_{\mathcal{Y}} K'^q = \Omega^q_{Y'/S} \otimes p^j E' \cap A^{k-q}_{\mathcal{Y}} E'_{Y'} = \Omega^q_{Y'/S} \otimes p^j A^{k-q-j}_{\mathcal{Y}} E'_{Y'}.$$

As $F^*_{Y/S}$ is divisible by p^q in degree q, we see that $\Psi(x)$ lies in

$$\Omega^q_{Y/S} \otimes p^{j+q} M^{k-q-j} E''_Y = \Omega^q_{Y/S} \otimes (M^k E'' \cap p^{j+q} E''_Y),$$

and $d\Psi(x) = \Psi(dx)$ lies in $p^{j+1+q}\Omega^q_{Y/S} \otimes E''$. This shows that

$$\Psi(x) \in F_{Y/S*} M^k K'' \cap (\operatorname{Dec} P)^j K'',$$

as required.

Here is the main step in the proof.

7.2.4 Lemma: *For any j and k, the map Ψ induces filtered quasi-isomorphisms:*

$$(\operatorname{Gr}_P \breve{K}'_{Y'/S}, A_{\mathcal{Y}}) \to F_{Y/S*}(\operatorname{Gr}_{\operatorname{Dec} P} \breve{K}''_{Y/S}, M) = F_{Y/S*}(E_0(\breve{K}''_{Y/S}, \operatorname{Dec} P), M)$$

and hence quasi-isomorphisms:

$$\psi_{j,k} \colon \operatorname{Gr}^j_P \operatorname{Gr}^k_{A_{\mathcal{Y}}} K'_{Y'/S} \to F_{Y/S*} \operatorname{Gr}^j_{\operatorname{Dec} P} \operatorname{Gr}^k_M K''_{Y/S} \cong F_{Y/S*} \operatorname{Gr}^j_{\operatorname{Dec} N} \operatorname{Gr}^k_P K_{Y/S}.$$

Proof: Lemma (7.2.3.1) shows that the map

$$u\colon (E_0^{j,n-j}(\check{K}''_{Y/S}, \operatorname{Dec} P), M) \to (E_1^{j+n,-j}(\check{K}''_{Y/S}, P), M)$$

is a filtered quasi-isomorphism, so the lemma will follow if we show that the composite $u \circ \psi$ is also. We have the following commutative diagram, in which the vertical arrows are isomorphisms:

$$\begin{array}{ccc}
(E_0^{j,n-j}(\check{K}', P), A_Y) & \xrightarrow{u\circ\psi} & F_{Y/S*}(E_1^{j+n,-j}(\check{K}'', P), M) \\
\downarrow & & \downarrow \\
(\Omega^n_{Y'/S} \otimes \operatorname{Gr}^j_P \check{E}', A_Y[-n]) & \longrightarrow & F_{Y/S*}\underline{H}^n(\operatorname{Gr}^{j+n}_P \check{K}''_{Y''}, M) \\
\downarrow{\scriptstyle p^{-j}} & & \downarrow{\scriptstyle p^{-n-j}} \\
(\Omega^n_{Y'/S} \otimes \operatorname{Gr}^0_P \check{E}', A_Y[-n-j]) & \longrightarrow & F_{Y/S*}\underline{H}^n(\operatorname{Gr}^0_P \check{K}''_{Y/S}, M[-j-n])
\end{array}$$

Now Mazur's formula (1.2.7) for the Cartier operator shows that the bottom arrow is just the Cartier isomorphism (7.2.2.1), and the lemma follows immediately. ∎

Since the filtrations with which we are working are not finite, we still have a little work to do to finish the proof of the theorem. Note that $P^{j-a+1}\operatorname{Gr}^j_{A_Y}\check{E}' = 0$, and $P^{j-b}\operatorname{Gr}^j_{A_Y}\check{E}' = \operatorname{Gr}^j_{A_Y}\check{E}'$. It follows that

$$P^{j-a+1+d}\operatorname{Gr}^j_{A_Y}\check{K}' = 0 \quad \text{and} \quad P^{j-b}\operatorname{Gr}^j_{A_Y}\check{K}' = \operatorname{Gr}^j_{A_Y}\check{K}',$$

so that the filtration induced by P on $\operatorname{Gr}^j_{A_Y}\check{K}'$ is finite. Similarly, the filtration induced by $\operatorname{Dec} P$ on $\operatorname{Gr}^j_M \check{K}''$ is finite. We conclude that Ψ induces quasi-isomorphisms:

$$\Psi\colon P^j \operatorname{Gr}^k_{A_Y}\check{K}' \to \operatorname{Dec} P^j \operatorname{Gr}^k_M \check{K}''$$

for all j and k. Taking $k > b+d$ and $j = 0$ and noting that $M^k\check{K}'' \subseteq P^0\check{K}' = K'$ and $\operatorname{Dec} P^0\check{K}'' \subseteq K''$, we find quasi-isomorphisms:

$$\Psi\colon \operatorname{Gr}^k_{A_Y} K' \to F_{Y/S*}\operatorname{Gr}^k_M K''$$

Moreover, for these large k, $A_Y^{k+1}E' = pA^k_{Y/S}E'$ and $M^{k+1}E'' = pM^kE''$, so we see that the map

$$\Psi\colon A^k_{Y/S}K' \to F_{Y/S*}M^kK''$$

induces a quasi-isomorphism when reduced modulo p. As both the source and target complexes are p-adically separated and complete, we conclude that the arrow itself is a quasi-isomorphism for large k. In other words, we have shown that for large k, the map $\Psi: P^0 A^k_{Y/S}\check{K}' \to \operatorname{Dec} P^0 M^k \check{K}''$ is a quasi-isomorphism. Now it is clear from (4.4.1) that Ψ is a bifiltered quasi-isomorphism. ∎

Lemma (4.4.1) implies that for any $\sigma \subseteq \mathbf{Z} \times \mathbf{Z}$ we obtain a quasi-isomorphism

$$\Psi_\sigma: (K', A_Y, P)_\sigma \to F_{Y/S*}(K'', M, \operatorname{Dec} P)_\sigma \cong F_{Y/S*}(K, P, \operatorname{Dec} N)_\sigma.$$

In fact, Ψ induces an isomorphism of $\mathbf{L}$-filtered objects, as described in (4.1.5). Note that $(K, P, \operatorname{Dec} N)_\sigma \cong (K, \operatorname{Dec} N, P)_{\sigma'}$, where σ' is the transpose of σ. As $\operatorname{Dec} N$ is G-transversal to (p), we can also express this in terms of 1-gauges, using equation (4.3.4.3). Recall from (4.2) that ϵ' is defined by $\epsilon'(i) =: \epsilon(-i) - i$. We find the following result, which will be useful when we attempt to globalize Theorem (7.2.1).

7.2.5 Corollary: *For any $\sigma \subseteq \mathbf{Z} \times \mathbf{Z}$, and any 1-gauge ϵ we obtain quasi-isomorphisms*

$$\Psi_\sigma: (K'_{Y'/S}, A_Y, P)_\sigma \to F_{Y/S*}(K''_{Y/S}, M, \operatorname{Dec} P)_\sigma \cong (K_{Y/S}, \operatorname{Dec} N, P)_{\sigma'}.$$

$$\Psi^\epsilon: A^\epsilon_Y K'_{Y'/S} \to F_{Y/S*} \operatorname{Dec} N^{\epsilon'} K_{Y/S}.$$

These maps are compatible with inclusions given by $\tau \subseteq \sigma$ (resp. $\epsilon \geq \delta$), and in fact define isomorphisms of $\mathbf{L}$-filtered objects.

7.3 The main theorem

We are now ready for a global formulation of Theorem (7.2.1). In this section, "tame" will mean with respect to the control function $\langle\ \rangle$, and we let $\mathbf{G}_\gamma$ denote the set of all $\langle\ \rangle$-tame gauges.

7.3.1 Theorem: *Let $\Phi: F^*_{X/S}E' \to E$ be an admissible F-span on X/S, and let (E', A) be the associated T-crystal on X'/S. For any tame gauge ϵ, let*

$$\operatorname{Dec}' N^\epsilon Ru_{X/S*}E =: \operatorname{Dec} N^{\epsilon'} Ru_{X/S*}E.$$

Then $F_{X/S}$ induces a natural isomorphism in the derived category of $\mathbf{G}_\gamma$-filtered objects:

$$\Psi_{\mathbf{G}_\gamma}: (Ru_{X'/S*}E', A_{\mathbf{G}_\gamma}) \to F_{X/S*}(Ru_{X/S*}E, \operatorname{Dec}' N_{\mathbf{G}_\gamma}).$$

In particular, for each tame ϵ*, we have a natural isomorphism in the derived category:*

$$\Psi^{\epsilon}: A^{\epsilon} Ru_{X'/S*}E' \to F_{X/S*} \operatorname{Dec} N^{\epsilon'} Ru_{X/S*}E,$$

compatible with the distinguished triangles associated with any inequality $\delta > \epsilon$*.*

Proof: The first step is the construction of the arrow. We begin by doing this in a generalization of a lifted situation, which we shall call a "fundamental thickening of Frobenius." Suppose that $X \subseteq Y$ is an embedding of X into a logarithmic formal scheme which is integral and logarithmically smooth over S, and let $Y_{/X}$ denote the exact formal completion of Y along X (*c.f.* section (1.1)). Let $F_{Y_0/S}: Y_0 \to Y_0'$ be the exact relative Frobenius morphism of Y_0/S, and suppose that Y' is a lifting of Y_0'. Then a morphism F from $Y_{/X}$ to $Y'_{/X'}$ induces a morphism F_D on divided power envelopes $D \to D'$. If the reduction of F modulo p is the exact relative Frobenius morphism of D_0/S we shall say that $F_D: D \to D'$ is a "fundamental thickening of Frobenius."

Note that if $X \subseteq Y$ is an exact embedding of X into an integral logarithmic formal scheme over S, then it is easy to see from Kato's construction of the exact relative Frobenius map that the square

$$\begin{array}{ccc} X' & \xrightarrow{\pi_{X/S}} & X \\ \downarrow & & \downarrow \\ Y_0' & \xrightarrow{\pi_{Y/S}} & Y_0 \end{array}$$

is Cartesian.

7.3.2 Lemma: *Suppose that* D *and* D' *are fundamental thickenings of* X *and* X' *and that* $F: D \to D'$ *is a fundamental thickening of Frobenius. Then the natural map*

$$\eta: E'_{D'} \to F_*F^*E'_D \cong E''_D$$

takes $A^i_{D'}E'_{D'}$ *into* $M^i_{\gamma}E''_D$*.*

Proof: This statement can be checked locally, so we may and shall assume that all our schemes are affine. In the case of the constant span, our lemma just says that F^* takes the ideal $J^{[i]}_{D'}$ to $(p)^{[i]}$. This is well-known but its proof is worth repeating. Because F^* is a PD-homomorphism, it suffices to show that it maps $J_{D'}$ into $p\mathcal{O}_D$. But the ideal $J_{D'}$ is the PD-ideal generated by the ideal of X' in Y', and this ideal is generated by $\pi^*_{Y/S}J_X$. If a is any section of J_X, $F^*_{Y/S}\pi^*_{Y/S}a = a^p$. But $a^p = p!a^{[p]}$ in $\mathcal{O}_D$, as required.

For a general admissible F-span Φ, our lemma is true by construction when (D, F) is a lifted situation. In general, suppose D is the PD-envelope of an exact closed immersion $j\colon X \to Z$, with Z/S smooth. Choose a map $g\colon Z_{/X} \to Y$ such that $g \circ j$ is the inclusion $X \to Y$; since $X = Y_0$ the map g is necessarily log smooth. In fact it is even classically smooth, since $Z_{/X}$ has the log structure induced from Y. Let $g'_0\colon Z'_{0/X'} \to X'$ be the pullback of g_0 via $\pi_{Y_0/S}$; then the natural map $Z_{0/X} \to Z'_{0/X'}$ is the exact relative Frobenius map of Z_0. Because Y'/S is log smooth, we can find a lifting $g'\colon Z'_{/X'} \to Y'$ of g'_0, and g' will also be formally smooth. Then because g' is smooth we can find a map $G\colon Z_{/X} \to Z'_{/X'}$ lifting the relative Frobenius morphism of Z_0/S_0 and such that the following diagram commutes:

$$\begin{array}{ccccc} & & Z_{/X} & \xrightarrow{G} & Z'_{/X'} \\ & \nearrow j & \downarrow g & & \downarrow g' \\ X & \longrightarrow & Y & \xrightarrow{F_{Y/S}} & Y' \end{array}$$

Let us use the same letter G to denote the map $D \to D'$ it induces. We have the following commutative diagram:

$$\begin{array}{ccc} g^* F^*_{Y/S} A^i_{Y'} E'_{Y'} & \longrightarrow & g^* M^i_\gamma E''_Y \\ \downarrow \cong & & \downarrow \cong \\ G^* g'^* A^i_{Y'} E'_{D'} & \longrightarrow & M^i_\gamma E''_D \\ \downarrow & & \downarrow \\ G^* g'^* E'_{D'} & \xrightarrow{\theta} & E''_D \end{array}$$

The filtration $A_{D'}$ of $E'_{D'}$ is the saturation of the filtration $g'^* A_{Y'}$ with respect to the PD-ideal $J_{D'}$ of X' in D', so it suffices to check that for any i and j, θ takes $J^{[j]}_{D'} g'^* A^i_{Y'} E'_{D'}$ into $M^{i+j}_\gamma E''_D$. As we have observed, G^* maps $J^{[j]}_{D'}$ into $(p)^{[j]}$, and hence $J^{[j]}_{D'} A^i_{D'} E'_{D'}$ into $(p)^{[j]} M^i_\gamma E''_D \subseteq M^{i+j}_\gamma E''_D$. Thus, the lemma is true when $F = G$.

To prove the general case, we use the commutative diagram

$$\begin{array}{ccc} F^* E'_{D'} & \xrightarrow{\epsilon} & G^* E'_{D'} \\ \downarrow \theta_F & & \downarrow \theta_G \\ E''_D & \xrightarrow{id} & E''_D \end{array}$$

as well as the explicit formula (1.1.8.6) for ϵ in terms of a logarithmic coordinate system $\{m'_i : i = 1, \dots n\}$ on Y. Because F and G are both fundamental thickenings of relative Frobenius, $F^*m_i' = \lambda(u_i)G^*m_i'$ with $u_i \in 1 + p\mathcal{O}_D$ for all i. Write $u_i = ps_i + 1$. Then if e is a local section of $E'_{D'}$ we have, in the notation of (1.1.8):

$$\epsilon F^*(e) = \sum_I (ps)^{[I]} G^*(\nabla(\partial_I)e) = \sum_I p^{[I]} s^I G^*(\nabla(\partial_I)e) = \sum_I s^I G^*(p^{[I]}\partial_I e).$$

Now suppose that $e \in A^k E'_{D'}$. Since the filtration $A_{D'}$ is Griffiths transversal to ∇',

$$\nabla(\partial_I)e =: \prod_i \prod_{I_i' < I_i} (\nabla(\partial_i) - I_i')e \in A^{k-|I|}E'_{D'},$$

and since $p^{[I]} \in (p)^{[|I|]}$ and $A_{D'}$ is saturated with respect to (p,γ), it follows that $p^{[I]}\partial_I e \in A^k_{D'}E'_{D'}$. As we showed above, $\theta_G G^* A^k_{D'}E'_{D'} \subseteq M^k_\gamma E''_D$, and it follows that $\theta_G(\epsilon F^*(e)) \in M^k_\gamma E''_{D'}$. The diagram now shows that $\theta_F F^*(e) \in M^k_\gamma E''_D$. ∎

Warning: It is not true that the image of $(F_{X/S})^*_{cris} A^i E' \to E''$ is contained in $M^i_\gamma E''$. This would require that the above lemma be true for any pair of objects $T \in \mathrm{Cris}(X/S)$ and $T' \in \mathrm{Cris}(X'/S)$ and *any* morphism $F: T \to T'$ covering the relative Frobenius map $U \to U'$, not just for for fundamental thickenings of Frobenius.

We first construct the arrow $\Psi_{\mathbf{G}_\gamma}$ when X admits a fundamental thickening of Frobenius, coming from $F_{Y/S}: Y_{/X} \to Y'_{/X'}$. Let T and T' be the PD envelopes of X in Y and Y', respectively, and let us also denote by $F_{Y/S}$ the morphism $T \to T'$ induced by $F_{Y/S}$. We obtain a map

$$\eta_Y: E'_{T'} \to F_{Y/S*}E''_T.$$

Now in the derived category, the morphism induced by $F_{X/S}$ on crystalline cohomology can be identified with the map:

$$F^*_{Y/S}: K'_{Y'/S} \to F_{Y/S*}K''_{Y/S}$$

7.3.3 Claim: *Let $\sigma =: \sigma(\epsilon)$ (c.f (4.2.1)) and let σ' be its transpose. Then $F^*_{Y/S}$ sends the subcomplex $A^\epsilon K_{Y'/S}$ of the complex $K'_{Y'/S}$ into*

$$\begin{aligned}(F_{Y/S*}K''_{Y/S}, M, \mathrm{Dec}^*\, P)_\sigma &\cong (F_{Y/S*}K_{Y/S}, P, \mathrm{Dec}^*\, N)_\sigma \\ &\cong (F_{Y/S*}K_{Y/S}, \mathrm{Dec}^*\, N, P)_{\sigma'}\end{aligned}$$

It suffices to prove that each $p^{\epsilon(k)}A^kK_{Y'}$ maps into the aforementioned complex. This will follow from Lemma (7.3.2) and the fact that $F^*_{Y/S}$ is divisible by p^q on $\Omega^q_{Y'/S}$.

$$\begin{aligned} F^*_{Y/S}(p^{\epsilon(k)}A^kK')^q &= F^*_{Y/S}(p^{\epsilon(k)}A^{k-q}K'^q) \\ &\subseteq p^{\epsilon(k)}p^qM^{k-q}_\gamma K''^q \\ &\subseteq p^{\epsilon(k)}p^q\sum_{j=0}^{\infty}p^{\langle j\rangle}M^{k-q-j}K''^q \\ &\subseteq \sum M^{k-j+\langle j\rangle+\epsilon(k)}\cap p^{q+\langle j\rangle+\epsilon(k)}K''^q. \end{aligned}$$

Since ϵ is tame, we have $\epsilon(k)+\langle j\rangle \geq \epsilon(k-j)$, and hence

$$\begin{aligned} M^{k-j+\langle j\rangle+\epsilon(k)}\cap p^{q+\langle j\rangle+\epsilon(k)}K''^q &\subseteq \sum_{j=0}^{\infty}M^{k-j+\epsilon(k-j)}\cap p^{q+\epsilon(k-j)}K''^q \\ &\subseteq \sum_{i=-\infty}^{k}M^{i+\epsilon(i)}\cap p^{q+\epsilon(i)}K''^q \\ &\subseteq (K'',M,\operatorname{Dec}^*P)^q_\sigma. \end{aligned}$$

This proves the claim.

To construct the arrow of the theorem, we note that we have a natural bifiltered morphism

$$(K_{Y/S},\operatorname{Dec}N,P)\to(K_{Y/S},\operatorname{Dec}^*N,P) \qquad (7.3.3.1)$$

which I claim is a bifiltered quasi-isomorphism. This is a local question, and is independent of the choice of Y/S. As we have proved it in the case of a lifted situation (7.2.3), it is true in general. Because the filtration $\operatorname{Dec}N$ is G-transversal to p, we can identify $(K_{Y/S},\operatorname{Dec}N,P)_{\sigma'}$ with $\operatorname{Dec}N^{\epsilon'}K_{Y/S}$ for every ϵ (4.3.4.3). Making this identification and composing the map in the claim with the inverse of (7.3.3.1), we obtain a map

$$\Psi_{\mathbf{G}_\gamma}\colon(Ru_{X'/S*}E',A_{\mathbf{G}_\gamma})\to(K_{Y/S},\operatorname{Dec}'N_{\mathbf{G}_\gamma}).$$

This completes the construction of the arrow $\Psi_{\mathbf{G}_\gamma}$ when X admits a fundamental thickening of Frobenius. Let us review explicitly the sense in which it is independent of the choices involved. If $X\to Y_i$ for $i=1,2$ are two closed immersions into integral log smooth S-schemes with fundamental thickenings of Frobenius $F_i\colon Y_{i/X}\to Y'_{i/X'}$, then we can also embed X into $Y=:Y_1\times_SY_2$, and it is clear that $F_1\times F_2$ induces a fundamental thickening of Frobenius on the divided power envelopes. The projection morphisms $Y \to Y_i$ then induce quasi-isomorphisms compatible with the arrows constructed above. Now quite generally, we can find an open cover of X such that each open subset admits a fundamental thickening with a lifting of Frobenius, and the compatibilities allow us to construct $\Psi_{\mathbf{G}_\gamma}$ for the associated simplicial scheme $X_\bullet$. Thus the arrow exists in general. Moreover, the question of whether or not it is a quasi-isomorphism is local and independent of the choice of the embeddings. Thus, to prove that our global arrow is a quasi-isomorphism we may work locally, in a lifted situation $\mathcal{Y}$. We are therefore reduced to the situation of (7.2.5) and the theorem is proved. ∎

7.3.4 Example: Suppose that E' is a locally free crystal of $\mathcal{O}_{X/W}$-modules, let $E =: E'' =: F^*_{X/S}E'$, and take Φ to be the identity map. Then $\Phi\colon E'' \to E$ is an F-span, and it is apparent that M and N are just the usual p-adic filtrations. In this case, Theorem (7.3.1) says that there is a canonical filtered quasi-isomorphism

$$Ru_{X'/S*}(E', P) \to F_{X/S*}(Ru_{X/S*}F^*_{X/S}E', \operatorname{Dec} P).$$

This case has been used recently (and independently) by K. Kato in his forthcoming study [18] of the cohomology of F-gauges, which is closely related to our results here.

7.3.5 Corollary: *Let $\Phi\colon F^*_{X/S}E' \to E$ be an admissible F-span on X/S, of level within $[a, b]$.*

1. *In the filtered derived category we have natural isomorphisms:*

$$\psi^0\colon (\operatorname{Gr}^0_P Ru_{X'/S}\breve{E}', A) \to (\operatorname{Gr}^0_{\operatorname{Dec} N} Ru_{X/S*}\breve{E}, P)$$

$$\psi_i\colon (\operatorname{Gr}^i_A \operatorname{Gr}^0_P Ru_{X'/S*}E') \to (\operatorname{Gr}^{-i}_{\operatorname{Dec} N} \operatorname{Gr}^0_P Ru_{X/S*}E).$$

2. *If $j < p + a$, we have an isomorphism:*

$$\begin{aligned}(Ru_{X'/S*}A^jE', P) &\to ((\operatorname{Dec} N)^{-j}Ru_{X/S*}E, \operatorname{Dec} N[-j])\\ (Ru_{X'/S*}(A^jE') \otimes \mathbf{F}_p, P) &\to (((\operatorname{Dec} N)^{-j}Ru_{X/S*}E) \otimes \mathbf{F}_p, \operatorname{Dec} N[-j])\end{aligned}$$

3. *If $j < p + a - 1$, then we also have an isomorphism:*

$$Ru_{X'/S*}(\operatorname{Gr}^j_A E', P) \to (\operatorname{Gr}^{-j}_{\operatorname{Dec} N} Ru_{X/S*}E, \operatorname{Dec} N[-j])$$

Proof: Write K' and K for $Ru_{X'/S*}E'$ and $F_{X/S*}Ru_{X/S*}E$, respectively. For each $k \in \mathbf{Z}$ let

$$1_k(i) =: \begin{cases} 1 & \text{if } i < k \\ 0 & \text{if } i \geq k/ \end{cases}$$

It is easy to compute the corresponding subsets of $\mathbf{Z} \times \mathbf{Z}$ (4.2.1):

$$\overline{\sigma}(1'_k) = \overline{\sigma}'(1_k) = 0 \times [k, \infty) \cup [1, \infty) \times \mathbf{Z},$$

and

$$\overline{\sigma}(c'_1) = [1, \infty) \times \mathbf{Z}.$$

Using equation (4.3.4.3) we find that

$$\operatorname{Dec} N^{c'_1} K = \operatorname{Dec} N^1 \check{K} \text{ and } \operatorname{Dec} N^{1'_k} K = \operatorname{Dec} N^0 \check{K} \cap P^k \check{K} + \operatorname{Dec} N^1 \check{K}.$$

We find a commutative diagram

$$\begin{array}{ccccc} A^{c_1} K' & \longrightarrow & A^{1_k} K' & \longrightarrow & A^k \operatorname{Gr}_P^0 K' \\ \downarrow \psi & & \downarrow \psi & & \downarrow \psi_0^k \\ F_* \operatorname{Dec} N^1 K & \longrightarrow & F_* \operatorname{Dec} N^0 K \cap P^k \check{K} & \longrightarrow & F_* P^k \operatorname{Gr}^0_{\operatorname{Dec} N} \check{K} \end{array}$$

Passing to associated graded complexes, we find isomorphisms:

$$\operatorname{Gr}_A^k \operatorname{Gr}_P^0 K' \to \operatorname{Gr}_P^k \operatorname{Gr}^0_{\operatorname{Dec} N} \check{K}.$$

Now using the G-transversality of $\operatorname{Dec} N$ to (p) we see that

$$\operatorname{Gr}_P^k \operatorname{Gr}^0_{\operatorname{Dec} N} \check{K} \cong \operatorname{Gr}_P^0 \operatorname{Gr}^{-k}_{\operatorname{Dec} N} \check{K} \cong \operatorname{Gr}_P^0 \operatorname{Gr}^{-k}_{\operatorname{Dec} N} K.$$

This completes the proof of (7.3.5.1).

For any $\epsilon \in \mathbf{G}$, let $\overline{\epsilon} =: \epsilon \wedge c_{\epsilon(a)}$ and $\underline{\epsilon} =: \epsilon \wedge c'_{\epsilon(a)}$. Then $\epsilon \mapsto \overline{\epsilon}$ and $\epsilon \mapsto \underline{\epsilon}$ are lattice homomorphisms and $\overline{\epsilon}' = \underline{\epsilon}'$. Using the fact that Φ has level within (a, ∞) one verifies easily that

$$A^{\overline{\epsilon}} K' = A^{\epsilon} K' \text{ and } \operatorname{Dec} N^{\underline{\epsilon}} K \cong \operatorname{Dec} N^{\epsilon} K$$

Let ϵ_j denote the maximal tame gauge which vanishes at j. If $j < p + a$,

$$\overline{\epsilon}_j(i) = ((j - i) \vee 0) \wedge (j - a) = (\overline{c'_j} \vee c_0)(i)$$

so $\overline{\epsilon}_j = \underline{c}'_j \vee c_0$. Hence our involution $\epsilon \mapsto \epsilon'$ takes

$$\begin{array}{ccc} \overline{\epsilon}_j \vee c_k & \mapsto & \underline{c}_j \vee c'_0 \vee c'_k \\ \overline{\epsilon}_j + 1 \vee c_k & \mapsto & \underline{c}_j + 1 \vee c'_0 \vee c'_k \end{array}$$

But we have

$$\operatorname{Dec} N^{\underline{c}_j \vee c'_0 \vee c'_k} K \cong p^j \operatorname{Dec} N^{-j} K \cap \operatorname{Dec} N^{k-j} K.$$

For fixed j and varying k, $\{A^{\bar{\epsilon}_j \vee c_k} E'\}$ cuts out the filtration P induced on $A^k E'$, while $\{\operatorname{Dec} N^{\underline{c}_j \vee c'_0 \vee c'_k}\}$ cuts out the filtration N induces on $p^j \cap \operatorname{Dec} N^0 Ru_{X/S*}E$. Using the G-transversality of $\operatorname{Dec} N$ to p, we can identify this with the filtration induced by $N[-j]$ on $\operatorname{Dec} N^{-j} Ru_{X/S*}E$. The remaining statements follow easily. ∎

7.3.6 Corollary: *If $\Phi\colon F^*_{X/S}E' \to E$ is an admissible F-span on X/S, then Φ induces an isomorphism*

$$Ru_{X'/S*}E' \to F_{X/S*}(\operatorname{Dec} N)^0 Ru_{X/S*}E.$$

Proof: This follows by applying (7.3.5.2) with $j = 0$. ∎

7.3.7 Corollary: *Let $\Phi\colon F^*_{X/S}E' \to E$ be an admissible F-span on X/S, of level within $[0, b]$, and let $\Psi\colon Ru_{X'/S*}E' \to F_{X/S*}Ru_{X/S*}E$ be the natural map induced by Φ and $F_{X/S}$, as in (7.3.1). Then if d is the relative dimension of X/S, there is a canonical morphism in the derived category*

$$V\colon F_{X/S*}Ru_{X/S*}E \longrightarrow Ru_{X'/S*}E'$$

such that $V \circ \Psi$ and $\Psi \circ V$ are each multiplication by p^{b+d}. For any integer $q \geq 0$, we can also find a morphism

$$V_q\colon F_{X/S*}\left(\tau_{\leq q} Ru_{X/S*}E\right) \longrightarrow \tau_{\leq q}\left(Ru_{X'/S*}E'\right)$$

such that $V_q \circ (\tau_{\leq q}\Psi)$ and $(\tau_{\leq q}\Psi) \circ V_q$ are multiplication by p^{q+b}.

Proof: Applying (7.3.6), we may and shall identify the morphism Ψ with the morphism $(\operatorname{Dec} N)^0 Ru_{X/S*}E \to Ru_{X/S*}E$. To simplify the notation we shall write K for $Ru_{X/S*}E$.

Notice that since Φ has level within $[0, \infty)$, $N^0 \check{E} \subseteq P^0 \check{E}$, and it follows that $(\operatorname{Dec} N)^0 \check{K} \subseteq P^0 \check{K} = K$. For any $j \geq 0$, multiplication by p^j induces a map $\gamma\colon (\operatorname{Dec} N)^{-j} K \to (\operatorname{Dec} N)^0 K$. Thus, for any $e \geq 0$, we find a commutative diagram

$$\begin{array}{ccccc}
(\operatorname{Dec} N)^0 K & \longrightarrow & (\operatorname{Dec} N)^{-b-e} K & \xrightarrow{\beta_e} & K \\
\big\downarrow{\scriptstyle p^{b+e}} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle p^{b+e}} \\
(\operatorname{Dec} N)^0 K & \xrightarrow{id} & (\operatorname{Dec} N)^0 K & \longrightarrow & K
\end{array}$$

in which the composite map along the top has been identified with Ψ. We shall see below that when $e = d$, the map β_e is an isomorphism, and we can define V to be $\gamma \circ \beta_e^{-1}$. Taking $e = q$, we find a similar diagram after applying the truncation functor $\tau_{\leq q}$, and we shall also see that $\tau_{\leq q}\beta_q$ is an isomorphism. Thus our corollary follows from the following lemma.

7.3.8 Lemma: *The inclusion maps*

$$\begin{array}{rcl}(\operatorname{Dec} N)^{-b-d} Ru_{X/S*}E & \longrightarrow & Ru_{X/S*}E \\ \tau_{\leq q}\Big[(\operatorname{Dec} N)^{-b-q} Ru_{X/S*}E\Big] & \longrightarrow & \tau_{\leq q} Ru_{X/S*}E\end{array}$$

are isomorphisms, in the derived category.

Proof: Let us consider the map $\beta_e\colon (\operatorname{Dec} N)^{-b-e}K \to K$, where $e \geq 0$. To prove that β_d is an isomorphism, it will suffice to prove that each $\operatorname{Gr}^i_{\operatorname{Dec} N} K$ is acyclic if $i < -b-d$; to prove that $\tau_{\leq q}(\beta_q)$ is an isomorphism, it will suffice to prove that $\tau_{\leq q} \operatorname{Gr}^i_{\operatorname{Dec} N} K$ is acyclic if $i < -b-q$. It even suffices to prove that each $\operatorname{Gr}^i_{\operatorname{Dec} N} \operatorname{Gr}^j_P K \cong \operatorname{Gr}^{i-j}_{\operatorname{Dec} N} \operatorname{Gr}^0_P K$ is acyclic if $j \geq 0$ and $i < -b-d$, and evidently it suffices to prove this if $j = 0$. A similar statement holds for the truncations. But (7.3.5.1) tells us that

$$\operatorname{Gr}^i_{\operatorname{Dec} N} \operatorname{Gr}^0_P K \cong \operatorname{Gr}^{-i}_A \operatorname{Gr}^0_P K'.$$

which is the Kodaira-Spencer complex (6.2.1) of E' on X'/S_0. This complex looks like:

$$\operatorname{Gr}^b_A E_{X'} \otimes \Omega^{-i-b}_{X'/S_0} \to \operatorname{Gr}^{b-1}_A E'_{X'} \otimes \Omega^{-i-b+1}_{X'/S_0} \to \ldots \to \operatorname{Gr}^{-i-d}_A E'_{X'} \otimes \Omega^d_{X'/S_0}$$

This complex vanishes when $i < -b-d$. If $i < -b-q$ the complex begins beyond degree q, and in particular is killed by $\tau_{\leq q}$ ∎

7.4 Cohomological consequences

Suppose now that X/S_0 is proper in addition to being logarithmically smooth and integral, and that Φ is an admissible F-span on X/S; for simplicity of notation we suppose that S is affine. We have a natural map

$$\Psi\colon Ru_{X'/S*}E' \to F_{X/S*}Ru_{X/S*}E.$$

Passing to cohomology, we obtain an $\mathcal{O}_S$-linear map

$$\Phi^q\colon H^q(X'/S, E') \to H^q(X/S, E) \qquad (7.4.0.1)$$

Let $e =: \min(d, q)$; then it follows from (7.3.7) that there is a natural map

$$V_q\colon H^q(X/S, E) \to H^q(X'/S, E')$$

such that $V_q \circ \Phi^q$ and $\Phi^q \circ V_q$ are multiplication by p^{e+b}. Letting $\overline{H}^q$ denote the quotient of H^q by its p^∞-torsion, we see that Φ^q defines a p-isogeny of $\mathcal{O}_S$-modules. We have proved

7.4.1 Proposition: *If X/S_0 is proper and of relative dimension d, with S affine, and if $\Phi\colon F^*_{X/S}E' \to E$ is an admissible F-span on X/S of level within $[0, b]$, then we obtain a p-isogeny*

$$\overline{\Phi}^q\colon \overline{H}^q(X'/S, E') \to \overline{H}^q(X/S, E)$$

of level within $[0, e]$, *where* $e =: \min(d, q)$. ∎

Our main result in this section tells us that, under suitable conditions, formation of the filtration A and N is compatible with passage to cohomology. It can be regarded as the cohomological version of our generalization of Mazur's fundamental theorem to the case of coefficients in an F-span. First we must investigate the behavior of two spectral sequences associated to an F-span, the analogs of the Hodge and conjugate spectral sequences [23] for ordinary De Rham cohomology in characteristic p. Recall from (6.2.2) that the spectral sequence associated with the filtered object (E'_0, A) on $\mathrm{Cris}(X/S_0)$ can be written

$$E_1^{i,j}(E', A) \cong H^{i+j}(X'/S_0, \mathrm{Gr}^i_A E'_0) \cong H^j(X, \Omega^{i\cdot}_{E'/S}) \Rightarrow H^{i+j}(X/S*, E'_0),$$

where by definition (6.2.2.1) $\Omega^{i\cdot}_{E'/S} =: Ru_{X/S*} \mathrm{Gr}^i_A E'_0[i]$. We call this spectral sequence the "Hodge spectral sequence" associated to the F-span (E, Φ).

We can also consider the filtered sheaf (E_0, N) on $\mathrm{Cris}(X/S)$ and the associated spectral sequence

$$E_1^{i,j}(E_0, N) \cong H^{i+j}(X/S, \mathrm{Gr}^i_N E_0) \Rightarrow H^{i+j}(X/S, E_0).$$

However it is more useful for us to look at the spectral sequence associated to $(Ru_{X/S*}E_0, \mathrm{Dec}\, N)$:

$$E_1^{i,j}(Ru_{X/S*}E_0, \mathrm{Dec}\, N) \cong H^{i+j}(X/S, \mathrm{Gr}^i_{\mathrm{Dec}\, N} Ru_{X/S*}E_0) \Rightarrow H^{i+j}(X/S, E_0).$$

Using (7.3.5.1) and (6.2.2.1), we see that

$$F_{X/S*} \mathrm{Gr}^i_{\mathrm{Dec}\, N} Ru_{X/S*}E_0 \cong \mathrm{Gr}^{-i}_A Ru_{X'/S*}E'_0 \cong \Omega^{-i\cdot}_{E'_0/S}[i].$$

Thus the $E_1^{i,j}$ term of the above spectral sequence is canonically isomorphic to $H^{2i+j}(X', \Omega^{-i\cdot}_{E'_0/S})$. If we set $i' =: 2i+j$, $j' =: -i$, and $\overline{E}_r^{i',j'} =: E_{r-1}^{i,j}$, then we can write our spectral sequence as

$$\overline{E}_2^{i',j'} \cong H^{i'}(X', \Omega^{j'\cdot}_{E'_0/S}) \Rightarrow H^{i'+j'}(X/S, E_0).$$

We shall call this spectral sequence the "conjugate spectral sequence" associated to an F-span. To summarize:

7.4.2 Proposition: *Suppose X/S_0 is proper and that $\Phi: F^*_{X/S}E' \to E$ is an admissible F-span on X/S, with S affine. Then there are two spectral sequences (with corresponding filtrations on the abutments), called respectively the "Hodge" and "conjugate" spectral sequences (and filtrations):*

$$\begin{aligned} E_1^{i,j}(E,\Phi) \cong H^j(X', \Omega^{i\cdot}_{E'_0/S}) &\Rightarrow H^{i+j}(X'/S, E'_0) \\ \overline{E}_2^{i,j}(E,\Phi) \cong H^i(X', \Omega^{j\cdot}_{E'_0/S}) &\Rightarrow H^{i+j}(X/S, E_0). \end{aligned}$$

∎

Here is the main result of this section.

7.4.3 Theorem: *Suppose X/S_0 is proper and that $\Phi: F^*_{X/S}E' \to E$ is an admissible F-span on X/S, with S affine. Suppose that n is an integer for which the cohomology groups $H^n(X/S, E)$ and $H^{n+1}(X/S, E)$ are p-torsion free and that in the conjugate spectral spectral sequence (7.4.2), $\overline{E}_2^{i,j} = \overline{E}_\infty^{i,j}$ whenever $i+j=n$. Let $(H^n(X/S,E), N)$ and $(H^n(X/S,E'), A)$ denote the filtered objects attached as in (5.1.2) and (5.2.13) to the F-span on S/S: $\Phi_n: H^n(X/S,E') \to H^n(X/S,E)$.*

1. *The groups $H^n(X'/S, E')$ and $H^{n+1}(X'/S, E')$ are p-torsion free, and in the Hodge spectral sequence we have $E_1^{i,j}(E,\Phi) = E_\infty^{i,j}(E,\Phi)$ whenever $i+j=n$.*

2. *There are natural isomorphisms:*

$$\begin{aligned} H^n(X/S, \operatorname{Dec} N^i E) &\cong N^i H^n(X/S, E) \\ H^n(X'/S, A^i E') &\cong A^i H^n(X'/S, E') \end{aligned}$$

Proof: Since $\operatorname{Dec} N$ is G-transversal to (p), we can view $(K, \operatorname{Dec} N) =: (Ru_{X/S*}E, \operatorname{Dec} N)$ as a **G**-structure; it has level within $[-b, \infty)$. The torsion hypothesis says that $H^q(X, Ru_{X/S*}E)$ is p-torsion free for $q=n$ and

$n+1$, and our hypotheses on the conjugate spectral sequence is equivalent to the assertion that for each i, the sequence

$$0 \to H^n((\mathrm{Dec}\, N)^i K_0) \to H^n(K_0) \to H^n(K_0/(\mathrm{Dec}\, N)^i K_0) \to 0$$

is exact, or equivalently, that the maps

$$H^q((\mathrm{Dec}\, N)^i K_0) \to H^q(K_0)$$

are injective for $q = n$ and $n+1$. Since X is quasi-compact, our functors also commute with direct limits, and so the hypotheses (4.4.3) hold, with $t = p$, $A = \mathrm{Dec}\, N$, and H the hypercohomology functor. We conclude that the maps

$$H^q(X, (\mathrm{Dec}\, N)^i Ru_{X/S*}E) \to H^q(X/S, E)$$

are injective for $q = n$ and $n+1$. Furthermore, the corresponding filtration on $H^n(X/S, E)$ defines a **G**-structure, and in particular is G-transversal to (p).

Recalling that (7.3.6) tells us that ψ induces an isomorphism

$$E' \to F_{X/S*}(\mathrm{Dec}\, N)^0 Ru_{X/S*}E,$$

we see that the map $\Phi^q \colon H^q(X'/S, E') \to H^q(X/S, E)$ is injective for $q = n$ and $n+1$. It follows that $H^n(X/S, E')$ and $H^{n+1}(X/S, E')$ are p-torsion free.

By corollary (7.3.5.1), we have for every i a commutative diagram

$$\begin{array}{ccccc} H^n(P^i \operatorname{Gr}^0_{\mathrm{Dec}\, N} K) & \to & H^n(gr^0_{\mathrm{Dec}\, N} K) & \to & H^n(\operatorname{Gr}^0_{\mathrm{Dec}\, N} K / P^i \operatorname{Gr}^0_{\mathrm{Dec}\, N} K) \\ \downarrow & & \downarrow & & \downarrow \\ H^n(A^i \operatorname{Gr}^0_P K') & \to & H^n(\operatorname{Gr}^0_P K') & \to & H^n(\operatorname{Gr}^0_P K' / A^i \operatorname{Gr}^0_P K') \end{array}$$

in which the vertical arrows are isomorphisms induced by ψ^0. But Corollary (4.4.7) tells us that the top sequence is short exact, and it follows that the bottom sequence is also. This implies that $E_1^{i,j}(E, \Phi) = E_\infty^{i,j}(E, \Phi)$ whenever $i + j = n$. Now we can again apply (4.4.3), this time to the $\mathbf{G}_\gamma$ structure defined by (E', A). We conclude that the maps $H^q(X'/S, A^i E') \to H^q(X'/S, E')$ are injective and define a filtration on $H^n(X'/S, E')$ which is G-transversal to (p, γ).

We have now proved that $H^n(X, (Ru_{X/S*}E, \mathrm{Dec}\, N_{\mathbf{G}}))$ defines a gauge structure and that $H^n(X/S, (E', A_{\mathbf{G}_\gamma}))$ defines a tame gauge structure. Furthermore, our main theorem (7.3.1) identifies the restriction of the former to

$\mathbf{G}_\gamma$ with the latter. Our formulas follow formally from this. It is not difficult to be explicit. First of all, we can identify the image of

$$\Phi_n : H^n(X'/S, E') \to H^n(X/S, E)$$

with the image of $H^n(X, \operatorname{Dec} N^0 Ru_{X/S*}E)$, and hence $N^0H^n(X/S, E)$ with $H^n(X, \operatorname{Dec} N^0 Ru_{X/S*}E)$. As the filtration on $H^n(X/S, E)$ induced by the filtration $\operatorname{Dec} N$ on $H^n(X, Ru_{X/S*}E,)$ is G-transversal to p and agrees in level 0 with the filtration N, the two must coincide. Furthermore, we can identify $H^n(X'/S, A^kE')$ with $H^n(X'/S, A^{\epsilon_k}E')$. Now

$$\begin{aligned}
H^n(X'/S, A^{\epsilon_k}E') &\cong H^n(X/S, \operatorname{Dec} N^{\epsilon'_k} Ru_{X/S*}E)\\
&\cong \sum_i p^{\epsilon_k(i)+i} N^{-i} H^n(X/S, E)\\
&\cong \sum_i p^{\epsilon_k(i)} \left[p^i H^n(X/S, E) \cap N^0 H^n(X/S, E)\right]\\
&\cong \sum_i p^{\epsilon_k(i)} (\Phi_n)^{-1} \left[p^i H^n(X/S, E)\right]\\
&\cong \sum_i p^{\langle k-i \rangle} M^i H^n(X'/S, E')
\end{aligned}$$

This is precisely the definition of the filtration M_γ on $H^n(X/S, E')$. Since $F_{S/S}$ is the identity map, $M_\gamma = A$, and our proof is complete. ∎

When we are working over a field it possible to make a statement even in the presence of p-torsion. For the sake of simplicity in our statements we shall assume that X/S_0 is perfectly smooth (*c.f* 1.2.3).

7.4.4 Corollary: *Suppose that X is logarithmically smooth and proper over a perfect field k endowed with a fine logarithmic structure and that $\Phi: F^*_{X/W}E' \to E$ is an admissible F-span on X/W. Suppose n is an integer such that the p-torsion subgroups of $H^q(X'/W, E')$ and of $H^q(X/W, E)$ have the same length when $q = n$ and when $q = n + 1$. Then the Hodge spectral sequence of (E, Φ) degenerates at E_1 in degree n if and only if the conjugate spectral sequence of (E, Φ) degenerates at E_2 in degree n. In particular, this equivalence holds if $\Phi: F^*_{X/W}E \to E$ is an F-crystal and if X/k is perfectly smooth.*

Proof: Let us note first that if Φ is an F-crystal and X/k is perfectly smooth, then the exact Frobenius diagram is Cartesian and $E' \cong \pi^*_{X/S}E$. Hence $H^i(X'/W, E') \cong F^*_W H^i(X/W, E)$, so that in this case the two cohomology groups certainly have isomorphic p-torsion. In any case, we know from (7.4.1) that the cohomology groups $H^n_{cris}(X'/W, E')$ and $H^n_{cris}(X/W, E)$ have the same rank, and if they also have the same torsion it follows from the universal coefficient theorem that the k-vector spaces $H^n(X'/k, E'_0)$ and $H^n(X/k, E_0)$ have the same dimension. Now the Hodge spectral sequence degenerates at E_1 in degree n if and only if

$$h^n(X'/k, E'_0) = \sum_{i+j=n} h^j(X', \Omega^{i\cdot}_{E'_0/S}),$$

and the conjugate spectral sequence degenerates at E_2 if and only if

$$h^n(X/k, E_0) = \sum_{i+j=n} h^i(X', \Omega^{j\cdot}_{E'_0/S}).$$

These are now obviously equivalent. ∎

7.4.5 Remark: It is clear from the proof of (7.4.4) that if both the Hodge and conjugate spectral sequences of an F-span degenerate, then conversely the p-torsion submodules of $H^q(X/W, E)$ and $H^q(X'/W, E')$ have the same length. Thus the hypothesis on p-torsion is necessary; in (8.4.3) we give an example to show that it is not superfluous.

7.5 Higher direct images

Suppose that $f: X \to Y$ is a smooth morphism of log smooth and integral S_0-schemes and let $\Phi: F^*_{X/S}E' \to E$ be an F-span on X/S. From the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_{X/S}} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{F_{Y/S}} & Y' \end{array}$$

one obtains a morphism

$$F^*_{Y/S} R^q f'_{cris*} E' \to R^q f_{cris*} F^*_{X/S} E'$$

of sheaves on Cris (Y/S). Composing with the map obtained by applying the functor $R^q f_{cris*}$ to the morphism Φ, we obtain a map

$$\Phi^q: F^*_{Y/S} R^q f'_{cris*} E' \to R^q f_{cris*} E.$$

With further hypotheses, we shall see that Φ^q defines an admissible F-span on Y/S.

7.5.1 Theorem: *Suppose that X and Y are as above and that $f\colon X \to Y$ is perfectly smooth. Let $\Phi\colon F^*_{X/S}E' \to E$ be a locally free F-span on X/S which is uniform in a neighborhood of $R_{X/S}$ (and hence admissible by (5.2.9)). Fix an integer n, and consider the following conditions:*

1. *The sheaves $R^q f'_{cris*}E'$ are locally free for $q \geq n$.*
2. *The conjugate spectral sequence for the F-span $\Phi_{X/Y}$ on X/Y degenerates at E_2 in degree n,* i.e. $\overline{E}_2^{ij} = \overline{E}_\infty^{ij}$ *whenever $i + j = n$.*
3. *The sheaf $R^q f'_{X'/Y'*} \operatorname{Gr}_A E'_0$ is acyclic for $F^*_{Y/S}$,* i.e.

$$\operatorname{Tor}_i^{\mathcal{O}_{Y'}}(R^q f'_{X'/Y'} \operatorname{Gr}_A E'_0, F_{Y/S*}\mathcal{O}_Y)$$

vanishes for $i > 0$.

Condition (1) implies that Φ_n defines a nondegenerate F-span on Y/S, of width at most the width of $\Phi_{X/S}$ plus $\dim(X/Y)$. *If all three conditions are satisfied, then the T-crystal $(E', A) =: \alpha_{X/S}(\Phi)$ on X'/Y and the morphism $f'\colon X' \to Y'$ satisfy the hypotheses of (6.3.2), and $(R^n f'_{cris*}E', A)$ defines an admissible T-crystal on Y'. Furthermore, the F-span Φ_n is admissible, and*

$$\alpha_{Y/S}(R^n(\Phi)) \cong R^n f_{cris*}(\alpha_{X/S}(\Phi)).$$

Proof: We work locally on Y, with the aid of local liftings $(\mathcal{Y}, F_{\mathcal{Y}/S})$ of Y and its relative Frobenius morphism. Let $\tilde{\Phi}$ denote the F-span $F^*_{X/Y}\tilde{E} \to E$ deduced from Φ as in (5.2.14), and let $\pi =: \pi_{X'/Y'/S}$. As explained in (5.2.14), $\tilde{\Phi}$ is again admissible, and its corresponding T-crystal $(\tilde{E}, A_{X/Y})$ is $\pi^*(E', A_{X/S})$. We have a commutative diagram:

$$\begin{array}{ccccc} F^*_{Y/S}R^q f'_{cris*}E' & \longrightarrow & R^q f_{cris*}F^*_{X/S}E' & \longrightarrow & R^q f_{cris*}E \\ \downarrow{\scriptstyle \beta^q} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ R^q \tilde{f}_{cris*}\tilde{E} & \longrightarrow & R^q f_{cris}F^*_{X/Y}\tilde{E} & \longrightarrow & R^q f_{cris*}E \end{array}$$

Then (7.4.1), with S replaced by $\mathcal{Y}$, shows that the arrow along the bottom is a p-isogeny of level at most the level of Φ plus the relative dimension of X/Y.

Condition (1) implies that formation of $R^n f_{cris*}$ commutes with base change and defines a locally free crystal of $\mathcal{O}_{Y/S}$-modules *c.f.* [4, 7.12] and [20, 6.12]. Since $f\colon X \to Y$ is perfectly smooth, the square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X' \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

is Cartesian. Thus the base changing arrow β: $F^*_{Y/S}R^n f'_{cris*}E' \to R^n\tilde{f}_{cris*}\tilde{E}$ is an isomorphism. It follows that Φ_n is a p-isogeny, whose level is as described.

Now suppose that conditions (2) and (3) are also satisfied. We know from (7.4.2) that, in the spectral sequence attached to the filtered object $(\mathrm{Gr}^0_P \tilde{E}, A_{X/Y})$ and the morphism $\tilde{f}_{\tilde{X}/y}$, we have $E_1^{i,j} = E_\infty^{i,j}$ when $i+j \geq n$, and hence we can apply (6.3.2) to conclude that the maps

$$R^q\tilde{f}_{\tilde{X}/Y}A^i\tilde{E}_0 \xrightarrow{\tilde{\iota}^q} R^q\tilde{f}_{\tilde{X}/Y*}\tilde{E}_0$$

are injective when $q \geq n$. We need to descend this information to Y', *i.e.*, to prove that the maps

$$R^q f'_{X'/Y'}A^iE'_0 \xrightarrow{\iota^q} R^q f'_{X'/Y'*}E'_0$$

are also injective. Outside the support of $R_{Y'/S}$, the map $F_{Y/S}$ is faithfully flat, and so the only difficulty is local around the support of $R_{Y'/S}$. Since the inverse image by f'^{-1} of this support is contained in the support of $R_{X'/S}$, the T-crystal $(E', A_{X/S})$ is uniform there, and in particular its associated graded is locally free. By descending induction we may and shall suppose that ι^q is injective when $q > n$.

Thanks to our assumption that (1) holds for $q \geq n$, we know that the sheaf $R^n f'_{X'/Y'*}E'_0$ is acyclic for $F^*_{Y/S}$. Let us assume, by induction on i, that the same is true of $R^n f'_{X'/Y'*}A^iE'_0$; we may also assume that $R^{n+1}f'_{X'/Y'*}A^{i+1}E'_0$ is acyclic. The injectivity of ι^{n+1} implies that we have an exact sequence

$$0 \to K \to R^n f_{X'/Y'*}A^{i+1}E'_0 \to R^n f'_{X'/Y'*}A^iE'_0 \to R^n f'_{X'/Y'*}\mathrm{Gr}^i_A E'_0 \to 0.$$

We can conclude by condition (3) that the image I of $R^n f'_{X'/Y'*}A^{i+1}E'_0$ in $R^n f'_{X'/Y'*}A^iE'_0$ is also acyclic for $F^*_{Y/S}$. It follows that the top row in the commutative diagram below is exact.

$$\begin{array}{ccccccc} 0 & \to & F^*_{Y/S}K & \to & F^*_{Y/S}R^n f'_{X'/Y'*}A^{i+1}E'_0 & \to & F^*_{Y/S}R^n f'_{X'/Y'*}A^iE'_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \to & R^n\tilde{f}_{\tilde{X}/Y*}A^{i+1}\tilde{E}_0 & \to & R^n\tilde{f}_{\tilde{X}/Y*}A^i\tilde{E}_0 \end{array}$$

Our acyclicity assumptions for $n+1$ imply that the vertical arrows are isomorphisms, and it follows that $F^*_{Y/S}K$ is zero. Since $F_{Y/S}$ is faithful, $K = 0$ and the map $R^n f'_{X'/Y'*}A^{i+1}E' \to R^n f'_{X'/Y'*}A^iE'$ is injective. It follows that $R^n f'_{X'/Y'*}A^{i+1}E'$ is also acyclic for $F_{Y/S}$, and that the spectral sequence of (E'_0, A) and the morphism $f'_{X'/Y'}$ degenerates at E_1 in degrees $\geq n$.

Let us choose a local lifting $F_{\mathcal{Y}}$ of $F_{Y/S}$. An argument similar to the one we just made, but simpler, shows that the sheaves $R^q f'_{X'/Y'*}E'$ are p-torsion free for $q \geq n$. Furthermore, we see by descending induction on q and (1.3.7) that the sheaves $R^q f_{X'/Y'*}E'$ are locally free and commute with base change for $q \geq n$, and in particular that they form crystals on Y'/S. Thus the hypotheses of (6.3.2) are satisfied, and we conclude that the maps $R^q f'_{X'/Y'*}A^i E' \to R^q f'_{X'/Y'*}E'$ are injective and define a filtration which is G-transversal to (p, γ). In fact, an application of (4.4.3) shows that they remain injective when pulled back by $F_{\mathcal{Y}}$, and that the filtration they induce is G-transversal to (p, γ). It is now easy to conclude that the map

$$F_{\mathcal{Y}}^* R^q f'_{X'/Y'*}A^i E' \to R^q \tilde{f}_{\tilde{X}/Y} A^i \tilde{E}$$

is an isomorphism. But (7.4.3) tells us that $R^q \tilde{f}_{\tilde{X}/Y}(\tilde{E}, A)$ is just the filtration M_γ attached to Φ^q. This shows that the F-span Φ^q is admissible and completes the proof of the theorem. ∎

7.5.2 Corollary: *Suppose $f: X \to Y$ is a perfectly smooth and proper morphism of smooth and integral log schemes schemes over S_0, with Y reduced. Suppose further that (E, Φ) is a uniform F-T-crystal on X/S (5.3.1) such that for every n the sheaves $R^n f_{cris*}E$ are locally free on $\mathcal{O}_{Y/S}$ and for every closed point y of Y, $H^n_{cris}(X(y)/W(y), E(y))$ is torsion free, and that the Hodge spectral sequence of $(E(y), \Phi(y))$ degenerates at E_1. Then the hypotheses and conclusion of (7.5.1) are satisfied for all n.*

Proof: Corollary (7.4.4) tells us that the conjugate spectral sequence of the restriction of (E, Φ) to every fiber of f degenerates at E_2. Arguing as in the proof of (6.3.4), we see that, locally on Y, the terms of the conjugate spectral sequence of (E, Φ) are locally free and commute with base change, and that the spectral sequence degenerates at E_2. ∎

The mixed characteristic analog of Corollary (7.5.1) concerns higher direct images of F-T-crystals. Recall (5.3.1) that if X/S_μ is a smooth and integral log scheme, then an F-T-crystal on X/S_μ consists of an F-crystal (E, Φ) on (X_0/S) together with a T-crystal (E, B) on X/S and an isomorphism $\pi^*_{X/S}((E, B)|_{X/S}) \cong \alpha_{X/S}(\Phi)$.

7.5.3 Corollary: *Let $f: X \to Y$ be a perfectly smooth and proper morphism of smooth integral S_μ-schemes with log structure, and let (E, Φ, B) be a locally free F-T-crystal on X/S which is uniform in a neighborhood of the support of R_X. Suppose that for some integer n, the F-span (E, Φ) satisfies the following conditions:*

1. *The sheaves $R^q f_{cris*}E$ are locally free for $q \geq n$.*

2. *The conjugate spectral sequence for the F-span $\Phi_{X/Y}$ on X/Y degenerates at E_2 in degree n,* i.e. $\overline{E}_2^{ij} = \overline{E}_\infty^{ij}$ *whenever $i + j = n$.*

3. *The sheaf $R^q f_{X'/Y'*} \operatorname{Gr}_A E_0$ is locally free in a neighborhood of the support of $R_{Y/S}$ for $q \geq n$.*

Then (E, B) satisfies the hypotheses of (6.3.2), so that $(R^n f_{cris*}E, B)$ is a T-crystal on Y/S, and

$$\Phi_n \colon F_{Y/S}^* R^n f_{cris*}E \to R^n f_{cris*}E$$

is an F-crystal on Y/S. Furthermore, $(R^n f_{cris*}E, \Phi_n, B)$ defines an F-T-crystal on Y/S, and in particular we have $\alpha_{Y_0/S}(\Phi_n) \cong R^n f_{cris*}\alpha_{X_0/S}(\Phi)$.

Proof: Let (E, A) be the restriction of (E, B) to X_0/S, so that by definition, its pullback (E', A) to X'/S is $\alpha_{X/S}(\Phi)$. We argue the same way as in the proof of (7.5.1), using F_X and F_Y instead of $F_{X/S}$ and $F_{Y/S}$, to prove that $R^n f_{cris*}(\operatorname{Gr}_P^0 E, A)$ degenerates at E_1 in degree n. Hence we can apply (6.3.2) to conclude that the maps $R^n f_{cris*}B^i E \to R^n f_{cris*}E$ are injective and define a T-crystal on Y/S. It follows from the compatibility (6.3.2) of the formation of restriction with higher direct images that the restriction of $(R^n f_{cris*}E, B)$ to Y_0/S is the T-crystal $(R^n f_{cris*}E, A)$, and condition (3) implies that

$$\pi_{Y/S}^*(R^n f_{cris*}E, A) \cong R^n f'_{cris*}(E', A).$$

Now (7.5.1) tells us that $R^n f'_{cris*}(E', A) \cong \alpha_{Y/S}(\Phi_n)$, and we conclude that the triple $(R^n f_{cris*}E, \Phi_n, B)$ forms an F-T-crystal on Y/S. ∎

We shall describe some applications of our results to Hodge and Newton polygons (Katz's conjecture with coefficients in an F-crystal) in Section (8.3).

8 Examples and applications

8.1 Liftings and splittings

The idea that a lifting of a variety over k to W should cause its Hodge spectral sequence to degenerate may be originally due to Raynaud (oral communication). The first results along these lines were obtained by Kato [19] and further developed by Fontaine-Messing [12], Deligne, and Illusie [7] and [17]. Faltings has also proved such results, with coefficients in a Fontaine module [9]. Here we attempt to generalize and unify these results using the language of F-T-crystals.

We begin with a rudimentary discussion of deformations of T-crystals. Let us fix the following notation. Let S be a flat formal scheme over Spf W, endowed with a fine logarithmic structure (*e.g.* the trivial one). (Later we shall have to assume that the absolute Frobenius morphism of S_0 lifts, at least locally, to S, but we do not need this now.) Write S_μ for the reduction of S modulo p^μ for any $\mu \in \mathbf{Z}^+$, and let $S_0 = S_1$ and $S_\infty = S$. Let (X'/S_μ) be a logarithmically smooth and integral morphism of fine log schemes, and suppose that $\nu \in \mathbf{Z}^+ \cup \{\infty\}$ is greater than or equal to μ and that Y'/S_ν is a lifting of X'/S_μ) (formal if $\nu = \infty$)). If (E', C) is a T-crystal on Y'/S and (E', B) is its restriction to X'/S, as defined in (3.3.1), we say that "(E', C) is a lifting of (E', B) to Y'/S." Let $i: X' \to Y'$ be the inclusion, and let I be the ideal of X' in Y' (*i.e.* (p^μ)), with its natural divided power structure γ.

Recall that a pair of filtrations (P, Q) on an object E is said to be "n-opposed" if and only if for each j the natural map $P^j E \oplus Q^{n-j+1} E \to E$ is an isomorphism. If this is the case, there are natural isomorphisms:

$$\bigoplus_j P^j E \cap Q^{n-j} E \cong E \cong \bigoplus_j \mathrm{Gr}^j_P E \cong \bigoplus_j \mathrm{Gr}^{n-j}_Q E.$$

If we start with a filtered object (E, P), giving a filtration Q which is n-opposed to P is often called "splitting the filtration P." If (E, P, Q) is graded, we say that the pair of filtrations (P, Q) on E is "opposed" if and only if for each n the bifiltered object (E^n, P, Q) is n-opposed. Finally, one says that a triple (W, P, Q) of filtrations on E is "opposed" if and only if the pair of filtrations induced on $\mathrm{Gr}_W E$ by (P, Q) is opposed. (Of course, all these definitions are taken from [6].)

Recall from (6.2.1) that if (E', B) is a T-crystal on Y'/S, we call $\mathrm{Gr}_B E'$ the Kodaira-Spencer sheaf of (E', B). We endow it with the filtrations $I_\gamma =: (I, \gamma)$ and I, induced by the (I, γ)-adic and I-adic filtrations of E', respectively. Furthermore, we write E'_m for the reduction of E' modulo p^m, for any $m \in \mathbf{Z}^+$.

Proposition 8.1.1 (Kodaira-Spencer Decomposition) *Let (E', B) be a T-crystal on X'/S and (E', C) a lifting of (E', B) to Y'/S_ν. Suppose that $\nu = \mu + \delta$ and let $\mu' =: \min(\mu, \delta)$. Then, on the restricted crystalline site of Y'/S,*

1. *If $\nu = \infty$, the triple of filtrations $i_{cris*}(B, C, I_\gamma)$ is opposed.*

2. *For any ν, the lifting (E', C) of (E', B) determines a functorial splitting of the I_γ-adic filtration on $(\mathrm{Gr}_B E') \otimes W_{\mu'}$. In particular, it defines canonical isomorphisms*

$$\bigoplus_{i+j=n} \mathrm{Gr}^j_{I,\gamma} \mathcal{O}_S \otimes \mathrm{Gr}^i_B E'_{\mu'} \to (\mathrm{Gr}^n_B E') \otimes W_{\mu'}.$$

3. *If $n < p+a$, then the lifting (E', C) determines a splitting of the filtered sheaf $((B^n E') \otimes W_{\mu'}, I)$, and in particular an isomorphism*

$$\left[B^n E'_{\mu'}\right] \oplus \left[\bigoplus_{i<n} \mathrm{Gr}^i_B E'_{\mu'}\right] \cong (B^n E') \otimes W_{\mu'}.$$

Proof: Let us first observe that there is a natural isomorphism of sheaves on $\mathrm{Cris}(Y'/S)$:

$$i_{cris*} B^n E' \cong C^n E + I C^{n-1} E' + \cdots I^{[n-i]} C^i E' + I^{[n-i+1]} C^{i-1} E' + \cdots. \quad (8.1.1.1)$$

Indeed, if $V \subseteq T$ is an object of Cris(Y'/S), the divided powers of the ideal J_T are by definition compatible with (p,γ), and hence the ideal $J_T + p^\mu \mathcal{O}_T$ of X' in T is a divided power ideal. Thus we can regard T as an object of Cris(X'/S), and in fact the value of $i_{cris*}B^nE'$ on T is just the value of B^nE' on this object [4, p. 5.17]. Equation (8.1.1.1) now follows from the definition (3.3.1) of pullback of T-crystals.

Now suppose that Z is any object of $Rcris(Y'/S)$. It is clear from equation (8.1.1.1) that the natural map:

$$\left[I^{[n-i+1]}E'_Z \cap B^nE'_Z\right] \oplus \left[C^iE'_Z \cap B^nE'_Z\right] \to B^nE'_Z$$

is surjective. If $\nu = \infty$, (E'_Z, C) is normally transversal to each (p^k), by (3.3.3). Hence

$$I^{[n-i+1]}E'_Z \cap C^iE'_Z \subseteq I^{[n-i+1]}C^iE'_Z \subseteq B^{n+1}E'_Z,$$

proving the first statement.

Of course when $\mu = \infty$, (8.1.1.2) follows easily from (8.1.1.1). Namely, if we combine (8.1.1.2) and (6.2.3), we find

$$\begin{aligned} \operatorname{Gr}^n_B E' &\cong \bigoplus_{i+j=n} C^i \operatorname{Gr}^n_B E' \cap I^j_\gamma \operatorname{Gr}^n_B E' \\ &\cong \bigoplus_j \operatorname{Gr}^j_{I,\gamma} \operatorname{Gr}^n_B E' \\ &\cong \bigoplus_{i+j=n} \operatorname{Gr}^j_{I,\gamma} \mathcal{O}_S \otimes \operatorname{Gr}^i_B E'_\mu \end{aligned}$$

In the more general situation of (8.1.1.2) we have to use a slightly different argument: the filtration C itself is too coarse. For each n, let we define a new filtration C_n on E' by setting $j =: n - i$ and

$$C^i_nE' =: C^nE' + IC^{n-1}E' + I^{[2]}C^{n-2}E' + \cdots I^{[j]}C^iE' \subseteq B^nE'$$

Equation (8.1.1.1) shows that we still have

$$B^nE' \subseteq C^i_nE' \cap B^nE' + I^{[j+1]}E' \cap B^nE'.$$

We can apply compute $I^{[j+1]}E' \cap C^i_nE'$ using the calculus of gauges. Namely, let $(J,\gamma) =: ((p^\nu),\gamma)$, and let g and h be the control functions (4.2.2) defined respectively by (I,γ) and (J,γ); note that $h(i) = g(i) + i\delta$. The filtration C on E'_Z is G-transversal to (J,γ) and compatible with (p) by (3.3.3). By

(4.3.4) it defines a $\mathbf{G}_h$-structure, where h is the control function associated with (J,γ). Define

$$\epsilon(k) =: \begin{cases} g(n-k) & \text{if } k \geq i \\ \infty & \text{if } k < i, \end{cases}$$

and let ϵ_h be the h-tame closure (4.2.4) of ϵ. Then $C_n^i E' = C^\epsilon E' = C^{\epsilon_h} E'$. Let c denote the constant gauge whose value is always $g(j+1)$, so that $C^c E' = I^{[n-i+1]} E'$. Then

$$I^{[n-i+1]} E' \cap C_n^i E' = C^{c \vee \epsilon_h} E' = \sum_k p^{g(j+1) \vee \epsilon_h(k)} C^k E',$$

which I claim is contained in $p^\delta B^n E' + B^{n+1} E'$. If $k \geq i$,

$$p^{g(j+1)\vee \epsilon_h(k)} C^k E' \subseteq I^{[j+1]} C^i E' \subseteq B^{n+1} E'.$$

If $k < i$, we have

$$\begin{aligned} \epsilon_h(k) &=: \inf\{\epsilon(k') + h(k'-k) : k' \geq k\} \\ &= \inf\{g(n-k') + h(k'-k) : k' \geq i\} \\ &= \inf\{g(n-k') + g(k'-k) + (k'-k)\delta : k' \geq i\} \\ &\geq g(n-k) + (i-k)\delta \\ &\geq g(n-k) + \delta \end{aligned}$$

It therefore follows that

$$p^{g(j+1)\vee\epsilon_h(k)} C^k E' \subseteq p^{\delta + g(n-k)} C^k E' \subseteq p^\delta I^{[n-k]} C^k E' \subseteq p^\delta B^n E'.$$

We conclude that the filtrations C_n^i and (I,γ) are n-opposed on $(\mathrm{Gr}_B^n E) \otimes W_\delta$, and (8.1.1) follows from (6.2.3) as before.

If $n < p+a$, we use the same method to show that $\tilde{C}_n^i$ (defined analogously, but with the I in place of (I,γ)) splits the filtration induced on $B^n E'$ by the I-adic filtration on E'. Everything is the same except when $j = n-a = p-1$. In this case we find

$$I^{j+1} E' \cap \tilde{C}_n^i E' \subseteq I I^j \tilde{C}_n^i E' + p^\delta B^n E' \subseteq I B^n E' + p^\delta B^n E'.$$

Thus the two filtrations are n-opposed on $(B^n E'/I B^n E') \otimes W_{\mu'}$, and again we are reduced to (6.2.3). ∎

8.1.2 Corollary: *Suppose that $b + \dim X'/S_\mu \leq n < p+a$, and suppose that $\nu \geq 2\mu$. Then a lifting (E',C) of (E,B) to Y' determines an isomorphism:*

$$Ru_{X'_\mu/S*}(B^n E' \otimes W_\mu) \cong Ru_{X'/S*} \mathrm{Gr}_B E'_\mu \cong \bigoplus_i \Omega^i_{E_\mu/S}[-i]$$

Proof: Since $\delta \geq \mu$, we have by (8.1.1.3) an isomorphism

$$B^n E'_\mu \oplus \bigoplus_{i=a}^{n-1} \mathrm{Gr}^i_B E'_\mu \cong B^n E' \otimes W_\mu$$

By (6.2.1), $Ru_{X'/S*} \mathrm{Gr}^j_B E'_\mu$ is quasi-isomorphic to the Kodaira-Spencer complex whose term in degree q is $\mathrm{Gr}^{j-q}_B E_X \otimes \Omega^q_{X/S}$, hence is zero when $j > b + \dim X'/S$. Since $n \geq b + \dim X/S$, this implies that $Ru_{X'/S*} \mathrm{Gr}^j_B E'_\mu$ is acyclic if $j > n$ and that the natural map

$$Ru_{X'/S*} B^n E'_\mu \to Ru_{X'/S*} \mathrm{Gr}^n_B E'_\mu$$

is an isomorphism. ∎

8.2 Decomposition, degeneration, and vanishing theorems

We begin by discussing the conjugate filtration. This takes place in characteristic p, so we shall suppose that $\mu = 1$. Let $F_{X/S}: X \to X'$ be the exact relative Frobenius morphism (1.2.3), with $\pi_{X/S}: X' \to X$ the natural projection. Let Y'/S_ν be a smooth lifting of X'/S_0. Notice that our result does not require a lifting of (E, B) to a lifting of Y of X/S, but rather a lifting of the pull-back (E', A) of (E, B) to Y'/S.

Theorem 8.2.1 (conjugate decomposition) *Suppose that $\Phi: F^*_{X/S} E' \to E$ is an admissible F-span on X/S of width strictly less than $p - \dim(X/S_0)$. Then if $\nu \geq 2$, a lifting (E', C) of the T-crystal $(E', A) =: \alpha_{X/S}(\Phi)$ to Y'/S defines a canonical splitting of the filtration $(Ru_{X/S*} E_0, \mathrm{Dec}\, N)$. In particular,*

1. *The conjugate spectral sequence (7.4.2) of (E, Φ) degenerates at E_1, and the associated filtration on the abutment is canonically split.*

2. *There is a canonical isomorphism:*

$$Ru_{X'/S*} \mathrm{Gr}_A E'_0 \cong F_{X/S*} Ru_{X/S*} E_0$$

3. *Suppose also that X/S is proper and that the cohomology sheaves $R^q f_{X/S*} E$ are p-torsion free when $q = n$ and $n + 1$. Then the cohomology spectral sequence of the filtered object (E'_ν, C) degenerates at E_1 in degree n.*

Proof: Without loss of generality we may assume that the level of Φ is within $[0,b]$, with $b < p - \dim(X/S_0)$. When $n =: b + \dim(X/S_0)$, we know by Lemma (7.3.8) that $Ru_{X/S*} \operatorname{Dec} N^{-n} E \cong Ru_{X/S*} E$, so the isomorphism (7.3.5.2) gives us a filtered quasi-isomorphism

$$(Ru_{X'/S*}(A^n E' \otimes \mathbf{F}_p, P)) \rightarrow F_{X/S*}(Ru_{X/S*} E_0, \operatorname{Dec} N[-n]).$$

As we have seen in (8.1.1.3), the lifting of the T-crystal $(E', A) =: \alpha_{X/S}(\Phi)$ determines a splitting of the filtered object $(Ru_{X/S*} A^n E' \otimes \mathbf{F}_p, P)$, which translates into a splitting of the filtered object $(Ru_{X/S*} E_0, \operatorname{Dec} N[-n])$. This proves (8.2.1.1), and provides us with a canonical isomorphism

$$Ru_{X/S*}(\operatorname{Gr}_{\operatorname{Dec} N} E_0) \rightarrow F_{X/S*}(Ru_{X/S*} E_0).$$

Composing with the isomorphism $(\operatorname{Gr}_A Ru_{X'/S*} E'_0) \cong \operatorname{Gr}_{\operatorname{Dec} N} Ru_{X/S*} E_0)$ of (7.3.5.1), we obtain (8.2.2.2). Alternatively, we could could have used (7.3.5.2) and (8.1.2). The remaining statements (8.1.1.3) follow from (7.4.3) and (6.3.1). ∎

Our next result is a generalization of the theorem of Deligne and Illusie [7] to the case of cohomology with coefficients in an F-T-crystal (E, Φ, B) on X/S (5.3.1). We use the same notation and hypotheses as in the previous section, except where otherwise noted. We work locally on S, and assume that S is affine and that we are given a lifting F_S to S_μ of the absolute Frobenius morphism of S_0. Then we can form as usual a Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\pi_{X/S}} & X \\ \downarrow & & \downarrow \\ S_\mu & \xrightarrow{F_{S_\mu}} & S_\mu \end{array}$$

When X/S_μ is perfectly smooth, the corresponding relative Frobenius morphism on the reductions modulo p $X_0 \rightarrow X'_0$ is exact, so our notation will be consistent with that of the previous results.

First some terminology. If R is a ring and K is a complex of R-modules, we say that K is "strictly decomposed in degree n" if $d^n_K = d^{n-1}_K = 0$, and we say that K is "perfectly decomposed in degree n" if and only if it quasi-isomorphic to a flat complex K' such that K' is strictly decomposed in degree n. Similarly, if (K, B) is a filtered complex of R-modules, we say (K, B) is "perfectly decomposed in degree n" if it is quasi-isomorphic to a filtered complex (K', B) such that $\operatorname{Gr}_B K'$ and K' are flat with $d^n_{K'} = d^{n-1}_{K'} = 0$. Of course, this implies that each of the complexes $B^i K'$ and $\operatorname{Gr}_B K'$ is strictly decomposed in degree n also. Finally, we say that a filtered complex of quasi-coherent sheaves (K, B) on a scheme S is "perfectly decomposed in degree n" if there exists an affine open cover of S on which (K, B) becomes perfectly decomposed in degree n. Finally, we say that (K, B) is "perfectly decomposed" if it is so in all degrees.

Theorem 8.2.2 (Hodge decomposition) *Suppose that (E, Φ, B) is a 1-F-T-crystal (5.3.2) on a perfectly smooth X/S_0, of width less than $p - \dim(X/S_0)$. Then a lifting (E', C) of $(E', A_{X/S})$ to a lifting of X/S_0 to S_2 defines functorial isomorphisms:*

$$L\pi^*_{X/S_0} Ru_{X/S*} \operatorname{Gr}_B E_0 \cong F_{X/S*} Ru_{X/S*} E_0$$

$$LF^*_{S_0} Rf_{X/S*} \operatorname{Gr}_B E_0 \cong Rf_{X/S*} E_0.$$

Furthermore, if X/S is proper, $Rf_{X/S*}(E_0, B)$ is perfectly decomposed. Consequently the spectral sequence of the filtered object (E_0, B) and the functor $f_{X/S*}$ degenerates at E_1, and the sheaves of $\mathcal{O}_{S_0}$-modules $R^q f_{X/S*} \operatorname{Gr}_B E_0$ and $R^q f_{X/S*} E_0$ are locally free and commute with base change. ∎

Proof: First observe that (E, B) is automatically uniform by (5.3.12). The first isomorphism is an immediate consequence of (8.2.1.2) and the isomorphism

$$L\pi^*_{X/S_0} Ru_{X/S*} \operatorname{Gr}_B E_0 \cong Ru_{X'/S*} \operatorname{Gr}_{A_{X/S}} E'_0.$$

Applying Rf'_*, we obtain an isomorphism

$$Rf'_* L\pi^*_{X/S_0} Ru_{X/S*} \operatorname{Gr}_B E_0 \to Rf'_* F_{X/S*} Ru_{X/S*} E_0 \cong Rf_* Ru_{X/S*} E_0.$$

Since X/S is perfectly smooth, it is flat [20, 4.5] and X' is the pull-back of X by means of the absolute Frobenius endomorphism of S_0. Thus, the base change map

$$LF^*_{S_0} Rf_* Ru_{X/S*} \operatorname{Gr}_B E_0 \to Rf'_* L\pi^*_{X/S_0} Ru_{X/S*} \operatorname{Gr}_B E_0$$

is also an isomorphism. Since $Rf_{X/S*} \cong Rf_* Ru_{X/S*}$, we obtain the desired isomorphism

$$LF^*_{S_0} Rf_{X/S*} \operatorname{Gr}_B E_0 \cong Rf_{X/S*} E_0.$$

To prove that, in the proper case, $Rf_{X/S*}(E_0, B)$ is perfectly decomposed, we refine the argument of Deligne and Illusie, *c.f.* [17, 2.5]. Since (E, B) is uniform, so is $(E, A_{X/S})$, and it follows that $Ru_{X/S*}(E_{X/S_0}, A_{X/S})$ is perfect as a filtered complex. That is, locally on S it can be represented by a bounded filtered complex of finitely generated flat S_0-modules, whose associated graded is also flat. Thus our statement follows from the following result.

8.2.3 Lemma: *Let R be a noetherian local ring of characteristic p and with residue field k. Suppose that (K, B) is a filtered complex of R-modules such such that $\mathrm{Gr}_B K$ has finite tor-dimension and $H^i(\mathrm{Gr}_B K)$ is finitely generated for all $i \geq n$, and suppose that there exists an isomorphism in the derived category $R \overset{\mathbf{L}}{\otimes}_{F_R} \mathrm{Gr}_B K \to K$. Then (K, B) is perfectly decomposed in degrees $\geq n$.*

Proof: If R is a field, it is clear that any complex is perfectly decomposed. We shall see below that a filtered complex (K, B) over a field is perfectly decomposed in degree n if and only if $E_1^{i,j}(K, B) = E_\infty^{i,j}$ whenever $i + j = n$. If R is local with residue field k, we say that a complex is "minimal in degree n" if and only if $K \otimes k$ is strictly decomposed in degree n. We write $h^n(K)$ for the length of $H^n(K)$.

8.2.4 Claim: *Suppose R is an Artinian local k-algebra. Let (K, B) be a complex with a finite filtration which is bounded above and such that $H^n(\mathrm{Gr}_B K \overset{\mathbf{L}}{\otimes} k)$ has finite length. Then*

$$h^n(K) \leq h^n(\mathrm{Gr}_B K \overset{\mathbf{L}}{\otimes} k) \lg R,$$

and the following are equivalent:

1. *(K, B) is perfectly decomposed in degree n.*
2. *$H^n(K)$ and $H^n(\mathrm{Gr}_B K)$ are free, their formation commutes with all base change, and furthermore $E_1^{i,j}(K, B) = E_\infty^{i,j}$ whenever $i + j = n$.*
3. *$h^n(K) = h^n(\mathrm{Gr}_B K \overset{\mathbf{L}}{\otimes} k) \lg R$.*

Proof: The only facts that require proof are the implication 3 implies 1 and the inequality. Replace (K, B) by a quasi-isomorphic filtered complex whose Gr is free; then K is also free, and we can use K to calculate filtered derived tensor products. Note that we have $h^n(K \otimes k) \leq h^n(\mathrm{Gr}_B \otimes k)$, with equality if and only if the maps $d^n \otimes id_k$ and $d^{n-1} \otimes id_k$ are strictly compatible with the filtrations [6, 1.3.2]. If equality holds we say we are in the "strict case." Let $\overline{y}$ be a basis for the image of $d^n \otimes id_k$, compatible with the filtration induced by B. Choose $\overline{x}$ in $K^n \otimes k$ such that $d^n \overline{x} = \overline{y}$. If $d^n \otimes id_k$ is strictly compatible with the filtration B, we may choose $\overline{x}$ such that $\overline{x}_j \in B^i K^n \otimes k$ whenever $\overline{y}_j \in B^i K^{n+1} \otimes k$. Now choose a lifting x of $\overline{x}$ (with the same property in the strict case). Consider the complex which is free in degree

n with basis x and free in degree $n+1$ with basis dx. This is naturally a subcomplex of the complex K, is acyclic (resp., its Gr is acyclic), and the quotient K' is also free. Furthermore, the map $K \to K'$ is a (resp. filtered) quasi-isomorphism, and $d^n_{K'} \otimes id_k = 0$. Arguing the same way for d^{n-1}, we see that we may arrange for $K' \otimes k$ to be strictly decomposed in degree n. Let $r =: \lg R$. We have:

$$rh^n(\mathrm{Gr}_B K \overset{\mathbf{L}}{\otimes} k) \geq rh^n(K \overset{\mathbf{L}}{\otimes} k) = rh^n(K' \overset{\mathbf{L}}{\otimes} k) = r\lg(K'^n \otimes k) = \lg K'^n$$

$$= h^n(K') + \lg \operatorname{Im} d^{n-1}_{K'} + \lg \operatorname{Im} d^n_{K'} = h^n(K) + \lg \operatorname{Im} d^{n-1}_{K'} + \lg \operatorname{Im} d^n_{K'}$$

This implies that $h^n(K \otimes k)\lg R \geq h^n(K)$. Furthermore, if we have equality, $h^n(\mathrm{Gr}_B K \otimes k) = h^n(K \otimes k)$, and we are in the strict case. Thus, our map $(K, B) \to (K', B)$ is a filtered quasi-isomorphism. Furthermore, $d^n_{K'} = d^{n-1}_{K'} = 0$, so that K' is strictly decomposed in degree n.

It is now easy to prove the lemma. By a standard argument we may assume that $\mathrm{Gr}_B K^i$ is finitely generated and free if $i \geq n$. Let us write F^*_R for $R \overset{\mathbf{L}}{\otimes}_{F_R}$. Note first that both the hypothesis and the conclusion of the lemma are stable by derived base change $R \to R'$. In particular, $K' =: K \overset{\mathbf{L}}{\otimes}_R k$ satisfies the hypotheses of the lemma, and so $h^i(F^*_k \operatorname{Gr} K') = h^i(K')$ for $i \geq n$. But evidently $h^i(F^*_k \operatorname{Gr} K') = h^i(\operatorname{Gr} K')$, and so the previous claim implies that $(K, B) \overset{\mathbf{L}}{\otimes} k$ is perfectly decomposed in degrees $\geq n$. Thus we may assume without loss of generality that the boundary maps of $K \overset{\mathbf{L}}{\otimes} k$ vanish in degrees $\geq n-1$. We now prove that the same is true of $K \overset{\mathbf{L}}{\otimes} R'$ for every Artinian quotient R' of R, arguing by induction on the length r of R'. If $r > 1$, the homomorphism $F_{R'}$ can be factored $F_{R'} = f \circ g$, where $g\colon R' \to R''$ is a map to a proper quotient of R'. Hence the induction hypothesis applies to $K \overset{\mathbf{L}}{\otimes} R''$. It follows in particular that $g^* \mathrm{Gr}_B K$ is perfectly decomposed and hence so is $F^*_{R'} \mathrm{Gr}_B K$. Let $K' =: K \overset{\mathbf{L}}{\otimes} R'$; from the isomorphism $F^*_{R'} \mathrm{Gr}_B K' \cong K'$ we have

$$h^n(K') = h^n(F^*_{R'} \mathrm{Gr}_B K') = rh^n(F^*_{R'} \mathrm{Gr}_B K' \overset{\mathbf{L}}{\otimes} k) = rh^n(F^*_k(\mathrm{Gr}_B K' \overset{\mathbf{L}}{\otimes} k))$$

$$= rh^n(\mathrm{Gr}_B K' \overset{\mathbf{L}}{\otimes} k) \geq h^n(\mathrm{Gr}_B K') \geq h^n(K').$$

It follows that we have equalities everywhere, and in particular $h^n(K') = rh^n(\mathrm{Gr}_B K')$, so (K', B) is strictly decomposed in degree n. ∎

We next discuss an analog and generalization of Faltings' theorems [9, §IVb], which provide the cohomology of an F-T-crystal in mixed characteristic with the structure of a Fontaine-module. We begin with a derived category result which uses the filtered derived gauge construction described in Section (4.5).

8.2.5 Theorem: *Suppose that X/S_μ is perfectly smooth and that (E, Φ, B) is a μ-F-T-crystal on X/S of width less than $p-\dim(X/S_\mu)$. Then associated to a lifting F_{S_μ} of the Frobenius endomorphism of S_0 to S_μ and to liftings $Y/S_{\mu+1}$ of X/S_μ and (E, C) of (E, B) to Y/S, there is a natural isomorphism*

$$L\pi^*_{X/S_\mu} T^\iota(Ru_{X/S*}E_\mu, B) \to F_{X/S_\mu *} Ru_{X/S*}E_\mu.$$

Proof: Without loss of generality we may assume that Φ is effective, with level contained in $[0, b]$. Let m be b plus the relative dimension d of X/S_μ and let ϵ_m denote the maximal p-tame gauge which vanishes at m. Write K for $Ru_{X/S*}E$ and K' for $Ru_{X'/S*}E'$. Then, using the fact that $m < p$ as in the proof of (5.3.11.4), we find that $\operatorname{Dec} N^{\epsilon'_m} K \cong p^m \operatorname{Dec} N^{-m} K$. Furthermore, the map $\operatorname{Dec} N^{-m} K \to K$ is a quasi-isomorphism because $m \geq b + d$, by Lemma (7.3.8).

On the other hand, by (4.5.3) the liftings of (E, B) and X define a quasi-isomorphism

$$T^{\epsilon_m}(K_\mu, B) \to C^{\epsilon_m} K \otimes W_\mu \cong B^{\epsilon_m} K \otimes W_\mu$$

and hence

$$L\pi^*_{X/S} T^{\epsilon_m}(K_\mu, B) \to L\pi^*_{X/S} B^{\epsilon_m} K \otimes W_\mu \cong A^{\epsilon_m}_{X_0/S} K' \otimes W_\mu.$$

Furthermore, we may replace ϵ_m by ι, as in the proof of (5.3.11.4). Combining this with the quasi-isomorphism

$$A^{\epsilon_m}_{X_0/S} K' \otimes W_\mu \cong F_{X_0/S*} \operatorname{Dec} N^{\epsilon'_m} K$$

of the main theorem (7.3.1), we obtain a quasi-isomorphism

$$L\pi^*_{X/S} T^\iota(K_\mu, B) \to K_\mu.$$

∎

By passing to cohomology we obtain the following result, which is essentially due to Faltings, in the context of Fontaine modules. (It follows from his [9, §IVb], combined with [9, 2.1].) We do get a slight improvement, since he seems to require a $p-2$ in place of our $p-1$.

8.2.6 Theorem: *Suppose that, in the situation of (8.2.5), we also have that X/S_μ is proper. Then given liftings as above to data over $S_{\mu+1}$,*

1. *The filtered complex $(Rf_{X/S_\mu *}E_\mu, B)$ can be provided with the structure ψ of a Fontaine-complex (5.3.6), depending naturally on the liftings.*

2. *The inverse system $R^q f_{cris*}B$ can be endowed with a structure ϕ of an object of $MF^\nabla(S/W)$ (depending on the liftings to $S_{\mu+1}$).*

3. *Each $R^q f_{X/S_\mu *}B^i E_\mu$ is, locally on S_μ, isomorphic to a direct sum of sheaves of the form $\mathcal{O}_S \otimes W_{\mu'}$ for various $\mu' \leq \mu$, and its formation commutes with base change to any S'/S which is flat over W_μ.*

4. *The maps $R^q f_{X/S_\mu *}B^i E_\mu \to R^q f_{X/S_\mu *}B^{i-1}E_\mu$ are injective and locally split.*

Proof: It is immediate to see that formation of T^ι commutes with base change and derived functors. Thus applying $Rf'*$ to the isomorphism provided by (8.2.5), we obtain

$$\begin{array}{ccc}
Rf'_* L\pi^*_{X/S_\mu} T^\iota(Ru_{X/S_\mu *}E_\mu, B) & \longrightarrow & Rf'_* F_{X_0/S_0 *} Ru_{X/S_\mu *}E_\mu \\
\downarrow \cong & & \downarrow \cong \\
LF^*_{S_\mu} T^\iota(Rf_* Ru_{X/S_\mu *}E_\mu, B) & \longrightarrow & Rf_* Ru_{X/S_\mu *}E_\mu \\
\downarrow \cong & & \downarrow \cong \\
LF^*_{S_\mu} T^\iota(Rf_{X/S_\mu *}E_\mu, B) & \xrightarrow{\psi} & Rf_{X/S_\mu *}E_\mu
\end{array}$$

This is the Fontaine-complex structure we are seeking; the rest of the theorem follows from Corollary (5.3.7). ∎

8.2.7 Corollary: *Suppose we are in the situation of Theorem (8.2.6) with $\mu = \infty$. Then the sheaves E^q (obtained by killing the p^∞-torsion of $R^q f_{cris*}E$) are locally free, and if B is the filtration on E^q induced from the filtration B on E, the triple (E^q, Φ, B) forms a uniform F-T-crystal on S/S.*

Proof: This follows immediately from Corollary (5.3.14) and Theorem (8.2.6). ∎

It is well-known that degeneration results as above imply Kodaira-type vanishing theorems, *c.f.* [7] and [17], for example. We obtain the following vanishing theorem with coefficients in the Kodaira-Spencer sheaves of an F-T-crystal. Trying to be careful with the logarithmic structures, we obtain two results which are not quite dual to each other.

Theorem 8.2.8 (Kodaira vanishing) *Suppose X/S_0 is perfectly smooth and proper and (E, Φ, B) is an effective 1-F-T-crystal on X/S. Suppose that the pull-back (E', A) of (E, B) to X'/S admits a lifting (E', C) to a smooth lifting Y'/S_2 of X' over S_2. Suppose that the width of (E, Φ) plus the dimension d of X/S_0 is less than p and that S is affine and L is ample on X. Then*

1. *If $i + j > d$, then $H^j(X, L \otimes \Omega^{i\cdot}_{E/S}) = 0$*
2. *Suppose also that S is regular and that X is Cohen-Macaulay and purely of dimension d. Then if $i + j < d$, $H^j(X, L^{-1} \otimes \Omega^{i\cdot}_{E/S}) = 0$.*

Proof: Recall that $H^j(X, L \otimes \Omega^{i\cdot}_{E/S}) \cong H^{i+j}(X, L \otimes \mathrm{Gr}^i_B E^\cdot_{X/S})$. Since X/S_0 is perfectly smooth it is flat, and since (E, B) is automatically uniform (5.3.12), it follows that $L \otimes \Omega^{i\cdot}_{E/S}$ is a complex of flat $\mathcal{O}_{S_0}$-modules. Thus by standard base-changing arguments, it suffices to prove the vanishing of the cohomology groups of these sheaves along the fibers of X/S_0. Notice that when S_0 is regular, these fibers are, locally on S_0, defined by a regular sequence, and hence are Cohen-Macaulay if X is. Thus we may assume without loss of generality that S_0 is the spectrum of a field.

Now $\mathrm{Gr}_B E^\cdot_{X/S}$ is a complex of locally free $\mathcal{O}_X$-modules which is concentrated in degrees $[0, d]$. Hence Serre's vanishing theorem tells us that $H^q(X, L^n \otimes \mathrm{Gr}_B E^\cdot_{X/S}) = 0$ for $n >> 0$ and for $q > 0$. When X is Cohen-Macaulay of pure dimension d, we can apply Serre duality and the same vanishing to conclude that $H^q(X, L^n \otimes \mathrm{Gr}_A E^\cdot_{X/S}) = 0$ for $q < d$ and $n << 0$. Thus the above theorem will follow from the following result:

8.2.9 Claim: *Suppose that $H^q(X, L^p \otimes \mathrm{Gr}_B E^\cdot_{X/S}) = 0$ for all $q > 0$. or that S is regular and that we have this vanishing for one value of q. Then $H^q(X, L \otimes \mathrm{Gr}_B E^\cdot_{X/S}) = 0$ in the same range.*

Proof: The sheaf $L^p = F^*_X L$ inherits a canonical integrable connection, and endowed with the zero filtration it becomes a T-crystal on X/S_0. Then $L^p \otimes E$ is again a T-crystal, and its filtration is just the filtration induced by the filtration B of E. The E_1 term of the spectral sequence associated to this filtration is $H^q(X, L^p \otimes \mathrm{Gr}_B E^\cdot_{X/S})$, and hence our assumption implies that $H^q(X/S, F^*_X L \otimes E_0) = 0$ for all $q > 0$. Now consider the isomorphism $Ru_{X'/S*} \mathrm{Gr}_A E'_{X'} \cong F_{X/S*} Ru_{X/S*} E_0$ provided by (8.2.1.2), thanks to our lifting. This can be written as an isomorphism $\mathrm{Gr}_A E^\cdot_{X'/S} \to F_{X/S*} E^\cdot_{X/S}$ in the derived category of $\mathcal{O}_{X'}$-modules and $\mathcal{O}_{X'}$-linear maps. Let $L' =: \pi^*_{X/S}L$, and tensor the above isomorphism with L' to obtain an isomorphism:

$$L' \otimes \mathrm{Gr}_A E^{\cdot}_{X'/S} \cong L' \otimes F_{X/S*} E^{\cdot}_{X/S} \cong F_{X/S*} F^*_X L \otimes E^{\cdot}_{X/S}$$

Taking cohomology of both sides, we see that there is an isomorphism

$$H^q(X', L' \otimes \mathrm{Gr}_A E^{\cdot}_{X'/S}) \cong H^q(X, F^*_X L \otimes E^{\cdot}_{X/S}) = 0.$$

Since S is a field, F_S is faithfully flat, so the base changing map

$$F^*_S H^q(X, L \otimes \mathrm{Gr}_A E^{\cdot}_{X/S}) \to H^q(X', L' \otimes \mathrm{Gr}_A E^{\cdot}_{X'/S})$$

are isomorphisms, and we conclude that $H^q(X, L \otimes \mathrm{Gr}_A E^{\cdot}_{X/S})$ vanishes as well. ∎

It is well-known that degeneration and vanishing results in characteristic p can be used to deduce similar results in characteristic zero. By way of example, we include only the following result, for smooth proper logarithmic schemes over $\mathrm{Spec}\, W$. We note that if W is endowed with a hollow log structure, the absolute Frobenius endomorphism of $\mathrm{Spec}\, k$ with its induced log structure lifts to $\mathrm{Spec}\, W$, so we can apply our theory. Our methods do not apply, however, to the "canonical" log structure (corresponding to $1 \mapsto p$), and in particular do not give us information about semi-stable reduction over W.

8.2.10 Corollary: *Let X/S be a perfectly smooth and proper logarithmic scheme over $S =: \mathrm{Spec}\, W$ (where W is endowed with a hollow log structure). Let (E, B, Φ) be an F-T-crystal on X/S of width less than $p - \dim(X/S)$.*

1. *The Hodge spectral sequence*

$$E_1^{ij} = H^i(X, \Omega^{i\cdot}_{E/S}) \cong H^{i+j}(X/S, \mathrm{Gr}^i_B E) \Rightarrow H^n(X/W, E)$$

degenerates at E_1.

2. *If L is ample and $i + j > \dim(X/S)$, then $H^j(X, L \otimes \Omega^{i\cdot}_{E/S}) = 0$.*

3. *If L is ample, if X is Cohen-Macaulay of pure dimension d, and if $i + j < d$, then $H^j(X, L^{-1} \otimes \Omega^{i\cdot}_{E/S}) = 0$.*

Proof: Our lifting F_S of frobenius allows us to pull-back X/S and (E, B) to obtain liftings of X'_0 and (E', A). Thus the first statement follows from (8.2.6), and the next two statements follow from (8.2.8), the universal coefficient theorem, and Nakayama's lemma. ∎

It may be instructive to compare our results to those of Deligne and Illusie by considering the case of the constant F-T crystal $\mathcal{O}_{X/S}$. In this case N is just the p-adic filtration, and $Ru_{X/S_0*}\mathcal{O}_{X/S} \cong \Omega^\cdot_{X/S_0}$. Thus

$$N^i Ru_{X/S_0*}\mathcal{O}_{X/S_0} = \begin{cases} 0 & \text{if } i > 0 \\ \Omega^\cdot_{X/S_0} & \text{if } i \leq 0 \end{cases}$$

Then $\operatorname{Dec} N$ is just the canonical filtration, and so

$$\operatorname{Gr}\operatorname{Dec} N .... \cong \oplus \underline{H}^q(\Omega^\cdot)$$

According to (8.2.1), a lifting modulo p^2 of X'/S provides us with an isomorphism in the derived category

$$\oplus \underline{H}^q(\Omega^\cdot_{X/S}) \cong \Omega^\cdot_{X/S}.$$

This tells us that the complex $\Omega^\cdot_{X/S}$ is perfectly decomposed—exactly as the statement [7, 2.3], except that in [7] a clever use of duality allows an extension of the result to dimension p. Note that the isomorphism above takes place in the unfiltered derived category, and hence tells us nothing about the Hodge filtration. However, when X/S is proper, (8.2.2) tells us that the filtered complex (Ω, B) is perfectly decomposed, which implies (2.4) and (4.1.2) of [7].

8.3 Hodge and Newton polygons

Let S_0 denote $\operatorname{Spec} k$ endowed with a fine saturated logarithmic structure; recall that there exists a finitely generated integral monoid P with $P^* = 0$ such that the map $P \to k$ sending every nonzero element of P into zero is a chart for S_0. Let S denote an element of $\operatorname{Cris}(S_0/W)$ whose underlying scheme is $\operatorname{Spec} W$. For example, we could consider the "hollow" logarithmic structure associated to the prelog structure $P \to W$ sending every nonunit to 0. We let $F_{X/S}: X \to X'$ denote the exact relative Frobenius morphism of X/S_0.

Suppose that X/S_0 is proper and logarithmically smooth and integral, and suppose that (E', A) is a T-crystal on X'/S. As in Remark (4.4.8), for each $n \in \mathbf{N}$, we set

$$h^{i,n-i}(E'_0, A) =: \dim_k H^n(X'/S_0, \operatorname{Gr}^i_A E'_0) = \dim_k H^{n-i}(X', \Omega^{i\cdot}_{E'_0/S_0}),$$

and we denote the Hodge polygon [4, p. 8.43] attached to the sequence of numbers $h^i =: h^{i,n-i}(E'_0, A)$ by $\mathrm{Hdg}^n(E', A)$. For example, the Hodge polygons of $(\mathcal{O}_{X'/W}, J_{X/W}, \gamma)$ are the usual Hodge polygons of X'/S_0. If $\Phi\colon F^*_{X/S}E' \to E$ is an admissible F-span on X/S, we write $\mathrm{Hdg}^n(X/S, \Phi)$ for $\mathrm{Hdg}^n(\alpha_{X/S}(\Phi))$. For example, an F-span on k/W is just a p-isogeny (5.1.1) $\Phi\colon E' \to E$ between two finitely generated free W-modules, and its Hodge polygon Hdg^n is trivial if $n \neq 0$, so we may drop the superscript from the notation. It is clear that our polygon in this case is the same as Mazur's "abstract Hodge polygon of an F-span" [23]. Recall that such an F-span on k/W is determined up to isomorphism by its Hodge polygon.

Let $\Phi\colon F^*_X E' \to E$ be an admissible F-span on X/S, and fix an integer n. Recall from (7.4.1) that we obtain an F-span $\overline{\Phi}_n\colon E'^n \to E^n$ on S_0/S by taking the map $H^n(X'/S, E') \to H^n(X/S, E)$ induced by $F_{X/S}$ and Φ and dividing by the torsion of $H^n(X/W, E')$ and $H^n(X/W, E)$. Our aim is to compare the Hodge polygons of $\overline{\Phi}_n$ and of Φ.

8.3.1 Theorem: *Suppose X/S_0 is proper and logarithmically smooth and integral, and let $\Phi\colon F^*_{X/S}E' \to E$ be an admissible F-span on X/S. For each $n \in \mathbf{N}$, let $\overline{\Phi}_n$ denote the F-span on* $\mathrm{Spec}\, S_0/S$ *obtained as above by taking crystalline cohomology and killing torsion. Then the Hodge polygon* $\mathrm{Hdg}(\mathrm{Spec}\, S_0/S, \overline{\Phi}_n)$ *lies on or above the Hodge polygon* $\mathrm{Hdg}^n(X/S, \Phi)$. *Furthermore, the following conditions are equivalent:*

1. *The two polygons* $\mathrm{Hdg}(\mathrm{Spec}\, S_0/S, \overline{\Phi}_n)$ *and* $\mathrm{Hdg}^n(X/S, \Phi)$ *have the same projection to the x-axis.*
2. *The rank of $H^n(X/S, E')$ is the sum of the Hodge numbers $h^{i,n-i}(E'_0, A)$.*
3. *For $q = n$ and $q = n+1$, the groups $H^q(X'/S, E')$ are torsion free, and the maps*
$$H^q(X'/S_0, A^i E'_0) \to H^q(X'/S_0, E'_0)$$
are injective for all i.

Proof: To prove that $\mathrm{Hdg}(S_0/S, \overline{\Phi}_n)$ lies on or above the Hodge polygon $\mathrm{Hdg}^n(X/S, \Phi)$ we follow the method of [4, 8.36]. In particular, the following estimate will suffice [4, 8.37]. Let us write H^n for $H^n(X/S, E)$ and H'^n for $H^n(X'/S, E')$; we shall identify $\overline{H}'^n$ with its image in $\overline{H}^n$ under $\overline{\Phi}_n$.

8.3.2 Claim: *For all $i \geq 1$,*

$$\lg(\overline{H}'^n / p^i \overline{H}^n \cap \overline{H}'^n) \leq h^n(X'/S_0, \mathrm{Gr}_A^{i-1} E'_0) + 2h^n(X'/S_0, \mathrm{Gr}_A^{i-2} E'_0) + \cdots$$

Proof: In the commutative diagram below and what follows it we have written D in place of $\operatorname{Dec} N$, K in place of $Ru_{X/S*}E$, and H^n for $H^n(X, \ )$.

$$\begin{array}{ccccc} H^n(p^iK \cap D^0K) & \longrightarrow & H^n(D^0K) & \xrightarrow{\beta} & H^n(D^0K/p^iK \cap D^0K) \\ \downarrow & & \downarrow\gamma & & \downarrow \\ H^n(p^iK) & \longrightarrow & H^n(K) & \longrightarrow & H^n(K/p^iK) \end{array}$$

Because the rows are exact, we find a surjective map from the image of β to the image of $H^n(D^0K)$ in $H^n(K)/p^iH^n(K)$. Corollary (7.3.6) allows us to identify $H^n(X'/S, E')$ with $H^n(X/S, D^0K)$, and hence we have a surjective map from the image of β to the image of $\overline{H}'^n$ in $\overline{H}^n/p^i\overline{H}^n$. It follows that

$$\lg \overline{H}'^n/(\overline{H}'^n \cap p^i\overline{H}^n) \leq \lg H^n(D^0K/p^iK \cap D^0K).$$

For each $j \in \mathbf{N}$ we have an exact sequence:

$$H^n(\operatorname{Gr}_P^{j-1} D^0K) \to H^n(D^0K/p^jK \cap D^0) \to H^n(D^0K/p^{j-1}K \cap D^0K),$$

and it follows that

$$h^n(D^0K/p^iK \cap D^0) \leq h^n(\operatorname{Gr}_P^{i-1} D^0K) + h^n(\operatorname{Gr}_P^{i-2} D^0K) + \cdots h^n(\operatorname{Gr}_P^0 D^0K).$$

Using the G-transversality of D to (p), we see that multiplication by p^j induces an isomorphism $D^{-j}\operatorname{Gr}_P^0 K \to \operatorname{Gr}_P^j D^0K$, so that

$$h^n(\operatorname{Gr}_P^j D^0K) = h^n(D^{-j}\operatorname{Gr}_P^0 K).$$

From the exact sequences

$$H^n(D^{1-j}K \operatorname{Gr}_P^0 K) \to H^n(D^{-j}\operatorname{Gr}_P^0 K) \to H^n(\operatorname{Gr}_D^{-j}\operatorname{Gr}_P^0 K)$$

we see that

$$h^n(D^{-j}\operatorname{Gr}_P^0 K) \leq h^n(\operatorname{Gr}_D^{-j}\operatorname{Gr}_P^0 K) + h^n(\operatorname{Gr}_D^{1-j}\operatorname{Gr}_P^0 K) + \cdots h^n(\operatorname{Gr}_D^0 \operatorname{Gr}_P^0 K).$$

But Corollary (7.3.5) implies that $h^n(\operatorname{Gr}_D^{-j}\operatorname{Gr}_P^0 K) = h^n(\operatorname{Gr}_A^j \operatorname{Gr}_P^0 E')$. Now our (8.3.2) follows immediately.

Now the length of the projection of $\operatorname{Hdg}(\operatorname{Spec} S_0/S, \overline{\Phi}_n)$ to the x-axis is the rank of $H^n(X/S, E')$ which is the same as the rank of $H^n(X/S, E)$, so it is clear that conditions (1) and (2) are equivalent. According to (4.4.8), conditions (2) and (3) are also equivalent. ∎

We now suppose that S is endowed with the hollow logarithmic structure. It is clear that then we can choose a lifting F_S of the absolute Frobenius endomorphism of S_0 to S; in fact, the set of such liftings is a torsor under the group $\mathrm{Hom}(P, W^*)$.

If X/S_0 is perfectly smooth and proper and $\Phi\colon F_X^*E \to E$ is an admissible F-crystal on X/S, then because F_S is flat, the base changing morphism $F_S^*H^n(X/S,E) \to H^n(X'/S, \pi_{X/S}^*E)$ is an isomorphism, and we obtain an isogeny

$$\overline{\Phi}_n\colon F_S^*\overline{H}^n(X/S,E) \to \overline{H}^n(X/S,E).$$

Ignoring the logarithmic structure on S_0, we can regard this isogeny as defining an F-crystal on $\mathrm{Spec}\, k/W$. Let $\mathrm{Nwt}(\overline{\Phi}_n)$ denote its Newton polygon [22].

Corollary 8.3.3 (Katz's Conjecture) *Let X/S_0 be perfectly smooth and proper, and let $\Phi\colon E \to E$ be an admissible F-crystal on X/S. Then each*

$$\Phi_n\colon F_S^*\overline{H}^n(X/S,E) \to \overline{H}^n(X/S,E)$$

defines an F-crystal on $\mathrm{Spec}\, k/W$, and we have inequalities of Hodge polygons:

$$\mathrm{Nwt}(\overline{\Phi}_n) \geq \mathrm{Hdg}(\overline{\Phi}_n) \geq \mathrm{Hdg}^n(X/S,\Phi).$$

The polygons $\mathrm{Nwt}(\Phi_n)$ and $\mathrm{Hdg}(\overline{\Phi}_n)$ have the same endpoint, and the following conditions are equivalent:

1. *The two polygons $\mathrm{Nwt}(\overline{\Phi}_n)$ and $\mathrm{Hdg}^n(X/S,\Phi)$ have the same projection to the x-axis.*

2. *The rank of $H^n(X/S,E)$ is the sum of the Hodge numbers $h^{i,n-i}(E_0', A)$.*

3. *For $q = n$ and $q = n+1$, the groups $H^q(X/S,E)$ are torsion free, and the maps*

   $$H^q(X'/S_0, A^iE_0') \to H^q(X'/S_0, E_0')$$

   are injective for all i.

4. *The two polygons $\mathrm{Hdg}(\overline{\Phi}_n)$ and $\mathrm{Hdg}^n(X/S,\Phi)$ coincide.*

Proof: It is of course a general fact that the Newton polygon of an F-crystal on k/W lies above its Hodge polygon and has the same endpoint [23], so our inequalities follow from (8.3.1). The equivalence of (1)–(3) follows from the analogous equivalence in (8.3.1) as well as the isomorphism

$$F_S^*H^n(X/S,E) \to H^n(X'/S,E').$$

If these conditions are satisfied, then by (7.4.4) we see that $H^q(X/S, E)$ is torsion free for $q = n$ and $n+1$ and that the conjugate spectral sequence degenerates at E_2 in degree n. Then Theorem (7.4.3) tells us that the abstract Hodge filtration A of the F-span $\overline{\Phi}_n$ can be identified with the cohomological filtration on $H^n(X'/S, E')$ induced by the filtration A of E'_0 and so the two polygons $\mathrm{Hdg}(\mathrm{Spec}\, S_0, \overline{\Phi}_n)$ and $\mathrm{Hdg}^n(X/S, \Phi)$ coincide. ∎

Suppose now that Y/S is proper and logarithmically smooth and integral and that (E, B) is a T-crystal on Y/S. Then we can also speak of the Hodge polygons of (E_K, B) on the generic fiber Y_{S_η}/S_η of Y/S. Thus, $\mathrm{Hdg}^n(E_K, B)$ is the Hodge polygon attached to the numbers $h^{i,j}(E, B) : i + j = n$, computed in characteristic 0: $h^{i,j}(E, B)$ is the rank of the n^{th} hypercohomology group of the Kodaira-Spencer complex

$$\mathrm{Gr}^j_B E_Y \to \mathrm{Gr}^{j-1}_B E_Y \otimes \Omega^1_{Y/S} \to \ldots \mathrm{Gr}^{j-q}_B E_Y \otimes \Omega^q_{Y/W} \to \ldots$$

8.3.4 Theorem: *Suppose Y/S is proper and perfectly smooth and (E, Φ, B) is an F-T-crystal on Y/S of width less than $p - \dim Y/S$. Then for all n,*

$$\mathrm{Hdg}^n(E_K, B) = \mathrm{Hdg}(\overline{\Phi}_n).$$

Proof: When $Y = S$, it is obvious from the definitions that $\mathrm{Gr}^i_B E$ is a free W-module of finite rank and that

$$\mathrm{Gr}^i_B E \otimes_W k \cong \mathrm{Gr}^i_A E_0.$$

Thus the theorem is trivial in this case. In general, we know from (8.2.7) that (E^n, Φ_n, B) is an F-T-crystal on $\mathrm{Spec}\, S/S$. As

$$\mathrm{Gr}^i_B E^n_K \cong H^n_{cris}(Y_K/K, \mathrm{Gr}^i_B E^n \otimes K),$$

the theorem follows. ∎

The following corollary, at least in the case of constant coefficients, is due to Deligne [5].

8.3.5 Corollary: *With the hypotheses of the previous theorem, the Newton polygon of Φ_n lies on or above $\mathrm{Hdg}^n(E_K, B)$.*

∎

8.3.6 Remark: The previous result is a special case of a much more general consequence of the work of Faltings in [9] (although with a somewhat more restricted notion of logarithmic structure.) We can only give a rough idea.

Let S be the spectrum of the ring of integers V in any finite extension K of the fraction field K_0 of W, and let Y/S be smooth and proper. Then Faltings proves that the representation of $\mathrm{Gal}(\overline{K}/K)$ in $H^n_{\acute{e}t}(Y_{\overline{K}}, \mathbf{Q}_p)$ is associated (à la Fontaine) to the data obtained by combining the action of Frobenius on the crystalline cohomology of the special fiber of Y/S with the Hodge filtration on the cohomology of the generic fiber. In particular, it follows that these data are "weakly admissible," and hence that the Newton polygon of the F-crystal lies above the Hodge polygon of the generic fiber. Faltings even has a similar result with coefficients, provided one starts with a p-adic representation of the fundamental group of the generic fiber. Notice that there are no restrictions on the ramification of K, the dimension of Y, or the level of the Hodge filtrations involved.

8.3.7 Remark: Suppose that S is a scheme of finite type and geometrically connected over Spec $\mathbf{Z}$ and that U/S is smooth and separated. After replacing S by some dense open subset, we may find a smooth $\overline{X}/S$ and an open immersion $U \to \overline{X}$ over S such that the complement of U in $\overline{X}$ is a relative divisor with normal crossings over S. Let X be the logarithmic scheme obtained by pushing forward the trivial logarithmic structure on U to $\overline{X}$. Then X/S is logarithmically smooth and proper. If σ is a $\mathbf{C}$-valued point of S, then the Hodge filtration of $H^n_{DR}(X(\sigma)/\mathbf{C}) \cong H^n_{DR}(U(\sigma)/\mathbf{C})$ is precisely the filtration used by Deligne in [6] in the construction of the mixed Hodge structure on $H^n_{DR}(U(\sigma)/\mathbf{C})$. Localizing S some more, we may assume that for every closed point s of S, the characteristic $p(s)$ of the residue field $k(s)$ satisfies the hypotheses of (8.2.5). Thus, the Hodge polygon of the F-crystal $H^n_{cris}(X(s)/W(s))$ agrees with the Hodge polygon of the mixed Hodge structure of the geometric generic fiber.

8.4 F-crystals on curves

In this subsection we study the cohomology of F-T-crystals in the simplest possible nontrivial case, namely F-T-crystals of rank two and width one on curves, as well as the cohomology of the symmetric powers of such F-crystals. As an application, we give a simple proof of some results of Ulmer in the theory of modular forms [28].

As usual, we let S denote a p-adic formal scheme which is flat over W and endowed with a fine logarithmic structure and, eventually, a lifting of its absolute Frobenius endomorphism. Suppose that X/S_0 is a perfectly smooth connected logarithmic morphism of relative dimension one, so that $\Omega^i_{X/S_0} = 0$ unless $i = 0$ or 1.

For an important case to keep in mind, consider $S = \operatorname{Spf} W$, $\overline{X}$ the complete modular curve classifying semi-stable elliptic curves with a suitable (prime-to-p) level structure, and let X be the logarithmic scheme obtained from $\overline{X}$ by logarithmically cutting out the "cusps." More precisely, if $U \subseteq \overline{X}$ is the open subscheme of X classifying smooth elliptic curves, X has the logarithmic structure obtained by taking the direct image on $\overline{X}$ of the trivial logarithmic [20, (1.4)] structure on U. Then the universal elliptic curve over U prolongs to a perfectly smooth morphism $f: Z \to X$ of logarithmic schemes over S. Then by (7.5.2) and (7.5.3), we see that $R^1 f_{cris*}(\mathcal{O}_{Z/S}, J_{Z/S}, \Phi)$ defines a uniform F-T-crystal (E, Φ, B) on X/S. Poincaré duality for f defines a principal polarization (3.3.2) on (E, Φ, B).

If (E, B) is a T-crystal on X/S of level within $[0, b]$, let $E^{\cdot}_{X/S_0}$ be the De Rham complex of its restriction to X/S_0, and let

$$\xi_i \colon \operatorname{Gr}^i_B E_X \to \operatorname{Gr}^{i-1}_B E_X \otimes \Omega^1_{X/S}$$

be the Kodaira-Spencer mapping, *i.e.* the map induced by the connection ∇. Then the Kodaira-Spencer complexes of (E, B) are as follows:

$$\operatorname{Gr}^i_B E^{\cdot}_{X/S_0} = \begin{cases} \operatorname{Gr}^0_B E_X & \text{if } i = 0 \\ \operatorname{Gr}^i_B E_X \xrightarrow{\xi_i} \operatorname{Gr}^{i-1}_B E_X \otimes \Omega^1_{X/S} & \text{if } 0 < i < b+1 \\ \operatorname{Gr}^b_B E_X \otimes \Omega^1_{X/S}[-1] & \text{if } i = b+1. \end{cases}$$

Assume now that (E, B) is uniform of level one and rank two, so that $\omega =: B^1 E_X$ is an invertible sheaf on E_X. Assume also that (E, B) is endowed with a principal polarization of weight one (3.3.2). The polarization induces an isomorphism $B^1 E_X \cong \operatorname{Hom}(\operatorname{Gr}^0_B E_X, \mathcal{O}_X)$, and we shall identify these two sheaves. Let us also assume that the Kodaira-Spencer mapping

$$\xi \colon \omega \longrightarrow \omega^{-1} \otimes \Omega^1_{X/S}$$

is not identically zero. Then there is a unique effective divisor R such that ξ defines an isomorphism

$$\omega \cong \omega^{-1} \otimes \Omega^1_{X/S}(-R), \tag{8.4.0.1}$$

which we can also regard as an isomorphism $\omega^2(R) \cong \Omega^1_{X/S}$. We shall call R the "ramification divisor" of (E, B). In the case of the modular curve discussed above, $R = 0$, and if (E, B) is obtained by pullback via a morphism π to the modular curve, then R is just the ramification divisor of π. For example, in [28] Ulmer considers the Igusa curve of level p^n, and if $n > 0$ R is a nonzero divisor which is supported at the supersingular points.

Consider the m^{th} symmetric product $S^m(E,B)$ of (E,B), with filtration B induced in the usual way by the filtration B on E. It is clear that (S^mE,B) forms a T-crystal on (X/S). The polarization on E defines a polarization on (S^mE,B), but this polarization will not be principal if $m \geq p$.

8.4.1 Proposition: *Suppose as above that X/S_0 is perfectly smooth of relative dimension one, proper, and connected, and let (E,B) be a rank two uniform T-crystal on X/S endowed with a principal polarization of weight one. Suppose that the degree of $\omega =: B^1E_X$ is positive and that the Kodaira-Spencer mapping*

$$\xi\colon \omega \to \omega^{-1} \otimes \Omega^1_{X/S_0}$$

is nonzero, with ramification divisor R.

1. *If $m > 0$, the Hodge numbers of (S^mE,B) in degree one (4.4.8) are as follows:*

$$h^{i,1-i}(S^mE_0,B) = \begin{cases} h^1(\omega^{-m}) & \text{if } i = 0 \\ \deg R & \text{if } 0 < i < m+1,\ p \not| i \\ h^0(\omega^{2i-m}(R)) + h^1(\omega^{2i-m}) & \text{if } 0 < i < m+1,\ p | i \\ h^0(\omega^{m+2}(R)) & \text{if } i = m+1. \end{cases}$$

2. *If $0 < m < p$, $H^i(X/S,S^mE) = 0$ if $i \neq 1$, and the hypotheses of (6.3.4) hold. Thus, the maps $H^1(X/S,B^iS^mE) \to H^1(X/S,S^mE)$ are injective and define a T-crystal on (S_0/S).*

Proof: Locally on X we may choose a basis $\{x,y\}$ for E_X such that $\{x\}$ is a basis for B^1E_X. Then $B^iS^mE_X$ has a basis $\{x^{i'}y^{m-i'} : i' \geq i\}$, and the Gauss-Manin connection acts by the rule

$$\nabla x^{i'}y^{m-i'} = i'x^{i'-1}y^{m-i'}\nabla x + (m-i')x^{i'}y^{m-i'-1}\nabla y$$

It is clear that for $i \in [0,m]$,

$$\mathrm{Gr}^i_B S^mE_X \cong (\mathrm{Gr}^1_B E_X)^{\otimes i} \otimes (\mathrm{Gr}^0_B E_X^{\otimes m-i}) \cong \omega^i \otimes \omega^{i-m} \cong \omega^{2i-m}$$

Furthermore, the Kodaira-Spencer map

$$\mathrm{Gr}^i_B S^mE_X \to \mathrm{Gr}^{i-1}_B S^mE_X \otimes \Omega^1_{X/S}$$

can be identified with i times the standard map $\omega^{2i-m} \to \omega^{2i-m}(R)$. We find quasi-isomorphisms

$$\mathrm{Gr}^i_B S^mE^{\cdot}_{X/S} \cong \begin{cases} \omega^{-m} & \text{if } i = 0 \\ \omega^{2i-m}(R)|_R[-1] & \text{if } p \not| i \text{ and } 0 < i < m+1 \\ \omega^{2i-m} \oplus \omega^{2i-m}(R)[-1] & \text{if } p|i \text{ and } 0 < i < m+1 \\ \omega^{m+2}(R)[-1] & \text{if } i = m+1. \end{cases}$$

The proof of the proposition is now straightforward. ∎

Suppose now that (E, B, Φ) is an F-T-crystal on X/S (5.3.1), uniform and endowed with a principal polarization (5.2.2)

$$(E, \Phi) \to \mathrm{Hom}\Big((E, \Phi), \mathcal{O}_{X/S}(-1)\Big)$$

Because $\alpha_{X/S}(\Phi)$ is isomorphic to the pullback of (E, B) to X'/S, we find that $\Phi_X \colon F_X^* E_X \to E_X$ annihilates $F_X^* E_X$, and there is a commutative diagram:

$$\begin{array}{ccc} F_X^* E_X & \xrightarrow{\Phi_X} & E_X \\ \downarrow & \nearrow & \downarrow \\ F_X^* \operatorname{Gr}_B^0 E_X & \xrightarrow{h} & F_X^* \operatorname{Gr}_B^0 E_X \end{array}$$

Here $h \colon F_X^* \operatorname{Gr}_B^0 E_X \to \operatorname{Gr}_B^0 E_X$ is the "Hasse-Witt" morphism, and can be interpreted as a map $\omega^{-p} \to \omega^{-1}$, or a section of ω^{p-1}. If it is identically zero, then the image of Φ_0 is $B^1 E_X$, and we have an isomorphism $\omega^{-p} \to \omega$, *i.e.* a trivialization of ω^{p+1}. Furthermore, in this case we see that the filtration $B^0 E_X$ and $N^0 E_X$ coincide, and since the latter is horizontal, it follows that the Kodaira-Spencer mapping is zero. In any case, we see that ω has nonnegative degree.

Let us assume from now on that the Kodaira-Spencer and Hasse-Witt mappings are nonzero. Then there is an effective divisor Σ such that the image of h is $I_\Sigma \otimes \omega^{-1}$, and we have $\mathcal{O}_X(\Sigma) \cong \omega^{p-1}$. In particular, $\deg(\Sigma) = (p-1)\deg \omega$. Applying the previous result and (8.3.3), we find:

8.4.2 Corollary: *Suppose that X/S_0 is perfectly smooth of relative dimension one, and (E, Φ, B) is a rank two F-T-crystal on X/S which is uniform of level one and endowed with a principal polarization of weight one. Suppose that the degree of $\omega =: B^1 E_X$ is positive and that the Kodaira-Spencer mapping is nonzero, with ramification divisor R. Then if $0 < m < p$, $(H^1(X/S, E), \Phi)$ defines an F-crystal on S_0/S, and its Hodge numbers are given by the formulas (8.4.1.1).*

∎

Let us remark that by (8.3.3), the Newton polygon of $(H^1(X/S, E), \Phi)$ lies above the Hodge polygon we have described; this gives the result of Ulmer [28]. We should also point out that for simple enough logarithmic structures, at least (for example, for the trivial log structure on $S = \operatorname{Spec} W$ and the "omit the cusps" log structure on X), one can use Faltings' results to extend these calculations to the case of $m \geq p$. Namely, one can always find a smooth lifting Y of X to S and then a lifting (E, Φ, C) of the F-T-crystal to Y; if p is odd then Faltings' theory guarantees the existence of a p-adic representation associated to (E, Φ, C). Taking the symmetric powers of this representation and applying (8.3.6), we still find that the Newton polygon lies above the Hodge polygon given by the obvious extension of (8.4.1).

We close by discussing the cohomology of "horizontal" F-spans on elliptic curves.

8.4.3 Example: We shall see that there exists an F-span $\Phi\colon F_{X/S}E' \to E$ on an elliptic curve whose Hodge spectral sequence does not degenerate but whose conjugate spectral sequence does. As explained in (7.4.4), this F-span cannot be an F-crystal, and the torsion subgroups of $H^1(X'/W, E')$ and $H^1(X/W, E)$ have different lengths.

Let X/k be an elliptic curve, and let Ξ be an extension of $\mathcal{O}_{X'/W}$ by $\mathcal{O}_{X'/W}$ in the category of crystals of $\mathcal{O}_{X'/W}$-modules. Let ξ denote its isomorphism class, which we can view as an element of $H^1(X'/W, \mathcal{O}_{X'/W})$. Then Ξ corresponds to an exact sequence

$$0 \longrightarrow \mathcal{O}_{X'/W} \longrightarrow E' \longrightarrow \mathcal{O}_{X'/W} \longrightarrow 0$$

We can view the subcrystal $\mathcal{O}_{X'/W}$ of E' as defining a filtration on E'. Since the filtration is horizontal it *a fortiori* satisfies Griffiths transversality, and hence if we let A be the PD-saturation of this filtration we obtain a T-crystal on X'/W. Then $\mathrm{Gr}_A E'_{X'} \cong \mathcal{O}_{X'} \oplus \mathcal{O}_{X'}$, and the Kodaira-Spencer mapping is zero. Thus we obtain

$$\begin{aligned} \Omega^{0\cdot}_{E'_0/k} &\cong \mathcal{O}_{X'} \\ \Omega^{1\cdot}_{E'_0/k} &\cong \mathcal{O}_{X'}[1] \oplus \Omega^1_{X'/k} \\ \Omega^{2\cdot}_{E'_0/k} &\cong \Omega^1_{X'/k}[1] \end{aligned}$$

This shows that the sum of the Hodge numbers in degree 1 associated to the two spectral sequences is 4. In particular, the Hodge spectral sequence degenerates if and only if $h^1(X'/k, E'_0) = 4$, and this holds if and only if the reduction modulo p of our extension is split. In other words, the Hodge spectral sequence degenerates at E_1 if and only if ξ is divisible by p in $H^1(X'/W)$.

If we now follow the procedure of (5.1.2) we find an F-span $\Phi\colon F^*E' \to E$, which fits into the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X/W} & \longrightarrow & F^*_{X/W}E' & \longrightarrow & \mathcal{O}_{X/W} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathcal{O}_{X/W} & \longrightarrow & E & \longrightarrow & \mathcal{O}_{X/W} & \longrightarrow & 0 \end{array}$$

This diagram shows that the isomorphism class of the extension corresponding to the sequence along the bottom is $pF^*_{X/W}\xi$. Thus, the filtration N on $\mathrm{Gr}^0_P(E)$ is just the subobject $\mathcal{O}_{X/k}$ defined by the extension in the diagram; note that this (mod p) extension is split, so that $h^1(X/k, E_0) = 4$ and the conjugate spectral sequence always degenerates at E_2.

Now suppose that ξ is not divisible by p. Then we see that the Hodge spectral sequence does not degenerate, and the conjugate spectral sequence does. Furthermore, $H^1(X'/W, E')$ is p-torsion free, while $H^1(X/W, E)$ is not.

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