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CONFIGURATIONS OF REAL AND COMPLEX POLYNOMIALS

by
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This article is dedicated to the memory of Mario Raimondo.

§0. Introduction.

The purpose of this article is to give a geometric explanation of the surprising equality (cf. [C-P],[Ar1],[Ar3]) between, on one hand, the number of configurations of (complex) lemniscate generic polynomials, and , on the other hand, the number of configurations of real monic Morse polynomials with the maximal number of (real) critical points.

This discovery occurred when Arnold gave a series of talks at the Scuola Normale in 1989 on the subject of catastrophe theory, and there was somehow a bet whether there could be a geometrical correspondence between the two sets.

Afterwards, Arnold developed a quite general theory concerning the ubiquity of Euler, Bernoulli and Springer numbers (cf.[Ar1], [Ar2], [Ar3]) in the realm of singularity theory.

In this article, among other things, we prove the equality of the above two numbers by geometric methods.

It would of course be very interesting to extend the type of correspondence introduced here to a more general context, like the case of spaces of universal deformations of 0-modular isolated singularities. In a different direction, we plan to extend these type of results to the case of real algebraic functions, using the results of [B-C].

Let us explain now in some detail what are our present results.

We adopt here the notation and terminology of [C-P] and [C-W] : given a polynomial $P(z)$ we consider $|P(z)|$ as a (weak) Morse function , and we define the ***big lemniscate configuration*** of P to be equal to the union of the singular level sets of $|P|$ (the so called lemniscates). P is said to be ***lemniscate generic*** if P has distinct roots and every level set $\Gamma_c = \{z : |P(z)| = c\}$ has at most one ordinary quadratic singularity. Two big lemniscate configurations Γ_1, Γ_2 are said to be isotopic if there is a path σ in the space of diffeomorphisms of $\mathbb{C}$ such that $\sigma(0)$ is the identity and $\sigma(1)(\Gamma_1) = \Gamma_2$.

One of the main results of [C-P] was that there is a bijective correspondence between isotopy classes of big lemniscate configurations and connected components of the space $\mathcal{L}_n$ of lemniscate generic polynomials. Assume now that $P \in \mathbb{R}[z]$: then, if P is lemniscate generic, automatically all the critical points of P are real; thus, letting $(n+1)$ be the degree of P, P has n distinct real critical values which are different from zero.

Let $\mathcal{L}_n$ be the open set of complex lemniscate generic polynomials of degree $(n+1)$, let $\mathcal{L}_{n,\mathbb{R}}$ be the set of real lemniscate generic polynomials (an open set in the space of real polynomials), let finally $\mathbb{G}\mathcal{M}_n$ (which is called the "Set of generic maximally real polynomials") be the open set of real polynomials with n real and distinct critical values : thus $\mathcal{L}_{n,\mathbb{R}} \subset \mathbb{G}\mathcal{M}_n$, and every component of $\mathbb{G}\mathcal{M}_n$ is the closure (in $\mathbb{G}\mathcal{M}_n$) of a finite number of components of $\mathcal{L}_{n,\mathbb{R}}$.

If P is in $\mathbb{G}\mathcal{M}_n$ and $y_1 < \dots < y_n$ are the critical points, we associate to P the sequence $u_1 = P(y_1), \dots, u_n = P(y_n)$, a ***snake sequence*** (cf. [Da], [Ar3]), what simply means that $(-1)^i(u_i - u_{i+1})$ has constant sign.

If P is lemniscate generic and real, there is another way of ordering the critical values, namely by increasing absolute values : we let $Y_{n,\mathbb{R}} = \{(w_1, \dots, w_n) \in \mathbb{R}^n : 0 < |w_1| < \dots < |w_n|\}$ be the space of admissible critical values. Clearly $Y_{n,\mathbb{R}}$ has exactly 2^n connected components homeomorphic to $\mathbb{R}^n$.

Main Theorem.

(a) Each connected component of $\mathcal{L}_n$ contains exactly 2^{n+1} connected components of $\mathcal{L}_{n,\mathbb{R}}$.

(b) The number of connected components of $\mathcal{L}_{n,\mathbb{R}}$ mapping to a fixed component of $Y_{n,\mathbb{R}}$ equals the number of components of $\mathbb{G}\mathcal{M}_n$, whence the number of connected components of $\mathbb{G}\mathcal{M}_n$ equals twice the number K_n of connected components of $\mathcal{L}_n$; the number instead of components of $\mathbb{G}\mathcal{M}_n \cap$ {monic polynomials} equals K_n.

(c) (cf. Arnold [Ar1]) The number of components of $\mathbb{G}\mathcal{M}_n$ equals the number of *snake sequences* (this means, for fixed $w_1, ..., w_n$, the number of snake sequences $u_1, ..., u_n$ that can be obtained by permuting $w_1, ..., w_n$).

(d) (cf. [Ar1],[C-P]) The number of components b_n of $\mathcal{L}_{n,\mathbb{R}}$ gives rise to the following exponential generating function :

$$2\Sigma_n(b_n/n!)t^n = \int 4/(1-\sin(2t)) = 2(\sec(2t)+\tan(2t)).$$

e) the number of snake sequences equals the number of isotopy classes of lemniscate configurations multiplied by 2.

The above result is related to a curious rediscovery of Riemann's existence theorem, done by Thom in 1960 ([Thom]) In fact, in 1957 C. Davis ([Da]) showed in particular that for each choice of n distinct real numbers there is a real polynomial of degree $(n+1)$ having those as critical values (in fact, up to affine transformations in the source, a unique one for each snake sequence formed with those numbers), and a similar question was asked for complex polynomials.

Thom remarked that by Riemann's existence theorem the answer is that for each choice of n distinct complex numbers and an equivalence class of admissible monodromy there exists exactly one polynomial,up to affine transformations in the source, having those points as critical values and the given monodromy.

In this paper we link the two answers by describing explicitly, even when the branch points are not all real, the monodromies which come from real polynomials.

In fact, in [C-P] it was shown also that every big lemniscate configuration occurs for some real polynomial for which the monodromy tree (cf.[C-W]) is linear (that is, homeomorphic to a segment).

Here, in a similar vein, we establish another result (which is essential in order to establish our main theorem), which allows us to understand the lemniscate configurations which come from real polynomials as the ones obtained from "snake" linear trees (theorem B stated below is an abridged version of theorems 2.1, 2.3 and 2.12) :

Theorem B.

Given $w_1, ..., w_n \in \mathbb{R}$ with $0 < |w_1| < ... < |w_n|$, there is a canonical choice of a geometric basis of $\pi_1(\mathbb{C} - \{w_1, ..., w_n\}, 0)$ such that the real lemniscate generic polynomials P having $w_1, ..., w_n$ as critical values, correspond exactly to the monodromy trees which are "snake" linear trees.

Also, for each fixed choice of $w_1, ..., w_n$ as above, if $n \geq 4$ there is some lemniscate configuration which cannot be obtained with a real polynomial.

To get the flavour of the second statement one should remark that the monodromies which come from real monic polynomials, (whose number is $K_n \sim O((2/\pi)^n(n)!)$) are quite few compared with all the possible monodromies, whose number is $(n+1)^{n-2}$. Nevertheless, since the number of lemniscate configurations is exactly K_n, we initially hoped that there would be a bijection between the set of real monodromies and the set of lemniscate configurations.

From theorem 2.1 it is then easily seen that, fixing the (real) critical values, and a linear tree in the canonical basis, the snake condition is equivalent to the condition that the associated polynomial is real.

In this way part b) of the main theorem is proven.

Finally, the proof of a) of the main theorem is a straightforward consequence of Lefschetz' fixed points theorem, while c) follows from the quoted result of Davis, which we reprove (in 2.3) with a small precision, for the sake of completeness.

Parts d),e) follow then from a),b),c) and the results of [Ar1],[C-P].

Section 2 contains also other miscellaneous results.

In the third section we employ the branch points map used by several authors ([Da],[Lo],[Ly],[C-W],[C-P],[Ar3]) in order to give a quick proof of a generalization of Davis' theorem along the same lines. Later on, we prove in theorem 3.7 a much more precise result, namely that the monodromies of real generic polynomials are given, in a canonical basis, by trees obtained from a snake linear trees by adding, in a symmetric way, pairs of isomorphic trees

(see section 3 for a more precise statement). From this result it is possible to calculate the number of connected components of the space of real generic polynomials of degree equal to $n+1$, but we have not yet found a simple formula for it.
The proof that we give of 3.7 is completely algebraic, implies in particular a third proof of the quoted theorem of Davis (after the ones given in [Da], [Ar2], and in 2.1, 2.3), and is susceptible of generalizations.

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1. Notation, set-up, preliminary results.

(1.1) Definition.
a) Let $P \in \mathbb{R}[z]$ be a polynomial of degree $(n+1)$: P is said to be maximally real if all the critical points of P (the roots $y_1, ..., y_n$ of its derivative) are real. We let $\mathcal{M}_n$ be the closed set of maximally real polynomials. Its interior $\mathcal{M}'_n$ corresponds to the polynomials with real distinct critical points and contains the open set $\mathbb{G}\mathcal{M}_n$ of the maximally real polynomials which are also generic, i.e., are such that the branch points of P, wiz., the real numbers $u_i = P(y_i)$, are distinct.
b) If P is maximally real there is a standard ordering $y_1 \leq ... \leq y_n$ of the critical points, hence we have, for P as above, also a canonical ordering $u_1 = P(y_1), ..., u_n = P(y_n)$ of the branch points, which we shall call the *source ordering.*

(1.2) Definition-remark.
i) A sequence $u_1, ..., u_n$ of real numbers is said to be a *weak up-down sequence* if $(-1)^i(u_i - u_{i+1}) \leq 0$, a *weak down-up sequence* if $(-1)^i(u_i - u_{i+1}) \geq 0$, a *weak snake sequence* if one of the two above holds. A *snake sequence* will be a weak snake sequence where $u_i \neq u_{i+1}$ for each i.

ii) if P is a maximally real polynomial, then its branch points $u_1 = P(y_1), \ldots, u_n = P(y_n)$, taken with the source ordering, yield a weak snake sequence.

(1.3) Definition.

a) A polynomial $P \in \mathbb{C}[z]$ of degree $(n+1)$ is said to be *generic* iff it has n distinct branch points.

b) P is moreover said to be lemniscate-generic if the branch points have n distinct absolute values different from 0.

c) If P is lemniscate generic, then there is a standard ordering for the branch points, by which we get another sequence $w_1, \ldots, w_n$ with $0 < |w_1| < \ldots < |w_n|$.

(1.4) Remark.

A polynomial $P \in \mathbb{R}[z]$ which is lemniscate generic is automatically maximally real, and there are three distinct orderings for the set of its branch points, the source, the standard and the target ordering (the first never coincides with the last).

We want to define the *Hurwitz space* $\mathcal{H}_n$ of polynomials. To do this, we consider the notion of source equivalence.

(1.5) Definition.

i) Two polynomials $P, Q \in \mathbb{C}[z]$ are said to be *source equivalent* $(P \sim Q)$ iff there exists an isomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ $(\varphi \in A(1, \mathbb{C}))$ such that $Q = P \circ \varphi$.

ii) The Hurwitz space $\mathcal{H}_n$ of polynomials is the quotient $V_n / A(1, \mathbb{C})$ of the space V_n of polynomials of degree $(n+1)$ in $\mathbb{C}[z]$, by the relation of source equivalence.

We want now to define the real part of the Hurwitz space.

In order to do it, let us observe that the operation $P \to \bar{P}$ of complex conjugation of coefficients of P passes to the quotient, since if $Q = P \circ \varphi$, then $\bar{Q} = \bar{P} \circ \bar{\varphi}$, as it is easy to verify.

The fixed locus for complex conjugation is given by $V_n \cap \mathbb{R}[z]$, and the next proposition determines the fixed locus inside $\mathcal{H}_n$.

(1.6) Proposition.

If $Q \in \mathbb{C}[z]$ is (source-) equivalent to $\bar{Q}$, then Q is equivalent to a real polynomial $P \in \mathbb{R}[z]$. Whence the real part $\mathcal{H}_{n,\mathbb{R}}$ of the Hurwitz space $\mathcal{H}_n$ is indeed the image to the quotient of $V_n \cap \mathbb{R}[z]$.

Proof. We can clearly replace Q by any other polynomial which is equivalent to Q, and therefore we can assume that Q is of the form

$$Q = z^{n+1} + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots\dots + a_0.$$

If $\varphi(z) = \alpha z + \beta$, and $\bar{Q} = Q \circ \varphi$, then we immediately get $\alpha^{n+1} = 1$, and $\beta = 0$.

Let a be a square root of α, so that $\alpha = a/\bar{a}$, and set $P(z) = Q(az)$.

Then $\bar{P}(z) = \bar{Q}(\bar{a}z) = Q(\alpha\bar{a}z) = Q(az) = P(z)$.

Q.E.D.

Unfortunately, it is not true that two real polynomials P,Q are equivalent iff there exists a φ in $A(1,\mathbb{R})$ with $Q = P \circ \varphi$, as it is shown by the example of $P = z^4 + z^2 + 1$, $Q = z^4 - z^2 + 1$.
But this holds true if the polynomials are generic :

(1.7) Proposition.

If $P, Q \in \mathbb{R}[z]$ are $A(1,\mathbb{C})$ equivalent, then they are $A(1,\mathbb{R})$ equivalent if they cannot be written as a composition of two polynomial maps of strictly lower degree. In particular, if U_n is the open set of generic polynomials, then the image to the quotient of $U_n \cap \mathbb{R}[z]$ is the quotient $(U_n \cap \mathbb{R}[z])/A(1,\mathbb{R})$.

Proof. We can clearly assume, replacing P and Q by $A(1,\mathbb{R})$ equivalent polynomials, that P and Q are of the following *"Tschirnhausen"* form

$$P = \pm z^{n+1} + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots\dots + a_0,$$

$$Q = \pm z^{n+1} + b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \dots\dots + b_0.$$

Since there are $\alpha \neq 0$, $\beta \in \mathbb{C}$ such that $Q(z) = P(\alpha z + \beta)$ we get as before $\beta = 0, \alpha^{n+1} = \pm 1$.

Then $b_i = \alpha^i a_i$, whence $\alpha^i \in \mathbb{R}$ whenever $a_i \neq 0$.

If $\alpha \in \mathbb{R}$, we are done, else, there is a minimal m such that $\alpha^m \in \mathbb{R}$, and $a_i = 0$ if i is not divisible by m. In the latter case there is a polynomial R of degree $(n+1)/m$ such that $P(z) = R(z^m)$.
The proof is over, since a polynomial of the form $R(z^m)$ can only be generic if $m = 2$ and R is linear : but in this case $\alpha \in \mathbb{R}$. Q.E.D.

(1.8) Corollary.

The generic real Hurwitz space of polynomials, that is, the quotient $(U_n \cap \mathbb{R}[z])/A(1,\mathbb{R})$, is isomorphic to the quotient of the space $T_{n,\mathbb{R}}$ of generic real Tschirnhausen polynomials

$\{P | P(z) = \pm z^{n+1} + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots\dots + a_0\ ,\ a_i \in \mathbb{R},$

and P is a generic polynomial$\}$

by the involution ι which sends $P(z)$ to $P(-z)$. In particular, for n even, the generic real Hurwitz space is isomorphic to the space of monic generic real Tschirnhausen polynomials.
The quotient $\mathbb{R}[z]/A(1,\mathbb{R})$ is also isomorphic to the quotient of the space $N_{n,\mathbb{R}}$ of normalized real polynomials

$$\{P | P(z) = \pm z^{n+1} + a_n z^n + a_{n-1}z^{n-1} + \dots\dots + a_1 z, a_i \in \mathbb{R}\}$$

by the involution ι which sends $P(z)$ to $P(-z)$.

Proof.

The first assertion was already proven, the second follows in an entirely similar way.

(1.9) Remark.

$U_n/A(1,\mathbb{C})$ is an open set in $V_n/A(1,\mathbb{C}) \cong T_n/\mu_{n+1}$, where T_n is the space of complex (Tschirnhausen) polynomials of the form

$$P(z) = z^{n+1} + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots\dots + a_0,$$

and where μ_{n+1} is the group of $(n+1)^{\text{th}}$ roots of unity in $\mathbb{C}$. The difference $T_n - U_n$ is called the bifurcation hypersurface Δ_n (cf. [C–W]).
The group extension associated to the Galois cover $T_n - \Delta_n \to U_n/A(1,\mathbb{C}) \cong (T_n - \Delta_n)/\mu_{n+1}$ is described in the main theorem of [C-W].

(1.10) Definition-remark.

i) Let us denote by $\mathcal{H}'_n$ be the generic Hurwitz space, i.e., the quotient $U_n/A(1,\mathbb{C})$, and by $\mathcal{H}'_{n,\mathbb{R}}$ its real part. If we set moreover $U_{n,\mathbb{R}} = U_n \cap \mathbb{R}[z]$, then $\mathcal{H}'_{n,\mathbb{R}} \cong U_{n,\mathbb{R}}/A(1,\mathbb{R})$.

ii) Inside $U_{n,\mathbb{R}}$ we let $\mathbb{G}\mathcal{M}_n$ be the subset of those generic polynomials for which the critical points (or, equivalently, the critical values) are real. It is clear that $\mathbb{G}\mathcal{M}_n$ is a union of connected components of $U_n \cap \mathbb{R}[z]$, and that each component of $\mathbb{G}\mathcal{M}_n$ is made up of $A(1,\mathbb{R})$–orbits. The polynomials in $\mathbb{G}\mathcal{M}_n$ are said to be *maximally real and generic.*

iii) Let $\mathcal{L}_n$ be the open set in U_n consisting of lemniscate generic complex polynomials, and let $\mathcal{L}_{n,\mathbb{R}}$ be its real part. Clearly these open sets are made of equivalence classes, whence one can define the lemniscate generic Hurwitz space $\mathcal{L}\mathcal{H}_n$, and similarly its real part $\mathcal{L}\mathcal{H}_{n,\mathbb{R}}$.

In [C-W] and [C-P] (where, though, $\mathcal{H}'_n$ was denoted Z_n) a key importance had the study of the critical value fibration, associating to a generic polynomial P the unordered set of its n critical values :

(1.11)

$$\psi_n : \mathcal{H}'_n \to W_n = \{B = \{w_1, ..., w_n\} | w_i \in \mathbb{C}, \text{ and } w_i \neq w_j \text{ for } i \neq j\}.$$

We recall some definitions and results from the two cited papers, which are a consequence of Riemann's existence theorem

(1.12) Results and definitions concerning the critical value fibration.

a) $\psi_n : \mathcal{H}'_n \to W_n$ is an unramified covering space whose fibre over B is the set of conjugacy classes $[\mu]$ of monodromies $\mu : \pi_1(\mathbb{C} - B, x_0) \to \mathcal{S}_{n+1}$, such that the image of μ is a transitive subgroup, and each element of a geometric basis of $\pi_1(\mathbb{C} - B, x_0)$ is mapped to a transposition. Here, two homomorphisms μ and μ' as above are said to be in the same conjugacy class iff there exists an inner automorphism φ of $\mathcal{S}_{n+1}$, such that $\mu = D\varphi \circ \mu' \circ \varphi^{-1}$; and a geometric basis is a basis of n loops γ_i $(i = 1, ..., n)$ formed by a segment joining x_0 with a small circle around w_i.

b) Since the group $\mathcal{B}_n = \pi_1(W_n, \{1, ..., n\})$, called Artin's braid group, acts (cf.[Bir]) as a group of automorphisms of $\pi_1(\mathbb{C} - \{1, ..., n\}, x_0)$, the monodromy of ψ_n is such that σ sends $[\mu]$ to $[\mu \circ \sigma^{-1}]$.

c) the elements in a fibre of ψ_n, once a geometric basis $\gamma_1, ..\gamma_n$ for $\pi_1(\mathbb{C} - B, x_0)$ has been fixed, can be put in a bijective correspondence with E_n, the set of isomorphism classes of edge labelled trees with n edges (we take $(n+1)$ unlabelled vertices which represent the set $P^{-1}(x_0)$, and we adjoin n edges, labelled by an integer from 1 to n, corresponding to the transpositions $\mu(\gamma_i)$, and joining the two vertices moved by the above transposition).

d) Let $Y_n \subset W_n$ be the subset $\{\{w_1, ..., w_n\} | 0 < |w_1| < ... < |w_n|\}$, so that $\mathcal{L}_n = \psi_n^{-1}(Y_n)$.
Let Λ_n be the image of $\pi_1(Y_n, \{1, ..., n\}) \to \pi_1(W_n, \{1, ..., n\})$: then the connected components of $\mathcal{L}_n$ correspond to the Λ_n–orbits on E_n.

(1.13) Remark.
i) Writing $r_i = |w_i| - |w_{i-1}|$ and $\eta_i = w_i/|w_i|$, we see that Y_n is homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$, hence $\pi_1(Y_n) \cong \mathbb{Z}^n$. The images T_j of the generators of $\pi_1(Y_n)$ are the braids, which keep fixed the points $1, ..., n$ different from j, and move j in a circle around the origin $(t \mapsto e^{2\pi i t} j)$.

ii) each connected component of $\mathcal{LH}_n$, being a finite connected covering of Y_n, is also homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$.

iii) the real part of $Y_n, Y_{n,\mathbb{R}} = \{(w_1, ..., w_n) \in \mathbb{R}^n : 0 < |w_1| < ... < |w_n|\}$ is homeomorphic to $\{-1, +1\}^n \times (\mathbb{R}^+)^n$.

iv) ψ_n commutes with complex conjugation.

From the last part of the previous remark it follows that ψ_n carries the real part $\mathcal{LH}_{n,\mathbb{R}}$ of the lemniscate generic Hurwitz space to $Y_{n,\mathbb{R}}$, but we are going to see soon that $\mathcal{LH}_{n,\mathbb{R}}$ is far from being the full inverse image of $Y_{n,\mathbb{R}}$, which consists of $(n+1)^{n-2}2^n$ disjoint copies of $\mathbb{R}^+$.

(1.14) Lemma.
Each connected component A of $\mathcal{LH}_n$ contains exactly 2^n connected components of $\mathcal{LH}_{n,\mathbb{R}}$.

Proof. Each connected component A of $\mathcal{LH}_n$ is homeomorphic to $(S^1)^n \times (\mathbb{R}^+)^n$, and for each point $r = (r_1, ..., r_n) \in (\mathbb{R}^+)^n$, the set $A_r \cong (S^1)^n \times \{r\}$, which is invariant by conjugation, contains only a finite number of self conjugate points.

We apply now the Lefschetz's fixed point formula to f = complex conjugation on A_r.
Since A_r is a covering of $(S^1)^n$, and f induces, via the covering projection, the standard conjugation on $(S^1)^n$, we see immediately that f acts as -1 on $H_1(A_r, \mathbb{Z})$.
Thus, the number of fixed points of f on A_r is exactly 2^n.
Now, the real part of A is a closed submanifold of A, whence, it union of components of $A \cap \psi_n^{-1}(Y_{n,\mathbb{R}})$, which is a trivial covering of $Y_{n,\mathbb{R}}$. Our assertion follows then immediately.

Q.E.D.

(1.15) Corollary

For each connected component A of $\mathcal{LH}_n$ the restriction φ_n of ψ_n to the real part of A is injective to $Y_{n,\mathbb{R}}$ if and only if it maps surjectively to $Y_{n,\mathbb{R}}$.

§2. Statement and proof of the main theorems.

In this section, before giving a proof of the main theorem, we will give a characterization of the monodromies of maximally real polynomials as "snake" linear trees.

This will be done geometrically, whereas a second proof, of algebraic nature, will be given in section three, where we will more generally characterize the monodromies of real generic polynomials (identifying them as self conjugate monodromies).

(2.1) Theorem.

Let $(w_1, ..., w_n) \in Y_{n,\mathbb{R}}$ (thus $w_i \in \mathbb{R}$, and $0 < |w_1| < ... < |w_n|$). Then there is a canonical choice of a geometric basis of $\pi_1(\mathbb{C} - \{w_1, ..., w_n\}, 0)$ such that for each real lemniscate generic polynomial P having $w_1, ..., w_n$ as critical values, the edge labelled monodromy tree $\mathcal{T}$ of P can be determined as follows.
Let $y_1 < ... < y_n$ be the critical points of P, let $u_1 = P(y_1), ..., u_n = P(y_n)$ be the snake of its critical values, and let moreover σ be the permutation such that $u_i = w_{\sigma(i)}$.
Then the tree $\mathcal{T}$ is a linear tree consisting of n consecutive segments with

labels (from left to right) $\sigma(1), \ldots, \sigma(n)$.

(2.2) Choice of the canonical basis (see figure 1)

Let Ξ be the planar graph consisting of the union of n circumpherences χ_i, of radius $\epsilon << 1$ and with centres in the n points w_i, together with the complement in $\mathbb{R}$ of n open intervals of radius ϵ centered around the n points w_i.

Clearly $\mathbb{C} - \{w_1, \ldots, w_n\}$ is homotopy equivalent to Ξ, thus it suffices to choose the geometric basis inside $\pi_1(\Xi, 0)$.

Let γ_i be the loop based at 0 which consists of a *"right turning"* symplicial path δ_i from 0 to $P_i = (w_i - \epsilon[\operatorname{sign} w_i])$, followed by χ_i run counterclockwise, and finally followed by the inverse of δ_i. Here, a symplicial path is said to be right turning if, whenever the path, after following an edge, comes to a node, then takes as next edge the one to the right. We might observe that the inverse of a right turning path is left turning.

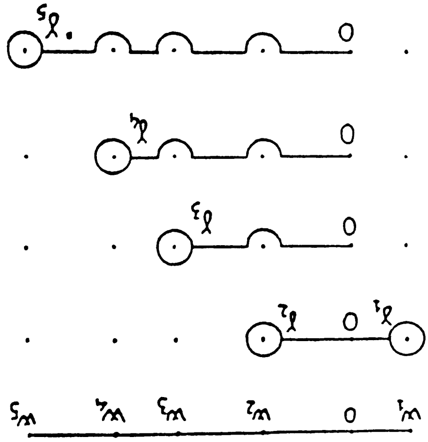

Figure 1 : Choice of the canonical basis for $\pi_1(\mathbb{C} - \{w_1, \ldots, w_5\}, 0)$

Proof of theorem 2.1

Consider P as a map of $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself, and consider the graph $\Theta = P^{-1}(\mathbb{P}^1_{\mathbb{R}})$. We consider $\mathbb{P}^1_{\mathbb{R}}$ as a graph with vertices $w_1, \ldots, w_n$ and ∞, and therefore also $(n+1)$ edges. Letting as usual the weight of a vertex be the number of edges stemming from it,Θ has one vertex (∞) of weight $2(n+1)$, n vertices of weight 4 at the critical points $y_1, \ldots, y_n$ and all the other vertices of weight 2.

Whence, an easy calculation yields the number $(-2n)$ for the topological Euler-Poincare's characteristic $\chi(\Theta)$.

Let us now disregard the vertices of weight 2 in Θ, and remark that Θ contains $\mathbb{P}^1_{\mathbb{R}}$. We are left then with $(3n+1)$ edges, $(n+1)$ of which are intervals in $\mathbb{P}^1_{\mathbb{R}}$. The remaining $2n$ are in conjugate pairs, each contained either in the upper or in the lower half plane. Therefore through each critical point y_i passes exactly one edge E_i contained in the upper half plane.

Claim : We contend that the other end point of E_i must be ∞ (compare figure 2).

In fact, otherwise, the other end point of E_i should be a critical point y_j, with $i \neq j$. We can clearly assume $i < j$, and we shall see that if $j = i+1$ we have a contradiction. In fact, in this case we would have three edges, namely E_i, its conjugate, and the interval $[y_i, y_j]$ mapping to the interval with ends u_i, u_j and not containing ∞. But this contradicts the local structure of the map P at the simple critical point y_i (the local degree is 2). If instead, $j > i+1$, since the y_i's and ∞ are the only singular points of yΘ, the other end point of E_{i+1} must be a critical point y_k, with $i+1 < k < j$. By induction on $|j-i|$, we finally find a contradiction.

Q.E.D for the claim.

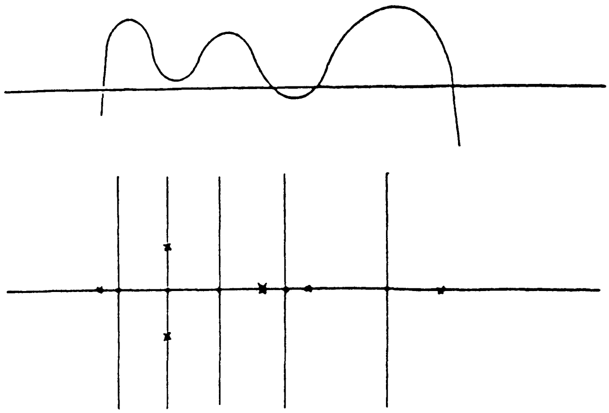

Figure 2: A polynomial of degree 6 and its graph $\Theta = P^{-1}(\mathbb{P}^1_{\mathbb{R}})$.

The critical points of P are partitioned into two sets : the set of local minima for $|P|_{|\mathbb{R}}$, and the set of local maxima for $|P|_{|\mathbb{R}}$. If y_i is a local minimum, then the edge E_i maps bijectively to the interval with ends u_i and ∞ which contains 0 ; if instead y_i is a local maximum, then the edge E_i maps bijectively to the interval with ends u_i and ∞ which does not contain 0. We clearly have a pair of conjugate roots of P for each local minimum of $|P|_{|\mathbb{R}}$, and all the remaining roots are real.

Observe moreover that we have a real root exactly in each interval in $\mathbb{R}$ between two consecutive maxima of $|P|_{|\mathbb{R}}$, and that one cannot have two consecutive minima.

In order to describe the monodromy μ of P, we want to determine explicitly the transposition τ_i of the roots of P obtained by the liftings of the path γ_i described above.

Clearly a lifting of γ_i is contained in the graph $P^{-1}(\Xi)$ (see figure 3) . We remark that since P is orientation preserving, the lifting of a right turning path will be right turning too.

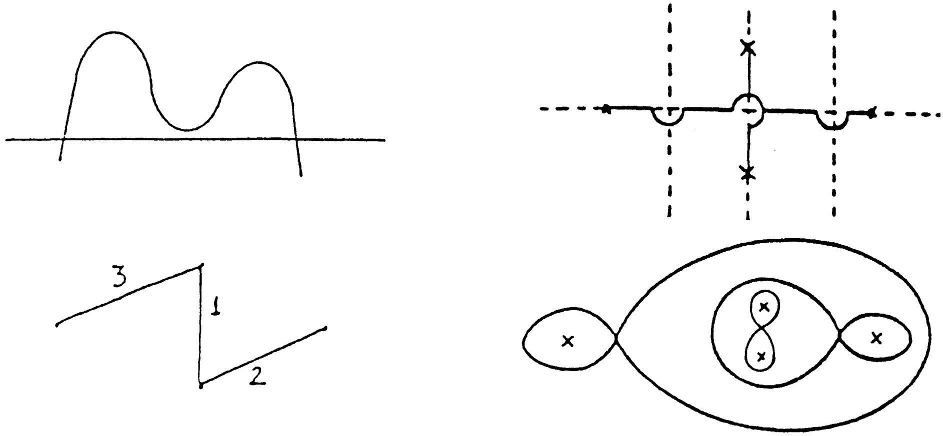

Figure 3 : A polynomial of degree 4, part of the graph $P^{-1}(\Xi)$, the associated monodromy tree, the lemniscate configuration.

(2.2) Sublemma

1) If y_i is a local minimum of $|P|_{|\mathbb{R}}$, the corresponding $\tau_{\sigma(i)}$ permutes the pair of conjugate roots of P lying on E_i and its conjugate.

2) If y_i is a local maximum, as well as y_{i+1}, y_{i-1}, $\tau_{\sigma(i)}$ permutes the pair of real roots lying in the two intervals with endpoint y_i.

3) If y_i is a local maximum, but y_{i+1}, y_{i-1} are local minima, $\tau_{\sigma(i)}$ permutes the non real root in E_{i+1} with the non real root in the conjugate of E_{i-1}.

4) In the remaining cases, $\tau_{\sigma(i)}$ permutes the neighbouring real root with the non real root in the union of E_{i+1} with the conjugate of E_{i-1}.

Proof.

Let ξ_i be the small circle around y_i which is the local inverse image of χ_i. Then $P^{-1}(\Xi)$ has four nodes on ξ_i which partition it into 4 arcs, each mapping to a semicircle in χ_i.
These nodes are called upper, left, lower, right, with obvious meaning.

1) Lifting the path δ_i with initial point the root on E_i, we end up to the upper node, then lifting χ_i we end up in the lower node, finally the lifting of the inverse of δ_i gives the conjugate of the first part of the path, therefore the end point is the conjugate root of the one we started with.

2) Lifting the path δ_i with initial point the real root on the left of y_i, we end up to the left node, then lifting χ_i we end up in the right node, finally lifting the inverse of δ_i we get to the real root on the right of y_i.
In the remaining two cases the situation changes since we have to lift some semicircles to a neighbourhood of a critical point, whence the lifts will be one of the above mentioned 4 arcs around the critical points.

3) We lift the path δ_i with initial point the root on the conjugate of E_{i-1},thus we end in the left node around y_i, since when we approach y_{i-1} we have to turn right, then we proceed to the right node : when we approach y_{i+1} we have to turn left, thus we end up to the non real root in E_{i+1}.

4) The proof is similar to case 3 : if we approach y_{i-1} we have to turn right, if instead we approach y_{i+1} we have to turn left.

Q.E.D. for the Sublemma.

In order to finish the proof of theorem 2.1, we recall that the roots of our polynomial P are partitioned as follows :

a) a conjugate pair is associated to critical points which are local minima of

$|P|_{|\mathbf{R}}$,

b) a single real root is associated to a sequence of consecutive (in the source ordering) local maxima of $|P|_{|\mathbf{R}}$,

c) a single real root is associated to any local maximum of $|P|_{|\mathbf{R}}$, which is either y_1 or y_n.
Recall also that one cannot have two consecutive local minima. We associate to P the linear edge labelled tree $\mathcal{T}$ consisting of n consecutive segments with labels (from left to right) $\sigma(1), ..., \sigma(n)$.
We take a bijection of the roots of P with the vertices of $\mathcal{T}$ as follows

β) assume that y_i, y_{i+1} are local maxima for $|P|_{|\mathbf{R}}$: then to the root corresponding according to b) we associate the vertex v lying between the edges labelled $\sigma(i)$ and $\sigma(i+1)$

α) if y_i is a local minimum for $|P|_{|\mathbf{R}}$, we take any bijection between the two roots associated according to a) and the two vertices of the edge labelled $\sigma(i)$

γ) if y_1 (resp. y_n) is a local maximum for $|P|_{|\mathbf{R}}$, the root corresponding according to c) will be associated to the end of $\sigma(1)$ (resp. : $\sigma(n)$).
According to the meaning of the monodromy graph, and by sublemma 2.2 the monodromy of $\gamma_{\sigma(i)}$ is the transposition $\tau_{\sigma(i)}$ permuting the two vertices of the edge labelled $\sigma(i)$.

Q.E.D.

For the reader's convenience, we reformulate in our context the result of Davis quoted in the introduction (with essentially the same proof).

(2.3) Theorem (C.Davis, cf. [Da])

For each weak snake sequence $u_1, ..., u_n$ of real numbers, there exists exactly one maximally real Tschirnhausen polynomial, and exactly one (maximally real) normalized polynomial whose snake sequence of critical values is the given one.
In particular, if $u_1, ..., u_n$ are "lemniscate generic" (i.e., there is a permutation σ such that if $u_i = w_{\sigma(i)}$, then $0 < |w_1| < ... < |w_n|$) each linear monodromy

tree as in theorem 2.1 comes from a real polynomial.

Proof. Let us prove the assertion first in the case where we have a snake sequence (thus u_i, u_{i+1} are distinct).
A first remark is that the snake sequence associated to $P(-z)$ is the reverse of the snake sequence associated to P (that is, $u_n, ..., u_1$) whence the snake sequence is only $A^+(1,\mathbb{R})$–invariant (moreover, by corollary 1.8, every normalized polynomial is $A^+(1,\mathbb{R})$–equivalent to a unique Tschirnhausen one.
Remark that the space of normalized maximally real polynomials has two components, mapping to the space of up-down, respectively down-up sequences.
The crucial point is that a maximally real polynomial determines a natural source ordering of the critical points $y_1, ..., y_n$ thus the space of monic normalized maximally real polynomials is isomorphic to a subspace of the space of complex monic normalized polynomials taken together with an ordering of the critical points.
More precisely, we have

$$\mathcal{C} = \{(y_1, ..., y_n) | y_i \in \mathbb{R} \ , \ y_1 < y_2 ... < y_n\} \cong \mathbb{R} \times (\mathbb{R}^+)^{n-1}$$

inside $\{(y_1, ..., y_n) | y_i \in \mathbb{C}\} \cong \mathbb{C}^n$.
There is a surjective polynomial map, homogeneous of degree $(n + 1)$, $\beta : \mathbb{C}^n \to \mathbb{C}^n$ associating to $(y_1, ..., y_n)$ the branch points $u_i = P_y(y_i)$ of the normalized polynomial $P_y = \int(\prod_{i=1,..n}(z - y_i))$. The claim is that β (or $-\beta$, if n is even) maps the above $\mathcal{C} = \mathbb{R} \times (\mathbb{R}^+)^{n-1}$ to the space $\mathcal{V}$ of up-down sequences, which is again $\mathbb{R} \times (\mathbb{R}^+)^{n-1}$. A first remark is that β is unramified on the open set of $\mathbb{C}^n$ where all the y_i's are distinct . This follows from Riemann's existence theorem, which shows indeed more, as follows.
If all the y_i's are distinct, once a geometric basis for $\pi_1(\mathbb{C} - \{u_1, ..., u_n\})$ is fixed, to each y_i is associated a transposition τ_i, and if $u_i = u_j$, the transpositions τ_i and τ_j are disjoint. Conversely, given $(u_1^\circ, ..., u_n^\circ)$, we can give in a continuous way, for each $(u_1, ..., u_n)$ in a neighbourhood of $(u_1^\circ, ..., u_n^\circ)$, a homomorphism of a free group in n elements to $\pi_1(\mathbb{C} - \{u_1, ..., u_n\})$ taking the i-th generator to a geometric loop around u_i.
For each choice of the τ_i's as above, we can use the monodromy determined by the products of the τ_i's, to construct a continuous family (parametrized

by the points $(u_1, ..., u_n)$) of Riemann surfaces isomorphic to $\mathbb{C}$, together with source classes of pointed polynomial maps (this means, with a choice of a fixed point over the base point for the fundamental group), and an ordering of the (distinct) critical points.

Therefore the intersection of the inverse image of the given neighbourhood U of $(u_1^\circ, ..., u_n^\circ)$ with the open set where the y_i's are distinct is homeomorphic to a product of U with a finite discrete space. A second remark is that β is closed (in fact, being homogeneous, it induces a map between the corresponding projective spaces, and we can use the compactness of projective space).

A third remark is that β is finite, as it follows from Riemann's existence theorem.

Since β is unramified on $\mathcal{C}, \beta(\mathcal{C})$ is an open set in $\mathcal{V}$. Since β is closed, the image of the closure of $\mathcal{C}$ is closed. If $\beta(\mathcal{C})$ would not be the entire $\mathcal{V}$, there would be a point in the closure of $\mathcal{C}$ mapping to the interior of $\mathcal{V}$. But this is a contradiction, since obviously if u_i, u_{i+1} are distinct, also y_i, y_{i+1} are distinct.

We have thus proven that $\beta : \mathcal{C} \to \mathcal{V}$ is unramified, surjective, closed, whence it is a covering map. Since $\mathcal{V}$ is simply connected, $\beta : \mathcal{C} \to \mathcal{V}$ is a homeomorphism.

In the general case when some u_i, u_{i+1} are not distinct, observe that since β is closed, β maps the closure of $\mathcal{C}$, $\bar{\mathcal{C}}$ to the closure $\bar{\mathcal{V}}$ of $\mathcal{V}$; thus surjectivity is proven in general. Unicity follows since $\bar{\mathcal{C}}$ maps surjectively via a proper and finite map to $\bar{\mathcal{V}}$: the general fibre is one point, thus connected, therefore any fibre is connected, thus reduced to one point.

The last assertion follows immediately from theorem 2.1.

Q.E.D.

(2.4) Definition.

Given distinct real numbers $t_1 < t_2 < ... < t_n$ we consider the number K_n of up–down sequences that can be formed out of $t_1, ..., t_n$. It is easy to see that this number is independent of the choice of $t_1, ..., t_n$. In fact there is a bijection of the above up–down sequences with the set of permutations σ of $\{1, ..., n\}$ such that $\sigma(1) > \sigma(2) < \sigma(3) > ...$, which we will call *up-down (abstract) snakes.* Similarly we can define down-up snakes, snakes, and then the number for down-up sequences formed with $t_1, ...t_n$ is equal to K_n, and the number of snakes is $2K_n$.

(2.5) Remark.

The number K_n of up-down snakes equals the number of connected components of the open set $\mathcal{W}$, in the space $\mathcal{V}$ ($\mathcal{V} \cong (\mathbf{R} \times (\mathbf{R}^+)^{n-1})$) of up–down sequences, given by the sequences $u_1, ..., u_n$ where all the u_i's are distinct.

(2.6) Main Theorem.

(a) Each connected component of $\mathcal{L}_n$ contains exactly 2^{n+1} connected components of $\mathcal{L}_{n,\mathbf{R}}$.

(b) The number of connected components of $\mathcal{L}_{n,\mathbf{R}}$ mapping to a fixed component of $Y_{n,\mathbf{R}}$ equals the number $2K_n$ of snakes, whence the number of connected components of $\mathcal{L}_n$ equals K_n.

(c) (cf. Arnold [Ar1]) the number of connected components of $\mathbb{G}\mathcal{M}_n \cap$ {monic polynomials} equals the number K_n of up-down snakes ; the number of connected components of $\mathbb{G}\mathcal{M}_n$ equals $2K_n$.

(d) (cf. [Ar1],appendix to [C-P]) the number of components b_n of $\mathcal{L}_{n,\mathbb{R}}$ gives rise to the following exponential generating function :

$$2\Sigma_n(b_n/n!)t^n = \int 4/(1 - \sin(2t)) = 2(\sec(2t) + \tan(2t)).$$

(e) (cf. appendix to [C-P]) the number K_n of up-down snakes (which by b) equals the number of components of $\mathcal{L}_n$) is equal to the number of sequences $x_0, ..., x_{n-2}$, such that x_i is an integer with $0 \leq x_i \leq i$, and such that for each integer m there are at most two i's with $x_i = m$.

Proof. Recall that the number of connected components of $\mathcal{L}_n$ equals the number of connected components of the quotient $\mathcal{LH}_n$, whereas the inverse image of any connected component of $\mathcal{LH}_{n,\mathbf{R}}$ consists (cf. 1.8) of 2 connected components of $\mathcal{L}_{n,\mathbf{R}}$.

Therefore a) is an immediate consequence of lemma (1.14).

To prove b), it suffices to show that the number of connected components of $\mathcal{LH}_{n,\mathbb{R}}$ mapping to a fixed component of $Y_{n,\mathbb{R}}$ equals K_n. But this follows from the last assertion of theorem (2.3), since two snake sequences yield isomorphic linear trees (according to 2.1) if and only if they are the reverse of each other (although not needed, we recall that if $P(z)$ yields a snake sequence, $P(-z)$ yields the reverse snake sequence, and that $P(z)$ and $P(-z)$ are source equivalent).
Then the number α_n of connected components of $\mathcal{L}_n$ equals K_n since by a) the number of connected components of $\mathcal{L}_{n,\mathbb{R}}$ equals $2^{n+1}\alpha_n$, while, by what we have just seen, it equals $2K_n$ times the number of components of $Y_{n,\mathbb{R}}$, which is 2^n. To prove c), recall that Davis' theorem 2.3 shows that the map which associates to a polynomial with distinct real critical points its snake sequence of branch points yields a homeomorphism of the quotient $\mathbb{G}\mathcal{M}_n/A^+(1,\mathbb{R})$ with the space $\mathcal{W}'$ of snake sequences formed of n distinct points. Whence, the number of connected components of $\mathbb{G}\mathcal{M}_n$ equals the number of connected components of $\mathcal{W}'$. But $\mathcal{W}'$ is just given by two disjoint copies of the open set $\mathcal{W}$ considered in remark 2.5, which has K_n components. Thus c) is proven.
d) and e) follow immediately from a), b),c) and the cited papers. To avoid confusion, we only remark that K_n is denoted by a_{n-1} in [C-P], where it is proven that $\Sigma_n(a_n/n!)t^n = 1/(1-\sin(t))$, whereas [Ar1] shows that $\Sigma_n(K_n/n!)t^n = \sec(t)+\tan(t)$: a baby calculus verification shows d).

Q.E.D.

We want now to consider, for a given choice of critical values, the lemniscate configurations that can be obtained from real polynomials. Before doing this, we recall the connection between monodromy trees and lemniscate configurations.
The big lemniscate configuration Γ_P of a polynomial $P : \mathbb{C} \to \mathbb{C}$ is the union of the preimages of 0 under P together with the singular level sets of $|P|$.
Denoting by $\Delta_c = \{z \in \mathbb{C} : |P(z)| = c\}$, we have

$$\Gamma_P = P^{-1}(0) \cup \bigcup_{i=1,\dots,k} \Delta_{|w_i|}, \tag{2.7}$$

where $w_1, ..w_k$ are the critical values of P. If p_i is a critical point of multiplicity $m_i - 1$, the lemniscate $\Delta_{|w_i|}$ has a singularity consisting of m_i smooth curves

intersecting with angles π/m_i. In the case $m_i = 2$ this singularity is called a *node*.
If P is lemniscate generic, let $w_1, ..., w_n$ be the critical values of P with the usual order $0 < |w_1| < ... < |w_n|$. We have a monodromy edge labelled tree once a geometric basis $\gamma_1, ..., \gamma_n$ of $\pi_1(\mathbb{C} - \{w_1, ..., w_n\}, 0)$ is fixed.
In [C-P] (cf. also [B-C]) it was proven that the isotopy class of the embedding of Γ_P in $\mathbb{C}$ is completely determined by a rooted (connected) tree g whose vertices correspond to the connected components of Γ_P (the root corresponds to $\Delta_{|w_n|}$), and whose edges correspond to the connected components of $\cup_{i=0,...,k}\Delta_{|w_i|+\epsilon}$ (if we set $w_0 = 0$, and we choose $\epsilon > 0$ a sufficiently small real number such that $\epsilon < |w_1|$ and $|w_i| + \epsilon < |w_{i+1}|$).
The main theorem of [C-P] would in particular describe the class of graphs obtained from lemniscate configurations and show that there is a bijection between connected components of $\mathcal{L}_n$ and the isomorphism class of such trees.
To describe abstractly the correspondence associating to an edge labelled tree $\mathcal{T}$ the associated lemniscate rooted tree g, it was convenient (cf. ibidem) to give the following

(2.8) Definition

Given an edge labelled tree $\mathcal{T}$ with n edges, the k–skeleton $\mathcal{T}_k$ of $\mathcal{T}$ is the subgraph of $\mathcal{T}$ with the same vertices and with the edges whose label is $\leq k$.
To $\mathcal{T}$ one associates a rooted graph g, whose vertices correspond to the connected components $\mathcal{C}$ of the various skeleta $\mathcal{T}_k$, with $\mathcal{T}$ corresponding to the root, and with an edge connecting $\mathcal{C}$ and $\mathcal{C}'$ if $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{C}$ is a component of $\mathcal{T}_k$, $\mathcal{C}'$ is a component of $\mathcal{T}_{k+1}$.
The k–partition $\mathcal{P}_k$ of $\mathcal{T}$ is the partition of $\{1, ..k\}$ determined by the components of $\mathcal{T}_k$ of dimension 1.

(2.9) Remark.

Given two edge labelled trees $\mathcal{T}, \mathcal{T}'$ with n edges, they determine the same lemniscate tree g if and only if they determine, for each k, the same k–partition $\mathcal{P}_k$ (the proof of this statement is phrased in slightly different terms in [C-P], pages 630-631).

We restrict from now on to linear trees $\mathcal{T}$.

(2.10) Remarks.

1) There is a natural correspondence which associates to a permutation τ the linear edge labelled tree $\mathcal{T}$ with n edges labelled $\tau(1), ..., \tau(n)$ from left to right. This correspondence induces a bijection between the set of isomorphism classes of linear edge labelled trees with n edges and the family of left cosets in the symmetric group $\mathcal{S}_n$ for the subgroup of order two generated by the reflection r sending i to $n-i$.

2) Let $w_1, ..., w_n$ be "lemniscate generic" real numbers taken with the standard order $(0 < |w_1| < ... < |w_n|)$. Then there is a permutation ψ giving the target ordering of the given numbers $(t_1 < t_2 < ... < t_n)$, and $t_i = w_{\psi(i)}$.

Notice that ψ depends only upon the sign of w_i. The condition that the monodromy associated to τ comes from a real polynomial can thus be phrased by the condition that $\sigma = \psi^{-1} \circ \tau$ *is an abstract snake* (cf. 2.4).

(2.11) Corollary.

For each sequence of critical values $w_1, ..., w_n$ such that $0 < |w_1| < ... < |w_n|$, the lemniscate configurations of real polynomials are the image of the map from the set of n-snakes to the set of those nested partitions $\mathcal{P}_1, ...\mathcal{P}_n$ coming from lemniscate trees g, which associates to a snake σ (cf. 2.4) the nested partitions corresponding as in 2.8 to $\tau = \psi \circ \sigma$ (ψ is the permutation, as in 2.10)2, comparing the standard with the target ordering) .

In the statement of 2.11 we did not bother so much about specifying the image set : the main reason for this is that we know a priori that the above map factors through the equivalence relation $\sigma \sim \sigma \circ r$, thus we can view it as a map between $\{n\text{–snakes modulo reflection}\} \to \{\text{lemniscate configurations}\}$, where we know that both sets have cardinality K_n. Thus the lack of surjectivity will be measured by the lack of injectivity (1.15 states the same principle from the opposite point of view of fixing the lemniscate configuration and asking whether all choices of signs are achieved by a real polynomial yielding the given configuration).

(2.12) Theorem.

For $n \geq 4$ and for each sequence of real numbers $w_1, ..., w_n$ such that $0 < |w_1| < ... < |w_n|$, the lemniscate configurations of real polynomials having

$w_1, \ldots, w_n$ as critical values are not all the possible lemniscate configurations.

Proof. It will suffice, by the remarks we have just made, to exhibit two non-isomorphic edge labelled linear trees $\mathcal{T}, \mathcal{T}'$ yielding the same nested partitions. The rest of the proof follows by several steps :

2.13) Define an *inner reflection* to be the operation associating to a linear tree f as above the tree $\mathcal{T}'$ obtained by picking up a 1-dimensional component B of the k-skeleton and reversing it (that is, if B is a segment with labels $h_1, \ldots, h_b$ from left to right, B^r will be the segment with labels $h_b, \ldots, h_1$ from left to right).
Define an inner reflection to be even iff the number b is even, odd otherwise.

2.14) an inner reflection does not affect the associated nested partitions (whence, the associate lemniscate configuration remains the same)

2.15) We claim that if $\psi^{-1} \circ \tau$ is a snake σ, applying an inner reflection we get τ' and then $\psi^{-1} \circ \tau' = \sigma'$ is a snake if and only if the reflection is odd. In fact, defining $\psi^{-1}(B)$ as the segment with labels $\psi^{-1}(h_1), \ldots, \psi^{-1}(h_b), \psi^{-1}(B^r) = \psi^{-1}(B)^r$, therefore if $\psi^{-1}(B)$ is a snake also $\psi^{-1}(B)^r$ is a snake. The only problem to check whether σ' is a snake comes by comparing $\psi^{-1}(h_0)$ with $\psi^{-1}(h_1)$, and $\psi^{-1}(h_b)$ with $\psi^{-1}(h_{b+1})$, where h_0, h_{b+1} are the labels respectively preceding and following B. But since B is a component of the k skeleton, h_0, h_{b+1} are $> k$, whence for instance $\psi^{-1}(h_0)$ is either bigger than all of $\psi^{-1}(h_1), \ldots, \psi^{-1}(h_b)$, or smaller. Thus we have a snake if and only if either $\psi^{-1}(B)$ and $\psi^{-1}(B)^r$ are both up-down, or they are both down-up. But this clearly holds if and only if b is odd.

2.16) For each j with $1 \leq j \leq n$, there exists a snake σ with $\sigma(j) = n$. In fact w_n is either the biggest or the smallest of $w_1, \ldots, w_n$, and it suffices to observe that for each i, given arbitrary i distinct real numbers, it is possible to form with them an up-down sequence and also a down-up sequence (this applies after dividing $w_1, \ldots, w_{n-1}$ in two sets of respective cardinalities $(j-1)$ and $(n-j)$).

2.17) Let j be even, take τ such that the associated snake σ has $\sigma(j) = n$, and operate on the corresponding $\mathcal{T}$ the odd inner reflection corresponding to the segment B of $\mathcal{T}_{n-1}$ lying to the left of n. The resulting $\mathcal{T}'$ is obviously non isomorphic to $\mathcal{T}$, but it yields a snake.

Q.E.D.

(2.18) Remark. For $n = 3$ and for each sequence of real numbers $w_1, ..., w_n$ such that $0 < |w_1| < ... < |w_n|$, the 2 possible configurations are achieved by real polynomials.
For $n = 4$ we get 4 out of 5 for each choice of the signs of $w_1, ..., w_n$. The missing configuration varies.
For $n = 5$ we get 11 out of 16 configurations for each choice of the signs.
For $n = 6$ there is a choice for which we get 34 configurations out of 61, and a choice for which we get 37 ones.

From 1.15 and 2.12 follows immediately the following

(2.19) Remark. For $n \geq 4$ there exist lemniscate configurations g such that the signs of real numbers $w_1, ..., w_n$ (with $0 < |w_1| < ... < |w_n|$) which are the critical values of a real polynomial yielding the given configuration g are subject to some restrictions.
For $n \leq 5$ there exist configurations g such that no such restriction occurs.

Question : for which n does there exist a configuration g for which all the possible signs for $w_1, ..., w_n$ can be realized?

(2.20) Example. Given a lemniscate generic real polynomial P, we can compose P with a real affinity in the target. Clearly, if we replace P by $aP (a \in \mathbb{R}^*)$, then aP remains lemniscate generic and with the same lemniscate configuration. If instead we replace P by $P + c$ $(c \in \mathbb{R})$, we remain in the same component of $\mathcal{M}_n$, but $P + c$ is lemniscate generic if and only if, assuming without loss of generality that the critical values of P are positive, $c \neq -w_i$, or, for $i < j, c \neq 1/2(w_i - w_j)$.
Therefore, it is easy to see that we range in $(1/2)n(n+1)+1$ distinct components of $\mathcal{L}_{n,\mathbb{R}}$, thus a natural question is whether one hits $(1/2)n(n+1)+1$ distinct components of $\mathcal{L}_n$.

The answer is negative, as it is shown by the case where $n = 4$, the critical values are 1,2,3,4, and the snake linear tree is $-1-4-2-3-$. In fact, for $c = -(2-\epsilon)$, we get the same configuration as for P.

(2.21) Definition. A real polynomial of degree $n+1$ is said to be ***totally real*** if it has $n+1$ distinct real roots. Clearly, a totally real polynomial is maximally real.

(2.22) Proposition. The space of totally real lemniscate generic polynomials of degree $n+1$ has $2(m!)^2$ components for $n = 2m$, and $2(m!)((m+1)!)$ components for $n = 2m+1$.

Proof. Remark that a lemniscate generic polynomial is totally real if and only if the associated snake of critical values $u_1, ..., u_n$ has alternating signs. Therefore the critical values $w_1, ..., w_n$ must be partitioned according to their sign into two disjoint sets of respective cardinalities $m, n-m$. It is easy now to count the number of snakes obtainable by $w_1, ..., w_n$, and we conclude by theorem 2.6.

Q.E.D.

§3. Components of the space of real generic polynomials.

We start this section by generalizing the theorem of Davis to the case of non maximally real polynomials.

We begin by setting up some notation.
Assume that P is a polynomial with k real critical points $y_1 < ... < y_k$ and m pairs $(\zeta_1, \bar{\zeta}_1)...(\zeta_m, \bar{\zeta}_m)$ of complex conjugate critical points ($n = k + 2m$). As usual, the $A^+(1, \mathbb{R})$ source -equivalence class of P (or of $-P$) is uniquely represented by the normalized polynomial

$$(3.1) \qquad P_{y,\zeta} = \int \left(\prod_{i=1,..k} (z - y_i) \prod_{j=1,..m} (z - \zeta_j)(z - \bar{\zeta}_j) \right).$$

We consider the critical values $u_i = P_{y,\zeta}(y_i)$, which form a weak snake sequence, and the conjugate pairs of critical values $(v_j, \bar{v}_j) = (P_{y,\zeta}(\zeta_j), P_{y,\zeta}(\bar{\zeta}_j))$. Let $\mathbb{H}$ be the upper half plane in $\mathbb{C}$, and $\bar{\mathbb{H}}$ its closure. Naturally, conjugate pairs of complex numbers are parametrized by points of $\bar{\mathbb{H}}$.
Similarly to the proof of 2.3, we let

$$\mathcal{C}'' = \{(y_1, ..., y_k) | y_i \in \mathbf{R}, y_1 \leq y_2 ... \leq y_k\} \times \bar{\mathbb{H}}^m \cong \mathbf{R} \times (\mathbf{R}^{\geq 0})^{k-1} \times \bar{\mathbb{H}}^m$$

embedded inside $\mathbb{C}^n$, by associating to $(y_1, ..., y_k)(\zeta_1, ..., \zeta_m)$ the n–tuple $(y_1, ..., y_k, \zeta_1, \bar{\zeta}_1, ..., \zeta_m, \bar{\zeta}_m)$. We consider again the branch point map, the surjective polynomial map, homogeneous of degree $(n+1)$, $\beta : \mathbb{C}^n \to \mathbb{C}^n$ associating to $(y_1, ..., y_k, \zeta_1, \bar{\zeta}_1, ..., \zeta_m, \bar{\zeta}_m)$ the ordered set $(u_1, ..., u_k, v_1, \bar{v}_1, ..., v_m, \bar{v}_m)$.

Finally, we let $\beta'' : \mathcal{C}'' = \mathbf{R} \times (\mathbf{R}^{\geq 0})^{k-1} \times \bar{\mathbb{H}}^m \to \mathcal{V}'' = \mathbf{R} \times (\mathbf{R}^{\geq 0})^{k-1} \times \bar{\mathbb{H}}^m$ the composition with the projection associating to $(u_1, ..., u_k, v_1, \bar{v}_1, ..., v_m, \bar{v}_m)$ the point $[(u_1, ..., u_k), \{v_1, \bar{v}_1\}, ... \{v_m, \bar{v}_m\}]$, where we view now $(u_1, ..., u_k)$ as a point of the space $\mathcal{V}$ of weak up-down sequences (down-up if n is odd).
We have the following analogue of the theorem of C. Davis (except for unicity, which does not hold) :

(3.2) Proposition. The map $\beta'' : \mathcal{C} \to \mathcal{V}''$ is surjective.

Proof. The map $\beta'' : \mathcal{C}'' \to \mathcal{V}''$ is closed and finite and the boundary of $\mathcal{C}''$ maps to the boundary of $\mathcal{V}''$. Let $\mathcal{C}'$ be the open set in $\mathcal{C}''$ where $y_1 < y_2 ... < y_k, \zeta_j \notin \mathbf{R}$, $\zeta_i \neq \zeta_j$, for all i, j.
Define similarly $\mathcal{V}'$. Then $\mathcal{C}'' - \mathcal{C}'$ maps to $\mathcal{V}'' - \mathcal{V}'$. Moreover, as we know, β'' is unramified, whence open on $\mathcal{C}'$. If the open set $\beta''(\mathcal{C}')$ would not contain $\mathcal{V}'$, there would be a point in $\mathcal{V}'$ which belongs to $\beta''(\mathcal{C}'')$ =closure of $\beta''(\mathcal{C}')$, a contradiction again.
Therefore $\beta''(\mathcal{C}'')$ =closure of $\beta''(\mathcal{C}')$ contains the closure of $\mathcal{V}'$, that is, $\mathcal{V}''$.

Q.E.D.

(3.3) Remark. The $A^+(1, \mathbf{R})$ source -equivalence classes of generic monic real polynomials with exactly k real critical values correspond to the inverse image $\beta''^{-1}(\mathcal{V}^*)$, where $\mathcal{V}^*$ is the open set in $\mathcal{V}'$ where all the u_i's are distinct. We shall not pursue this point of view, since we shall determine the connected components of $\beta''^{-1}(\mathcal{V}^*)$ by a different method.

(3.4) Remark. A necessary condition for an algebraic function $f : C \to \mathbb{P}^1$ to be real is that the branch locus B is self conjugate (hence, the branch points will be k real critical values $w_1, ...w_k$ and m pairs $(v_1, \bar{v}_1)...(v_m, \bar{v}_m)$ of complex conjugate critical values where v_i lies in the upper half plane). If B is self conjugate, moreover, it is easy to see that f is real if and only if complex conjugation on $\mathbb{P}^1$ lifts to C. This means that complex conjugation sends the class of the monodromy μ to itself (of course we have to express both monodromies in a fixed basis of π_1).
Assuming that 0 is not a critical value, we choose a geometric basis of $\pi_1(\mathbb{P}^1 - B, 0)$ by choosing loops $\gamma_1, .., \gamma_k$ around the w_i's as in 2.2, and by choosing pairs of self conjugate loops $(\delta_j, \bar{\delta}_j)$ around the pairs $(v_j, \bar{v}_j)$.

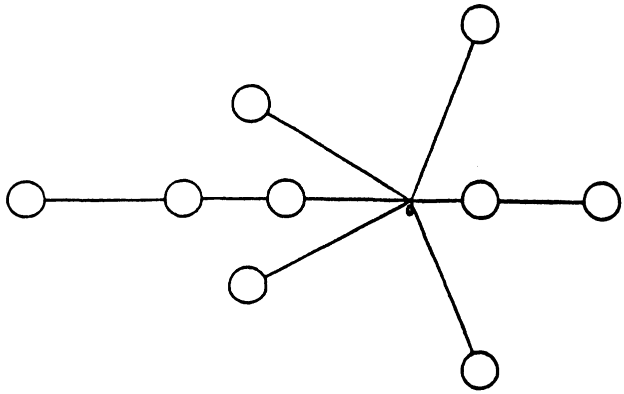

Figure 4 : choice of the canonical basis for a real polynomial

For use in the calculation, we observe that, if we separate the real branch points into the set of negative ones $w_s^- < .. < w_1^- < 0$ and the set of positive ones $0 < w_1^+ < .. < w_r^+$, we have

$$\bar{\gamma}_i^+ = (\gamma_1^+)^{-1}(\gamma_2^+)^{-1}....(\gamma_{i-1}^+)^{-1}(\gamma_i^+)\gamma_{i-1}^+....\gamma_1^+ \tag{3.5}$$

and similarly for the $\bar{\gamma}_i^-$'s. We can thus rephrase 3.4 as follows :

(3.6) μ is the monodromy of a real algebraic function if and only if there exists a permutation α of period 2 (induced by conjugation on $f^{-1}(0)$) such that, setting $\tau_i = \mu(\gamma_i^+), \tau_i' = \mu(\gamma_i^-), \nu_j = \mu(\delta_j), \nu_j' = \mu(\bar{\delta}_j), \rho_{i-1} = \tau_1^{-1}\tau_2^{-1}....\tau_{i-1}^{-1}$, and similarly ρ_{i-1}', we have :

$$\alpha\tau_i\alpha = \rho_{i-1}\tau_i\rho_{i-1}^{-1}\ ,\ \alpha\tau_i'\alpha = \rho_{i-1}'\tau_i'\rho_{i-1}'^{-1}\ ,\ \alpha\nu_j\alpha = \nu_j'.$$

We can now characterize the monodromies of generic real polynomials

(3.7) Theorem.

Let $w_s^- < .. < w_1^- < 0 < w_1^+ < .. < w_r^+$, be distinct real numbers $\neq 0$, and let $(v_1, \bar{v}_1)...(v_m, \bar{v}_m)$ m distinct pairs of conjugate complex numbers with v_i in the upper half plane.
Set $k = s + r, n = 2m + k, B = \{w_s^-, .., w_1^-, w_1^+, .., w_r^+\} \cup \{v_1, \bar{v}_1\} ... \cup \{v_m, \bar{v}_m\}$.

Then there is a canonical choice of a geometric basis of $\pi_1(\mathbb{C} - B, 0)$ (as in 3.4), such that the edge labelled monodromy trees $\mathcal{T}$ (in E_n, and with the branch points as labels) coming from generic real polynomials are exactly those obtained as follows.
Take a snake linear edge labelled tree $\mathcal{T}'$ in E_k, having $w_s^-, .., w_1^-, w_1^+, .., w_r^+$, as labels (snake with respect to the ordering of the $w_i^{\pm}$'s in $\mathbf{R}$), and let α' be the canonical permutation on the vertices of $\mathcal{T}'$ which is the product of all the transpositions corresponding to the "local minima" edges, i.e., the edges which have a label of the same sign of its neighbours .

Then $\mathcal{T}$ is made out of a subtree isomorphic to $\mathcal{T}'$ and of the union $\mathcal{T}^*$ of an unordered pair of edge labelled graphs $\mathcal{T}_1^*, \mathcal{T}_2^*$, (simply connected but not necessarily connected), with respective labels obtained by choosing m among the labels $v_1, ..v_m\ , \bar{v}_1, ..., \bar{v}_m$, which are isomorphic under the natural isomorphism α^* which exchanges the edge v_i and the edge $\bar{v}_i$ in such a way that α^* agrees with α' on the common vertices of the subgraphs $\mathcal{T}', \mathcal{T}^*$ (thus α' and α^* together define an involution α on $\mathcal{T}$).

(3.8) Remark.
A more efficient way to label $\mathcal{T}$ is to use the target ordering for the edges of the snake linear tree (hence those labels are numbers from 1 to k), and numbers i'' for v_i, numbers i' for $\bar{v}_i$ (cf. figure 5).

Proof of theorem 3.7.

Let $\mathcal{T}$ be the monodromy tree associated to μ, let E^+ be the subgraph consisting of the edges labelled by the w_j^+'s, define E^- analogously and let finally $\mathcal{T}' = E^+ \cup E^-$.
We shall show later that $\mathcal{T}'$ is a tree.
Let moreover S_i be the the subgraph consisting of the edges labelled by the w_j^+'s with $j \leq i$ (resp. S_i' for the w_j^-'s with $j \leq i$). Define moreover, for a subgraph $\mathcal{S}$, supp$(\mathcal{S})$ as the union of the vertices of $\mathcal{S}$.

By the formulae 3.6. α carries supp (S_i) into itself, and since μ is the monodromy of a polynomial, ρ_i acts on S_i as a product of cycles corresponding to the supports of the connected components of S_i. In particular, 3.6 implies that the support of the connected component of S_j containing the edge w_j^+ is sent to itself, thus by induction α leaves the support of every component of S_j invariant for each j.

Recall that the edge w_j^+ corresponds to the transposition τ_j and let $\{a, b\} =$ supp (τ_j). We have three cases :

1) $\{a, b\} \cap$ supp $(S_{j-1}) = \emptyset$,

2) $\{a, b\} \cap$ supp $(S_{j-1}) = \{b\}$

3) $\{a, b\} \subset$ supp (S_{j-1}).
Since by 3.6 we have an equality $\alpha\{a, b\} = \rho_{i-1}\{a, b\}$, in case 1) $\alpha(\{a, b\}) = \{a, b\}$, in case 2) $\alpha(a) = a, \alpha(b) = \rho_{i-1}(b) \neq b$, in case 3) a and b belong to different components of S_{j-1}, whence by our previous remark $\alpha(a) = \rho_{i-1}(a), \alpha(b) = \rho_{i-1}(b)$.
Let a be such that $\alpha(a) = c \neq a$. If j is minimum such that $a \in$ supp (S_j), we must be in case 1), and then $\alpha(a) = b = \tau_j(a)$.
Conversely, if case 1) holds, and the edge τ_j is not a component of E^+, then τ_j appears in the cycle decomposition of α (in fact, if a, b are the vertices of τ_j, we can assume then that there is a smallest $i > j$ such that a belongs to supp (S_i)), and then $\alpha(a) = \rho_{i-1}(a) = \tau_j(a) = b$.
In order to consider the case where the edge τ_j is a component of E^+ (note that the argument for E^- is completely analogous) we first prove that $\mathcal{T}'$ is

connected. In fact, we saw that α preserves the connected components of E^+, E^-,whence if A, B are two connected components of $\mathcal{T}'$, they are left invariant by α and there exists an edge $\nu_j = \mu(\delta_j)$ connecting A and B : but then also $\nu'_j = \mu(\bar{\delta}_j)$ connects A and B, contradicting the fact that $\mathcal{T}$ is a tree.

If now the edge τ_j is a component of E^+, then there exists a component A of E^- intersecting the edge τ_j in a vertex a, which must then be a fixed point for α.

The conclusion is that α acts on supp $(\mathcal{T}')$ as the product of those transpositions τ_j, τ'_j such that the edge corresponding to τ_j is a connected component of S_j but not of E^+ (similarly for τ'_j).

We prove now that $\mathcal{T}'$ is a snake linear tree.

E^+ is a union of disjoint linear trees : else, there is b belonging to edges τ_j, τ_h, τ_k, with $j < h < k$, and j, h, k minimal with this property.

But then,$\alpha(b)$ must equal $\rho_{h-1}(b)$ and $\rho_{k-1}(b)$. By our choice of $k, \rho_{k-1}(b) = \rho_{h-1}\tau_h(b)$, thus $b = \tau_h(b)$, a contradiction.

Using that the transpositions giving the cycle decomposition of α are disjoint, we immediately see that the components of E^+ are snake linear trees. In fact, if the edge τ_i intersects S_{i-1}, then either both of its vertices lie in S_{i-1}, or α fixes one of two vertices.

$\mathcal{T}'$ is linear : otherwise, since the intersection points of E^+ and E^- are left fixed by α, if a vertex a would belong to, say, two edges of E^+ and one edge of E^-, then α would act as the identity on the vertices of two adjacent edges of E^+, what is easily seen to be impossible.

Since if an edge of E^+ intersects E^- then its vertices are left fixed by α, it follows that $\mathcal{T}'$ is also snake linear.

We set then $\mathcal{T}^*$ to be the union of the edges of $\mathcal{T}$ not in $\mathcal{T}'$.

If A is a connected component of $\mathcal{T}^*$, then A intersects $\mathcal{T}'$ in a vertex a. Assume that $\alpha(A) = A$: then, since $\alpha(a) = a$ in this case, α would have a fixed edge in A, contradicting 3.6.

Therefore $\alpha(A)$ and A have disjoint edges, are clearly canonically isomorphic, and $\alpha(A)$ intersects $\mathcal{T}'$ in $\alpha(a)$.

The rest of the proof is now straightforward.

In fact, conversely, a tree $\mathcal{T}$ with the stated properties defines an involution α satisfying 3.6, and we conclude by remark 3.4. Q.E.D.

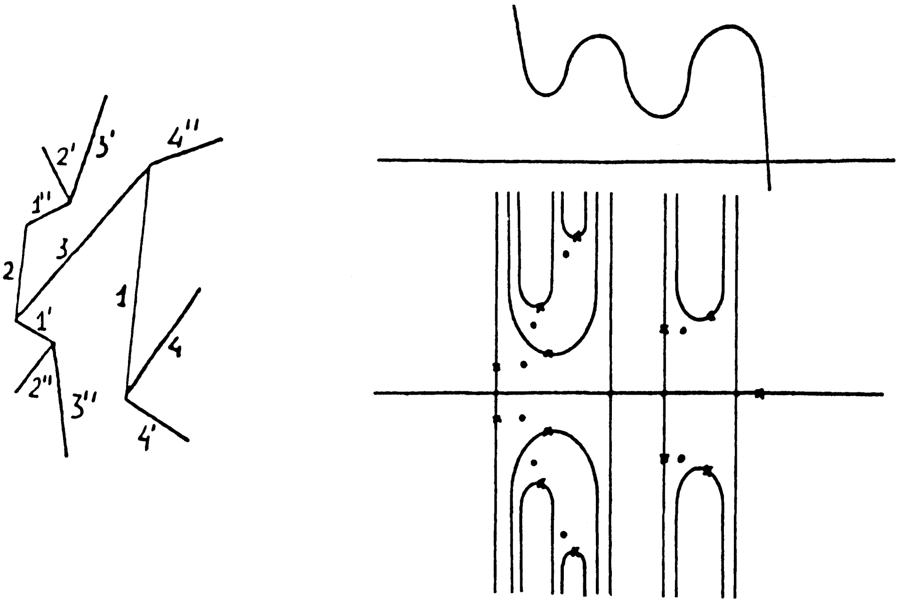

Figure 5 : A generic polynomial of degree 13, its monodromy tree , and its graph $\Theta = P^{-1}(\mathbb{P}^1_{\mathbb{R}})$.

(3.9) Remark.

A first observation is that for each generic real polynomial P, there is an equivalent polynomial $Q(z) = P(z + c)$ such that all the real critical values are positive. Therefore the connected components of the open set of real generic polynomials of degree $n + 1$ with k real critical values correspond to the set of orbits of the braid group $\mathcal{B}_m(2m + k = n)$ on the isomorphism classes of edge labelled trees $\mathcal{T}$ as in theorem 3.7 (where, though, the role of $\mathcal{T}_1^*, \mathcal{T}_2^*$ cannot be interchanged). Here the braid group acts in the standard way (cf. [C-W]) on the labels i' and i'' (that is, the standard generators $\sigma_j, j = 1,..m-1$, of $\mathcal{B}_m$ act by letting ν_j become ν_{j+1}, whereas the new ν_j is the old ν_{j+1} conjugated by the old ν_j, and similarly for ν'_j, ν'_{j+1}). For each subgraph , say $\mathcal{T}_1^*$, we have two more subgraphs, $\mathcal{T}_1^{*'}, \mathcal{T}_1^{*''}$, whose connected components (which can be reduced to a vertex) correspond to the connected components of the complement of $P^{-1}(\mathbb{P}^1_{\mathbb{R}})$ which are contained in the upper half plane and map to the lower half plane (resp.: to the upper half plane). Notice that in this case the roles of $\mathcal{T}_1^*, \mathcal{T}_2^*$, are distinguished since we only look at $A^+(1,\mathbb{R})$–orbits.

The geometric picture is illustrated in figure 5.

It is clear that the action of the braid group respects the subtrees given by these connected components , and that it can transform any such tree to any other with the same number of edges.

Using the above remarks one can find , for each snake linear tree $\mathcal{T}'$ with k edges, the number of the braid group orbits on the set of trees $\mathcal{T}$ which have $\mathcal{T}'$ as the "snake" part.

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