Astérisque

MEI-CHI SHAW

Semi-global existence theorems of $\bar{\partial}_b$ for $(0, n-2)$ forms on pseudo-convex boundaries in $\mathbb{C}^n$

Astérisque, tome 217 (1993), p. 227-240

<http://www.numdam.org/item?id=AST_1993__217__227_0>

SEMI-GLOBAL EXISTENCE THEOREMS OF $\bar{\partial}_b$ FOR $(0, n-2)$ FORMS ON PSEUDO-CONVEX BOUNDARIES IN $\mathbb{C}^n$

MEI-CHI SHAW

INTRODUCTION

Table of contents:

Let M be the boundary of a pseudo-convex domain D in $\mathbb{C}^n$, $n \geq 2$. We consider the tangential Cauchy-Riemann equations

$$\bar{\partial}_b u = \alpha \tag{0.1}$$

on an open subset $\omega \subset M$, where α is a (p,q) form in ω, $0 \leqq p \leqq n$, $1 \leqq q < n-1$. Since $\bar{\partial}_b^2 = 0$, in order for Eq.(0.1) to be solvable, α must satisfy the compatibility condition

$$\bar{\partial}_b \alpha = 0 \quad \text{in} \quad \omega. \tag{0.2}$$

Recently, the semi-global existence results have been obtained by the author for any (p,q) form α, where $1 \leqq q \leqq n-3$, such that ω is a pseudo-convex boundary of finite type as defined in D'Angelo [8]. It is proved in [22] that when $\partial\omega$ lies in a flat or a Levi-flat hypersurface which has a Stein neighborhood basis, then Eq.(0.1) is solvable for all (p,q) forms α satisfying condition (0.2), where $1 \leqq q \leqq n-3$. When $q = n-1$ Eq.(0.1) corresponds to the Lewy equation and it is well-known that for most α it is not solvable locally unless ω is Levi-flat (see Hörmander [11]).

Partially supported by NSF grant DMS 91-01161

In this paper we shall discuss the local and semi-global existence theorems for the remaining case, i.e., when $q = n-2$. It was observed by Rosay [19] that when $q = n-2$, condition (0.2) is not sufficient for Eq.(0.1) to be solvable in ω. In fact, there is an additional compatibility condition that α must satisfy in order for Eq.(0.1) to be solvable. This additional condition, called condition (A), is a condition on the boundary of $\partial\omega$ and will be derived in Section I. Our main purpose in this paper is to show that condition (0.2) and condition (A) are the necessary and sufficient conditions for Eq.(0.1) to be solvable when α is a smooth $(p, n-2)$ form. We also characterize those domains on which condition (0.2) always implies condition (A) (see Proposition 1.2). This condition (A), though easy to derive, does not seem to have been observed before.

The local solvability of Eq.(0.1) has been studied by many people when M is *strongly* pseudo-convex (see [1,4,10,19,20,21,23,24]). In this case it was proved in Henkin [10] that one can construct explicit solution kernels for $1 \leqq q \leqq n-2$ when $\partial\omega$ lies in a hyperplane. When $1 \leqq q < n-2$, he actually derived a homotopy formula for $\bar{\partial}_b$. When $q = n-2$, such a homotopy formula will not hold (see Nagel-Rosay [17]) and polynomial approximation arguments were used to construct the solution kernels. In fact, Henkin [10] showed that if $\partial\omega$ is Runge, then condition (0.2) is sufficient for Eq.(0.1) to be solvable when $q = n-2$. In this paper we shall characterize those domains such that condition (0.2) is sufficient for Eq.(0.1) to be solvable. These domains are more general than Runge ones. In the strongly pseudo-convex case, it is especially important to study the case when $n = 3$ and α is a (0,1) form, since this will give us some insight into the problem of embeddability of abstract CR structures of real dimension 5 (see Webster [25]).

The plan of the paper is as follows. In Section I we define the notation and state our main results in Theorems 1 and 2. In Section II we use the Cauchy problem for $\bar{\partial}$ for the top degree forms to prove Theorems 1 and 2. The Cauchy problem is different in this case from the lower degree cases and this is when the second compatibility condition was used. The rest of the proof is similar to the case when $1 \leqq q \leqq n-3$. In the end of this paper we give an example by Rosay [19] which shows that condition (0.2) is not sufficient for Eq.(0.1) to be solvable. The author would like to thank Professor Catlin for helpful discussions and to thank professor So-Chin Chen for pointing out the reference [3].

1. Notation and the main results

Let M be the boundary of a pseudo-convex domain D and ρ be its defining function, i.e., $M = \{z \in \mathbb{C}^n | \rho(z) = 0\}$ and $|d\rho| = 1$ on M. We assume that M is of finite type in the sense of D'Angelo [8]. Let $\omega \subset M$ such that $\omega = M \cap \{z \in \mathbb{C}^n | r(z) < 0\}$ and $d\rho \wedge dr \neq 0$ on the boundary of $\partial\omega$. The space

$C^\infty_{(p,q)}(\bar\omega)$ denotes all the (p,q) forms in ω with coefficients in $C^\infty(\bar\omega)$. For any $\alpha \in C^\infty_{(p,q)}(\bar\omega)$, there exists a smooth (p,q) form $\tilde\alpha$ in C^n such that $\tau\tilde\alpha = \alpha$ where τ is the pointwise restriction operator to the boundary and projection to the parts which are orthogonal to the ideal generated by $\bar\partial\rho$. Similarly we define the space $C^\infty_{(p,q)}(\omega)$ for (p,q) forms in ω with $C^\infty(\omega)$ coefficients. The $\bar\partial_b$ operator is defined to be as follows: for any $\alpha \in C^\infty_{(p,q)}(\omega)$ and $\tilde\alpha$ that is any extension of α such that $\tau\tilde\alpha = \alpha$, then we define $\bar\partial_b\alpha = \tau(\bar\partial\tilde\alpha)$. It is easy to see that the definition of $\bar\partial_b$ is independent of the choice of $\tilde\alpha$. For other definitions and the basic properties of the $\bar\partial_b$ complex, we refer the readers to Kohn-Rossi [16] or Folland-Kohn [9]. Since p plays no role in the discussion of $\bar\partial_b$, we shall assume that $p = n$ for simplicity.

Let K be a compact set in $\mathbb{C}^n$. We shall use the notation $\mathcal{O}(K)$ to denote the set of functions which are defined and holomorphic in some open neighborhood of K. Let $\omega_\epsilon \subset\subset \omega$ such that ω_ϵ increases to ω as $\epsilon \searrow 0$ and each $\partial\omega_\epsilon$ is smooth. For any $\alpha \in C^\infty_{(n,n-2)}(\bar\omega)$ such that Eq. (0.1) is solvable for some $u \in C^\infty_{(n,n-3)}(\omega)$, then for any $g \in \mathcal{O}(\partial\omega)$ we have, for small $\epsilon > 0$,

$$
\begin{aligned}
\int_{\partial\omega} \alpha \wedge g &= \lim_{\epsilon\to 0}\int_{\partial\omega_\epsilon} \alpha\wedge g \\
&= \lim_{\epsilon\to 0}\int_{\partial\omega_\epsilon} \bar\partial_b u\wedge g \\
&= \lim_{\epsilon\to 0}\int_{\partial\omega_\epsilon} \bar\partial u\wedge g \\
&= \lim_{\epsilon\to 0}\int_{\partial\omega_\epsilon} \bar\partial (u\wedge g) \\
&= \lim_{\epsilon\to 0}\int_{\partial\omega_\epsilon} d(u\wedge g) \\
&= 0
\end{aligned} \tag{1.1}
$$

The third equality in (1.1) holds since the difference of $\bar\partial u$ and $\bar\partial_b u$ is a multiple of $\bar\partial\rho$ and $\bar\partial\rho = d\rho - \partial\rho$. Thus another necessary condition for Eq.(0.1) to be solvable for $\alpha \in C^\infty_{(n,n-2)}(\bar\omega)$ is that

$$\int_{\partial\omega} \alpha\wedge g = 0 \qquad \text{for all } g \in \mathcal{O}(\partial\omega). \tag{A}$$

The following proposition characterizes all the domains ω such that condition (0.2) will imply condition (A). At the end of this paper we shall give an example of a $\bar\partial_b$-closed form which does not satisfy condition (A).

Proposition 1.2. *If $\mathcal{O}(\bar{\omega})$ is dense in $\mathcal{O}(\partial\omega)$ (in the $C(\partial\omega)$ norm), for any α satisfying condition (0.2), α satisfies condition (A). In particular, if polynomials are dense in $\mathcal{O}(\partial\omega)$, then condition (0.2) implies condition (A).*

Proof. From our assumption, for any $g \in \mathcal{O}(\partial\omega)$, there exists a sequence of holomorphic functions $g_n \in \mathcal{O}(\bar{\omega})$ such that g_n converges to g in $C(\partial\omega)$. We have, for any α satisfying condition (2),

$$\begin{aligned}\int_{\partial\omega} \alpha \wedge g &= \lim_{n\to\infty} \int_{\partial\omega} \alpha \wedge g_n \\ &= \lim_{n\to\infty} \int_{\omega} \bar{\partial}(\alpha \wedge g_n) \\ &= \lim_{n\to\infty} \int_{\omega} \bar{\partial}_b\alpha \wedge g_n \\ &= 0.\end{aligned}$$

Thus condition (0.2) implies condition (A). If one can approximate any function $g \in \mathcal{O}(\partial\omega)$ by holomorphic polynomials, it is obvious that (2) implies (A) and the proposition is proved.

Our main results in this paper are the following theorems.

Theorem 1. *Let M be the boundary of a smooth pseudo-convex domain in $\mathbb{C}^n$, $n \geqq 3$ and M is of finite type. Let $\omega \subset M$ be a connected subset such that the boundary $\partial\omega$ is the transversal intersection of M with a simply connected Levi-flat hypersurface M_0 which has a Stein neighborhood basis. Let ω' be any relatively compact subset of ω. For any $\alpha \in C^\infty_{(n,n-2)}(\bar{\omega})$ such that α satisfies the compatibility conditions (0.2) and (A), there exists a $u \in C^\infty_{(n,n-3)}(\omega')$ such that $\bar{\partial}_b u = \alpha$ in ω'.*

If one assumes that ω can be exhausted by subsets whose boundaries lie in Levi-flat hypersurfaces, then we have the following semi-global existence result.

Theorem 2. *Let M and ω be the same as in Theorem 1. Furthermore we assume $\omega = \cup\omega_i$ such that $\omega_i \subset\subset \omega_{i+1} \subset\subset \omega$ and $\partial\omega_i$ lies in a Levi-flat hypersurface for each i. For any $\alpha \in C^\infty_{(n,n-2)}(\bar{\omega})$ such that α satisfies the conditions (0.2) and (A), there exists a $u \in C^\infty_{(n,n-3)}(\omega)$ such that $\bar{\partial}_b u = \alpha$ in ω.*

Corollary 2.1. *If M_0 is simply connected and defined by a pluriharmonic function, then the assertions in Theorem 2 holds. In particular, if M_0 is a hyperplane, then the assertions in Theorem 2 hold*

We also have the following local solvability result near a point of finite type.

Theorem 3. *Let M be a smooth be a smooth pseudo-convex hypersurface in $\mathbb{C}^n$, $n \geqq 3$ and $z_0 \in M$. Suppose z_0 is a point of finite type, then there exists a local neighborhood basis $\{\omega_\epsilon\}_{\epsilon>0}$ of z_0 for M such that the following holds: for any $\epsilon > 0$, if $\alpha \in C^\infty_{(n,n-2)}(\bar\omega_\epsilon)$ such that α satisfies the conditions (0.2) and (A), there exists a $u \in C^\infty_{(n,n-3)}(\omega_\epsilon)$ such that $\bar\partial_b u = \alpha$ in ω_ϵ.*

We note that Bedford-Fornaess (see the example on P. 21 in [3]) has given an example of an levi-flat hypersurface which does not have a Stein neighborhood basis. We mention that Bedford-de Bartolomeis [2] showed that Levi-flat hypersurfaces can not always be flattened locally even from one side. Thus our theorems generalize the results of Henkin [9] even in the strongly pseudo-convex case.

2. Proof of the theorems

To prove Theorems 1 and 2, we need to solve the Cauchy problem for $\bar\partial$ on the top degree forms. Let $L^2_{(p,q)}(G)$ denote (p,q) forms on a domain G with $L^2(G)$ coefficients. We denote the space of square integrable holomorphic functions by $H^2(G)$ and the spce of holomorphic functions in $C^\infty(\overline{G})$ by $A^\infty(\overline{G})$. We have the following lemma.

Lemma 2.1. *Let G be a bounded pseudo-convex domain in $\mathbb{C}^n$, $n \geqq 2$. For any $f \in L^2_{(n,n)}(\mathbb{C}^n)$, such that f is supported in $\overline{G}$ and*

$$\int_G f \wedge g = 0 \qquad \text{for any} \quad g \in H^2(\overline{G}), \tag{2.2}$$

we can find a $u \in L^2_{(n,n-1)}(\mathbb{C}^n)$ such that u is supported in $\overline{G}$ and $\bar\partial u = f$ in the distribution sense in $\mathbb{C}^n$. Furthermore, we have the following estimates:

$$\| u \|^2_G \leqq C \| f \|^2_G \tag{2.3}$$

where the constant C depends only on the diameter of the domain G.

If we assume that G is a bounded pseudo-convex domain with smooth boundary, then we can substitute (2.2) by the condition

$$\int_G f \wedge g = 0 \qquad \text{for any} \quad g \in A^\infty(\overline{G}), \tag{2.2'}$$

and the same conclusion holds.

If we assume that G is a bounded pseudo-convex domain with a Stein neighborhood basis, then we can substitute (2.2) by the condition

$$\int_G f \wedge g = 0 \qquad \textit{for any} \quad g \in \mathcal{O}(\overline{G}), \tag{2.2"}$$

and the same conclusion holds.

Proof. We shall first prove the lemma assuming that G is a bounded pseudoconvex domain. Following Hörmander's theory for $\bar\partial$, the $\bar\partial$-Neumann operators for $(0,1)$ forms, denoted by N_1, exist on G. One can also define the $\bar\partial$-Neumann operator on functions (denoted by N_0) using N_1. In fact, let ϑ be the formal adjoint of $\bar\partial$, then it follows from Theorem 3.1.19 in Folland-Kohn [9] that

$$N_0 = \vartheta N_1^2 \bar\partial \tag{2.4}$$

whenever the formula is defined. We also have that

$$\vartheta\bar\partial N_0 = I - H \tag{2.5}$$

where ϑ is the adjoint operator of $\bar\partial$ and H is the Bergman projection operator from square-integrable functions into square integrable holomorphic functions $H^2(G)$.

In fact, the formula (2.4) and (2.5) hold on all of $L^2(G)$. To see this, we use the fact that N_1 is a bounded operator on $L^2_{(0,1)}(G)$ and the bounds only depend on the diameter of G. In fact, using the precise estimates obtained by Hörmander [12], we can have the following estimates:

$$\|N_1\alpha\|_G^2 \leqq e\delta^2 \|\alpha\|_G^2 \tag{2.6}$$

where δ is the diameter of the domain G (for details of estimate (2.6), see Hörmander[12] and the proposition 2.3 in [21]). For any $v \in C^\infty(\overline{G})$, we have from (2.4)

$$\begin{aligned} \|N_0 v\|^2 &= (\bar\partial\vartheta N_1^2\bar\partial v, N_1^2\bar\partial v) \\ &= (N_1\bar\partial v, N_1^2\bar\partial v) \\ &\leqq \|N_1\bar\partial v\|\|N_1^2\bar\partial v\| \\ &\leqq e^{\frac{1}{2}}\delta\|N_1\bar\partial v\|^2 \end{aligned} \tag{2.7}$$

On the other hand, we have

$$\begin{aligned} (N_1\bar\partial v, N_1\bar\partial v) &= (N_1^2\bar\partial v, \bar\partial v) \\ &= (\vartheta N_1^2\bar\partial v, v) \\ &\leqq \|N_0 v\|\|v\| \end{aligned} \tag{2.8}$$

Combining (2.10) and (2.11), we have

$$\|N_0 v\|^2 \leqq e\delta^2 \|v\|^2 \tag{2.9}$$

Thus N_0 defined by (2.4) is a bounded on all smooth functions and it extends to $L^2(G)$. We also have that (2.5) holds for all functions in $L^2(G)$. Also, it follows from (2.5) that

$$\begin{aligned} \|\bar{\partial} N_0 v\|^2 =& (\vartheta\bar{\partial} N_0 v, N_0 v) \\ &= ((I-H)v, N_0 v) \leqq \|v\|\|N_0 v\| \leqq e^{\frac{1}{2}}\delta\|v\|^2 \end{aligned} \tag{2.10}$$

Using N_0, we define

$$u = - \star \overline{\bar{\partial} N_0 \overline{\star f}} \tag{2.11}$$

where $\star$ is the Hodge star operator extended naturally to the $L^2(G)$ forms. Using the relations that $\vartheta = - \star \partial \star$ and $\star\star = I$, we have from (2.5),

$$\begin{aligned} \bar{\partial} u &= -\bar{\partial}(\star \overline{\partial N_0 \overline{\star f}}) \\ &= \star\overline{\vartheta\bar{\partial} N_0 \overline{\star f}} \\ &= \star\overline{(\star f - \overline{H(\star f)})} \\ &= f - \star\overline{H(\overline{\star f})} \end{aligned} \tag{2.12}$$

For any $g \in H^2(G)$, from (2.2), we have

$$(\star f, \bar{g}) = \int_G f \wedge g = 0 \tag{2.13}$$

Thus from (2.8), $H(\overline{\star f}) = 0$ and $\bar{\partial} u = f$ in G.

Extending u to be zero outside G, we have for any $\phi \in C^\infty_{(n,n)}(\mathbb{C}^n)$, that

$$\begin{aligned} (u, \vartheta\phi)_{\mathbb{C}^n} &= (\star\overline{\vartheta\phi}, \star\bar{u})_G \\ &= (-\bar{\partial} \star \bar{\phi}, \star\bar{u})_G \\ &= (- \star \bar{\phi}, \bar{\partial}^* \star \bar{u})_G \\ &= (\star\bar{\phi}, \star\partial\bar{u})_G \\ &= (\star\bar{\phi}, \star\bar{f})_G \\ &= (f, \phi)_{\mathbb{C}^n} \end{aligned}$$

where the third equality holds since $\star\bar{u} \in \mathrm{Dom}(\bar{\partial}^*)$. This implies that $\bar{\partial}u = f$ in $\mathbb{C}^n$ in the distribution sense. The estimate (2.3) holds from (2.11) with the constant $C = e^{\frac{1}{2}}\delta$.

When G is a bounded pseudo-convex domain with smooth boundary, following Kohn's theorem in [9] on the global regularity of the solutions of $\bar{\partial}$ on strongly pseudo-convex domains, we have that $A^\infty(\overline{G})$ is dense in $H^2(G)$ (in the $L^2(G)$ norm). Thus if f satisfies condition (2.2'), it also satisfies condition (2.2) and the same results holds.

When G is a pseudo-convex domain with a Stein neighborhood basis , we can assume that $\overline{G} = \cap G_i$ such that each G_i is strongly pseudo-convex with smooth boundary. We note that on each G_i, the space $\mathcal{O}(\overline{G}_i)$ is dense in $H^2(G_i)$ since G_i is strongly pseudo-convex (this essentially follows from the Kohn's $\bar{\partial}$-Neumann theory on strongly pseudo-convex domains). Since

$$\int_{G_i} f \wedge g = \int_G f \wedge g = 0$$

for every $g \in \mathcal{O}(\overline{G}_i)$, thus condition (2.2) is satisfied for any $g \in H^2(G_i)$ and there exists a solution u_i which is compactly supported in $\overline{G}_i$ and $\bar{\partial}u_i = f$ in $\mathbb{C}^n$. Furthermore, it follows from (2.3) one has that

$$\| u_i \|_{G_i} \leqq C \| f \|_G$$

where the constant C can be chosen independent of i. Thus there exist a weak convergent subsequence of u_i which converges to an element u in $\mathbb{C}^n$ and the support of u is contained in $\overline{G}$. One easily sees that u satisfies (2.3) and $\bar{\partial}u = f$ in the distribution sense in $\mathbb{C}^n$ and the lemma is proved.

Proof of Theorem 1. Let $D_0 = \{z \in \mathbb{C}^n | r(z) < 0\}$ and $\Omega = D \cap D_0$. From our assumption that M_0 has a Stein neighborhood basis, we have that $\bar{\omega}_0 = M_0 \cap D$ also has a Stein neighborhood basis, since any domain of finite type has a Stein neighborhood basis. Let G_i be a sequence of smooth decreasing pseudo-convex domains G_i such that $\bar{\omega}_0 = \cap G_i$. We define $\Omega_i = \Omega \setminus \bar{G}_i$, $\Omega \cap G_i = D_i$ and $\bar{\Omega}_i \cap \omega = \bar{\omega}_i$. Then each D_i is pseudo-convex since it is the intersection of two pseudo-convex domains. Also each D_i has a Stein neighborhood basis since D and D_0 both have Stein neighborhood basis. We also denote the boundary of G_i by ∂G_i and $\partial G_i \cap \Omega$ by ω_i^0. Thus $\Omega_i \nearrow \Omega$ and $\omega_i \nearrow \omega$.

For any α that satisfies the conditions (0.2) and (A), we have, for any $g \in$

$\mathcal{O}(\overline{D}_i)$,

$$\begin{aligned}
\int_{\partial\omega_i^0} \alpha \wedge g &= \int_{\partial\omega_0} \alpha \wedge g - \int_{\omega\setminus\omega_i} d(\alpha \wedge g) \\
&= \int_{\partial\omega_0} \alpha \wedge g - \int_{\omega\setminus\omega_i} \bar{\partial}\alpha \wedge g \\
&= \int_{\partial\omega_0} \alpha \wedge g - \int_{\omega\setminus\omega_i} \bar{\partial}_b\alpha \wedge g \qquad (2.14)\\
&= \int_{\partial\omega} \alpha \wedge g \\
&= 0.
\end{aligned}$$

For a fixed i, we extend α to $\tilde{\alpha}$ on Ω such that $\bar{\partial}\tilde{\alpha}$ vanishes to high order on ω. Let $\zeta \in C_0^\infty(\mathbb{C}^n)$ be a cut-off function such that $\zeta = 1$ on $\overline{\Omega}_i$ and $\zeta = 0$ on $\Omega_i \setminus \Omega_{i+1}$. We define $\alpha_1 = \zeta\bar{\partial}\tilde{\alpha}$ and extend it to be zero outside Ω, then $\alpha_1 \in C^\infty_{(n,n-1)}(\mathbb{C}^n)$ and $\bar{\partial}\alpha_1 = \bar{\partial}\zeta \wedge \bar{\partial}\tilde{\alpha}$. Setting $\alpha_2 = \bar{\partial}\alpha_1$. Then $\alpha_2 \in C^\infty_{(n,n)}(\mathbb{C}^n)$ and α_2 is supported in $\overline{D}_i$.

For any $g \in \mathcal{O}(\overline{D}_i)$, it follows from (2.14) that

$$\begin{aligned}
\int_{D_i} \alpha_2 \wedge g &= \int_{D_i} \bar{\partial}\zeta \wedge \bar{\partial}\tilde{\alpha} \wedge g \\
&= \int_{D_i} \bar{\partial}(\zeta \wedge \bar{\partial}\tilde{\alpha} \wedge g) \\
&= \int_{\partial D_i} \zeta\bar{\partial}\tilde{\alpha} \wedge g \\
&= \int_{\omega_i^0} \bar{\partial}\tilde{\alpha} \wedge g \qquad (2.15)\\
&= \int_{\omega_i^0} d(\tilde{\alpha} \wedge g) \\
&= \int_{\partial\omega_i^0} \alpha \wedge g \\
&= 0
\end{aligned}$$

Thus α_2 satisfies (2.2). Using lemma 2.1 on the domain D_i, there exists a $u_1 \in L^2_{(n,n-1)}(\mathbb{C}^n)$ such that $\bar{\partial}u_1 = \alpha_2$ in $\mathbb{C}^n$ and the support of u_1 is contained in $\bar{D}_i$. We set

$$\beta_1 = \alpha_1 - u_1 \qquad (2.16)$$

then we have $\beta_1 = \bar\partial\tilde\alpha$ on Ω_i and $\bar\partial\beta_1 = \bar\partial\alpha_1 - \bar\partial u_1 = 0$ in $\mathbb{C}^n$. Using the $\bar\partial$-Neumann operator on Ω to solve the Cauchy problem for $\bar\partial$ for $(n, n-1)$ forms with support in $\bar\Omega$ (see Proposition 2.7 in [22] for details), we have that there exists a β_0 such that

$$\bar\partial\beta_0 = \beta_1 \qquad \text{in} \quad \mathbb{C}^n$$

and the support of β_0 is in $\bar\Omega$. Furthermore, β_0 is smooth up to the boundary ω_i. We define $\alpha_0 = \tilde\alpha - \beta_0$, then $\bar\partial\alpha_0 = \bar\partial\tilde\alpha - \bar\partial\beta_0 = 0$ in Ω_i and $\alpha_0 = \alpha$ in ω_i. This shows that we have extended α to be $\bar\partial$-closed on Ω_i.

Since $\Omega_i \nearrow \Omega$, any compact subset ω' can be contained in the boundary of a pseudo-convex set Ω' and $\Omega' \subset \Omega_i$ for some i sufficiently large. Thus one can use the $\bar\partial$-Neumann operator on $(n, n-2)$ forms to solve $\bar\partial u = \alpha_0$ on Ω' and the solution u will be smooth to the part of boundary ω_i following the regularity of the $\bar\partial$-Neumann problem up to the boundary points of finite type proved by Kohn [15] and Catlin [7]. Since the method is similar to the case in [22] we omit the details. Restricting u to ω', Theorem 1 is proved.

Proof of Theorem 2. Applying the argument of Theorem 1, we can construct a solution u_i on every ω_i. To extract the convergent subsequence u_i, we assume first that $n - 2 > 1$. Since $\bar\partial_b(u_i - u_{i+1}) = 0$ in ω_i, from the results of [22], there exists a v_i in ω_i such that $\bar\partial_b v_i = u_i - u_{i+1}$ in ω_i. Extending v_i smoothly to $\tilde v_i$ outside ω_{i-1}, we have that $\bar\partial_b\tilde v_i = 0$ in ω_{i-1}. Letting $\tilde u_{i+1} = u_{i+1} + \bar\partial_b\tilde v_i$, we have that $\bar\partial_b\tilde u_{i+1} = \alpha$ in ω_{i+1} and $\tilde u_{i+1} = u_i$ in ω_{i-1}. Continuing this way one can construct a solution u for Eq.(0,1) in ω and Theorem 2 is proved for $n > 3$. The case when $q = 1, n = 3$ is more involved and the argument involved holomorphic approximation. We refer the readers to the proof of lemma 3.1 in [22] and omit the details.

Proof of Corollary 2.1. If M_0 is defined by a pluriharmonic function $r(z) = 0$ and M_0 is simply connected, then there exists a holomorphic function h such that $r(z) = \mathrm{Im}h(z)$. After a holomorphic change of coordinates it is easy to see that the conditions of Theorem 2 are satisfied and the corollary is proved.

Proof of Theorem 3.

We shall construct a neighborhood basis $\{\omega_\epsilon\}$ of z_0 such that $\partial\omega_\epsilon$ lies in a holomorphically flat hypersurface. The following arguments were kindly provided by Catlin. Since z_0 is a point of finite type, it follows from Catlin [6] that z_0 is weakly regular. Thus there exist a family of strictly plurisubharmonic functions $\{\lambda_k\}_{k\in\mathbb{N}}$ defined in a neighborhood U of z_0 such that $0 \leqq \lambda_k \leqq 1$ and

$$(2.17) \qquad \sum_{i,j} \frac{\partial^2 \lambda_k}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j \geqq k|a|^2 \qquad \text{on} \quad M \cap U$$

for all $a \in \mathbb{C}^n$.

Let Ω be the pseudo-convex domain with boundary M. Let $B_\epsilon(z_0)$ denotes a ball of radius ϵ with center z_0 and $K_\epsilon = \overline{B}\epsilon(z_0) \setminus B_{\frac{\epsilon}{2}}(z_0)$. We claim that there exists a strictly plurisubharmonic function ϕ_ϵ defined on $\overline{\Omega \cap B_\epsilon(z_0)}$ such that

$$(2.18) \qquad \phi_\epsilon(z_0) > \sup_{K_\epsilon} \phi_\epsilon(z).$$

Let $\eta \in C_o^\infty(B_\epsilon(z_0))$ such that $\eta = 3$ on $B_{\frac{\epsilon}{4}}(z_0)$ and $\eta = 0$ on K_ϵ. We define $g_k = \eta + \lambda_k$. We have $g_k(z_0) \geqq 2$ and

$$g_k(z_0) > \sup_{K_\epsilon} g_k.$$

If we choose k sufficiently large, from (2.17), we have that g_k is strictly plurisubharmonic near $\overline{M \cap \overline{B}_\epsilon(z_0)}$. Using Proposition 3.16 in Catlin [5], there exists function ϕ_ϵ on $\overline{\Omega \cap B_\epsilon(z_0)}$ such that

$$\phi_\epsilon = g_k \qquad \text{on} \quad M \cap B_\epsilon(z_0)$$

and

$$\phi_\epsilon \leqq g_k \qquad \text{on} \quad \overline{\Omega \cap B_\epsilon(z_0)}.$$

One easily sees that ϕ_ϵ satisfies (2.18). It follows from Theorem 3.15 in [5] that the holomorphic convex hull of K_ϵ is the same as the hull of K_ϵ with respect to plurisubharmonic functions. Thus there exists a holomorphic function $f_\epsilon \in A^\infty(\overline{\Omega \cap B_\epsilon})$ such that

$$f_\epsilon(z_0) > \sup_{K_\epsilon} |f(z)|$$

Multiplying f_ϵ by a nonzero constant and raise to high order if necessary, we can assume that $f_\epsilon(z_0) = 1$ and $\sup_{K_\epsilon} |f(z)| \leqq \frac{1}{2}$. Applying Sard's theorem, one can find a regular value μ of the level set $\{Ref_\epsilon = \mu$, where $\frac{1}{2} < \mu < 1\} = H_\epsilon$ is a smooth hypersurface and M intersects H_ϵ transversally. Letting $\omega_\epsilon \equiv M \cap \{Ref_\epsilon > \mu\}$, it is easy to see that $\omega_\epsilon \subset B_\epsilon(z_0) \cap M$. Thus we have constructed a neighborhood basis $\{\omega_\epsilon\}$ which satisfies all the hypothesis of Theorem 2 and Corollary 2.1 and Theorem 3 is proved.

3. AN EXAMPLE

In this section we shall examine some examples which were due to Rosay [19] and construct an explicit example of a $\bar{\partial}_b$-closed form which does not satisfy condition (A). Let S_n be the unit sphere in $\mathbb{C}^n$, $n \geqq 3$ and $\Sigma_1 = S_n \cap \{|z_1|^2 < \frac{1}{2}\}$, $\Sigma_2 = S_n \cap \{|z_1|^2 > \frac{1}{2}\}$. It is proved in [19] that one can solve Eq.(0.1) for any (p,q) form α satisfying condition (0.2) in Σ_2 for all $1 \leqq q \leqq n-2$. While on Σ_1, this is only true when $1 \leqq q < n-2$. We note that the boundary of Σ_1 and Σ_2 lie in the Levi flat hypersurface $M_0 = \{|z|^2 = \frac{1}{2}\}$ which has a Levi-flat Stein neighborhood basis. Let $n = 3$ and $\zeta(z_3)$ be a cut-off function such that $\zeta(z_3) = 0$ when $|z_3|^2 \geqq \frac{1}{4}$ and $\zeta(z_3) > 0$ when $|z_3|^2 < \frac{1}{4}$. We define

$$f = \frac{\zeta(z_3)}{z_2} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_3$$

and

$$\alpha = \tau f$$

where τ is the projection operator from $(3,1)$ form in $\mathbb{C}^3$ to Σ_1 defined in Section I. It is easy to see that α is a smooth (3,1) form on Σ_1, since for $z \in S_3$, $|z_2|^2 = 1 - |z_1|^2 - |z_3|^2 \geqq \frac{1}{2} - |z_3|^2 > 0$ on the support of ζ. Also $\bar{\partial} f = 0$ on Σ_1 which implies that $\bar{\partial}_b \alpha = 0$ on Σ_1. If we set $h(z) = \frac{1}{z_1}$, then $h \in \mathcal{O}(\partial\Sigma_1)$ and

$$\begin{aligned}
\int_{\partial\Sigma_1} \alpha \wedge h &= \int_{\partial\Sigma_1} \frac{1}{z_1} \frac{\zeta(z_3)}{z_2} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_3 \\
&= \iint_{|z_3|^2 \leqq \frac{1}{4}} \zeta(z_3) \int_{|z_2|^2 = \frac{1}{2} - |z_3|^2} \int_{|z_1|^2 = \frac{1}{2}} \frac{1}{z_1 z_2} dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_3 \\
&= (2\pi i)^2 \iint_{|z_3| \leqq \frac{1}{2}} \zeta(z_3) dz_3 \wedge d\bar{z}_3 \\
&\neq 0
\end{aligned}$$

Thus α does not satisfy the necessary condition (A) and thus can not be solved on Σ_1 (or on arbitrarily large subset of Σ_1). We note that α is not smooth on Σ_2. On the other hand, since $\mathcal{O}(\Sigma_2)$ is dense in $\mathcal{O}(\partial\Sigma_2)$, Any $\bar{\partial}_b$-closed (3,1) form can be solved on Σ_2 from Theorem 2. We also note that the boundary of Σ_2 is not Runge.

References

1. A. Andreotti and C.D. Hill, *Convexity and the H. Lewy problem. I and II.*, Ann. Scuola Norm. Sup. Pisa **26** (1972), 325-363, ibid,747-806.
2. E. Bedford and P. de Bartolomeis, *Levi flat hypersurfaces which are not holomprphically flat*, Pro. A. M. S. (1981), 575-578.
3. E. Bedford and J. E. Fornaess, *Domains with pseudoconvex neiborhood systems*, Invent. Math. **47** (1978), 1-27.
4. A. Boggess and M.-C. Shaw, *A kernel approach to local solvability of the tangential Cauchy-Riemann equations*, Trans. Amer. Math. Society **289** (1985), 643-659.
5. D. Catlin, *Boundary behavior of holomorphic functions on pseudoconvex domains*, J. of differential Geometry **15** (1980), 605-625.
6. D. Catlin, *Global regularity of the $\bar{\partial}$-Neumann problem*, Pro. Symp. pure Math. **41** (1984), 39-49.
7. D. Catlin, *Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudo-convex domains*, Ann. of Math. **126** (1987), 131-191.
8. J. P. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. **115** (1982), 615-637.
9. G.B. Folland and J. J. Kohn, *The Neumann problem for the CauchyRiemann complex*, vol. 75, Ann. of Math. Studies, Princeton Univ. Press, Princeton, N.J., 1972.
10. G. M. Henkin, *The Lewy equation and analysis on pseudo-convex manifolds*, Uspheki Mat. Nauk **32** (1977), 57-118 English transl. in Russ. Math. Surv., 32, 1977, 59-130.
11. L. Hörmander, *Linear partial differential operators*, Springer-verlag, New york, 1963.
12. L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113** (1965), 89-152.
13. H. Lewy, *On the local character of the solution of an atypical differential equation in three variables and related proble for regular functions of two complex variables*, Ann. Math. **64** (1956), 514-522.
14. J. J. Kohn, *Global regularity for $\bar{\partial}$ on weakly pseudo-convex manifolds*, Trans. of A. M. S. **181** (1973), 273-292.
15. J. J. Kohn, *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains: Sufficient conditions*, Acta Math. **142** (1979), 79-122.
16. J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. Math. **81** (1965), 451-472.
17. A. Nagel and J.-P. Rosay, *Nonexistence of homotopy formula for (0,1) forms on hypersurfaces in $\mathbb{C}^3$*, Duke Math. Jour. **58** (1989), 823-827.
18. J.-P. Rosay, *Equation de Lewy-résolubilite globale de l'équation $\partial_b u = f$ sur la frontiére de domaines faiblement pseudo-convexes de $\mathbb{C}^2$ (ou $\mathbb{C}^n$)*, Duke Math. J. **49** (1982), 121-128.
19. J.-P. Rosay, *Some applications of Cauchy-Fantappie forms to (local) problems on $\bar{\partial}_b$*, Ann. Scuola Normale Sup. Pisa **13** (1986), 225-243.
20. M.-C. Shaw, *L^p estimates for local solutions of $\bar{\partial}_b$ on strongly pseudo-convex CR manifolds*, Math. Ann. **288** (1990), 35-62.
21. M.-C. Shaw, *L^2 existence theorems for the $\bar{\partial}_b$-Neumann problem on strongly pseudo-convex CR manifolds*, Jour. Geometric Analysis **1** (1991), 139-163.
22. M.-C. Shaw, *Local existence theorems with estimates for $\bar{\partial}_b$ on weakly pseudoconvex CR manifolds*, Math. Ann. **294** (1992), 677-700.
23. F. Treves, *Homotopy formulas in the tangential Cauchy-Riemann complex*, Memoirs of the Amer. Math. Society, Providence, Rhode Island.

24. S. M. Webster, *On the local solution of the tangential CauchyRiemann equations*, Ann. Inst Henri Poicaré 6 (1989), 167-182.
25. S. M. Webster, *On the proof of Kuranishi's embedding theorem*, Ann. Inst H. Poincaré 6 (1989), 183-207.

Mei-Chi Shaw
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556 USA
e-mail address: mei.chi.shaw1@nd.edu