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Straightening of Arcs

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pour son oeuvre sur l' hibernation des marmottes.

Introduction.

Any smooth arc γ in $\mathbb{C}^n$ is polynomially convex ([4], or [5]). And one can approximate any continuous function on γ by polynomials. Our goal is to show that if $n \geq 2$, under a global biholomorphic change of variables, an arc can always be "straightened" (approximately mapped to a line segment). This makes polynomial convexity and polynomial approximation trivial, unfortunately we need to use polynomial convexity in our proof. Here is a precise statement.

Proposition *Let Γ be a smooth ($\mathcal{C}^\infty$) arc in $\mathbb{C}^n (n \geq 2)$, parametrized by $\gamma : [0,1] \to \mathbb{C}^n$. There exists (T_j) a sequence of automorphisms of $\mathbb{C}^n$ so that: $T_j \circ \gamma$ converges, in $\mathcal{C}^\infty$ topology, to the map $t \mapsto (t, 0, \ldots, 0)$, and the restriction of $\overset{-1}{T}_j$ to $[0,1] \times \{0, \ldots, 0\})$ (identified with $[0,1]$) converges to γ in $\mathcal{C}^\infty$ topology.*

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Remarks.

1) The Proposition above can be generalized to smooth totally real disks (of real dimension $k, k \leq n$) which are polynomially convex. In [2], a joint work with F. Forstnerič, the case of totally real manifolds is studied in much greater generality (in the real analytic setting).
2) There is a reason why we want not only the convergence of T_j on Γ, but also the convergence of $\overset{-1}{T}_j$ on $[0,1] \times \{(0,\ldots,0)\}$. This is explained in II.

I. Proof of the Proposition

In I.1 we consider the case of real analytic arcs. For real analytic arcs, the Proposition follows very easily from a recent theorem by Andersen and Lempert [1], (see also [2]). In I.2 the case of smooth arcs is considered.

The theorem by Andersen and Lempert is this following:

Theorem (Andersen - Lempert) *Let T be a biholomorphic map from a star shaped domain Ω onto a Runge domain Ω' (in $\mathbb{C}^n, n \geq 2$). Then T can be uniformly approximated on compact sets in Ω by (global) automorphisms of $\mathbb{C}^n$.*

I.1 The real analytic case

We suppose that γ is a real analytic map from $[0,1\}$ into $\mathbb{C}^n, \dot{\gamma} \neq 0$ and γ is $1-1$. Set $J = [0,1] \times \{(0,\ldots,0)\} \subset \mathbb{C}^n$. We identify $[0,1]$ with J. Then we can extend γ to a holomorphic map $\tilde{\gamma}$ defined in some neighborhood U of J in $\mathbb{C}^n$, and $1-1$. For example, we can set

$$\tilde{\gamma}(z_1,\ldots,z_n) = \gamma(z_1) + \sum_{j=2}^{n} z_j \alpha_j(z_1).$$

In this formula γ denotes the holomorphic extension of γ to a neighborhood of $[0,1]$ in $\mathbb{C}$, and one has to take the ($\mathbb{C}^n$ valued) maps α_j holomorphic and so that for any $t \in [0,1]$ the vectors $\left(\dot{\gamma}(t), \alpha_2(t), \ldots, \alpha_n(t)\right)$ are linearly independent.

Let φ be a function defined on $\mathbb{R}$, which is 0 in $[0,1]$ and strictly positive and convex off $[0,1]$. Set $\rho(z_1, \ldots, z_n) = \varphi(x_1) + |y_1|^2 + \sum_{j=2}^{n} |z_j|^2$. For $\epsilon > 0$ let $U_\epsilon = \{z \in \mathbb{C}^n,\ \rho(z) < \epsilon\}$. This is a convex and strictly pseudoconvex neighborhood of J. For ϵ smaller than some ϵ_0 positive, $U_\epsilon \subset U$. We claim that for ϵ small enough $\tilde{\gamma}(U_\epsilon)$ is Runge. Indeed, since Γ is polynomially convex, Γ has a basis of Stein neighborhood which are Runge. Fix such a neighborhood $V \subset \tilde{\gamma}(U)$. Take ϵ so small that $\tilde{\gamma}(U_\epsilon) \subset V$. Then $\tilde{\gamma}(U_\epsilon)$ is Runge in V since it is defined by the inequality $\rho \circ \tilde{\gamma}^{-1} < \epsilon$ and $\rho \circ \tilde{\gamma}^{-1}$ is strictly plurisubharmonic ([3] Theorem 4.3.2).

Since V is Runge in $\mathbb{C}^n, \tilde{\gamma}(U_\epsilon)$ is Runge in $\mathbb{C}^n$, as desired.

We can apply the Andersen Lempert theorem to approximate the restriction of $\tilde{\gamma}$ to U_ϵ, uniformly on compact sets, by a sequence (S_j) of biholomorphisms of $\mathbb{C}^n$. Finally, set $T_j = \overset{-1}{S}_j$. We have better than convergence in the $\mathcal{C}^\infty$ topology on Γ or J. We have uniform convergence on neighborhoods of J and Γ, $\overset{-1}{T}_j$ converges to $\tilde{\gamma}$ on U_ϵ and (by the implicit function theorem) T_j converges, uniformly on compact sets to $\overset{-1}{\tilde{\gamma}}$ on $\tilde{\gamma}(U_\epsilon)$. Hence $T_j \circ \gamma$ converges to the map $t \mapsto (t, 0, \ldots, 0)$.

II.2 The smooth case

A very natural idea is to approximate smooth arcs by real analytic arcs, and then to straighten the real analytic arcs following I.1. This is what we are going to do, but it needs to be done with care (I did not find a trivial trick). In particular, one cannot expect uniform convergence on neighborhoods of Γ and $[0,1]$ as in the real analytic case. We first prove a Lemma to handle this question of convergence of maps defined on shrinking neighborhoods.

II.2.1 Lemma: *Let $K \subset \Omega \subset \mathbb{R}^p$ K convex and compact, Ω open. Set $\Omega_j = \{z \in \Omega, \operatorname{dist}(z, K) < \frac{1}{j}\}$. Let χ be a diffeomorphism from Ω into $\mathbb{R}^p$. Set $K' = \chi(K)$. Let χ_j be a sequence of smooth maps $\chi_j : \Omega_j \to \mathbb{R}^p$ so that $\|\chi_j - \chi\|_{C^1(\Omega_j)} \leq \frac{C}{j^2}$. Then, for j large enough, $K' \subset \chi_j(\Omega_j)$, and χ_j is a diffeomorphism. And if $\|\chi_j - \chi\|_{C^k(\Omega_j)}$ tends to 0 as j tends to ∞, so does $\|\overset{-1}{\chi}_j - \overset{-1}{\chi}\|_{C^k(K')}$.*

(By $\|\Psi\|_{\mathcal{C}^k(K')}$, we mean $\sup\limits_{\substack{|\alpha|\leq k \\ x\in K'}} |D^\alpha \Psi(x)|$, i.e. the $\mathcal{C}^k$ norm on jets which is stronger than the $\mathcal{C}^k$ norm of the restrictions)

Proof: Shrinking Ω if needed, we can assume that Ω is convex and that there is a constant $A > 0$ so that for all x and $y \subset \Omega$

$$\frac{1}{A}|x-y| \leq |\chi(x) - \chi(y)| \leq A|x-y|.$$

For j large enough (so that $\frac{C'}{j^2} < \frac{1}{2A}$) one gets

$$\frac{1}{2A}|x-y| \leq |\chi_j(x) - \chi_j(y)| \leq 2A|x-y|.$$

Hence χ_j is a diffeomorphism (whose Jacobian together with its inverse is bounded). Let $a \in K$, $S_j = \{z \in \mathbb{C}^n \ \ |z-a| = \frac{1}{j}\}$. The image of S_j under χ is a topological sphere which "contains" the ball of radius $\frac{1}{2Aj}$ centered at $\chi(a)$. For any $z \in S_j$, $|\chi(z) - \chi_j(z)| \leq \frac{C}{j^2}$.

If $j \geq 4AC$, so $\frac{1}{2Aj} - \frac{C}{j^2} \geq \frac{1}{4Aj}$, one sees that the image, under χ_j, of the ball $\{z \in \mathbb{C}^n \,|z - a| \leq \frac{1}{2j}\}$ will contain the ball of radius $\frac{1}{4Aj}$ centered at $\chi(a)$. Hence $\chi(a) \in \chi_j(\Omega_j)$, $K' \subset \chi_j(\Omega_j)$. The reader may dislike this topological argument and indeed one can clearly use other methods (like the contraction principle) for more constructive proofs.

We get also from the proof the following information if $b \in K'$ (and j is large enough):

$$|\overset{-1}{\chi}(b) - \overset{-1}{\chi}_j(b)| \leq \frac{1}{2j}.$$

Indeed, set $b = \chi(a)$ above.

To estimate $\|\overset{-1}{\chi} - \overset{-1}{\chi}_j\|_{C^k(K')}$ one has to compare the derivatives of order $\leq k$ of $\overset{-1}{\chi}$ and $\overset{-1}{\chi}_j$ at points $b \in K'$. These are obtained from the partial derivatives of χ at $\overset{-1}{\chi}(b)$ and χ_j at $\overset{-1}{\chi}_j(b)$. One compares the partial derivatives of χ at $\overset{-1}{\chi}(b)$ with the corresponding derivatives of χ at $\overset{-1}{\chi}_j(b)$, the difference is at most $\frac{C_k}{j}\|\overset{-1}{\chi}\|_{C^{k+1}}$. Then one compares the derivatives of χ and χ_j at $\overset{-1}{\chi}_j(b)$.
The Lemma is thus established.

II.2.2 Proof of the Proposition (smooth case)

Again $J = [0,1] \times \{(0,\ldots,0)\} \subset \mathbb{C}^n$. We extend γ to a map $\tilde{\gamma}$ defined on $\mathbb{C}^n$ with the following properties:

$$\begin{cases} \tilde{\gamma}(t,0,\ldots,0) = \gamma(t) \text{ for } t \in [0,1] \\ \bar{\partial}\tilde{\gamma} \text{ vanishes to infinite order along } R \times \{0,\ldots,0\} \\ \tilde{\gamma} \text{ defines a diffeomorphism from a neighborhood of } J \text{ onto } \mathbb{C}^n \end{cases}$$

If one wishes to keep the most elementary tools, it is better to do it more precisely by taking:

$$\tilde{\gamma}(z_1, \ldots, z_n) = \gamma(z_1) + \sum_{j=2}^{n} z_j \alpha_j(z_1).$$

Now γ denotes an almost analytic ($\overline{\partial}$ vanishing to infinite order along $\mathbb{R}$) extension of γ to $\mathbb{C}$, that we can take with compact support. And the α_j's are holomorphic polynomials maps so that for any $t \in [0,1]$ the vectors $(\dot{\gamma}(t), \alpha_2(t), \ldots, \alpha_n(t))$ are linearly independent.

So we have

$$\begin{cases} \tilde{\gamma} & \text{is holomorphic in } (z_2, \ldots, z_n) \\ \bar{\partial}\tilde{\gamma} & \text{vanishes to infinite order along } \mathbb{R} \times \mathbb{C}^{n-1}. \end{cases}$$

These facts will facilitate our proof.

Let λ be a $\mathcal{C}^\infty$ function on $\mathbb{R}$ so that $\lambda(s) = 0$ if $x \geq 3$ and $\lambda(x) = 1$ if $x \leq 2$. For $j \in \mathbb{N}$, let λ_j be the function defined on $\mathbb{C}$ by $\lambda_j(x + jy) = \lambda(jy)\lambda(-jy)$. This is a function which is equal to 1 if $|y| < \frac{2}{j}$, equal to 0 if $|y| > \frac{3}{j}$, and with polynomial (in j) estimates on the derivatives as $j \to +\infty$.

We solve the problem $\bar{\partial} u_j = \lambda_j(z_1)\bar{\partial}\tilde{\gamma}$ by elementary means: $u_j(z_1, \ldots, z_n) = \frac{1}{\pi z_1} * \lambda_j \frac{\partial \tilde{\gamma}}{\partial \bar{z}_1}$, where the convolution takes place in the z_1 variable. Remember that $\tilde{\gamma}$ is holomorphic in $(z_2, \ldots, z_n)$. Due to the fast decay of $\frac{\partial \tilde{\gamma}}{\partial \bar{z}_1}$ and its derivatives, when approaching $\mathbb{R} \times \mathbb{C}^{n-1}$, u_j tends to 0 in $\mathcal{C}^\infty$ topology. And the convergence is fast: for every k and $l \in \mathbb{N}$ and every compact set $H \subset \mathbb{C}^n$:

$$\|u_j\|_{\mathcal{C}^k(H)} = \mathcal{O}(j^{-l}) \quad (j \to \infty).$$

We take the same notations as in I.1

$$\rho(z_1,\dots,z_n) = \varphi(x_1) + |y_1|^2 + \sum_{j=2}^{n} |z_j|^2$$

and

$$U_\epsilon = \{z \in \mathbb{C}^n,\ \rho(z) < \epsilon\}.$$

We fix U a bounded neighborhood of J in $\mathbb{C}^n$, small enough in order that $\tilde{\gamma}$ defines a diffeomorphism from a neighborhood of $\bar{U}$ into $\mathbb{C}^n$ and $\rho \circ \tilde{\gamma}^{-1}$ is strictly plurisubharmonic of $\tilde{\gamma}(U)$. This is possible since $\tilde{\gamma}$ has maximum rank along J and $\bar{\partial}\tilde{\gamma}$ vanishes to infinite (therefore second) order along J.

We fix V a Stein neighborhood of Γ, which is Runge and so that $\bar{V} \subset \tilde{\gamma}(U)$. For j large enough $\tilde{\gamma} - u_j$ is a diffeomorphism from U into $\mathbb{C}^n$ so that $\tilde{\gamma} - u_j(U) \supset V$, and on V $\rho \circ (\tilde{\gamma} - u_j)^{-1}$ is strictly plurisubharmonic. Here we do not use any holomorphicity of $\tilde{\gamma} - u_j$ (which is true only on a small neighborhood of J), but only the fact that $\tilde{\gamma} - u_j$ is a small ($\mathcal{C}^\infty$) perturbation of $\tilde{\gamma}$, on U which is fixed.

Now the same argument as in I.1 (where we used the plurisubharmonicity of $\rho \circ \tilde{\gamma}^{-1}$) shows that there exist $j_\circ \in \mathbb{N}$ and $\epsilon_\circ > 0$ so that for any $j \geq j_\circ$ and $\epsilon \in (0, \epsilon_\circ)$, $(\tilde{\gamma} - u_j)(U_\epsilon)$ is a Runge domain in $\mathbb{C}^n$.

In particular, for j large enough, $(\tilde{\gamma} - u_j)$ is $1-1$ on U_{2/j^2} and $(\tilde{\gamma} - u_j)(U_{2/j^2})$ is a Runge domain. Remember

$$U_{2/j^2} = \left\{ z \in \mathbb{C}^n, \rho(z) = \varphi(x_1) + |y_1|^2 + \sum_{j=2}^{n} |z_j|^2 \leq \frac{2}{j^2} \right\}.$$

Due to the choice of u_j, $\tilde{\gamma} - u_j$ is holomorphic on U_{2/j^2}. By the Andersen Lempert theorem one can approximate $\tilde{\gamma} - u_j$ on U_{2/j^2} by global biholomorphisms of $\mathbb{C}^n$.

Hence one can find a sequence (S_j) of biholomorphisms of $\mathbb{C}^n$ so that for every $k \in \mathbb{N}$ and $l \in \mathbb{N}$:

$$\|\tilde{\gamma} - S_j\|_{\mathcal{C}^k(U_{1/j^2})} = \mathcal{O}(j^{-l}) \quad (j \to \infty).$$

The set U_{1/j^2} contains the $1/j$ neighborhood of J in $\mathbb{C}^n$. The Lemma yields the Proposition, by taking $T_j = \overset{-1}{S}_j$.

Note: To generalize the Proposition to polynomially convex totally real disks, instead of arcs, one needs to use more sophisticated ways to solve $\bar{\partial}$, and one cannot use cutoffs, as given by λ_j.

III Discussion

We would like to say that two sets K and K' in $\mathbb{C}^n$ are "equivalent" if one can approximately move K to K' by an automorphism of $\mathbb{C}^n$. However one should be careful. We certainly do not intend to say that the polynomially convex circle $\Gamma_1 = \{(e^{i\theta}, e^{-i\theta})\}$ is equivalent (in $\mathbb{C}^n$) to the circle $\Gamma_2 = \{(e^{i\theta}, 0)\}$. But Γ_1 can be approximately mapped to Γ_2 by the automorphism $(z, w) \to (z, \frac{w}{n})$ (n large). This is why we propose the definition:

Definition *K and K' are equivalent if and only if there exists (T_j) a sequence of automorphisms of $\mathbb{C}^n$ and T a homeomorphism (or diffeomorphism) from K onto K' so that:*

$$T_j \mid K \text{ tends to the map } T$$

$$\text{and } \overset{-1}{T}_j \mid K \text{ tends to the map } \overset{-1}{T}, \text{ uniformly (or in } \mathcal{C}^k \text{ norm, or } \ldots).$$

The convergence of $\overset{-1}{T}_j$ on K does not follow from the convergence of T_j on K. And even if one requires that T_j converges on K to T and $\overset{-1}{T}_j$ converges on

K', this does not force the limit of $\overset{-1}{T}_j$ to be $\overset{-1}{T}$ (take K and K' consisting of two points, not the same points).

The definition given above has at least the following advantage: If K and K' are equivalent, K is polynomially convex if and only if K' is also polynomially convex. More generally, the bijection T from K onto K' extends to a bijective map from $\hat{K}$ (the polynomial hull of K) onto $\hat{K}'$.

Indeed: Since T is a limit of polynomials maps T extends to $\hat{K}$, in the same way $\overset{-1}{T}$ extends to $\hat{K}'$. The map T extended to $\hat{K}$ maps $\hat{K}$ into $\hat{K}'$, because any polynomial map P maps $\hat{K}$ into the polynomial hull of $P(K)$, and by a limit argument. If we compose the extended maps $\overset{-1}{T}$ and T, one gets the identity map on K, therefore on $\hat{K}$ . . . .

References:

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