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# Graded $C^*$-algebras and many-body perturbation theory : II. The Mourre estimate

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# Graded $C^*$-Algebras and Many-Body Perturbation Theory: II. The Mourre Estimate

Anne Boutet de Monvel-Berthier and Vladimir Georgescu [1]

## 1. Introduction

We have introduced in [BG 1,2] the notion of graded $C^*$-algebra with the purpose of obtaining a natural framework for the description and study of hamiltonians with a many-channel structure. If H is a self-adjoint operator in a Hilbert space $\mathscr{H}$, the expression "H has a many-channel structure" is not mathematically well defined, although in examples of physical interest the meaning is rather obvious. Spectral theory alone is not enough in order to decide whether H is a many-channel hamiltonian or not. Usually the distinction is acquired with the help of scattering theory through the introduction of the channel wave operators. However, there are results (like the HVZ theorem which describes the essential spectrum of a N-body hamiltonian in terms of the spectra of the subsystems) which are outside the scope of scattering theory but should belong to a general theory of "many-channel hamiltonians". Our proposal in [BG 1,2] was to define the many-channel character of a self-adjoint operator H by its affiliation to a $C^*$-algebra provided with a graduation which allows one to describe a "subsystem structure" for the system whose hamiltonian is H. From our point of view, the main object associated to the physical system is a graded $C^*$-algebra, the possible dynamics are given by self-adjoint operators H affiliated to it, and we are interested in assertions independent of the explicit form of H.

Our purpose here is to show that the Mourre estimate fits very nicely in such a framework. Given two self-adjoint operators H, A such that the commutator [H,A] is a continuous sesquilinear form on D(H), we associate to them a function $\rho:\mathbb{R}\to]-\infty,+\infty]$ in terms of which the property of A of being locally conjugated to H is easily described. If the action of the unitary group associated to A is compatible in some sense with the grading of the $C^*$-algebra and if this algebra has a property which we call reducibility, then the ρ-function associated to H can be estimated in terms of the ρ-functions associated to "sub-hamiltonians". Our arguments are inspired from those of Froese and Herbst [FH], but the main point here is that the explicit form of H is never used, but only its affiliation to the algebra. In particular, in the N-body case H could be of the form described in

[1] Lecture delivered by V. Georgescu

Proposition 7 of [BG 2] (see also section 2 below; this class is more general than the class of dispersive hamiltonians of [D2] and [G]) or it could be a hamiltonian with hard core interactions (this situation is treated in a joint work with A.Soffer, paper in preparation). We shall explicitely calculate the ρ-function (and so get the result of [PSS] and [FH]) for Agmon hamiltonians using theorem 3.4 which gives the ρ-function of an operator $H = H_1 \otimes 1 + 1 \otimes H_2$ in terms of those of $H_j$ assuming that A is similarly decomposable. Theorems 3.4 and 4.4 are, technically speaking the main results of this paper, the applications to hamiltonians affiliated to the N-body $C^*$-graded algebra, being only an example (in this context theorem 2.1 being important)

In the rest of this section we shall recall the framework introduced in [BG1,2]. Some more specific properties of what we call the N-body $C^*$-graded algebra are studied in section 2. In section 3 we introduce in a more general setting the ρ-functions (which are more systematically studied in [ABG 2]) and prove the first important result, formula (3.8). Finally, in section 4 we define the reducible algebras and show how a Mourre estimate is proved for hamiltonians affiliated to such algebras.

We recall now the definition of a $C^*$-graded algebra as introduced in [BG1,2]. Let $\mathscr{A}$ be a $C^*$-algebra and $\mathscr{L}$ a finite lattice, i.e. a finite partially ordered set such that the upper bound $Y \vee Z$ and the lower bound $Y \wedge Z$ of each pair $Y,Z \in \mathscr{L}$ exists. We shall denote **O** (resp. X) the least (resp. the biggest) element of $\mathscr{L}$. We say that *$\mathscr{A}$ is a $\mathscr{L}$-graded $C^*$-algebra* if a family $\{\mathscr{A}(Y)\}_{Y \in \mathscr{L}}$ of $C^*$-subalgebras of $\mathscr{A}$ is given such that

(i) $\mathscr{A} = \Sigma\{\mathscr{A}(Y) \mid Y \in \mathscr{L}\}$, the sum being direct (as linear spaces);

(ii) $\mathscr{A}(Y)\mathscr{A}(Z) \subset \mathscr{A}(Y \vee Z)$ for all $Y,Z \in \mathscr{L}$.

One can introduce such a notion for infinite $\mathscr{L}$ also (then $\Sigma\{\mathscr{A}(Y) \mid Y \in \mathscr{L}\}$ is only dense in $\mathscr{A}$) and an interesting example of such an object will appear in the next section.

We can put in evidence a *filtration* of $\mathscr{A}$ by a family $\{\mathscr{A}_Y\}_{Y \in \mathscr{L}}$ of $C^*$-subalgebras by defining $\mathscr{A}_Y = \Sigma\{\mathscr{A}(Z) \mid Z \leq Y\}$. Then $\mathscr{A}_Y \subset \mathscr{A}_Z$ if $Y \leq Z$ and $\mathscr{A}_X = \mathscr{A}$. If we denote $\mathscr{L}(Y) = \{Z \in \mathscr{L} \mid Z \leq Y\}$, then $\mathscr{L}(Y)$ is a finite lattice also and $\mathscr{A}_Y$ is a $\mathscr{L}(Y)$-graded $C^*$-algebra in a canonical way. Finally, observe that $\mathscr{A}(X)$ is a $*$-ideal in $\mathscr{A}$ (so $\mathscr{A}(Y)$ is a $*$-ideal in $\mathscr{A}_Y$), and if we denote $\mathscr{B}_Y = \Sigma\{\mathscr{A}(Z) \mid Z \nleq Y\}$, then $\{\mathscr{B}_Y\}_{Y \in \mathscr{L}}$ is a decreasing family of closed $*$-ideals in $\mathscr{A}$ such that $\mathscr{A} = \mathscr{A}_Y + \mathscr{B}_Y$ (algebraic direct sum) for all $Y \in \mathscr{L}$.

For each $Y \in \mathscr{L}$ we shall denote $\mathscr{P}(Y)$, $\mathscr{P}_Y$ the projection operators of $\mathscr{A}$ onto $\mathscr{A}(Y)$, resp. $\mathscr{A}_Y$, associated to the direct sum decompositions $\mathscr{A} = \Sigma\{\mathscr{A}(Y) \mid Y \in \mathscr{L}\}$

resp. $\mathscr{A}=\mathscr{A}_Y+\mathscr{B}_Y$. More precisely, if $S\in\mathscr{A}$, then one can write it in a unique way as a sum $S=\Sigma\{S(Y)\mid Y\in\mathscr{L}\}$ with $S(Y)\in\mathscr{A}(Y)$. Then $\mathscr{P}(Y)(S)=S(Y)$. Obviously $\mathscr{P}_Y=\Sigma\{\mathscr{P}(Z)\mid Z\leq Y\}$, which is equivalent to $\mathscr{P}(Y)=\Sigma\{\mathscr{P}_Z\,\mu(Z,Y)\mid Z\leq Y\}$, where $\mu:\mathscr{L}\times\mathscr{L}\to\mathbb{Z}$ is the Möbius function of $\mathscr{L}$. Clearly each $\mathscr{P}(Y):\mathscr{A}\to\mathscr{A}$ is a linear, continuous projection (i.e.$\mathscr{P}(Y)^2=\mathscr{P}(Y)$) which commutes with the involution. But the main point is that $\mathscr{P}_Y:\mathscr{A}\to\mathscr{A}$ *is a linear, continuous projection which is also a* $*$*-homomorphism of* $\mathscr{A}$ *onto* $\mathscr{A}_Y$. In particular, if $S\in\mathscr{A}$ is a normal element and f is a complex continuous function on the spectrum of S (which vanishes at zero if $\mathscr{A}$ has not unit) then $\mathscr{P}_Y(f(S))=f(\mathscr{P}_Y(S))$. Observe that $\mathscr{B}_Y=\ker\mathscr{P}_Y$, which gives a new proof of the fact that $\mathscr{B}_Y$ is a closed $*$-ideal in $\mathscr{A}$.

Let $\mathscr{A}$ be an arbitrary $C^*$-algebra realised on a Hilbert space $\mathscr{H}$ (i.e. $\mathscr{A}$ is a $C^*$-subalgebra of $B(\mathscr{H})$, the space of bounded linear operators in $\mathscr{H}$) and H a self-adjoint operator in $\mathscr{H}$. Denote $C_\infty(\mathbb{R})$ the abelian $C^*$-algebra of complex continuous functions on $\mathbb{R}$ which tend to zero at infinity (with the sup norm). Then $(\lambda-H)^{-1}\in\mathscr{A}$ for some complex $\lambda$ if and only if $f(H)\in\mathscr{A}$ for all $f\in C_\infty(\mathbb{R})$. If this is fulfilled, we shall say that H *is affiliated to* $\mathscr{A}$. In some applications it is useful to work with *self-adjoint but non-densely defined operators in* $\mathscr{H}$. By this we mean that a closed subspace $\mathscr{K}$ of $\mathscr{H}$ and a self-adjoint densely defined operator H in $\mathscr{K}$ are given (so $\mathscr{K}$ is the closure of the domain of H in $\mathscr{H}$; think, formally, that $H=\infty$ on $\mathscr{H}\ominus\mathscr{K}$). Let then $R(\lambda)=(\lambda-H)^{-1}$ on $\mathscr{K}$ and $R(\lambda)=0$ on $\mathscr{H}\ominus\mathscr{K}$, for $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Clearly, the family $\{R(\lambda)\mid\lambda\in\mathbb{C}\setminus\mathbb{R}\}$ of bounded operators in $\mathscr{H}$ is a pseudo-resolvent, i.e. $R(\lambda)^*=R(\lambda^*)$ and $R(\lambda_1)-R(\lambda_2)=(\lambda_2-\lambda_1)R(\lambda_1)R(\lambda_2)$. In fact, as shown in [HP], there is a bijective correspondence between (not necessarily densely defined) self-adjoint operators in $\mathscr{H}$ and pseudo-resolvents on $\mathscr{H}$ (or spectral measures E such that $E(\mathbb{R})\neq 1$). Using Stone-Weierstrass theorem, it is trivial to establish a bijective correspondence between pseudo-resolvents and $*$-homomorphisms $\phi:C_\infty(\mathbb{R})\to B(\mathscr{H})$ (put $R(\lambda)=\phi(r_\lambda)$ where $r_\lambda(x)=(\lambda-x)^{-1}$). Clearly $\phi(f)|_{\mathscr{K}}=f(H)$ and $\phi(f)|_{\mathscr{H}\ominus\mathscr{K}}=0$.

As a conclusion of this discussion, if $\mathscr{A}$ is an arbitrary C*-algebra, a $*$-homomorphism $\phi:C_\infty(\mathbb{R})\to\mathscr{A}$ will be called *self-adjoint operator affiliated to* $\mathscr{A}$. As above, to give $\phi$ is equivalent to giving a pseudo-resolvent $\{R(\lambda)\mid\lambda\in\mathbb{C}\setminus\mathbb{R}\}$ with $R(\lambda)\in\mathscr{A}$. We shall use in such a case a symbol H and denote $\phi(f)=f(H)$ for $f\in C_\infty(\mathbb{R})$ and $R(\lambda)=(\lambda-H)^{-1}$. When $\mathscr{A}$ is realised in a Hilbert space $\mathscr{H}$, then H is realised as a (non-densely defined in general) self-adjoint operator in $\mathcal{H}$. If $\mathcal{A}_1$ is another $C^*$-algebra and $\mathcal{P}:\mathcal{A}\to\mathcal{A}_1$ is a $*$-homomorphism then $\mathcal{P}\phi:C_\infty(\mathbb{R})\to\mathcal{A}_1$ is a $*$-homomorphism which defines a self-adjoint operator $H_1$ affiliated to $\mathcal{A}_1$. We shall denote $H_1=\mathcal{P}(H)$.

Let us go back now to our $\mathcal{L}$-graded $C^*$-algebra $\mathcal{A}$. For each self-adjoint operator H affiliated to $\mathcal{A}$ and each $Y\in\mathcal{L}$ we may consider the self-adjoint operator $H_Y$ affiliated to $\mathcal{A}_Y$ defined by $H_Y=\mathcal{P}_Y(H)$ (i.e. $f(H_Y)=\mathcal{P}_Y(f(H))$ for all $f\in C_\infty(\mathbb{R})$). Observe that $H_X=H$. If H is just an element of $\mathcal{A}$, then $H_Y=\mathcal{P}_Y(H)$ is just the projection of H onto $\mathcal{A}_Y$. If $\mathcal{A}$ is realised on a Hilbert space $\mathcal{H}$ and H is the hamiltonian of a system (i.e. $e^{-iHt}$ describes the time evolution of the system), then the $H_Y$'s will be called sub-hamiltonians (they describe the evolution of the system when parts of the interaction have been suppressed). Observe that each $H_Y$ (and $H=H_X$) has its own domain $D(H_Y)$ which is not dense in $\mathcal{H}$ in general. In the many-body case with hard-core interactions, D(H) is not dense, $D(H_O)$ is dense and $D(H_Y)$ for $Y\neq\mathbf{O}$, X is sometimes dense and sometimes not. If $H_Y$ is densely defined for all Y, we shall say that the densely defined self-adjoint operator H in $\mathcal{H}$ is $\mathcal{L}$-affiliated to $\mathcal{A}$. Such operators are easy to construct using the following criterion. Let $H_O=H(\mathbf{O})$ by a densely defined self-adjoint operator in $\mathcal{H}$ affiliated to $\mathcal{A}_O=\mathcal{A}(\mathbf{O})$. For each $Y\neq\mathbf{O}$, let H(Y) be a symmetric, $H_O$-bounded operator in $\mathcal{H}$ with relative bound zero and such that $H(Y)(H_O+i)^{-1}\in\mathcal{A}(Y)$. Then $H=\Sigma\{H(Y)\,|\,Y\in\mathcal{L}\}$ is self-adjoint and $\mathcal{L}$-affiliated to $\mathcal{A}$. Moreover, for all $Y\in\mathcal{L}$, we have $H_Y=\mathcal{P}_Y(H)=\Sigma\{H(Z)\,|\,Z\leq Y\}$. If $H_O$ is bounded below, then it is enough that H(Y) be $H_O$-form bounded with relative bound zero and for c large enough $(H_O+c)^{-1/2}H(Y)(H_O+c)^{-1/2}\in\mathcal{A}(Y)$.

We stop here this accumulation of definitions. In [BG 2] these notions are used in the spectral theory of N-body systems. For example, we show that the Weinberg-Van Winter equation and the HVZ theorem are very natural in this framework (both the statements and the proofs).

## 2. The N-body Algebra

In this section we shall describe some important properties of a graded $C^*$-algebra canonically associated to an Euclidean space (in place of the usual N-body formalism, we prefer to work in the geometrical setting first considered by Agmon, Froese and Herbst and systematically developed in [ABG 1]).

Let E be an Euclidean space (finite dimensional real Hilbert space). We provide it with the unique translation invariant Borel measure such that the volume of a unit cube is $(2\pi)^{-(\dim E)/2}$. Then $\mathscr{H}(E)$ is the Hilbert space $L^2(E)$ and the Fourier transform $(\mathscr{F} f)(x)=\int_E e^{-i(x|y)}f(y)dy$ induces a unitary operator in $\mathscr{H}(E)$ . Denote $\mathbf{B}(E)=B(\mathscr{H}(E))$. If $E=\mathbf{O}\equiv\{0\}$ then $\mathscr{H}(\mathbf{O})=\mathbb{C}$ and $\mathscr{F}=1$. For any Borel function $f:E\to\mathbb{C}$ we denote $f(Q)$ the operator of multiplication by f and $f(P)=\mathscr{F}^*f(Q)\mathscr{F}$. Then $\mathbf{K}(E)$ will be the $C^*$-algebra of compact operators on $\mathscr{H}(E)$ and $\mathbf{T}(E)$ the $C^*$-algebra of operators of the form $f(P)$ with $f:E\to\mathbb{C}$ continuous and convergent to zero at infinity (i.e. $f\in C_\infty(E)$). By convention $\mathbf{K}(\mathbf{O})=\mathbf{T}(\mathbf{O})=\mathbb{C}$.

If E,F are Euclidean spaces and $G=E\oplus F$ is their euclidean direct sum, then there is a canonical isomorphism of $\mathscr{H}(E)\otimes\mathscr{H}(F)$ (Hilbert tensor product) with $\mathscr{H}(G)$. For $S\in\mathbf{B}(E)$, $T\in\mathbf{B}(F)$ we write $S\otimes_E^G T$ for the operator in $\mathbf{B}(G)$ corresponding to $S\otimes T$ by the preceding isomorphism. Finally, if $\mathscr{M}\subset\mathbf{B}(E)$, $\mathscr{N}\subset\mathbf{B}(F)$ are $*$-subalgebras, then we denote $\mathscr{M}\hat{\otimes}_E^G\mathscr{N}$ the $C^*$-algebra on $\mathscr{H}(G)$ obtained as the norm-closure of the linear space generated by the operators of the form $S\otimes_E^G T$ with $S\in\mathscr{M}, T\in\mathscr{N}$.

Now let us fix an Euclidean space X and denote $\Pi(X)$ the set of all subspaces of X provided with the natural order relation (inclusion). Then $\Pi(X)$ is a complete lattice with $\mathbf{O}$, resp. X, as least, resp. biggest, element. For $Y,Z\in\Pi(X)$ we have $Y\vee Z=Y+Z$ and $Y\wedge Z=Y\cap Z$. Let $Y\in\Pi(X)$ and $Y^\perp\in\Pi(X)$ its orthogonal. Then $Y,Y^\perp$ are Euclidean spaces, $X=Y\oplus Y^\perp$ and we abbreviate $\otimes_Y^X=\otimes_Y$. We shall be interested in the $C^*$-subalgebras of $\mathbf{B}(X)$ defined by

$$(2.1)\qquad \mathscr{T}(Y)=\mathbf{K}(Y)\hat{\otimes}_Y\mathbf{T}(Y^\perp).$$

The family $\{\mathscr{T}(Y)\mid Y\in\Pi(X)\}$ has the following properties:

(i) The algebraic sum $\Sigma\{\mathscr{T}(Y)\mid Y\in\Pi(X)\}$ is direct i.e. each element S in the linear subspace of $\mathbf{B}(X)$ generated by $\cup\{\mathscr{T}(Y)\mid Y\in\Pi(X)\}$ can be *uniquely* written as a sum $S=\Sigma\{S(Y)\mid Y\in\Pi(X)\}$ with $S(Y)\in\mathscr{T}(Y)$ and $S(Y)\neq 0$ only for a finite number of Y's.

(ii) For all $Y,Z\in\Pi(X)$ we have: $\mathscr{T}(Y)\mathscr{T}(Z)\subset\mathscr{T}(Y+Z)$.

For proofs of the first, resp. second, assertion, see [BG 2], resp. [ABG 1].

In particular, the $*$-subalgebra $\Sigma\{\mathscr{T}(Y)\mid Y\in\Pi(X)\}$ of $\mathbf{B}(X)$ is $\Pi(X)$-graded in a natural sense. It is obviously not norm-closed, and *we shall denote* $\mathscr{T}$ *its closure.* This is the graded $C^*$-algebra canonically associated to X we were talking about at the beginning of this section.

For the N-body problem only subalgebras of $\mathscr{T}$ of the following type are needed. Let $\mathscr{L}\subset\Pi(X)$ be a finite family of subspaces of X such that $\mathbf{O},X\in\mathscr{L}$ and $Y+Z\in\mathscr{L}$ if $Y,Z\in\mathscr{L}$ (so $\mathscr{L}$ is *not* a sub-lattice of $\Pi(X)$, because $Y\cap Z\notin\mathscr{L}$ in general; however, $\mathscr{L}$ is a lattice for the order relation induced by $\Pi(X)$). Denote:

$$\mathscr{A}=\Sigma\{\mathscr{T}(Y)\,|\,Y\in\mathscr{L}\}. \tag{2.2}$$

Then $\mathscr{A}$ is a $C^*$-subalgebra of $\mathscr{T}$ which is also a $\mathscr{L}$-graded $C^*$-algebra (in the notation we do not mention the dependence on $\mathscr{L}$, which is considered fixed from now on). Let us mention that the projection $\mathscr{P}_Y$ of $\mathscr{T}$ onto the subalgebra $\mathscr{A}_Y=\Sigma\{\mathscr{T}(Z)\,|\,Z\in\mathscr{L},Z\subset Y\}$ can be explicitely described as follows. Assume $Y\neq X$ and denote $Y^+$ the set of elements of $Y^\perp$ which do not belong to any $Z^\perp$ with $Z\in\mathscr{L}$, $Z\not\subset Y$. Then $Y^+$ is a dense cone in $Y^\perp$ and for any $\omega\in Y^+$ we have $\mathscr{P}_Y(S)=\text{s-lim}_{\lambda\to\infty}e^{-i\lambda(P,\omega)}Se^{i\lambda(P,\omega)}$ for all $S\in\mathscr{A}$.

Let us explain in what sense the choice of $\mathscr{L}$ corresponds to the N-body problem. Define, inductively, $\mathscr{L}_1=\mathscr{L}$, $\mathscr{L}^1=\{X\}$; $\mathscr{L}_2=\mathscr{L}_1\setminus\mathscr{L}^1$, $\mathscr{L}^2$=the set of maximal elements of $\mathscr{L}_2$; $\mathscr{L}_3=\mathscr{L}_2\setminus\mathscr{L}^2$, $\mathscr{L}^3$=the set of maximal elements of $\mathscr{L}_3$; etc.... Then, N is the integer defined by $\mathscr{L}_N=\{\mathbf{O}\}$. For example, the two-body problem corresponds to $\mathscr{L}=\{\mathbf{O},X\}$ and the characteristic $C^*$-algebra is $\mathscr{A}=\mathbf{T}(X)+\mathbf{K}(X)$ (direct sum). The (generalized) three-body problem is described by $\mathscr{L}=\{\mathbf{O},Y_1,\ldots,Y_n,X\}$ where $Y_j$ are subspaces such that $\mathbf{O}\neq Y_j\neq X$ and $Y_i+Y_j=X$ if $i\neq j$ (hence $Y_i\not\subset Y_j$ for $i\neq j$; observe that one could have $Y_i\cap Y_j\neq\mathbf{O}$, but $Y_i\wedge Y_j=\mathbf{O}$ in $\mathscr{L}$). The characteristic $C^*$-algebra in such a case is $\mathscr{A}=\mathbf{T}(X)+\mathscr{T}(Y_1)+\ldots+\mathscr{T}(Y_n)+\mathbf{K}(X)$ (direct sum) and if $S_i\in\mathscr{T}(Y_i)$ then $S_iS_j\in\mathbf{K}(X)$ if $i\neq j$. The complications which appear for $N\geq4$ are due to the "nested" structure of $\mathscr{L}$.

A large class of (densely defined) self-adjoint operators $\mathscr{L}$-affiliated to the algebra $\mathscr{A}$ is described in Proposition 7 of [BG 2]. Very roughly, they are of the form $H=h(P)+\Sigma\{V_Y(Q_Y,P)\,|\,Y\in\mathscr{L},Y\neq\mathbf{O}\}$ where $h:X\to\mathbb{R}$ is a continuous function divergent at infinity and $Q_Y$ is the projection on Y of Q, so that $\mathscr{F}V_Y\mathscr{F}^*$ is, in the representation $\mathscr{H}(X)\cong L^2(Y^\perp,\mathscr{H}(Y))$, the operator of multiplication by an operator-valued function.

In the rest of this section we shall isolate some properties of the algebra $\mathscr{T}$ related to the *"geometric"* methods introduced by Simon [S] in the N-body problem and further refined in [PSS] and [FH].

THEOREM 2.1: *Let* $\chi:X\to\mathbb{C}$ *be continuous and homogeneous of degre zero outside the unit sphere* (i.e. $\chi(x)=\chi(x/|x|)$ *if* $|x|\geq 1$). *Then* $[S,\chi(Q)]$ *is a compact operator for each* $S\in\mathscr{T}$. *If* $S\in\mathscr{T}(Z)$ *for some* $Z\in\Pi(X)$ *and* $\chi(e)=0$ *for* $e\in Z^{\perp}$, $|e|=1$, *then both* $S\chi(Q)$ *and* $\chi(Q)S$ *are compact operators.*

**Proof**: Observe first that for each $M<\infty$ there is $c>0$ such that if $|y|\leq M$ :

$$(2.3)\qquad |\chi(x+y)-\chi(x)| = |\chi(\tfrac{x+y}{|x+y|})-\chi(\tfrac{x}{|x|})| \leq w(\tfrac{c}{|x|})$$

for x large enough, where $w$ is the modulus of continuity of the restriction of $\chi$ to the unit sphere. It is clearly enough to prove the theorem for S of the form $K\otimes_Z T$ with $K\in\mathbf{K}(Z)$ and $T\in\mathbf{T}(Z^{\perp})$. Let $\chi_0(x)=\chi(\pi_Z^{\perp}(x))$ where $\pi_Z^{\perp}$ is the orthogonal projection of X onto $Z^{\perp}$. Then $\chi_0(Q)=1\otimes_Z\Phi$ where $\Phi$ is the operator of multiplication by $\chi|_{Z^{\perp}}$ in $\mathscr{H}(Z^{\perp})$. From (2.3) and a result of Cordes [C] it follows that $[T,\Phi]$ is compact in $\mathscr{H}(Z^{\perp})$. So $[S,\chi_0(Q)]=K\otimes_Z[T,\Phi]$ is compact in $\mathscr{H}(X)$.

Writing $[S,\chi(Q)]=[S,\chi(Q)-\chi_0(Q)]+[S,\chi_0(Q)]$ and observing that $\chi(x)-\chi_0(x)=0$ if $x\in Z^{\perp}$, it follows that it is enough to prove the second part of the proposition for bounded uniformly continuous functions $\chi:X\to\mathbb{C}$ such that $|\chi(z+z')|\to 0$ as $|z'|\to\infty$, $z'\in Z^{\perp}$, uniformly in z when z runs over any compact subset of Z (use (2.3) to show that this is fulfilled by $\chi-\chi_0$ or by the initial $\chi$ if $\chi(e)=0$ for $e\in Z^{\perp}$, $|e|=1$). Let us show for example that $S\chi(Q)=(1\otimes_Z T)(K\otimes_Z 1\cdot\chi(Q))$ is compact. In the representation $\mathscr{H}(X)\cong L^2(Z^{\perp};\mathscr{H}(Z))$, the operator $K\otimes_Z 1\cdot\chi(Q)$ becomes the operator of multiplication by the function $z'\mapsto K\psi(z')\in\mathbf{B}(Z)$ where $(\psi(z')u)(z)=\chi(z+z')u(z)$. Since $\chi$ is bounded and uniformly continuous, $\psi: Z^{\perp}\to\mathbf{B}(Z)$ is bounded and norm-continuous. The last condition we put on $\chi$ is equivalent to $\text{s-lim}_{|z'|\to\infty}\psi(z')=0$ . Since K is compact in $\mathscr{H}(Z)$ we get $\|K\psi(z')\|_{\mathbf{B}(Z)}\to 0$ as $|z'|\to\infty$. It is standard now to show that $K\otimes_Z 1\cdot\chi(Q)$ is the norm-limit in $\mathbf{B}(X)$ of operators of the form $\sum K_j\otimes_Z\Phi_j$ where $K_j\in\mathbf{K}(Z)$ and $\Phi_j$ is the operator of multiplication by a $C_0^{\infty}$ function in $\mathscr{H}(Z^{\perp})$. Since $(1\otimes_Z T)(K_j\otimes_Z\Phi_j)=K_j\otimes_Z(T\Phi_j)$ and $T\Phi_j\in\mathbf{K}(Z^{\perp})$, the proof is finished.■

**Remark**: The fact that $[S,\chi(Q)]$ is compact for $S\in\mathscr{T}$ shows that $\mathscr{T}$ is a non-trivial subalgebra (and a rather small one) of $\mathbf{B}(X)$.

Let us go back to the N-body algebra $\mathscr{A}$ associated to some semi-lattice $\mathscr{L}\subset\Pi(X)$ as in (2.2). Let $Y\in\mathscr{L}$, $Y\neq X$. Following [FH] and [ABG 1], we shall call a function $\chi_Y:X\to\mathbb{R}$ *Y-reducing*, if it is continuous, homogeneous of degree zero outside the unit sphere and if $\chi_Y(e) = 0$ for all e such that $|e|=1$ and $e \in Z^\perp$ for some $Z \in \mathcal{L}$ with $Z\not\subset Y$. Recall that $\mathcal{B}_Y=\Sigma\{\mathcal{T}(Z)\,|\,Z\in\mathcal{L},Z\not\subset Y\}$ is a norm-closed $*$-ideal in $\mathcal{A}$ and $\mathcal{A}=\mathcal{A}_Y+\mathcal{B}_Y$ direct sum. It follows from theorem 2.1 that for a Y-reducing function $\chi_Y$ we have (remark that $\mathcal{A}(X) = \mathbf{K}(X)$):

(i) $[S,\chi_Y(Q)] \in \mathcal{A}(X)$ for all $S \in \mathcal{A}$;

(ii) $S\chi_Y(Q)$ and $\chi_Y(Q)S$ belong to $\mathcal{A}(X)$ for all $S \in \mathcal{B}_Y$.

A family $\{\chi_Y\}_{Y\in\mathcal{L}}$ of functions $\chi_Y:X\to\mathbb{R}$ is called $\mathcal{L}$-reducing if $\chi_X=0$, each $\chi_Y$ is Y-reducing for $Y\neq X$ and $\Sigma\{\chi_Y^2\,|\,Y\in\mathcal{L}\}=1$ on X. It is easy (see [ABG 1]), to construct such families having the supplementary properties:

(iii) $\chi_Y=0$ if Y is not a maximal element in $\mathcal{L}\setminus\{X\}$ (i.e. $\chi_Y\neq0$ only for $Y\in\mathcal{L}^2$);

(iv) $\chi_Y \in C^\infty(X)$ and $\chi_Y(x)=0$ on a neighbourhood on the unit sphere of the set $\cup\{S_X\cap Z^\perp\,|\,Z \in\mathcal{L},\ Z\not\subset Y\}$.

Let us make a final remark concerning the structure of the algebra $\mathcal{T}$. It is convenient now to indicate explicitly the dependence on the space X by denoting $\mathcal{T}(Y)=\mathcal{T}^X(Y)$, $\mathcal{T}=\mathcal{T}^X$. Let $\mathcal{T}_Y^X$ be the norm-closure of $\Sigma\{\mathcal{T}^X(Z)\,|\,Z\in\Pi(X), Z\subset Y\}$ and $\mathcal{S}_Y^X$ the norm-closure of $\Sigma\{\mathcal{T}^X(Z)\,|\,Z\in\Pi(X),\ Z\not\subset Y\}$ . So $\mathcal{T}_Y^X$ is a $C^*$-subalgebra of $\mathcal{T}^X$, $\mathcal{S}_Y^X$ is a norm-closed $*$-ideal and $\mathcal{T}_X^X = \mathcal{T}^X$. We would like to point out the following relations: for $Z,Y \in\Pi(X)$ such that $Z\subset Y$ we have

$$\mathcal{T}^X(Z)=\mathcal{T}^Y(Z)\hat{\otimes}_Y\mathbf{T}(Y^\perp). \tag{2.4}$$

In fact, if we denote $E=Y\cap Z^\perp$, then $Y=Z\oplus E$ and $Z^\perp=E\oplus Y^\perp$. It is clear that $\mathbf{T}(Z^\perp)=\mathbf{T}(E)\hat{\otimes}_E^{Z^\perp}\mathbf{T}(Y^\perp)$ so we get:

$$\mathcal{T}^X(Z)=\mathbf{K}(Z)\hat{\otimes}_Z^X(\mathbf{T}(E)\hat{\otimes}_E^{Z^\perp}\mathbf{T}(Y^\perp))=(\mathbf{K}(Z)\hat{\otimes}_Z^Y\mathbf{T}(E))\hat{\otimes}_Y^X\mathbf{T}(Y^\perp)=$$

$$=\mathcal{T}^Y(Z)\hat{\otimes}_Y^X\mathbf{T}(Y^\perp).$$

From (2.4) we also obtain:

$$\mathcal{T}_Y^X=\mathcal{T}^Y\hat{\otimes}_Y\mathbf{T}(Y^\perp). \tag{2.5}$$

Here we may specialize to the N-body algebra $\mathscr{A}\equiv\mathscr{A}^X$ associated to $\mathscr{L}$ with the convention that $\mathscr{A}_Y$ is constructed using $\mathscr{L}(Y)$. So:

(2.6) $\qquad \mathscr{A}_Y^X = \mathscr{A}_Y \hat{\otimes}_Y T(Y^\perp).$

To $\mathscr{A}_Y$, if $Y\neq 0$, we may associate a family $\{\chi_Z^Y\}_{Z\in\mathscr{L}(Y)}$ (with $\chi_Z^Y\neq 0$ only if $Z<\bullet Y$, i.e. Y covers Z) with $\chi_Z^Y:Y\to\mathbb{R}$ and then we may extend $\chi_{ZY}=\chi_Z^Y\otimes_Y 1$ (i.e. $\chi_{ZY}:X\to\mathbb{R}$ is given by $\chi_{ZY}(x)=\chi_Z^Y(\pi_Y(x))$. Observe that for each fixed $Y\in\mathscr{L}\setminus\{0\}$ we shall have $\Sigma\{\chi_{ZY}^2 \mid Z\in\mathscr{L}, Z<\bullet Y\}=1$.

## 3. General Considerations on the Mourre Estimate

In this section we shall quit the graded C*-algebra setting in order to present certain notions and results related to Mourre theory in the framework introduced in [ABG 1,2] and [BGM]. We do this step hopping that so we shall put in a better light the proof of the Mourre estimate for hamiltonians with a many channel structure.

Let $\mathscr{H}$ be a (complex, separable) Hilbert space and A a self-adjoint operator in $\mathscr{H}$. Denote $W_\alpha = e^{iA\alpha}$ the unitary group in $\mathscr{H}$ generated by A. We shall say that *a closed operator* T *in* $\mathscr{H}$ *is of class* $C^1(A)$, and we shall write $T\in C^1(A)$, if its domain D(T) is invariant under the group W and if for all $u\in D(T)$ the function $\alpha\mapsto \langle W_\alpha u | T W_\alpha u\rangle$ is of class $C^1$. In this case we denote [T,A] the sesquilinear form on D(T) given by $\langle u | i[T,A]u\rangle = \frac{d}{d\alpha}\langle W_\alpha | T W_\alpha\rangle|_{\alpha=0}$. Let $\mathscr{G}$ =D(T) equipped with the graph-norm. Then [T,A] is a continuous sesquilinear form on $\mathscr{G}$ and it is often useful to think of it as a continuous linear operator from $\mathscr{G}$ to its adjoint space $\mathscr{G}^*$. It is shown in [ABG 1] that, if $\mathscr{G}$ is invariant under W, then $T\in C^1(A)$ if and only if the sesquilinear forms $\left[T, \frac{1}{\alpha} W_\alpha\right]$ on $\mathscr{G}$ converge weakly when $\alpha\to 0$ and in this case:

(3.1) $\qquad i[T,A] = \text{s-}\lim_{\alpha\to 0}\left[T, \frac{1}{\alpha} W_\alpha\right],$

the strong limit being in $B(\mathscr{G},\mathscr{G}^*)$. If the limit exists in norm in this space, then we write $T\in C_n^1(A)$; this is equivalent to the norm-derivability of $\alpha\mapsto W_\alpha^* T W_\alpha \in B(\mathscr{G},\mathscr{G}^*)$. For bounded T we identify $B(\mathscr{G},\mathscr{G}^*)=B(\mathscr{H})$.

We shall now associate to each self-adjoint operator H in $\mathscr{H}$ of class $C^1(A)$ two functions $\hat{\rho} \equiv \hat{\rho}_H^A$ and $\rho = \rho_H^A$ defined on $\mathbb{R}$ with values in $]-\infty,+\infty]$, according to the following rule. Denote $E(\lambda;\varepsilon)=E((\lambda-\varepsilon,\lambda+\varepsilon))$ for $\lambda \in \mathbb{R}$ and $\varepsilon>0$. Then

(3.2) $\quad \hat{\rho}_H^A(\lambda) = \sup\{a\in\mathbb{R} \mid \text{there is } \varepsilon>0 \text{ and a compact operator K such that } E(\lambda;\varepsilon)[iH,A]E(\lambda;\varepsilon) \geq aE(\lambda;\varepsilon)+K\}$,

(3.3) $\quad \rho_H^A(\lambda)= \sup\{a\in\mathbb{R} \mid \exists\varepsilon>0 \text{ such that } E(\lambda;\varepsilon)[iH,A]E(\lambda;\varepsilon) \geq aE(\lambda;\varepsilon)\}$.

Another way of defining $\rho_H^A$ is as follows. For $\varepsilon>0$, let

$$\rho_\varepsilon(\lambda) = \inf\{\langle u \mid [iH,A]u\rangle \mid u=E(\lambda;\varepsilon)u,\ \|u\|=1\}$$

(with the convention $\inf \emptyset=\infty$).Then $\rho_\varepsilon(\lambda)\to\rho(\lambda)$ as $\varepsilon\to+0$. Let us also mention the following fact. If $\lambda_0 \in \mathbb{R}$, then the spectral measure of the operator $H-\lambda_0$ is $S\mapsto E(S+\lambda_0)$ ($S\subset\mathbb{R}$ Borel set). Hence we get $\rho_{H-\lambda_0}^A(\lambda)=\rho_H^A(\lambda+\lambda_0)$ and similarly for $\hat{\rho}$.

A systematic study of the functions $\rho$ and $\hat{\rho}$ is presented in [ABG 2], from which we quote now some results. It is easy to show that $\rho$ and $\hat{\rho}$ are lower semicontinuous (l.s.c.) functions, $\hat{\rho}(\lambda)<\infty$ if and only if $\lambda\in\sigma_{ess}(H)$ and $\hat{\rho}(\lambda)<\infty$ if and only if $\lambda\in\sigma(H)$. From the virial theorem we get that, if $\hat{\rho}(\lambda)>0$, then $\lambda$ has a neighbourhood in which there is at most a finite number of eigenvalues (counting multiplicities). A deeper consequence of this theorem is the following result (implicitly contained in [FH] and explicitly isolated and proved in [ABG 2]).

PROPOSITION 3.1: *If $\lambda$ is an eigenvalue of* H *and* $\rho(\lambda)>0$, *then* $\rho(\lambda)=0$. *Otherwise,* $\hat{\rho}(\lambda)=\rho(\lambda)$.

The next result is easy, but very useful in applications.

PROPOSITION 3.2: *Let* $\Lambda\subset\mathbb{R}$ *be a compact set and* $\theta: \Lambda\to\mathbb{R}$ *an upper semicontinuous function (u.s.c.) such that* $\theta(\lambda)<\rho(\lambda)$ *for all* $\lambda\in\Lambda$. *Then there is* $\varepsilon>0$ *such that for all* $\lambda\in\Lambda$:

(3.4) $\quad E(\lambda;\varepsilon)[iH,A]E(\lambda;\varepsilon) \geq \theta(\lambda)E(\lambda;\varepsilon)$.

Functions $\theta$ as in the last proposition play a role in the proof of the propagation theorems (see [D1] and [T]) but we shall need them in the proof of the theorem below. They are very easy to construct, as the next example shows (this explains corollary 4.3 from [D1]). For any $\nu>0$, let

(3.5) $\quad \theta_\nu(\lambda) = \inf_{|\mu-\lambda|<\nu} \rho(\mu)-\nu.$

Then $\theta_v:\mathbb{R}\to]-\infty,+\infty]$ is upper semicontinuous, $\theta_{v_1}(\lambda)<\theta_{v_2}(\lambda)$ if $v_2<v_1$ and $\theta_{v_1}(\lambda)\neq\infty$, and $\lim_{v\to 0}\theta_v(\lambda)=\rho(\lambda)$ for all $\lambda\in\mathbb{R}$. Moreover, $\theta_v(\lambda)<\infty$ if $\mathrm{dist}(\lambda,\sigma(H))<v$. This choice is useful in abstract considerations, but a better one can be made in the case of Agmon hamiltonians. Let us mention that

$$\rho_v(\lambda)\leq \inf_{|\mu-\lambda|<v}\rho(\mu)\leq\rho(\lambda)$$

with $\rho_v$ defined after (3.3).

We can introduce now the main concept of Mourre theory.

DEFINITION: *Let* H *be a self-adjoint operator in the Hilbert space* $\mathcal{H}$. *We shall say that a self-adjoint operator* A is conjugated to H at some point $\lambda\in\mathbb{R}$ *if* $H\in C^1(A)$ *and* $\hat{\rho}_H^A(\lambda)>0$.

In the graded $C^*$-algebra setting it is better to work only with bounded operators. So it is useful to be able to express the preceding property in terms of the resolvent of H.

PROPOSITION 3.3: *Let* H *and* A *be self-adjoint operators,* $\lambda_0$ *a complex number outside the spectrum of* H *and* $R=(\lambda_0-H)^{-1}$. *Assume that* $e^{iA\alpha}$ *leaves invariant the domain of* H. *Then* $H\in C^1(A)$ (resp. $H\in C^1_n(A)$) *if and only if* $R\in C^1(A)$ (resp. $R\in C^1_n(A)$). *In this case*

(3.6) $\quad [R,A]=R[H,A]R.$

*Assume, moreover, that* $\lambda_0\in\mathbb{R}$ *(so H has to have a spectral gap). Then, for all real* $\lambda\neq\lambda_0$, *we shall have*

(3.7) $\quad \hat{\rho}_R^A((\lambda_0-\lambda)^{-1})=(\lambda_0-\lambda)^{-2}\hat{\rho}_H^A(\lambda).$

*In particular,* A *is conjugated to* H *at some* $\lambda\in\mathbb{R}\setminus\{\lambda_0\}$ *if and only if it is conjugated to* R *at* $(\lambda_0-\lambda)^{-1}$.

**Proof:** Since $W_\alpha=e^{iA\alpha}$ leaves invariant the domain of H, it is easy to show that $[R,1/\alpha W_\alpha]=R[H,1/\alpha W_\alpha]R$. Denote $\mathcal{G}$ the domain of H (assumed dense without loss of generality) provided with the graph norm; then $\mathcal{G}\subset\mathcal{H}$ continuously and densely and, after identification of $\mathcal{H}$ with its adjoint space $\mathcal{H}^*$ using Riesz lemma, we get $\mathcal{G}\subset\mathcal{H}\subset\mathcal{G}^*$. Using (3.1) and the fact that R is an isomorphism of $\mathcal{H}$ onto $\mathcal{G}$ and of $\mathcal{G}^*$ onto $\mathcal{H}$, we see that $[R,1/\alpha W_\alpha]$ is weakly convergent in $B(\mathcal{H})$ (i.e. R is of class $C^1(A)$) if and only if $H\in C^1(A)$ and then (3.6) is true.

In order to prove (3.7), we may assume $\lambda_0=0$. Let $\varphi:\mathbb{R}\setminus\{0\}\to\mathbb{R}\setminus\{0\}$ be the diffeomorphism $\varphi(\lambda)=-\lambda^{-1}$. Then the spectral measure of R is $E_R(S) = E(\varphi(S))$. Using (3.6) we get for $\lambda>0$ (for example) and $0<\varepsilon<\lambda$:

$$E_R(\lambda;\varepsilon)[iR,A]E_R(\lambda;\varepsilon) = H^{-1}E(I_\varepsilon)[iH,A]E(I_\varepsilon)H^{-1},$$

where we have denoted $I_\varepsilon=(-(\lambda-\varepsilon)^{-1},-(\lambda+\varepsilon)^{-1})$. For each $a<\hat{\rho}_H^A(-\lambda^{-1})$ there are $\varepsilon_0>0$ and a compact operator K such that

$$E(-\lambda^{-1};\varepsilon_0)[iH,A]E(-\lambda^{-1};\varepsilon_0) \geq aE(-\lambda^{-1};\varepsilon_0)+K.$$

If $\varepsilon$ is small enough, $I_\varepsilon$ is a neighbourhood of $-\lambda^{-1}$ contained in $(-\lambda^{-1}-\varepsilon_0,-\lambda^{-1}+\varepsilon_0)$, hence $E(I_\varepsilon)[iH,A]E(I_\varepsilon) \geq aE(I_\varepsilon)+E(I_\varepsilon)KE(I_\varepsilon)$. We get

$$H^{-1}E(I_\varepsilon)[iH,A]E(I_\varepsilon)H^{-1} \geq aH^{-2}E(I_\varepsilon)+H^{-1}E(I_\varepsilon)KE(I_\varepsilon)H^{-1} \geq$$
$$\geq a(\lambda-\varepsilon)^2E(I_\varepsilon)+H^{-1}E(I_\varepsilon)KE(I_\varepsilon)H^{-1}.$$

Since the last term here is compact, we obtain $\hat{\rho}_R^A(\lambda)\geq\lambda^2\hat{\rho}_H^A(-\lambda^{-1})$. For the reverse inequality one has to start from $E(\lambda;\varepsilon)[iH,A]E(\lambda;\varepsilon) = HE(\lambda;\varepsilon)[iR,A]E(\lambda;\varepsilon)H$. ■

**Remark:** The preceding proposition is not true if D(H) is not assumed invariant under $e^{iA\alpha}$. For N-body hamiltonians with hard-core interactions, if A is the generator of dilations, then R is of class $C^1(A)$ but D(H) is not invariant under $e^{iA\alpha}$.

We pass now to the main result of this section, namely the calculation of the $\rho$-function for an operator H of the form $H^1\otimes 1+1\otimes H^2$ assuming that A admits a similar decomposition. Assume that two self-adjoint *bounded from below* operators $H^1$, $H^2$ are given in Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. We denote $\mathcal{G}^j=D(H^j)$ provided with the graph norm, so that $\mathcal{G}^j$ is a Hilbert space continuously embedded in $\mathcal{H}_j$. Let $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$, $\mathcal{G}_1=\mathcal{G}^1\otimes\mathcal{H}_2$ and $\mathcal{G}_2=\mathcal{H}_1\otimes\mathcal{G}^2$ (Hilbert tensor products). It is known that there are continuous embeddings $\mathcal{G}_1\subset\mathcal{H}$ and $\mathcal{G}_2\subset\mathcal{H}$, $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) being the domain of the self-adjoint operator $H_1= H^1\otimes 1$ (resp. $H_2=1\otimes H^2$) in $\mathcal{H}$. Moreover, the operator $H=H_1+H_2$ is self-adjoint on the domain $\mathcal{G}=\mathcal{G}_1\cap\mathcal{G}_2$ and its spectrum is given by $\sigma(H)=\sigma(H^1)+\sigma(H^2)$ (these assertions depend on the boundedness from below of the operators, see section 2.1 in [ABG 1]). Consider now a self-adjoint operator $A^j$ in $\mathcal{H}_j$ such that $H^j$ is of class $C^1(A^j)$. Recall that the self-adjoint operator $A=A^1\otimes 1+1\otimes A^2\equiv A_1+A_2$ can be defined by the property $e^{iA\alpha}=e^{iA^1\alpha}\otimes e^{iA^2\alpha}$ for all $\alpha\in\mathbb{R}$. It is then obvious that $D(H)=\mathcal{G}$ is invariant under $e^{iA\alpha}$. By hypothesis,

$B^j=[iH^j,A^j]$ is a continuous sesquilinear form on $\mathscr{G}^j$. It is well-known that $B^1$ will extend to a continuous sesquilinear form $B_1=B^1\otimes 1$ on $\mathscr{G}_1$ and similarly $B^2$ to $B_2=1\otimes B^2$ on $\mathscr{G}_2$. Now it is easy to show that $H_j$ is of class $C^1(A_j)$ and of class $C^1(A)$ and $[iH_j,A_j]=[iH_j,A]=B_j$ (use $e^{-iA\alpha}H_1e^{+iA\alpha}=e^{-iA_1\alpha}H_1e^{+iA_1\alpha}=(e^{-iA^1\alpha}H^1e^{+iA^1\alpha})\otimes 1$). Since $\mathscr{G}=\mathscr{G}_1\cap\mathscr{G}_2$ (with the intersection topology), the sesquilinear form $B=B_1+B_2$ is continuous on $\mathscr{G}$. It follows that H is of class $C^1(A)$ and $[iH,A]=B$.

These arguments prove the first part of the next theorem:

THEOREM 3.4: *Let* $H^1$, $H^2$ *be two self-adjoint, bounded from below operators in the Hilbert spaces* $\mathscr{H}_1$, $\mathscr{H}_2$ *.Assume that* $A^j$ *is a self-adjoint operator in* $\mathscr{H}_j$ *such that* $H^j$ *is of class* $C^1(A^j)$. *Let* $H=H^1\otimes 1+1\otimes H^2$ *and* $A=A^1\otimes 1+1\otimes A^2$, *self-adjoint operators in* $\mathscr{H}=\mathscr{H}_1\otimes\mathscr{H}_2$. *Then* H *is of class* $C^1(A)$ *and for all* $\lambda\in\mathbb{R}$:

$$\rho_H^A(\lambda)=\inf_{\lambda=\lambda_1+\lambda_2}[\rho_{H^1}^{A^1}(\lambda_1)+\rho_{H^2}^{A^2}(\lambda_2)] \tag{3.8}$$

**Proof:** (i) We have to prove only the preceding formula. Denote $\rho=\rho_H^A$, $\rho_j=\rho_{H^j}^{A^j}$. Since $\sigma(H)=\sigma(H^1)+\sigma(H^2)$, (3.8) is obvious if $\lambda\notin\sigma(H)$, both members being equal to $+\infty$. Moreover, by adding to $H^j$ a constant and taking into account that $\rho_{H-\lambda_0}^A(\lambda)=\rho_H^A(\lambda+\lambda_0)$, we can assume $H^j\geq 0$, so that H is positive too. Hence, when we prove (3.9), we may assume without loss of generality that $\lambda\in\sigma(H)$, $\lambda\geq 0$ and we may consider only decompositions $\lambda=\lambda_1+\lambda_2$ with $\lambda_j\in\sigma(H_j)$, so that $\lambda_j\geq 0$.

(ii) Let us first prove that the function $f(\lambda)$ defined by the r.h.s. of (3.8) on $\mathbb{R}_+=[0,+\infty[$ is l.s.c. (then its extension by $+\infty$ for $\lambda<0$ will be l.s.c. on $\mathbb{R}$). Let $f_j=\rho_j|_{\mathbb{R}_+}$ and $F(\lambda_1,\lambda_2)=f_1(\lambda_1)+f_2(\lambda_2)$. Then $F:\mathbb{R}_+^2\to]-\infty,\infty]$ is l.s.c.. For $\lambda\geq 0$ denote $I_\lambda=\{(\lambda_1,\lambda_2)\mid\lambda_j\geq 0$ and $\lambda_1+\lambda_2=\lambda\}$. $I_\lambda$ is a compact subset of $\mathbb{R}_+^2$ and $f(\lambda)=\inf\{F(\lambda_1,\lambda_2)\mid(\lambda_1,\lambda_2)\in I_\lambda\}$. Assume $f(\lambda)>a$; we have to show that $f(\mu)>a$ for $\mu$ in a neighbourhood of $\lambda$. We have $F(\lambda_1,\lambda_2)>a$ for all $(\lambda_1,\lambda_2)\in I_\lambda$, hence each such $(\lambda_1,\lambda_2)$ has a neighbourhood $U(\lambda_1,\lambda_2)$ in $\mathbb{R}_+^2$ on which F is strictly greater than a. $I_\lambda$ being compact, it may be covered by a finite set $U_1,\dots,U_n$ of such neighbourhoods. Then $U=U_1\cup U_2\cup\dots\cup U_n$ is a neighbourhood of $I_\lambda$. Since $I_\lambda$ is compact, U will contain a set of the form $I_\lambda(\varepsilon)=\{(\lambda_1,\lambda_2)\in\mathbb{R}_+^2\mid\lambda-\varepsilon\leq\lambda_1+\lambda_2\leq\lambda+\varepsilon\}$. So $F(\lambda_1,\lambda_2)>a$ on $I_\lambda(\varepsilon)$. Since F attains its lower bound on compacts, we shall have $f(\mu)>a$ for $\lambda-\varepsilon\leq\mu\leq\lambda+\varepsilon$.

(iii) For each $\nu>0$ we denote $\theta^j_\nu$ the function $\mathbb{R}\to]-\infty,+\infty]$ associated to $\rho_j$ according to the rule (3.5). Then $\theta^j_\nu$ is u.s.c. and is finite on the open neighbourhood $\{\mu\in\mathbb{R} \mid \mathrm{dist}(\mu;\sigma(H^j))<\nu\}$ of $\sigma(H^j)$. On this set we also have $\theta^j_\nu(\mu)<\rho_j(\mu)$.

Let us fix some arbitrary $\lambda\in\sigma(H)$ (so $\lambda\geq 0$) and some (small) numbers $\nu>\nu'>0$. The set of all $\mu\leq\lambda$ such $\mathrm{dist}(\mu;\sigma(H^j))\leq\nu'$ is a compact and the restriction of $\theta^j_\nu$ to it is a finite-valued u.s.c. function such that $\theta^j_\nu(\mu)<\rho_j(\mu)$. According to Proposition 3.2, there is $\varepsilon\in(0,\nu')$ such that for all $\mu\leq\lambda$ with $\mathrm{dist}(\mu;\sigma(H^j))\leq\nu'$ :

$$(3.9)\qquad E^j(\mu;\varepsilon)B^jE^j(\mu;\varepsilon)\geq\theta^j_\nu(\mu)E^j(\mu;\varepsilon).$$

Here $E^j$ is the spectral measure of $H^j$. But, if $\mathrm{dist}(\mu;\sigma(H^j))>\nu'$, then $E^j(\mu;\varepsilon)=0$, because we assumed $\varepsilon<\nu'$. Hence (3.9) is valid for all $\mu\leq\lambda$ if we give an arbitrary finite value to $\theta^j_\nu(\mu)$ for $\mathrm{dist}(\mu;\sigma(H^j))>\nu'$.

It will be convenient to define $\theta^j_\nu(\mu)$ for $\mu\leq\lambda$ and $\mathrm{dist}(\mu;\sigma(H^j))>\nu'$ as equal to a finite constant bigger than $\sup\left\{\theta^j_\nu(\tau)\mid\tau\leq\lambda,\ \mathrm{dist}(\tau;\sigma(H^j))\leq\nu'\right\}$ (observe that the function $\theta^j_\nu$ being u.s.c. is bounded from above on this compact set). We shall, however, keep the same notation for this new function.

(iv) Let us work in a spectral representation of the operator $H^2$. Then there is a measure space $S_2$ and a Borel function $\omega_2:S_2\to\mathbb{R}_+$ such that $\mathcal{H}_2\cong L^2(S_2)$ and $H^2$ is the operator of multiplication by $\omega_2$. We then identify $\mathcal{H}_1\otimes\mathcal{H}_2\cong L^2(S_2;\mathcal{H}_1)$ so that $H$ becomes the operator of multiplication by the operator-valued function $s\mapsto H^1+\omega_2(s)$. From (3.9) we get for all $s\in S_2$:

$$(3.10)\qquad E^1(\lambda-\omega_2(s);\varepsilon)B^1E^1(\lambda-\omega_2(s);\varepsilon)\geq\theta^1_\nu(\lambda-\omega_2(s))E^1(\lambda-\omega_2(s);\varepsilon).$$

If $f$ is a bounded Borel function, then $f(H)$ is in $L^2(S_2;\mathcal{H}_1)$ the operator of multiplication by the operator-valued function $s\mapsto f(H^1+\omega_2(s))$. Hence, if $E$ is the spectral measure of $H$, then $E(\lambda;\varepsilon)$ is just the operator of multiplication by $s\mapsto E^1(\lambda-\omega_2(s);\varepsilon)$ (take $f$ equal to the characteristic function of $(\lambda-\varepsilon,\lambda+\varepsilon)$). So (3.10) is equivalent to

$$(3.11)\qquad E(\lambda;\varepsilon)B_1E(\lambda;\varepsilon)\geq[1\otimes\theta^1_\nu(\lambda-H^2)]E(\lambda;\varepsilon).$$

Observe that $1\otimes\theta^1_\nu(\lambda-H^2)=\theta^1_\nu(\lambda-H_2)$. Writing an estimate similar to (3.11) with $H^1$ and $H^2$ interchanged, we obtain ($B=B_1+B_2$):

$$(3.12)\qquad E(\lambda;\varepsilon)BE(\lambda;\varepsilon)\geq[\theta^1_\nu(\lambda-H_2)+\theta^2_\nu(\lambda-H_1)]E(\lambda;\varepsilon).$$

(v) We have to find the lower bound of the operator $\theta_v^1(\lambda-H_2)+\theta_v^2(\lambda-H_1)$ on the subspace $E(\lambda;\varepsilon)\mathcal{H}$. Let us work in a spectral representation of both $H^1$ and $H^2$, so that $\mathcal{H} \cong L^2(S_1 \times S_2)$, $H_j$ is the operator of multiplication by $(s_1,s_2) \mapsto \omega_j(s_j)$ and $H$ is $(s_1,s_2) \mapsto \omega_1(s_1)+\omega_2(s_2)$. Hence $\theta_v^1(\lambda-H_2)+\theta_v^2(\lambda-H_1)$ is the operator of multiplication by $\theta_v^1(\lambda-\omega_2(s_2))+\theta_v^2(\lambda-\omega_1(s_1))$ and the subspace $E(\lambda;\varepsilon)\mathcal{H}$ is the set of functions in $\mathcal{H}$ which are zero outside the set $\{(s_1,s_2) \mid \lambda-\varepsilon<\omega_1(s_1)+\omega_2(s_2)<\lambda+\varepsilon\}$. In conclusion:

$$[\theta_v^1(\lambda-H_2)+\theta_v^2(\lambda-H_1)]E(\lambda;\varepsilon) \geq$$

$$\geq [\inf \{\theta_v^1(\lambda-\tau_2)+\theta_v^2(\lambda-\tau_1) \mid \lambda-\varepsilon<\tau_1+\tau_2<\lambda+\varepsilon,\ \tau_j \in \sigma(H_j)\}]\, E(\lambda;\varepsilon).$$

Let us consider the inf in the r.h.s. and replace the variables $\tau_1$, $\tau_2$ by $\lambda_1=\lambda-\tau_2$, $\lambda_2=\lambda-\tau_1$. Then we must have $\lambda_1\leq\lambda$, $\lambda_2\leq\lambda$, $|\lambda_1+\lambda_2-\lambda|<\varepsilon$, and we are interested in $\inf(\theta_v^1(\lambda_1)+\theta_v^2(\lambda_2))$. Taking into account the way $\theta_v^j$ has been chosen in (iii), we may also assume $\mathrm{dist}(\lambda_j,\sigma(H^j))\leq v'$. But then clearly this infimum is minorated by

$$\inf\left\{ \inf_{|\mu_1-\lambda_1|<v} \rho_1(\mu_1) + \inf_{|\mu_2-\lambda_2|<v} \rho_2(\mu_2)-2v \,\middle|\, \lambda_1,\lambda_2\leq\lambda,\ |\lambda_1+\lambda_2-\lambda|<\varepsilon \right\}.$$

The numbers $\mu_1$, $\mu_2$ which appear here satisfy $\mu_j\leq\lambda_j+v$ and $|\mu_1+\mu_2-\lambda|<\varepsilon+2v$. Hence we can bound by below the above quantity by:

$$\inf \{\rho_1(\mu_1) + \rho_2(\mu_2)-2v \mid \mu_1,\mu_2\leq\lambda+v,\ |\mu_1+\mu_2-\lambda|<\varepsilon+2v\} \geq$$

$$\geq \inf \{\rho_1(\mu_1) + \rho_2(\mu_2) -2v \mid |\mu_1+\mu_2-\lambda|<\varepsilon+2v\} =$$

$$= \inf_{|\mu-\lambda|<\varepsilon+2v} \ \inf_{\mu=\mu_1+\mu_2} [\rho_1(\mu_1) + \rho_2(\mu_2)] - 2v =$$

$$= \inf_{|\mu-\lambda|<\varepsilon+2v} f(\mu) - 2v \geq \inf_{|\mu-\lambda|<3v} f(\mu)-2v := \theta_v.$$

From (3.12) we obtain then: $E(\lambda;\varepsilon)BE(\lambda;\varepsilon) \geq \theta_v E(\lambda;\varepsilon)$. So $\rho(\lambda)\geq\theta_v$. Since $v>0$ is arbitrary and $\theta_v \to f(\lambda)$ as $v\to+0$ due to the lower semicontinuity of f, we get $\rho(\lambda)\geq f(\lambda)$.

(vi) It remains to be shown that the equality is in fact realised in $\rho(\lambda)\geq f(\lambda)$. Of course, only the case $\lambda \in \sigma(H)$ is non trivial. By the lower semi-continuity of F and the compactness of $I_\lambda$ (see (ii)) it follows that there are $\lambda_1 \in \sigma(H^1)$ and $\lambda_2 \in \sigma(H^2)$ such that $\lambda=\lambda_1+\lambda_2$ and $f(\lambda)=\rho_1(\lambda_1)+\rho_2(\lambda_2)$. By the definition of $\rho_j(\lambda_j)$ (see the remark after (3.3)) there is a sequence $\{u_n^j\}_{n\in\mathbb{N}}$ (j=1,2) such that $u_n^j = E^j(\lambda_j;\frac{1}{n})u_n^j$,

$\|u_n^j\|=1$ and $<u_n \mid B^j u_n^j> \to \rho_j(\lambda_j)$. Let $u_n=u_n^1 \otimes u_n^2$. Then $\|u_n\|=1$ and $E(\lambda;\frac{2}{n})u_n=u_n$. Moreover

$$<u_n \mid Bu_n> = <u_n^1 \mid B^1 u_n^1> + <u_n^2 \mid B^2 u_n^2> \to \rho_1(\lambda_1)+\rho_2(\lambda_2).$$

This finishes the proof. ■

**Remark**: Assume that $H^2$ has a purely continuous spectrum. Then $\rho_{H^2}^{A^2} = \hat{\rho}_{H^2}^{A^2}$, H has also a purely continuous spectrum and

$$\hat{\rho}_H^A(\lambda) = \rho_H^A(\lambda) = \inf_{\lambda=\lambda_1+\lambda_2} [\rho_{H^1}^{A^1}(\lambda_1) + \rho_{H^2}^{A^2}(\lambda_2)]. \tag{3.13}$$

As an example, let us see how the theorem should be used for the case of Agmon hamiltonians (cf. [ABG 1]). Let $Y \in \mathscr{L}\setminus\{X\}$ and $H_Y = H^Y \otimes_Y 1 + 1 \otimes_Y \Delta^{Y^\perp}$. We take $A=A^Y \otimes_Y 1 + 1 \otimes_Y A^{Y^\perp}$ where A is the generator of the dilation group normalised such that $[i\Delta,A]=\Delta$. Obviously, for $X\neq 0$:

$$\rho_\Delta^A(\lambda) = \begin{cases} +\infty & \text{if } \lambda<0 \\ \lambda & \text{if } \lambda\geq 0. \end{cases}$$

Let $\rho^Y=\rho_{H^Y}^{A^Y}$ and $\rho_Y=\rho_{H_Y}^A$. For $Y\neq X$ we have $\rho_Y=\hat{\rho}_Y := \hat{\rho}_{H_Y}^A$ by Proposition 3.1. In conclusion:

$$\hat{\rho}_Y(\lambda) = \inf_{\mu\geq 0} [\rho^Y(\lambda-\mu)+\mu] \text{ for all } Y \in \mathscr{L}\setminus\{X\} \text{ and } \lambda \in \mathbb{R}. \tag{3.14}$$

## 4. Reducible Graded C*-Algebras

In this section we shall introduce a class of graded C*-algebras so that the ρ-function of a hamiltonian affiliated to such an algebra can be easily estimated in terms of the ρ-functions of sub-hamiltonians if the action of the conjugate operator is compatible with the graduation. The definition below is motivated by Theorem 2.1 and the existence of $\mathscr{L}$-reducing families (mentioned after the proof of theorem 2.1) for the N-body algebra.

Let us consider a finite lattice $\mathscr{L}$ and a $\mathscr{L}$-graded C*-algebra $\mathscr{A}$. Recall that for each $Y \in \mathscr{L}$ we have a canonical decomposition $\mathscr{A}=\mathscr{A}_Y+\mathscr{B}_Y$ such that $\mathscr{A}_Y$ is a C*-subalgebra of $\mathscr{A}$, $\mathscr{B}_Y$ is a closed $*$-ideal, $\mathscr{A}_Y \cap \mathscr{B}_Y=\{0\}$ and the projection $\mathscr{P}_Y : \mathscr{A} \to \mathscr{A}_Y$ is a $*$-homomorphism. Moreover, $\mathscr{A}_Y \subset \mathscr{A}_Z$ if $Y\leq Z$ and $\mathscr{A}_X=\mathscr{A}$. In this section we shall furthermore assume that $\mathscr{A}$ is realised on a (separable) Hilbert space $\mathscr{H}$ (i.e. $\mathscr{A}\subset B(\mathscr{H})$ is a C*-subalgebra).

DEFINITION: *A family* $\{J_Y\}_{Y\in\mathscr{L}}$ *of bounded, symmetric operators in* $\mathscr{H}$ *is called* $\mathscr{A}$-reducing *if:*

(a) $J_X=0$ *and* $\Sigma\{J_Y^2 \mid Y\in\mathscr{L}\}=1$;

(b) *for each* $S\in\mathscr{A}$ *and* $Y\in\mathscr{L}$, *we have* $[S,J_Y]\in\mathscr{A}(X)$;

(c) *if* $Y\in\mathscr{L}$ *and* $S\in\mathscr{B}_Y$, *then* $SJ_Y$ *and* $J_YS$ *belong to* $\mathscr{A}(X)$.

If such a family exits, we shall say that $\mathscr{A}$ *is a reducible* $\mathscr{L}$*-graded* C*-*algebra.* Recall that $\mathscr{A}_Y$ is canonically a $\mathscr{L}(Y)$-graded C*-algebra; if each $\mathscr{A}_Y$ is reducible , we shall say that $\mathscr{A}$ *is completely reducible.*

In connection with this definition, recall that $\mathscr{A}(X)$ is also a closed $*$-ideal in $\mathscr{A}$ (and $\mathscr{A}(Y)$ in $\mathscr{A}_Y$), hence at (c) we could have required only $SJ_Y\in\mathscr{A}(X)$. If $\mathscr{L}=\{\mathbf{O},X\}$, then $\mathscr{A}$ is (completely) reducible: it is enough to take $J_O=1$, $J_X=0$. The remarks which end section 2 prove that the N-body algebra is completely reducible (take $J_Y=\chi_Y(Q)$).

For two operators $S,T\in\mathscr{A}$ we shall write $S\sim T$ if $S-T\in\mathscr{A}(X)$. Since $\mathscr{A}(X)$ is a closed $*$-ideal, this relation is equivalent with equality in the quotient $\mathscr{A}/\mathscr{A}(X)$ C*-algebra, so it is compatible with the algebraic operations and with continuous functional calculus for normal elements. This can also be seen from the fact that $S\sim T$ if and only if $\mathscr{P}_Y(S)=\mathscr{P}_Y(T)$ for all $Y\neq X$ (and if and only if $\mathscr{P}(Y)(S)=\mathscr{P}(Y)(T)$ for all $Y\neq X$).

PROPOSITION 4.1: *Let* $\{J_Y\}_{Y\in\mathscr{L}}$ *be an* $\mathscr{A}$*-reducing family. For each* $S\in\mathscr{A}$ *and* $Y\in\mathscr{L}$ *denote* $S_Y=\mathscr{P}_Y(S)$. *Then for* $S^1,\ldots,S^n\in\mathscr{A}$ *we have*

(4.1) $$S^1S^2\ldots S^n \sim \Sigma_Y\, J_YS_Y^1S_Y^2\ldots S_Y^nJ_Y.$$

**Proof**: Since $\mathscr{P}_Y$ is a homomorphism, we have $(S^1S^2\ldots S^n)_Y=S_Y^1S_Y^2\ldots S_Y^n$, so we may assume that there is only one factor. Then, using $[S,J_Y]\in\mathscr{A}(X)$ and $(S-S_Y)J_Y\in\mathscr{A}(X)$ (because $S-S_Y\in\mathscr{B}_Y$) we get

$$S=\Sigma_Y\, SJ_Y^2=\Sigma_Y\,([S,J_Y]J_Y+J_Y(S-S_Y)J_Y+J_YS_YJ_Y)\sim\Sigma_YJ_YS_YJ_Y. \blacksquare$$

COROLLARY 4.2: *If* H *is a self-adjoint operator affiliated to* $\mathscr{A}$, $\varphi \in C_\infty(\mathbb{R})$ *and* $S \in \mathscr{A}$, *then*

(4.2) $\qquad \varphi(H) \sim \Sigma_Y J_Y \varphi(H_Y) J_Y,$

(4.3) $\qquad \varphi(H) S \varphi(H) \sim \Sigma_Y J_Y \varphi(H_Y) S_Y \varphi(H_Y) J_Y.$

**Remark**: In the N-body case considered in section 2 we have $\mathscr{A}(X)=\mathbf{K}(X)$. Taking into account theorem 2.1, if H is a self-adjoint operator in $\mathscr{H}(X)$ affiliated to $\mathscr{T}$, $\varphi \in C_\infty(\mathbb{R})$ and $\chi: X \to \mathbb{C}$ is continuous and homogeneous of degree zero for $|x| \geq 1$, we shall have $[\varphi(H), \chi(Q)] \in \mathbf{K}(X)$. If H is affiliated to the algebra $\mathscr{A}$ described by (2.2) and $\chi = \chi_Y$ is Y-reducing, then we also have $\chi_Y(Q)(\varphi(H) - \varphi(H_Y)) \in \mathbf{K}(X)$. These assertions are generalisations of some of the results from section 2.6 of [ABG 1].

We arrive, finally, to what we call "Mourre theory in a graded C*algebra setting". From now on we assume that a densely defined, self-adjoint operator A in $\mathscr{H}$ is given such that the group of automorphisms associated to $W_\alpha = e^{iA\alpha}$ leaves $\mathscr{A}$ invariant and its action is compatible with the grading, i.e.

(4.4) $\qquad W_\alpha^* \mathscr{A}(Y) W_\alpha \subset \mathscr{A}(Y)$ for all $Y \in \mathscr{L}$ and $\alpha \in \mathbb{R}$.

If we denote $\{\mathscr{W}_\alpha\}$ the group of automorphisms of $B(\mathscr{H})$ given by $\mathscr{W}_\alpha(S) = W_\alpha^* S W_\alpha$, then the preceding requirements are fulfilled if and only if $\mathscr{W}_\alpha(\mathscr{A}) = \mathscr{A}$ and $\mathscr{W}_\alpha \mathscr{P}(Y) = \mathscr{P}(Y) \mathscr{W}_\alpha$ for all $Y \in \mathscr{L}$, $\alpha \in \mathbb{R}$ (the second condition being equivalent to $\mathscr{W}_\alpha \mathscr{P}_Y = \mathscr{P}_Y \mathscr{W}_\alpha$ for all $Y, \alpha$).

Let us remark that if we consider the algebra $\mathscr{T}$ of section 2.2 and if $W_\alpha$ is the dilation group, then these conditions are fulfilled (so for $\mathscr{A}$ given by (2.2) also). Moreover, in this case $\{\mathscr{W}_\alpha\}_{\alpha \in \mathbb{R}}$ induces a *norm-continuous* group of automorphisms of $\mathscr{T}$ (in particular, its generator, which is formally $[\cdot, iA]$, is norm-densely defined).

We will be interested in the spectral analysis of a self-adjoint operator H affiliated to $\mathscr{A}$ by the conjugate operator method. Proposition 3.3 shows that, if H has a spectral gap (i.e. $\exists \lambda_0 \in \mathbb{R} \setminus \sigma(H)$; in fact we shall be interested only in H bounded from below), then it is better to study $R = (\lambda_0 - H)^{-1}$, which is bounded, self-adjoint and belongs to $\mathscr{A}$. In particular, we shall not have to put any condition of invariance under $W_\alpha$ of D(H), which could be non-dense (in hard-core case for example). So, for the moment we consider an arbitrary self-adjoint operator $R\in\mathscr{A}$.

PROPOSITION 4.3: *Let* $R\in\mathscr{A}$ *and denote* $R(Y)=\mathscr{P}(Y)(R)$, $R_Y=\mathscr{P}_Y(R)$. *Then* $R\in C_n^1(A)$ *if and only if* $R(Y)\in C_n^1(A)$ *for all* $Y\in\mathscr{L}$ *and also if and only if* $R_Y\in C_n^1(A)$ $(\forall Y\in\mathscr{L})$. *In this case we shall have*

$$[iR,A]\in\mathscr{A} \text{ and } \mathscr{P}(Y)([iR,A])=[iR(Y),A],\ \mathscr{P}_Y([iR,A]=[iR_Y,A] \text{ for all } Y\in\mathscr{L}.$$

The proof is trivial because $\mathscr{A}$, $\mathscr{A}(Y)$, $\mathscr{A}_Y$ are norm-closed and $\mathscr{P}(Y)$, $\mathscr{P}_Y$ commute with $\mathscr{W}_\alpha$. The problem which we would like to study now is the relation between $\hat{\rho}_R^A$ and $\hat{\rho}_{R_Y}^A$ with $Y\neq X$. Since A is fixed in this section, we shall leave it out in the notations of the $\hat{\rho}$-functions.

As an example, let us consider the "two-body" case $\mathscr{L}=\{O,X\}$. Let $R\in\mathscr{A}$ self-adjoint with $R\in C_n^1(A)$. Then $R=R_O+R(X)$ with $R_O\in\mathscr{A}_O=\mathscr{A}(O)$ and $R(X)\in\mathscr{A}(X)$ (which is $\mathbf{K}(X)$ in the N-body case). If $R\in C_n^1(A)$, then according to proposition 4.3

$$[iR,A] = [iR_O,A] + [iR(X),A] \sim [iR_O,A]$$

(because $[iR(X),A]=\mathscr{P}(X)([iR,A])\in\mathscr{A}(X)$). If $\varphi:\mathbb{R}\to\mathbb{C}$ is continuous (and $\varphi(0)=0$ if $\mathscr{A}$ has not a unit) then $\varphi(R)=\mathscr{P}_O(\varphi(R))+\mathscr{P}(X)(\varphi(R))=\varphi(R_O)+\mathscr{P}(X)(\varphi(R))\sim\varphi(R_O)$. Hence $\varphi(R)[iR,A]\varphi(R)\sim\varphi(R_O)[iR_O,A]\varphi(R_O)$ and $\varphi^2(R)\sim\varphi^2(R_O)$ if $\varphi\in C_0^\infty(\mathbb{R})$ (and $\varphi(0)=0$ if $\mathscr{A}$ has not unit). If $\mathscr{A}(X)$ contains only compact operators, it is easy to get from this that $\hat{\rho}_R(\lambda)=\hat{\rho}_{R_0}(\lambda)$ for all $\lambda$ ($\neq0$ if $\mathscr{A}$ has not unit) . In particular, A is conjugated to R at $\lambda$ ($\neq0$ if $\mathscr{A}$ has not unit) if and only if it is conjugated to $R_O$ at $\lambda$.

Recall that if $\mathscr{L}=\{O,X\}$, then $\mathscr{A}$ is automatically reducible. Let us go back now to a general $\mathscr{A}$, but assume it reducible. Let $\{J_Y\}$ be an $\mathscr{A}$-reducible family. Consider a self-adjoint element $R\in\mathscr{A}$ of class $C_n^1(A)$ and a continuous function $\varphi:\mathbb{R}\to\mathbb{R}$ which vanishes at zero if $\mathscr{A}$ has not a unit. Then $\varphi(R)\in\mathscr{A}$ and $\mathscr{P}_Y(\varphi(R))=\varphi(R_Y)$. Let us take $S=[iR,A]$ in (4.3). Then corollary 4.2 and proposition 4.3 give:

$$\varphi(R)[iR,A]\varphi(R)\sim\Sigma_Y J_Y\varphi(R_Y)[iR_Y,A]\varphi(R_Y)J_Y, \tag{4.5}$$

$$\varphi^2(R)\sim\Sigma_Y J_Y\varphi^2(R_Y)J_Y. \tag{4.6}$$

Let us write $S \lesssim T$ for $S,T \in \mathscr{A}$, if we have this inequality modulo $\mathscr{A}(X)$ (i.e. $\exists K \in \mathscr{A}(X)$ such that $S \leq T+K$). It follows from (4.5), (4.6) that we have $\varphi(R)[iR,A]\varphi(R) \gtrsim a\varphi^2(R)$ for some $a \in \mathbb{R}$ if and only if

(4.7) $\quad \Sigma_Y J_Y[\varphi(R_Y)[iR_Y,A]\varphi(R_Y) - a\varphi^2(R_Y)]J_Y \gtrsim 0.$

In conclusion, the following result has been proved (observe that $J_X = \mathbf{O}$):

THEOREM 4.4: *Let $\mathscr{A}$ be a reducible $\mathscr{L}$-graded $C^*$-algebra such that $\mathscr{A}(X)$ contains only compact operators. Let A be a densely defined self-adjoint operator in $\mathscr{H}$ such that $e^{-iA\alpha}\mathscr{A}(Y)e^{iA\alpha} \subset \mathscr{A}(Y)$ for all $Y \in \mathscr{L}$ and $\alpha \in \mathbb{R}$. Consider a self-adjoint operator $R \in \mathscr{A}$ of class $C^1_n(A)$. Then for all $\lambda \in \mathbb{R} \setminus \{0\}$ we have*

(4.8) $\quad \hat{\rho}^A_R(\lambda) \geq \min\left\{\hat{\rho}^A_{R_Y}(\lambda) \mid Y \in \mathscr{L} \setminus \{X\}\right\}.$

*In particular, if A is conjugated at some $\lambda \neq 0$ to all $R_Y$ with $Y < X$, then A is also conjugated at $\lambda$ to R.*

**Remarks**:

(a) If $\mathscr{A}$ has unit, the condition $\lambda \neq 0$ is not necessary.

(b) Only $J_Y \neq 0$ really appear in (4.7); hence in (4.8) the minimum has to be taken only over these Y's. For example, if there is an $\mathscr{A}$-reducing family $\{J_Y\}$ with $J_Y \neq 0$ only for $Y \in \mathscr{L}^2$ (as in the N-body situation considered in section 2), then:

(4.9) $\quad \hat{\rho}^A_R(\lambda) \geq \min\left\{\hat{\rho}^A_{R_Y}(\lambda) \mid Y \in \mathscr{L}^2\right\}.$

In the next corollary we use the obvious fact that if $\lambda_0 \notin \sigma(H)$ then $\lambda_0 \notin \sigma(H_Y)$ for any $Y \in \mathscr{L}$ (because $\mathscr{P}_Y$ are $*$-homomorphisms).

COROLLARY 4.5: *Assume that H is a self-adjoint unbounded operator in $\mathscr{H}$ which has a spectral gap and which is affiliated to $\mathscr{A}$. Moreover, assume that the domain of $H_Y$ is invariant under $e^{iA\alpha}$ (all $Y \in \mathscr{L}$, $\alpha \in \mathbb{R}$). If H is of class $C^1_n(A)$, then each $H_Y$ is of class $C^1_n(A)$ and*

(4.10) $\quad \hat{\rho}^A_H \geq \min\left\{\hat{\rho}^A_{H_Y} \mid Y \in \mathscr{L} \setminus \{X\}\right\}.$

(*Remark* (b) *above applies here too). In particular, if A is conjugated to each $H_Y$ with $Y < X$ at some $\lambda \in \mathbb{R}$, then H is conjugated to H at $\lambda$.*

Combining theorem 3.4 (more precisely fromula (3.14)) with corollary 4.5 one easily gets the results of [PSS], [FH] and [ABG 1] for N-body or Agmon hamiltonians (much more general situations may be considered, as we shall show in a later publication). In fact (3.14) shows by induction over Y that $\hat{\rho}_Y \geq 0$ for all Y. Hence, using again (3.14) and proposition 3.1 we see that for Y<X we have $\hat{\rho}_Y(\lambda)=0$ only if $\hat{\rho}^Y(\lambda)=0$ or if $\hat{\rho}^Y(\lambda)>0$ but $\lambda$ is an eigenvalue of $H^Y$. So we get by induction that $\hat{\rho}_Y(\lambda)>0$ if $\lambda$ is not a threshold or eigenvalue of $H^Y$. Then (4.10) implies $\hat{\rho}_X(\lambda)>0$ if $\lambda$ is not a threshold of H.

## REFERENCES


[ABG1] Amrein W.O., Boutet de Monvel-Berthier A. and Georgescu V., *Notes on the* N-*Body Problem, I*, Preprint Université de Genève, UGVA-DPT 1988/11-598.

[ABG2] Amrein W.O., Boutet de Monvel-Berthier A. and Georgescu V., *Notes on the* N-*Body Problem, II*, Preprint Université de Genève, UGVA-DPT 1991/04-718, p.164-424.

[BG1] Boutet de Monvel-Berthier A. and Georgescu V., *Graded C*-Algebras and Many-Body Perturbation Theory.* I. *The N-Body Problem,* C.R.Acad.Sci. Paris, **312**, Série I, p.477-482, 1991.

[BG2] Boutet de Monvel-Berthier A. and Georgescu V., *Graded C*-Algebras in the N-Body Problem*, J.Math.Phys., **32**, n°11 (1991), p.3101-3110.

[BGM] Boutet de Monvel-Berthier A., Georgescu V. and Mantoiu M., *Locally Smooth Operators and the Limiting Absorption Principle for N-Body Hamiltonians,* Rev. in Math. Phys. (1992), to appear — preprint Universität Bielefeld, BIBOS, n°433/1990, p.1-101].

[C] Cordes H.O., *On the Compactness of Commutators of Multiplications and Convolutions, and Boundedness of Pseudo-Differential Operators,* J.Funct.Analysis, **18** (2) (1975), p.115-132.

[D1] Derezinski J., *A New Proof of the Propagation Theorem for N-Body Quantum Systems*, Comm. Math Phys. ,**122** (1989), p.203-231.

[D2] Derezinski J., *The Mourre Estimate for Dispersive N-Body Schrödinger Operators*, Preprint Virginia Polytechnic Institute, 1988.

[C] Gérard C., *The Mourre Estimate for Regular Dispersive Systems*, Ann. Inst. Henri Poincaré, **54** (1) (1991), p.59-88.

[FH] Froese R.G. and Herbst I., *A New Proof of the Mourre Estimate*, Duke Math J., **49** (1982), p.1075-1085.

[HP] Hille E. and Phillips R.S., *Functional Analysis and Semi-groups,* Amer. Math. Soc., Providence, R.I. 1957.

[PSS] Perry P., Sigal. I.M. and Simon B., *Spectral Analysis of N-Body Schrödinger Operators,* Ann. Math., **114** (1981), p.519-567.

[S] Simon B., *Geometric Methods in Multiparticle Quantum Systems,* Comm. Math. Phys., **55** (1977), p.259-274.

[T] Tamura R., *Propagation Estimates for N-body Quantum Systems,* Bull. Fac. Sci. Ibaraki Univ., **22** (1990), p.29-48.

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