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INTRODUCTION

Avec le soutien du C.N.R.S et de la D.R.E.D, l'année académique 1990/91 fût une année spéciale consacrée aux méthodes semiclassiques.

A l'origine les méthodes semi-classiques désignaient les techniques utilisées par les physiciens pour essayer de comprendre les relations subtiles existant entre la mécanique classique de Newton et la mécanique quantique de Heisenberg-Schrödinger (lorsque la constante de Planck $\hbar$ devient négligeable par rapport aux autres grandeurs physiques: masse, énergie, distances, ...). L'exemple fondamental est la méthode B.K.W (Brillouin, Kramers, Wentsel) qui consiste à construire des solutions asymptotiques, par rapport à la constante de Planck, de l'équation de Schrödinger. Cette méthode est restée longtemps formelle. La justification mathématique rigoureuse a nécessité l'élaboration de théories sophistiquées qui ont vu le jour dans les années 1970 (indice de Maslov, opérateurs intégraux de Fourier-Hörmander). A partir de ces travaux de base, de nombreux mathématiciens se sont attaqués avec succès à divers problèmes issus de la physique et se traduisant par l'étude spectrale d'opérateurs pseudo-différentiels, dépendant de paramètres. Citons quelques exemples parmi les plus connus:

- le comportement du spectre de l'opérateur de Schrödinger lorsque la constante de Planck tend vers zéro (régle de Bohr-Sommerfeld, effet tunnel)
- le comportement asymptotique des grandes valeurs propres (formules du type Weyl)
- la trace du noyau de la chaleur lorsque la température tend vers zéro et les invariants géométriques associés
- diffusion quantique ou acoustique: problèmes à plusieurs corps, problèmes inverses, résonances
- systèmes périodiques: analyse du spectre de bande, problèmes inverses
- description de certains systèmes quantiques désordonnés: potentiels quasi périodiques, équation de Harper, chaos quantique
- limite thermodynamique.

Durant ces quinze dernières années, les méthodes semi-classiques se sont beaucoup enrichies avec le développement de l'analyse microlocale des équations aux dérivées partielles et de leurs solutions. De nombreux mathématiciens (et physiciens!) ont participé à ce développement. Parmi les travaux que l'on peut considérer comme fondamentaux mentionnons en particulier ceux de S. Agmon, Y. Colin de Verdière, J. Chazarain, L. Hörmander, V. Ivrii, J. Leray,V. Maslov, R. Melrose, J. Sjöstrand, A. Voros (je cite ces noms car il me sem-

ble bien représenter le rapprochement fructueux qui s'est effectué durant cette période entre l'analyse des équations aux dérivées partielles et la physique-mathématique).
Deux volumes de la collection **Astérisque** regroupent les actes de l'Ecole d'Eté et du Colloque International organisés à Nantes, en Juin 1991. L'Ecole d'Eté était centrée sur quatre cours: V. Ivrii (Asymptotiques Spectrales); M.A Shubin (Théorie spectrale sur les variétés non compactes); A. Soffer (Problèmes à N-corps) et G. Ulhmann (Problèmes inverses). Le Colloque International comportait vingt conférences portant sur des thèmes variés, illustrant la puissance des méthodes semi-classiques appliquées aux équations de la mécanique quantique ou à l'équation des ondes acoustiques. Les sujets abordés concernent principalement l'équation de Schrödinger sous différents aspects: N-corps, champs magnétiques, limite thermodynamique, solitons, cristaux. Deux exposés sont consacrés à la diffusion acoustique par un obstacle et à la conjecture de Lax-Philips sur les résonances.

En conclusion, je voudrais remercier les institutions et les personnes qui ont permis le succès de cette année spéciale sur les méthodes semiclassiques, en premier lieu le C.N.R.S en la personne de J.P Ferrier et la D.R.E.D en la personne de J. Giraud. Je remercie également tous ceux qui ont participé à l'organisation des différents colloques qui se sont déroulés entre Novembre 1990 et Juin 1991, en particulier les collègues suivants: J. Bellissard, J.M.Bismut, A. Ben Arous, J.M. Combes, C. Gérard , A. Grigis , J.C Guillot, B. Helffer, A. Martinez, J.F.Nourrigat, F. Pham, J. Sjöstrand, A.Unterberger, A. Voros.
Je remercie l'université de Nantes et le conseil général de Loire-Atlantique pour le soutien qu'ils nous ont apporté.
D. Macé-Ramette a assuré avec dévouement et compétence le secrétariat de cette année spéciale, je l'en remercie.

Nantes, le 21 Décembre 1992

D. Robert

RESUMES

1. AGMON Shmuel . *A representation theorem for solutions of Schrödinger type equations on non compact Riemannian manifolds*

Let X be a real analytic Riemannian manifold with a boundary ∂X. Denote its interior by X and its metric by g. Introduce on X a conformal metric h defined by $h = p^{-2}g$ where $p(x)$ is a real analytic on X such that $p(x) > 0$ in $X, p(x) = 0$ and $dp \neq 0$ on ∂X. Under the metric h, X becomes a complete non-compact Riemannian manifold with a corresponding Laplacian Γ_h. Consider solutions of the differential equation.

$$(*)\ \ \Gamma_h u + \lambda q(x) u = 0 \quad \text{on} X$$

where $q(x)$ is a real analytic function on ∂X and $\lambda \in \mathbf{C}$.
Our main result is a representation theorem for all solutions of equation (*). The theorem is a generalization of a representation formula established by Helgason and Minemura for solutions of the Helmholtz equation on hyperbolic space.

2. BOUTET de MONVEL Anne-Marie; GEORGESCU Vladimir. *Some developments and applications of the abstract Mourre theory*

Our aim is to present several applications of a version of Mourre theory that we have recently developed. We can easily deduce from it, for example, a very precise form of the limiting absorption principle for perturbations $H = h(P) + V_S + V_L$ of a constant coefficient pseudo-differential operator $h(P)$ by short-range and long-range *non local* potentials V_S and V_L. The perturbations V_S, V_L are quite singular locally (the sum above is required to exist only in form-sense) and the assumptions concerning their behaviour at infinity are essentially optimal (e.g V_S is of Enss type). Furthermore, if such an H is perturbed by another short-range potential, the relative wave operators exist and are complete. The theory works aslo for systems (like Dirac operators). Other applications are to division theorems, i.e.properties of the operators of multiplication by $(h(x) \pm io)^{-1}$, under minimal regularity assumptions on h. In particular these examples show that the regularity assumptions we make in our abstract version of Mourre theory are essentially optimal.

3. BUSLAEV Vladimir; PERELMAN Gregor. *On nonlinear scattering of states which are close to a soliton*
Under some conditions on the function F the nonlinear Schroedinger equation

$$i\psi_t = -\psi_{xx} + F(|\psi|^2)\psi, \psi = \psi(x,t) \in \mathbf{C},$$

admits a class of bounded solutions $w(x|\sigma(t))$, which parameters $\sigma = \sigma(t) \in \mathbf{R}^4$ depend explicitely on time t. The Cauchy problem for the Schroedinger equation with the initial data

$$\psi(x,0) = w(x|\sigma_0(0)) + \chi_0(x)$$

is considered where χ_0 is assumed to have the sufficiently small norm

$$N = \|(1+x^2)\chi_0\|_2 + \|\chi_0'\|_2.$$

If the spectrum of the linearization of the Schroedinger equation on the soliton $w(\cdot|\sigma_0(0))$ has the simplest structure in some natural sense, the asymptotic behavior of ψ as $t \longrightarrow +\infty$ is given by the formula (in $\mathbf{L}_2$-norm):

$$\psi = w(\cdot|\sigma_+(t)) + \exp(-il_0 t)f_+ + o(1),$$

here $\sigma_+(0)$ is close to $\sigma_0(0), l_0 = -\partial_x^2, f_+ \in \mathbf{L}_2(\mathbf{R})$ and is sufficiently small.

4. BRUNING Jochen; SUNADA Toshikazu. *On the spectrum of gauge-periodic elliptic operators*
We consider a symmetric elliptic operator, D, on a complete Riemannian manifold which admits a properly discontinuous action of a group Γ, with compact quotient. We assume that D is "gauge periodic" i.e. commutes with the group action twisted by a gauge; a typical example is the Schrödinger operator with constant magnetic field. We associate a C^*-algebra with this situation and prove that the spectrum of (the closure) D has band sructure if this C^*-algebra has the "Kadison property". For the magnetic Schrödinger operator, we can derive an optimal upperbound on the number of gaps for rational flux.

5. GEORGESCU Vladimir; BOUTET de MONVEL Anne-Marie. *Graded C^*-algebras and many-body perturbation theory:* II. *The Mourre estimate*
Let $\mathcal{L}$ be a finite lattice with largest element X and $\mathcal{A}$ a C^*-algebra. We say that $\mathcal{A}$ is $\mathcal{L}$-graded if a family $\{\mathcal{A}(Y)\}_{Y\in\mathcal{L}}$ of C^*-subalgebras has been given such that $\mathcal{A} = \sum_{Y\in\mathcal{L}} \mathcal{A}(Y)$ (direct sum) and $\mathcal{A}(Y)\mathcal{A}(Z) \subset \mathcal{A}(Y \vee Z)$ for $Y, Z \in \mathcal{Z}$. The Hamiltonians usually considered in the many-body problems are affiliated to such an algebra. If $\mathcal{A}$ is realized on a Hilbert space $\mathcal{H}$, the many-channel structure of a self-adjoint operator H(in general non densely defined) affiliated to $\mathcal{A}$ may be described as follows : for each $Y \in \mathcal{L}, \mathcal{A}_Y = \sum_{Z\leq Y} \mathcal{A}(Z)$ is a C^*-algebra, the natural projection $\mathcal{P}_Y : \mathcal{A} \longrightarrow \mathcal{A}_Y$ is a $*$-homomorphism and there is a unique self-adjoint operator H_Y such that $\mathcal{P}_Y(f(H)) = f(H_Y)$ for all

$f \in C_\infty(\mathbf{R})$. Let A be a self-adjoint operator such that $e^{-iA\alpha}\mathcal{A}(Y)e^{iA\alpha} \subset \mathcal{A}(Y)$ for all Y and α. Assume that $D(H_Y)$ is invariant under $e^{iA\alpha}$ for all Y and $\frac{d}{d\alpha}e^{-iA\alpha}He^{iA\alpha}$ exists in norm in $B(D(H), D(H)^*)$ and H has a spectral gap. Our main result is that,under a further assumption on $\mathcal{A}$ which is independent of H and trivially verified in the N-body case. A is conjugate to H at a point $\lambda \in \mathbf{R}$ if it is conjugate to each H_Y with $Y \neq X$ at λ.

6. GUILLEMIN Victor. *The homogeneous Monge-Ampere equation on a pseudoconvex domain*
In the first three sections of this article I give a new proof of a theorem of Jack Lee which says that if M is a compact strictly pseudoconvex domain with a real-analytic boundary, one can find a defining function on the boundary which satisfies the homogeneous complex Monge-Ampere equation. The proof involves complexifying a solution of a related real Monge-Ampere equation.
The rest of this article is devoted to a generalization of a theorem of L. Boutet de Monvel. Boutet's theorem says that if X is a compact manifolf equipped with a real-analytic Riemannian metric and f is a real-analytic function of M then the following are equivalent

(1) f can be extended holomorphically to a Grauert of radius r, about X.

(2) The diffusion equation, $\frac{\partial u}{\partial t} = \Delta^{\frac{1}{2}}u$, can be solved *backwards* in time over the interval,$-r \leq t \leq 0$ with initial data :

$$u(0, x) = f(x).$$

In the second half of this article I show that this theorem has a generalization in which Grauert tubes are replaced by a family ,$\phi = r$, of strictly pseudoconvex domains, ϕ satisfying homogeneous Monge-Ampere.

7. HAGEDORN George. *Classification and normal forms for quantum mechanical eigenvalue crossings*
In the analysis of molecular systems, one is led to the study of a quantum mechanical Hamiltonian for the electrons that is a function of n parameters that describe the positions of the nuclei. As the parameters are varied, the spectrum of the electron Hamiltonian can change.The way in which the graphs of the discrete eigenvalues cross one another depends on the symmetry group of the Hamiltonian function. We classify generic crossings of minimal multiplicity eigenvalues under all possible symmetry circumstances. For each of the eleven types of crossings, we derive a normal form for the Hamiltonian function near the crossing.

8. HELFFER Bernard; SJÖSTRAND Johannes. *Semiclassical expansions of the thermodynamic limit for a Schrödinger equation*
We give a proof of the semi-classical expansion of the thermodynamic limit for a model introduced in statistical mechanics by M.Kac. For this family

(parametrized by m) of Schrödinger operators $P^{(m)}(h) = -\sum_{k=1}^{m} h^2\partial^2/\partial x_k^2 + V^{(m)}(x)$ defined on $\mathbf{R}^m$, this corresponds to the study of the expansion in power of h of $lim_{m\to\infty}\lambda(m,h)/m$ where $\lambda(m,h)$ is the first eigenvalue of $P^{(m)}(h)$.

9. HEMPEL Reiner. *Eigenvalue asymptotics related to impurities in crystals*

As a mathematical model for energy levels produced by impurities in a crystal, we study perturbations of a (periodic) Schrödinger operator $H = -\Delta + V$ by a potential λW, where λ is a real coupling constant and W decays at infinity. Assuming that H has a spectral gap, we ask for the number of eigenvalues which are moved into the gap and cross a fixed level E in the gap, as λ increases. Such "impurity levels" are a basic ingredient in the quantum mechanical theory of the color of crystals (insulators) and of the conductivity of (doped) semi-conductors in solid state physics.

In the general case where W is allowed to change its sign, we discuss upper and lower asymptotic bounds for the eigenvalue counting function.

We also provide bounds for the total number of eigenvalues crossing E as the height of a repulsive "barrier", living on a compact set K, tends to ∞. While quasi-classical arguments give some useful hints, it turns out that, in particular, lower bounds are very sensitive and depend highly on the structure of the set K. Here decoupling via natural Dirichlet boundary conditions tends to play a dominating rôle, e.g. if the set K has many small holes ("swiss cheese").

10. HISLOP Peter. *Singular perturbations of Dirichlet and Neumann domains and resonances for obstacle scattering*

We consider the problem of proving the existence of and estimating the location of scattering poles for a class of trapping obstacles known as Helmholtz resonators with both Dirichlet and Neumann boundary conditions. We treat the case when the diameter of the tube linking the cavity to the exterior is made small and the high energy behavior of resonances when the tube diameter is fixed. The latter case gives an example of the Lax-Phillips conjecture.

11. IKAWA Mitsuru. *Singular perturbation of symbolic flows and the modified Lax-Phillips conjecture*

In order to consider the modified Lax-Phillips conjecture for scattering by obstacles consisting of several convex bodies, the zeta functions of a dynamical system in the exterior of the obstacle play an important role.

In this paper we develope a theory for singular perturbations of symbolic dynamics and consider the zeta functions associated with dynamical systems. We give a sufficient condition for the existence of poles of the zeta functions of the singularity perturbed dynamics.

As the application of this theory, the validity of the modified Lax-Philipps conjecture for obstacles consisting of small balls is proved.

12. LIEB Elliott. *Large atoms in large magnetic fields*
The ground state energy of an atom of nuclear charge Ze and in a magnetic field B is evaluated exactly in the asymptotic regime $Z \to \infty$. We present the results of a rigorous analysis that reveals the existence of 5 regions as $Z \to \infty$: $B \ll Z^{4/3}$, $B \approx Z^{4/3}$, $Z^{4/3} \ll B \ll Z^3$, $B \approx Z^3$, $B \gg Z^3$. Different regions have different physics and different asymptotic theories. Regions 1,2,3,5 are described exactly by a simple density functional theory, but only in regions 1,2,3 is it of the semiclassical Thomas-Fermi form. Region 4 cannot be described exactly by any simple density functional theory; surprisingly, it can be described by a simple *density matrix* functional theory, as found after this talk was presented. [There are two more recent references: Phys. Rev. Lett. **69**, 749-752 (1992) and Commun. Pure Appl. Math. (in press for the McKean issue).] A surprising conclusion is that although the magnetic field has a profound effect on the atomic energy in regions 2,3,4 and 5, the atom remains spherical (to leading order) in regions 2 and 3.

13. NAKAMURA Shu. *Resolvent estimates and time-decay in the semiclassical limit*
We consider resolvent estimates for Schrödinger operators in the semiclassical limit. We construct a semiclassical analogue of the theory of multiple commutator estimates by Jensen, Mourre and Perry [JMP]. Then we apply it to the barrier-top energy and nontrapping energies to obtain semiclassical estimates for powers of the resolvent. As a consequence, we also obtain estimates for the time-decay in the semiclassical limit.

14. RALSTON James. *Magnetic breakdown*
This article constructs time-dependent asymptotic solutions to the magnetic Schrödinger equation in the weak magnetic field limit in the case of "interband magnetic breakdown". This means that there is an eigenvalue crossing in the (Bloch) spectrum of the zero magnetic field operator and interband tunnelling effects occur.

15. SHUBIN Michael; GROMOV Michael. *Near-cohomology of Hilbert complexes and topology of non-simply connected manifolds*
Near cohomologies of Hilbert complexes are obtained heuristically by taking cochains with small coboundaries modulo cochains which are close to cocycles. Rigorously this leads to a family of closed cones depending on a small real parameter up to an equivalence relation. It is proved that the near cohomologies are homotopy invariants of a Hilbert complex with respect to the chain homotopy equivalence defined by morphism and homotopy operators which are bounded linear operators. Applying this to the Hilbert de Rham complex on the universal covering of a non-simply connected manifold gives homotopy invariants of this manifold. A von Neumann algebra stucture on a Hilbert complex allows to convert near-cohomologies to number homotopy invariants of the

complex. For the Hilbert de Rham complex they coincide with the invariants introduced and investigated by the authors in an earlier paper and include heat kernel decay exponents by S.P. Novikov and M.A. Shubin.

16. SIMON Barry. *The Scott correction and the quasi-classical limit*
The Scott correction is the second term in a large Z asymptotic expansion of the total binding energy of an atom with nuclear charge Z. The atom is complicated system with multiparticle correlations among the electrons. Nevertheless, the proof of the Scoot correction can be reduced to the study of the semi-classical limit of a one-body system where the electron-electron interaction is replaced by an averaged self-consistent potential.

17. SJÖSTRAND Johannes. *Exponential convergence of the first eigenvalue divided by the dimension, for certain sequences of Schrödinger operators*
We consider certain sequences of Schrödinger operators

$$-h^2\Delta + V^{(m)}(x), x \in \mathbf{R}^m, m = 1, 2, ...$$

Our assumptions imply that $V^{(m)}$ is strictly convex. If $\mu(m, h)$ denotes the lowest eigenvalue, we study the exponential convergence of $\mu(m, h)/m$ when m tends to ∞.

18. VAINBERG Boris. *Scattering of waves in a medium depending periodically on time*
The asymptotic behaviour as $t \to \infty, |x| \leq a < \infty$ of solutions of exterior mixed problems for hyperbolic equations and systems is obtained when the boundary of a domain and coefficients of the equations depend periodically on time. It is supposed that the coefficients are constant in a neigborhood of infinity and that the non-trapping condition is fulfilled. The method of the research is based on using a special parametrix, Fourier-Bloch transform and analytical properties of an integral eguation which arises. This method can be regarded as an alternative one to the Lax-Phillips scattering theory. Then the asymptotic behavior of the solutions is used to prove existence of the wave operators and of the scattering operator, if the general energy of any solution is uniformly bounded for $t \geq 0$ provided that it is bounded at $t = 0$.

19. WHITE Denis. *Long range scattering and the Stark effect*
We prove the completeness of Dollard's modified wave operators for the Stark effect Hamiltonians $H_0 = -(1/2\Delta - x_1$ and $H = H_0 + V$ where V is a general long range potential. As a consequence, the "unmodified" wave operators do not exist if V is not short range. In one space dimension this quantum mechanical result differs from the classical result : Jensen and Ozawa have shown that the usual wave operators in classical mechanics do exist. We show however that

this mathematical difference cannot be detected by any quantum mechanical observable. We derive the existence and completeness of the modified wave operators (in arbitrary space dimensions) from the comparable result for two Hilbert space wave operators by a stationary phase argument.

20. YAFAEV Dimitri. *Radiation conditions and scattering theory for three-particle Hamiltonians*
The correct form of radiation conditions is found in scattering problem for three-particle Hamiltonians H. For example, in a cone Γ of the configuration space where all pair potentials are vanishing the radiation conditions-estimate has the following forme. Let $\nabla^{(s)}$,

$$\nabla^{(s)}u(x) = \nabla u(x) - |x|^{-2} < \nabla u(x), x > x,$$

be the projection of the gradient ∇ on the plane, orthogonal to x, and let ξ be the characteristic function of Γ. Then the operator

$$\xi(|x|) + 1)^{-1/2}\nabla^{(s)}$$

is locally (away from thresholds and eigenvalues of H) H-smooth (in the sense of T.Kato). In cones where some of pair potentials are not vanishing radiation conditions-estimates have similar (though weaker) form with the gradient replaced by its projection on a certain subspace. Such estimates allows us to give an elementary proof of the asymptotic completeness for three-particle systems in the framework of the theory of smooth perturbations.