RUDOLF SCHARLAU
CLAUDIA WALHORN

Integral lattices and hyperbolic reflection groups

Astérisque, tome 209 (1992), p. 279-291
<http://www.numdam.org/item?id=AST_1992__209__279_0>

INTEGRAL LATTICES AND HYPERBOLIC REFLECTION GROUPS

Rudolf SCHARLAU and Claudia WALHORN

1. Reflective lattices and their groups

In this paper we wish to contribute to the problem of giving a full, explicit classification of all arithmetic groups of isometries of hyperbolic space which are generated by reflections in hyperplanes.

We consider integral quadratic lattices L, that is, L is a free $\mathbb{Z}$-module of finite rank, $L \cong \mathbb{Z}^r$, together with a symmetric bilinear form. The value of the form at vectors $x, y \in V := \mathbb{R}L$ is denoted by $(x, y) \in \mathbb{R}$. "Integral" means that $(x, y) \in \mathbb{Z}$ for all $x, y \in L$. We shall deal with the following two cases: L is **Euclidean**, i.e. the form is positive definite, or L is **Lorentzian**, i.e. the form is of signature $(n, 1)$, $n + 1 = r$.

For $v \in V$ with $(v, v) \neq 0$, the reflection

$$s_v : x \longmapsto x - \tfrac{2(v,x)}{(v,v)} v$$

is an isometry of the quadratic vector space V. A primitive vector $v \in L$ (that is, $v/m \notin L$ for all integers $m > 1$) is called a **root** of L if $(v, v) > 0$ and if s_v maps L into itself. By the above formula, this holds if and only if $(v, L) \subseteq \mathbb{Z}(v, v)/2$. In particular, $(v, v)/2$ divides the exponent of the finite group $L^\#/L$, where $L^\# = \{y \in V \mid (L, y) \subseteq \mathbb{Z}\}$ is the dual lattice. Notice that a vector v with $(v, v) = 1$ or 2 always is a root, but if L is not unimodular, then other values may occur. In the Lorentz case, the restriction to vectors with $(v, v) > 0$ is made for the following reason. The set of vectors $x \in V$ such that $(x, x) = -1$ falls into two connected components. If $O^+(V)$ denotes the subgroup of index 2 of the orthogonal group $O(V)$ mapping each of these into itself, then $(v, v) > 0$ is equivalent to $s_v \in O^+(V)$. Each of the connected components is a model of hyperbolic n-space H^n, and $O^+(V)$ induces the full isometry group of H^n. Notice that if $(v,v) > 0$, then the orthogonal complement $v^\perp$ is of signature $(n-1,1)$. Thus the fixed point set of s_v really "is" a hyperbolic subspace (hyperplane) in our hyperbolic space.

A basic (though trivial) observation now is that in the Euclidean case, the set $R(L)$ of all roots of L is a root system in the usual sense of Lie algebra theory. Indeed, if $v, v' \in R(L)$, then $s_v v' \in R(L)$ since $s_{s_v v'} = s_v s_{v'} s_v$. The "crystallographic condition" $2(v,v')/(v,v) \in \mathbb{Z}$ also holds; we have just seen that it holds for all $v' \in L$. Strictly speaking, $R(L)$ should be considered as a root system R together with a fixed quadratic form invariant under the Weyl group $W(R) = \langle s_v \mid v \in R \rangle$, and the notion of isomorphism is that of an isometric bijection. Thus, $R(L)$ is even a finer invariant than a usual root system.

We now come to the basic definition of this paper, which was first introduced explicitly by E. Vinberg [Vi2].

1.1. **Definition.** A quadratic lattice L is called **reflective** if

a) in the Euclidean case: $R(L)$ has the maximum possible rank $\dim L$,

b) in the Lorentz case: the normal subgroup $W(L)$ of $O^+(L)$ generated by all reflections s_v, v a root, is of finite index.

See [BS] for general results about the structure of reflective Euclidean lattices and for a full classification in small dimensions. If the signature is $(n,1)$, any group $W(L)$ acts as a discrete group, generated by reflections in hyperplanes, on hyperbolic n-space. By definition, $W(L)$ is "crystallographic" in the sense that it has a fundamental domain of finite volume if and only if L is reflective. (Recall that $O^+(L)$ always has a fundamental domain of finite volume.) Like in the case of reflection groups on the sphere or on Euclidean space, these "hyperbolic reflection groups" satisfy the axioms of abstract Coxeter groups (after one has fixed a fundamental domain and thus a fundamental set of generating reflections) and are described by Coxeter diagrams; see [Vi1]. Contrary to the spherical or Euclidean case, a full classification of hyperbolic reflection groups is not known, not even in the case of "arithmetic" groups $W(L)$.

A basic lemma of Vinberg's [Vi2] relates the two notions of reflectiveness of lattices to each other:

1.2. VINBERG'S LEMMA. *A necessary condition for a Lorentzian lattice L to be reflective is that, for all primitive isotropic vectors $c \in L$, the Euclidean lattice $c^\perp/\mathbb{Z}c$ is reflective.*

In particular, if L is of the form $L = \mathbb{H}\perp M$, $\mathbb{H} = (\mathbb{Z}^2, \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))$ the hyperbolic plane and M a positive definite lattice, then L is reflective only if M is reflective. (Take $c = (1,0) \in \mathbb{H}$ in Vinberg's lemma.) Now notice that a lattice of the form $\mathbb{H}\perp M$ of dimension ≥ 3 has only one class in its genus (since it is indefinite and has at least one two dimensional Jordan component at each prime; we refer for instance to [OM] for general facts about quadratic forms). Thus $\mathbb{H}\perp M$ only depends on the genus $\mathcal{G}(M)$, and Vinberg's lemma automatically sharpens to the following statement:

1.3. LEMMA. *If a lattice $L = \mathbb{H}\perp M$, where M is positive definite of rank at least 2 is reflective, then the genus $\mathcal{G}(M)$ (which only depends on L) is* **totally reflective** *in the sense that every lattice $M' \in \mathcal{G}(M)$ is reflective.*

It is true under quite general circumstances that L of signature $(n,1)$ is of the shape $\mathbb{H}\perp M$. In particular, this holds if $n \geq 4$ and L is **strongly square free** in the following sense: The exponent of $L^{\#}/L$ is square free (we say that L itself is **square free** for short), and the p-exponent of $\det L$ is at most half the dimension of L, for all p. In terms of Jordan decompositions, this means that for each p, the localized lattice has only a unimodular and a p-modular component $L_{0,p}$ resp. $L_{1,p}$, and $\dim(L_{1,p}) \leq \dim(L_{0,p})$. It is well known that to any lattice M (of whatever signature), one can associate a strongly square free lattice $\widetilde{M}$ (on the same space, but possibly scaling the quadratic form) such that $O(M) \subseteq O(\widetilde{M})$. In particular, if M is reflective, then $\widetilde{M}$ is reflective. The genus of $\widetilde{M}$ only depends on the genus of M, and (in the positive definite case) the mapping $M \mapsto \widetilde{M}$ induces a surjection of genera. This technique has been used by many people and goes back at least to Watson [Wa]. The construction of $\widetilde{M}$ breaks into two parts. One first makes the lattice square free by (iterated) application of the substitution $M \mapsto p^{-1}(M \cap p^2 M^{\#}) + M$, for the prime divisors p of $\det M$; then one applies, if necessary, the "local dualizing" operation $M \mapsto D_p M := {}^p(M^{\#} \cap p^{-1} M)$ to interchange the unimodular and the p-modular component of a Jordan decomposition.

The following theorem is an immediate consequence of an unpublished result of J. Biermann [Bie].

1.4. THEOREM. *There are only finitely many totally reflective genera of positive definite lattices of dimension ≥ 3.*

Proof: It is well known and easy to prove that any genus (w.l.o.g. strongly square free) of large enough dimension $n \geq n_0$ contains a non-reflective lattice. (It suffices to embed a lower-dimensional non-reflective lattice M_0 as an orthogonal summand; for $n_0 = 57$, the Leech lattice always works.) For lattices of fixed dimension $n \geq 3$, the main result of [Bie] says that any genus of sufficiently large determinant contains a lattice with trivial automorphism group which *a fortiori* is non-reflective. □

Remark 1. The result of Biermann's we have just referred to depends on the mass formula as well as on various algebraic reduction techniques and requires a long and complicated proof. Only a part of the arguments is actually needed if one only wants to prove the existence of a non-reflective lattice in a genus. In the case of signature $(4,1)$, we shall come back to this question in Section 3 below.

Remark 2. The first good upper bound $n_0 = 30$ has been given (implicitly) by Vinberg [Vi4], §4. Recently F. Esselmann [Es] proved that 20 is the precise value of the largest dimension of totally reflective genera, and $n = 21$ is the largest value for which a reflective lattice of signature $(n,1)$ exists; R. Borcherds had proved already several years ago that the unique even lattice of this signature and determinant 4 is reflective; [Bor], §8, Example 5.

We now observe that Theorem 1.4 together with Vinberg's lemma gives a new and –modulo the use of Biermann's result– very short proof of the following finiteness result for arithmetic hyperbolic reflection groups due to V.V. Nikulin.

1.5. COROLLARY. *There are only finitely many strongly square free reflective lattices of signature* $(n,1)$, $n \geq 4$.

This is a special case of [Ni], Theorem 5.2.1. The result holds without the assumption that the lattices should be strongly square free. (Of course one must then restrict to primitive lattices.) In fact, for Nikulin's proof it is irrelevant whether or not they are. In the above proof, one could avoid the assumption with some additional technical effort. More important is that Nikulin proves the result also for $n = 2, 3$ where our method completely breaks down for the following trivial reason: If $L = \mathbb{H} \perp M$ is strongly square free of dimension 3 or 4, then M is not necessarily primitive but can be scaled by an arbitrary square free factor, of course without affecting the reflectiveness.

Nikulin's proof of the finiteness result is geometrical, dealing with combinatorial and metrical properties of the fundamental polytopes of hyperbolic reflection groups. It also applies to lattices (with the appropriate signatures) over number fields. However, it gives only poor bounds on the discriminants of reflective lattices and apparently no other arithmetical restrictions..

2. Normal forms for lattices

Apart from Vinberg's basic general work and Nikulin's fundamental finiteness result, the existing literature on hyperbolic reflection groups and reflective lattices deals mainly with examples. Complete classifications have been obtained only under very restrictive assumptions (e.g. for unimodular lattices [Vi3]). In this and the next section we want to show that our knowledge of reflective Lorentzian lattices can be considerably improved by making a systematic use of the arithmetic theory of integral quadratic forms. In this article, we shall restrict ourselves to the (hyperbolic) dimensions $n = 3, 4$. If $n = 3$, we shall only treat the case of isotropic forms, where $H^n/W(L)$ is non-compact. A third general condition is that we require the groups $W(L)$ to be maximal among groups of this kind. Under these assumptions, we now want to give certain normal forms for our lattices. (The term "normal form" is not to be understood in a completely rigorous sense.) The main ingredient of these normal forms will be a fact which was already mentioned above: we can assume the lattices to be strongly square free, and if the unimodular and the p-modular component have the same dimension, for some prime p, then we can interchange them by the operation D_p not affecting the group.

We first deal with the case $n = 3$. Here we shall drop the assumption of square freeness in certain cases for the benefit of dealing only with even lattices, i.e. $(v, v) \in 2\mathbb{Z}$ for all $v \in L$. The quadratic form on $\mathbb{Z}^4$ in the following proposition is $f(v) = (v, v)/2$, not (v, v). A left upper index means scaling the quadratic form.

2.1. PROPOSITION. *If L is a lattice of signature $(3,1)$ with $O(L)$ maximal, then we may assume that L has the shape $L = \mathbb{H} \perp {}^{s}[a, b, c]$, where $\mathbb{H}$ denotes the hyperbolic plane $(\mathbb{Z}^2, xy)$, ${}^{s}[a, b, c]$ the binary lattice $(\mathbb{Z}^2,\ s(az^2 + bzw + cw^2))$, the discriminant $-D_0 = -(4ac - b^2)$ of $[a, b, c]$ is a fundamental discriminant (discriminant of a quadratic field), and s is a square free natural number relatively prime to D_0.*

Proof: We may start with an L which is strongly square free. The main aim of the proof is to embed the hyperbolic plane $\mathbb{H}$ into L (after possibly replacing L by a lattice with the same or a larger group). Once this is proved, the precise shape as given in the proposition will be obvious from the fact that L is strongly square free or equal to the even sublattice $L^\circ = \{x \in L \mid (x, x) \in 2\mathbb{Z}\}$ of a strongly square free lattice. It is sufficient to embed $\mathbb{H}$ locally for all primes p. Choose a Jordan decomposition $\mathbb{Z}_p L = L_0 \perp {}^{p} L_1$, L_0, L_1 unimodular. First consider the case $p \neq 2$. If $\dim L_0 \geq 3$, it is clear that $\mathbb{H}$ embeds into L_0. If $\dim L_0 = \dim L_1 = 2$, then one of L_0, L_1 is isotropic, since $L_0 \perp {}^{p} L_1 = L$ is isotropic. After possibly interchanging L_0 and L_1, we may assume that

$L_0 \cong \mathbb{H}$.

Now we consider the case $p = 2$. There are various cases according to the dimensions and the parity even/odd of L_0, L_1. In some cases, we pass from L to the even sublattice $L^\circ = \{x \in L \mid (x,x) \in 2\mathbb{Z}_2\}$. We give the details only in the case where $\dim L_0 = \dim L_1 = 2$. If L_0, L_1 are both even, then the same proof as for $p \neq 2$ applies. If one of them is even and the other odd, then we may assume that L_0 is even and L_1 is odd, that is, L_1 is a diagonal form $\langle a, b\rangle$, a, b odd. We claim that the Jordan decomposition can be chosen such that $L_0 \cong \mathbb{H}$. Indeed, if $L_0 \not\cong \mathbb{H}$, then $L_0 \cong A_2$, the A_2-root-lattice with quadratic form [1,1,1]. But since A_2 over $\mathbb{Z}_2$ represents primitively any odd number, in particular $-a$, it is clear that already $A_2 \perp \langle 2a\rangle$ contains a primitive isotropic vector and thus splits off a hyperbolic plane. (In fact $A_2 \perp \langle 2a\rangle \cong \mathbb{H} \perp \langle 2(a+4)\rangle$.) If finally L_0, L_1 are both odd, then we may pass to the even sublattice L°, which is of the shape ${}^2M$, for M odd, unimodular and thus can be replaced by the odd unimodular lattice M. □

For the signature $(4,1)$, we have chosen the following normal form which, for reasons to be explained later, restricts the positive definite ternary constituent to a determinant not divisible by 4.

2.2. PROPOSITION. *If L is a lattice of signature $(4,1)$ with $O(L)$ maximal, then we may assume that L has the shape $L = \mathbb{H} \perp M$ or ${}^2\mathbb{H} \perp M$, where M is obtained from a ternary lattice $\widetilde{M}$ of square free determinant d which is odd in the case ${}^2\mathbb{H}$, by applying the dualizing operators D_p for some set of odd prime divisors $p \mid d$.*

Proof: First notice that the condition on M just means that M is primitive and square free (not necessarily strongly square free) and of determinant not divisible by 4, resp. odd determinant. We have chosen the formulation with the D_p to emphasize that we have essentially only to deal with square free determinants. The proof amounts to showing that L which we assume to be strongly square free splits off $\mathbb{H}$ if $4 \nmid \det L$ and splits off ${}^2\mathbb{H}$ if $4 \mid \det L$. This must be shown locally for each p. The claim is obvious for $p \neq 2$ since the unimodular Jordan component is of dimension at least 3 and therefore isotropic.

Now consider the case $p = 2$ and assume first that $4 \nmid \det L$. Let v be a primitive isotropic vector; it exists because $\dim L = 5$. Then $(L, v) = \mathbb{Z}_2$ or $2\mathbb{Z}_2$, since L is square free; $2\mathbb{Z}_2$ is impossible since the 2-modular Jordan component is at most one-dimensional and thus is anisotropic. Therefore, L splits off $\mathbb{H}$ or $\langle -1, 1\rangle$. We have to show that the case $\langle -1, 1\rangle$ can be reduced to the case $\mathbb{H}$. The only situation where there could be a problem occurs when L is odd, of the form $\langle -1, 1\rangle \perp M$ for an even lattice M. Over $\mathbb{Z}_2$, the lattice M is of the shape $B \perp \langle 2a \rangle$, where a is an odd number and the binary lattice is $\mathbb{H}$ or A_2, the unique anisotropic binary even unimodular lattice over $\mathbb{Z}_2$. We have already observed in the proof of 2.1 that in this situation we can always assume that $B = \mathbb{H}$, (i.e. M is isotropic over $\mathbb{Z}_2$). But then it is clear that we can replace M locally and hence also globally by an odd lattice M' of the same determinant on the same space. The lattices $L = \langle -1, 1 \rangle \perp M$ and $L' := \mathbb{H} \perp M'$ are both odd, hence in the same genus and therefore isometric over $\mathbb{Z}$.

Now we treat the case where $4 \mid \det L$. We claim that there exists a primitive isotropic vector v with $(L, v) = 2\mathbb{Z}_2$. If not, such a vector satisfies $(L, v) = \mathbb{Z}_2$ and L, being odd, is locally of the shape $\langle -1, 1 \rangle \perp \langle a \rangle \perp {}^2B$ for a binary unimodular lattice B. The lattice $2\langle a \rangle \perp {}^2B \cong \langle 4a \rangle \perp {}^2B$ certainly represents primitively some number $4b$, b odd. But $\langle -1, 1 \rangle$ represents every odd number, in particular $-b$, and we finally arrive at a primitive isotropic vector $v \in 2L^{\#}$, as desired. We now know that L splits off either ${}^2\mathbb{H}$ or $\langle -2, 2 \rangle$ (and there is no choice for a given L, since the parity of a 2-modular Jordan component is unique). In the case $\langle -2, 2 \rangle$, we replace L first by D_2L which locally is of the form $\langle -1, 1 \rangle \perp {}^2M$ for some ternary unimodular M and then by the even sublattice $(D_2L)^\circ$ which is ${}^2\mathbb{H} \perp {}^2M$. Scaling by $1/2$ we finally arrive at a unimodular lattice L' (which by the way carries the original, unscaled form) such that $O(L) \subseteq O(L')$. Thus we may remove L from the list of lattices to be considered. We want to remark that in fact $O(L) \neq O(L')$ and thus $O(L)$ is not maximal and eliminating L in this case is not only a matter of normalization. To verify the last claim one equivalently shows that $O(D_2L) \subset O((D_2L)^\circ)$ (proper inclusion). To see this, one checks that $(D_2L)^\circ$ has further odd integral over-lattices (in addition to D_2L itself) and these are necessarily permuted transitively by $O(L)$. □

3. On the classification of reflective Lorentzian lattices in dimensions 3 and 4

The following proposition considerably extends the list of hyperbolic reflection groups in dimensions 3 and 4 known previously. In particular, we have found quite a few new maximal groups.

3.1. PROPOSITION. *The* $49 + 42$ *quadratic lattices in tables* 1 *and* 2 *below are all reflective and give rise to maximal, pairwise non-conjugate arithmetic reflection groups on hyperbolic* 3*-space, resp.* 4*-space.*

More detailed information about the 49 + 42 groups is given in [SW]. The data collected in that paper were obtained, on the basis of our normal forms, by implementing Vinberg's algorithm for finding fundamental roots on a computer; they in particular set in evidence that the lattices listed in our tables 1 and 2 really are reflective.

We do not prove here that the groups are maximal within this class of groups.

Table 1. Reflective lattices of signature $(3,1)$.
$L = \mathbb{H}\perp^{s}[a,b,c] = (\mathbb{Z}^4, f)$, where $f(v) = (v,v)/2 = x_0x_1+s(ax_2^2+bx_2x_3+cx_3^2)$
$D = s^2(4ac-b^2)$ is the (negative of the) discriminant
r is the number of fundamental roots

No.	D	$^{s}[a,b,c]$	r	No.	D	$^{s}[a,b,c]$	r
1.	3	$[1,1,1]$	4	26.	72	$^{3}[1,0,2]$	8
2.	4	$[1,0,1]$	4	27.	75	$^{5}[1,1,1]$	6
3.	7	$[1,1,2]$	6	28.	84	$[1,0,21]$	11
4.	8	$[1,0,2]$	5	29.	84	$[3,0,7]$	8
5.	11	$[1,1,3]$	6	30.	84	$[5,4,5]$	11
6.	12	$^{2}[1,1,1]$	4	31.	100	$^{5}[1,0,1]$	6
7.	15	$[1,1,4]$	8	32.	120	$[1,0,30]$	11
8.	15	$[2,1,2]$	6	33.	120	$[2,0,15]$	14
9.	19	$[1,1,5]$	7	34.	120	$[3,0,10]$	13
10.	20	$[1,0,5]$	6	35.	120	$[5,0,6]$	14
11.	20	$[2,2,3]$	6	36.	132	$[1,0,33]$	15
12.	24	$[1,0,6]$	6	37.	140	$^{2}[3,1,3]$	11
13.	24	$[2,0,3]$	6	38.	147	$^{7}[1,1,1]$	6
14.	28	$^{2}[1,1,2]$	8	39.	168	$[3,0,14]$	18
15.	35	$[3,1,3]$	8	40.	168	$[6,0,7]$	18
16.	36	$^{3}[1,0,1]$	5	41.	180	$^{3}[1,0,5]$	15
17.	39	$[1,1,10]$	10	42.	196	$^{7}[1,0,1]$	9
18.	40	$[1,0,10]$	9	43.	200	$^{5}[1,0,2]$	13
19.	40	$[2,0,5]$	7	44.	300	$^{10}[1,1,1]$	7
20.	44	$^{2}[1,1,3]$	7	45.	360	$^{3}[1,0,10]$	20
21.	52	$[1,0,13]$	10	46.	360	$^{3}[2,0,5]$	20
22.	56	$[1,0,14]$	9	47.	507	$^{13}[1,1,1]$	12
23.	60	$^{2}[1,1,4]$	10	48.	588	$^{14}[1,1,1]$	12
24.	63	$^{3}[1,1,2]$	10	49.	900	$^{15}[1,0,1]$	18.
25.	68	$[1,0,17]$	13				

Table 2. Reflective lattices of signature $(4,1)$.
$\langle a_1, ..., a_r\rangle$ denotes a diagonal form
$\langle a_b c\rangle$ denotes the binary form $(v,v) = ax^2 + 2bxy + cy^2$
$-D$ is the determinant
r is the number of fundamental roots

No.	D	bilinear form	r
1.	1	$\mathbb{H}\perp\langle 1,1,1\rangle$	5
2.	3	$\mathbb{H}\perp\langle 1\rangle\perp\langle 2_1 2\rangle$	6
3.	3	$\mathbb{H}\perp\langle 1,1,3\rangle$	7
4.	4	${}^2\mathbb{H}\perp\langle 1,1,1\rangle$	5
5.	5	$\mathbb{H}\perp\langle 1,1,5\rangle$	7
6.	5	$\mathbb{H}\perp\langle 1\rangle\perp\langle 2_1 3\rangle$	8
7.	6	$\mathbb{H}\perp\langle 1,1,6\rangle$	7
8.	6	$\mathbb{H}\perp\langle 2\rangle\perp\langle 2_1 2\rangle$	6
9.	7	$\mathbb{H}\perp\langle 1,1,7\rangle$	10
10.	9	$\mathbb{H}\perp\langle 3\rangle\perp\langle 2_1 2\rangle$	7
11.	9	$\mathbb{H}\perp\langle 1,3,3\rangle$	7
12.	11	$\mathbb{H}\perp\langle 1\rangle\perp\langle 2_1 6\rangle$	9
13.	12	${}^2\mathbb{H}\perp\langle 1\rangle\perp\langle 2_1 2\rangle$	6
14.	14	$\mathbb{H}\perp\langle 1,2,7\rangle$	13
15.	14	$\mathbb{H}\perp\langle 2\rangle\perp\langle 2_1 4\rangle$	8
16.	15	$\mathbb{H}\perp\langle 3\rangle\perp\langle 2_1 3\rangle$	10
17.	15	$\mathbb{H}\perp\langle 1,3,5\rangle$	15
18.	15	$\mathbb{H}\perp\langle 1,1,15\rangle$	12
19.	15	$\mathbb{H}\perp\langle 5\rangle\perp\langle 2_1 2\rangle$	10
20.	18	$\mathbb{H}\perp\langle 2,3,3\rangle$	8
21.	18	$\mathbb{H}\perp\langle 6\rangle\perp\langle 2_1 2\rangle$	7
22.	20	${}^2\mathbb{H}\perp\langle 1,1,5\rangle$	7
23.	21	$\mathbb{H}\perp\langle 7\rangle\perp\langle 2_1 2\rangle$	14
24.	21	$\mathbb{H}\perp\langle 1,3,7\rangle$	18
25.	25	$\mathbb{H}\perp\langle 1,5,5\rangle$	9
26.	25	$\mathbb{H}\perp\langle 5\rangle\perp\langle 2_1 3\rangle$	14
27.	30	$\mathbb{H}\perp\langle 1,2,15\rangle$	19
28.	30	$\mathbb{H}\perp\langle 2\rangle\perp\langle 2_1 8\rangle$	11
29.	30	$\mathbb{H}\perp\langle 10\rangle\perp\langle 2_1 2\rangle$	10
30.	36	${}^2\mathbb{H}\perp\langle 3\rangle\perp\langle 2_1 2\rangle$	7
31.	42	$\mathbb{H}\perp\langle 14\rangle\perp\langle 2_1 2\rangle$	11
32.	45	$\mathbb{H}\perp\langle 15\rangle\perp\langle 2_1 2\rangle$	16
33.	45	$\mathbb{H}\perp\langle 3\rangle\perp\langle 2_1 8\rangle$	21
34.	60	${}^2\mathbb{H}\perp\langle 3\rangle\perp\langle 2_1 3\rangle$	11
35.	60	${}^2\mathbb{H}\perp\langle 5\rangle\perp\langle 2_1 2\rangle$	12
36.	75	$\mathbb{H}\perp\langle 1\rangle\perp\langle 10_5 10\rangle$	14
37.	75	$\mathbb{H}\perp\langle 1,5,15\rangle$	51
38.	84	${}^2\mathbb{H}\perp\langle 7\rangle\perp\langle 2_1 2\rangle$	15
39.	100	${}^2\mathbb{H}\perp\langle 1,5,5\rangle$	11
40.	150	$\mathbb{H}\perp\langle 10\rangle\perp\langle 2_1 8\rangle$	33
41.	180	${}^2\mathbb{H}\perp\langle 15\rangle\perp\langle 2_1 2\rangle$	17
42.	300	${}^2\mathbb{H}\perp\langle 1\rangle\perp^5\langle 2_1 2\rangle$	19

It is likely that our list of maximal (arithmetic, non-cocompact) reflection groups in dimensions 3 and 4 is complete. We now want to give a brief sketch of what we can prove so far in this direction. The complete determination of all reflective lattices (with maximal group, say) in some fixed dimension breaks up into two steps, which, although they are logically not strictly separate from each other, turn out to require quite different methods. In the first step one has to produce a finite (and not too large) list of lattices which one can show contains all the reflective lattices. In particular, this list of candidates should only contain lattices which satisfy the conclusion of Vinberg's lemma. In an intermediate step one applies Vinberg's algorithm to each lattice in the list. This shows immediately (at least in principle, in small dimensions also in practice) that a number of the candidates from the list are indeed reflective. In a second step one must show that all remaining lattices are non-reflective. For this, Vinberg's algorithm can only give numerical evidence, but no proof. Of course, the task of the second step depends on how good, that is, how small the list from the first step was. One can hope that for larger dimensions, starting with about 10, any lattice satisfying the necessary condition of Vinberg's lemma is indeed reflective and thus the second step completely disappears.

We now have to treat the dimensions 3 and 4 separately. For $n = 3$ we do not say anything about the first step here except that it is of an essentially geometric nature; the arithmetic theory of quadratic forms does not seem to help much here. Vinberg's lemma does for $n = 3$ not even imply the finiteness of the number of reflective lattices: the number s ocurring in the normal form cannot be bounded. The solution of the second step is given for $n = 3$ by the theorem below which is a consequence of the following two propositions. We refer to the normal form and notation for the lattices as introduced in Proposition 2.1.

3.2. PROPOSITION. *If D_0 is odd or if $8 \nmid D_0$ and $f(v)$ is even, then any two roots v, v' such that $f(v) = f(v') > 0$ are conjugate under $O^+(L)$. In any case, there are at most two conjugacy classes of roots v for fixed $f(v) > 0$.*

This proposition which by the way is of obvious independent interest is proved along lines which are essentially known: using strong approximation, one reduces to the local case; the local situation has been studied for instance in [Kn,Tr]. A special case of 3.2 has been treated (with a different proof) in [EGM], Theorem 11.3.

3.3. PROPOSITION. *Each of the following conditions on the lattice L and the value q implies that two distinct fundamental roots v, v' s.th. $f(v) = f(v') = q$ are never conjugate under $W(L)$:*

i) $D_0 \neq 3$, *and there exists a prime number* $p \neq 2$ *such that* $p \mid q$, $p \mid D_0$.

ii) *There exists a prime number* $p \neq 2$ *such that* $p \mid q$, $p \mid s$, *and* $(\frac{3D_0}{p}) \neq 1$.

iii) $2 \mid q$, $2 \mid D_0$.

iv) $D_0 = 3$, $q \not\equiv s(3)$.

v) $D_0 = 4$, *there exists a prime number* $p \neq 2$ *such that* $p \mid q$, $p \equiv \pm 5\ (12)$.

The proof of this proposition combines arithmetical facts with properties of abstract, combinatorial Coxeter groups [Ti]. Detailed proofs of the last two results will appear elsewhere. The next theorem is an obvious consequence of the two previous propositions, taking into account the fact that the stabilizer

$A = A(L)$ in $O^+(L)$ of any fundamental domain for $W(L)$ is a complement to $W(L)$ in $O^+(L)$.

3.4. THEOREM. *Suppose that L and q satisfy one of the conditions stated in Proposition 3.3. Let $o(L, q) = 1, 2$ be the number of orbits of $O^+(L)$ on roots v with $f(v) = q$. If L has more than $4 \cdot o(L, q)$ fundamental roots v with $f(v) = q$, then L is non-reflective.*

The number 4 in the last proposition is an upper bound for the order of A which comes from the fact that any rotation of order 3 or 4 in $O^+(L)$ is a product of two reflections in $O^+(L)$. This last result will also be published elsewhere; it was suggested by an unpublished result of Mennicke who treated rotations of order 3 for lattices $L = \mathbb{H} \perp [1, 0, d]$, d square free. Modulo extensive use of a computer the trivial bound $|A| \leq 24$ is by the way sufficient for most applications of 3.4.

We now turn to dimension $n = 4$ and outline the first step of the general procedure, the compilation of a list of candidates containing lattices for all maximal groups. In all dimensions $n \geq 4$, this step relies basically on Vinberg's lemma and the finiteness of totally reflective genera (Theorem 1.4). The task is to give strong bounds on the determinant (even better: on the largest prime factor of the determinant) of totally reflective genera of dimension $m = n - 1$ which are square free and without essential loss of generality even strongly square free. For $n = 4$, the verification of the existence of at least one non-reflective lattice in almost every genus in question is facilitated by the following result.

3.5. LEMMA. *Let M be a ternary lattice of determinant not divisible by 4. If M is indecomposable, then M is even non-reflective.*

Recall that by Proposition 2.2 it is really sufficient to deal with lattices of determinant d with $4 \nmid d$, and even d square free. Lemma 3.5 is completely false for lattices of determinant divisible by 4. There is no reason to prove the lemma here; it is an immediate consequence (verifying case-by-case) of the full classification of all ternary reflective lattices in dimension 3 given in [BS]. The existence of indecomposable lattices in each genus of sufficiently large determinant is a consequence of the mass formula. The mass of the decomposable lattices is a sum of essentially binary masses which grows more slowly than the whole mass. The determination of an upper bound for the determinant of a "totally decomposable" genus finally amounts to comparing certain values of L-series. Once one has obtained such a bound, the determination of all totally reflective genera will reduce to checking all genera up to this bound by computer.

For the second step in the proof of the completeness of table 2, we use a method of Bugaenko [Bu]. If a lattice L obtained in the first step is such that Vinberg's algorithm appears not to stop, one tries to find involutionary symmetries $\sigma_1, ..., \sigma_m$, $m \geq 2$, between appropriate subsets of the set of fundamental roots, extending to the whole lattice L and to H^4, such that their common fixed point set in H^4 is empty. Then the group $S = \langle \sigma_1, ..., \sigma_m \rangle$ must be infinite. On the other hand, this group preserves a fundamental domain of $W(L)$ and thus injects into $O(L)/W(L)$. Thus, the lattice is non-reflective.

Acknowledgement. This work was partly supported by the Wissenschaft and Forschung Ministry of the State Government of Nordrhein–Westfalen.

Literature

[Bie] J. Biermann: Gitter mit kleiner Automorphismengruppe in Geschlechtern von $\mathbb{Z}$-Gittern mit positiv-definiter quadratischer Form. Dissertation, Göttingen 1981.

[BS] B. Blaschke and R. Scharlau: Reflective integral quadratic forms. In preparation.

[Bor] R. Borcherds: Automorphism Groups of Lorentzian Lattices. J. Algebra **111** (1987), 133–153.

[Bu] V.O. Bugaenko: Arithmetic crystallographic groups generated by reflections, and reflective hyperbolic lattices. Advances in Soviet Mathematics **8** (1992), 33–55.

[EGM] J. Elstrodt, F. Grunewald, and J. Mennicke: Discontinuous Groups on Three-dimensional Hyperbolic Space: Analytical Theory and Arithmetic Applications. Russian Math. Surveys **38** (1983), 137–168.

[Es] F. Esselmann: Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe. Universität Bielefeld, Sonderforschungsbereich 343, Preprint No. 1992 – 23.

[Kn] M. Kneser: Witts Satz für quadratische Formen über lokalen Ringen. Nachr. Akad. Wiss. Göttingen, Math.-Phys. Klasse (1972), 195–203.

[Ni] V. V. Nikulin: On the Classification of Arithmetic Groups Generated by Reflections in Lobachevsky Spaces. Math. USSR Izvestija **18** (1982), 99–123.

[OM] O. T. O'Meara: *Introduction to quadratic Forms.* Springer Verlag, Berlin, Heidelberg, New York 1971.

[SW] R. Scharlau and C. Walhorn: Tables of hyperbolic reflection groups. Universität Bielefeld, Ergänzungsreihe des Sonderforschungsbereichs 343, Heft 1992 – 01.

[Ti] J. Tits: Le problème des mots dans les groupes de Coxeter. Istituto Nazionale die Alta Matematica, Symposia Mathematica **I** (1968), 175–185.

[Tr] A. Trojan: The Integral Extension of Isometries of Quadratic Forms over Local Fields. Canadian J. of Math. **18** (1966), 920–942.

[Vi1] E. B. Vinberg: Discrete Groups Generated by Reflections in Lobačevskiĭ Spaces. Math. USSR Sbornik **1** (1967), 429–444.

[Vi2] E. B. Vinberg: On Unimodular Integral Quadratic Forms. Functional Anal. Appl. **6** (1972), 24–31.

[Vi3] E. B. Vinberg: On Groups of Unit Elements of Certain Quadratic Forms. Math. USSR Sbornik **16** (1972), 17–35.

[Vi4] E. B. Vinberg: The Absence of Crystallographic Groups of Reflections in Lobachevsky Spaces of Large Dimension. Trans. Moscow Math. Soc. **47** (1985), 75–112.

[Wal] C. Walhorn: Reflektive Gitter der Signatur (4,1) und hyperbolische Spiegelungsgruppen. Diplom thesis, Bielefeld 1990.

[Wa] G. L. Watson: Transformations of a quadratic form which do not increase the class-number. Proc. London Math. Soc. **12** (1962), 577–587.

Rudolf Scharlau & Claudia Walhorn
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
D–4800 Bielefeld 1
Germany

electronic mail:
scharlau@mathematik.uni-bielefeld.de