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JOHN KNOPFMACHER
ARNOLD KNOPFMACHER

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# METRIC PROPERTIES OF ALGORITHMS INDUCING LÜROTH SERIES EXPANSIONS OF LAURENT SERIES

**John KNOPFMACHER & Arnold KNOPFMACHER**

## 1. Introduction

Recently the present authors [**8**] introduced and studied some properties of various unique expansions of formal Laurent series over a field $F$, as the sums of reciprocals of polynomials, involving "digits" $a_1, a_2, \ldots$ lying in a polynomial ring $F[X]$ over $F$. In particular, one of these expansions (described below) turned out to be analogous to the so–called *Lüroth* expansion of a real number, discussed in Perron [**15**] Chapter 4.

In a partly parallel way, Artin [**1**] and Magnus [**11,12**] had earlier studied a Laurent series analogue of simple *continued fractions* of real numbers, involving "digits" $x_1, x_2, \ldots$ in a polynomial ring as above. In addition to sketching elementary properties of an $n$–dimensional "Jacobi–Perron" variant of this, Paysant–Leroux and Dubois [**13, 14**] also briefly outlined certain "metric" theorems analogous to some of Khintchine [**7**] for real continued fractions, in the case when $F$ is a finite field. The main aim of this paper is to state or derive some similar metric or ergodic results for the Laurent series Lüroth–type expansion referred to above. (These results were introduced at the Geneva conference by the first–named author, and are *partly* based on his forthcoming paper [**9**]. For analogous results concerning Lüroth expansions of *real* numbers, see Jager and de Vroedt [**5**] and Salát [**16**].)

In order to explain the conclusions, we first fix some notation and describe the inverse–polynomial Lüroth–type representation to be considered:

Let $\mathcal{L} = F((z))$ denote the field of all *formal Laurent series* $A = \sum_{n=v}^{\infty} c_n z^n$ in an indeterminate $z$, with coefficients $c_n$ all lying in a given field $F$. Although the main case of importance usually occurs when $F$ is the field $\mathbb{C}$ of complex numbers, certain interest also attaches to other ground fields $F$ and most of the results of [8] hold for arbitrary $F$. It will be convenient to write $X = z^{-1}$ and also consider the ring $F[X]$ of *polynomials* in $X$, and the field $F(X)$ of *rational functions* in $X$, with coefficients in $F$.

If $c_v \neq 0$, we call $v = v(A)$ the *order* of $A$ above, and define the *norm* (or *valuation*) of $A$ to be $\|A\| = q^{-v(A)}$, where initially $q > 1$ may be an arbitrary constant, but *later will be chosen* as $q = \text{card}(F)$, if $F$ is finite. Letting $v(0) = +\infty$, $\|0\| = 0$, one then has (cf. Jones and Thron [6] Chapter 5):

$$\|A\| \geq 0 \text{ with } \|A\| = 0 \text{ iff } A = 0,$$

(1.1)

$$\|AB\| = \|A\| \cdot \|B\|, \text{ and}$$

$\|\alpha A + \beta B\| \leq \max(\|A\|, \|B\|)$ for non–zero $\alpha, \beta \in F$, with equality when $\|A\| \neq \|B\|$. By (1.1), the norm $\|\ \|$ is non–Archimedean, and it is well known that $\mathcal{L}$ forms a *complete* metric space relative to the metric $\rho$ such that $\rho(A, B) = \|A - B\|$.

In terms of the notation $X = z^{-1}$ above, we shall make frequent use of the polynomial $[A] = \sum_{v \leq n \leq 0} c_n X^{-n} \in F[X]$, and refer to $[A]$ as the *integral part* of $A \in \mathcal{L}$. Then $v = v(A)$ is the *degree* deg$[A]$ of $[A]$ relative to $X$, and the same function [ ] was used by Artin [1] and Magnus [11, 12] for their continued fractions. (For a recent application of Artin's algorithm, $F$ finite, see Hayes [4].)

Given $A \in \mathcal{L}$ now note that $[A] = a_0 \in F[X]$ iff $v(A_1) \geq 1$ where $A_1 = A - a_0$. As in [8], if $A_n \neq 0 (n > 0)$ is already defined, we then let $a_n = \left[\frac{1}{A_n}\right]$ and put $A_{n+1} = (a_n - 1)(a_n A_n - 1)$. If some $A_m = 0$ or $a_n = 0$, this recursive process stops. It was shown in [8] that this algorithm leads to a finite or convergent (relative to $\rho$) *Lüroth–type* series expansion

$$A = a_0 + \frac{1}{a_1} + \sum_{r \geq 2} \frac{1}{a_1(a_1 - 1) \ldots a_{r-1}(a_{r-1} - 1) a_r}, \tag{1.2}$$

where $a_r \in F[X]$, $a_0 = [A]$, and $\deg(a_r) \geq 1$ for $r \geq 1$. Furthermore this expansion is *unique* for $A$ subject to the preceding conditions on the "digits" $a_r$.

If $I$ denotes the ideal in the power series ring $F[[z]]$, consisting of all power series $x$ such that $x(0) = 0$, then another way of looking at this expansion algorithm is in terms of operators $a : I-\{0\} \to F[X]$, $T : I \to I$ such that $a(x) = \left[\frac{1}{x}\right]$, $T(0) = 0$ and otherwise $T(x) = \big(a(x)-1\big)\big(xa(x)-1\big)$. Then, for $x = A_1 \in I$, $a_1 = a_1(x) = a(x)$, and more generally $a_n = a_n(x) = a_1(T^{n-1}x)$ if $0 \neq T^{n-1}x \in I$.

*From now on* it will be *assumed* that $F = \mathbb{F}_q$ is a *finite* field with exactly $q$ elements. For that case it was shown in **[9]** that $T : I \to I$ is *ergodic* relative to the Haar measure $\mu$ on $I$ such that $\mu(I) = 1$, and that this fact implies in particular:

**Theorem 1.** (i) *For any given polynomial $k \in \mathbb{F}_q[X]$, deg$(k) \geq 1$, and all $x \in I$ outside a set of Haar measure* 0, *the digit value $k$ has asymptotic frequency*

$$\lim_{n\to\infty} \frac{1}{n}\#\{r \leq n : a_r(x) = k\} = \|k\|^{-2} = q^{-2\deg(k)}.$$

(ii) *For all $x \in I$ outside a set of Haar measure* 0 *there exists a single asymptotic mean–value*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \deg\big(a_r(x)\big) = \frac{q}{q-1}.$$

(iii) *For all $x \in I$ outside a set of Haar measure* 0,

$$\|x - w_n\| = q^{\left(-\frac{2q}{q-1}+o(1)\right)n} \quad \text{as } n \to \infty,$$

*where*

$$w_n = w_n(x) := \sum_{r=1}^{n} \frac{\lambda_{r-1}}{a_r},\ \lambda_0 = 1,\ \lambda_r = \frac{1}{a_1(a_1-1)\ldots a_r(a_r-1)}.$$

Regarding (iii), a similar but more elementary algebraic conclusion **[8]** states that

$$\|x - w_n\| \leq q^{-2n-1} \ \text{ for all } x.$$

Our main aim in the present article will be to state and prove various further metric results concerning polynomial "digits" $a_r(x)$ and their limiting distributions.

## 2. Sharper Metric Conclusions

A useful description of the Haar measure $\mu$ on $I$ is given in Sprindžuk [**17**]. In particular $\mu(C) = q^{-r}$ for any "circle", "disc" or "ball"

$$C = C(x, q^{-r-1}) := \left\{ y \in \mathcal{L} : \|x - y\| \leq q^{-r-1} \right\}.$$

Using this, the proof of Theorem 1 (i) in [**9**] includes:

$$\mu\left\{ x \in I : a_r(x) = k \right\} = \|k\|^{-2} \tag{2.1}$$

for any $k \in \mathbb{F}_q[X]$, $\deg(k) \geq 1$, and $r \geq 1$, and

(2.2) the Lüroth–type digits $a_r(x)$ define identically–distributed independent random variables $a_r$ relative to $\mu$ on $I$.

More precisely, by Theorem 3.16 and the law of the iterated logarithm (Theorem 3.17) in Galambos [**3**], we obtain:

**Theorem 2.** *Let* $A_{n,k}(x) = \#\{r \leq n : a_r(x) = k\}$. *Then for almost all* $x \in I$

$$\limsup_{n\to\infty} \frac{A_{n,k}(x) - n\|k\|^{-2}}{\sqrt{n \log\log n}} = \sqrt{2\|k\|^{-2}(1 - \|k\|^{-2})}.$$

*Further, for any real* $s$,

$$\lim_{n\to\infty} \mu \left\{ x \in I : A_{n,k}(x) - n\|k\|^{-2} < \frac{s}{\|k\|} \sqrt{\frac{n}{(1 - \|k\|^{-2})}} \right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-u^2/2} du.$$

Next we note that Theorem 1 (ii) is equivalent to the existence of a *Khintchine–type* constant

$$\lim_{n\to\infty} \left\| a_1(x) a_2(x) \ldots a_n(x) \right\|^{1/n} = q^{q/(q-1)} \text{ a.e.}$$

This conclusion can be refined to:

**Theorem 3.** *For almost all $x \in I$,*

$$\limsup_{n\to\infty} \frac{\sum_{r=1}^{n} \deg\big(a_r(x)\big) - c_1 n}{\sqrt{n \log\log n}} = \sqrt{2c_2},$$

*where $c_1 = q/(q-1)$, $c_2 = q/(q-1)^2$. Hence as $n \to \infty$*

$$\Big\| a_1(x) a_2(x) \ldots a_n(x) \Big\|^{1/n} = q^{q/(q-1)} + O\left(\sqrt{\frac{\log\log n}{n}}\right) \text{ a.e.}$$

*Proof.* Define a sequence $(t_n)$ of independent random variables $t_n$ on $I$ by

$$t_n(x) = \begin{cases} \deg\big(a_n(x)\big) & \text{if } \big\|a_n(x)\big\| \le n^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then the expected value

$$E(t_n) = \sum_{q^r \le n^2} q^{-2r} r(q-1)q^r = E\Big(\deg\big(a_n(\cdot)\big)\Big) + O\left(\frac{\log n}{n^2}\right),$$

and

$$E(t_n^2) = \sum_{q^r \le n^2} q^{-2r} r^2 (q-1)q^r = E\Big(\deg^2\big(a_n(\cdot)\big)\Big) + O\left(\frac{\log^2 n}{n^2}\right).$$

Hence the variance

$$\operatorname{var}(t_n) = \operatorname{var}\Big(\deg\big(a_n(\cdot)\big)\Big) + O\left(\frac{\log^2 n}{n^2}\right),$$

and

$$B_n := \sum_{r=1}^{n} \operatorname{var}(t_r) = \frac{qn}{(q-1)^2} + O(1).$$

Next, since

$$t_n(x) \le 2\log_q n = o\left(\sqrt{\frac{B_n}{\log\log B_n}}\right),$$

the law of the iterated logarithm implies:

$$\limsup_{n\to\infty} \frac{\sum_{r=1}^{n} t_r - \sum_{r=1}^{n} E(t_r)}{\sqrt{2B_n \log\log B_n}} = 1 \ a.e.$$

Hence

$$\limsup_{n\to\infty} \frac{\sum_{r=1}^{n} t_r - \sum_{r=1}^{n} E\Big(\deg\big(a_r(\cdot)\big)\Big)}{\sqrt{2\frac{q}{(q-1)^2} n \log\log n}} = 1 \ a.e.$$

Now let $U_n = \Big\{x \in I : t_n(x) \neq \deg\big(a_n(x)\big)\Big\}$. Then

$$\mu(U_n) = \sum_{\|k\|>n^2} \|k\|^{-2} < \frac{1}{n^2},$$

and the Borel–Cantelli lemma yields $\mu\big(\limsup_{n\to\infty} U_n\big) = 0$. Thus, for almost all $x \in I$, there exists $n_0(x)$ with

$$t_n(x) = \deg\big(a_n(x)\big) \quad \text{for} \quad n \geq n_0(x),$$

and hence Theorem 3 follows.

The next theorem sharpens part (iii) of Theorem 1 above:

**Theorem 4.** *If $w_n = w_n(x)$ is defined as in Theorem 1 (iii), then*

$$\limsup_{n\to\infty} \frac{\deg(x - w_n) + \frac{2qn}{q-1}}{\sqrt{n \log\log n}} = \frac{\sqrt{8q}}{q-1} \ a.e.$$

*Hence*

$$\frac{1}{n}\deg(x - w_n) = -\frac{2q}{q-1} + O\left(\sqrt{\frac{\log\log n}{n}}\right) \ a.e.$$

*Proof.* From Theorem 3, by symmetry as in Feller [**2**] page 205, we have

$$\liminf_{n\to\infty} \frac{\sum_{r=1}^{n} \deg\big(a_r^2(x)\big) - \frac{2qn}{q-1}}{\sqrt{n \log\log n}} = -2\sqrt{2c_2} \ a.e.$$

Also [**9**] shows that

$$1 - \sum_{r=1}^{n+1} \deg\big(a_r^2(x)\big) \leq \deg(x - w_n) \leq -1 - \sum_{r=1}^{n} \deg\big(a_r^2(x)\big),$$

which then leads to Theorem 4.

The next theorem sharpens the conclusion [**9**] that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \big\|a_r(x)\big\| = \infty \ a.e.$$

**Theorem 5.** *For any fixed* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left\{x \in I : \frac{1}{n\log_q n}\left|\sum_{r=1}^{n}\big\|a_r(x)\big\| - (q-1)\right| > \varepsilon\right\} = 0,$$

*i.e.* $\frac{1}{n\log_q n}\sum_{r=1}^{n}\big\|a_r(x)\big\| \to q-1$ *in probability over* $I$.

*Proof.* We write $s = \log_q y$ iff $y = q^s$, and use the truncation method of Feller [**2**], Chapter 10, §2, applied to the random variables $U_r, V_r$ $(r \leq n)$ defined by

$$\begin{aligned} &U_r(x) = \big\|a_r(x)\big\|,\ V_r(x) = 0 && \text{if } \big\|a_r(x)\big\| \leq n\log_q n,\\ &U_r(x) = 0, \qquad V_r(x) = \big\|a_r(x)\big\| && \text{if } \big\|a_r(x)\big\| > n\log_q n. \end{aligned}$$

Then

$$\begin{aligned} \mu&\left\{x \in I : \frac{1}{n\log_q n}\left|\sum_{r=1}^{n}\big\|a_r(x)\big\| - (q-1)\right| > \varepsilon\right\}\\ &\leq \mu\left\{x : \Big|U_1 + \cdots + U_n - (q-1)n\log_q n\Big| > \varepsilon n\log_q n\right\} +\\ &\quad + \mu\Big\{x : V_1 + \cdots + V_n \neq 0\Big\}, \end{aligned}$$

and

$$\begin{aligned} \mu\Big\{x : V_1 + \cdots + V_n \neq 0\Big\} &\leq n\mu\Big\{x : \big\|a_1(x)\big\| > n\log_q n\Big\}\\ &= n \sum_{\|k\| > n\log_q n} \|k\|^{-2} < \frac{1}{\log_q n} = o(1). \end{aligned}$$

Now note that

$$E(U_1 + \cdots + U_n) = nE(U_1), \text{ var}(U_1 + \cdots + U_n) = n \text{ var}(U_1),$$

where

$$E(U_1) = \sum_{\|k\| \le n \log_q n} \|k\|^{-1} = \sum_{q^r \le n \log_q n} q^{-r}(q-1)q^r$$
$$= (q-1)\log_q([n \log_q n]),$$

and

$$\text{var}(U_1) < E(U_1^2) = \sum_{\|k\| \le n \log_q n} 1 = \sum_{q^r \le n \log_q n} (q-1)q^r$$
$$< qn \log_q n.$$

Chebyshev's inequality then yields

$$\mu\left\{x : \left|U_1 + \cdots + U_n - nE(U_1)\right| > \varepsilon nE(U_1)\right\}$$
$$\le \frac{n \text{ var}(U_1)}{(\varepsilon nE(U_1))^2} < \frac{qn^2 \log_q n}{\left(\varepsilon(q-1)n \log_q([n \log_q n])\right)^2} = o(1).$$

Since $E(U_1) \sim (q-1)\log_q n$ as $n \to \infty$, Theorem 5 follows.

**Remark.** Since a theorem in Galambos [**3**, p. 46], implies that **either**

$$\limsup_{n\to\infty} \frac{1}{n \log_q n} \sum_{r=1}^{n} \|a_r(x)\| = \infty \text{ } a.e. \textbf{ or } \liminf_{n\to\infty} \frac{1}{n \log_q n} \sum \|a_r(x)\| = 0 \text{ } a.e.,$$

the conclusion of Theorem 5 does not carry over to validity with probability one.

## 3. Estimates for Individual Digits

Without attempting to be exhaustive, we conclude this article with two asymptotic results emphasizing the sizes of individual digits $a_n(x)$ *a.e.* as $n \to \infty$.

**Theorem 6.** *Let $\psi(n)$ be a positive increasing function of $n$. Then*

$$\|a_n(x)\| = O(\psi(n)) \text{ a.e.} \iff \sum_{n=1}^{\infty} \frac{1}{\psi(n)} < \infty.$$

*In fact $\|a_n(x)\| = O(\psi(n))$ is false a.e. if the series diverges.*

*Proof.* Let $V_n = \{x \in I : \|a_n(x)\| > \psi(n)\}$. Since $\mu\{x : a_n(x) = k\} = \|k\|^{-2}$, it follows that

$$\mu(V_n) = \sum_{\|k\|>\psi(n)} \|k\|^{-2} = \sum_{q^r>\psi(n)} q^{-2r}(q-1)q^r \le \frac{1}{\psi(n)}.$$

If $\sum_{n=1}^{\infty} \psi(n)^{-1} < \infty$, then the Borel–Cantelli lemma (cf. [**3**], page 36) now yields $\mu(\limsup V_n) = 0$. Hence $\|a_n(x)\| > \psi(n)$ for at most finitely many $n$, for almost all $x \in I$. Thus $\|a_n(x)\| = O(\psi(n))$ *a.e.*

If $\sum_{n=1}^{\infty} \psi(n)^{-1}$ diverges, the Abel–Dini theorem (Knopp [**10**], page 290) implies that there exists a positive increasing function $\theta(n)$ with $\theta(n) \to \infty$ as $n \to \infty$, such that $\sum_{n=1}^{\infty} \psi(n)^{-1}\theta(n)^{-1}$ also diverges. Then let $W_n = \{x \in I : \|a_n(x)\| > \psi(n)\theta(n)\}$. The independence of the random variables $a_n$ implies the independence of the sets $W_n$. Also

$$\sum_{n=1}^{\infty} \mu(W_n) = \sum_{n=1}^{\infty} \sum_{\|k\|>\psi(n)\theta(n)} \|k\|^{-2} > \frac{1}{q} \sum_{n=1}^{\infty} \frac{1}{\psi(n)\theta(n)} = \infty.$$

Thus the Borel–Cantelli lemma yields $\mu(\limsup W_n) = 1$, and so $\|a_n(x)\| > \psi(n)\theta(n)$ holds with probability one, for infinitely many $n$. Thus $\|a_n(x)\| = O(\psi(n))$ is false *a.e.*

Theorem 6 implies for example that $\|a_n(x)\| = O(n(\log n)^{\alpha})$ *a.e.* for any $\alpha > 1$, while $\|a_n(x)\| = O(n(\log n)^{\beta})$ is false *a.e.* for any $\beta \le 1$. This leads to the

**Corollary.** *For almost all $x \in I$*

$$\limsup_{n\to\infty} \frac{\log\|a_n(x)\| - \log n}{\log\log n} = 1.$$

**Remark.** The corresponding lower limit is not finite *a.e.*, since (2.1) earlier shows that $\|a_n(x)\|$ can take any particular constant value $q^N$ $(N \geq 1)$ for all $n$, and all $x$ in a set of positive measure.

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J. Knopfmacher
Department of Mathematics
University of the Witwatersrand
Johannesburg, 2050
South Africa.

A. Knopfmacher
Dept. of Computational &
Applied Mathematics
University of the Witwatersrand
Johannesburg, 2050
South Africa.