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1. Introduction and statement of results

In 1853 Chebyshev asserted in a letter to M. Fuss that there are more primes $p \equiv 3 \pmod 4$ than $p \equiv 1 \pmod 4$. S. Knapowski and P. Turán in their well-known series of papers on comparative prime number theory [5] write, after quoting Littlewood's result that $\pi(x,4,1)-\pi(x,4,3)$ changes sign infinitely many times as $x \to \infty$, the following lines: *one feels that Chebyshev's vague formulation could also be interpreted so as*

$$\lim_{Y\to\infty} N(Y)/Y = 0, \tag{1.1}$$

where $N(Y)$ denotes the number of integers $m \le Y$ with the property

$$\pi(m,4,1) \ge \pi(m,4,3) \tag{1.2}$$

(cf. also [6], page 26). They support this conjecture by referring to Shanks [7], who found that (1.2) is not fulfilled for $m \le 26860$, is then fulfilled for $m = 26861$ and $m = 26862$, and is again false for $26863 \le m \le 616768$. They also ask the following general question ([5], Problem 7).

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For fixed positive integers a, b and q such that $(a,q) = (b,q) = 1$, $a \not\equiv b \pmod{q}$, what is the asymptotical behaviour of $N_{a,b}(Y)$ for $Y \to \infty$, where $N_{a,b}(Y)$ denotes the number of integers $m \leq Y$ with

$$\pi(m,q,a) \geq \pi(m,q,b) \quad ?$$

Our aim is to prove a general result concerning boundary values of Dirichlet series and to show its relevance to Chebyshev's problem. As a corollary we obtain the following theorem.

THEOREM 1. *Suppose a and q are positive integers satisfying $(a,q) = 1$, $a \not\equiv 1 \pmod{q}$ and let the Generalized Riemann Hypothesis (G.R.H.) be true for Dirichlet L-series $\pmod{q}$. Then there exist two constants $0 < c_1 < c_2 < 1$ such that the inequalities*

$$c_1 Y \leq N_{a,1}(Y) \leq c_2 Y$$

hold for all sufficiently large Y.

This shows that the Knapowski-Turán conjecture (1.1) is false at least when we accept the G.R.H.

The basic tool used in the proof of Theorem 1 is a result concerning generalized Dirichlet series which seems to be of an independent interest. For the sake of brevity, let $\mathcal{A}$ denote the set of all functions

$$F(z) = \sum_{n=1}^{\infty} a_n e^{iw_n z}, \quad z = x + iy, \quad y > 0 \tag{1.3}$$

satisfying the following conditions:

1. $0 \leq w_1 < w_2 < \ldots$ are real numbers.
2. $a_n \in \mathbb{C}$, $n = 1, 2, 3, \ldots$
3. There exists a non-negative integer B such that

$$\sum_{n=2}^{\infty} |a_n| w_n^{-B} < \infty. \tag{1.4}$$

4. There exists a non-negative number L_0 such that for every x, $|x| \geq L_0$, the limit

$$P(x) = \lim_{y \to 0+} \operatorname{Re} F(x + iy)$$

exists and represents a locally bounded function of $x \in \mathbb{R} \setminus [-L_0, L_0]$.

Moreover, let

$$\alpha(F) = \inf_{\substack{y>0 \\ x\in\mathbb{R}}} \operatorname{Re} F(x+iy), \qquad \beta(F) = \sup_{\substack{y>0 \\ x\in\mathbb{R}}} \operatorname{Re} F(x+iy).$$

It was proved in [4] that if $F \in \mathcal{A}$ and $\alpha(F) < u < \beta(F)$ then there exists a positive number $l = l(u, F)$ such that

$$\inf_{x\in I} P(x) < u < \sup_{x\in I} P(x) \tag{1.5}$$

for every interval $I \subset \mathbb{R} \setminus [-L_0, L_0]$ of length $\geq l$.

This result is of importance to the prime number theory being a substitute for Ingham's method [1], [2]. Now we impose somewhat stronger conditions on F and we estimate the measure of the set of x satisfying (1.5).

THEOREM 2. *Let $F \in \mathcal{A}$ and suppose that*

$$\|P\|^2 = \sup_{|t|>L_0+1} \int_0^1 |P(x+t)|^2 \, dx < \infty. \tag{1.6}$$

Then for every real number u satisfying $\alpha(F) < u < \beta(F)$ there exist positive constants $l = l(u, F)$ and $d_1 = d_1(u, F)$ such that

$$|\{x \in I \;:\; P(x) > u\}| \geq d_1 \tag{1.7}$$

and

$$|\{x \in I \;:\; P(x) < u\}| \geq d_1 \tag{1.8}$$

for every interval $I \subset \mathbb{R} \setminus [-L_0, L_0]$ of length $\geq l$ (where $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$).

We apply this theorem to the function

$$\begin{aligned} F_{a,b}(z) = -2e^{-z/2}\frac{1}{\phi(q)} &\sum_{\chi \pmod q} (\overline{\chi(a)} - \overline{\chi(b)})\, K(z, \chi') \\ -\frac{2}{\phi(q)} &\sum_{\chi \pmod q} (\overline{\chi(a)} - \overline{\chi(b)})\, m(\tfrac{1}{2}, \chi), \end{aligned} \tag{1.9}$$

where $q \geq 2$, $0 < a, b < q$, $(a, q) = (b, q) = 1$, $a \not\equiv b \pmod q$ are integers, K denotes the K-function as introduced in [3]:

$$K(z, \chi') = \sum_{\gamma>0} e^{\rho z}, \quad z = x + iy, \quad y > 0$$

(the summation being taken over all non-trivial $L(s,\chi')$ zeros ρ with positive imaginary parts γ); χ' is the primitive Dirichlet character induced by χ, and $m(\frac{1}{2},\chi)$ is the multiplicity of a zero of $L(s,\chi)$ at $s=\frac{1}{2}$ (we put $m(\frac{1}{2},\chi)=0$ when $L(s,\chi)\neq 0$). We obtain the following corollaries.

COROLLARY 1. *Suppose the G.R.H. is true for Dirichlet L-functions* (mod q). *Then for every real number u satisfying $\alpha(F_{a,b}) < u < \beta(F_{a,b})$ there exist positive constants $c_0 = c_0(u,q)$ and $d_0 = d_0(u,q)$ such that*

$$\left|\left\{T \le t \le c_0 T \;:\; \psi(t,q,a) - \psi(t,q,b) > u\sqrt{t}\right\}\right| \ge d_0 T \tag{1.10}$$

and

$$\left|\left\{T \le t \le c_0 T \;:\; \psi(t,q,a) - \psi(t,q,b) < u\sqrt{t}\right\}\right| \ge d_0 T \tag{1.11}$$

for sufficiently large T.

COROLLARY 2. *Suppose the G.R.H. is true for Dirichlet L-functions* (mod q) *and let* $(a,q)=1$, $a \not\equiv 1$ (mod q). *Then for every positive u there exist $c_1 = c_1(u,q) > 0$ and $d_1 = d_1(u,q) > 0$ such that*

$$\#\left\{Y \le m \le c_1 Y \;:\; \psi(m,q,a) - \psi(m,q,1) > u\sqrt{m}\right\} \ge d_1 Y, \tag{1.12}$$

$$\#\left\{Y \le m \le c_1 Y \;:\; \psi(m,q,a) - \psi(m,q,1) < -u\sqrt{m}\right\} \ge d_1 Y, \tag{1.13}$$

$$\#\left\{Y \le m \le c_1 Y \;:\; \pi(m,q,a) - \pi(m,q,1) > u\sqrt{m}/(\log m)\right\} \ge d_1 Y, \tag{1.14}$$

and

$$\#\left\{Y \le m \le c_1 Y \;:\; \pi(m,q,a) - \pi(m,q,1) < -u\sqrt{m}/(\log m)\right\} \ge d_1 Y, \tag{1.15}$$

for all sufficiently large Y.

Let us remark that our Theorem 1 follows at once from Corollary 2; it is sufficient therefore to prove this corollary only.

Applying Theorem 2 to the function

$$F_a(z) = -2e^{-z/2}\frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)}\, K(z,\chi') - \frac{2}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi(a)}\, m(\tfrac{1}{2},\chi),$$

$$(z = x+iy, \quad y>0, \quad (a,q)=1)$$

in place of $F_{a,b}$, we can prove results analogous to Corollaries 1 and 2 for the remainders $\psi(t,q,a)-t/\phi(q)$, $\psi(m,q,1)-m/\phi(q)$, and $\pi(m,q,1)-\operatorname{li}x/\phi(q)$.

2. Proof of Theorem 2

For a real $\delta>0$ we consider the subsidiary function

$$F_\delta(z)=\sum_{n=1}^{\infty}a_nS^N(\delta w_n)e^{iw_nz},\quad z=x+iy,\quad y>0,$$

where

$$S(\nu)=\begin{cases}(\sin\nu)/\nu, & \nu\neq 0,\\ 1, & \nu=0\end{cases}$$

and $N=B+2$. Since $S(\nu)\le\min(1,1/|\nu|)$, the sum F_δ is absolutely convergent for $y\ge 0$. Moreover, $F_\delta\to F$ as $\delta\to 0$ almost uniformly on the upper half-plane and thus $\alpha(F_\delta)\to\alpha(F)$ and $\beta(F_\delta)\to\beta(F)$ as $\delta\to 0$. Let us fix a δ_0, $0<\delta_0<\frac{1}{2}$, so small that $\alpha(F_{\delta_0})<u<\beta(F_{\delta_0})$.

From (1.4) it follows that the sum in (1.3) absolutely and uniformly converges in every closed half-plane $y\ge y_0$ with $y_0>0$. Hence we can integrate $F(z)$ term by term. Thus for $y>0$

$$F_{\delta_0}(z)=\frac{1}{(2\delta_0)^N}\int_{-\delta_0}^{\delta_0}\cdots\int_{-\delta_0}^{\delta_0}F(z+t_1+\ldots+t_N)\,dt_1\ldots dt_N.$$

We take real parts and make $y\to 0+$. Using the Lebesgue bounded integration theorem we obtain

$$\operatorname{Re}F_{\delta_0}(x)=\frac{1}{(2\delta_0)^N}\int_{-\delta_0}^{\delta_0}\cdots\int_{-\delta_0}^{\delta_0}P(x+t_1+\ldots+t_N)\,dt_1\ldots dt_N \tag{2.1}$$

for $|x|>L_0+N\delta_0$.

Let us consider now the following two cases.

Case 1. $\alpha(F)\beta(F)<0$. Then of course $\alpha(F)<0$ and $\beta(F)>0$. Obviously it suffices to prove (1.7) for positive u only. Moreover, (1.8) follows from (1.7) by considering $-F$ instead of F. Let hence u be positive and let us fix u_1 satisfying

$$u<u_1<\alpha(F_{\delta_0}). \tag{2.2}$$

$\operatorname{Re}F_{\delta_0}(x)$ is almost periodic in the sense of Bohr. Hence there exists a positive constant $l_1=l_1(u_1,F,\delta_0)$ such that every interval of length $\ge l_1$ contains a real number x such that

$$\operatorname{Re} F_{\delta_0}(x) \geq u_1. \tag{2.3}$$

Let now $I \subset \mathbb{R} \setminus [-L_0, L_0]$ be an interval of length $\geq l_1 + N\delta_0$. Let x_0 be its middle point and let x satisfying (2.3) be such that $|x - x_0| \leq l_1/2$. Let

$$\mathbf{A} = \{t \in I \ : \ P(t) > u\}$$

$$\mathbf{B} = \{(t_1, \ldots, t_N) \ : \ |t_j| < \delta_0 \ (j = 1, 2, \ldots, N), \ x + t_1 + \ldots + t_N \in \mathbf{A}\}$$

$$\mathbf{C} = [-\delta_0, \delta_0]^N \setminus \mathbf{B}.$$

Using (2.1) and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
u_1 &\leq \operatorname{Re} F_{\delta_0}(x) \\
&= \frac{1}{(2\delta_0)^N} \left(\int_{\mathbf{B}} \!\cdots\! \int + \int_{\mathbf{C}} \!\cdots\! \int \right) P(x + t_1 + \ldots + t_N)\, dt_1 \ldots dt_N \\
&\leq u + \frac{1}{(2\delta_0)^N} \int_{-\delta_0}^{\delta_0} \cdots \int_{-\delta_0}^{\delta_0} \int_{\substack{\mathbf{A}-(x+t_1+\ldots+t_{N-1}) \\ |t_N| < \delta_0}} P(x + t_1 + \ldots + t_N)\, dt_N \ldots dt_1 \\
&\leq u + \frac{1}{(2\delta_0)^N} \int_{-\delta_0}^{\delta_0} \cdots \int_{-\delta_0}^{\delta_0} \int_{\substack{\mathbf{A}-(x+t_1+\ldots+t_{N-1}) \\ |t_N| < \delta_0}} |\mathbf{A}|^{\frac{1}{2}} \, \|P\|\, dt_1 \ldots dt_N \\
&= u + |\mathbf{A}|^{\frac{1}{2}} \, \|P\|/(2\delta_0).
\end{aligned}$$

Hence

$$|\mathbf{A}| \geq \left(2\delta_0(u_1 - u)\|P\|^{-1}\right)^2.$$

Hence it is enough to take

$$d(u, F) = \left(2\delta_0(u_1 - u)\|P\|^{-1}\right)^2$$

and

$$l(u, F) = l_1(u_1, F, \delta_0) + N\delta_0. \tag{2.4}$$

Case 2. $\alpha(F)\beta(F) \geq 0$. Replacing if necessary F by $-F$ we can assume that $\alpha(F) \geq 0$. Then (1.7) can be proved exactly in the same way as in Case 1. To prove (1.8) let us fix u_1 satisfying $\max(\alpha(F), \alpha(F_{\delta_0})) < u_1 < u$. Let l_1, I and x have the same meaning as previously with (2.3) replaced by

$$(2.3)' \qquad \operatorname{Re} F_{\delta_0}(x) \le u_1.$$

Let moreover

$$\mathbf{A}_1 = \{t \in I \; : \; P(t) < u\}$$
$$\mathbf{B}_1 = \{(t_1, \ldots, t_N) \; : \; |t_j| < \delta_0 \; (j = 1, 2, \ldots, N), \; x + t_1 + \ldots + t_N \in \mathbf{A}_1\}$$
$$\mathbf{C}_1 = [-\delta_0, \delta_0]^N \setminus \mathbf{B}_1.$$

Then

$$\begin{aligned} u_1 \ge \operatorname{Re} F_{\delta_0}(x) &\ge \frac{1}{(2\delta_0)^N} \int \underset{\mathbf{C}_1}{\cdots} \int P(x + t_1 + \ldots + t_N)\, dt_1 \ldots dt_N \\ &\ge u(2\delta_0)^{-N} \mu(\mathbf{C}_1) \\ &= u(2\delta_0)^{-N} \left((2\delta_0)^N - \mu(\mathbf{B}_1)\right), \end{aligned}$$

μ being the N-dimensional Lebesgue measure. Hence

$$\mu(\mathbf{B}_1) \ge (2\delta_0)^N (1 - u_1/u).$$

But

$$\begin{aligned} \mu(\mathbf{B}_1) &= \int_{-\delta_0}^{\delta_0} \cdots \int_{-\delta_0}^{\delta_0} \int_{\substack{\mathbf{A}_1 - (x + t_1 + \ldots + t_{N-1}) \\ |t_N| < \delta_0}} dt_N \ldots dt_1 \\ &\le |\mathbf{A}_1| \, (2\delta_0)^{N-1} \end{aligned}$$

and consequently

$$|\mathbf{A}_1| \ge 2\delta_0 (1 - u_1/u).$$

We obtain (1.8) with the same $l(u, F)$ as in (2.4) and

$$d_1(u, F) = 2\delta_0 (1 - u_1/u).$$

The proof is complete.

3. Proof of the corollaries

We apply Theorem 2 to the function $F_{a,b}$ defined by (1.9). It belongs to the class $\mathcal{A}$; condition 4. is satisfied with $L_0 = 0$ (cf. [3]). Condition (1.6) can be proved as follows. For positive y we have by term-by-term integration

$$\begin{aligned}\int_0^1 &|\operatorname{Re} F_{a,b}(x+t+iy)|^2\,dx \\ &\ll 1 + \sum_{\gamma>0}\sum_{\gamma'>0} \frac{1}{\gamma}\frac{1}{\gamma'} e^{-(\gamma+\gamma')y} \left|\int_0^1 e^{i(\gamma-\gamma')x}\,dx\right| \\ &\ll 1 + \sum_{\gamma>0}\sum_{\gamma'>0} \frac{1}{\gamma}\frac{1}{\gamma'} \min(1, |\gamma-\gamma'|^{-1}) \ll 1\end{aligned}$$

uniformly in $t \in \mathbb{R}$ (γ and γ' denote imaginary parts of non-trivial zeros of all Dirichlet L-functions $\pmod q$); (1.6) therefore follows making $y \to 0+$ and using the Lebesgue bounded integration theorem. Finally it is proved in [4], page 242, that

$$P(x) = e^{-x/2}(\psi(e^x, q, a) - \psi(e^x, q, b)) + O(x\, e^{-x/2}).$$

Hence (1.10) and (1.11) follow from (1.7) and (1.8) by the change of variable $t = e^x$; this proves Corollary 1.

To prove Corollary 2 observe that by (3.3), (4.3) and (8.11) of [3] we have

$$\operatorname{Re} F_{1,a}(re^{i\phi}) = \frac{1}{\pi}(\phi - \pi/2)\log r + O(1) \quad \text{for} \quad 0 < r < 1,\ 0 < \phi < \pi,$$

and hence $\alpha = -\infty$ and $\beta = +\infty$. Using this, Corollary 1 and the obvious remark that $\psi(t,q,a) - \psi(t,q,1) = \psi([t],q,a) - \psi([t],q,1)$ we obtain (1.12) and (1.13). Inequalities for $\pi(x,q,a) - \pi(x,q,1)$ follow from what we have just proved and the partial summation.

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