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ON THE STRUCTURE AND THE NUMBER OF SUM-FREE SETS

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1. Introduction

A finite set A of positive integers is called *sum-free*, if $A \cap (A+A) = \emptyset$. In this note we study the structure of sum-free sets. For n odd, $\{1, 3, 5, \ldots, n\}$ and $\{\frac{n+1}{2}\ \frac{n+3}{2}, \ldots, n\}$ are important examples of such sets.

For any non-empty finite set $K \subset \mathbb{Z}$, we denote by $\ell(K)$ and $m(K)$, respectively, the largest and smallest element of K, by $d(K)$ the greatest common divisor of the elements of K, and by $|K|$ the cardinality of K. For the sets A considered below, we set $m := m(A)$, $\ell := \ell(A)$, $a := |A|$, $2A := A+A$ and $A - m := \{x - m \mid x \in A\}$, $\ell - A := \{\ell - x \mid x \in A\}$. Denote $[m, n] = \{x \in \mathbb{Z} \mid m \leq x \leq n\}$. There is a general property of sum-free sets (from [CE], page 63) which we will use later: If B is a sum-free subset of $\{1, \ldots, n\}$ then B contains at most one of i and $\ell(B) - i$, for each positive integer $i < \ell(B)$; and if $\ell(B)$ is even, then $\frac{1}{2}\ell(B) \notin B$. Hence

$$|B| \leq \lceil \frac{1}{2}\ell(B) \rceil \leq \lceil \frac{1}{2}n \rceil \ . \tag{1}$$

We will show that if the cardinality of a sum-free set A does not differ much from $\frac{1}{2}\ell(A)$, then A does not differ much from one of the two examples mentioned above. More precisely, we will prove

Theorem 1. *Let A be a sum-free set of positive integers for which $a \geq \frac{5}{12}\ell+2$. Then either*

1) *All elements of A are odd, or*
2) *A contains both odd and even integers, $m \geq a$, and for $A_1 := A \cap [1, \frac{1}{2}\ell]$ we have*

$$|A_1| \leq \frac{\ell - 2a + 3}{4} .$$

Let $f(n)$ denote the number of sum-free subsets of $\{1, \ldots, n\}$.

P.J. Cameron and P. Erdös in their talk at the First Conference of the Canadian Number Theory Association [CE, page 64] conjectured that

$$f(n) = O(2^{\frac{n}{2}}) .$$

P. Erdös and A. Granville, and independently N. Calkin as well as N. Alon [Al] showed that

$$f(n) = 2^{\left(\frac{1}{2}+ o(1)\right)n} .$$

The proof in [Al] is more general and in particular applies to any group.

As a simple corollary of Theorem 1 we will prove that the number of sum-free sets $A \subset [1, n]$ for which $a \geq \frac{5}{12}\ell + 2$ has the bound $O\left(2^{\frac{n}{2}}\right)$.

2. The Structure of Sum-Free Sets of Large Cardinality

As a main tool in the proof of Theorem 1 we will use the following two theorems from [F1].

Let M and N be finite sets of non-negative integers such that $m(M) = m(N) = 0$.

Theorem 2. *If $\ell(M) = \max(\ell(M), \ell(N))$ and $\ell(M) \leq |M| + |N| - 3$, then $|M + N| \geq \ell(M) + |N|$.*

Theorem 3. *If $\max(\ell(M), \ell(N)) \geq |M| + |N| - 2$ and $d(M \cup N) = 1$, then*

$$|M + N| \geq |M| + |N| - 3 + \min(|M|, |N|) .$$

We shall also use the following result from [F2]:

Lemma. *If $A \subset \mathbb{Z}$ is finite, then*

$$|2A| \geq 2|A| - 1 . \tag{2}$$

Proof of Theorem 1. Let us call a set A difference-free if $A \cap (A - A) = \emptyset$. Note first that the notions of sum-free set and of difference-free set coincide. For if $x, y, z \in A$, then $x = y + z \Longleftrightarrow y = x - z$. Thus if A is not sum-free then A is not difference-free and conversely.

In the set $A - A$, to each positive difference $x - y$ there corresponds the negative difference $y - x$. Denote by $(A - A)_+$ and $(A - A)_-$, respectively, the set of positive and negative differences.

Since $A - A = (A - A)_+ \cup (A - A)_- \cup \{0\}$ and $|(A - A)_+| = |(A - A)_-|$, we have

$$|A - A| = 2|(A - A)_+| + 1 . \tag{3}$$

The sets A and $(A - A)_+$ are both contained in the interval $[1, \ell]$. Since A is difference-free, it follows that

$$|A| + |(A - A)_+| \leq \ell . \tag{4}$$

This inequality is very restrictive for large $a = |A|$, and we will use it in conjunction with a lower bound for $|(A - A)_+|$ to be obtained from Theorems 2 and 3, to prove Theorem 1.

Let us study various cases according to the value of $d(A - m)$.

We first observe that $d(A - m) \leq 2$, for if $d(A - m) \geq 3$ then $a \leq \frac{\ell}{3} + 1$ which contradicts the condition $a \geq \frac{5}{12}\ell + 2$.

In case $d(A - m) = 2$ first consider the subcase when m is odd. Then all the numbers of A are odd and we have Case 1 of Theorem 1.

If $d(A - m) = 2$, then m cannot be even, under the hypothesis of Theorem 1. Indeed, if m is even and $d(A - m) = 2$ then all the integers in A are even and the set $\frac{A}{2} := \{x \mid x = \frac{a}{2},\ a \in A\}$ is sum-free, with largest element $\ell_1 = \frac{\ell}{2}$. Also if $a \geq \frac{5}{12}\ell + 2$ then (1), applied to $B = \frac{A}{2}$, would yield $\frac{5}{12}\ell + 2 \leq a = |A| = |\frac{A}{2}| = |B| \leq \frac{\ell_1 + 1}{2} = \frac{\ell + 2}{4}$, which is absurd.

The only case left is that in which $d(A - m) = 1$. Clearly the elements of A cannot then all be of the same parity. We define sets M and N by $M := A - m$ and $N := \ell - A$. Then $m(M) = m(N) = 0$, $\ell(M) = \ell(N) = \ell - m$, $|M| = |N| = a$, $|M + N| = |A - A|$; and $d(M \cup N) = 1$ since $d(M) = 1$. If we had

$$\ell - m \geq 2a - 2 , \tag{5}$$

Theorem 3 would apply, giving $|A - A| = |M + N| \geq 3a - 3$, whence $|(A - A)_+| \geq \frac{3a}{2} - 2$ by (3). Using this in (4) together with $a \geq \frac{5}{12}\ell + 2$ would yield the absurd

$$\ell \geq |(A - A)_+| + a \geq \frac{5a}{2} - 2 > \frac{25}{24}\ell .$$

Hence (5) is impossible: $\ell - m < 2a - 2$ if $d(A-m) = 1$ and $a \geq \frac{5}{12}\ell + 2$.

Theorem 2 applies, and gives $|A - A| \geq \ell - m + a$, whence $|(A-A)_+| \geq \frac{1}{2}(\ell - m + a - 1)$ by (3).

Using this inequality, (4) and $a \geq \frac{5}{12}\ell + 2$, we get

$$m > \frac{\ell}{4} \, . \tag{6}$$

Having obtained this lower bound for m, we can strengthen it as follows.

For any positive integer i, the integers i and $m+i$ cannot both belong to $A(m \in A$ and A is sum-free). Hence the union $[\ell - 2m + 1, \ell]$ of the intervals $I = [\ell - 2m + 1,\ \ell - m]$ and $I + m$ contains at most m elements of A. Recall that $A_1 = A \cap \left[1, \frac{\ell}{2}\right]$. Let $A_2 = A \setminus A_1 = A \cap \left[\frac{\ell+1}{2}, \ell\right]$. Then by (6), $A_2 \subset \left[\frac{\ell+1}{2}, \ell\right] \subset [\ell - 2m + 1, \ell]$, and therefore

$$|A_2| \leq m \, . \tag{7}$$

Now $2A_1 \cap A_2 = \emptyset(A_2 \subset A$, and $2A_1 \cap A = \emptyset$ since A is sum-free) and by (6), $2A_1 \subset \left[\frac{\ell+1}{2}, \ell\right]$. Hence

$$|2A_1| + |A_2| \leq \left| \left[\frac{\ell+1}{2}, \ell\right] \right| \leq \frac{\ell+1}{2} \, .$$

By adding this inequality to (7) and using (2) and $|A_1| + |A_2| = a$ we get $2a \leq \frac{1}{2}(\ell + 3) + m$. Hence with $a \geq \frac{5}{12}\ell + 2$ we get

$$m > \frac{\ell}{3} + 2 \, . \tag{8}$$

From (8) we have $A \subset [m, \ell] \subset [\ell - 2m + 1, \ell]$. We have seen that this last interval contains at most m integers from A; it follows that $m \geq a$, which proves the first inequality in Case 2 of Theorem 1.

To establish the second inequality of Case 2, we observe that $\ell - A_1$, $2A_1$, and A_2 are pairwise disjoint subsets of $\left[\frac{\ell+1}{2}, \ell\right]$. We have already verified this for $2A_1$ and A_2. Also, $(\ell - A_1) \cap A_2 = \emptyset$ since A is sum-free and $(\ell - A_1) \cap 2A_1 = \emptyset$ because $\ell - A_1 \subset [0, \ell - m]$, $2A_1 \subset [2m, \ell]$ and $\ell - m < 2m$ by (8). Finally, $\ell - A_1 \subset \left[\frac{\ell}{2},\ \ell - 1\right]$ since $A_1 \subset \left[1,\ \frac{\ell}{2}\right]$; and $\frac{\ell}{2} \notin A$ if ℓ is even, because A is sum-free.

It now follows that $|\ell - A_1| + |2A_1| + |A_2| \leq \frac{\ell+1}{2}$, whence by (2), $3|A_1| + |A_2| - 1 \leq \frac{\ell+1}{2}$, or $|A_1| \leq \frac{\ell}{4} - \frac{a}{2} + \frac{3}{4}$. This completes the proof of Theorem 1.

3. Maximal Sum-Free Sets

We will call a sum-free subset of $[1,n]$ *maximal* if it is of maximal cardinality.

We will now prove the following theorem, stated without proof in [CE] on page 63.

Theorem 2. *For $n \geq 24$, the only maximal sets are*
(i) the set C of all odd numbers in $[1,n]$;
(ii) the set D of all numbers in $[1,n]$ which are greater than $\frac{n}{2}$,
(iii) if n is even, the set $E = D - 1 = \left[\frac{n}{2},\ n-1\right]$.

Proof: Clearly, the sets C, D and E are sum-free of cardinality $\lceil\frac{n}{2}\rceil$. Hence, by (1), a sum-free set A is maximal if and only if $|A| = \lceil\frac{n}{2}\rceil$. Let A be any maximal set. Then $a = |A| = \lceil\frac{n}{2}\rceil \geq \frac{n}{2} \geq \frac{5}{12}n + 2$ (if $n \geq 24$) $\geq \frac{5}{12}\ell + 2$, so the condition of Theorem 1 is satisfied.

In Case 1, $A \subseteq C$, so $A = C$.

In Case 2, $m \geq a$, so $A \subseteq \left[\lceil\frac{n}{2}\rceil,\ n\right] = F$, say. If n is odd, then $F = D$, so that $A = D$. If n is even, then $F = \left\{\frac{n}{2},\ \frac{n}{2}+1, \ldots, n\right\}$ which is a set of cardinality $\lceil\frac{n}{2}\rceil + 1$, so precisely one of its elements does not belong to A. If $\frac{n}{2} \notin A$, then $A = D$, and if $\frac{n}{2} \in A$, then $n \notin A$, therefore, $A = E$.

4. Some Examples of Sum-Free Sets

We now give two examples to show that each of the inequalities in Theorem 1 is best possible.

Example 1. Let us consider positive integers m and n such that $n \geq 36$ and $5n + 24 \leq 12m < 6n$ ($n \geq 36$ ensures the existence of at least one such m). Then define the set $A = ([n-m+1, n] \cup \{m\}) \setminus \{2m\}$. Then one has

1) A is a sum-free set,
2) $|A| = m$ and $\ell(A) = n$ so that the condition $a \geq \frac{5\ell}{12} + 2$ is fulfilled,
3) A contains an even number (because we have $n > 24$, so that $m > 12$ and $[n-m+1, n]$ contains at least two even numbers),
4) $m = a$.

Example 2. Let us consider two positive integers m and n satisfying $11n + 18 \leq 24m \leq 12n - 12$. (such that n is odd and $n \geq 53$), and let us define

$$A = \left[m, \frac{n-1}{2}\right] \cup ([n-m+1, n] \setminus [2m, n-1]) \ .$$

Then one has

1) A is a sum-free set,
2) $|A| = \frac{4m-n+1}{2}$ and $\ell(A) = n$, so that condition $a \geq \frac{5\ell}{12} + 2$ is fulfilled,

3) A contains an even number (because $[n-m+1,\ 2m-1]$ contains at least one even number),
4) therefore, we have $A \cap [1,\ \frac{\ell}{2}] = [m,\ \frac{n-1}{2}]$ so that $|A \cap [1,\ \frac{\ell}{2}]\,| = \frac{\ell}{4} - \frac{a}{2} + \frac{3}{4}$.

It can be shown that when m is sufficiently large, both equalities $m = a$ and $|A_1| = \frac{\ell}{4} - \frac{a}{2} + \frac{3}{4}$ cannot hold at the same time; and indeed deeper results can be established correlating the lower bound of m and the upper bound of $|A_1|$.

The hypothesis $a \geq \frac{5}{12}\ell + 2$ in Theorem 1 cannot be replaced by $a \geq \frac{2}{5}\ell$, as is seen from the example (for $n \in \mathbb{N}$ divisible by 5) of the set $[\frac{n}{5}+1,\ \frac{2n}{5}] \cup [\frac{4n}{5}+1, n]$. Furthermore, this set is locally maximal in the sense of the following definition.

Definition. A set A in $[1,n]$ is *locally maximal* if A is sum-free, but if $A \subseteq A' \subseteq [1,n]$ and$A' \neq A$, then A' is not sum-free.

There naturally arises the problem of determining all locally maximal sets.

5. On the Number of Sum-Free Sets

Theorem 1 immediately gives an upper bound for the number of sum-free sets for which $a \geq \frac{5}{12}\ell + 2$.

In Case 1 the number of sum-free sets with $|A| = a$ is less than or equal to $\binom{[\frac{n+1}{2}]}{a}$.

In Case 2 the number of sum-free sets is less than or equal to $\binom{n-a+1}{a}$. These upper bounds confirm the conjecture of Cameron and Erdös for the number of sum-free sets for which $a \geq \frac{5}{12}\ell + 2$.

It may be conjectured that the number of sum-free sets in $[1,n]$ of cardinality a is $O\left(\binom{\frac{n}{2}}{a}\right)$.

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