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An explicit construction of the Grothendieck residue complex

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0 Introduction

Background In his talk at the Edinburgh congress in 1958 [Gr], A. Grothendieck described a duality theory for coherent sheaves. For the special case of a scheme X of finite type over a field k, this duality theory is based on a certain canonical complex of quasi-coherent sheaves called the *residue complex*. It replaces the sheaf of top degree differential forms which appears in Serre duality on projective space. The residue complex $\mathcal{K}^{\cdot}_X$ is a direct sum of sheaves $\mathcal{D}(X/Y)$, where Y runs over the irreducible closed subsets of X. Say Y has generic point y and $L \subset \mathcal{O}_{X,y}$ is a field such that L/k is separable and $k(y)/L$ is finite. Then $\mathcal{D}(X/Y)$ is canonically isomorphic to $\mathrm{Hom}_L^{\mathrm{cont}}(\mathcal{O}_{X,y}, \Omega^p_{L/k})$, where $\mathcal{O}_{X,y}$ has the $\mathfrak{m}_y$-adic topology and $p = \mathrm{rank}_L \, \Omega^1_{L/k}$.

The full treatment of Grothendieck's duality theory, namely the text "Residues and Duality" [RD] by R. Hartshorne, places the theory in the abstract setting of *derived categories*. Instead of the single dualizing object $\mathcal{K}^{\cdot}_X$, one has a functor $f^! : \mathsf{D}^+_{\mathrm{c}}(Y) \to \mathsf{D}^+_{\mathrm{c}}(X)$ assigned to every morphism $f : X \to Y$ (in a suitable category of schemes), and when f is proper, $f^!$ is right adjoint to $\mathrm{R}f_*$. If X is a scheme of finite type over a field k, with structural morphism π, then the residue complex is obtained as the *Cousin complex* associated to $\pi^! k \in \mathsf{D}^+_{\mathrm{c}}(X)$. In [RD] ch. VI it is denoted by $\pi^{\triangle} k = \mathrm{E}(\pi^! k)$. What is lost in this general, yet natural, approach, is the module structure of the residue complex. The summands of $\pi^{\triangle} k$ are not expressed in a concrete form (they are local cohomologies of $\pi^! k$), and a fortiori, neither is the coboundary operator. Moreover, $\pi^{\triangle} k$ is only determined up to isomorphism, and in order to make this isomorphism unique one must introduce a substantial amount of extra data (cf. [RD] ch. VI thm. 3.1).

There have been many efforts since then to state parts of the theory in terms more *accessible to computation*. All these efforts utilize some sort of residue map defined on differential forms. For curves this is a classical construction, used by J.P. Serre in [Se]. For higher dimensions one had Grothendieck's residue symbol ([RD] ch. III §9). Out of that grew two types of residue maps. Say X is an n-dimensional variety over a perfect field k. The first type is a residue map on local cohomology groups, Res : $\mathrm{H}^n_x(\Omega^n_{X/k}) \to k$, for closed points $x \in X$. This is the approach taken by E. Kunz, J. Lipman and others (see [Li1], [Li2], [HK] and [Hu]). The second type resembles Serre's residue map, in that it uses differential forms with values in local fields (only this time of dimension n). This approach was developed by A.N. Parshin and V.G. Lomadze, who were influenced by the work of F. El Zein [EZ]. Let us mention that in [Be], A. Beilinson shows (among other things) how to get Parshin's residue map using a generalization of J. Tate's construction (cf. [Ta2] and [AK] ch. VIII §2). We shall make use of the Parshin residue map here.

The objective of this monograph is to give an explicit construction of the Grothendieck residue complex $\mathcal{K}^{\cdot}_X$ when X is a *reduced* scheme of finite type over a *perfect field* k. By "explicit" we mean a construction that involves concrete realizations of the complex as an $\mathcal{O}_X$-module (differential forms etc.) and straightforward formulas for the coboundary operator. Thus on the one hand the complex $\mathcal{K}^{\cdot}_X$ should be constructed in some direct fashion, and on the other hand, an isomorphism $\mathcal{K}^{\cdot}_X \cong \pi^! k$ in $\mathsf{D}(X)$ should be exhibited. By the very nature of $\pi^!$, getting such an isomorphism, not to mention making this isomorphism canonical, requires "going outside of X", i.e. considering morphisms between schemes and the variance of $\mathcal{K}^{\cdot}_X$ (cf. remark 4.5.10).

Outline of the construction As in Grothendieck's original description, our complex $\mathcal{K}^{\cdot}_X$ is a direct sum of *dual modules* $\mathcal{K}(x)$, $x \in X$ (denoted $\mathcal{D}(X/Y)$ in [Gr]). For any coefficient field $\sigma : k(x) \to \hat{\mathcal{O}}_{X,x}$ (i.e. a k-algebra lifting) we set $\mathcal{K}(\sigma) := \mathrm{Hom}^{\mathrm{cont}}_{k(x)}(\hat{\mathcal{O}}_{X,x}, \omega(x))$, where $\hat{\mathcal{O}}_{X,x}$ has the $\mathfrak{m}_x$-adic topology, $d := \mathrm{rank}_{k(x)} \Omega^1_{k(x)/k}$ and $\omega(x) := \Omega^d_{k(x)/k}$. Our first task is to find, for any two coefficient fields σ, σ', a canonical isomorphism $\Phi_{\sigma,\sigma'} : \mathcal{K}(\sigma) \xrightarrow{\simeq} \mathcal{K}(\sigma')$, such that for three coefficient fields $\sigma, \sigma', \sigma''$, one has $\Phi_{\sigma,\sigma''} = \Phi_{\sigma',\sigma''} \circ \Phi_{\sigma,\sigma'}$. This will give us a module $\mathcal{K}(x)$ together with isomorphisms $\Phi_\sigma : \mathcal{K}(\sigma) \xrightarrow{\simeq} \mathcal{K}(x)$. The second task is, given a pair of points $x, y \in X$ with $y \in \{x\}^-$ of codimension 1 (i.e. y is an immediate specialization of x), and given coefficient fields σ, τ for x, y respectively, to find a coboundary homomorphism $\delta_{(x,y),\sigma/\tau} : \mathcal{K}(\sigma) \to \mathcal{K}(\tau)$. The homomorphisms $\delta_{(x,y),\sigma/\tau}$ should commute with the isomorphisms $\Phi_{\sigma,\sigma'}$ and $\Phi_{\tau,\tau'}$, thus defining a coboundary homomorphism $\delta_{(x,y)} : \mathcal{K}(x) \to \mathcal{K}(y)$.

It turns out that both tasks are accomplished simultaneously, once formulated properly. Let us assume for simplicity that X is integral, of dimension n. A *saturated chain* of length l in X is a sequence $\xi = (x_0, \ldots, x_l)$ of points of X with each x_{i+1} and immediate specialization of x_i. A pair of *compatible coefficient fields* for ξ is a pair of coefficient fields $\sigma : k(x_0) \to \hat{\mathcal{O}}_{X,x_0}$ and

$\tau : k(x_l) \to \hat{\mathcal{O}}_{X,x_l}$ such that "upon completion along ξ, σ becomes a $k(x_l)$-algebra homomorphism, via τ" - see def. 4.1.5 for the precise statement. Given a saturated chain ξ and compatible coefficient fields σ/τ for it, there is a naturally defined homomorphism $\delta_{\xi,\sigma/\tau}$ which we shall describe later on in the introduction. Now observe that if x is the generic point of X then x has a unique coefficient field ρ, and $\mathcal{K}(\rho) = \omega(x) = \Omega^n_{k(X)/k}$. We prove that for any $y \in X$ there is a finite set of saturated chains S of the form $\xi = (x, \ldots, y)$, such that for any coefficient field σ for y, the map $\sum_{\xi \in S} \delta_{\xi,\rho/\sigma} : \omega(x) \to \mathcal{K}(\sigma)$ is surjective (Internal Residue Isomorphism, thm. 4.3.13). Moreover, the kernel $\omega(x)_{\mathrm{hol}:S}$ of this map is independent of σ. This provides the sought after isomorphism $\Phi_{\sigma,\sigma'}$. Since for any saturated chain $\eta = (y, \ldots, z)$ there are many compatible coefficient fields σ/τ, and since $\delta_{(x,\ldots,y,\ldots,z),\rho/\tau} = \delta_{(y,\ldots,z),\sigma/\tau} \circ \delta_{(x,\ldots,y),\rho/\sigma}$, we get the commutation between the Φ's and the δ's. Thus the coboundary homomorphism $\delta_\xi : \mathcal{K}(y) \to \mathcal{K}(z)$ is defined.

The collection $(\{\mathcal{K}(x)\}, \{\delta_\xi\}, \{\Phi_\sigma\})$ is called a system of residue data on X. It is unique up to a unique isomorphism. The passage to the residue complex is easy. Define $X_q := \{x \in X \mid \dim \{x\}^- = q\}$, $\mathcal{K}_X^{-q} := \bigoplus_{x \in X_q} \mathcal{K}(x)$ and $\delta_X := \sum_{(x,y)} \delta_{(x,y)}$ (see thm. 4.3.20). The fact that $\delta_X^2 = 0$ is an immediate consequence of the Parshin-Lomadze theorem on the sum of residues (thm. 4.2.15; cf. [Pa1] §1 prop. 7 and [Lo] §3 thm. 3).

Some properties of the complex $\mathcal{K}_X^{\cdot}$ can be deduced directly from its construction. For an open immersion $i : U \to X$ there is a canonical isomorphism $\gamma_i^* : \mathcal{K}_U^{\cdot} \xrightarrow{\simeq} i^*\mathcal{K}_X^{\cdot}$ (prop. 4.4.1). For a finite morphism $f : X \to Y$ there is a canonical isomorphism $\gamma_f^\flat : \mathcal{K}_X^{\cdot} \xrightarrow{\simeq} f^\flat \mathcal{K}_Y^{\cdot}$ (see def. 4.4.3 and thm. 4.4.5), and hence a trace map $\mathrm{Tr}_f : f_*\mathcal{K}_X^{\cdot} \to \mathcal{K}_Y^{\cdot}$. Let $\pi : X \to \operatorname{Spec} k$ be the structural morphism. There is a nonzero homomorphism $\mathrm{Tr}_\pi : \pi_*\mathcal{K}_X^0 \to k$ (cor. 4.4.13), which for proper π induces a homomorphism of complexes $\mathrm{Tr}_\pi : \pi_*\mathcal{K}_X^{\cdot} \to k$ (thm. 4.4.14). If X is integral of dimension n then $\tilde{\omega}_X := \mathrm{H}^{-n}\mathcal{K}_X^{\cdot}$ is the sheaf of regular differential forms of Kunz (thm. 4.4.16).

Although the complex $\mathcal{K}_X^{\cdot}$ is canonical, it is somewhat difficult to identify it with $\pi^! k$ in $\mathsf{D}(X)$. For X smooth irreducible of dimension n we show that the fundamental class $\mathrm{C}_X : \Omega^n_{X/k}[n] \to \mathcal{K}_X^{\cdot}$ is a quasi-isomorphism, thus giving an isomorphism $\mathcal{K}_X^{\cdot} \cong \pi^! k$ in $\mathsf{D}(X)$ (thm. 4.5.2). From this it follows that on any reduced X, $\mathcal{K}_X^{\cdot}$ is a residual complex (see def. 4.3.1 and cor. 4.5.6). If π is proper and some isomorphism $\mathcal{K}_X^{\cdot} \cong \pi^! k$ exists, then there is a unique isomorphism $\zeta_X : \mathcal{K}_X^{\cdot} \xrightarrow{\simeq} \pi^! k$ in $\mathsf{D}(X)$ such that our trace morphism $\mathrm{Tr}_\pi : \pi_*\mathcal{K}_X^{\cdot} \to k$ corresponds to that of [RD] ch. VII cor. 3.4 b) (thm. 4.5.9). We prove existence of such an isomorphism only when π factors into $\pi = \rho f$ with f finite and ρ smooth (cor. 4.5.8); note that this includes all quasi-projective varieties. In the appendix (by P. Sastry) the existence of a canonical isomorphism $\zeta_X : \mathcal{K}_X^{\cdot} \xrightarrow{\simeq} \pi^! k$

in $\mathsf{D}(X)$ is established in general (see remark 4.5.10). A complete treatment of the identification $\mathcal{K}^{\cdot}_X \cong \pi^{\triangle}k$ shall appear in [SY], where both $\mathcal{K}^{\cdot}_X$ and $\pi^{\triangle}k$ are considered as sheaves on the site $\mathcal{V}_{\mathrm{Zar}}$ of [Li1].

The explicit construction of the residue complex shows that it carries a canonical structure of a complex of right $\mathcal{D}_X$-modules, regardless of singularities or the characteristic of the field k. We indicate how the bigraded $\mathcal{O}_X$-module $\mathcal{K}^{\cdot,*}_X := \mathcal{H}om_X(\Omega^*_{X/k}, \mathcal{K}^{\cdot}_X)$ of [EZ] ch. II §2.1 can be made into a double complex, without having to embed X in a smooth scheme. These issues are discussed in digressions 4.5.12 and 4.5.13.

Let us briefly explain the contents of the various chapters.

Semi-Topological Rings The topologized rings one runs across in this area (e.g. Beilinson completions of $\mathcal{O}_X$-algebras) usually do not have adic topologies. Thus the conventional methods (say, those of [EGA I] ch. 0 §7) are not applicable. To complicate matters even further, these aren't topological rings in the usual sense : the multiplication map $A \times A \to A$ is not continuous. It was not at all clear what can be done with such rings (take completion for instance, remark 1.2.10). Since our work relies heavily on topological considerations, we undertook to develop the theory of semi-topological rings.

A *semi-topological* (ST) *ring* is a ring A, equipped with a linear topology on its additive group, such that for all $a \in A$ the multiplications $x \mapsto ax$ and $x \mapsto xa$ are continuous endomorphisms (def. 1.2.1). Similarly we define ST A-modules (def. 1.2.2). Relaxing the continuity requirement enables an unexpectedly rich structure. Let us denote by $\mathsf{STMod}(A)$ the category of left ST A-modules and continuous A-linear homomorphisms. In $\mathsf{STMod}(A)$ there are direct sums, products, limits and tensor products. Given an indeterminate t one defines new ST rings $A[t]$, $A[[t]]$, $A((t))$, etc., of polynomials, power series and Laurent series respectively. (Note that even if A is a topological ring (in the usual sense), $A((t))$ needn't be - remark 1.3.8.) A continuous homomorphism of ST rings $A \to B$ determines a base change functor $\mathsf{STMod}(A) \to \mathsf{STMod}(B)$, $M \mapsto B \otimes_A M$, which is left adjoint to "restriction of scalars" (prop. 1.2.14). In particular, if A^{d} is the ring A with the discrete topology and M^{d} is a discrete A^{d}-module, then $M := A \otimes_{A^{\mathrm{d}}} M^{\mathrm{d}}$ is said to have the fine A-module topology (def. 1.2.3 and remark 1.2.16).

Given a ST ring A and an ideal $I \subset A$, one can define a ST ring $\hat{A} := \lim_{\leftarrow n} A/I^{n+1}$ (having the usual I-adic topology when A is discrete). Suppose A is a commutative noetherian ST ring and M is a finitely generated A-module with the fine A-module topology. Generalizing the I-adic case we have an isomorphism of ST $\hat{A}$-modules $\hat{A} \otimes_A M \cong \lim_{\leftarrow n} M/I^{n+1}M$ (prop. 1.2.20).

In section 1.5 we examine the differential calculus over commutative ST rings. Let A be a commutative ST k-algebra (def. 1.2.17). It turns out that continuous k-derivations of A into separated ST A-modules are represented by a universal derivation $\mathrm{d} : A \to \Omega^{1,\mathrm{sep}}_{A/k}$. One defines topologically smooth and étale homomorphisms relative to k, extending the usual notions of formally smooth and étale homomorphisms (see def. 1.5.7 and thm. 1.5.11). For instance, if $A \to B$ is topologically étale relative to k then $(B \otimes_A \Omega^{1,\mathrm{sep}}_{A/k})^{\mathrm{sep}} \cong \Omega^{1,\mathrm{sep}}_{B/k}$. Suppose A is a noetherian commutative ST k-algebra, differentially of finite type over k (def. 1.5.16). Let $I \subset A$ be an ideal and let $\hat{A}$ be the ST k-algebra $\lim_{\leftarrow n} A/I^{n+1}$. Then $\hat{A}$ is topologically étale over A relative to k (thm. 1.5.18). This implies that for such A, $A[[t]]$ is topologically étale over $A[t]$ relative to k.

We think that ST rings can be used to generalize the work of R. Hübl on traces of differential forms [Hu]. Another possible application is for calculations involving Beilinson's sheaf of adeles (with values in $\mathcal{O}_X$), which can be made into a sheaf of ST rings.

Topological Local Fields An n-dimensional local field consists of a field K, together with complete discrete valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_n$, such that for $i = 1, \ldots, n-1$ the residue field κ_i of $\mathcal{O}_i$ is the fraction field of $\mathcal{O}_{i+1}$, and K is the fraction field of $\mathcal{O}_1$. Let k be a fixed perfect field. A *topological local field* (TLF) K over k is a local field which is also a ST k-algebra. We require that there will be some isomorphism $K \cong F((t_n)) \ldots ((t_1))$ (a parametrization) with F discrete and $\mathrm{rank}_F \, \Omega^1_{F/k} < \infty$ (def. 2.1.10). The category of TLFs over k is denoted by $\mathsf{TLF}(k)$. Changing the parametrization involves continuous differential operators, and this process is explored in thm. 2.1.17. We show that if $K \to L$ is a finite morphism in $\mathsf{TLF}(k)$ then L has the fine K-module topology. In characteristic p the topology is, in a sense, superfluous - see prop. 2.1.21. This is because a differential operator of order $\leq p^n - 1$ is linear over the field $K^{(p^n/k)}$ and is therefore continuous (thm. 2.1.14). We give an example of a TLF K of dimension 2 in characteristic 0 and many automorphisms of it (as a local field) which aren't continuous (example 2.1.22). Thus K has many equally "natural" topologies. This example refutes the claim made by Lomadze, that a local field has a canonical topology on it ([Lo] p. 502).

At this point the reader, accustomed to the classical (i.e. 1-dimensional) situation, where the topology is determined by the valuation, may ask: which is the "correct" topology on a local field? The answer is that the same algebro-geometric data that defines the local field (a chain of points $\xi = (x_0, \ldots, x_n)$ in a scheme, see §3.3) also defines the topology.

In section 2.1 we define a base change operation for TLFs. To do this it is necessary to introduce clusters of topological local fields, which are artinian ST algebras whose residue fields are TLFs. The prototypical example of *finitely* *ramified base change* is the morphism $k((s)) \to k((s))((t))$, which is gotten from the morphism $k(s) \to k(s)((t))$ by the base change $k(s) \to k((s))$. In section 2.3 we prove the existence of traces of differential forms, using results of E. Kunz [Ku1].

Our approach to the residue functor is axiomatic (§2.4). Theorem 2.4.3 is an improved version of [Lo] thm. 1, adapted to the setup of topological local fields. It says that there is a contravariant functor Res on the the category $\mathsf{TLF}(k)$, such that $\operatorname{Res} K = \Omega^{*,\mathrm{sep}}_{K/k}$ for a TLF K. Given a morphism $K \to L$ the map $\operatorname{Res}_{L/K} : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}$ is a homomorphism of differential graded ST left $\Omega^{*,\mathrm{sep}}_{K/k}$-modules. The proof uses the notion of topological smoothness and the separated de Rham cohomology algebra $\mathrm{H}^*\Omega^{*,\mathrm{sep}}_{L/K}$. The residue functor is actually defined on the category $\mathsf{CTLF}_{\mathrm{red}}(k)$ of reduced clusters of TLFs. We are able to prove the following: let $A \to B$ be a morphism in $\mathsf{CTLF}_{\mathrm{red}}(k)$. Then the residue pairing $\langle -,- \rangle_{B/A}$ is a perfect pairing of semi-topological A-modules (Topological Duality, thm. 2.4.22). We also prove: the residue maps commute with topologically smooth, finitely ramified base change (thm. 2.4.23).

We wish to point out that in characteristic 0, the residue theory for local fields developed in [Lo] is faulty, since it does not take the topology into account. This rather surprising fact is clearly demonstrated by example 2.4.24. Also included in this section are digressions on residues in Milnor K-theory and on de Rham cohomology.

Beilinson Completions Given any chain $\xi = (x_0, \ldots, x_l)$ in X and a quasi-coherent sheaf $\mathcal{M}$, the *Beilinson completion* $\mathcal{M}_\xi$ *of* $\mathcal{M}$ *along* ξ is defined (def. 3.1.1). The completion operation $(-)_\xi$ is a special case of Beilinson's *adeles*, described in [Be] (see [Hr] for a discussion and proofs). We introduce a topology on the completion $\mathcal{M}_\xi$ in a natural way (def. 3.2.1). If $\xi = (x)$ and $\mathcal{M}$ is coherent then $\mathcal{M}_\xi$ is just the $\mathfrak{m}_x$-adic completion of $\mathcal{M}_x$ with the $\mathfrak{m}_x$-adic topology. For chains of length ≥ 1 the topology is more complicated.

It turns out that given a chain ξ in X, the completion $\mathcal{O}_{X,\xi} := (\mathcal{O}_X)_\xi$ is a commutative ST k-algebra, and for any quasi-coherent sheaf $\mathcal{M}$, $\mathcal{M}_\xi$ is a ST $\mathcal{O}_{X,\xi}$-module. Any differential operator $D : \mathcal{M} \to \mathcal{N}$ extends to a continuous DO $D_\xi : \mathcal{M}_\xi \to \mathcal{N}_\xi$. If η is a face of ξ (i.e. a subchain), the face map $\mathcal{M}_\eta \to \mathcal{M}_\xi$ is continuous. We prove that for a saturated chain ξ of length $n \geq 1$ the face map $\mathcal{M}_{\mathrm{d}_n \xi} \to \mathcal{M}_\xi$ is dense (Approximation Theorem, thm. 3.2.11) and the face map $\mathcal{M}_{\mathrm{d}_0 \xi} \to \mathcal{M}_\xi$ is strict (thm. 3.2.14). We also prove that the completion $\mathcal{O}_{X,\xi}$ is a Zariski ST ring (see def. 3.2.10 and thm. 3.3.8), so the functor $(-)_\xi$ is exact (in the topological sense). For any face η of ξ, $\mathcal{O}_{X,\xi}$ is topologically étale over $\mathcal{O}_{X,\eta}$ relative to k (cor. 3.2.8). This, with the Zariski property, shows that the completion $(\Omega^*_{X/k})_\xi$ is isomorphic, as a ST differential graded k-algebra, to the separated algebra of differentials $\Omega^{*,\mathrm{sep}}_{\mathcal{O}_{X,\xi}/k}$ (def. 1.5.3).

Let $\xi = (x, \ldots, y)$ be a saturated chain of length n. Then $k(\xi) := k(x)_\xi$ is an n-dimensional reduced cluster of TLFs, whose spectrum is determined by repeated normalizations (thm. 3.3.2, cor. 3.3.7). This shows that $\mathcal{O}_{X,\xi}$ is a semi-local ring with Jacobson radical $\mathfrak{m}_\xi := (\mathfrak{m}_x)_\xi$. On the other hand, these results connect the geometry to the theory of topological local fields and residues.

Residues on Schemes Given a coefficient field $\sigma : k(y) \to \mathcal{O}_{X,(y)} = \hat{\mathcal{O}}_{X,y}$ the induced map $\bar{\sigma} : k(y) \to k(\xi)$ is a morphism in $\mathsf{CTLF}_{\mathrm{red}}(k)$. Thus we obtain Parshin's residue map

$$\mathrm{Res}_{\xi,\sigma} : \Omega^*_{k(x)/k} \to \Omega^{*,\mathrm{sep}}_{k(\xi)/k} \xrightarrow{\mathrm{Res}_{k(\xi)/k(y),\sigma}} \Omega^*_{k(y)/k}$$

(def. 4.1.3). Using thm. 4.1.12 which compares completion to finitely ramified base change we prove the transitivity of the residue maps for compatible coefficient fields. Given saturated chains $(x, \ldots, y)$ and $(y, \ldots, z)$, and compatible coefficient fields σ/τ for $(y, \ldots, z)$, one has (cor. 4.1.16):

$$\mathrm{Res}_{(x,\ldots,y,\ldots,z),\tau} = \mathrm{Res}_{(y,\ldots,z),\tau} \circ \mathrm{Res}_{(x,\ldots,y),\sigma} : \Omega^*_{k(x)/k} \to \Omega^*_{k(z)/k} \,.$$

We can now define the coboundary homomorphism $\delta_{\xi,\sigma/\tau}$. Let $\xi = (x, \ldots, y)$ be a saturated chain and let σ/τ be compatible coefficient fields for ξ. For any $\phi \in \mathcal{K}(\sigma)$ consider the diagram:

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[r, "\phi"] & \omega(x) \arrow[d, "\mathrm{Res}_{\xi,\tau}"] \\
\mathcal{O}_{X,y} \arrow[u, "\mathrm{loc}"] \arrow[r, dotted, "\delta(\phi)"] & \omega(y)
\end{tikzcd}$$

Since $\mathrm{Res}_{\xi,\tau}$ is a locally differential operator (def. 3.1.8) it follows that $\delta(\phi)$ is continuous for the $\mathfrak{m}_y$-adic topology, and its completion $\delta(\phi)_{(y)} : \mathcal{O}_{X,(y)} \to \omega(y)$ is $k(y)$-linear (via τ). Thus we get $\delta_{\xi,\sigma/\tau} : \mathcal{K}(\sigma) \to \mathcal{K}(\tau)$.

Let us say a few words about holomorphic forms. Say $\xi = (x, \ldots, y)$ is a saturated chain and τ is a coefficient field for y. Define $\delta_{\xi,\tau} : \omega(x) \to \mathcal{K}(\tau)$ by $\delta_{\xi,\tau}(\alpha)(a) := \mathrm{Res}_{\xi,\tau}(a\alpha)$, $\alpha \in \omega(x)$, $a \in \mathcal{O}_{X,y}$. Using a base change argument we prove that $\omega(x)_{\mathrm{hol}:\xi} := \ker(\delta_{\xi,\tau}) \subset \omega(x)$ is independent of τ (lemma 4.2.1). The elements of $\omega(x)_{\mathrm{hol}:\xi}$ are said to be *holomorphic along* ξ. The quotient $\omega(x)/\omega(x)_{\mathrm{hol}:\xi}$ is a cofinite $\mathcal{O}_{X,y}$-module, with socle canonically isomorphic to $\omega(y)$. This allows us to define the order of pole along ξ of a form $\alpha \in \omega(x)$ (def. 4.2.10).

Finally, we wish to stress the role of topological considerations in this work. Take the Parshin residue map $\mathrm{Res}_{\xi,\sigma} : \Omega^{*}_{k(x)/k} \to \Omega^{*}_{k(y)/k}$. Even though it is a map between *algebraic* objects, it is defined using *topological* methods (viz. TLF's). Moreover, its important properties (e.g. being a locally differential operator, prop. 4.1.4; or transitivity, cor. 4.1.16) are proved topologically. The main result of the paper, the internal residue isomorphism (thm. 4.3.13), is also proved using topological arguments.

Problems Here is a list of some problems related to the present construction.

1) Let k be a perfect field, and let $f : X \to Y$ be a smooth morphism of relative dimension n between reduced k-schemes of finite type. Describe explicitly the derived category isomorphism $\mathcal{K}^{\cdot}_{X} \cong \omega_{X/Y}[n] \otimes_{\mathcal{O}_X} f^{*}\mathcal{K}^{\cdot}_{Y}$.

2) Remove the hypothesis that X is reduced.

3) Remove the hypothesis that k is a perfect field. Allow k to be any field, or a complete DVR with perfect residue field, or $\mathbb{Z}$. This may require a more sophisticated theory of topological local fields.

4) Equivariant case: let $f : X \to Y$ be an equivariant morphism for the action of some algebraic group G over k (an algebraically closed field). Relate the complexes of invariants $\Gamma(X, \mathcal{K}^{\cdot}_{X})^{G(k)}$ and $\Gamma(Y, \mathcal{K}^{\cdot}_{Y})^{G(k)}$.

5) Explore connections with de Rham homology and intersection homology, especially when $k = \mathbb{C}$ (cf. digressions 4.5.12 and 4.5.13).

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1 Semi-Topological Rings

1.1 Preliminaries on Linearly Topologized Abelian Groups

Let M be an abelian group. Given a nonempty collection $\{U_\alpha\}_{\alpha \in I}$ of subgroups of M, let $\mathcal{T}$ be the topology on M generated by the subbasis $\{x + U_\alpha\}_{x \in M, \alpha \in I}$. With this topology M becomes a topological group. We call $\mathcal{T}$ the linear topology generated by $\{U_\alpha\}_{\alpha \in I}$, and we say that M is a linearly topologized abelian group. Let us begin with an elementary but useful lemma (cf. [GT] ch. I §2.3 and §2.4).

Lemma 1.1.1 *Let M be an abelian group, let $\{N_\alpha\}$ be a collection of linearly topologized abelian groups, and for each α, let $\phi_\alpha : M \to N_\alpha$ (resp. $\phi_\alpha : N_\alpha \to M$) be a homomorphism.*

a) *There exists a coarsest (resp. finest) linear topology $\mathcal{T}$ on M such that all the homomorphisms ϕ_α are continuous.*

b) *Let L be a linearly topologized abelian group and let $\psi : L \to M$ (resp. $\psi : M \to L$) be a homomorphism. Suppose that all the composed homomorphisms $\phi_\alpha \circ \psi : L \to N_\alpha$ (resp. $\psi \circ \phi_\alpha : N_\alpha \to L$) are continuous. Then ψ is continuous relative to $\mathcal{T}$.*

Proof First consider homomorphisms $\phi_\alpha : M \to N_\alpha$. Let $\{U_\beta\}$ be the collection of subgroups of M of the form $U_\beta = \phi_\alpha^{-1}(V_\alpha)$, with V_α an open subgroup of N_α for some α. The linear topology $\mathcal{T}$ generated by $\{U_\beta\}$ has the required properties.

Next consider homomorphisms $\phi_\alpha : N_\alpha \to M$. Here we take for $\{U_\beta\}$ the collection of all subgroups of M such that for all α, $\phi_\alpha^{-1}(U_\beta)$ is open in N_α, and we let $\mathcal{T}$ be the linear topology generated by $\{U_\beta\}$. $\square$

Note that in the case $\phi_\alpha : N_\alpha \to M$ the subgroups of the form $U_\beta = \sum_\alpha \phi_\alpha(V_\alpha)$, with $V_\alpha \subset N_\alpha$ open, are a fundamental system of neighborhoods of 0 for the topology $\mathcal{T}$.

Denote by TopAb the category of linearly topologized abelian groups and continuous homomorphisms. This is an additive category, but not an abelian one.

Definition 1.1.2 *A sequence of homomorphisms $M' \xrightarrow{\phi} M \xrightarrow{\psi} M''$ in TopAb is called exact if it is exact in the category Ab of abelian groups and if ϕ and ψ are strict.*

(See [GT] ch. III §2.8 for the definition of a strict homomorphism.)

It follows from lemma 1.1.1 that the category TopAb has direct and inverse limits; the underlying abelian groups are just the corresponding limits in Ab. In particular, TopAb has infinite direct sums and products. Note that the topology on $\prod_\alpha M_\alpha$ is the usual product topology.

Definition 1.1.3 *Let M be a linearly topologized abelian group.*

a) *The associated separated topological group of M is defined to be the quotient $M^{\mathrm{sep}} := M/\{0\}^-$, where $\{0\}^-$ is the closure of $\{0\}$ in M.*

b) *The completion of M is defined to be the inverse limit $M^{\mathrm{cpl}} := \lim_{\leftarrow \alpha} M/U_\alpha$ in TopAb, where $\{U_\alpha\}$ is the collection of open subgroups of M.*

c) *M is said to be separated (resp. separated and complete) if the canonical homomorphism $M \to M^{\mathrm{sep}}$ (resp. $M \to M^{\mathrm{cpl}}$) is bijective.*

Note that the canonical homomorphisms $M \to M^{\mathrm{sep}}$, $M^{\mathrm{sep}} \to M^{\mathrm{cpl}}$ and $M \to M^{\mathrm{cpl}}$ are all strict. Both functors $M \mapsto M^{\mathrm{sep}}$ and $M \mapsto M^{\mathrm{cpl}}$ are additive idempotent endo-functors on TopAb.

Lemma 1.1.4 *Let M be a linearly topologized abelian group. Then M is separated and complete in the sense of definition 1.1.3 iff every Cauchy net in M has a unique limit.*

Proof This is an immediate consequence of [GT] ch. III §7.3 cor. 2 to prop. 2, and of [Ko] §2.3 and §5.4. $\square$

It turns out that separated modules are more interesting, from the point of view of semi-topological rings, than complete ones; consider remark 1.2.10 and theorem 1.5.11.

Proposition 1.1.5 a) *An inverse limit of separated (resp. separated and complete) linearly topologized abelian groups is separated (resp. separated and complete).*

b) *A direct sum of separated (resp. separated and complete) linearly topologized abelian groups is separated (resp. separated and complete).*

c) *Let $M = \bigoplus_{n\in\mathbb{N}} M_n$ be a countable direct sum of separated linearly topologized abelian groups and let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy net (i.e. a Cauchy sequence) in M. Then there is some n_0 such that $x_i \in \bigoplus_{n=0}^{n_0} M_n$ for all i.*

Proof a) See [GT] ch. II §3.5 cor. to prop. 10, and ch. I §8.2 cor. 2 to prop. 7.

b) See [Ko] §10.2 (8) and §13.4 (2); the proofs there are for vector spaces but work also for linearly topologized abelian groups.

c) This is an easy exercise using a "diagonal" argument and the fact that the subgroups of the form $\bigoplus U_n$, with $U_n \subset M_n$ open, are a fundamental system of neighborhoods of 0 in $\bigoplus M_n$. $\square$

Generalizing the result on inverse limits in **Ab** we have:

Proposition 1.1.6 *Let*

$$(0 \to M_i' \xrightarrow{\phi_i} M_i \xrightarrow{\psi_i} M_i'' \to 0)_{i\in\mathbb{N}}$$

be an inverse system of exact sequences in **TopAb**. *Assume that $M_{i+1}' \to M_i'$ is surjective for all $i \in \mathbb{N}$. Then the sequence*

$$0 \to \varprojlim_i M_i' \xrightarrow{\phi} \varprojlim_i M_i \xrightarrow{\psi} \varprojlim_i M_i'' \to 0$$

is exact in **TopAb**.

Proof Consider the commutative diagram in **TopAb**

$$\begin{tikzcd}
0 \arrow[r] & \prod M_i' \arrow[r, "(\phi_i)"] & \prod M_i \arrow[r, "(\psi_i)"] & \prod M_i'' \arrow[r] & 0 \\
0 \arrow[r] & \lim_{\leftarrow} M_i' \arrow[r, "\phi"] \arrow[u] & \lim_{\leftarrow} M_i \arrow[r, "\psi"] \arrow[u] & \lim_{\leftarrow} M_i'' \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The top row is exact in **TopAb** and the vertical maps are strict monomorphisms. Also, the bottom row is exact in **Ab** by [Ha] ch. II prop. 9.1. Hence ϕ is a strict monomorphism.

In order to show that ψ is strict it suffices to check that for every open subgroup $V \subset \lim_{\leftarrow} M_i$, $\psi(V)$ is open in $\lim_{\leftarrow} M''_i$. Let $\gamma_{j,i} : M_j \to M_i$ be the maps in the system (M_i). We may assume that

$$V = (\lim_{\leftarrow} M_i) \cap (V_0 \times \cdots \times V_n \times M_{n+1} \times M_{n+2} \times \cdots)$$

where $V_i \subset M_i$ are open subgroups and $\gamma_{j,i}(V_j) \subset V_i$ for $0 \leq i \leq j \leq n$. Let

$$W := (\lim_{\leftarrow} M''_i) \cap (\psi_0(V_0) \times \cdots \times \psi_n(V_n) \times M''_{n+1} \times M''_{n+2} \times \cdots)$$

which is open in $\lim_{\leftarrow} M''_i$. We claim that $\psi(V) = W$. Clearly $\psi(V) \subset W$. Given $\underline{x}'' = (x''_0, x''_1, \ldots) \in W$, choose any $\underline{x} \in \psi^{-1}(\underline{x}'')$. Now $x_n \in V_n + \phi_n(M'_n)$, so there is some $\underline{x}' \in \lim_{\leftarrow} M'_i$ with $x_n - \phi_n(x'_n) \in V_n$. Setting $\underline{y} := \underline{x} - \phi(\underline{x}')$ we get $\underline{y} \in V \cap \psi^{-1}(\underline{x}'')$. □

Proposition 1.1.7 *Let $(M_i)_{i \in \mathbb{N}}$ be a direct system in* TopAb *s.t. all the homomorphisms $M_i \to M_{i+1}$ are strict monomorphisms, and let $M := \lim_{i \to} M_i$. Then for all i, $M_i \to M$ is a strict monomorphism. If moreover all the groups M_i are separated, then so is M.*

Proof We may assume $i = 0$. The injectivity of $M_0 \to M$ is known. Let $U_0 \subset M_0$ be any open subgroup. By hypothesis we can choose for every $j \geq 1$ an open subgroup $U_j \subset M_j$ s.t. $U_{j-1} = U_j \cap M_{j-1}$. Then $U := \bigcup U_j$ is an open subgroup of M and $U_0 = U \cap M_0$.

Now suppose all the M_i are separated, and let $x \in M$, $x \neq 0$. Then $x \in M_i$ for some i, and there is an open subgroup $U_i \subset M_i$ s.t. $x \notin U_i$. Let $U \subset M$ be an open subgroup s.t. $U_i = U \cap M_i$; then $x \notin U$. Hence M is separated. □

Proposition 1.1.8 *(Sufficient Conditions for Density)*

a) *Let $M^{\cdot} = (0 \to M^0 \to M^1 \to M^2 \to 0)$ and $N^{\cdot} = (0 \to N^0 \to N^1 \to N^2 \to 0)$ be two complexes in* TopAb*, with $N^{\cdot}$ exact, and let $\phi^{\cdot} : M^{\cdot} \to N^{\cdot}$ be a homomorphism of complexes. Suppose $\phi^0 : M^0 \to N^0$ and $\phi^2 : M^2 \to N^2$ are dense. Then $\phi^1 : M^1 \to N^1$ is dense too.*

b) *Let $(\phi_i : M_i \to N_i)_{i \in \mathbb{N}}$ be an inverse system of dense homomorphisms in* TopAb*, with $M_{i+1} \to M_i$ surjective for all i. Then $\phi : \lim_{\leftarrow i} M_i \to \lim_{\leftarrow i} N_i$ is dense.*

c) *Let $(\phi_\alpha : M_\alpha \to N_\alpha)_{\alpha \in I}$ be a direct system of dense homomorphisms in* TopAb*. Then $\phi : \lim_{\alpha \to} M_\alpha \to \lim_{\alpha \to} N_\alpha$ is dense.*

Proof a) Given any open subgroup $U \subset N^1$, let $\bar{N}^1 := N^1/U$ and let $\bar{\phi}^1 : M^1 \to \bar{N}^1$ be the induced homomorphism. We must show that $\bar{\phi}^1$ is surjective. Set $\bar{N}^0 := N^0/N^0 \cap U$ and $\bar{N}^2 := N^2/\mathrm{im}(U \to N^2)$. So $\bar{N}^{\cdot}$ is an exact complex of discrete groups. By assumption $\bar{\phi}^0 : M^0 \to \bar{N}^0$ and $\bar{\phi}^2 : M^2 \to \bar{N}^2$ are surjective; hence so is $\bar{\phi}^1$.

b) Let $U \subset \lim_{\leftarrow i} N_i$ be any open subgroup. Then U is the preimage of an open subgroup $U_j \subset N_j$ for some j. Thus $(\lim_{\leftarrow i} N_i)/U \to N_j/U_j$ is injective. By assumption $\lim_{\leftarrow i} M_i \to M_j$ and $\bar{\phi}_j : M_j \to N_j/U_j$ are surjective. Hence $\bar{\phi} : \lim_{\leftarrow i} M_i \to (\lim_{\leftarrow i} N_i)/U$ is surjective too.

c) Let $U \subset \lim_{\alpha \to} N_\alpha$ be any open subgroup. For $\beta \in I$ let $U_\beta \subset N_\beta$ be the preimage of U. Then $(\lim_{\alpha \to} N_\alpha)/U \cong \lim_{\alpha \to}(N_\alpha/U_\alpha)$. Since $\bar{\phi}_\alpha : M_\alpha \to N_\alpha/U_\alpha$ are assumed to be surjective, so is $\bar{\phi} : \lim_{\alpha \to} M_\alpha \to (\lim_{\alpha \to} N_\alpha)/U$. $\square$

1.2 Semi-Topological Rings

We will be considering topologized rings in which multiplication is continuous only in one argument. To distinguish these rings from ordinary topological rings we adopt the name "semi-topological ring". The following notation will be used throughout this section. Given a ring A and an element $a \in A$, left and right multiplication by a will be denoted by $\lambda_a : b \mapsto ab$ and $\rho_a : b \mapsto ba$, $b \in A$. Similarly given a left A-module M and elements $a \in A$ and $x \in M$, we set $\lambda_a : y \mapsto ay$, $y \in M$, and $\rho_x : b \mapsto bx$, $b \in A$. In order to emphasize where $a \in A$ acts we may indicate the module in superscript, e.g.: $\lambda_a^M : M \to M$. All rings under consideration have 1.

Definition 1.2.1 *A semi-topological (ST) ring is a ring A together with a topology on it satisfying the following conditions:*

i) *The additive group of A is a linearly topologized abelian group.*

ii) *For every $a \in A$ the multiplications $\lambda_a, \rho_a : A \to A$ are continuous.*

Definition 1.2.2 *Let A be a semi-topological ring. A semi-topological (ST) left A-module is a left A module M together with a topology on it satisfying the following conditions:*

i) *M is a linearly topologized abelian group.*

ii) *For every $a \in A$ and every $x \in M$ the multiplications $\lambda_a : M \to M$ and $\rho_x : A \to M$ are continuous.*

Similarly one defines semi-topological right modules and bimodules.

Denote by $\mathsf{STMod}(A)$ the category of semi-topological left A-modules and continuous A-linear homomorphism. It is an additive subcategory of TopAb, closed under direct and inverse limits. We define exact sequences in $\mathsf{STMod}(A)$ to be those which are exact in $\mathsf{TopAb}(A)$ (see def. 1.1.2). Given $M, N \in \mathsf{STMod}(A)$, we denote the group of morphisms between them by $\mathrm{Hom}_A^{\mathrm{cont}}(M, N)$. (The category of right modules we denote by $\mathsf{STMod}(A^\circ)$.)

Suppose M is a left A-module. Consider it as an abelian group with homomorphisms $\rho_x : A \to M$, $x \in M$. Let $\mathcal{T}$ be the finest topology on M making all the ρ_x continuous (see lemma 1.1.1). We claim that with this topology M becomes a semi-topological A-module. It suffices to show that for every $a \in A$ the endomorphism $\lambda_a^M : M \to M$ is continuous. Choose such a. For each $x \in M$ we have $\lambda_a^M \circ \rho_x = \rho_x \circ \lambda_a^A : A \to M$, which is continuous by definition. From lemma 1.1.1 it follows that λ_a^M is continuous.

Definition 1.2.3 *The above topology on M is called the fine A-module topology.*

The next proposition gives a characterization of this topology.

Proposition 1.2.4 *Let A be a semi-topological ring and let M be a semi-topological A-module. Then M has the fine A-module topology iff for every semi-topological A-module N*

$$\mathrm{Hom}_A^{\mathrm{cont}}(M, N) = \mathrm{Hom}_A(M, N). \tag{1.2.5}$$

Proof Suppose M has the fine A-module topology. Let $\psi : M \to N$ be an A-linear homomorphism; we must show that it is continuous. For any $x \in M$, one has $\psi \circ \rho_x = \rho_{\psi(x)} : A \to N$, which is continuous by definition. According to lemma 1.1.1 ψ is also continuous. In particular, Taking N to be the same module as M but with various topologies (satisfying the conditions of def. 1.2.2), and taking $\psi : M \to N$ to be the identity map, we see that the fine A-module topology is indeed the finest of them all.

Conversely, suppose that equality holds in (1.2.5). Then using the same setup as above, but this time N is the module M with the fine A-module topology, we see that the topology on M is finer than the fine A-module topology, and hence equal to it. □

Corollary 1.2.6 *Let $\{M_\alpha\}$ be a direct system of ST A-modules. If every M_α has the fine A-module topology then so does $\lim_{\alpha \to} M_\alpha$.*

Definition 1.2.7 *Let M be a semi-topological left A-module and let $\{m_\alpha\}$ be a subset of M. M is said to be free with basis $\{m_\alpha\}$ if for any semi-topological A-module N and any subset $\{n_\alpha\} \subset N$ there is a unique continuous A-linear homomorphism $\phi : M \to N$ with $\phi(m_\alpha) = n_\alpha$. Similarly for right modules.*

Clearly M is free iff $M \cong \bigoplus_\alpha A$ in $\mathsf{STMod}(A)$. We have another corollary to prop. 1.2.4:

Corollary 1.2.8 *Suppose $\phi : M \to N$ is a continuous surjective homomorphism of semi-topological A-modules, where M has the fine A-module topology. Then ϕ is a strict epimorphism iff N has the fine A-module topology. In particular, this is the case when M is free.*

A ring homomorphism $f : A \to B$ is called centralizing if $B = f(A) \cdot \mathrm{C}_B(A)$, where $\mathrm{C}_B(A)$ is the centralizer of A in B.

Proposition 1.2.9 *Let A be a semi-topological ring and let $f : A \to B$ be a centralizing ring homomorphism. Put on B, considered as a left A-module via f, the fine A-module topology. Then the following hold:*

a) *As a right A-module via f, B has the fine A-module topology. In particular, B is a semi-topological A-A-bimodule.*

b) *B is a semi-topological ring.*

c) *Let M be a left B-module. The fine B-module topology on M coincides with the fine A-module topology on it.*

Proof a) Choose a subset $\{c_\alpha\} \subset \mathrm{C}_B(A)$ such that the bimodule homomorphism $\phi : \bigoplus_\alpha A \to B$, $\phi(\sum a_\alpha) = \sum a_\alpha c_\alpha = \sum c_\alpha a_\alpha$, is surjective. By corollary 1.2.8, used twice, ϕ is a strict epimorphism and B has the fine A-module topology as a right A-module.

b) For every $b \in B$ the map λ_b (resp. ρ_b) is an endomorphism of the right (resp. left) A-module B. By part a) and proposition 1.2.4 both λ_b and ρ_b are continuous.

c) This follows from cor. 1.2.8 and the fact that a direct sum of strict homomorphisms is strict. $\square$

Observe that the proposition includes the case of a surjective ring homomorphism. Given a semi-topological ring A, let I be the closure of 0. Then

I is an ideal and $A^{\mathrm{sep}} = A/I$ is a semi-topological ring. Similarly, if M is a semi-topological left A-module, then M^{sep} is an A^{sep}-module. Thus $M \mapsto M^{\mathrm{sep}}$ is a functor $\mathsf{STMod}(A) \to \mathsf{STMod}(A^{\mathrm{sep}})$ and $A \mapsto A^{\mathrm{sep}}$ is a functor on semi-topological rings.

Suppose A is a semi-topological ring and $B \subset A$ a subring. Then B is also a semi-topological ring, and the same is true of its closure B^{-}. Similarly for a submodule.

Remark 1.2.10 The author does not know whether the completion M^{cpl} is, in general, a semi-topological A-module. The difficulty is in establishing the continuity of $\rho_x : A \to M^{\mathrm{cpl}}$ for x in the "boundary" $M^{\mathrm{cpl}} - M$. Of course, if the topology on M is generated by A-submodules there is no difficulty.

Definition 1.2.11 *Let A be a semi-topological ring and let M and N be right and left semi-topological A-modules, respectively. The tensor product topology on $M \otimes_A N$ is by definition the finest linear topology such that for every $x \in M$ and every $y \in N$ the homomorphisms $\lambda_x : N \to M \otimes_A N$, $y' \mapsto x \otimes y'$, and $\rho_y : M \to M \otimes_A N$, $x' \mapsto x' \otimes y$, are continuous (see lemma 1.1.1).*

Whenever a tensor product of semi-topological modules is encountered, it will be endowed with this topology by default. We state the following lemma whose proof is an application of lemma 1.1.1.

Lemma 1.2.12 *Suppose L is a linearly topologized abelian group and $\phi : M \otimes_A N \to L$ is a homomorphism such that for every $x \in M$, $y \in N$ the composed homomorphisms $\phi \circ \lambda_x : N \to L$ and $\phi \circ \rho_y : M \to L$ are continuous. Then ϕ is continuous relative to the tensor product topology on $M \otimes_A N$.*

Suppose $A_1, \ldots, A_n$ are semi-topological rings and $M_0, \ldots, M_n$ are semi-topological bimodules such that the tensor product $M := M_0 \otimes_{A_1} \cdots \otimes_{A_n} M_n$ makes sense. Then the tensor product topology on M is independent of the binary grouping of the factors (associativity of the tensor product topology). It is described directly as being the finest linear topology such that for every i, $0 \leq i \leq n$, and every $x_j \in M_j$, $j \neq i$, the homomorphism $M_i \to M$, $y \mapsto x_0 \otimes \cdots \otimes x_{i-1} \otimes y \otimes x_{i+1} \otimes \cdots \otimes x_n$ is continuous.

Another observation is that taking tensor products of semi-topological modules commutes with passing to the associated separated module. To be precise,

$$(M \otimes_A N)^{\mathrm{sep}} \cong (M^{\mathrm{sep}} \otimes_{A^{\mathrm{sep}}} N^{\mathrm{sep}})^{\mathrm{sep}} \tag{1.2.13}$$

as quotients of $M \otimes_A N$.

Semi-topological modules admit a useful base-change operation.

Proposition 1.2.14 *Let $A \to B$ be a continuous homomorphism of semi-topological rings and let M be a semi-topological left A-module. Then the tensor product topology on $B \otimes_A M$ makes it into a semi-topological left B-module. This topology is characterized by the following properties:*

i) *The canonical homomorphism of A-modules $M \to B \otimes_A M$ is continuous.*

ii) *(Adjunction) For any $N \in \mathsf{STMod}(B)$ the canonical homomorphism*

$$\mathrm{Hom}_B^{\mathrm{cont}}(B \otimes_A M, N) \to \mathrm{Hom}_A^{\mathrm{cont}}(M, N)$$

is bijective.

Proof First we must verify that the maps $\lambda_b^{B \otimes M} : B \otimes_A M \to B \otimes_A M$, $b \in B$, and $\rho_u : B \to B \otimes_A M$, $u \in B \otimes_A M$, are continuous. The continuity of $\lambda_b^{B \otimes M}$ follows from lemma 1.2.12. As for ρ_u, we may assume that $u = b \otimes x$, so $\rho_u = \rho_x \circ \rho_b^B$, which is continuous by definition. Therefore $B \otimes_A M$ is a semi-topological B-module. Properties i) and ii) are similarly checked.

Finally, we show that the two properties determine the topology on $B \otimes_A M$. Let N_1 and N_2 be two ST B-modules with the same underlying B-module $B \otimes_A M$, and both enjoying properties i) and ii). Then the identity map $N_1 \to N_2$ is a homeomorphism. □

Corollary 1.2.15 *If M has the fine A-module topology then $B \otimes_A M$ has the fine B-module topology.*

Remark 1.2.16 The ST A-modules with the fine A-module topologies are precisely those induced from discrete modules. To see this, let M be an A-module with the fine topology. Define A^{d} and M^{d} to be A and M, respectively, with the discrete topologies. Then $A \otimes_{A^{\mathrm{d}}} M^{\mathrm{d}} \to M$ is a homeomorphism.

Definition 1.2.17 *Let k be a commutative semi-topological ring. A semi-topological k-algebra is a semi-topological ring A together with a continuous centralizing homomorphism $k \to A$.*

Given two semi-topological k-algebras A and B their tensor product $A \otimes_k B$ is again a semi-topological k-algebra. Let us denote by $\mathsf{STComAlg}(k)$ the category of commutative semi-topological k-algebras and continuous k-algebra homomorphisms. An immediate consequence of prop. 1.2.14 is:

Corollary 1.2.18 *Let A and B be commutative ST k-algebras. Then $A \otimes_k B$ is the fibred coproduct of A and B in the category* $\mathsf{STComAlg}(k)$.

Discrete rings and modules are semi-topological. A more interesting example is provided by:

Lemma 1.2.19 *Let A be a ST ring and let $I \subset A$ be an ideal. For each $n \geq 0$ put on A/I^{n+1} the fine A-module topology, and put on $\hat{A} := \lim_{\leftarrow n} A/I^{n+1}$ the $\lim_{\leftarrow}$ topology. Then $\hat{A}$ is a ST ring.*

Proof According to prop. 1.2.9, every A/I^{n+1} is a ST ring. Let $a \in \hat{A}$. Then for every n the homomorphisms $\lambda_a, \rho_a : A/I^{n+1} \to A/I^{n+1}$ are continuous. Passing to the inverse limit it follows that $\lambda_a, \rho_a : \hat{A} \to \hat{A}$ are continuous. Therefore $\hat{A}$ is a ST ring. □

Of course if A is discrete we recover the I-adic topology on $\hat{A}$. In general $\hat{A}$ need not be separated nor complete topologically. Extending the standard result on I-adic completions of finitely generated modules over a noetherian commutative ring, we have:

Proposition 1.2.20 *Let A be a noetherian commutative ST ring and let $I \subset A$ be an ideal. Put on $\hat{A} := \lim_{\leftarrow n} A/I^{n+1}$ the topology of the previous lemma. Let M be a finitely generated A-module. For each $n \geq 0$ put on $M/I^{n+1}M$ the fine A-module topology, and put on $\hat{M} = \lim_{\leftarrow n} M/I^{n+1}M$ the $\lim_{\leftarrow}$ topology. Then the topology on $\hat{M}$ is the fine $\hat{A}$-module topology.*

Proof As in the proof of lemma 1.2.19, $\hat{M}$ is a ST $\hat{A}$-module. By corollary 1.2.8 it suffices to produce a strict epimorphism $\hat{A}^r \twoheadrightarrow \hat{M}$. Choose any exact sequence of $\hat{A}$-modules

$$0 \to \hat{K} \to \hat{A}^r \xrightarrow{\psi} \hat{M} \to 0.$$

For every n we get, in virtue of cor. 1.2.8, an exact sequence in $\mathsf{STMod}(A)$

$$0 \to K_n \to (A/I^{n+1})^r \xrightarrow{\psi_n} M/I^{n+1}M \to 0,$$

where $K_n := \operatorname{im}(\hat{K} \to (A/I^{n+1})^r)$ with the subspace topology. By prop. 1.1.6, $\psi = \lim_{\leftarrow} \psi_n$ is strict. □

Corollary 1.2.21 *Suppose M has the fine A-module topology. Then the natural homomorphism $\hat{A} \otimes_A M \to \hat{M}$ is an isomorphism in* $\mathsf{STMod}(\hat{A})$.

Observe that if $I^{n+1}M = 0$ for some $n \geq 0$, then the fine A-module topology on $M = \hat{M}$ and the fine $\hat{A}$-module topology on it coincide.

Proposition 1.2.22 *Let A be a ST ring, let M and N be ST A-modules and let $\phi : M \to N$ be a continuous A-linear homomorphism. Suppose that $M \cong \lim_{\leftarrow \alpha} M_\alpha$ in $\mathsf{STMod}(A)$ for some inverse system $(M_\alpha)_{\alpha \in I}$. Suppose also that N is finitely generated, separated and semi-simple in $\mathsf{STMod}(A)$. Then ϕ factors through some M_α.*

Proof We have $N \cong \bigoplus_{j=1}^{r} N_j$, where each N_j is a separated, simple, ST A-module. For each j let $U_j \subset N_j$ be a proper open subgroup, and define $U := \bigoplus_{j=1}^{r} U_j$. Then $U \subset N$ is an open subgroup, but the only A-submodule contained in U is 0. For $\alpha \in I$ set $K_\alpha := \ker(M \to M_\alpha)$. By the definition of the $\lim_{\leftarrow}$ topology there is some α_0 s.t. $K_{\alpha_0} \subset \phi^{-1}(U)$. So $\phi(K_{\alpha_0}) \subset U$, and being an A-module it must be 0. □

Let A be a ST ring and let $A_0 \subset A$ be a subring. A ST A-module is said to have an A_0-linear topology if there is a basis of neighborhoods of 0 consisting of A_0-submodules (e.g. take for A_0 the image of $\mathbb{Z}$).

Proposition 1.2.23 *Let A be a ST ring and let $A_0 \subset A$ be a subring. The full subcategory of $\mathsf{STMod}(A)$ consisting of modules with A_0-linear topologies is closed under quotients, subobjects, sums, products, direct limits and inverse limits.*

Proof Immediate from lemma 1.1.1, since the maps are A_0-linear. □

1.3 Rings of Laurent Series

An important class of semi-topological rings is that of rings of iterated Laurent series, which we will examine in this section.

Definition 1.3.1 *Let A be a commutative semi-topological ring and let $\underline{t} = (t_1, \ldots, t_n)$ be a sequence of indeterminates. We put on the A-algebras $A[\underline{t}] := A[t_1, \ldots, t_n]$, $A[\underline{t}]/(\underline{t})^{i+1}$, $i \geq 0$ and $A[\underline{t}, \underline{t}^{-1}] := A[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$ the fine A-module topologies. We put on $A[[\underline{t}]] := \lim_{\leftarrow i} A[\underline{t}]/(\underline{t})^{i+1}$ the inverse limit topology.*

Lemma 1.3.2 *$A[\underline{t}]$, $A[\underline{t}]/(\underline{t})^{i+1}$, $A[\underline{t}, \underline{t}^{-1}]$ and $A[[\underline{t}]]$ are all semi-topological A-algebras.*

Proof Immediate from prop. 1.2.9 and lemma 1.2.19. □

Definition 1.3.3 *Let A be a commutative ST ring and let t be an indeterminate. For every $j \geq 0$ put on $t^{-j}A[[t]]$ the fine $A[[t]]$-module topology. Put on $A((t)) := A[[t]][t^{-1}] = \lim_{j\to} t^{-j}A[[t]]$ the direct limit topology.*

Lemma 1.3.4 *The topology on $A((t))$ is the fine $A[[t]]$-module topology. Therefore $A((t))$ is a semi-topological A-algebra.*

Proof This follows from cor. 1.2.6 and prop. 1.2.9. □

Proposition 1.3.5 *The homomorphisms $A \hookrightarrow A[[t]]$, $A[[t]] \twoheadrightarrow A$ and $A[[t]] \hookrightarrow A((t))$ are all continuous and strict. If A is separated (resp. separated and complete) then so are $A[\underline{t}]$, $A[\underline{t}]/(\underline{t})^{i+1}$, $A[\underline{t}, \underline{t}^{-1}]$, $A[[\underline{t}]]$ and $A((t))$.*

Proof For every $i, h \geq 0$ consider the exact sequence of semi-topological A-modules

$$0 \to A[t]/t^{i+1}A[t] \to t^{-h}A[t]/t^{i+1}A[t] \to \bigoplus_{j=1}^{h} t^{-j}A \to 0$$

with its obvious splitting (in $\mathsf{STMod}(A)$). Passing to the inverse limit in i and then to the direct limit in h we get

$$A((t)) \cong \left[\bigoplus_{j=1}^{\infty} t^{-j}A\right] \oplus A[[t]]. \tag{1.3.6}$$

Therefore $A[[t]] \hookrightarrow A((t))$ is strict. A similar consideration shows that $A[[t]] \cong A \oplus tA[[t]]$, so the other two homomorphisms are strict.

The statements regarding separatedness and completeness follow from formula (1.3.6) and prop. 1.1.5. □

The topology on the ring of iterated Laurent series defined below generalizes Parshin's topology on a local field, see [Pa3] §1 def. 2.

Definition 1.3.7 *Let A be a commutative semi-topological ring and let $\underline{t} = (t_1, \ldots, t_n)$ be a sequence of indeterminates. The Laurent series ring in $\underline{t}$ over A is the semi-topological A-algebra $A((\underline{t})) = A((t_1, \ldots, t_n))$ defined recursively by*

$$A((t_1, \ldots, t_n)) := A((t_2, \ldots, t_n))((t_1)).$$

From proposition 1.3.5 it follows that the inclusion $A \hookrightarrow A((\underline{t}))$ is a strict monomorphism. Evidently the operations $A \mapsto A[\underline{t}]$, $A \mapsto A[\underline{t}]/(\underline{t})^{i+1}$, etc. are functors on the category $\mathsf{STComAlg}(\mathbb{Z})$ of commutative ST rings, sending the full subcategory of separated (resp. separated and complete) rings into itself.

Remark 1.3.8 As noticed by Parshin, if k is a discrete field and $n \geq 2$, the field of Laurent series $k((\underline{t})) = k((t_1, \ldots, t_n))$ is *not* a topological ring; i.e., multiplication is not a continuous function $k((\underline{t})) \times k((\underline{t})) \to k((\underline{t}))$ (see [Pa3] remark 1). Also, in this case $k((\underline{t}))$ is not a metrizeable topological space.

Lemma 1.3.9 *The image of $A[\underline{t}, \underline{t}^{-1}]$ in $A((\underline{t}))$ is dense.*

Proof Let $\underline{t}' := (t_2, \ldots, t_n)$. By induction for every $i \geq 0$ the map $A[\underline{t}', \underline{t}'^{-1}][t_1] \to A((\underline{t}'))[t_1]/(t_1^{i+1})$ is dense, so by prop. 1.1.8 b) we have that $A[\underline{t}', \underline{t}'^{-1}][t_1] \to A((\underline{t}'))[[t_1]]$ is dense. Hence for every $j \geq 0$, $t_1^{-j} A[\underline{t}', \underline{t}'^{-1}][t_1] \to t_1^{-j} A((\underline{t}'))[[t_1]]$ is dense, and finally by prop. 1.1.8 c), $A[\underline{t}, \underline{t}^{-1}] \to A((\underline{t}))$ is dense. □

Suppose A is separated and complete. An element $a(t) \in A((t))$ determines a function $a : \mathbb{Z} \to A$, $i \mapsto a_i$, in the usual way. The support of the function $a : \mathbb{Z} \to A$ is bounded below, and $a(t) = \sum_{i \in \mathbb{Z}} a_i t^i$ in the sense of [GT] ch. III §5.1. From the recursive definition of the ring $A((\underline{t}))$ one sees that any $a(\underline{t}) \in A((\underline{t}))$ determines a function $a : \mathbb{Z}^n \to A$, $\underline{i} \mapsto a_{\underline{i}}$, such that $a(\underline{t}) = \sum_{\underline{i} \in \mathbb{Z}^n} a_{\underline{i}} \underline{t}^{\underline{i}}$. There are certain conditions on the support of $a : \mathbb{Z}^n \to A$, and in fact one can show that these conditions are precisely equivalent to the summability of the collection of monomials $(a_{\underline{i}} \underline{t}^{\underline{i}})_{\underline{i} \in \mathbb{Z}^n}$.

Given another sequence $\underline{s} = (s_1, \ldots, s_n)$ of indeterminates and a sequence $\underline{e} = (e_1, \ldots, e_n)$ of positive integers, the homomorphism of ST A-algebras $A((\underline{s})) \to A((\underline{t}))$, $s_j \mapsto t_j^{e_j}$, makes $A((\underline{t}))$ into a free ST $A((\underline{s}))$-module, with basis $\{\underline{t}^{\underline{i}}\}$, $0 \leq i_j < e_j$. By abuse of notation we denote the image of $A((\underline{s}))$ by $A((\underline{t}^{\underline{e}}))$.

1.4 Preliminaries on Differential Operators

Let k be a commutative ring and let A be a commutative k-algebra. Given A-modules M and N, we use the following notation for the action of A on $\mathrm{Hom}_{\mathbb{Z}}(M, N)$: for $a, b \in A$, $\phi \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$ and $x \in M$, we set $(a\phi b)(x) = a\phi(bx) \in N$.

Recall the definition of differential operators (DOs) over A from M to N ([EGA IV] §16.8). Given $D \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$ and $a \in A$, set $[D, a] := Da - aD \in \mathrm{Hom}_{\mathbb{Z}}(M, N)$. We say that D is a differential operator of order $\leq n$ over

A, and denote this by $\mathrm{ord}_A(D) \leq n$, if for all $a_0, \ldots, a_n \in A$ it holds that $[\ldots[D, a_0], \ldots, a_n] = 0$. (We set $\mathrm{ord}_A(0) := -1$.) If D is k-linear, it is said to be a differential operator relative to k. Set

$$\mathrm{Diff}^n_{A/k}(M, N) := \{D \in \mathrm{Hom}_k(M, N) \mid \mathrm{ord}_A(D) \leq n\}$$

$$\mathrm{Diff}_{A/k}(M, N) := \bigcup_n \mathrm{Diff}^n_{A/k}(M, N).$$

Evidently, for any $n \geq 0$, $\mathrm{ord}_A(D) \leq n$ iff $\mathrm{ord}_A([D, a]) \leq n - 1$ for every $a \in A$.

Let I_A be the kernel of the multiplication map $A \otimes_k A \to A$, $a \otimes b \mapsto ab$, and define $\mathcal{P}^n_{A/k} := A \otimes_k A / I_A^{n+1}$. Consider $\mathcal{P}^n_{A/k}$ as an A-algebra via the first factor: $a \mapsto a \otimes 1$, and set $\mathrm{d}^n(a) := 1 \otimes a \pmod{I_A^{n+1}}$. d^n defines a right A-module structure on $\mathcal{P}^n_{A/k}$. Given an A-module M set $\mathcal{P}^n_{A/k}(M) := \mathcal{P}^n_{A/k} \otimes_A M$. The map $\mathrm{d}^n_M : M \to \mathcal{P}^n_{A/k}(M)$, $x \mapsto (1 \otimes 1) \otimes x$, is a universal differential operator of order $\leq n$; for any A-module N it induces an isomorphism

$$\mathrm{Hom}_A(\mathcal{P}^n_{A/k}(M), N) \xrightarrow{\simeq} \mathrm{Diff}^n_{A/k}(M, N). \tag{1.4.1}$$

If $A = k[\underline{t}] = k[t_1, \ldots, t_m]$ is a polynomial ring then I_A is generated as an $A \otimes_k A$-module by $t_i \otimes 1 - 1 \otimes t_i$, $i = 1, \ldots, m$. Therefore

$$\mathcal{P}^n_{k[\underline{t}]/k} = \bigoplus_{0 \leq i_1, \ldots, i_m \leq n} k[\underline{t}] \cdot \mathrm{d}^n(\underline{t}^{\underline{i}}) \ , \tag{1.4.2}$$

and this implies

Proposition 1.4.3 *If A is a finitely generated k-algebra and M is a finitely generated A-module, then $\mathcal{P}^n_{A/k}(M)$ is a finitely generated A-module.*

Proposition 1.4.4 *Suppose that the k-algebra A admits an augmentation $\eta : A \to k$, and let $J = \ker(\eta)$ be the augmentation ideal. Let M and N be A-modules which are annihilated by J^{m+1} and J^{n+1} respectively. Then*

$$\mathrm{Diff}^{m+n}_{A/k}(M, N) = \mathrm{Hom}_k(M, N).$$

Proof We have a k-module decomposition $A = k \oplus J$ induced by η. Write any $a \in A$ as $a = \lambda + x$, $\lambda \in k$, $x \in J$. Let $D \in \mathrm{Hom}_k(M, N)$. For any $a \in A$ one has

$$[D, a] = [D, \lambda] + [D, x] = [D, x] \in \mathrm{Hom}_k(M, N).$$

Choose arbitrary $a_0, \ldots, a_{m+n} \in A$ and define $D_0 := D$, $D_{i+1} := [D_i, a_i]$. If we write $a_i = \lambda_i + x_i$ as above, we also get $D_{i+1} = [D_i, x_i]$. So for $i = m + n + 1$,

$$D_{m+n+1} = \ldots \pm (x_{i_0} \ldots x_{i_j} D x_{i_{j+1}} \ldots x_{i_{m+n}}) \pm \ldots \ .$$

But either $j \geq n$ or $m+n-j-1 \geq m$. Therefore all the terms in the sum are 0 and $D_{m+n+1} = 0$. Working backwards we see that for all i, $0 \leq i \leq m+n+1$, $D_i \in \mathrm{Diff}^{m+n-i}_{A/k}(M, N)$ (cf. [EGA IV] prop. 16.8.8). □

The proposition has a noteworthy corollary:

Corollary 1.4.5 *Assume that k is a perfect field and that A is a local k-algebra. Then any short exact sequence of finite length A-modules can be split by a differential operator over A relative to k.*

Proof Let $\mathfrak{m}$ be the maximal ideal of A and let K be its residue field. Say we are given an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of finite length A-modules. Then these are $A/\mathfrak{m}^l$-modules for sufficiently large l. Since K is formally smooth over k, there exists a k-algebra lifting of K into $A/\mathfrak{m}^l$ (see [Ma] theorem 62). Let $D : M'' \to M$ be any K-linear splitting of the exact sequence. By the proposition, D is a differential operator over A relative to K (and hence relative to k). In fact, if $\mathfrak{m}^{i+1}M = 0$ and $\mathfrak{m}^{j+1}M'' = 0$, then $\mathrm{ord}_A(D) \leq i + j$. □

The next proposition is probably well known, but for lack of suitable reference we supply a proof here.

Proposition 1.4.6 *Let M and N be A-modules, let $D \in \mathrm{Diff}^n_{A/k}(M, N)$, and let $J \subset A$ be an ideal. Then for every $i \geq 0$*

$$D(J^{i+n}M) \subset J^i D(M) \subset N.$$

Proof It suffices to check the universal DO $\mathrm{d}^n : A \to \mathcal{P}^n_{A/k}$. We prove by induction on i that $\mathrm{d}^n(J^{n+i}) \subset J^i \mathcal{P}^n_{A/k}$. For $i = 0$ there is nothing to prove, so let $i \geq 1$. Choose $a_1, \ldots, a_{n+i} \in J$. Since each of its factors is in I_A the product $(1 \otimes a_1 - a_1 \otimes 1) \cdots (1 \otimes a_{n+i} - a_{n+i} \otimes 1) = 0$ in $\mathcal{P}^n_{A/k}$. Expanding this product we get

$$\mathrm{d}^n(a_1 \cdots a_{n+i}) = 1 \otimes a_1 \cdots a_{n+i} \in \sum_{j=1}^{i} J^j \mathrm{d}^n(J^{n+i-j}).$$

But by the induction hypothesis $\mathrm{d}^n(J^{n+i-j}) \subset J^{i-j}\mathcal{P}^n_{A/k}$ for $1 \leq j \leq i$. □

Suppose the ring k has characteristic p (a prime number). Let $\mathrm{F} : k \to k$ be the absolute Frobenius homomorphism, $\mathrm{F}(\lambda) = \lambda^p$. Define

$$A^{(p/k)} := k \otimes_k A \,, \tag{1.4.7}$$

where k acts on the first factor via F; thus $1 \otimes \lambda a = \lambda^p \otimes a$ in $A^{(p/k)}$ for all $\lambda \in k$ (see diagram below; cf. [Il] §2.1). Let $\mathrm{F}_{A/k} : A^{(p/k)} \to A$, $\lambda \otimes a \mapsto \lambda a^p$, be the relative Frobenius homomorphism. We make $A^{(p/k)}$ into a k-algebra via $\lambda \mapsto \lambda \otimes 1$. Hence $\mathrm{F}_{A/k}$ is a k-algebra homomorphism and its image is the k-subalgebra of A generated by $\{a^p \mid a \in A\}$. Recursively define $A^{(p^{n+1}/k)} := (A^{(p^n/k)})^{(p/k)}$. Observe that if k is a perfect field then the homomorphism $W : a \mapsto 1 \otimes a$ is a ring isomorphism $A \xrightarrow{\simeq} A^{(p/k)}$. For $k = \mathbb{F}_p$ we simply write $A^{(p)}$ instead of $A^{(p/k)}$.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{W}"] & A^{(p/k)} \arrow[r, "\mathrm{F}_{A/k}"] & A \\
k \arrow[u] \arrow[r, "\mathrm{F}"] & k \arrow[u] \arrow[ur] &
\end{tikzcd}$$

The next lemma generalizes a result of Chase (see [Ch] lemma 3.3).

Lemma 1.4.8 *Suppose k has characteristic p. Let M and N be A-modules and let $D \in \mathrm{Hom}_k(M, N)$.*

a) *If $\mathrm{ord}_A(D) \leq p^n - 1$ for some $n \geq 0$ then D is $A^{(p^n/k)}$-linear.*

b) *Assume that A is generated by r elements as an $A^{(p/k)}$-algebra. If D is $A^{(p^n/k)}$-linear for some n then $\mathrm{ord}_A(D) \leq r(p^n - 1)$.*

Proof a) Set $B := A^{(p^n/k)}$. For any $m \geq 0$ consider the homomorphism of B-bimodules $\psi^m : \mathcal{P}^m_{B/k} \to \mathcal{P}^m_{A/k}$ induced by the k-algebra homomorphism $B \to A$ (the iteration of the relative Frobenius). If ψ^m factors through $\mathcal{P}^0_{B/k} = B$ then every $D \in \mathrm{Diff}^m_{A/k}(M, N)$ has $\mathrm{ord}_B(D) = 0$, i.e. it is B-linear.

Set $I_A := \ker(A \otimes_k A \to A)$ and $I_B := \ker(B \otimes_k B \to B)$. Now I_B is generated as a left B-module by $\{b \otimes 1 - 1 \otimes b \mid b \in B\}$, and hence by elements of the form $(1 \otimes a) \otimes 1 - 1 \otimes (1 \otimes a)$, $a \in A$ (write $b = \sum \lambda_i \otimes a_i$ and factor out $\lambda_i \in k$). We get

$$\psi^m((1 \otimes a) \otimes 1 - 1 \otimes (1 \otimes a)) = a^{p^n} \otimes 1 - 1 \otimes a^{p^n} = (a \otimes 1 - 1 \otimes a)^{p^n}$$

so $\psi^m(I_B) \subset I_A^{p^n} \mathcal{P}^m_{A/k}$ (ideal power). Taking $m = p^n - 1$ we have $I_A^{p^n} \mathcal{P}^m_{A/k} = 0$ and ψ^m factors through $\mathcal{P}^0_{B/k}$.

b) Say A is generated as an $A^{(p/k)}$-algebra by $a_1, \ldots, a_r$. Then these elements also generate A as an $A^{(p^n/k)}$-algebra for all $n \geq 0$. Choose $n \geq 1$ and set

$l := r(p^n - 1)$, $B := A^{(p^n/k)}$. The ideal $J := \ker(A \otimes_B A \to A)$ is generated as a left A-module by $\{a_i \otimes 1 - 1 \otimes a_i\}_{i=1}^r$. Since

$$(a_i \otimes 1 - 1 \otimes a_i)^{p^n} = a_i^{p^n} \otimes 1 - 1 \otimes a_i^{p^n} = 0$$

we get $J^{l+1} = 0$. Therefore $\mathcal{P}^l_{A/B} = A \otimes_B A$ and

$$\begin{aligned} \mathrm{Hom}_B(M, N) &\cong \mathrm{Hom}_A(A \otimes_B M, N) \\ &\cong \mathrm{Hom}_A(\mathcal{P}^l_{A/B}(M), N) \\ &\cong \mathrm{Diff}^l_{A/B}(M, N). \end{aligned}$$

$\square$

As an immediate consequence of the lemma we obtain:

Theorem 1.4.9 *Assume that k has characteristic p and that the relative Frobenius homomorphism $\mathrm{F}_{A/k} : A^{(p/k)} \to A$ is finite. Then for any pair of A-modules M and N*

$$\mathrm{Diff}_{A/k}(M, N) = \bigcup_{n=0}^{\infty} \mathrm{Hom}_{A^{(p^n/k)}}(M, N).$$

This is the so-called p-filtration on $\mathrm{Diff}_{A/k}(M, N)$, cf. [Wo], proof of theorem 1.

1.5 Differential Properties of Semi-Topological Rings

Throughout this section k is a commutative semi-topological ring and A is a commutative semi-topological k-algebra. Recall that a derivation of degree i of a graded ring $\Omega^* = \bigoplus_{n=0}^{\infty} \Omega^n$ is an additive homomorphism $\mathrm{d} : \Omega^* \to \Omega^*$ of degree i satisfying $\mathrm{d}(\alpha\beta) = \mathrm{d}(\alpha)\beta + (-1)^{i|\alpha|}\alpha\mathrm{d}(\beta)$ for all $\alpha, \beta \in \Omega^*$ with α homogeneous of degree $|\alpha|$.

Definition 1.5.1 *A differential graded (DG) semi-topological k-algebra is a graded k-algebra $\Omega^* = \bigoplus_{n=0}^{\infty} \Omega^n$ (where $k \to \Omega^0$), together with:*

i) *A linear topology on each homogeneous component Ω^n, such that Ω^* with the direct sum topology is a semi-topological k-algebra.*

ii) *A continuous k-derivation of degree 1 of Ω^* satisfying $\mathrm{d}^2 = 0$.*

We denote by $\mathsf{STDGA}(k)$ the category of semi-topological differential graded algebras over k, with the obvious morphisms.

Let $\mathrm{T}_k^* A = \bigoplus_{n=0}^{\infty} \mathrm{T}_k^n A$ be the tensor algebra of A considered as a k-module. Put on $\mathrm{T}_k^n A = A \otimes_k \cdots \otimes_k A$ (n times) the tensor product topology and put on $\mathrm{T}_k^* A$ the direct sum topology. The associativity of the tensor product topology shows that $\mathrm{T}_k^* A$ is a semi-topological k-algebra. Therefore $A \otimes_k \mathrm{T}_k^* A$ is a semi-topological A-algebra (the multiplication is $(a_0 \otimes a_1 \otimes \cdots \otimes a_m)(b_0 \otimes b_1 \otimes \cdots \otimes b_n) = (a_0 b_0 \otimes a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n)$). Define a continuous k-linear homomorphism of degree 1, $\tilde{\mathrm{d}} : a_0 \otimes \cdots \otimes a_n \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_n$.

Now let $\Omega_{A/k}^* = \bigwedge_A^* \Omega_{A/k}^1$ be the algebra of differential forms over A relative to k, also known as the de Rham complex, and let d be the exterior derivative. The map $A \otimes_k \mathrm{T}_k^* A \to \Omega_{A/k}^*$ given by $a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto a_0 \mathrm{d}(a_1) \wedge \cdots \wedge \mathrm{d}(a_n)$ is a surjective A-algebra homomorphism, sending $\tilde{\mathrm{d}}$ to d. Put on $\Omega_{A/k}^*$ the quotient topology. Recall the notation λ_a, ρ_a used in §1.2 for left and right multiplication by $a \in A$.

Lemma 1.5.2 *$\Omega_{A/k}^*$ is a differential graded semi-topological k-algebra. The homomorphism $A \to \Omega_{A/k}^0$ is an isomorphism of semi-topological k-algebras. The topology on $\Omega_{A/k}^1$ is the finest linear topology such that for every $a \in A$ the homomorphisms $\lambda_a \circ \mathrm{d}, \rho_{\mathrm{d}(a)} : A \to \Omega_{A/k}^1$ are continuous.*

Proof According to prop. 1.2.9, $\Omega_{A/k}^*$ is a semi-topological k-algebra. Since $\Omega_{A/k}^n$ is the quotient of $A \otimes_k \mathrm{T}_k^n A$ it follows that $\Omega_{A/k}^*$ is the direct sum of the $\Omega_{A/k}^n$ in $\mathsf{STMod}(A)$. The continuity of d is due to the continuity of $\tilde{\mathrm{d}}$.

Now let N be the module $\Omega_{A/k}^1$ with any linear topology such that the homomorphisms $\lambda_a \circ \mathrm{d}, \rho_{\mathrm{d}(a)} : A \to \Omega_{A/k}^1$ are continuous. These homomorphisms factor through $A \otimes_k A$ as $a' \mapsto a \otimes a'$ and $a' \mapsto a' \otimes a$. By lemma 1.2.12 the homomorphism $A \otimes_k A \to N$ is continuous relative to the tensor product topology, so the identity map $\Omega_{A/k}^1 \xrightarrow{\simeq} N$ is continuous. □

Definition 1.5.3 *The separated algebra of differentials of A relative to k is the semi-topological differential graded k-algebra*

$$\Omega_{A/k}^{*,\mathrm{sep}} := (\Omega_{A/k}^*)^{\mathrm{sep}} = \bigoplus_{n=0}^{\infty} (\Omega_{A/k}^n)^{\mathrm{sep}}.$$

(We are using the fact that the functor $M \mapsto M^{\mathrm{sep}}$ commutes with infinite direct sums, which is a consequence of prop. 1.1.5 b).)

Proposition 1.5.4 *The continuous derivation* $\mathrm{d} : A \to \Omega^{1,\mathrm{sep}}_{A/k}$ *has the following universal property: given any separated semi-topological* A*-module* M*, the map*

$$\mathrm{Hom}^{\mathrm{cont}}_A(\Omega^{1,\mathrm{sep}}_{A/k}, M) \to \mathrm{Der}^{\mathrm{cont}}_k(A, M)$$

induced by d *is bijective.*

Proof The injectivity is true because $\Omega^{1,\mathrm{sep}}_{A/k}$ is generated as an A-module by $\mathrm{d}(a)$, $a \in A$. Given a continuous derivation $D : A \to M$, there is a corresponding A-linear homomorphism $\phi : \Omega^1_{A/k} \to M$. For any $a \in A$, $\phi \circ \lambda_a \circ \mathrm{d} = \lambda_a \circ D : A \to M$ and $\phi \circ \rho_{\mathrm{d}(a)} = \rho_{D(a)} : A \to M$ are continuous. By lemmas 1.5.2 and 1.1.1 it follows that ϕ is continuous. But M is separated, so ϕ factors through $\Omega^{1,\mathrm{sep}}_{A/k}$. □

By universality, $A \mapsto \Omega^{*,\mathrm{sep}}_{A/k}$ is a functor $\mathsf{STComAlg}(k) \to \mathsf{STDGA}(k)$. Given a homomorphism $f : A \to B$ in $\mathsf{STComAlg}(k)$ we use the same name for the induced DGA homomorphism.

For $n \geq 0$ the tensor product topology on $A \otimes_k A$ induces a topology on $\mathcal{P}^n_{A/k} = A \otimes_k A / I_A^{n+1}$. Set $\mathcal{P}^{n,\mathrm{sep}}_{A/k} := (\mathcal{P}^n_{A/k})^{\mathrm{sep}}$. This is a semi-topological A-algebra. Given a semi-topological A-module M set $\mathcal{P}^{n,\mathrm{sep}}_{A/k}(M) := (\mathcal{P}^n_{A/k} \otimes_A M)^{\mathrm{sep}}$.

Proposition 1.5.5 *Let* M *be a semi-topological* A*-module and let* $n \geq 0$*. The continuous differential operator of order* $\leq n$*,* $\mathrm{d}^n_M : M \to \mathcal{P}^{n,\mathrm{sep}}_{A/k}(M)$ *has the following universal property: given any separated semi-topological* A*-module* N *the map*

$$\mathrm{Hom}^{\mathrm{cont}}_A(\mathcal{P}^{n,\mathrm{sep}}_{A/k}(M), N) \to \mathrm{Diff}^{n,\mathrm{cont}}_{A/k}(M, N)$$

induced by d^n_M *is bijective.*

Proof This is a consequence of the universal property of $\mathcal{P}^n_{A/k}(M)$ and lemma 1.2.12 (cf. previous proposition). □

For $n = 1$ we get an isomorphism of semi-topological A-algebras

$$\mathcal{P}^{1,\mathrm{sep}}_{A/k} \cong A^{\mathrm{sep}} \oplus \Omega^{1,\mathrm{sep}}_{A/k} \tag{1.5.6}$$

the latter being a quotient of $\Omega^{*,\mathrm{sep}}_{A/k}$. The formula is $\mathrm{d}^1(a) = 1 \otimes a \mapsto a + \mathrm{d}(a)$.

Definition 1.5.7 *Let* $u : A \to B$ *be a homomorphism in* $\mathsf{STComAlg}(k)$*. We say that* B *is topologically smooth (resp. topologically étale) over* A *relative to* k *(or equivalently,* u *is topologically smooth relative to* k*, or* u *is smooth in* $\mathsf{STComAlg}(k)$*, etc.) if, given any commutative diagram in* $\mathsf{STComAlg}(A)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & C \arrow[d, "\pi"] \\ B \arrow[r, "g"] \arrow[ur, dotted, "\tilde{g}"] & C_0 \end{tikzcd}$$

with C and C_0 separated and π a surjection such that $\ker(\pi)^2 = 0$*, the homomorphism $g : B \to C_0$ can be lifted (resp. lifted uniquely) to a homomorphism $\tilde{g} : B \to C$ in* $\mathsf{STComAlg}(A)$ *whenever it can be lifted to a homomorphism $\tilde{g} : B \to C$ in* $\mathsf{STComAlg}(k)$*.*

When these algebras have discrete topologies this definition coincides with that of formally smooth and formally étale algebras relative to k (cf. [Ma] §30.A). Moreover, we have:

Proposition 1.5.8 *If B has the fine A-module topology and is formally smooth (resp. formally étale) over A relative to k for the discrete topologies, then it is also topologically smooth (resp. topologically étale).*

Proof Any A-algebra homomorphism $\tilde{g} : B \to C$ is automatically continuous (see prop. 1.2.4). □

Proposition 1.5.9 a) *(Transitivity of topological smoothness) Let $A \xrightarrow{u} B \xrightarrow{v} C$ be homomorphisms in* $\mathsf{STComAlg}(k)$*. If u and v are smooth (resp. étale) in* $\mathsf{STComAlg}(k)$*, then so is $v \circ u : A \to C$.*

b) *(Base change) Let $A \to B$ and $A \to A'$ be homomorphisms in* $\mathsf{STComAlg}(k)$*. If $A \to A'$ is smooth (resp. étale) in* $\mathsf{STComAlg}(k)$*, then so is $B \to B \otimes_A A'$.*

Proof Just like the proofs for formally smooth and formally étale homomorphisms (see [Ma] §28.E - 28.G) plus, in part b), the universal property of base change (cor. 1.2.18). □

Lemma 1.5.10 *Let $u : A \to B$ be a homomorphism in* $\mathsf{STComAlg}(k)$*. Then u is smooth (resp. étale) in* $\mathsf{STComAlg}(k)$ *iff the conditions of definition 1.5.7 are satisfied for all diagrams with* $\ker(\pi)$ *nilpotent (not necessarily of square 0).*

Proof One direction is trivial. For the other direction, suppose that u is smooth (resp. étale) and that we are given a diagram of continuous homomorphisms with $N^{n+1} = 0$, where $N = \ker(\pi)$. For $1 \leq i \leq n$ define $C_i := (C/N^{i+1})^{\mathsf{sep}}$ (so $C_n = C$). The intermediate diagrams involving $\pi_i : C_{i+1} \to C_i$ have $\ker(\pi_i)^2 = 0$, so g can be lifted (resp. uniquely lifted) step by step. $\square$

The following theorem is an adaptation of well known results to the context of semi-topological rings.

Theorem 1.5.11 *Given a homomorphism $A \to B$ in $\mathsf{STComAlg}(k)$ the following are equivalent:*

i) *B is topologically smooth (resp. topologically étale) over A relative to k.*

ii) *For every separated semi-topological B-module N the natural map $\mathrm{Der}_k^{\mathrm{cont}}(B, N) \to \mathrm{Der}_k^{\mathrm{cont}}(A, N)$ is surjective (resp. bijective).*

iii) *The natural homomorphism $(B \otimes_A \Omega^{1,\mathsf{sep}}_{A/k})^{\mathsf{sep}} \to \Omega^{1,\mathsf{sep}}_{B/k}$ in $\mathsf{STMod}(B)$ has a left inverse (resp. is an isomorphism).*

iv) *For every separated semi-topological B-module N, for every semi-topological A-module M and for every $n \geq 0$ the natural map $\mathrm{Diff}^{n,\mathrm{cont}}_{B/k}(B \otimes_A M, N) \to \mathrm{Diff}^{n,\mathrm{cont}}_{A/k}(M, N)$ is surjective (resp. bijective).*

v) *For every $n \geq 0$ the natural homomorphism $(B \otimes_A \mathcal{P}^{n,\mathsf{sep}}_{A/k})^{\mathsf{sep}} \to \mathcal{P}^{n,\mathsf{sep}}_{B/k}$ in $\mathsf{STMod}(B)$ has a left inverse (resp. is an isomorphism).*

Proof
ii) $\Rightarrow$ i): Say we are given the data of definition 1.5.7 and a continuous k-algebra lifting $h : B \to C$ of g. Then $\delta := f - h : A \to N = \ker(\pi)$ is a continuous k-derivation. Let $\tilde{\delta} : B \to N$ be an extension of δ. Define $\tilde{g} := h + \tilde{\delta} : B \to C$; this is a continuous A-algebra lifting of g. The uniqueness of $\tilde{g}$ comes from the uniqueness of $\tilde{\delta}$.

iii) $\Rightarrow$ ii): We first observe that $(B \otimes_A \Omega^{1,\mathsf{sep}}_{A/k})^{\mathsf{sep}}$ represents $\mathrm{Der}_k^{\mathrm{cont}}(A, N)$ for separated semi-topological B-modules N. So an isomorphism in iii) implies a bijection in ii), and a left inverse allows the extension of any continuous k-derivation $\delta : A \to N$ to a derivation $\tilde{\delta} : B \to N$.

iv) $\Rightarrow$ ii): Trivial, take $n = 1$ and $M = A$, and make use of the canonical splitting of $\mathrm{Der}^{\mathrm{cont}} \to \mathrm{Diff}^{1,\mathrm{cont}}$.

v) $\Rightarrow$ iii): Trivial, take $n = 1$ and use the splitting (1.5.6).

v) $\Rightarrow$ iv): Use prop. 1.5.5 and formula (1.2.13).

i) $\Rightarrow$ v): See [Sw], proof of theorem 13.12. In [Sw] the assumption is that B is a finite separable A-algebra, and there is no topology involved. However the same arguments can be applied to our more general and topologized setup, because the bimodules $\mathcal{P}^{n,\text{sep}}_{-/-}$ have the appropriate universal properties. $\square$

Condition iii) implies that when $A \to B$ is smooth in $\mathsf{STComAlg}(k)$, the canonical sequence

$$0 \to (B \otimes_A \Omega^{1,\text{sep}}_{A/k})^{\text{sep}} \to \Omega^{1,\text{sep}}_{B/k} \to \Omega^{1,\text{sep}}_{B/A} \to 0 \tag{1.5.12}$$

is split-exact in $\mathsf{STMod}(B)$.

Corollary 1.5.13 *If B is topologically étale over A relative to k then the natural homomorphism of semi-topological B-algebras $(B \otimes_A \Omega^{*,\text{sep}}_{A/k})^{\text{sep}} \to \Omega^{*,\text{sep}}_{B/k}$ is an isomorphism.*

Proof By the theorem we have an isomorphism in degrees ≤ 1. The homomorphism is surjective because $\Omega^{*,\text{sep}}_{B/k}$ is generated as a B^{sep}-algebra by $\Omega^{1,\text{sep}}_{B/k}$. Since $\Omega^{*}_{B/k}$ is an exterior algebra the B-linear homomorphism $\Omega^{1}_{B/k} \twoheadrightarrow \Omega^{1,\text{sep}}_{B/k} \xrightarrow{\simeq} (B \otimes_A \Omega^{1,\text{sep}}_{A/k})^{\text{sep}}$ induces a graded B-algebra homomorphism $\Omega^{*}_{B/k} \to (B \otimes_A \Omega^{*,\text{sep}}_{A/k})^{\text{sep}}$. By lemma 1.2.12 this homomorphism is continuous, so it passes to $\Omega^{*,\text{sep}}_{B/k}$, providing a continuous left inverse to the natural homomorphism. $\square$

Corollary 1.5.14 *(Cancellation) Let $A \xrightarrow{u} B \xrightarrow{v} C$ be homomorphisms in $\mathsf{STComAlg}(k)$. If u and $v \circ u$ are étale then so is v.*

Proof Use condition iii) of the theorem and the fact that $C \otimes_B B \cong C$. $\square$

If B happens to be separated and $\Omega^{*,\text{sep}}_{A/k}$ happens to be a free ST A-module, we have the simple formula:

$$(B \otimes_A \Omega^{*,\text{sep}}_{A/k})^{\text{sep}} \cong B \otimes_A \Omega^{*,\text{sep}}_{A/k} \tag{1.5.15}$$

and the same for $\mathcal{P}^{n,\text{sep}}_{A/k}$.

Definition 1.5.16 *Let A be a ST k-algebra. If $\Omega^{1,\text{sep}}_{A/k}$ is a finitely generated A-module with the fine A-module topology, we say that A is differentially of finite type over k. If moreover $\Omega^{1,\text{sep}}_{A/k}$ is free over A, its rank is called the differential degree of A over k.*

Here are a few examples of topological smoothness.

Example 1.5.17 Let $A \in \mathsf{STComAlg}(k)$ and let $\underline{t} = (t_1, \ldots, t_n)$ be a sequence of indeterminates. Put on $A[\underline{t}]$ and on $A[\underline{t}, \underline{t}^{-1}]$ the fine A-module topologies. Then $A[\underline{t}] \to A[\underline{t}, \underline{t}^{-1}]$ is étale and $A \to A[\underline{t}]$ is smooth in $\mathsf{STComAlg}(k)$. In fact,

$$\Omega^{1,\mathrm{sep}}_{A[\underline{t}]/k} \cong \left(A[\underline{t}] \otimes_A \Omega^{1,\mathrm{sep}}_{A/k}\right)^{\mathrm{sep}} \oplus \left(\bigoplus_{i=1}^{n} A[\underline{t}]^{\mathrm{sep}} \mathrm{d}t_i\right).$$

(Of course k is unimportant here.)

A useful result is:

Theorem 1.5.18 *Let $A \in \mathsf{STComAlg}(k)$. Assume that A is noetherian and differentially of finite type over k. Given an ideal $I \subset A$, put on $\hat{A} = \lim_{\leftarrow n} A/I^n$ the topology of lemma 1.2.19. Then $A \to \hat{A}$ is étale in* $\mathsf{STComAlg}(k)$.

Proof Let $\hat{N}$ be a separated ST $\hat{A}$-module and let $\delta : A \to \hat{N}$ be a continuous k-derivation. Then δ factors through some finitely generated A-module M which we may assume has the fine A-module topology.

By prop. 1.4.6, for every $n \geq 0$ we get a derivation $\delta_n : A/I^{n+1} \to M/I^n M$. Since the projection $A \twoheadrightarrow A/I^{n+1}$ is strict, δ_n is continuous. Passing to the inverse limit there is a continuous derivation $\hat{\delta} : \hat{A} \to \hat{M} \cong \hat{A} \otimes_A M \to \hat{N}$ (see cor. 1.2.21). Since $\hat{N}$ is separated and $A \subset \hat{A}$ is dense (prop. 1.1.8), this $\hat{\delta}$ is unique. □

Corollary 1.5.19 *Let A be as in the theorem and let $\underline{t} = (t_1, \ldots, t_n)$ be a sequence of indeterminates. Then $A[\underline{t}] \to A[[\underline{t}]]$ and $A[\underline{t}] \to A((\underline{t}))$ are étale in* $\mathsf{STComAlg}(k)$.

Proof The ST k-algebra $A[\underline{t}]$ also satisfies the assumptions of the theorem, so $A[\underline{t}] \to A[[\underline{t}]]$ is étale. Now take $n = 1$. By prop. 1.5.8 and lemma 1.3.4, $A[[t]] \to A((t))$ is étale, so by transitivity (prop. 1.5.9) $A[t] \to A((t))$ is also étale in $\mathsf{STComAlg}(k)$.

Therefore for every $1 \leq i \leq n$ the homomorphism

$$A((t_{i+1}, \ldots, t_n))[t_i] \to A((t_{i+1}, \ldots, t_n))((t_i)) = A((t_i, \ldots, t_n))$$

is étale. By base change

$$\begin{aligned} &A((t_{i+1}, \ldots, t_n))[t_1, \ldots, t_i] \\ &\qquad \to A((t_i, \ldots, t_n)) \otimes_{A((t_{i+1}, \ldots, t_n))[t_i]} A((t_{i+1}, \ldots, t_n))[t_1, \ldots, t_i] \\ &\qquad = A((t_i, \ldots, t_n))[t_1, \ldots, t_{i-1}] \end{aligned}$$

is étale, and finally by transitivity $A[\underline{t}] \to A((\underline{t}))$ is also étale in $\mathsf{STComAlg}(k)$. $\square$

A fact to be used later is:

Proposition 1.5.20 *Let $u : A \to B$ and $f : B \to C$ be homomorphisms in $\mathsf{STComAlg}(k)$, with u étale. Suppose C is separated, and consider it as a ST B-module via f. Let $g : B \to C$ be a continuous DO over B relative to k, s.t. $g \circ u : A \to C$ is a ring homomorphism. Then g is also a ring homomorphism.*

Proof For any $a, b \in B$ set $D_a(b) := g(ab) - g(a)g(b)$. We must show that $D_a(b) = 0$. Now for any $a \in B$, $D_a : B \to C$ is a continuous DO over B. First fix some $a \in A$. Then $D_{u(a)} \circ u = 0$, and by the uniqueness of extension of DOs, $D_{u(a)} = 0$. Next fix some $b \in B$. By symmetry $D_b \circ u = 0$, and again by invoking uniqueness we get $D_b = 0$. $\square$

2 Topological Local Fields

2.1 Definitions and Basic Properties

In this section we define topological local fields and examine their structure. The definition below is due to Parshin (see [Pa1] p. 697, [Pa3] §1 def. 1 and also [Ka] part II §3.1).

Definition 2.1.1 *An n-dimensional local field is a field K, together with complete discrete valuation rings (DVRs) $\mathcal{O}_1, \ldots, \mathcal{O}_n$, such that:*

i) *For $i = 1, \ldots, n-1$, the residue field of $\mathcal{O}_i$ equals the fraction field of $\mathcal{O}_{i+1}$.*

ii) *The fraction field of $\mathcal{O}_1$ equals K.*

The fraction field (resp. residue field) of $\mathcal{O}_i$ is denoted by κ_{i-1} (resp. κ_i). The number n is called the dimension of K and is denoted by $\dim(K)$. For $1 \leq i \leq n$ the fibred product $\mathcal{O}_1 \times_{\kappa_1} \cdots \times_{\kappa_{i-1}} \mathcal{O}_i$ is the largest subring of K on which the projection to κ_i is defined. Let $\mathcal{O} := \mathcal{O}_1 \times_{\kappa_1} \cdots \times_{\kappa_{n-1}} \mathcal{O}_n$. When dealing with a few local fields $K, L, \ldots$ we will write $\mathcal{O}_1(K), \mathcal{O}_1(L), \ldots$ etc.

Remark 2.1.2 The ring $\mathcal{O}$, being a valuation ring, is integrally closed, but unless $n = 1$ it is not noetherian (see [CA] ch. VI §1.4 cor. 1 and §3.6 prop. 9).

Example 2.1.3 Let F be a field and let $K := F((\underline{t}))$ be the field of Laurent series over F in the sequence of indeterminates $\underline{t} = (t_1, \ldots, t_n)$ (see §1.3). Then K is an n-dimensional local field with

$$\mathcal{O}_i = F((t_{i+1}, \ldots, t_n))[[t_i]]$$

$$\kappa_i = F((t_{i+1}, \ldots, t_n)).$$

By the Cohen structure theorem ([Ma] thm. 60), this is the general situation in the equal characteristics case, i.e. when char K = char $\kappa_1 = \ldots =$ char κ_n (or equivalently, when $\mathcal{O}$ contains a field).

Example 2.1.4 Let p be a prime number and consider the complete DVR with p-adic valuation $A := \lim_{\leftarrow i}(\mathbb{Z}/(p^i))((t))$. Let K be the fraction field of A. Then K is a 2-dimensional local field with $\mathcal{O}_1(K) = A$ and $\kappa_1(K) = \mathbb{F}_p((t))$ (where $\mathbb{F}_p := \mathbb{Z}/(p)$).

Definition 2.1.5 *Let K be an n-dimensional local field. A sequence $\underline{a} = (a_1, \ldots, a_n)$ of elements of $\mathcal{O}$ is called a system of parameters (resp. a regular system of parameters) in K if for all i the image of a_i in $\mathcal{O}_i$ is a parameter (resp. a regular parameter) in this DVR. A subsequence $(a_1, \ldots, a_j)$ of a system of parameters $(a_1, \ldots, a_n)$ is called an initial system of parameters of length j.*

Choose a regular system of parameters $\underline{a}$ in K. Every $a \in K^\times$ (units of K) can be written uniquely as

$$a = u\underline{a}^{\underline{i}} := u a_1^{i_1} \cdots a_n^{i_n}$$

with $\underline{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ and $u \in \mathcal{O}^\times$. Thus $\underline{a}$ gives rise to an isomorphism of ordered groups $K^\times/\mathcal{O}^\times \cong (\mathbb{Z}^n, \text{lex})$.

Let L/K be a finite extension of fields. Then any structure of n-dimensional local field on K extends uniquely to one on L. Conversely, any n-dimensional local field structure on L restricts to one on K. These statements follow from repeated applications of [CA] ch. VI §8.5 cor. 2, and §8.1 lemma 2 (cf. [Lo] §1.2).

Definition 2.1.6 *A finite homomorphism of local fields between the n-dimensional local fields K and L is a ring homomorphism $f : K \to L$ such that $[L : K] < \infty$ and f respects the local field structures.*

Let $f : K \to L$ be a finite homomorphism of n-dimensional local fields. Then for every $1 \leq i \leq n$, $\mathcal{O}_i(L)$ is a free $\mathcal{O}_i(K)$-module of finite rank. One has the following identity:

$$[L : K] = [\kappa_n(L) : \kappa_n(K)]\, e(L/K) \tag{2.1.7}$$

where

$$e(L/K) := [(L^\times/\mathcal{O}(L)^\times) : (K^\times/\mathcal{O}(K)^\times)]$$

is the ramification index.

Definition 2.1.8 *Let K and L be local fields of dimensions m and n, respectively ($m \leq n$). A homomorphism of local fields $f : K \to L$ is a ring homomorphism such that $f(K) \subset \mathcal{O}_1(L) \times_{\kappa_1(L)} \cdots \times_{\kappa_{n-m-1}(L)} \mathcal{O}_{n-m}(L)$ and such that the induced homomorphism $K \to \kappa_{n-m}(L)$ in a finite homomorphism of local fields. Define* $\dim(f) = \dim(L/K) := n - m$.

Example 2.1.9 Let F be any field and let t be an indeterminate. Then the inclusion $F \to F((t))$ is a homomorphism of dimension 1. Let K be the local field of example 2.1.4. Then the natural homomorphism $\mathbb{Q}_p \to K$ is *not* a homomorphism of local fields, because $\mathbb{Q}_p \not\subset A = \mathcal{O}_1(K)$.

We shall only be concerned with local fields of equal characteristics. Fix for the remainder of this section a perfect field k with the discrete topology.

Definition 2.1.10 *A topological local field (TLF) over k is a field K, together with the following structures on it:*

i) *A structure of an n-dimensional local field, for some $n \geq 0$.*

ii) *A ring homomorphism $k \to \mathcal{O}(K)$.*

iii) *A structure of a semi-topological ring.*

The two conditions below must be satisfied:

a) *If $n = 0$ the topology on K is discrete and* $\operatorname{rank}_K \Omega^1_{K/k} < \infty$.

b) *If $n > 0$, then there is a topological local field over k of dimension 0, F, and an isomorphism $K \cong F((t_1, \ldots, t_n))$ which respects the structures i),ii) and iii) above. Here $F((t_1, \ldots, t_n))$ has the topology of definition 1.3.7 and the local field structure of example 2.1.3.*

A morphism of topological local fields $f : K \to L$ is a continuous k-algebra homomorphism which is also a homomorphism of local fields.

An isomorphism $K \cong F((t_1, \ldots, t_n))$ as in condition b) is called a para - metrization of K. The condition $\operatorname{rank}_F \Omega^1_{F/k} < \infty$ (finiteness of differential degree) is equivalent to $\operatorname{tr.deg}_k F < \infty$ if $\operatorname{char} k = 0$. If $\operatorname{char} k = p > 0$ then it is equivalent to $[F : F^{(p)}] < \infty$. Note that given a parametrization $K \cong F((\underline{t}))$ the topology on K is $F[\underline{t}]$-linear (see prop. 1.2.23). A finite morphism of TLFs is a morphism $K \to L$ s.t. $[L : K] < \infty$. Denote the category of topological local fields by $\mathsf{TLF}(k)$.

Example 2.1.11 Let K be a TLF and let $\underline{t} = (t_1, \ldots, t_n)$ be a sequence of indeterminates. Then $K((\underline{t}))$ is a topological local field (of dimension $\dim(K) + n$) and $K \to K((\underline{t}))$ is a morphism in $\mathsf{TLF}(k)$.

Let K be a topological local field over k. Put topologies on $\mathcal{O}_i$ and κ_i such that the canonical homomorphisms $\mathcal{O}_i \hookrightarrow \kappa_{i-1}$ and $\mathcal{O}_i \twoheadrightarrow \kappa_i$ become strict. Thus if $K \cong F((t_1, \ldots, t_n))$ is a parametrization, then $\mathcal{O}_i \cong F((t_{i+1}, \ldots, t_n))[[t_i]]$ and $\kappa_i \cong F((t_{i+1}, \ldots, t_n))$ are homeomorphisms. By prop. 1.3.5 all these ST k-algebras are separated and complete.

Assume that $\operatorname{char} k = p$. Let K be an n-dimensional local field over k, and let $K \cong F((t_1, \ldots, t_n))$ be a parametrization. Let $d := \operatorname{rank}_F \Omega^1_{F/k}$. The field $K^{(p/k)}$ maps isomorphically to the subfield $F^{(p/k)}((\underline{t}^p)) := F^{(p/k)}((t_1^p, \ldots, t_n^p)) \subset K$. (See §1.4 for the definition of $K^{(p/k)}$.) Choose a p-basis $\underline{u} = (u_1, \ldots, u_d)$ for F. Looking at the definition of the topology on $F((\underline{t}))$ we see that

$$\bigoplus_{0 \leq i_1, \ldots, i_d, j_1, \ldots, j_n < p} F^{(p/k)}((\underline{t}^p))\, \underline{u}^{\underline{i}} \underline{t}^{\underline{j}} \xrightarrow{\simeq} F((\underline{t})) \tag{2.1.12}$$

is a homeomorphism. If we let $K^{(p/k)} \cong F^{(p/k)}((\underline{t}^p))$ be a parametrization, then the relative Frobenius $\mathrm{F}_{K/k} : K^{(p/k)} \to K$ becomes a finite morphism of TLFs. Iteration gives:

Proposition 2.1.13 *Let* $\operatorname{char} k = p$ *and let* $K \in \mathsf{TLF}(k)$. *Then for any* $m \geq 0$, *the field* $K^{(p^m/k)}$ *admits a unique structure of TLF over* k *s.t. the iterated relative Frobenius map* $K^{(p^m/k)} \to K$ *is a finite morphism in* $\mathsf{TLF}(k)$, *and s.t.* K *has the fine* $K^{(p^m/k)}$*-module topology.*

Theorem 2.1.14 *Let* $\operatorname{char} k = p$ *and let* $K \in \mathsf{TLF}(k)$. *Suppose* M *and* N *are semi-topological* K*-modules, with* M *having the fine* K*-module topology. Then any differential operator over* K, $D : M \to N$ *is continuous.*

Proof According to thm. 1.4.9 applied to $A = K$, D is $K^{(p^m)}$-linear for some $m \geq 0$. Since $k^{(p^m)} = k$, D is even $K^{(p^m/k)}$-linear. Now by prop. 1.2.9 c), M has the fine $K^{(p^m/k)}$-module topology, so D is continuous. □

Corollary 2.1.15 *If* $\operatorname{char} k = p$ *then* $\Omega^{*,\mathrm{sep}}_{K/k}$ *has the fine* K*-module topology, the differential* d *is* $K^{(p/k)}$*-linear, and*

$$\Omega^{*,\mathrm{sep}}_{K/k} \cong \Omega^*_{K/K^{(p/k)}} \cong \Omega^*_{K/k} \cong \Omega^*_K$$

(the last denoting absolute differentials).

Proof Put on $\Omega^*_{K/K^{(p/k)}}$ the fine $K^{(p/k)}$-module topology. It is a separated ST K-module and d is continuous, so $\Omega^*_{K/K^{(p/k)}} = \Omega^{*,\mathrm{sep}}_{K/K^{(p/k)}}$. But by lemma 1.4.8 a) every derivation of K is $K^{(p/k)}$-linear. □

Corollary 2.1.16 *Suppose* $\operatorname{char} k = p$. *Let* $K, L \in \mathsf{TLF}(k)$ *and let* $K \to L$ *be a continuous* k*-algebra homomorphism. Then* L *is topologically smooth over* K *relative to* k *iff* L *is a separable* K*-algebra.*

(See [Ma] §27.D for a definition of separability.)

Proof By the previous corollary we can erase the superscript "sep" in condition iii) of thm. 1.5.11. Now use [Ma] theorems 66 and 62. □

The next theorem is the key to the structure of topological local fields.

Theorem 2.1.17 *Let* L *be an* n*-dimensional local field, let* $K \in \mathsf{TLF}(k)$ *be* n*-dimensional, and let* $K \to L$ *be a finite homomorphism of local fields. Put on* L *the fine* K*-module topology. Let* $A \in \mathsf{STComAlg}(k)$ *be noetherian and differentially of finite type over* k*, and let* $\underline{s} = (s_1, \ldots, s_m)$ *be a sequence of indeterminates* ($m \leq n$)*. Suppose* $g : A[\underline{s}] \to L$ *is a homomorphism in* $\mathsf{STComAlg}(k)$ *such that* $g(A) \in \mathcal{O}_1(L) \times_{\kappa_1(L)} \cdots \times_{\kappa_{m-1}(L)} \mathcal{O}_m(L)$ *and* $(g(s_1), \ldots, g(s_m))$ *is an initial system of parameters in* L*. Then* g *has a unique extension to a homomorphism* $\hat{g} : A((\underline{s})) \to L$ *in* $\mathsf{STComAlg}(k)$.

Proof We use induction on m which we assume is at least 1. Choose a parametrization $K \cong F((\underline{t})) = F((t_1, \ldots, t_n))$, and a regular parameter r_1 of the DVR $\mathcal{O}_1(L)$. Define sequences of indeterminates $\underline{t}' := (t_2, \ldots, t_n)$ and $\underline{s}' := (s_2, \ldots, s_m)$ and ST k-algebras $B := A[\underline{s}]$ and $\hat{B} := A((\underline{s}'))[[s_1]]$. For every $i \geq 0$ set $B_i := B/(s_1^{i+1})$ and $\hat{B}_i := \hat{B}/(s_1^{i+1})$. Put on $\mathcal{O}_1(L)/(r_1^{i+1})$ the fine $F((\underline{t}'))$-module topology. Since $\mathcal{O}_1(L)$ has the fine $F((\underline{t}'))[[t_1]]$-module topology, there is an isomorphism of ST k-algebras $\mathcal{O}_1(L) \cong \lim_{\leftarrow i} \mathcal{O}_1(L)/(r_1^{i+1})$ (see prop. 1.2.20). We get induced continuous k-algebra homomorphisms $g_i : B_i \to \mathcal{O}_1(L)/(r_1^{i+1})$.

Now fix $i \geq 0$. The $\hat{B}_0$-linear map

$$\Phi : (\hat{B}_0)^{i+1} \xrightarrow{\simeq} \hat{B}_i,\ \Phi(b_0, \ldots, b_i) = \sum_{\mu=0}^{i} b_\mu s_1^\mu$$

and its inverse are continuous DOs over $\hat{B}$, by prop. 1.4.4. Similarly for L: if $\operatorname{char} k = 0$ (resp. $\operatorname{char} k = p$), let $\sigma : \kappa_1(L) \to \mathcal{O}_1(L)/(r_1^{i+1})$ be any $F((\underline{t}'))$-algebra (resp. k-algebra) lifting. The $\kappa_1(L)$-linear map, with respect to σ,

$$\Psi : \kappa_1(L)^{i+1} \xrightarrow{\simeq} \mathcal{O}_1(L)/(r_1^{i+1}),\ \Psi(c_0, \ldots, c_i) = \sum_{\nu=0}^{i} \sigma(c_\nu) r_1^\nu$$

and its inverse are continuous DOs over $\mathcal{O}_1(L)$. In characteristic 0 the continuity follows from the $F((\underline{t}'))$-linearity. In characteristic p we are using the fact that Ψ is a DO over $F((\underline{t}'))$ (of order $\leq i$) and thm. 2.1.14. In particular σ is continuous.

By induction on m, g_0 extends (uniquely) to a homomorphism $\hat{g}_0 : \hat{B}_0 \to \kappa_1(L)$ in $\mathsf{STComAlg}(k)$. Define $f : \hat{B} \twoheadrightarrow \hat{B}_0 \xrightarrow{\hat{g}_0} \kappa_1(L) \xrightarrow{\sigma} \mathcal{O}_1(L)/(r_1^{i+1})$, and consider $\mathcal{O}_1(L)/(r_1^{i+1})$ as a ST $\hat{B}$-module via f. Then Ψ is $\hat{B}$-linear. On the other hand g_i is a DO over B of order $\leq i$, since $[\ldots[g_i, a_0], \ldots, a_i](b) = g_i(b)\prod_{\mu=0}^{i}(g_i(a_\mu) - f(a_\mu)) = 0$ for any $a_0, \ldots, a_i \in B$, $b \in B_i$.

For every $0 \leq \mu, \nu \leq i$ there is a DO over B relative to k, $D_{\mu,\nu} : B_0 \to \kappa_1(L)$, such that

$$[D_{\mu,\nu}] = \Psi^{-1} \circ g_i \circ \Phi : B_0^{i+1} \to \kappa_1(L)^{i+1}$$

in matrix notation. Since $B \to \hat{B}$ is étale in $\mathsf{STComAlg}(k)$, $D_{\mu,\nu}$ extends to a continuous DO $\hat{D}_{\mu,\nu} : \hat{B}_0 \to \kappa_1(L)$ over $\hat{B}$. Putting the $\hat{D}_{\mu,\nu}$ together we get a continuous DO over $\hat{B}$

$$\hat{g}_i = \Psi \circ [\hat{D}_{\mu,\nu}] \circ \Phi^{-1} : \hat{B}_i \to \mathcal{O}_1(L)/(r_1^{i+1})$$

extending g_i. By prop. 1.5.20, $\hat{g}_i$ is a k-algebra homomorphism. The $\hat{g}_i$ form an inverse system. Passing to the inverse limit we get $\hat{g} : \hat{B} \to \mathcal{O}_1(L)$, and by lemma 1.3.4 it extends to a homomorphism $\hat{g} : A((\underline{s})) \to L$ in $\mathsf{STComAlg}(k)$. The uniqueness of $\hat{g}$ is obvious. □

Corollary 2.1.18 *Let $K \to L$ be a finite morphism in $\mathsf{TLF}(k)$. Then the topology on L is the fine K-module topology.*

Proof Let L' be the field L with the fine K-module topology. Then the identity map $h : L' \to L$ is continuous. Let $L \cong F((\underline{t}))$ be a parametrization of L, and let $g : F[\underline{t}] \to L'$ be the inclusion. By the theorem there is a continuous homomorphism $\hat{g} : L \cong F((\underline{t})) \to L'$ extending g. Since $h \circ \hat{g} : L \to L$ is continuous and $h \circ \hat{g}|_{F[\underline{t}]} = h \circ g$, uniqueness implies that $\hat{g}$ is the identity map. Therefore the two topologies on L are equal. □

Corollary 2.1.19 *Let $f : K \to L$ be a morphism in $\mathsf{TLF}(k)$ of dimension n, let $\underline{a} = (a_1, \ldots, a_n)$ be an initial system of parameters in L and let $\underline{s} = (s_1, \ldots, s_n)$ be a sequence of indeterminates. Then f extends uniquely to a finite morphism $\hat{f} : K((\underline{s})) \to L$ in $\mathsf{TLF}(k)$ with $\hat{f}(s_i) = a_i$.*

Proof To get $\hat{f}$ apply the theorem to $f : K[\underline{s}] \to L$. Let e_i be the order of a_i in $\mathcal{O}_i(L)$. From [CA] ch. III §2.11 prop. 14, used repeatedly, we get

$$[L : K((\underline{s}))] = e_1 \cdots e_n[\kappa_n(L) : K] < \infty.$$

□

We see that every morphism in $\mathsf{TLF}(k)$ factors as a Laurent series morphism (i.e. $K \to K((\underline{s}))$) followed by a finite morphism. This implies that a morphism is, topologically, a strict monomorphism.

Corollary 2.1.20 *Let K be a TLF and let $f : K \to L$ be a finite field extension. Then L admits a unique structure of TLF such that f becomes a morphism in* $\mathsf{TLF}(k)$.

Proof Say K has dimension n; then L has a unique structure of n-dimensional local field extending that of K. Put on L the fine K-module topology. We must exhibit a parametrization of L. Choose a k-algebra lifting $F = \kappa_n(L) \to \mathcal{O}(L)$ and a regular system of parameters $\underline{t}$ in it. By thm. 2.1.17 we get a continuous k-algebra homomorphism $g : F((\underline{t})) \to L$, which is in fact bijective (cf. previous cor.). Choose a parametrization $K \cong E((\underline{s}))$ and let $h : K \to F((\underline{t}))$ be the finite morphism extending $g^{-1} \circ f|_{E[\underline{s}]}$. Since $g \circ h : K \to L$ is continuous we have $g \circ h = f$, so in fact g is a homeomorphism, and it is the desired parametrization. □

Let $\mathsf{LF}(k)$ be the category of local fields over k, i.e. the objects have the structures i) and ii) of def. 2.1.10 and the morphisms are the homomorphisms which respect those structures. Let $\mathsf{unt} : \mathsf{TLF}(k) \to \mathsf{LF}(k)$ be the functor which forgets the topology. The behavior of this functor changes dramatically between characteristic 0 and positive characteristics.

Proposition 2.1.21 *Suppose* $\operatorname{char} k = p$. *Then the functor* unt *induces an equivalence between* $\mathsf{TLF}(k)$ *and the full subcategory of* $\mathsf{LF}(k)$ *consisting of the fields K such that* $\operatorname{rank}_K \Omega^1_{K/k} < \infty$.

Proof Clearly every such field $K \in \mathsf{LF}(k)$ lifts to some $\tilde{K} \in \mathsf{TLF}(k)$ (use the Cohen structure theorem). It suffices to show that unt is a full functor. So let $K, L \in \mathsf{TLF}(k)$ and let $f : \mathsf{unt}(K) \to \mathsf{unt}(L)$ be a morphism in $\mathsf{LF}(k)$. We must show that f is continuous.

First assume that $\dim(K) < \dim(L)$ and $L = E((s))$ for some TLF E. Let $f_i : K \to E[s]/(s^{i+1})$ be the induced homomorphism. By induction on the dimension we know that $f_0 : K \to E$ is continuous, so E is a ST K-module (via f_0). Choose $i \geq 0$ and let $\Psi : E^{i+1} \xrightarrow{\cong} E[s]/(s^{i+1})$, $\Psi(a_0, \ldots, a_i) = \sum a_\mu s^\mu$, which is an isomorphism of ST E-modules. The map $D_i := \Psi^{-1} \circ f_i : K \to E^{i+1}$ is a differential operator over K (of order $\leq i$), so by thm. 2.1.14 it is continuous. Hence f_i is continuous. Now pass to the inverse limit to conclude that f is continuous.

If $\dim(K) = \dim(L)$ write $K = F((t))$. Then by the discussion above the maps $f_i : F[t]/(t^{i+1}) \to E[s]/(s^{i+1})$ are continuous, and by passing to limits so if f. □

This is not the case in characteristic 0. If $\operatorname{char} k = 0$ and K is a local field over k of dimension ≥ 2, then K admits different topologies. Equivalently, given $K \in \mathsf{TLF}(k)$, there exist automorphisms of $\mathtt{unt}(K)$ in $\mathsf{LF}(k)$ which aren't continuous. Note that this contradicts the assertions in [Lo] pp. 501-502 regarding the uniqueness of the topology.

Example 2.1.22 Suppose $\operatorname{char} k = 0$. Let $K := k((t_1, t_2)) \in \mathsf{TLF}(k)$, and let $\{u_\alpha\}$ be a transcendency basis for $k((t_2))$ over $k(t_2)$, so $k((t_2))$ is separably algebraic over $k(t_2, \{u_\alpha\})$. Choose arbitrary $v_\alpha \in k((t_2))[[t_1]]$. Then there exists a unique $k(t_2)$-algebra lifting $\sigma : k((t_2)) \to k((t_2))[[t_1]]$ such that $\sigma(u_\alpha) = u_\alpha + t_1 v_\alpha$. Extend it to an automorphism σ of K such that $\sigma(t_1) = t_1$. σ is an automorphism of $\mathtt{unt}(K)$, but it is continuous precisely when all the v_α are 0.

To conclude this section we will show that the topology on a topological local field determines its local structure. Given $K \in \mathsf{TLF}(k)$, let $\pi_i : \mathcal{O}(K) \to \kappa_i(K)$ be the canonical continuous maps ($\mathcal{O}(K)$ has the topology induced from K).

Lemma 2.1.23 *Let $K \in \mathsf{TLF}(k)$ be n-dimensional.*

a) *Let s be an indeterminate and put on $\mathbb{Z}$ the discrete topology. If $f : \mathbb{Z}[[s]] \to K$ is a continuous ring homomorphism, then $f(s) \in \mathcal{O}(K)$ and $\pi_n \circ f(s) = 0$.*

b) *Let u and s be indeterminates and put on $\mathbb{Z}[u, u^{-1}]$ the discrete topology. Let $f : \mathbb{Z}[u, u^{-1}][[s]] \to K$ be a continuous ring homomorphism, with $f(u) \in \mathcal{O}(K)$. Then for every i, $0 \leq i \leq n-1$, either $\pi_i \circ f(s) = 0$ or $\pi_{i+1} \circ f(u) \neq 0$.*

Proof a) We prove by induction on i, $i \leq n-1$, that $\pi_i \circ f(s) \in \mathcal{O}_{i+1}(K)$. Since $\lim_{j\to\infty} s^j = 0$ in $\mathbb{Z}[[s]]$, we get $\lim_{j\to\infty} (\pi_i \circ f(s))^j = 0$ in $\kappa_i(K)$. Now $\kappa_i(K) \cong L((t))$ for some TLF L, so by the decomposition (1.3.6) and prop.

1.1.5 c) it follows that $\pi_i \circ f(s) \in \mathcal{O}_{i+1}(K) \cong L[[t]]$. Since $\kappa_n(K)$ is discrete it must be that $\pi_n \circ f(s) = 0$

b) Set $a := \pi_i \circ f(s)$ and $b := \pi_i \circ f(u)$. If $a \neq 0$ and $\pi_{i+1} \circ f(u) = 0$, then $b^{-h}a \in \kappa_i(K) - \mathcal{O}_{i+1}(K)$ for $h >> 0$. Now $\lim_{j\to\infty}(u^{-h}s)^j = 0$ in $\mathbb{Z}[u, u^{-1}][[s]]$, so $\lim_{j\to\infty}(b^{-h}a)^j = 0$ in $\kappa_i(K)$, which is impossible by prop. 1.1.5 c). □

Proposition 2.1.24 *Let $K \in \mathsf{TLF}(k)$ be n-dimensional and let $\underline{a} = (a_1, \ldots, a_n)$ be a sequence of elements of K. Then $\underline{a}$ is a system of parameters in K iff there exists a continuous ring homomorphism $\mathbb{Z}((\underline{t})) = \mathbb{Z}((t_1, \ldots, t_n)) \to K$, $t_i \mapsto a_i$.*

Proof Suppose $\underline{a}$ is a system of parameters. By thm. 2.1.17 there is a continuous ring homomorphism $\mathbb{Z}((\underline{t})) \to K$ sending $t_i \mapsto a_i$.

Conversely, suppose such a homomorphism $f : \mathbb{Z}((\underline{t})) \to K$ exists. Take i, $1 \leq i \leq n$, and consider the continuous ring homomorphism $\mathbb{Z}[[s]] \to \mathbb{Z}((\underline{t}))$, $s \mapsto t_i$. By lemma 2.1.23 a), $a_i \in \mathcal{O}(K)$ and $\pi_n(a_i) = 0$. Define l_i to be the smallest number such that $\pi_{l_i}(a_i) = 0$. Now take $i \leq n-1$, and consider the continuous ring homomorphism $\mathbb{Z}[u, u^{-1}][[s]] \to \mathbb{Z}((\underline{t}))$, $s \mapsto t_i$, $u \mapsto t_{i+1}$. By lemma 2.1.23 b), it follows that $l_{i+1} \geq l_i + 1$. Therefore $l_i = i$ and $\underline{a}$ is a system of parameters. □

2.2 Clusters of TLFs and Base Change

As before k is a fixed perfect field. In this section we define a category of algebras $\mathsf{CTLF}(k)$, which contains $\mathsf{TLF}(k)$ as a full subcategory. In this new category there is a convenient base change operation. Given an artinian k-algebra A let $A_{\text{red}} := A/(\text{radical})$. Since k is perfect, there exist k-algebra liftings $A_{\text{red}} \to A$.

Definition 2.2.1 *A cluster of topological local fields over k is an artinian, commutative semi-topological k-algebra A, together with a structure of a topological local field over k on each of its residue fields $A/\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec} A$. We require that there will exist some k-algebra lifting $A_{\text{red}} \to A$, relative to which A has the fine A_{red}-module topology.*

A is called equidimensional if the TLFs $A/\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec} A$, all have equal dimensions and equal differential degrees.

A morphism $f : A \to B$ of clusters of TLFs is a continuous k-algebra homomorphism such that for every $\mathfrak{q} \in \operatorname{Spec} B$ lying over some $\mathfrak{p} \in \operatorname{Spec} A$, the induced map on residue fields $A/\mathfrak{p} \to B/\mathfrak{q}$ is a morphism in $\mathsf{TLF}(k)$.

Denote the category of clusters of TLFs by $\mathsf{CTLF}(k)$. The topology on a cluster of TLFs A is local with respect to $\operatorname{Spec} A$, i.e. $A \cong \prod_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}}$ as ST rings. This is because the spectral decomposition is multiplication by idempotents, a continuous operation. Since A_{red} is a complete separated k-algebra, so is A.

The next proposition shows that the topology on A is independent of the lifting $A_{\mathrm{red}} \to A$ (provided this lifting is continuous). Given a morphism $f : A \to B$ and a maximal ideal $\mathfrak{q} \in \operatorname{Spec} B$, let $\dim_{\mathfrak{q}}(f) := \dim(A/\mathfrak{p} \to B/\mathfrak{q})$, where $\mathfrak{p} := f^{-1}(\mathfrak{q})$. We say that f is finite if $\dim_{\mathfrak{q}}(f) = 0$ for all $\mathfrak{q}$.

Proposition 2.2.2 a) *Let $\tau : A_{\mathrm{red}} \to A$ be a morphism in $\mathsf{CTLF}(k)$ (i.e. a continuous lifting). Then A has the fine A_{red}-module topology (via τ).*

b) *Let $A \to B$ be a finite morphism in $\mathsf{CTLF}(k)$. Then the topology on B is the fine A-module topology.*

c) *Let $A \in \mathsf{CTLF}(k)$ and let B be a finite A-algebra. Then there exists a unique structure of cluster of TLFs on B which makes $A \to B$ into a morphism in $\mathsf{CTLF}(k)$.*

Proof These questions are local on $\operatorname{Spec} B$, so we may assume that $\operatorname{Spec} B = \{\mathfrak{q}\}$ and $\operatorname{Spec} A = \{\mathfrak{p}\}$.

a) Set $K := A/\mathfrak{p}$ and let $K \cong F((\underline{t}))$ be a parametrization. Let A' be the algebra A with the fine K-module topology via τ, and let $h : A' \to A$ be the identity map. We must prove that h is a homeomorphism. Say $\sigma : K \to A$ is a lifting which determines the topology. Then it suffices to prove that $h^{-1} \circ \sigma : K \to A'$ is continuous. Now $g := h^{-1} \circ \sigma|_{F[\underline{t}]} : F[\underline{t}] \to A'$ is a DO over $F[\underline{t}]$ (cf. proof of thm. 2.1.17) so it extends to a continuous DO $\hat{g} : K \to A'$. But $h \circ \hat{g}$ is continuous, so $h \circ \hat{g} = \sigma$ and hence $h^{-1} \circ \sigma = \hat{g}$ is continuous.

b) (Cf. proof of cor. 2.1.18.) Let $\sigma : K = A/\mathfrak{p} \to A$ be a lifting which determines the topology. Let B' be the algebra B with the fine K-module topology and let $h : B' \to B$ be the identity map. As in the proof of thm. 2.1.17 there exists some continuous k-algebra lifting $\tau : L = B/\mathfrak{q} \to B'$. Now $h \circ \tau : L \to B$ is a morphism, so by part a), B has the fine L-module topology via $h \circ \tau$. This implies that h is a homeomorphism.

c) Denote the homomorphism $A \to B$ by f. Let $\sigma : K = A/\mathfrak{p} \to A$ be a lifting which defines the topology. Put on B the fine K-module topology, and put on $L = B/\mathfrak{q}$ the unique structure of TLF such that $K \to L$ is a finite morphism in $\mathsf{TLF}(k)$. It remains to exhibit a lifting $\tau : L \to B$ such that B has the fine L-module topology.

Choose a continuous k-algebra lifting $\tau : L \to B$ as before. Let B' be the algebra B with the fine L-module topology, so B' is a cluster of TLFs. Let $h : B' \to B$ be the identity map. Choose a parametrization $K \cong F((\underline{t}))$. The DO $g := h^{-1} \circ \sigma|_{F[\underline{t}]} : F[\underline{t}] \to B'$ extends to a continuous DO $\hat{g} : K \to B'$, and $h \circ \hat{g} = f \circ \sigma : K \to B$. Thus $\hat{g}$ is a morphism, and by part b) h is a homeomorphism. $\square$

Let k' be another perfect field, with discrete topology, and suppose there is a homomorphism $k \to k'$. Thus any ST k'-algebra is also a ST k-algebra.

Definition 2.2.3 *Let $A \in \mathsf{CTLF}(k)$ and let $A' \in \mathsf{CTLF}(k')$. A finitely ramified homomorphism $A \to A'$ is a continuous k-algebra homomorphism, such that for every $\mathfrak{p}' \in \operatorname{Spec} A'$ lying over some $\mathfrak{p} \in \operatorname{Spec} A$, the image of $(A/\mathfrak{p})^\times$ in the canonical valuation group $(A'/\mathfrak{p}')^\times / \mathcal{O}(A'/\mathfrak{p}')^\times$ is a subgroup of finite index.*

Theorem 2.2.4 *(Finitely Ramified Base Change) Let $f : A \to B$ be a morphism in $\mathsf{CTLF}(k)$, let $A' \in \mathsf{CTLF}(k')$ and let $u : A \to A'$ be a finitely ramified homomorphism. Then there exists an algebra $B' \in \mathsf{CTLF}(k')$, a morphism $f' : A' \to B'$ in $\mathsf{CTLF}(k')$ and a finitely ramified homomorphism $v : B \to B'$, satisfying:*

i) $\dim_{\mathfrak{q}'}(f') = \dim_{\mathfrak{q}}(f)$ *for every $\mathfrak{q}' \in \operatorname{Spec} B'$ lying over some $\mathfrak{q} \in \operatorname{Spec} B$, and the diagram below is commutative:*

$$\begin{tikzcd} B \arrow[r, "v"] & B' \\ A \arrow[u, "f"] \arrow[r, "u"] & A' \arrow[u, "f'"'] \end{tikzcd}$$

ii) *Suppose $g' : A' \to C'$ is a morphism in $\mathsf{CTLF}(k')$ and $w : B \to C'$ is a finitely ramified homomorphism, such that $w \circ f = g' \circ u$ and $\dim_{\mathfrak{q}'}(g') = \dim_{\mathfrak{q}}(f)$ for every $\mathfrak{q}' \in \operatorname{Spec} C'$ lying over some $\mathfrak{q} \in \operatorname{Spec} B$. Then there exists a unique finite morphism $h' : B' \to C'$ in $\mathsf{CTLF}(k')$ such that $g' = h' \circ f'$ and $w = h' \circ v$.*

Proof We can assume that $\operatorname{Spec} A = \{\mathfrak{p}\}$, $\operatorname{Spec} A' = \{\mathfrak{p}'\}$ and $\operatorname{Spec} B = \{\mathfrak{q}\}$. Choose a lifting $\sigma : A/\mathfrak{p} \to A$. Say $\dim(f) = m$ and pick $b_1, \ldots, b_m \in B$ such that their images form an initial system of parameters $\underline{b} = (b_1, \ldots, b_m)$ in $B/\mathfrak{q}$. Let $\underline{s} = (s_1, \ldots, s_m)$ be a sequence of indeterminates. As in the proof of prop. 2.2.2, we get a finite morphism $(A/\mathfrak{p})((\underline{s})) \to B$, extending $f \circ \sigma$

and sending $s_i \mapsto b_i$. Now $A((\underline{s})) \cong A \otimes_{A/\mathfrak{p}} (A/\mathfrak{p})((\underline{s}))$, giving rise to a finite morphism $A((\underline{s})) \to B$ extending f. There is also a continuous homomorphism $\tilde{u} : A((\underline{s})) \to A'((\underline{s}))$, which is finitely ramified. Define

$$B' := B \otimes_{A((\underline{s}))} A'((\underline{s}))$$

with the unique structure of CTLF to make $A'((\underline{s})) \to B'$ a finite morphism in $\mathsf{CTLF}(k')$. The maps f' and v are the obvious ones.

Suppose that an algebra C' and maps g', w are given as in ii). Let $\mathfrak{q}' \in \operatorname{Spec} C'$ be arbitrary, and set $\mathfrak{p}' := g'^{-1}(\mathfrak{q}')$ and $\mathfrak{q} := w^{-1}(\mathfrak{q}')$. Let $\underline{b}$ be as before. We claim that $(w(b_1), \ldots, w(b_m))$ is an initial system of parameters in $C'/\mathfrak{q}'$. If $\dim(C') = m$ this follows directly from prop. 2.1.24. Otherwise, recall that u is finitely ramified, so there is an element $a \in A$ with $u(a)$ a parameter of the DVR $\mathcal{O}_1(A'/\mathfrak{p}')$. From lemma 2.1.23 b) it follows that $((w(b_1), \ldots, w(b_m)), g' \circ u(a))$ is an initial system of parameters in $C'/\mathfrak{q}'$. As before we get a finite morphism $A'((\underline{s})) \to C'_{\mathfrak{q}'}$, $s_i \mapsto w(b_i)$. Thus a morphism $B' \to C'_{\mathfrak{q}'}$ exists, and it is clearly unique. $\square$

Example 2.2.5 Take $A := k(t_2)$, $B := k(t_2)((t_1))$ and $A' := k((t_2))$ with the standard homomorphisms. Then $B' = k((t_2))((t_1)) = k((t_1, t_1))$.

2.3 Differential Forms and Traces

As before k is a fixed perfect field with discrete topology. In this section we show that to each finite morphism $K \to L$ in $\mathsf{TLF}(k)$ there is attached a canonical trace map $\mathrm{Tr}_{L/K} : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}$.

Lemma 2.3.1 *Let K be a TLF over k. Then $\Omega^{*,\mathrm{sep}}_{K/k}$ is a free ST K-module of finite rank.*

Proof Let $K \cong F((t_1, \ldots, t_n)) = F((\underline{t}))$ be a parametrization of K. By cor. 1.5.19, $F[\underline{t}] \to K$ is topologically étale relative to k. By condition a) of def. 2.1.10, $\Omega^*_{F[\underline{t}]/k}$ is a finitely generated free $F[\underline{t}]$-module. Since K is separated, and using cor. 1.5.13, we get $\Omega^{*,\mathrm{sep}}_{K/k} \cong K \otimes_{F[\underline{t}]} \Omega^*_{F[\underline{t}]/k}$. $\square$

Recall that if $\operatorname{char} k = p$ then $\Omega^{*,\mathrm{sep}}_{K/k} = \Omega^*_{K/k} = \Omega^*_K$ (cor. 2.1.15).

Given a TLF K define the differential logarithm map $\operatorname{dlog} : K^\times \to \Omega^{1,\mathrm{sep}}_{K/k}$, $\operatorname{dlog}(a) := a^{-1}\mathrm{d}a$. This is a homomorphism of abelian groups, functorial with respect to continuous k-algebra homomorphisms.

Proposition 2.3.2 *There exists a unique functorial trace map, assigning to each finite morphism $K \to L$ in $\mathsf{TLF}(k)$ a map $\mathrm{Tr}_{L/K} : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}$, satisfying the following axioms:*

T1 $\mathrm{Tr}_{L/K}$ *is a homomorphism of semi-topological differential graded left $\Omega^{*,\mathrm{sep}}_{K/k}$-modules, of degree 0.*

T2 $\mathrm{Tr}_{L/K}$ *coincides with the field trace on $L = \Omega^{0,\mathrm{sep}}_{L/k}$.*

T3 $\mathrm{Tr}_{L/K} \circ \mathrm{dlog} = \mathrm{dlog} \circ \mathrm{N}_{L/K} : L^{\times} \to \Omega^{1,\mathrm{sep}}_{K/k}$, *where $\mathrm{N}_{L/K}$ is the field norm.*

(Cf. [Lo] props. 2 and 4, and [Ku1] §2.3 Satz 1.)

Proof 1) Assume k has characteristic 0. According to cor. 2.1.18 and prop. 1.5.8 any finite morphism $K \to L$ is topologically étale relative to k. Therefore $\Omega^{*,\mathrm{sep}}_{L/k} \cong \Omega^{*,\mathrm{sep}}_{K/k} \otimes_K L$. Let $\mathrm{Tr}_{L/K} : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}$ be the $\Omega^{*,\mathrm{sep}}_{K/k}$-linear extension of the field trace $\mathrm{Tr}_{L/K} : L \to K$. Because it is K-linear, $\mathrm{Tr}_{L/K}$ is continuous. The functoriality follows from the same property of the field trace.

Choose any finite Galois extension $g : K \to M$ containing L, and let $H := \mathrm{Hom}_{\mathsf{Alg}(K)}(L, M)$. By cor. 2.1.20 we may assume that g is a finite morphism in $\mathsf{TLF}(k)$. Then $g : \Omega^{*,\mathrm{sep}}_{K/k} \to \Omega^{*,\mathrm{sep}}_{M/k}$ is injective, and

$$g \circ \mathrm{Tr}_{L/K} = \sum_{h \in H} h : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{M/k} \ . \tag{2.3.3}$$

Since $h \circ \mathrm{d} = \mathrm{d} \circ h$ and $g \circ \mathrm{N}_{L/K} = \prod_{h \in H} h : L^{\times} \to M^{\times}$, it follows that $\mathrm{Tr}_{L/K}$ commutes with d and that axiom **T3** is satisfied.

2) Now assume that k has characteristic p. By [Ku1] §2.3 Satz 1 there is a functorial trace map $\mathrm{Tr}_{L/K} : \Omega^{*}_{L} \to \Omega^{*}_{K}$ for any finite extension $K \to L$. It is a homomorphism of DG Ω^{*}_{K}-modules; hence it is continuous. Axiom **T2** follows from [Ku1] (2.3.6) property d). In order to verify axiom **T3** we may assume (by transitivity) that L is either separable over K, or purely inseparable of degree p. In the first case formula (2.3.3) holds. In the second case it suffices to consider $a \in L^{\times} - K^{\times}$, and by [Ku1] (2.3.6) property e)

$$\mathrm{Tr}_{L/K} \circ \mathrm{dlog}(a) = a^{-p}\mathrm{Tr}_{L/K}(a^{p-1}\mathrm{d}a) = a^{-p}\mathrm{d}(a^{p}) = \mathrm{dlog} \circ \mathrm{N}_{L/K}(a) \ . \tag{2.3.4}$$

□

Remark 2.3.5 In [Ku1] E. Kunz proves the existence of a canonical trace map $\mathrm{Tr}_{L/K} : \Omega^{*}_{L/k} \to \Omega^{*}_{K/k}$ for any finite extension of fields $K \to L$ relative to any base field k. In characteristic 0 the proof is like part 1) of prop. 2.3.2, whereas in characteristic p it uses Tate's trace map, see [Ta1] p. 401. For TLFs the two trace maps $\mathrm{Tr}_{L/K}$ are compatible with the projections $\Omega^{*}_{-/k} \to \Omega^{*,\mathrm{sep}}_{-/k}$. (The author thanks R. Hübl for referring him to [Ku1].)

Remark 2.3.6 The multiplicative group $K^{\times}$ is considered here to be a *discrete* group, and the same holds for the Milnor ring $\mathrm{K}^{*}K$; see remark 1.3.8 and digression 2.4.25. When dealing with local class field theory one does topologize these groups appropriately; the reader is referred to [Pa3] §2, [Ka] part I §7.1 and [Kh] §2.3.

Definition 2.3.7 *Let $K \in \mathsf{TLF}(k)$ with $\mathrm{rank}_K \Omega^{1,\mathrm{sep}}_{K/k} = d$. The dual module of K is the free semi-topological K-module of rank 1, $\omega_K := \Omega^{d,\mathrm{sep}}_{K/k}$.*

The name is explained by the next proposition. First we define the trace pairing

$$\langle -,- \rangle_{L/K} : L \times \omega_L \xrightarrow{\text{mult}} \omega_L \xrightarrow{\mathrm{Tr}_{L/K}} \omega_K. \tag{2.3.8}$$

Proposition 2.3.9 *The trace pairing is a perfect pairing of semi-topological K-modules, i.e. it induces isomorphisms $\omega_L \cong \mathrm{Hom}^{\mathrm{cont}}_K(L, \omega_K)$ and vice-versa.*

Proof It suffices to show that $\mathrm{Tr}_{L/K}$ is non-zero, and we may assume $K \to L$ is either separable or inseparable of degree p. In the separable case this is well known. If $L = K[a]$ with a inseparable over K of degree p, then choose $b_1, \ldots, b_{d-1} \in K^{\times}$ such that $a, b_1, \ldots, b_{d-1}$ is a p-basis of L. Then

$$\begin{aligned}&\mathrm{Tr}_{L/K}(\mathrm{dlog}(b_1) \wedge \cdots \wedge \mathrm{dlog}(b_{d-1}) \wedge \mathrm{dlog}(a)) \\ &\quad = \mathrm{dlog}(b_1) \wedge \cdots \wedge \mathrm{dlog}(b_{d-1}) \wedge \mathrm{dlog}(a^p) \neq 0\end{aligned}$$

in ω_K. (Cf. [Ku1] (2.3.5)). □

Let $\mathsf{CTLF}_{\mathrm{red}}(k)$ be the full subcategory of $\mathsf{CTLF}(k)$ consisting of reduced algebras. It is an easy matter to extend the trace functor to $\mathsf{CTLF}_{\mathrm{red}}(k)$. For $A \in \mathsf{CTLF}_{\mathrm{red}}(k)$, we have $\Omega^{*,\mathrm{sep}}_{A/k} = \prod_{\mathfrak{p} \in \mathrm{Spec}\, A} \Omega^{*,\mathrm{sep}}_{(A/\mathfrak{p})/k}$. Given a finite morphism $A \to B$, the trace map is defined locally on $\mathrm{Spec}\, B$:

$$\mathrm{Tr}_{B/A} := \sum_{\mathfrak{q}|\mathfrak{p}} \mathrm{Tr}_{(B/\mathfrak{q})/(A/\mathfrak{p})} : \Omega^{*,\mathrm{sep}}_{B/k} \to \Omega^{*,\mathrm{sep}}_{A/k}. \tag{2.3.10}$$

Set $\omega_A := \bigoplus_{\mathfrak{p} \in \mathrm{Spec}\, A} \omega_{A/\mathfrak{p}}$, a free ST A-module of rank 1. Then the trace pairing $\langle -,- \rangle_{B/A} : B \times \omega_B \to \omega_A$ is a perfect pairing of ST A-modules.

The next proposition shows that the trace pairing commutes with finitely ramified base change (cf. [Lo] lemma 5 iii)).

Proposition 2.3.11 *Let the data of theorem 2.2.4 be given, and assume that f is a finite morphism and the algebras A, A' and B are reduced. For every $\mathfrak{q}' \in \operatorname{Spec} B'$ denote the length of the artinian local ring $B'_{\mathfrak{q}'}$ by $l(B'_{\mathfrak{q}'})$ and denote by $v_{\mathfrak{q}'}$ the induced map $B \to B'/\mathfrak{q}' = (B'_{\mathrm{red}})_{\mathfrak{q}'}$. Then*

$$u \circ \operatorname{Tr}_{B/A} = \operatorname{Tr}_{B'_{\mathrm{red}}/A'} \circ \left(\sum_{\mathfrak{q}'} l(B'_{\mathfrak{q}'})\, v_{\mathfrak{q}'} \right) : \Omega^{*,\mathrm{sep}}_{B/k} \to \Omega^{*,\mathrm{sep}}_{A'/k'}.$$

Proof We may assume that A, A', B are fields. Now $B' = B \otimes_A A'$. If $A \to B$ is separable then so is $A' \to B'$ and $l(B'_{\mathfrak{q}'}) = 1$ for all $\mathfrak{q}'$. All traces appearing are gotten by base change from the field trace $\operatorname{Tr}_{B/A} : B \to A$, so equality holds.

Next, assume that $B = A[b]$ with b inseparable of degree p over A. Then $\operatorname{Spec} B' = \{q'\}$, and either $u(b^p) \notin A'^{(p)}$, in which case $l(B') = 1$, $B' = A'[v(b)]$ and $\operatorname{Tr}_{B'/A'}(\operatorname{dlog} \circ v(b)) = u \circ \operatorname{dlog}(b^p)$; or $u(b^p) \in A'^{(p)}$, in which case $l(B') = p$, $B'_{\mathrm{red}} = A'$ and $u \circ \operatorname{dlog}(b^p) = 0$. Again equality holds.

The general situation now follows by transitivity. □

2.4 Residues in Topological Local Fields; Topological Duality

As before k is a fixed perfect field. Given $K \in \mathsf{TLF}(k)$, let $\mathrm{K}_* K = \bigoplus_{i=0}^{\infty} \mathrm{K}_i K$ be its Milnor ring (see [Mi]). As mentioned earlier (remark 2.3.6), $\mathrm{K}_* K$ has the discrete topology. For any $a_1, \ldots, a_i \in K^{\times}$, we denote the corresponding element (symbol) in $\mathrm{K}_i K$ by $(a_1, \ldots, a_i)$. Let $n := \dim(K)$. For every $1 \leq i \leq n$ there is a homomorphism of abelian groups

$$\operatorname{ord}^i_K := \underbrace{\partial \circ \cdots \circ \partial}_{i} : \mathrm{K}_i K \to \mathrm{K}_0 \kappa_i(K) = \mathbb{Z} \tag{2.4.1}$$

where $\partial : \mathrm{K}_* K \to \mathrm{K}_{*-1} \kappa_1(K)$ is the map of [BT] prop. 4.5. If $\underline{v} = (v_1, \ldots, v_n) : K^{\times} \to (\mathbb{Z}, \mathrm{lex})$ is a surjective valuation, one has $\operatorname{ord}^i_K(a_1, \ldots, a_i) = (-1)^{\binom{i}{2}} \det[v_{\mu}(a_{\nu})]$ (cf. [Lo] p. 501).

Remark 2.4.2 The original definition of ∂, namely the one in [Mi], differs from that of [BT] by a sign. We chose the latter since it permits $\mathcal{O}^{\times}$ to act from the *left*.

There is a canonical homomorphism of graded rings (called the Tate map on [Pa3] p. 166) $\operatorname{dlog} : \mathrm{K}_* K \to \Omega^{*,\mathrm{sep}}_{K/k}$, extending the differential logarithm $\operatorname{dlog} : K^{\times} \to \Omega^{1,\mathrm{sep}}_{K/k}$. Thus $\operatorname{dlog}(a_1, \ldots, a_n) = a_1^{-1} \mathrm{d}a_1 \wedge \cdots \wedge a_n^{-1} \mathrm{d}a_n$.

The following important theorem is due to Lomadze ([Lo] thm. 1). It generalizes the well known 1-dimensional case (see [Se] ch. II no. 11) and Parshin's result for 2-dimensional fields ([Pa1] §1 prop. 1). We present an improved version, in the framework of topological local fields.

Theorem 2.4.3 *Let k be a perfect field. There exists a unique functor* $\operatorname{Res} : \mathsf{TLF}(k)^{\circ} \to \mathsf{Ab}$, *such that* $\operatorname{Res} K = \Omega^{*,\mathrm{sep}}_{K/k}$ *for all* $K \in \mathsf{TLF}(k)$, *and satisfying the following axioms:*

R1 *Given a morphism $K \to L$ in $\mathsf{TLF}(k)$, the map* $\operatorname{Res}_{L/K} := \operatorname{Res}(K \to L) : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}$ *is a homomorphism of semi-topological differential graded left $\Omega^{*,\mathrm{sep}}_{K/k}$-modules of degree $-\dim(L/K)$.*

R2 *If $K \to L$ is a finite morphism then* $\operatorname{Res}_{L/K} = \operatorname{Tr}_{L/K}$.

R3 *If $K \to L$ is a morphism of dimension $n \geq 1$, then for any $a_1, \ldots, a_n \in L^{\times}$ it holds that*

$$\operatorname{Res}_{L/K} \circ \operatorname{dlog}(a_1, \ldots, a_n) = [\kappa_n(L) : K] \operatorname{ord}^n_L(a_1, \ldots, a_n).$$

The proof is postponed till later in this section.

Observe that for $L = K((t_1, \ldots, t_n))$, axiom **R3** yields

$$\operatorname{Res}_{L/K}(t_n^{-1} \mathrm{d}t_n \wedge \cdots \wedge t_1^{-1} \mathrm{d}t_1) = 1 \ .$$

Remark 2.4.4 Our residue map $\operatorname{Res}_{L/K}$ differs from the one defined on [Lo] p. 509 by a factor of $(-1)^{\binom{n}{2}}$, where $n = \dim(L/K)$.

Let $K \to L$ be a morphism in $\mathsf{TLF}(k)$. We call $K \to L$ smooth (resp. étale) if it is so in $\mathsf{STComAlg}(k)$. A Laurent series morphism $K \to K((\underline{t}))$ is smooth. Thus if $\operatorname{char} k = 0$, any morphism $K \to L$ is smooth, since it factors as $K \to K((\underline{t})) \to L$ with $K((\underline{t})) \to L$ finite separable. On the other hand, if $\operatorname{char} k = p$, $K \to L$ is a smooth morphism iff L is a separable K-algebra (see cor. 2.1.16). (One can actually show that any smooth morphism factors as $K \to K((\underline{t})) \to L$ with $K((\underline{t})) \to L$ finite separable.) Given a smooth morphism $K \to L$ of dimension n, any splitting $\Omega^{1,\mathrm{sep}}_{L/k} \cong \Omega^{1,\mathrm{sep}}_{L/K} \oplus \left(\Omega^{1,\mathrm{sep}}_{K/k} \otimes_K L\right)$ defines an isomorphism of left ST graded $\Omega^{*,\mathrm{sep}}_{K/k}$-modules

$$\Omega^{*,\mathrm{sep}}_{L/k} \cong \Omega^{*,\mathrm{sep}}_{K/k} \otimes_K \Omega^{*,\mathrm{sep}}_{L/K} \ . \tag{2.4.5}$$

This induces a canonical homomorphism of left ST graded $\Omega^{*,\mathrm{sep}}_{K/k}$-modules

$$\Omega^{*,\mathrm{sep}}_{L/k} \to \frac{\Omega^{*,\mathrm{sep}}_{L/k}}{\Omega^{*,\mathrm{sep}}_{K/k} \cdot \left(\Omega^{\leq n-1,\mathrm{sep}}_{L/k} + \mathrm{d}(\Omega^{n-1,\mathrm{sep}}_{L/k})\right)} \cong \Omega^{*,\mathrm{sep}}_{K/k} \otimes_K \mathrm{H}^n \Omega^{*,\mathrm{sep}}_{L/K} \ . \tag{2.4.6}$$

Hence any map $\Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}$ satisfying axioms **R1** - **R3** factors through the module on the right hand side of (2.4.6), and thus is completely determined by its restriction to $\Omega^{n,\mathrm{sep}}_{L/k}$ (if $K \to L$ is smooth !).

Note that by formula (2.4.6), $\Omega^{*,\mathrm{sep}}_{K/k} \otimes_K \mathrm{H}^n\Omega^{*,\mathrm{sep}}_{L/K}$ is a DG $\Omega^{*,\mathrm{sep}}_{K/k}$-module. Taking $L = K((t_1,\ldots,t_n))$, the action of the exterior derivative d on $\Omega^{*,\mathrm{sep}}_{K/k} \otimes_K \mathrm{H}^n\Omega^{*,\mathrm{sep}}_{K((\underline{t}))/K}$ is given by:

$$\mathrm{d}(\beta \otimes \underline{t}^{\underline{i}}\,\mathrm{dlog}(\underline{t})) = \mathrm{d}(\beta) \otimes \underline{t}^{\underline{i}}\,\mathrm{dlog}(\underline{t}) \ , \tag{2.4.7}$$

for $\beta \in \Omega^{*,\mathrm{sep}}_{K/k}$ and $\underline{i} \in \mathbb{Z}^n$. Define a K-linear map

$$\begin{cases} \Omega^{n,\mathrm{sep}}_{K((\underline{t}))/K} \to K \\ \sum_{\underline{i}\in\mathbb{Z}^n} a_{\underline{i}}\, \underline{t}^{\underline{i}}\,\mathrm{dlog}(\underline{t}) \mapsto a_{(0,\ldots,0)}. \end{cases} \tag{2.4.8}$$

An elementary calculation (say, using prop. 1.3.5 and lemma 1.3.9) shows that this map vanishes on $\mathrm{d}(\Omega^{n-1,\mathrm{sep}}_{K((\underline{t}))/K})$, inducing a map $\mathrm{H}^n\Omega^{*,\mathrm{sep}}_{K((\underline{t}))/K} \to K$. Extend it to a homomorphism of left $\Omega^{*,\mathrm{sep}}_{K/k}$-modules

$$\mathrm{Res}_{K((\underline{t}))/K,\underline{t}} : \Omega^{*,\mathrm{sep}}_{K((\underline{t}))/k} \to \Omega^{*,\mathrm{sep}}_{K/k} \tag{2.4.9}$$

using (2.4.6).

Definition 2.4.10 *Let $K \to L$ be a morphism of dimension n in $\mathsf{TLF}(k)$ and let $\underline{a} = (a_1,\ldots,a_n)$ be an initial system of parameters in L. Define*

$$\mathrm{Res}_{L/K,\underline{a}} := \mathrm{Res}_{K((\underline{a}))/K,\underline{a}} \circ \mathrm{Tr}_{L/K((\underline{a}))} : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}.$$

Suppose $K \to E$ is a finite morphism. Then $\Omega^{*,\mathrm{sep}}_{E((\underline{t}))/k} \cong \Omega^{*,\mathrm{sep}}_{E/k} \otimes_K \Omega^{*,\mathrm{sep}}_{K((\underline{t}))/K}$, implying that

$$\mathrm{Res}_{K((\underline{t}))/K,\underline{t}} \circ \mathrm{Tr}_{E((\underline{t}))/K((\underline{t}))} = \mathrm{Tr}_{E/K} \circ \mathrm{Res}_{E((\underline{t}))/E,\underline{t}}. \tag{2.4.11}$$

It immediate from the definition that

$$\mathrm{Res}_{K((\underline{t},\underline{s}))/K,(\underline{t},\underline{s})} = \mathrm{Res}_{K((\underline{s}))/K,\underline{s}} \circ \mathrm{Res}_{K((\underline{t},\underline{s}))/K((\underline{s})),\underline{t}} \tag{2.4.12}$$

where $(\underline{t},\underline{s}) := (t_1,\ldots,s_1,\ldots)$ is the concatenated sequence.

If $\mathrm{char}\, k = p$ we have $K((\underline{t}))^{(p/K)} = K((\underline{t}^p)) = K((t_1^p,\ldots,t_n^p))$, so

$$\mathrm{Tr}_{K((\underline{t}))/K((\underline{t}^p))}(\mathrm{dlog}(\underline{t})) = \mathrm{dlog}(\underline{t}^p) \ ,$$

and

$$\mathrm{Tr}_{K((\underline{t}))/K((\underline{t}^p))}(\underline{t}^{\underline{i}}\,\mathrm{dlog}(\underline{t})) = 0$$

if $0 \le i_j < p$ but $\underline{i} \ne (0,\ldots,0)$. Therefore we get an equality

$$\mathrm{Res}_{L/K,\underline{a}} = \mathrm{Res}_{L/K,\underline{a}^p} \tag{2.4.13}$$

for every initial system of parameters $\underline{a}$ in L.

Lemma 2.4.14 *The map $\mathrm{Res}_{L/K,\underline{a}}$ commutes with finitely ramified base change: given $K' \in \mathsf{TLF}(k')$ and a finitely ramified homomorphism $u : K \to K'$, let $K' \to L'$ be the resulting morphism in $\mathsf{CTLF}(k')$. Then in the notation of thm. 2.2.4 and prop. 2.3.11 one has*

$$u \circ \mathrm{Res}_{L/K,\underline{a}} = \sum_{\mathfrak{q}'} l(L'_{\mathfrak{q}'}) \mathrm{Res}_{(L'/\mathfrak{q}')/K',\underline{a}} \circ v_{\mathfrak{q}'} : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K'/k'} .$$

Proof In view of prop. 2.3.11 it suffices to show that

$$u \circ \mathrm{Res}_{K((\underline{a}))/K,\underline{a}} = \mathrm{Res}_{K'((\underline{a}))/K',\underline{a}} \circ v$$

which is immediate from the definitions. □

A homomorphism of fields $K \to L$ induces a homomorphism of graded rings $\mathrm{K}_* K \to \mathrm{K}_* L$, and this makes $\mathrm{K}_* L$ into a left $\mathrm{K}_* K$-module. If $K \to L$ is finite, there is a canonical transfer map $\mathrm{N}_{L/K} : \mathrm{K}_* L \to \mathrm{K}_* K$, which satisfies:

$$\begin{aligned} &\mathrm{N}_{L/K} \text{ is } \mathrm{K}_*K\text{-linear of degree } 0, \\ &\mathrm{N}_{L/K}(1) = [L : K] \text{ and } \mathrm{N}_{L/K}|_{\mathrm{K}_1 L} \text{ is the field norm.} \end{aligned} \tag{2.4.15}$$

(see [BT] p. 386 and [Ka] §1.7 prop. 5).

Now suppose that $K \to L$ is a finite morphism in $\mathsf{TLF}(k)$. Then according to [Lo] propositions 5 and 6 respectively we have

$$\mathrm{Tr}_{L/K} \circ \mathrm{dlog} = \mathrm{dlog} \circ \mathrm{N}_{L/K} : \mathrm{K}_* L \to \Omega^{*,\mathrm{sep}}_{K/k} \tag{2.4.16}$$

and

$$\mathrm{ord}^i_K \circ \mathrm{N}_{L/K} = [\kappa_i(L) : \kappa_i(K)] \, \mathrm{ord}^i_L : \mathrm{K}_i L \to \mathbb{Z}. \tag{2.4.17}$$

(Cf. also our digression 2.4.25.) According to [Lo] lemma 6 vii) b), it holds that

$$\mathrm{Res}_{K((\underline{a}))/K,\underline{a}} \circ \mathrm{dlog} = \mathrm{ord}^n_{K((\underline{a}))} : \mathrm{K}_n K((\underline{a})) \to K. \tag{2.4.18}$$

Lemma 2.4.19 *The map $\mathrm{Res}_{L/K,\underline{a}}$ satisfies axioms* **R1**, **R2** *and* **R3**.

Proof Axiom **R2** holds by definition. To verify axiom **R1** we may assume that $L = K((\underline{t}))$ and $\underline{a} = \underline{t}$; this is because of proposition 2.3.2. The $\Omega^{*,\mathrm{sep}}_{K/k}$-linearity is built into the definition. As for continuity, according to formula (2.4.12) we may assume that $L = K((t))$. Now use prop. 1.3.5. To see that $\mathrm{Res}_{L/K,\underline{t}}$ commutes with d, it suffices to look at the forms $\beta \wedge \underline{t}^{\underline{i}} \, \mathrm{dlog}(\underline{t})$, with $\beta \in \Omega^{*,\mathrm{sep}}_{K/k}$. For them we can use formula (2.4.7).

Finally, to prove axiom **R3** we consider the diagram

$$
\begin{tikzcd}
 & \Omega^{n,\mathrm{sep}}_{L/k} \arrow[dr, "\mathrm{Tr}_{L/K((\underline{a}))}"] & \\
\mathrm{K}_n L \arrow[ur, "\mathrm{dlog}"] \arrow[r, "\mathrm{N}_{L/K((\underline{a}))}"] \arrow[d, "\mathrm{ord}^n_L"] & \mathrm{K}_n K((\underline{a})) \arrow[r, "\mathrm{dlog}"] \arrow[d, "\mathrm{ord}^n_{K((\underline{a}))}"] & \Omega^{n,\mathrm{sep}}_{K((\underline{a}))/k} \arrow[d, "\mathrm{Res}_{K((\underline{a}))/K,\underline{a}}"] \\
\mathbb{Z} \arrow[r, "{[\kappa_n(L) : K]}"] & \mathbb{Z} \arrow[r] & K
\end{tikzcd}
$$

Looking at formulas (2.4.16), (2.4.17) and (2.4.18) we see that the three small diagrams commute. But the axiom is equivalent to the commutativity of the outer diagram. □

Proof (of thm. 2.4.3) First we show the uniqueness of the residue functor. Let $K \to L$ be a morphism of dimension n in $\mathsf{TLF}(k)$ and let $\underline{a}$ be an initial system of parameters in L of length n. We will show that $\mathrm{Res}_{L/K} = \mathrm{Res}_{L/K,\underline{a}}$. By functoriality and axiom **R2** we may assume that $L = K((\underline{a}))$. Using the factorization (2.4.6) and axiom **R1** it suffices to show that $\mathrm{Res}_{L/K}(\underline{a}^{\underline{i}}\,\mathrm{dlog}(\underline{a})) = \delta_{\underline{i},(0,\dots,0)}$ for all $\underline{i} \in \mathbf{Z}^n$. For $\underline{i} = (0,\dots,0)$ this is axiom **R3**. If $\mathrm{char}\, k = 0$ we are done, since we may "integrate" $\underline{a}^{\underline{i}}\,\mathrm{dlog}(\underline{a})$ if $\underline{i} \neq (0,\dots,0)$, showing it is a cocycle. If $\mathrm{char}\, k = p$ we get $\mathrm{Tr}_{L/L^{(p^j/K)}}(\underline{a}^{\underline{i}}\,\mathrm{dlog}(\underline{a})) = 0$ when j is large enough (take j s.t. $\underline{i} \notin p^j\mathbb{Z}^n$). Now apply functoriality and axiom **R2** to $K \to L^{(p^j/K)} \to L$.

To prove existence it suffices to show that $\mathrm{Res}_{L/K,\underline{a}} = \mathrm{Res}_{L/K,\underline{b}}$ for any two initial systems of parameters $\underline{a}$ and $\underline{b}$. Functoriality is then a consequence of formulas (2.4.11) and (2.4.12), and we already checked that the axioms are satisfied. The initial system of parameters $\underline{a}$ may be taken to be regular. Let us consider three cases:

<u>case 1</u> The map $K \to \kappa_n(L)$ is an isomorphism. Then $L = K((\underline{a}))$. Since the map $\mathrm{Res}_{L/K,\underline{b}}$ satisfies the axioms, the uniqueness proof above (and using formula (2.4.13) in characteristic p) tells us that $\mathrm{Res}_{L/K,\underline{a}} = \mathrm{Res}_{L/K,\underline{b}}$.

<u>case 2</u> The map $K \to \kappa_n(L)$ is separable. Then there is a factorization $K \to E \to L$ with $E \xrightarrow{\simeq} \kappa_n(L)$. Therefore $L = E((\underline{a}))$ and formula (2.4.11) reduces this to case 1.

<u>case 3</u> The map $K \to \kappa_n(L)$ is purely inseparable (and $\mathrm{char}\, k = p$). (Cf. [Lo] lemma 8.) First note that formula (2.4.13) allows us to assume that $K \to L$

is smooth: simply replace L with $K((\underline{a}))[\underline{b}^{p^j}]$ for j sufficiently large. Therefore we need only compare the two maps restricted to $\Omega^{n,\mathrm{sep}}_{L/k}$.

Let $\tilde{K}$ be the field K made into a 0-dimensional field in $\mathsf{TLF}(k)$ (so it has the discrete topology) and let $\tilde{L}$ be the field L with the TLF structure such that $\tilde{K}((\underline{a})) \to \tilde{L}$ is a finite morphism. Note that the original morphism $K \to L$ is then the finitely ramified base change obtained from $\tilde{K} \to \tilde{L}$ and $\tilde{K} \to K$. By lemma 2.4.14 we can assume that $\dim K = 0$.

Now let K' be an algebraic closure of K, considered as a 0-dimensional field in $\mathsf{TLF}(k)$, and let $K' \to B'$ be the morphism in $\mathsf{CTLF}(k)$ obtained by finitely ramified base change from $K \to L$ and $K \to K'$. Again appealing to lemma 2.4.14 it suffices to check that $\mathrm{Res}_{(B'/\mathfrak{q}')/K',\underline{a}} = \mathrm{Res}_{(B'/\mathfrak{q}')/K',\underline{b}}$ for every $\mathfrak{q}' \in \operatorname{Spec} B'$. Since K' is algebraically closed we are back to case 1. $\square$

Corollary 2.4.20 *There exists a unique functor* $\mathrm{Res} : \mathsf{CTLF}_{\mathrm{red}}(k)^{\circ} \to \mathsf{Ab}$ *extending* $\mathrm{Res} : \mathsf{TLF}(k)^{\circ} \to \mathsf{Ab}$, *s.t. for any morphism* $A \to B$, *the residue map* $\mathrm{Res}_{B/A} : \Omega^{*,\mathrm{sep}}_{B/k} \to \Omega^{*,\mathrm{sep}}_{A/k}$ *is (left)* $\Omega^{*,\mathrm{sep}}_{A/k}$*-linear.*

Proof Define $\mathrm{Res}_{B/A} := \sum_{\mathfrak{q}|\mathfrak{p}} \mathrm{Res}_{(B/\mathfrak{q})/(A/\mathfrak{p})}$. $\square$

Definition 2.4.21 *Let* $A \to B$ *be a morphism in* $\mathsf{CTLF}_{\mathrm{red}}(k)$. *The residue pairing is the map*

$$\langle -,- \rangle_{B/A} : B \times \omega_B \xrightarrow{\text{mult}} \omega_B \xrightarrow{\mathrm{Res}_{B/A}} \omega_A \ .$$

We can now state our main result on clusters of TLFs:

Theorem 2.4.22 *(Topological Duality) Let* $A \to B$ *be a morphism in* $\mathsf{CTLF}_{\mathrm{red}}(k)$. *The residue pairing is a perfect pairing of semi-topological* A*-modules.*

Proof We may assume that A and B are fields. Moreover, in view of prop. 2.3.9 we may assume that $B = A((\underline{t})) = A((t_1, \ldots, t_n))$ with $n \geq 1$. Set $\underline{t}' := (t_2, \ldots, t_n)$. Given $\phi \in \mathrm{Hom}^{\mathrm{cont}}_A(B, \omega_A)$ define, for every $i \in \mathbb{Z}$, $\phi_i : A((\underline{t}')) \to \omega_A$, $\phi_i(a) := \phi(a t_1^{-i})$. By induction on n there exists a unique $\alpha_i \in \omega_{A((\underline{t}'))}$ representing ϕ_i with respect to the pairing $\langle -,- \rangle_{A((\underline{t}'))/A}$. According to prop. 1.2.22, $\alpha_i = 0$ for $i << 0$. Then $\alpha := \sum_{i \in \mathbb{Z}} \alpha_i \wedge t_1^i \mathrm{dlog}(t_1) \in \omega_B$ represents ϕ, and it is unique. $\square$

Another important result is: (cf. [Lo] thm. 1 iv))

Theorem 2.4.23 *(Smooth Finitely Ramified Base Change) Let $f : A \to B$ be a morphism in $\mathsf{CTLF}_{\mathrm{red}}(k)$, let $A' \in \mathsf{CTLF}_{\mathrm{red}}(k)$ and let $u : A \to A'$ be a finitely ramified homomorphism, topologically smooth relative to k. Let $f' : A' \to B'$ be the morphism in $\mathsf{CTLF}(k)$ gotten by finitely ramified base change and let $v : B \to B'$ be the corresponding finitely ramified homomorphism (see thm. 2.2.4). Then B' is reduced, $u : \Omega^{*,\mathrm{sep}}_{A/k} \to \Omega^{*,\mathrm{sep}}_{A'/k}$ is injective and the diagram below commutes:*

$$\begin{array}{ccc} \Omega^{*,\mathrm{sep}}_{B/k} & \xrightarrow{v} & \Omega^{*,\mathrm{sep}}_{B'/k'} \\ \Big\downarrow{\scriptstyle \mathrm{Res}_{B/A}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{B'/A'}} \\ \Omega^{*,\mathrm{sep}}_{A/k} & \xrightarrow{u} & \Omega^{*,\mathrm{sep}}_{A'/k'} \end{array}$$

Proof We may assume that A, B and A' are fields. From formula (1.5.12) it follows that $\Omega^{*,\mathrm{sep}}_{A'/k} \cong \Omega^{*,\mathrm{sep}}_{A'/A} \otimes_A \Omega^{*,\mathrm{sep}}_{A/k}$, so u is injective.

Recall the proof of thm. 2.2.4. Since $u : A \to A'$ is smooth in $\mathsf{STComAlg}(k)$, so are $A[\underline{s}] \to A'[\underline{s}] \to A'((\underline{s}))$ (prop. 1.5.9 b) and cor. 1.5.19). But $A[\underline{s}] \to A((\underline{s}))$ is étale, so by diagram chasing in def. 1.5.7 we see that $A((\underline{s})) \to A'((\underline{s}))$ is smooth in $\mathsf{STComAlg}(k)$. By cor. 2.1.16, if $\operatorname{char} k = p$, this implies that $A'((\underline{s}))$ is a separable $A((\underline{s}))$-algebra. In characteristic 0 separability is automatic. Therefore $B' = B \otimes_{A((\underline{s}))} A'((\underline{s}))$ is reduced, which means that $l(B'_{\mathfrak{q}'}) = 1$ for all $\mathfrak{q}' \in \operatorname{Spec} B'$. By lemma 2.4.14 the diagram commutes. □

We saw that in characteristic p the topology on local fields is superfluous (prop. 2.1.21). The next example shows that in characteristic 0 and dimension ≥ 2, not only are there many topologies on a local field, there also are many residue maps, since these depend on the topology. Fix a field $K \in \mathsf{TLF}(k)$ and a morphism $f : \mathsf{unt}(K) \to L$ in $\mathsf{LF}(k)$, and let $\{f^\alpha : K \to L^\alpha\}$ be the morphisms in $\mathsf{TLF}(k)$ in the fibre over f relative to the functor unt. Then the maps $\Omega^*_{L/k} \twoheadrightarrow \Omega^{*,\mathrm{sep}}_{L^\alpha/k} \xrightarrow{\mathrm{Res}_{L^\alpha/K}} \Omega^{*,\mathrm{sep}}_{K/k}$ may change as α varies.

Example 2.4.24 Suppose $\operatorname{char} k = 0$. Let $L^\sigma := k((t_1, t_2)) \in \mathsf{TLF}(k)$ and let $\sigma : k((t_2)) \to L^\sigma$ be the standard morphism. Denote by L the untopologized local field $\mathsf{unt}(L^\sigma)$. Choose a transcendency basis $\{u_\alpha\}$ for $k((t_2))$ over $k(t_2)$ and fix some $u_0 \in \{u_\alpha\}$. Let $\tau : k((t_2)) \to k((t_2))[[t_1]]$ be the unique $k(t_2)$-algebra lifting such that $\tau(u_0) = u_0 + t_1$ and $\tau(u_\alpha) = u_\alpha$, $\alpha \neq 0$. Extend it to an isomorphism $k((t_1, t_2)) \xrightarrow{\simeq} L$ of local fields by sending $t_1 \mapsto t_1$, and let L^τ be this new TLF. The map $\tau : k((t_2)) \to L^\tau$ is then a morphism in $\mathsf{TLF}(k)$.

Consider the form $\beta := t_1^{-1}\mathrm{d}(u_0 + t_1) \wedge t_2^{-1}\mathrm{d}t_2 \in \Omega^*_{L/k}$ (the discrete, infinite dimensional space). Since we have a morphism $\tau : k((t_2)) \to L^\tau$, it follows that $\mathrm{d}(u_0+t_1)\wedge \mathrm{d}t_2 = \tau(\mathrm{d}u_0 \wedge \mathrm{d}t_2) = 0$ in $\Omega^{*,\mathrm{sep}}_{L^\tau/k}$. Therefore $\beta = 0$ and $\mathrm{Res}_{L^\tau/k}(\beta) = 0$.

On the other hand, in $\Omega^{*,\mathrm{sep}}_{L^\sigma/k}$ we have $\mathrm{d}u_0 \wedge \mathrm{d}t_2 = \sigma(\mathrm{d}u_0 \wedge \mathrm{d}t_2) = 0$, so $\beta = t_1^{-1}\mathrm{d}t_1 \wedge t_2^{-1}\mathrm{d}t_2$ and $\mathrm{Res}_{L^\sigma/k}(\beta) = -1$.

Digression 2.4.25 It is possible to define a residue map in Milnor K-theory. First one has the following result: Let $K \to L$ be a finite homomorphism of n-dimensional local fields (def. 2.1.8). Then for all $1 \leq i \leq n$,

$$\mathrm{N}_{\kappa_i(L)/\kappa_i(K)} \circ \partial^i_L = \partial^i_K \circ \mathrm{N}_{L/K} : \mathrm{K}_*L \to \mathrm{K}_{*-i}\kappa_i(K). \tag{2.4.26}$$

Here $\partial^i_K = \partial \circ \cdots \circ \partial$, so $\partial^i_K|_{\mathrm{K}_iK} = \mathrm{ord}^i_K$. The proof uses the same ideas found in [BT] ch. I §5.9 and [Ka] §1.7. This generalizes prop. 3.1 of [Kh].

Given a morphism $K \to L$ in $\mathsf{LF}(k)$, define

$$\mathrm{Res}^{\mathrm{M}}_{L/K} := \mathrm{N}_{\kappa_n(L)/K} \circ \partial^n_L : \mathrm{K}_*L \to \mathrm{K}_{*-n}K \tag{2.4.27}$$

where $n = \dim(L/K)$. Formula (2.4.26) implies that $\mathrm{Res}^{\mathrm{M}} := (\mathrm{K}_*(-), \mathrm{Res}^{\mathrm{M}}_{-/-})$ is a functor $\mathsf{LF}(k)^\circ \to \mathsf{Ab}$. It is not hard to verify that $\mathrm{dlog} : \mathrm{Res}^{\mathrm{M}} \circ \mathsf{unt} \to \mathrm{Res}$ is a natural transformation of functors on $\mathsf{TLF}(k)^\circ$. This is of particular interest in characteristic 0, when one takes into account example 2.4.24 and the preceding discussion.

Digression 2.4.28 We may define a version of de Rham cohomology in $\mathsf{TLF}(k)$ and get a Poincaré duality. Let us consider the easier case of a morphism $K \to L = K((\underline{t})) = K((t_1, \ldots, t_n))$. Set $\mathrm{Z}^*\Omega^*_{L/K} := \ker(\mathrm{d})$ and $\mathrm{B}^*\Omega^*_{L/K} := \mathrm{im}(\mathrm{d})$. If $\operatorname{char} k = p$ there is a relative Cartier operation, an $L^{(p/K)}$-algebra epimorphism $\mathrm{C}_{L/K} : \mathrm{Z}^*\Omega^*_{L/K} \twoheadrightarrow \Omega^*_{L^{(p/K)}/K}$ with kernel $\mathrm{B}^*\Omega^*_{L/K}$ (see [Il] §2.1). Define $\mathrm{Z}^*_1\Omega^*_{L/K} := \mathrm{Z}^*\Omega^*_{L/K}$, $\mathrm{B}^*_1\Omega^*_{L/K} := \mathrm{B}^*\Omega^*_{L/K}$ and by recursion

$$\mathrm{Z}^*_{i+1} = \mathrm{Z}^*_{i+1}\Omega^*_{L/K} := \mathrm{C}^{-1}_{L/K}(\mathrm{Z}^*_i\Omega^*_{L^{(p/K)}/K})$$

$$\mathrm{B}^*_{i+1} = \mathrm{B}^*_{i+1}\Omega^*_{L/K} := \mathrm{C}^{-1}_{L/K}(\mathrm{B}^*_i\Omega^*_{L^{(p/K)}/K}).$$

Then $\mathrm{B}^*_1 \subset \mathrm{B}^*_2 \subset \ldots \subset \mathrm{B}^*_i \subset \mathrm{Z}^*_i \subset \ldots \subset \mathrm{Z}^*_2 \subset \mathrm{Z}^*_1$. Set $\mathrm{Z}^*_\infty := \bigcap \mathrm{Z}^*_i$ and $\mathrm{B}^*_\infty :=$ closure of $\bigcup \mathrm{B}^*_i$ in $\Omega^{*,\mathrm{sep}}_{L/K}$. Having done so, we define

$$\mathrm{H}^{*,\mathrm{sep}}_{\mathrm{DR}}(L/K) := \begin{cases} \mathrm{H}^*\Omega^{*,\mathrm{sep}}_{L/K} & \text{if } \operatorname{char} k = 0 \\ \frac{\mathrm{Z}^*_\infty}{\mathrm{B}^*_\infty} & \text{if } \operatorname{char} k = p \end{cases}$$

Then one can show that

$$\mathrm{H}_{\mathrm{DR}}^{*,\mathrm{sep}}(L/K) \cong K \otimes_k \bigwedge\nolimits_k^* \left[\bigoplus_{i=1}^{n} k \cdot \mathrm{dlog} t_i\right]$$

and that the residue map induces a perfect pairing

$$\mathrm{H}_{\mathrm{DR}}^{i,\mathrm{sep}}(L/K) \times \mathrm{H}_{\mathrm{DR}}^{n-i,\mathrm{sep}}(L/K) \to \mathrm{H}_{\mathrm{DR}}^{n,\mathrm{sep}}(L/K) \xrightarrow{\mathrm{Res}_{L/K}} K.$$

3 The Beilinson Completion Functors

3.1 Definition of the Completions

Let X be a noetherian scheme. In [Be] A. Beilinson defines sheaves of adeles on X with values in any quasi-coherent sheaf. The "local factors" of the adeles are the completions discussed in this section.

Given a subset $S \subset X$ we denote its closure by S^-. If $x, y \in X$ are points s.t. y is a specialization of x, i.e. $y \in \{x\}^-$, we shall indicate this by writing $x \geq y$.

Definition 3.1.1 *Let n be a natural number. A chain of length n in X is a sequence $\xi = (x_0, \ldots, x_n)$ of points of X with $x_i > x_{i+1}$ for all i. If for every i, x_{i+1} is an immediate specialization of x_i (i.e. $\mathrm{codim}(\{x_{i+1}\}^-, \{x_i\}^-) = 1$), we call ξ a saturated chain.*

Let $\xi = (x_0, \ldots, x_n)$ be a chain. A face of ξ is any subchain η. For any integer $i = 0, \ldots, n$, the i-th face of ξ is the chain $\mathrm{d}_i\xi := (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. We say that ξ begins with x_0 and ends with x_n. Formally we introduce a chain $\underline{1}$ of length -1, and set $\mathrm{d}_0(x_0) := \underline{1}$. By convention whenever we write $\xi = (x, \ldots)$ etc. or specify that a chain ξ is saturated, it is implied that $\xi \neq \underline{1}$. If $\xi = (x_0, \ldots, x_n)$ and $\eta = (y_0, \ldots, y_m)$ are chains s.t. $x_n > y_0$, their concatenation is defined to be the chain $\xi \vee \eta := (x_0, \ldots, x_n, y_0, \ldots, y_m)$. For any chain ξ define $\xi \vee \underline{1} = \underline{1} \vee \xi := \xi$.

Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Its stalk at the point $x \in X$ is denoted by $\mathcal{M}_x$. Let $\mathfrak{m}_x \in \mathcal{O}_{X,x}$ be the maximal ideal. If $\mathcal{M}$ is quasi-coherent then for any $i \geq 0$, $\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x$ is a skyscraper quasi-coherent sheaf supported on $\{x\}^-$.

Definition 3.1.2 *To each chain ξ in X we associate an additive functor $(-)_\xi : \mathcal{M} \mapsto \mathcal{M}_\xi$ from the category $\mathsf{QCoh}(X)$ of quasi-coherent sheaves on X to the category Ab of abelian groups, called the Beilinson completion along ξ. The definition is by recursion on the length of ξ.*

a) *For any quasi-coherent sheaf* $\mathcal{M}$ *set* $\mathcal{M}_{\underline{1}} := \Gamma(X, \mathcal{M})$.

b) *Suppose* $\xi = (x, \ldots)$ *has length* ≥ 0.

  i) *Given a coherent sheaf* $\mathcal{M}$ *set* $\mathcal{M}_\xi := \lim_{\leftarrow i}(\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)_{\mathrm{d}_0\xi}$.

  ii) *Let* $\mathcal{M}$ *be a quasi-coherent sheaf and let* $(\mathcal{M}_\alpha)$ *be the direct system of its coherent subsheaves. Set* $\mathcal{M}_\xi := \lim_{\alpha \rightarrow}(\mathcal{M}_\alpha)_\xi$.

Let $\xi = (x_0, \ldots, x_n)$. For every $i = 0, \ldots, n$ there is a natural transformation $\partial_i : \mathcal{M}_{\mathrm{d}_i\xi} \rightarrow \mathcal{M}_\xi$, called the i-th face map. These satisfy the simplicial relations $\partial_j\partial_i = \partial_{i-1}\partial_j$ for all $i > j$. Thus for any face η of ξ there is a well defined face transformation $\partial : (-)_\eta \rightarrow (-)_\xi$. If $n = 0$ and $\mathcal{M}$ is coherent then $\mathcal{M}_{(x_0)}$ is nothing but the $\mathfrak{m}_{x_0}$-adic completion of $\mathcal{M}_{x_0}$. When $\mathcal{M}_{x_0}$ has finite length over the local ring $\mathcal{O}_{X,x_0}$, the face map $\partial_0 : \mathcal{M}_{\mathrm{d}_0\xi} \rightarrow \mathcal{M}_\xi$ is bijective. For the structure sheaf we abbreviate and write $\mathcal{O}_{X,\xi}$ instead of $(\mathcal{O}_X)_\xi$.

The group $\mathcal{M}_{\underline{1}}$ is only an auxiliary device introduced to simplify definitions and proofs. Completion along an actual chain $\xi = (x_0, \ldots, x_n)$ (as opposed to $\xi = \underline{1}$) is a local process - it depends only on the stalks at $x_n \in X$. Thus we can replace X with any open subscheme $U \subset X$ which contains x_n.

When convenient we shall consider the completion $\mathcal{M}_{(x_0,\ldots,x_n)}$ as a skyscraper sheaf supported on the closed set $\{x_n\}^-$. (It is seldom quasi-coherent !) Doing so the completion becomes a functor $(-)_\xi : \mathsf{QCoh}(X) \rightarrow \mathsf{Mod}(X)$ and the face maps ∂_i become $\mathcal{O}_X$-linear.

Consider the prototypical example:

Example 3.1.3 Let $X := \mathbb{A}^2_k = \operatorname{Spec} k[s,t]$, the affine plane over a field k. Take $x := (0)$, $y := (t)$ and $z := (s,t)$ in X, so $\xi := (x, y, z)$ is a saturated chain of length 2. We then have

$$\begin{aligned} \mathcal{O}_{X,(x)} &= k(s,t)\ , & \mathcal{O}_{X,(x,y)} &= k(s)((t))\ , \\ \mathcal{O}_{X,(y)} &= k(s)[[t]]\ , & \mathcal{O}_{X,(y,z)} &= k((s))[[t]]\ , \\ \mathcal{O}_{X,(z)} &= k[[s,t]]\ , & \mathcal{O}_{X,(x,y,z)} &= k((s))((t))\ . \end{aligned}$$

Proposition 3.1.4 *For any chain* ξ *(of length* ≥ 0*) the functor* $(-)_\xi$: $\mathsf{QCoh}(X) \rightarrow \mathsf{Ab}$ *is exact and commutes with direct limits.*

Proof The proof is by induction on the length of $\xi = (x, \ldots)$ and is divided into steps. We may assume that X is affine, so the functor $(-)_{\underline{1}}$ is exact.

1) Consider the functor $(-)_\xi : \mathsf{Coh}(X) \to \mathsf{Ab}$. To prove its exactness we modify the the proof for the usual adic completion over a noetherian ring. Given an exact sequence

$$\mathcal{M}^* = (0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0)$$

in $\mathsf{Coh}(X)$, define an inverse system $(\mathcal{M}_i^*)_{i\in\mathbb{N}}$, where $\mathcal{M}_i := \mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x$, $\mathcal{M}''_i := \mathcal{M}''_x/\mathfrak{m}_x^{i+1}\mathcal{M}''_x$ and $\mathcal{M}'_i := \mathrm{im}(\mathcal{M}'_x \to \mathcal{M}_i)$. Since $(-)_{\mathrm{d}_0\xi}$ is exact by induction, we get an inverse system of exact sequences $(\,(\mathcal{M}_i^*)_{\mathrm{d}_0\xi}\,)_{i\in\mathbb{N}}$ in Ab, and since $(\mathcal{M}'_{i+1})_{\mathrm{d}_0\xi} \to (\mathcal{M}'_i)_{\mathrm{d}_0\xi}$ is surjective for all i, we get an exact sequence

$$0 \to \lim_{\leftarrow i}(\mathcal{M}'_i)_{\mathrm{d}_0\xi} \to \mathcal{M}_\xi \to \mathcal{M}''_\xi \to 0\ .$$

But by the Artin-Rees lemma the filtration $(\,\mathcal{M}'_x \cap \mathfrak{m}_x^{i+1}\mathcal{M}_x\,)_{i\in\mathbb{N}}$ on $\mathcal{M}'_x$ is cofinal with the $\mathfrak{m}_x$-adic filtration on it. Therefore $\lim_{\leftarrow i}(\mathcal{M}'_i)_{\mathrm{d}_0\xi} \cong \mathcal{M}'_\xi$.

2) Now suppose $\lim_{\alpha\to} \mathcal{M}_\alpha = \mathcal{N}$ in $\mathsf{Coh}(X)$. Define $\mathcal{M}'_\alpha := \ker(\mathcal{M}_\alpha \to \mathcal{N})$ and $\mathcal{M}''_\alpha := \mathrm{im}(\mathcal{M}_\alpha \to \mathcal{N})$. Then $\lim_{\alpha\to} \mathcal{M}'_\alpha = 0$; since the category $\mathsf{Coh}(X)$ is noetherian, for each α_0 there exists some $\alpha_1 \geq \alpha_0$ such that $\mathrm{im}(\mathcal{M}'_{\alpha_0} \to \mathcal{M}'_{\alpha_1}) = 0$. This implies that $\lim_{\alpha\to}(\mathcal{M}'_\alpha)_\xi = 0$ too. Because $(-)_\xi$ and $\lim_\to$ are exact functors we have $\lim_{\alpha\to}(\mathcal{M}_\alpha)_\xi \xrightarrow{\simeq} \lim_{\alpha\to}(\mathcal{M}''_\alpha)_\xi$. Now there exists some α_0 s.t. $\mathcal{M}''_{\alpha_0} \xrightarrow{\simeq} \mathcal{M}''_\alpha \xrightarrow{\simeq} \mathcal{N}$ for all $\alpha \geq \alpha_0$; therefore $(\mathcal{M}''_{\alpha_0})_\xi \xrightarrow{\simeq} \lim_{\alpha\to}(\mathcal{M}''_\alpha)_\xi \xrightarrow{\simeq} \mathcal{N}_\xi$.

3) Suppose we are given a direct system $(\mathcal{M}_\alpha)_{\alpha\in I}$ in $\mathsf{QCoh}(X)$, with $\lim_{\alpha\to} \mathcal{M}_\alpha = \mathcal{N}$. Each $\mathcal{M}_\alpha$ is itself a direct limit of coherent sheaves; since direct limits commute we may assume that all $\mathcal{M}_\alpha$ are coherent. Let $(\mathcal{N}_\beta)_{\beta\in J}$ be the direct system of coherent subsheaves of $\mathcal{N}$. For each $(\alpha,\beta) \in I \times J$ let $\mathcal{L}_{\alpha,\beta} := \mathcal{M}_\alpha \times_{\mathcal{N}} \mathcal{N}_\beta$, a coherent sheaf. The direct system $(\mathcal{L}_{\alpha,\beta})_{(\alpha,\beta)\in I\times J}$ is a common refinement of $(\mathcal{M}_\alpha)_{\alpha\in I}$ and $(\mathcal{N}_\beta)_{\beta\in J}$, and by step 2)

$$\lim_{\alpha\to}(\mathcal{M}_\alpha)_\xi \cong \lim_{(\alpha,\beta)\to}(\mathcal{L}_{\alpha,\beta})_\xi \cong \lim_{\beta\to}(\mathcal{N}_\beta)_\xi = \mathcal{N}_\xi\ .$$

4) Finally any exact sequence $\mathcal{M}^*$ in $\mathsf{QCoh}(X)$ is a limit of some direct system of exact sequences $\mathcal{M}^*_\alpha$ of coherent sheaves. Since $(\mathcal{M}^*)_\xi \cong \lim_{\alpha\to}(\mathcal{M}^*_\alpha)_\xi$ it is exact. □

Just like for the adic completion of a noetherian ring we have:

Corollary 3.1.5 *For any chain ξ in X (of length ≥ 0) there is a natural isomorphism of $\mathcal{O}_X$-modules $\mathcal{M}_\xi \cong \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\xi}$. The $\mathcal{O}_X$-module $\mathcal{O}_{X,\xi}$ is flat.*

Proposition 3.1.6 *Let ξ be a chain in X (of length ≥ 0). Then the completion $\mathcal{O}_{X,\xi}$ is a commutative $\mathcal{O}_X$-algebra. Given a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$, the completion $\mathcal{B}_\xi$ is an $\mathcal{O}_{X,\xi}$-algebra. If $\mathcal{B}$ is coherent then $\mathcal{B}_\xi$ is a noetherian ring.*

Proof We may assume that X is affine, so $\mathcal{O}_{X,\underline{1}} = \Gamma(X, \mathcal{O}_X)$ is a noetherian ring. The proof is by induction on the length of $\xi = (x, \ldots)$.

1) For any $i \geq 0$ set $\mathcal{B}_i := \mathcal{O}_{X,x}/\mathfrak{m}_x^{i+1}$. This is a quasi-coherent $\mathcal{O}_X$-algebra. By induction $(\mathcal{B}_i)_{\mathrm{d}_0\xi}$ is an $\mathcal{O}_{X,\mathrm{d}_0\xi}$-algebra. Moreover, since $(\mathcal{B}_i)_{\mathrm{d}_0\xi} \cong \mathcal{B}_i \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\mathrm{d}_0\xi}$ is a localization of a quotient of the noetherian ring $\mathcal{O}_{X,\mathrm{d}_0\xi}$, it is noetherian too. Passing to the inverse limit we conclude that $\mathcal{O}_{X,\xi} = \lim_{\leftarrow i}(\mathcal{B}_i)_{\mathrm{d}_0\xi}$ is an $\mathcal{O}_{X,\mathrm{d}_0\xi}$-algebra. According to [CA] ch. III §2.10 cor. 5, this is a noetherian ring.

2) Let $\mathcal{B}$ be any quasi-coherent $\mathcal{O}_X$-algebra. By corollary 3.1.5 we have $\mathcal{B}_\xi \cong \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\xi}$. The right hand side exhibits $\mathcal{B}_\xi$ as an $\mathcal{O}_{X,\xi}$-algebra. If $\mathcal{B}$ is coherent then $\mathcal{B}_\xi$ is finite over $\mathcal{O}_{X,\xi}$, so it is noetherian. $\square$

Given a chain $\xi = (x, \ldots)$ we shall write $k(\xi) := k(x)_\xi$ and $\mathfrak{m}_\xi := (\mathfrak{m}_x)_\xi$. Thus $\mathfrak{m}_\xi \subset \mathcal{O}_{X,\xi}$ is an ideal and $\mathcal{O}_{X,\xi}/\mathfrak{m}_\xi = k(\xi)$.

Let $f : X \to Y$ be a finite morphism of noetherian schemes. If $y \in Y$ and $x \in f^{-1}(y)$, that is if x lies over y, we shall write $x|y$. This standard notation can be extended to chains: given chains $\eta = (y_0, \ldots, y_n)$ in Y and $\xi = (x_0, \ldots, x_n)$ in X s.t. $x_i|y_i$ for all i, we shall write $\xi|\eta$.

Proposition 3.1.7 *Let $f : X \to Y$ be a finite morphism of noetherian schemes, let $\mathcal{M} \in \mathsf{QCoh}(X)$ and let η be a chain in Y. Then there is a natural isomorphism $(f_*\mathcal{M})_\eta \cong \bigoplus_{\xi|\eta} \mathcal{M}_\xi$.*

Proof The proof is by induction on the length of η. To start the induction note that $(f_*\mathcal{M})_{\underline{1}} \cong \mathcal{M}_{\underline{1}}$. Say $\eta = (y, \ldots)$. First assume that $\mathcal{M}$ is coherent. Then $f_*\mathcal{M}$ is also coherent and the two inverse systems in $\mathsf{QCoh}(Y)$: $\left(\bigoplus_{x|y} f_*(\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)\right)_{i\in\mathbb{N}}$ and $\left((f_*\mathcal{M})_y/\mathfrak{m}_y^{i+1}(f_*\mathcal{M})_y\right)_{i\in\mathbb{N}}$ are equivalent. By induction we have for all $i \geq 0$ and all $x|y$:

$$f_*(\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)_{\mathrm{d}_0\eta} \cong \bigoplus_{\xi'|\mathrm{d}_0\eta} (\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)_{\xi'} .$$

Therefore

$$\begin{aligned} (f_*\mathcal{M})_\eta &= \lim_{\leftarrow i}((f_*\mathcal{M})_y/\mathfrak{m}_y^{i+1}(f_*\mathcal{M})_y)_{\mathrm{d}_0\eta} \\ &\cong \lim_{\leftarrow i}\left(\bigoplus_{x|y} f_*(\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)\right)_{\mathrm{d}_0\eta} \\ &\cong \bigoplus_{x|y} \bigoplus_{\xi'|\mathrm{d}_0\eta} \lim_{\leftarrow i}(\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)_{\xi'} \\ &\cong \bigoplus_{\xi|\eta} \mathcal{M}_\xi . \end{aligned}$$

Now let $\mathcal{M}$ be quasi-coherent. Then every coherent subsheaf $\mathcal{N}'_\alpha \subset f_*\mathcal{M}$ is contained in $f_*\mathcal{N}_\beta$ for some coherent $\mathcal{N}_\beta \subset \mathcal{M}$. According to prop. 3.1.4 we have $(f_*\mathcal{M})_\eta \cong \bigoplus_{\xi|\eta} \mathcal{M}_\xi$. $\square$

From here to the end of §3.1 we assume that X is a scheme of finite type over some noetherian ring k. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules and let $D : \mathcal{M} \to \mathcal{N}$ be a k-linear sheaf homomorphism. D is called a differential operator (DO) of order $\leq d$ (relative to k) if the following holds: for every open set $U \subset X$ the homomorphism $D : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})$ is a differential operator of order $\leq d$ over the k-algebra $\Gamma(U, \mathcal{O}_X)$. The set of all operators of order $\leq d$ is denoted by $\mathrm{Diff}^d_{X/k}(\mathcal{M}, \mathcal{N})$, and taking the union over all $d \geq 0$ we get $\mathrm{Diff}_{X/k}(\mathcal{M}, \mathcal{N})$. The composition of differential operators is again a differential operator, of a higher order. Differential operators of order $\leq d$ are represented by the sheaf of principal parts of order d, $\mathcal{P}^d_{X/k}(\mathcal{M}) = \mathcal{P}^d_{X/k} \otimes_{\mathcal{O}_X} \mathcal{M}$; that is to say, there is a canonical isomorphism

$$\mathrm{Diff}^d_{X/k}(\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_X(\mathcal{P}^d_{X/k}(\mathcal{M}), \mathcal{N}) \ .$$

(For more details see [EGA IV] §16.8 and §1.4 here.)

Definition 3.1.8 a) *Let A be a commutative k-algebra and let M and N be A-modules. A locally differential operator over A, relative to k, is a k-linear homomorphism $D : M \to N$ s.t. for every finitely generated A-submodule $M' \subset M$, $D|_{M'}$ is a differential operator over A.*

b) *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules and let $D : \mathcal{M} \to \mathcal{N}$ be a k-linear sheaf homomorphism. We call D a locally differential operator (relative to k) if for every open subset $U \subset X$ the homomorphism $D : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})$ is a locally differential operator over $\Gamma(U, \mathcal{O}_X)$.*

If $\mathcal{M}$ is coherent then any locally differential operator is actually a differential operator (because X is noetherian). The usage of the adverb "locally" is confusing, since it has nothing to do with the topology of X; however this usage is common in representation theory.

Lemma 3.1.9 a) *Let $\mathcal{M}$ a coherent sheaf, let $\mathcal{N}$ be a quasi-coherent sheaf and let $D \in \mathrm{Diff}_{X/k}(\mathcal{M}, \mathcal{N})$. Then $\mathrm{im}(D)$ is contained in some coherent subsheaf of $\mathcal{N}$.*

b) *The composition of two locally differential operators is again a locally differential operator.*

Proof a) Since X is of finite type over k the sheaf $\mathcal{P}^d_{X/k}(\mathcal{M})$ is coherent (cf. [EGA IV] prop. 16.8.6). If $\phi : \mathcal{P}^d_{X/k}(\mathcal{M}) \to \mathcal{N}$ represents D then $\mathrm{im}(D) \subset \mathrm{im}(\phi)$. (In fact $\mathrm{im}(\phi)$ is the $\mathcal{O}_X$-submodule of $\mathcal{N}$ generated by $\mathrm{im}(D)$.)

b) Given quasi-coherent sheaves $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$, and locally differential operators $\mathcal{L} \xrightarrow{D} \mathcal{M} \xrightarrow{E} \mathcal{N}$ we have to show that $E \circ D$ is a locally differential operator as well. Let $\mathcal{L}' \subset \mathcal{L}$ be a coherent subsheaf. Then $D|_{\mathcal{L}'}$ is a differential operator. By the preceding lemma, $\mathrm{im}(D) \subset \mathcal{M}'$ for some coherent $\mathcal{M}' \subset \mathcal{M}$. Since $E|_{\mathcal{M}'}$ is a differential operator, so is $E \circ D|_{\mathcal{L}'} = E|_{\mathcal{M}'} \circ D|_{\mathcal{L}'}$. □

Proposition 3.1.10 *Let $\mathcal{M}$ and $\mathcal{N}$ be quasi-coherent sheaves on X and let ξ be a chain in X. Any locally differential operator (relative to k) $D : \mathcal{M} \to \mathcal{N}$ extends to a locally differential operator over $\mathcal{O}_{X,\xi}$, $D_\xi : \mathcal{M}_\xi \to \mathcal{N}_\xi$. If D is has order $\leq d$, then so does D_ξ. The assignment $D \mapsto D_\xi$ is functorial.*

Proof The proof is by induction on the length of $\xi = (x, \ldots)$. For $\xi = \underline{1}$ the statement is trivial.

1) Assume $\mathcal{M}$ is coherent. According to lemma 3.1.9 a) there is a coherent subsheaf $\mathcal{N}' \subset \mathcal{N}$ such that D factors through $\mathcal{N}'$. Thus we may assume $\mathcal{N}$ to be coherent too. Let d be the order of D. According to prop. 1.4.6, for any integer $i \geq 0$ we have $D(\mathfrak{m}_x^{i+d+1}\mathcal{M}_x) \subset \mathfrak{m}_x^{i+1}\mathcal{N}_x$, so there are well defined differential operators

$$D_i : \mathcal{M}_x/\mathfrak{m}_x^{i+d+1}\mathcal{M}_x \to \mathcal{N}_x/\mathfrak{m}_x^{i+1}\mathcal{N}_x \ .$$

Upon applying $(-)_{\mathrm{d}_0\xi}$ we get:

$$(D_i)_{\mathrm{d}_0\xi} : (\mathcal{M}_x/\mathfrak{m}_x^{i+d+1}\mathcal{M}_x)_{\mathrm{d}_0\xi} \to (\mathcal{N}_x/\mathfrak{m}_x^{i+1}\mathcal{N}_x)_{\mathrm{d}_0\xi}$$

which has order $\leq d$ too. Passing to the inverse limit in i we obtain $D_\xi : \mathcal{M}_\xi \to \mathcal{N}_\xi$. If $E : \mathcal{N} \to \mathcal{L}$ is another locally DO then these considerations show that $(E \circ D)_\xi = E_\xi \circ D_\xi$.

2) Next assume $\mathcal{M}$ is quasi-coherent, and let $(\mathcal{M}_\alpha)$ be the collection of its coherent subsheaves. The functoriality of D_ξ on coherent sheaves shows that the differential operators $D_\xi : (\mathcal{M}_\alpha)_\xi \to \mathcal{N}_\xi$ patch together to a locally differential operator $D_\xi : \mathcal{M}_\xi \to \mathcal{N}_\xi$. □

3.2 Topologizing the Completions

In this section X is a scheme of finite type over a noetherian ring k. We introduce a canonical linear topology on the Beilinson completions $\mathcal{M}_\xi$. Recall that the category TopAb of linearly topologized abelian groups is additive and has direct and inverse limits (see §1.1). Repeating definition 3.1.2, but this time with TopAb instead of Ab, we get

Definition 3.2.1 *To each chain ξ in X, we associate an additive functor $(-)_\xi$: $\mathsf{QCoh}(X) \to \mathsf{TopAb}$. The definition is by recursion on the length of ξ.*

a) *For any quasi-coherent sheaf $\mathcal{M}$ set $\mathcal{M}_{\underline{1}} := \Gamma(X, \mathcal{M})$ with the discrete topology.*

b) *Suppose $\xi = (x, \dots)$ has length ≥ 0.*

   i) *Given a coherent sheaf $\mathcal{M}$, set $\mathcal{M}_\xi := \lim_{\leftarrow i} (\mathcal{M}_x / \mathfrak{m}_x^{i+1} \mathcal{M}_x)_{\mathrm{d}_0 \xi}$ in* TopAb.

   ii) *Let $\mathcal{M}$ be a quasi-coherent sheaf and let $(\mathcal{M}_\alpha)$ be the direct system of its coherent subsheaves. Set $\mathcal{M}_\xi := \lim_{\alpha \to} (\mathcal{M}_\alpha)_\xi$ in* TopAb.

Forgetting the topology we recover definition 3.1.2. Thus the completion $\mathcal{M}_\xi$ has many facets: a discrete abelian group, a linearly topologized abelian group, or an $\mathcal{O}_X$-module. There will be even more facets to $\mathcal{M}_\xi$, all depending on context.

Say ξ has length $n \geq 0$. The face maps $\partial_i : \mathcal{M}_{\mathrm{d}_i \xi} \to \mathcal{M}_\xi$, $0 \leq i \leq n$, are continuous. Later in this section we shall see that for $i = n$, ∂_i is a dense map, and for $i = 0$ it is strict. A special instance of this is $k(t) \hookrightarrow k((t))$ (dense) and $k[[t]] \hookrightarrow k((t))$ (strict).

The next two propositions are proved just like their counterparts in §3.1, using the recursive definition of the topology.

Proposition 3.2.2 *Let $\mathcal{M}$ and $\mathcal{N}$ be quasi-coherent sheaves and let $D : \mathcal{M} \to \mathcal{N}$ be a locally differential operator. Then the induced operator D_ξ of prop. 3.1.10 is continuous.*

Proposition 3.2.3 *Let $f : X \to Y$ be a finite morphism, let $\mathcal{M}$ be a quasi-coherent sheaf on X and let η be a chain in Y. Then the isomorphism $(f_* \mathcal{M})_\eta \cong \bigoplus_{\xi | \eta} \mathcal{M}_\xi$ of prop. 3.1.7 is a homeomorphism.*

Recall the definition of semi-topological (ST) rings and modules from §1.2. We put on the base ring k the discrete topology.

Proposition 3.2.4 *Let ξ be a chain in X. The completion $\mathcal{O}_{X,\xi}$ is a semi-topological k-algebra. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$, the completion $\mathcal{M}_\xi$ is a semi-topological $\mathcal{O}_{X,\xi}$-module. Given a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$, the completion $\mathcal{B}_\xi$ is a semi-topological $\mathcal{O}_{X,\xi}$-algebra.*

Proof The proposition amounts to the following statement: given an $\mathcal{O}_X$-bilinear pairing $\langle -, - \rangle : \mathcal{L} \times \mathcal{M} \to \mathcal{N}$ of quasi-coherent sheaves, the pairing

$\langle -,- \rangle_\xi : \mathcal{L}_\xi \times \mathcal{M}_\xi \to \mathcal{N}_\xi$ obtained by tensoring with $\mathcal{O}_{X,\xi}$ has the property that $\langle a,- \rangle : \mathcal{M}_\xi \to \mathcal{N}_\xi$ is continuous for all $a \in \mathcal{L}_\xi$. The statement is trivial for $\xi = \underline{1}$, so we can use induction on length.

By the definition of the topology one may assume that all three sheaves are coherent. Say $\xi = (x, \ldots)$. For every $i \geq 0$ there is a pairing

$$(\mathcal{L}_x/\mathfrak{m}_x^{i+1}\mathcal{L}_x)_{\mathrm{d}_0\xi} \times (\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)_{\mathrm{d}_0\xi} \to (\mathcal{N}_x/\mathfrak{m}_x^{i+1}\mathcal{N}_x)_{\mathrm{d}_0\xi} \,.$$

By induction for every $a \in \mathcal{L}_\xi$ the homomorphism

$$\langle a,- \rangle : (\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)_{\mathrm{d}_0\xi} \to (\mathcal{N}_x/\mathfrak{m}_x^{i+1}\mathcal{N}_x)_{\mathrm{d}_0\xi}$$

is continuous, and passing to the inverse limit shows that $\langle a,- \rangle : \mathcal{M}_\xi \to \mathcal{N}_\xi$ is continuous too. □

In this way we get a functor $(-)_\xi : \mathsf{QCoh}(X) \to \mathsf{STMod}(\mathcal{O}_{X,\xi})$, the latter being the category of ST $\mathcal{O}_{X,\xi}$-modules and continuous homomorphisms. The topology on the completion $\mathcal{M}_\xi$ is described below.

In §1.2 it was shown that a module M over a ST ring A has a finest topology with respect to which it becomes a ST module. This topology was called the fine A-module topology. Suppose $A_0 \subset A$ is a subring s.t. M has a basis of neighborhoods of 0 consisting of A_0-submodules; then we say that M has an A_0-linear topology.

Proposition 3.2.5 *Let ξ be a chain in X and let $\mathcal{M}$ be a quasi-coherent sheaf.*

a) *The topology on $\mathcal{M}_\xi$ is the fine $\mathcal{O}_{X,\xi}$-module topology.*

b) *Suppose $\xi = (\ldots, y)$; then the topology on $\mathcal{M}_\xi$ is $\mathcal{O}_{X,(y)}$-linear, and hence also k-linear.*

Proof a) The proof is by induction on the length of ξ. The statement is trivial for $\xi = \underline{1}$ since the discrete topology is the fine topology over a discrete ring. Say $\xi = (x, \ldots)$. First assume $\mathcal{M}$ is coherent. Replace X with a small enough neighborhood of ξ to get a surjection $\mathcal{O}_X^r \twoheadrightarrow \mathcal{M}$. For every $i \geq 0$ we have a surjection $(\mathcal{O}_{X,x}/\mathfrak{m}_x^{i+1}\mathcal{O}_{X,x})^r_{\mathrm{d}_0\xi} \twoheadrightarrow (\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x)_{\mathrm{d}_0\xi}$ in $\mathsf{STMod}(\mathcal{O}_{X,\mathrm{d}_0\xi})$. By induction both modules have the fine $\mathcal{O}_{X,\mathrm{d}_0\xi}$-module topology, so by cor. 1.2.8 this is a strict epimorphism. Passing to the inverse limit in i and using prop. 1.1.6 (cf. also the proof of prop. 3.1.4) we see that $\mathcal{O}_{X,\xi}^r \twoheadrightarrow \mathcal{M}_\xi$ is strict, so $\mathcal{M}_\xi$ has the fine $\mathcal{O}_{X,\xi}$-module topology.

Next let $\mathcal{M}$ be quasi-coherent and let $(\mathcal{M}_\alpha)$ be the direct system of its coherent subsheaves. By definition $\mathcal{M}_\xi = \lim_{\alpha \to} (\mathcal{M}_\alpha)_\xi$ in $\mathsf{STMod}(\mathcal{O}_{X,\xi})$, so by cor. 1.2.6 it has the fine $\mathcal{O}_{X,\xi}$-module topology.

b) All the limiting processes occurring in def. 3.2.1 involve $\mathcal{O}_{X,(y)}$-modules with $\mathcal{O}_{X,(y)}$-linear topologies and therefore remain within this subcategory of **TopAb** (cf. prop. 1.2.23). □

The tensor product of ST A-modules admits a canonical topology (see def. 1.2.11). By prop. 3.2.5 and cor. 1.2.15 we have:

Corollary 3.2.6 a) *If X is affine then for any quasi-coherent sheaf $\mathcal{M}$ and any chain ξ, $\mathcal{M}_\xi \cong \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{X,\underline{1}}} \mathcal{M}_{\underline{1}}$ as ST $\mathcal{O}_{X,\xi}$-modules.*

b) *If η is a face of ξ of length ≥ 0 then $\mathcal{M}_\xi \cong \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{X,\eta}} \mathcal{M}_\eta$ as ST $\mathcal{O}_{X,\xi}$-modules.*

Recall the notion of a topologically étale homomorphism relative to k, introduced in §1.5.

Proposition 3.2.7 *Assume $X = \operatorname{Spec} A$ is affine and let ξ be a chain in it. Then $\mathcal{O}_{X,\xi}$ is topologically étale over $\mathcal{O}_{X,\underline{1}} = A$, relative to k.*

Proof The proof is by induction on the length of ξ; if $\xi = \underline{1}$ the statement is trivial. Say $\xi = (x, \dots)$. Define $B := (\mathcal{O}_{X,x})_{\mathrm{d}_0\xi} \cong \mathcal{O}_{X,x} \otimes_A \mathcal{O}_{X,\mathrm{d}_0\xi}$ and let $I := (\mathfrak{m}_x)_{\mathrm{d}_0\xi} \subset B$. Then for all $i \geq 0$ we have $B/I^{i+1} \cong (\mathcal{O}_{X,x}/\mathfrak{m}_x^{i+1})_{\mathrm{d}_0\xi}$ as ST $\mathcal{O}_{X,\mathrm{d}_0\xi}$-algebras (all have the fine $\mathcal{O}_{X,\mathrm{d}_0\xi}$-module topologies), so $\lim_{\leftarrow i} B/I^{i+1} \cong \mathcal{O}_{X,\xi}$. Now B is noetherian. By induction $A \to \mathcal{O}_{X,\mathrm{d}_0\xi}$ is topologically étale rel. to k, so by prop. 1.5.8 and prop. 1.5.9 a), $A \to B$ is also topologically étale. Thus by thm. 1.5.11 we have $\Omega^{1,\mathrm{sep}}_{B/k} \cong \left(B \otimes_A \Omega^1_{A/k}\right)^{\mathrm{sep}}$ which is finitely generated over B and has the fine B-module topology. Since B is noetherian, thm. 1.5.18 implies that $B \to \mathcal{O}_{X,\xi}$ is topologically étale, and hence so is $A \to \mathcal{O}_{X,\xi}$. □

Corollary 3.2.8 *Let ξ be a chain in X and let η be a face of ξ of length ≥ 0. Then the face map $\mathcal{O}_{X,\eta} \to \mathcal{O}_{X,\xi}$ is topologically étale relative to k.*

Proof One can assume that X is affine and then use prop. 3.2.7 and the cancellation property of étale homomorphisms (cor. 1.5.14). □

Let $\Omega^*_{X/k}$ be the de Rham complex on X relative to k, with its differential d. By propositions 3.2.4 and 3.2.2 the completion $(\Omega^*_{X/k})_\xi$ with the differential d_ξ is a differential graded ST k-algebra (see def. 1.5.1).

Corollary 3.2.9 *Let ξ be a chain in X of length ≥ 0. The k-algebra homomorphism $\mathcal{O}_{X,\xi} \to (\mathcal{O}_{X,\xi})^{\mathrm{sep}}$ induces a canonical isomorphism of DG ST k-algebras $\Omega^{*,\mathrm{sep}}_{\mathcal{O}_{X,\xi}/k} \xrightarrow{\simeq} (\Omega^*_{X/k})^{\mathrm{sep}}_\xi$.*

Proof We may assume that $X = \operatorname{Spec} A$. By cor. 3.2.6 a) there is an isomorphism of ST $\mathcal{O}_{X,\xi}$-algebras $(\Omega^*_{X/k})_\xi \cong \mathcal{O}_{X,\xi} \otimes_A \Omega^*_{A/k}$. Now use prop. 3.2.7 and cor. 1.5.13. $\square$

We shall abbreviate $(\Omega^*_{X/k})^{\text{sep}}_\xi$ to $\Omega^{*,\text{sep}}_{X/k,\xi}$.

Definition 3.2.10 *A commutative noetherian ST ring A is called a Zariski ST ring if the following conditions hold:*

i) *Every finitely generated ST A-module with the fine A-module topology is separated.*

ii) *Every homomorphism $M \to N$ of finitely generated ST A-modules with the fine A-module topologies is strict.*

Condition ii) need only be checked for monomorphisms (cf. proof of prop. 3.2.5). Theorem 3.3.8 gives a sufficient condition for the completion $\mathcal{O}_{X,\xi}$ of the structure sheaf along a saturated chain ξ to be Zariski .

For any ST ring A the category $\mathsf{STMod}(A)$ is exact; a short exact sequence in it is a sequence of strict homomorphisms which is exact in the untopologized category $\mathsf{Mod}(A)$. Evidently, if $\mathcal{O}_{X,\xi}$ is a Zariski ST ring then the functor $(-)_\xi : \mathsf{Coh}(X) \to \mathsf{STMod}(\mathcal{O}_{X,\xi})$ is exact.

Assume $\mathcal{O}_{X,\xi}$ is a Zariski ST ring. Then $(\Omega^*_{X/k})_\xi$ is separated, so $\Omega^{*,\text{sep}}_{X/k,\xi} = (\Omega^*_{X/k})_\xi$. Another conclusion is the following. Let $\mathcal{M}$ and $\mathcal{N}$ be quasi-coherent sheaves and let $D : \mathcal{M} \to \mathcal{N}$ be a differential operator. Suppose that $\xi = (x, \dots)$ and $\mathcal{N}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module. Then the differential operator $D_\xi : \mathcal{M}_\xi \to \mathcal{N}_\xi$ of prop. 3.1.10 is the unique extension of D to a continuous differential operator. This is because $\mathcal{N}_\xi$ is separated and $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi}$ is topologically étale (see thm. 1.5.11).

Let $\phi : M \to N$ be a homomorphism in TopAb. We say that ϕ is dense if $\operatorname{im}(\phi)$ is (everywhere) dense in N.

Theorem 3.2.11 (Approximation) *Assume X is a separated excellent noetherian scheme. Let $\mathcal{M}$ be a quasi-coherent sheaf on X. Let $S \subset X$ be a finite subset and let $\xi = (\dots, x)$ be a chain s.t. for all $y \in S$, (x, y) is a saturated chain. Assume that the completions $\mathcal{O}_{X,(x,y)}$ are all Zariski ST rings. Then the face map $\partial : \mathcal{M}_\xi \to \bigoplus_{y \in S} \mathcal{M}_{\xi \vee (y)}$ is dense.*

Proof We break up the proof into 4 steps.

1) First suppose $\xi = (x)$ and $\mathcal{M}_x \cong k(x)$. For every $y \in S$ the completion $\mathcal{M}_{(x,y)} \cong k(x)_{(y)}$ which is a finite product of fields, the completions of $k(x)$ with respect to the discrete valuations with center y on the integral scheme $\{x\}^-_{\text{red}}$

(cf. thm. 3.3.2). If $y_1, y_2 \in S$ are distinct points, then the valuations centered on them are distinct, because X is separated. Thus if $\prod_{y\in S} k(x)_{(y)} = \prod_{i=1}^{r} L_i$, the valuations of $L_1, \ldots, L_r$ are pairwise independent. Since our topology on $\prod_{y\in S} k(x)_{(y)}$ coincides in this case with the usual valuative topology, the Artin-Whaples approximation theorem tells us that $\partial : \mathcal{M}_{(x)} \to \bigoplus_{y\in S} \mathcal{M}_{(x)\vee(y)}$ is dense.

2) Again let $\xi = (x)$ and now assume that $\mathcal{M}_x$ has finite length over $\mathcal{O}_{X,x}$. By induction on the length of $\mathcal{M}_x$, by the exactness of completion and by prop. 1.1.8 a) we reduce the problem to a module of length 1, which is treated in step 1.

3) Now let $\xi = (w, \ldots, x)$ be an arbitrary chain (possibly of length 0, i.e. with $w = x$) and assume that $\mathcal{M}$ is coherent. By induction (or by step 2 if $w = x$) the homomorphisms

$$(\mathcal{M}_w/\mathfrak{m}_w^{i+1}\mathcal{M}_w)_{\mathrm{d}_0\xi} \to \bigoplus_{y\in S} (\mathcal{M}_w/\mathfrak{m}_w^{i+1}\mathcal{M}_w)_{\mathrm{d}_0\xi\vee(y)}$$

are dense for all $i \in \mathbb{N}$, hence by prop. 1.1.8 b) so is the inverse limit $\partial : \mathcal{M}_\xi \to \bigoplus_{y\in S} \mathcal{M}_{\xi\vee(y)}$.

4) Let $\mathcal{M}$ be a quasi-coherent sheaf and let $(\mathcal{M}'_\alpha)$ be its coherent subsheaves. By step 3 we have a direct system of dense homomorphisms $(\mathcal{M}_\alpha)_\xi \to \bigoplus_{y\in S} (\mathcal{M}_\alpha)_{\xi\vee(y)}$ so by prop. 1.1.8 c) the limit homomorphism ∂ is dense. □

Corollary 3.2.12 *Let $G \subset X$ be a finite subset and let $S = \bigcup_{x\in G} S_x$ be a finite set of saturated chains s.t. each $\xi \in S_x$ begins with x. Assume that no chain in S is a face of any other chain. Assume also that $\mathcal{O}_{X,\eta}$ is a Zariski ST ring for all saturated chains η of length ≤ 1. Then for any quasi-coherent sheaf $\mathcal{M}$, the face map $\partial : \bigoplus_{x\in G} \mathcal{M}_{(x)} \to \bigoplus_{x\in G} \bigoplus_{\xi\in S_x} \mathcal{M}_\xi$ is dense.*

Proof We may assume that $G = \{x\}$. The proof is by induction on the maximal length of chains in S, using the transitivity of dense maps. □

Completion along saturated chains behaves very much like adic completion on a curve. The next lemma puts this into concrete terms. Given a point $x \in X$ and a germ t of $\mathcal{O}_X$ at x we write $t(x)$ for the image of t in the residue field $k(x)$. For a module M we denote its localization with respect to t by M_t.

Lemma 3.2.13 *Let $y \in X$ be a point and let $S \subset X$ be a finite subset such that for all $x \in S$, (x, y) is a saturated chain. Let M be a finitely generated $\mathcal{O}_{X,y}$-module supported on the closed subset $S^- \subset \operatorname{Spec} \mathcal{O}_{X,y}$. Suppose $t \in \mathcal{O}_{X,y}$ satisfies: $t(y) = 0$ but $t(x) \neq 0$ for all $x \in S$. Then the canonical map $M_t \to \bigoplus_{x\in S} M_x$ is bijective.*

Proof Let $I \subset \mathcal{O}_{X,y}$ be a defining ideal of $S^- \subset \operatorname{Spec} \mathcal{O}_{X,y}$. For sufficiently large n, M is an $\mathcal{O}_{X,y}/I^n$-module. The scheme $\operatorname{Spec}(\mathcal{O}_{X,y}/I^n)$ is a 1-dimensional noetherian scheme with only one closed point, namely y. Therefore $(\mathcal{O}_{X,y}/I^n)_t = \prod_{x \in S} (\mathcal{O}_{X,y}/I^n)_x$. $\square$

Theorem 3.2.14 *Let $\mathcal{M}$ be a coherent sheaf on X, let $\eta = (y, \ldots)$ be a chain and let $S \subset X$ be a finite subset s.t. for all $x \in S$, (x, y) is a saturated chain. Suppose the completion $\mathcal{O}_{X,\eta}$ is a Zariski ST ring. Then the face map $\partial : \mathcal{M}_\eta \to \bigoplus_{x \in S} M_{(x) \vee \eta}$ is a strict homomorphism of ST $\mathcal{O}_{X,\eta}$-modules.*

Proof For every $i \geq 0$ define $U_i := \bigoplus_{x \in S} \mathcal{M}_x / \mathfrak{m}_x^{i+1} \mathcal{M}_x$ and $V_i := \operatorname{im}(\mathcal{M}_y \to U_i)$. We shall prove the following statements:

$$\phi : \mathcal{M}_\eta \to \lim_{\leftarrow i} (V_i)_\eta \text{ is a strict epimorphism.} \tag{3.2.15}$$

$$\psi : \lim_{\leftarrow i} (V_i)_\eta \to \lim_{\leftarrow i} (U_i)_\eta = \bigoplus_{x \in S} \mathcal{M}_{(x) \vee \eta} \text{ is a strict monomorphism.} \tag{3.2.16}$$

The composition $\partial = \psi \circ \phi$ is then strict.

1) Choose $t \in \mathcal{O}_{X,y}$ as in lemma 3.2.13. Then for fixed $i \geq 0$ we have $U_i = \bigcup_{l \geq 0} t^{-l} V_i$. Since $\mathcal{O}_{X,\eta}$ is a Zariski ST ring, for every l, $(t^{-l} V_i)_\eta \hookrightarrow (t^{-(l+1)} V_i)_\eta$ is a strict monomorphism. According to prop. 1.1.7, $(V_i)_\eta \hookrightarrow (U_i)_\eta \cong \lim_{l \to} (t^{-l} V_i)_\eta$ is also a strict monomorphism, and by prop. 1.1.6 statement (3.2.16) holds.

2) For each $i, j \geq 0$ define $W_{i,j} := V_i / \mathfrak{m}_y^{j+1} V_i$. Fixing j, the length of the $\mathcal{O}_{X,y}$-modules $W_{i,j}$ is bounded (by the length of $\mathcal{M}_y / \mathfrak{m}_y^{j+1} \mathcal{M}_y$), so the inverse system $(W_{i,j})_{i \in \mathbb{N}}$ is constant for $i >> 0$. There is some i_j s.t. $i \geq i_j$ implies $W_{i,j} \cong W_{i,i_j}$. Moreover, we can assume that the sequence (i_j) is increasing. Thus $(W_{i_j,j})_{j \in \mathbb{N}}$ is an inverse system. Since inverse limits commute we have isomorphisms in $\mathsf{STMod}(\mathcal{O}_{X,\eta})$:

$$\lim_{\leftarrow j} (W_{i_j,j})_\eta \cong \lim_{\leftarrow j} \lim_{\leftarrow i} (W_{i,j})_\eta \cong \lim_{\leftarrow i} \lim_{\leftarrow j} (W_{i,j})_\eta = \lim_{\leftarrow i} (V_i)_\eta \, . \tag{3.2.17}$$

Define $K_j := \ker(\mathcal{M}_y / \mathfrak{m}_y^{j+1} \mathcal{M}_y \twoheadrightarrow W_{i_j,j})$. We claim that for all j, $K_{j+1} \to K_j$ is surjective. In fact, since $W_{i_{j+1},j} = W_{i_j,j}$, both K_{j+1} and K_j are quotients of $\ker(\mathcal{M}_y \to V_{i_{j+1}})$. Therefore upon applying the completion $(-)_\eta$ we get an inverse system of exact sequences

$$0 \to (K_j)_\eta \to (\mathcal{M}_y / \mathfrak{m}_y^{j+1} \mathcal{M}_y)_\eta \to (W_{i_j,j})_\eta \to 0 \tag{3.2.18}$$

in $\mathsf{STMod}(\mathcal{O}_{X,\eta})$ which satisfies the hypotheses of prop. 1.1.6. Passing to the inverse limit in j and using (3.2.17) we deduce statement (3.2.15). $\square$

3.3 The Geometry of Completion

In this section we shall give two geometric ways of looking at the Beilinson completion of the structure sheaf of a scheme X along a saturated chain $\xi = (x_0, \ldots, x_n)$. We are following Parshin's description as found in [Pa2] §1.1, although with new notation. Throughout most of the section the topology on the completion will not play any part. We assume X is an excellent noetherian scheme (e.g. a scheme of finite type over a field, over $\mathbb{Z}$, or over a complete semi-local noetherian ring; see [Ma] §34 and [EGA IV] §7.8).

Given a scheme Z we shall denote by $k(Z)$ its total ring of fractions (the global sections of the sheaf of total rings of fractions, see [Ha] p. 141). If Z is a reduced noetherian scheme with generic points $z_1, \ldots, z_r$ then $k(Z) = k(z_1) \times \cdots \times k(z_r)$.

Recall the definition of an n-dimensional local field (def. 2.1.1). Suppose A is a artinian ring s.t. for each $\mathfrak{m} \in \operatorname{Spec} A$, $A/\mathfrak{m}$ is an n-dimensional local field. Then for $i = 0, \ldots, n$ define $\mathcal{O}_i(A) := \prod_{\mathfrak{m} \in \operatorname{Spec} A} \mathcal{O}_i(A/\mathfrak{m})$ and similarly define $\kappa_i(A)$ and $\mathcal{O}(A)$. Observe that $\kappa_0(A)$ is simply the ring $A_{\mathrm{red}} = A/\mathrm{rad}(A)$.

Definition 3.3.1 *Let B be an artinian ring and suppose that for each $\mathfrak{n} \in \operatorname{Spec} B$, $B/\mathfrak{n}$ is an n-dimensional local field. Let A be an artinian ring and let $f : A \to B$ be a ring homomorphism. For each $\mathfrak{n} \in \operatorname{Spec} B$ lying over some $\mathfrak{m} \in \operatorname{Spec} A$, there is an induced valuation on the field $A/\mathfrak{m}$ into the ordered group $(B/\mathfrak{n})^{\times}/\mathcal{O}(B/\mathfrak{n})^{\times} \cong (\mathbb{Z}^n, \mathrm{lex})$. We say that f is unramified at $\mathfrak{n}$ if $\mathfrak{n} = B_{\mathfrak{n}} \cdot f(\mathfrak{m})$, and if the the ramification index and the residue degree of the (possibly infinite) field extension $A/\mathfrak{m} \to B/\mathfrak{n}$ are both 1. We say that f is unramified if it is unramified at all $\mathfrak{n} \in \operatorname{Spec} B$.*

Let ξ be a chain in X. We shall define, by recursion on the length of ξ, a scheme X^{ξ} together with a morphism $\pi^{\xi} : X^{\xi} \to X$. Let $X^{\underline{1}}$ be the normalization of X_{red} in its total ring of fractions $k(X_{\mathrm{red}})$, and let $\pi^{\underline{1}} : X^{\underline{1}} \to X$ be the canonical morphism. Next let $\xi = (\ldots, y)$ have length $n \geq 0$ and suppose that $\pi^{\mathrm{d}_n \xi} : X^{\mathrm{d}_n \xi} \to X$ has been defined. Let $Y := \{y\}^{-}_{\mathrm{red}} \subset X$, define $X^{\xi} := (X^{\mathrm{d}_n \xi} \times_X Y)^{\underline{1}}$, and let $\pi^{\xi} : X^{\xi} \to X$ be the canonical morphism. (See figure 1.)

Thus the scheme X^{ξ} is a disjoint union of normal excellent integral schemes, and the morphism π^{ξ} is finite. If $\xi = (\ldots, y)$ then X^{ξ} is equidimensional and $\pi^{\xi}(X^{\xi}) = \{y\}^{-}$. Given another chain $\eta = (y, \ldots)$, we have $X^{\xi \vee \mathrm{d}_0 \eta} \cong \coprod_{\tilde{\eta}|\eta} (X^{\xi})^{\tilde{\eta}}$ as schemes over X, where $\tilde{\eta}|\eta$ means $\tilde{\eta}$ is a chain in X^{ξ} lying over η.

The theorem below is essentially due to Beilinson; part a) of the theorem appears (without proof) in [Be]. See also [Pa2] prop. 1.

Theorem 3.3.2 *Let X be an integral excellent noetherian scheme and let $\xi = (x_0, \ldots, x_n)$ be a saturated chain in X, with x_0 being the generic point. Then:*

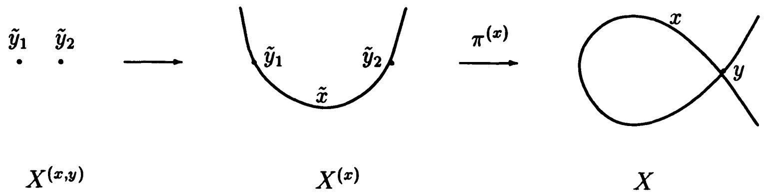

Figure 1: The morphism $\pi^{\xi} : X^{\xi} \to X$

a) *The completion* $k(X)_{\xi} = k(\xi)$ *is an artinian ring, and for each* $\mathfrak{m} \in \operatorname{Spec} k(X)_{\xi}$*, the field* $k(X)_{\xi}/\mathfrak{m}$ *has a canonical structure of an* n*-dimensional local field.*

b) *The homomorphism* $k(X) \to k(X)_{\xi}$ *is unramified; in particular* $k(X)_{\xi}$ *is reduced.*

c) *For every* $i = 0, \ldots, n$ *there is a canonical isomorphism of rings*

$$k(X^{(x_0,\ldots,x_i)})_{(x_i,\ldots,x_n)} \xrightarrow{\simeq} \kappa_i(k(X)_{\xi}) \ .$$

d) *For every* $i = 1, \ldots, n$ *the ring* $\mathcal{O}_i(k(X)_{\xi})$ *is the integral closure of* $\mathcal{O}_{X,(x_i,\ldots,x_n)}$ *in* $\kappa_{i-1}(k(X)_{\xi})$*. In particular, each* $\mathcal{O}_i(k(X)_{\xi})$ *is an* $\mathcal{O}_X$*-algebra (supported on* $\{x_n\}^-$*).*

Observe that taking $i = n$ in part c) we get a bijection between the factors of the artinian ring $k(X)_{\xi}$ and the irreducible components of X^{ξ}. We first need a lemma.

Lemma 3.3.3 *Let* A *be a ring and let* $t \in A$ *be an element satisfying the following conditions:* $A \cong \lim_{\leftarrow i} A/(t)^{i+1}$*;* $A/(t)$ *is a reduced artinian ring; and* t *is a non-zero-divisor on* A*. Then* A *is a finite product of complete DVRs with regular parameter* t*. To be precise, say* $Z := \operatorname{Spec} A/(t) \subset \operatorname{Spec} A$*, so that* $A/(t) = \prod_{z\in Z} k(z)$*. Then* $A = \prod_{z \in Z} A_z$*, each* A_z *is a complete DVR, and* $A_z/(t) = k(z)$*.*

Proof The ideal $(t) \subset A$ is its Jacobson radical $\operatorname{rad}(A)$: since for any $a \in (t)$ we have $1 - a \in A^{\times}$ (units), it follows that $(t) \subset \operatorname{rad}(A)$. On the other hand $A/(t)$ is semi-simple, so $\operatorname{rad}(A) \subset (t)$. Therefore A is a complete semi-local ring, and by [CA] ch. III §2.13 cor. to prop. 19 we get the decomposition $A = \prod_{z\in Z} A_z$.

In order to show that A_z is a DVR we can assume that $A = A_z$ is local. Since $\bigcap_{i\in\mathbb{N}}(t)^i = 0$ every nonzero $a \in A$ has the form $a = ut^i$, $u \in A^\times$, $i \in \mathbb{N}$. But t is a non-zero-divisor, so u and i are uniquely determined. Therefore A is an integral domain, and in fact a DVR with regular parameter t. $\square$

Proof (of the theorem) The theorem is trivially true for $n = 0$, so assume $n \geq 1$. By our hypothesis the normalization $\tilde{X} = X^{\underline{1}} \to X$ is a finite morphism. We have $k(X) = k(\tilde{X})$, so according to prop. 3.1.7, $k(X)_\xi = k(\tilde{X})_\xi \cong \prod_{\tilde{\xi}|\xi} k(\tilde{X})_{\tilde{\xi}}$, where $\tilde{\xi}|\xi$ means $\tilde{\xi}$ is a chain in $\tilde{X}$ lying over ξ.

Fix some chain $\tilde{\xi} = (\tilde{x}_0, \ldots, \tilde{x}_n)$ lying over $\xi = (x_0, \ldots, x_n)$. The local ring $\mathcal{O}_{\tilde{X},\tilde{x}_1}$ is a DVR of $k(\tilde{X})$; choose a regular parameter t in it. Since the sequence

$$0 \to \mathcal{O}_{\tilde{X},\tilde{x}_1} \xrightarrow{t} \mathcal{O}_{\tilde{X},\tilde{x}_1} \to k(\tilde{x}_1) \to 0$$

is exact, so is

$$0 \to \mathcal{O}_{\tilde{X},\mathrm{d}_0\tilde{\xi}} \xrightarrow{t} \mathcal{O}_{\tilde{X},\mathrm{d}_0\tilde{\xi}} \to k(\tilde{x}_1)_{\mathrm{d}_0\tilde{\xi}} \to 0 \; .$$

Directly from the definition one has $\mathcal{O}_{\tilde{X},\mathrm{d}_0\tilde{\xi}} \cong \lim_{\leftarrow i} \mathcal{O}_{\tilde{X},\mathrm{d}_0\tilde{\xi}}/(t)^{i+1}$. By induction on n, $k(\tilde{x}_1)_{\mathrm{d}_0\tilde{\xi}} = k(\mathrm{d}_0\tilde{\xi})$ is a finite product of $(n-1)$-dimensional local fields. Lemma 3.3.3 says that $\mathcal{O}_{\tilde{X},\mathrm{d}_0\tilde{\xi}}$ is a finite product of complete DVRs, each with parameter t. Upon inverting t we see that $k(\tilde{X})_{\tilde{\xi}} = k(\tilde{X}) \otimes_{\mathcal{O}_{\tilde{X},\tilde{x}_1}} \mathcal{O}_{\tilde{X},\mathrm{d}_0\tilde{\xi}}$ is a product of n-dimensional local fields. This proves part a). Now $t \in k(X)$ and by induction $k(\tilde{x}_1) \to k(\tilde{x}_1)_{\mathrm{d}_0\tilde{\xi}}$ is unramified, hence $k(X) \to k(\tilde{X})_{\tilde{\xi}}$ is also unramified, and part b) is verified.

The arguments presented above show that in fact

$$\kappa_1(k(X)_\xi) \cong \prod_{(\tilde{x}_1,\ldots,\tilde{x}_n)|(x_1,\ldots,x_n)} k(\tilde{x}_1)_{(\tilde{x}_1,\ldots,\tilde{x}_n)} = k(X^{(x_0,x_1)})_{(x_1,\ldots,x_n)} \; .$$

By induction, for every component Z of $X^{(x_0,x_1)}$ and every chain $(\tilde{x}_1, \ldots, \tilde{x}_n)$ in Z lying over $(x_1, \ldots, x_n)$, it holds

$$\kappa_{i-1}(k(Z)_{(\tilde{x}_1,\ldots,\tilde{x}_n)}) = k(Z^{(\tilde{x}_1,\ldots,\tilde{x}_i)})_{(\tilde{x}_i,\ldots,\tilde{x}_n)} \; .$$

Taking the product over all such Z and $(\tilde{x}_1, \ldots, \tilde{x}_n)$ we end up with

$$\kappa_{i-1}(k(X^{(x_0,x_1)})_{(x_1,\ldots,x_n)}) = k(X^{(x_0,\ldots,x_i)})_{(x_i,\ldots,x_n)} \; .$$

But $\kappa_{i-1}(\kappa_1) = \kappa_i$, so part c) is proved.

Since $\mathcal{O}_1(k(X)_\xi) = \prod_{\tilde{\xi}|\xi} \mathcal{O}_{\tilde{X},\mathrm{d}_0\tilde{\xi}}$ is a finite $\mathcal{O}_{X,\mathrm{d}_0\xi}$-algebra, it is its integral closure in $k(X)_\xi$. In order to prove part d) for $i > 1$ use induction and the fact that $\pi^{(x_0,x_1)} : X^{(x_0,x_1)} \to X$ is a finite morphism.

$\square$

Corollary 3.3.4 *Let X be an excellent noetherian scheme and let $(x_0, \ldots, x_n)$ be a saturated chain in X. Then for all $i = 0, \ldots, n$ there is a canonical isomorphism of $\mathcal{O}_X$-algebras*

$$\kappa_i(k(x_0)_{(x_0,\ldots,x_i)})_{(x_i,\ldots,x_n)} \xrightarrow{\simeq} \kappa_i(k(x_0)_{(x_0,\ldots,x_n)}) .$$

Corollary 3.3.5 *Let X be an excellent noetherian scheme and let $\xi = (x, \ldots)$ be a saturated chain in it. Then the completion $\mathcal{O}_{X,\xi}$ of $\mathcal{O}_X$ along ξ is a complete noetherian semi-local ring with Jacobson radical $\mathfrak{m}_\xi$. In particular $\mathcal{O}_{X,\xi}$ is an excellent ring, and a faithfully flat $\mathcal{O}_{X,x}$-algebra.*

Proof By definition $\mathcal{O}_{X,\xi} = \lim_{\leftarrow i} \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1}$, and by the theorem $k(\xi) = \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi$ is a semi-simple artinian ring. □

From now till further announcement we shall assume X is a scheme of finite type over a perfect field k. Given a chain $\xi = (x_0, \ldots, x_n)$ in X, a linearization of ξ is by definition a finite k-morphism $f : X \to \mathbb{A}^m_k$ s.t. $f(\xi) := (\ (f(x_0), \ldots, f(x_n))\ )$ is a linear chain in X (i.e. each $\{f(x_i)\}^-$ is a linear subspace of $\mathbb{A}^m_k$). By the strong form of Noether normalization (see [CA] ch. V §3.1 thm. 1) any chain has a linearization.

Let A be a semi-topological ring. In §1.3 a topology was introduced on the ring of Laurent series $A((t_1, \ldots, t_n)) := A((t_n)) \cdots ((t_1))$. Consider the affine space $\mathbb{A}^m_k = \operatorname{Spec} k[t_1, \ldots, t_m]$ and the linear chain $\eta = (y_0, \ldots, y_n)$, where y_i is the prime ideal $(t_1, \ldots, t_i)$. The completion of the function field $k(t_1, \ldots, t_m)$ along η is a field of Laurent series $k(t_{n+1}, \ldots, t_m)((t_1, \ldots, t_n))$, and its topology as a ring of Laurent series coincides with the topology specified in def. 3.2.1.

Recall the definitions of a topological local field (TLF) and of a cluster of TLFs (definitions 2.1.10 and 2.2.1).

Proposition 3.3.6 *Let X be a scheme of finite type over a perfect field k and let ξ be a saturated chain of length n in it. Then the ST k-algebra $k(\xi)$ is an equidimensional, n-dimensional reduced cluster of TLFs over k.*

Proof Say $\xi = (x, \ldots)$. We may assume that X is integral with generic point x. Fix some $\mathfrak{m} \in \operatorname{Spec} k(\xi)$ and set $L := k(\xi)/\mathfrak{m}$. We have to show that L is a TLF over k. Choose a linearization $f : X \to \mathbb{A}^m_k$ of ξ, with $f(\xi) = \eta = (y, \ldots, z)$. Then $K := k(\eta) \cong k(z)((t_1, \ldots, t_n))$ is a TLF over k. By prop. 3.2.3

$$k(x) \otimes_{k(y)} k(\eta) \cong (f_* k(x))_\eta \cong \prod_{\xi' | \eta} k(x)_{\xi'}$$

giving rise to a finite homomorphism $K \to L$, and L has the fine K-module topology. From the proof of thm. 3.3.2 we see that the valuation on L extends the valuation on K. According to cor. 2.1.20, L is a TLF. $\square$

Combining the last proposition with cor. 3.2.8 and thm. 3.3.2 we have

Corollary 3.3.7 *Let $\xi = (x, \ldots)$ be a saturated chain in X. Then the face map $\partial : k(x) \to k(x)_\xi = k(\xi)$ is a topologically étale (relative to k), dense, unramified homomorphism of clusters of TLFs over k.*

Conveniently, in working over a perfect field one can use coefficient fields. The next theorem uses them, and the fact that the ring $k(\xi)$, for ξ saturated, is semi-simple artinian.

Theorem 3.3.8 *Let X be a scheme of finite type over a perfect field k and let ξ be a saturated chain in it. Then the completion $\mathcal{O}_{X,\xi}$ of the structure sheaf along ξ is a Zariski ST ring. Moreover, every finitely generated ST $\mathcal{O}_{X,\xi}$-module with the fine $\mathcal{O}_{X,\xi}$-module topology is complete.*

Lemma 3.3.9 *Let $\xi = (x, \ldots)$ be a saturated chain in X and let $\sigma : k(x) \to \mathcal{O}_{X,(x)}$ be a coefficient field, i.e. a k-algebra lifting. Suppose the ST k-algebras $\mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1}$ are separated for all $i \geq 0$. Then σ extends uniquely to a continuous k-algebra lifting $\sigma_\xi : k(\xi) \to \mathcal{O}_{X,\xi}$.*

Proof Fix $i \in \mathbb{N}$. The homomorphism $\sigma_i : k(x) \to \mathcal{O}_{X,x}/\mathfrak{m}_x^{i+1}$ is a DO of order $\leq i$ over $\mathcal{O}_X$ relative to k. By prop. 3.2.2 it extends to a continuous DO $(\sigma_i)_\xi : k(\xi) \to \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1}$ over $\mathcal{O}_{X,\xi}$. Because $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi}$ is topologically étale relative to k, and because $\mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1}$ is separated, $(\sigma_i)_\xi$ is a ring homomorphism (prop. 1.5.20) and is unique. Passing to the inverse limit we get $\sigma_\xi : k(\xi) \to \mathcal{O}_{X,\xi}$. $\square$

Assume the hypotheses of the lemma. Let $D : k(x)^n \xrightarrow{\simeq} \mathcal{O}_{X,x}/\mathfrak{m}_x^{i+1}$ be any $k(x)$-linear isomorphism. Then D is a DO, and $D_\xi : k(\xi)^n \xrightarrow{\simeq} \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1}$ is an isomorphism of ST $k(\xi)$-modules (cf. proof of prop. 1.5.20). In particular $\mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1}$ has the fine $k(\xi)$-module topology. Since $k(\xi)$ is a semi-simple artinian ring, it follows that any finite length $\mathcal{O}_{X,\xi}$-module with the fine $\mathcal{O}_{X,\xi}$-module topology is a free ST $k(\xi)$-module (via σ_ξ).

Proof (of the theorem) The proof is by induction on the length of ξ. If $\xi = (x)$ this is a standard fact, since $\mathcal{O}_{X,(x)}$ has the $\mathfrak{m}_{(x)}$-adic topology. So we may assume $\xi = (x, y, \ldots)$ has length ≥ 1.

1) First let us prove that for any finite length $\mathcal{O}_{X,x}$-module M, M_ξ is a complete separated module. Choose $t \in \mathcal{O}_{X,y}$ as in lemma 3.2.13, so $M = \bigcup_{l \geq 0} t^{-l}V$, where $V := \operatorname{im}(\mathcal{O}_{X,y} \to M)$. Applying the completion $(-)_{\mathrm{d}_0\xi}$ we get an isomorphism of ST $\mathcal{O}_{X,\mathrm{d}_0\xi}$-modules $M_\xi \cong \lim_{l \to} t^{-l}V_{\mathrm{d}_0\xi}$. Since $\mathcal{O}_{X,\mathrm{d}_0\xi}$ is a Zariski ST ring the homomorphisms $t^{-l}V_{\mathrm{d}_0\xi} \hookrightarrow t^{-l-1}V_{\mathrm{d}_0\xi}$ are strict and these modules are separated. According to prop. 1.1.7, M_ξ is separated.

Now let us prove completeness. Choose a coefficient field $\tau : k(y) \to \mathcal{O}_{X,(y)}$. By lemma 3.3.9 it extends to $\tau_{\mathrm{d}_0\xi} : k(\mathrm{d}_0\xi) \to \mathcal{O}_{X,\mathrm{d}_0\xi}$. As mentioned above, $(V/tV)_{\mathrm{d}_0\xi}$ is a free ST $k(\mathrm{d}_0\xi)$-module. Thus we obtain an isomorphism of ST $k(\mathrm{d}_0\xi)$-modules $M_\xi \cong V_{\mathrm{d}_0\xi} \oplus \left[\bigoplus_{l \geq 0} t^{-l-1}(V/tV)_{\mathrm{d}_0\xi}\right]$. By assumption the summands are separated and complete, being finitely generated $\mathcal{O}_{X,\mathrm{d}_0\xi}$-modules. According to prop. 1.1.5, M_ξ is also separated and complete.

2) By step 1 we are in a position to use lemma 3.3.9. Choose a coefficient field $\sigma : k(x) \to \mathcal{O}_{X,(x)}$, and consider $\mathcal{O}_{X,\xi}$ as an augmented ST $k(\xi)$-algebra via σ_ξ. Let M be a finite length $\mathcal{O}_{X,\xi}$-module M with the fine $\mathcal{O}_{X,\xi}$-module topology. Then $M \cong k(\xi)^n$ for some n, so it is complete and separated. If $\phi : M \hookrightarrow N$ is any injection of finite length $\mathcal{O}_{X,\xi}$-modules with the fine $\mathcal{O}_{X,\xi}$-module topologies, then ϕ splits continuously over $k(\xi)$ and hence is strict.

3) Let M be a finitely generated $\mathcal{O}_{X,\xi}$-module with the fine $\mathcal{O}_{X,\xi}$-module topology. For each $i \geq 0$ put on $M/\mathfrak{m}_\xi^{i+1}M$ the fine $\mathcal{O}_{X,\xi}$-module topology, which makes it separated and complete. Since $\mathcal{O}_{X,\xi}$ is noetherian and $\mathfrak{m}_\xi$-adically complete, according to prop. 1.2.20 the map $M \to \lim_{\leftarrow i} M/\mathfrak{m}_\xi^{i+1}M$ is a homeomorphism. By prop. 1.1.5 a) it follows that M is separated and complete.

Now let $\phi : M \hookrightarrow N$ be an injection of finitely generated ST $\mathcal{O}_{X,\xi}$-modules with fine topologies. For $i \geq 0$ set $N_i := N/\mathfrak{m}_\xi^{i+1}N$ and $M_i := M/M \cap \mathfrak{m}_\xi^{i+1}N$. By step 2, $\phi_i : M_i \to N_i$ is a strict monomorphism, so by prop. 1.1.6 so is $\phi : M \cong \lim_{\leftarrow i} M_i \to N$. □

Remark 3.3.10 It seems plausible that the theorem is true for any noetherian scheme X; it certainly should hold for excellent schemes. However we could not find a proof which does not resort to splitting. The difficulty lies in showing that a direct limit of strict monomorphisms is strict.

We move on to the second geometric interpretation of completion, and once more X is any excellent noetherian scheme. Given a chain $\xi = (x_0, \ldots, x_n)$ in X set $X_\xi := \operatorname{Spec} \mathcal{O}_{X,\xi}$ and let $i_\xi : X_\xi \to X$ be the morphism corresponding to the ring homomorphism $\mathcal{O}_{X,x_n} \to \mathcal{O}_{X,\xi}$. Define $X_{\underline{1}} := X$ and $i_{\underline{1}} :=$ identity morphism. (See figure 2.) Note that X_ξ is also a noetherian excellent scheme (cor. 3.3.5).

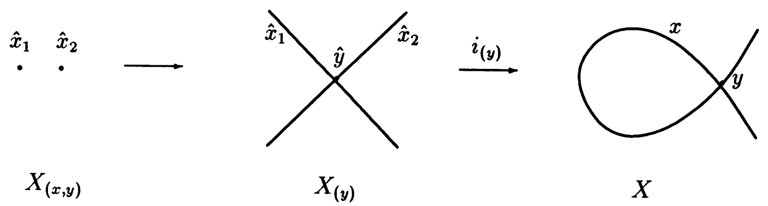

Figure 2: The morphism $i_\xi : X_\xi \to X$

For a quasi-coherent sheaf $\mathcal{M}$ on X we may identify the completion $\mathcal{M}_\xi$ with $\Gamma(X_\xi, i_\xi^* \mathcal{M})$. The morphism i_ξ is flat; it is also "quasi-finite", in the following restricted sense. Given points $x \in X$ and $\hat{x} \in X_\xi$ we say that $\hat{x}$ is minimal over x if $\{\hat{x}\}^-$ is an irreducible component of the fibre $i_\xi^{-1}(\{x\}^-)$. Then the set $\{\hat{x} \in X_\xi \mid \hat{x} \text{ is minimal over } x\}$ is finite. Note that if $\xi = (x, \ldots)$ is a saturated chain then every $\hat{x} \in i_{\mathrm{d}_0\xi}^{-1}(x)$ is minimal, because $\operatorname{Spec} k(x)_{\mathrm{d}_0\xi}$ is 0-dimensional.

Remark 3.3.11 We may think of minimal points as "algebraic", as the following example suggests. Take $X := \mathbb{A}_k^2 = \operatorname{Spec} k[s,t]$, $x := (0) \in X$, $z := (s,t) \in X$ and $\xi := (z)$, so $X_\xi = \operatorname{Spec} k[[s,t]]$. Choose any element $f \in k[[t]] \cdot t$ transcendental over $k[t]$. Then the point $\hat{y} := k[[s,t]] \cdot (f - s) \in X_\xi$ is in $i_\xi^{-1}(x)$ but is not minimal. The fibre $i_\xi^{-1}(x)$ consists of the generic point $\hat{x}$ of X_ξ and infinitely many "transcendental" points of codimension 1, such as $\hat{y}$.

Theorem 3.3.12 *Let X be an excellent noetherian scheme and let $\xi = (x, \ldots)$ be a saturated chain in it. Then there is a canonical isomorphism $X_\xi \cong \coprod_{\hat{x}|x} (X_{\mathrm{d}_0\xi})_{(\hat{x})}$ of schemes over X, where $\hat{x}|x$ stands for $\hat{x} \in i_{\mathrm{d}_0\xi}^{-1}(x)$.*

Proof Set $\hat{X} := X_{\mathrm{d}_0\xi}$ and $\hat{i} := i_{\mathrm{d}_0\xi}$. For $n \geq 0$ let $A_n := \mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}$. By definition we have $\mathcal{O}_{X,\xi} \cong \lim_{\leftarrow n} \Gamma\left(\hat{X}, \hat{i}^* A_n\right)$. Adjunction gives for every $n \geq 0$ a homomorphism $u_n : \hat{i}^* A_n \to \prod_{\hat{x}|x} \mathcal{O}_{\hat{X},\hat{x}}/\mathfrak{m}_{\hat{x}}^{n+1}$ of quasi-coherent sheaves on $\hat{X}$.

Since $\Gamma(\hat{X}, \hat{i}^* k(x)) = k(x)_{\mathrm{d}_0\xi} = k(\xi)$ is a reduced artinian ring (cf. thm. 3.3.2), for $n = 0$ we have an isomorphism $u_0 : \hat{i}^* k(x) \xrightarrow{\cong} \prod_{\hat{x}|x} k(\hat{x})$. Thus $\hat{i}$ is unramified at all points $\hat{x}|x$ and in particular $(\hat{i}^* \mathfrak{m}_x)_{\hat{x}} = \mathfrak{m}_{\hat{x}}$. For each $n \geq 0$ consider the exact sequence on X

$$0 \to \mathfrak{m}_x^{n+1} \to \mathcal{O}_{X,x} \to A_n \to 0 \ .$$

Upon applying $\hat{\imath}^*$ and taking stalks at any $\hat{x}|x$ one sees that $(u_n)_{\hat{x}} : (\hat{\imath}^* A_n)_{\hat{x}} \to \mathcal{O}_{\hat{X},\hat{x}}/\mathfrak{m}_{\hat{x}}^{n+1}$ is bijective. Therefore u_n itself is bijective, and in the inverse limit so is $\lim_{\leftarrow n} u_n : \mathcal{O}_{X,\xi} \to \prod_{\hat{x}|x} \mathcal{O}_{\hat{X},(\hat{x})}$. □

Corollary 3.3.13 *Let $\xi = (\ldots, y)$ and $\eta = (y, \ldots)$ be saturated chains in X. Then there is an isomorphism $X_{\xi \vee \mathrm{d}_0 \eta} \cong \coprod_{\hat{\xi}|\xi} (X_\eta)_{\hat{\xi}}$ of schemes over X, where $\hat{\xi}|\xi$ means $\hat{\xi}$ is a chain in X_η lying over ξ.*

Proof Use induction on the length of $\xi = (x, \ldots, y)$, noting that $X_\eta \cong \coprod_{\hat{y}|y} (X_\eta)_{(\hat{y})}$:

$$X_{\xi \vee \mathrm{d}_0 \eta} \cong \coprod_{\hat{x}|x} (X_{\mathrm{d}_0 \xi \vee \mathrm{d}_0 \eta})_{(\hat{x})} \cong \coprod_{\hat{x}|x} \coprod_{\hat{\xi}|\mathrm{d}_0 \xi} ((X_\eta)_{\hat{\xi}})_{(\hat{x})} \cong \coprod_{\hat{\xi}|\xi} (X_\eta)_{\hat{\xi}} \ .$$

□

Corollary 3.3.14 *If X is normal then so is X_ξ.*

Proof By the theorem and induction it suffices to consider $\xi = (x)$. Now $\mathcal{O}_{X,x}$ is a normal excellent integral noetherian local ring, so by analytic normality ([Ma] thm. 79) so is its $\mathfrak{m}_x$-adic completion $\mathcal{O}_{X,(x)}$. □

Lemma 3.3.15 *Let X be a normal scheme of finite type over a perfect field k and let $\xi = (\ldots, y)$ and $\eta = (y, \ldots)$ be saturated chains in it. Then the face map $\partial : \mathcal{O}_{X,\eta} \to \mathcal{O}_{X,\xi \vee \mathrm{d}_0 \eta}$ is a strict monomorphism.*

Proof By induction on the length of ξ it suffices to consider $\xi = (x, y)$. Set $\hat{X} := X_\eta$. For every $\hat{y}|y$ the homomorphism $\mathcal{O}_{X,y} \to \mathcal{O}_{\hat{X},\hat{y}}$ is faithfully flat, so there exists some $\hat{x} \in \hat{X}$ with $\hat{x} \geq \hat{y}$ and $\hat{x}|x$. Since $\mathcal{O}_{\hat{X},\hat{y}}$ is a noetherian integral domain we have injections

$$\mathcal{O}_{X,\eta} \cong \prod_{\hat{y}|y} \mathcal{O}_{\hat{X},\hat{y}} \hookrightarrow \prod_{\hat{x}|x} \mathcal{O}_{\hat{X},\hat{x}} \hookrightarrow \prod_{\hat{x}|x} \mathcal{O}_{\hat{X},(\hat{x})} \cong \mathcal{O}_{X,\xi \vee \mathrm{d}_0 \eta} \ .$$

Now use thm. 3.2.14 and thm. 3.3.8. □

In general we have:

Theorem 3.3.16 *Let X be a reduced scheme of finite type over a perfect field k and let $\eta = (y, \ldots)$ be a saturated chain in it. Then there exists a finite set S of chains in X satisfying:*

i) *Every $\xi \in S$ is saturated, begins with the generic point of some irreducible component of X and ends with y.*

ii) *The face map $\partial : \mathcal{O}_{X,\eta} \to \prod_{\xi \in S} \mathcal{O}_{X,\xi \vee \mathrm{d}_0 \eta}$ is a strict monomorphism of ST k-algebras.*

Note that the trivial case when y itself is a generic point is included, taking $S = \{(y)\}$.

Proof Let $\pi : \tilde{X} \to X$ be the normalization ($\pi^{\underline{1}} : X^{\underline{1}} \to X$ in the previous notation) and let $\tilde{\eta}_1, \ldots, \tilde{\eta}_r$ be the distinct chains in $\tilde{X}$ lying over η. Since $\mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}}$ is injective it follows that $\mathcal{O}_{X,\eta} \to (\pi_* \mathcal{O}_X)_\eta \cong \prod_{i=1}^r \mathcal{O}_{\tilde{X},\tilde{\eta}_i}$ is a strict monomorphism (remember that $\mathcal{O}_{X,\eta}$ is a Zariski ST ring).

For each $\tilde{\eta}_i = (\tilde{y}_i, \ldots)$ choose a saturated chain $\tilde{\xi}_i = (\tilde{x}_i, \ldots, \tilde{y}_i)$ in $\tilde{X}$, with $\tilde{x}_i$ being the generic point of the component of $\tilde{X}$ containing $\tilde{y}_i$. By lemma 3.3.15, $\mathcal{O}_{\tilde{X},\tilde{\eta}_i} \to \mathcal{O}_{\tilde{X},\tilde{\xi}_i \vee \mathrm{d}_0 \tilde{\eta}_i} = k(\tilde{\xi}_i \vee \mathrm{d}_0 \tilde{\eta}_i)$ is a strict monomorphism.

Let S be any finite set of chains in X as described in i) which contains all the chains $\pi(\tilde{\xi}_i)$, $i = 1, \ldots, r$. Since π is a birational morphism one gets $\prod_{\xi \in S} k(\xi \vee \mathrm{d}_0 \eta) \cong \prod_{\xi \in S} \prod_{\tilde{\zeta} | \xi \vee \mathrm{d}_0 \eta} k(\tilde{\zeta})$; and $\prod_{i=1}^r k(\tilde{\xi}_i \vee \mathrm{d}_0 \tilde{\eta}_i)$ is a direct factor of this ring. Therefore $\prod_{i=1}^r \mathcal{O}_{\tilde{X},\tilde{\eta}_i} \to \prod_{\xi \in S} \prod_{\tilde{\zeta} | \xi \vee \mathrm{d}_0 \eta} k(\tilde{\zeta})$ is a strict monomorphism. Putting it all together we see that $\partial : \mathcal{O}_{X,\eta} \to \prod_{\xi \in S} k(\xi \vee \mathrm{d}_0 \eta)$ is a strict monomorphism. $\square$

4 Residues on Schemes

4.1 The Parshin Residue Map

Parshin found a definition of a residue map that generalizes the residue map for curves used by Serre in [Se] ch. II no. 7. We present a variant of this residue map, which depends on geometric data (a chain in X) and algebraic data (a pseudo-coefficient field). The main result of this section is cor. 4.1.16, which establishes the transitivity of the residue maps with respect to compatible coefficient-fields. In this section X is a scheme of finite type over a perfect field k.

Definition 4.1.1 *Let $(A, \mathfrak{m})$ be a local k-algebra. A pseudo-coefficient field (resp. quasi-coefficient field, resp. coefficient field) for A is a k-algebra homomorphism $\sigma : K \to A$ where K is a field and the extension $\bar{\sigma} : K \to A/\mathfrak{m}$ is finite (resp. finite separable, resp. bijective). If $A = \mathcal{O}_{X,(x)} = \hat{\mathcal{O}}_{X,x}$ for some point $x \in X$, we say that σ is a pseudo-coefficient field (resp. quasi-coefficient field, resp. coefficient field) for x.*

By Hensel's lemma every quasi-coefficient field gives rise to a unique coefficient field. Thus if $K \subset k(x)$ is a subfield s.t. $K \to k(x)$ is finite separable there is a bijection $\sigma \mapsto \sigma|_K$ between the sets $\mathrm{Hom}_{\mathsf{Alg}(k)}(k(x), \mathcal{O}_{X,(x)})$ and $\mathrm{Hom}_{\mathsf{Alg}(k)}(K, \mathcal{O}_{X,(x)})$. In particular a closed point x has a unique coefficient field. If X is reduced and x is the generic point of an irreducible component then x has a unique coefficient field, since $\mathcal{O}_{X,x} \xrightarrow{\simeq} k(x)$.

Let $\xi = (x, \ldots, y)$ be a saturated chain of length n in X and let $\sigma : K \to \mathcal{O}_{X,(y)}$ be a pseudo-coefficient field. Let $\bar{\sigma}$ be the composed k-algebra homomorphism

$$\bar{\sigma} : K \xrightarrow{\sigma} \mathcal{O}_{X,(y)} \xrightarrow{\partial} \mathcal{O}_{X,\xi} \twoheadrightarrow k(\xi)$$

where ∂ is the face map. According to prop. 3.3.6 and thm. 3.3.2 c), $\bar{\sigma}$ is a morphism in $\mathsf{CTLF}_{\mathrm{red}}(k)$ of dimension n. Recall that given any morphism

$f : A \to B$ in $\mathsf{CTLF}_{\mathrm{red}}(k)$ there is a canonical residue map

$$\mathrm{Res}_f = \mathrm{Res}_{B/A} : \Omega^{*,\mathrm{sep}}_{B/k} \to \Omega^{*,\mathrm{sep}}_{A/k} . \tag{4.1.2}$$

It is a homomorphism of differential graded ST $\Omega^{*,\mathrm{sep}}_{A/k}$-modules (see §2.4).

The next definition is taken from [Lo] p. 516.

Definition 4.1.3 (Parshin's Residue Map) *Let $\xi = (x, \ldots, y)$ be a saturated chain in X and let $\sigma : K \to \mathcal{O}_{X,(y)}$ be a pseudo-coefficient field. Parshin's residue map is the composition*

$$\mathrm{Res}_{\xi,\sigma} = \mathrm{Res}_{\xi,K} : \Omega^{*}_{k(x)/k} \xrightarrow{\partial} \Omega^{*,\mathrm{sep}}_{k(\xi)/k} \xrightarrow{\mathrm{Res}_{k(\xi)/K}} \Omega^{*,\mathrm{sep}}_{K/k} .$$

The residue map $\mathrm{Res}_{\xi,\sigma}$ is a homomorphism of differential graded k-modules of degree $-n$, where n is the length of ξ.

Proposition 4.1.4 *Let $\xi = (x, \ldots, y)$ be a saturated chain in X and let σ be a coefficient field for y. Then the residue map $\mathrm{Res}_{\xi,\sigma} : \Omega^{*}_{k(x)/k} \to \Omega^{*}_{k(y)/k}$ is a locally differential operator over $\mathcal{O}_X$ relative to k.*

Proof Since these are skyscraper sheaves it suffices to check stalks at y. Given a form $\alpha \in \Omega^{*}_{k(x)/k}$ we will show that $\mathrm{Res}_{\xi,\sigma}|_{\mathcal{O}_{X,y}\cdot\alpha}$ is a differential operator. Consider the k-linear homomorphism $\phi : \mathcal{O}_{X,y} \to \Omega^{*}_{k(y)/k}$, $\phi(a) = \mathrm{Res}_{\xi,\sigma}(a\alpha)$. It factors through the continuous $k(y)$-linear homomorphisms

$$\mathcal{O}_{X,(y)} \xrightarrow{\partial} \mathcal{O}_{X,\xi} \twoheadrightarrow k(\xi) \xrightarrow{\cdot\alpha} \Omega^{*,\mathrm{sep}}_{k(\xi)/k} \xrightarrow{\mathrm{Res}_{\sigma}} \Omega^{*}_{k(y)/k} .$$

The module $\Omega^{*}_{k(y)/k}$ is discrete, so $\phi(\mathfrak{m}^{i+1}_{(y)}) = 0$ for $i >> 0$. Hence $\mathrm{Res}_{\xi,\sigma}|_{\mathcal{O}_{X,y}\cdot\alpha}$ factors $k(y)$-linearly through the finite length $\mathcal{O}_{X,(y)}$-module $\left(\mathcal{O}_{X,(y)} \cdot \alpha\right) / \left(\mathfrak{m}^{i+1}_{(y)} \cdot \alpha\right)$. According to prop. 1.4.4 it is a differential operator of order $\leq i$. $\square$

Definition 4.1.5 *Let $\xi = (x, \ldots, y)$ be a saturated chain in X and let σ and τ be coefficient fields for x and y respectively. We say that σ/τ are compatible coefficient fields for ξ if $\sigma_\xi : k(\xi) \to \mathcal{O}_{X,\xi}$ is a $k(y)$-algebra homomorphism; i.e. if the diagram below commutes:*

$$\begin{tikzcd} k(y) \arrow[r, "\tau"] \arrow[d, "\bar{\tau}"'] & \mathcal{O}_{X,\xi} \arrow[d] \\ k(\xi) \arrow[r, "="] \arrow[ur, "\sigma_\xi"] & k(\xi) \end{tikzcd}$$

Suppose $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ are saturated chains and suppose σ is a coefficient field for y. Consider the continuous k-algebra homomorphism

$$\bar{\sigma}_\eta : k(\eta) \xrightarrow{\sigma_\eta} \mathcal{O}_{X,\eta} \xrightarrow{\partial} \mathcal{O}_{X,\xi \vee \mathrm{d}_0\eta} \twoheadrightarrow k(\xi \vee \mathrm{d}_0\eta) \ .$$

Say ξ has length n. According to cor. 3.3.4 one has $[\kappa_n(\, k(\xi \vee \mathrm{d}_0\eta)\,) : k(\eta)] = [\kappa_n(\, k(\xi)\,) : k(y)] < \infty$. Therefore $\bar{\sigma}_\eta : k(\eta) \to k(\xi \vee \mathrm{d}_0\eta)$ is a morphism in $\mathsf{CTLF}_{\mathrm{red}}(k)$ of dimension n.

The next lemma shows that compatibility of coefficient fields is transitive.

Lemma 4.1.6 *Let $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ be saturated chains. Let ρ, σ and τ be coefficient fields for x, y and z respectively, s.t. ρ/σ and σ/τ are compatible for ξ and η respectively. Then ρ/τ are compatible coefficient fields for $\xi \vee \mathrm{d}_0\eta = (x, \ldots, y, \ldots, z)$.*

Proof It suffices to show that the diagram

$$\begin{tikzcd} k(\eta) \arrow[r, "\sigma_\eta"] \arrow[d, "\bar{\sigma}_\eta"'] & \mathcal{O}_{X,\xi \vee \mathrm{d}_0\eta} \arrow[d] \\ k(\xi \vee \mathrm{d}_0\eta) \arrow[r, "="] \arrow[ur, "\rho_{\xi \vee \mathrm{d}_0\eta}"] & k(\xi \vee \mathrm{d}_0\eta) \end{tikzcd} \tag{4.1.7}$$

is commutative. By assumption if we replace η with (y) everywhere in the diagram it becomes commutative. Hence $\rho_{\xi \vee \mathrm{d}_0\eta} \circ \bar{\sigma}_\eta$ and σ_η are $k(y)$-algebra homomorphisms. But $k(y) \to k(\eta)$ is topologically étale, so by uniqueness for every $i \geq 0$

$$\rho_{\xi \vee \mathrm{d}_0\eta} \circ \bar{\sigma}_\eta = \sigma_\eta : k(\eta) \to \mathcal{O}_{X,\xi \vee \mathrm{d}_0\eta}/\mathfrak{m}^{i+1}_{\xi \vee \mathrm{d}_0\eta} \ .$$

Now pass to the inverse limit in i. □

Proposition 4.1.8 *Let $\xi = (x_0, \ldots, x_n)$ be a saturated chain in X. There exist compatible coefficient fields $\sigma_i : k(x_i) \to \mathcal{O}_{X,(x_i)}$ s.t. each pair σ_i/σ_j, $i < j$, is compatible for $(x_i, \ldots, x_j)$.*

In characteristic 0 this is an immediate consequence of Noether normalization. In general we reduce this to a problem in linear algebra:

Proof It suffices to find quasi-coefficient fields K_i which fit into a diagram

$$\begin{array}{ccccccc}
K_n & \longrightarrow & K_{n-1} & \longrightarrow \cdots \longrightarrow & K_0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{X,x_n} & \longrightarrow & \mathcal{O}_{X,x_{n-1}} & \longrightarrow \cdots \longrightarrow & \mathcal{O}_{X,x_0} \\
\downarrow & & \downarrow & & \downarrow \\
k(x_n) & & k(x_{n-1}) & & k(x_0)
\end{array}$$

(cf. proof of lemma 4.1.6). To do so we find k-vector spaces $V_n \subset \cdots \subset V_0 \subset \mathcal{O}_{X,x_n}$ s.t. for all i,

$$1 \otimes \mathrm{d} : k(x_i) \otimes_k V_i \to \Omega^1_{k(x_i)/k} \tag{4.1.9}$$

is bijective. Then $\mathrm{tr.deg}_k\, k(x_i) = \mathrm{rank}_k\, V_i$ and the polynomial ring $k[V_i]$ embeds into $k(x_i)$. Letting K_i be the fraction field of $k[V_i]$ we see that $K_i \to \mathcal{O}_{X,x_i}$ is a quasi-coefficient field.

Suppose we succeeded in finding k-vector spaces $V'_n \subset V'_{n-1} \subset \ldots \subset V'_i \subset \mathcal{O}_{X,x_n}$ satisfying (4.1.9). The $\mathcal{O}_{X,x_i}$-module $\Omega^1_{X/k,x_i}$ is spanned by $\mathrm{d}(V'_i)$ and $\mathrm{d}(\mathfrak{p}_i)$, where $\mathfrak{p}_i \subset \mathcal{O}_{X,x_n}$ is the prime ideal of x_i. Hence $\mathrm{d}(V'_i) + \mathrm{d}(\mathfrak{p}_i)$ span $\Omega^1_{k(x_{i-1})/k}$. We can modify V'_j, $i \leq j \leq n$, to some subspace $V_j \subset V'_j \oplus \mathfrak{p}_i \subset \mathcal{O}_{X,x_n}$ s.t. $V_j \equiv V'_j \pmod{\mathfrak{p}_i}$, $\mathrm{rank}_k\, V_j = \mathrm{rank}_k\, V'_j$ and $1 \otimes \mathrm{d} : k(x_{i-1}) \otimes_k V_j \to \Omega^1_{k(x_{i-1})/k}$ is injective. Next extend V_i to an appropriate subspace $V_{i-1} \subset \mathcal{O}_{X,(x_n)}$. $\square$

Let $\sigma : K \to \mathcal{O}_{X,(x)}$ be a pseudo-coefficient field for $x \in X$ and let $\xi = (x, \ldots)$ be a saturated chain. Assume that $\bar{\sigma} : K \to k(x)$ is purely inseparable. If $\bar{\sigma}$ is bijective set $K_\xi := k(\xi)$, and let $u := \partial : K \to K_\xi$. Otherwise $\mathrm{char}\, k = p$ and we define K_ξ below, using "purely inseparable descent".

Suppose k has characteristic p. Given a k-algebra A let $A^{(p/k)}$ be the k-algebra defined in (1.4.7) and let $\mathrm{F}_{A/k} : A^{(p/k)} \to A$ be the relative Frobenius homomorphism. The map $\mathrm{F}_{k(\xi)/k} : k(\xi)^{(p/k)} \to k(\xi)$ is a finite morphism in $\mathsf{CTLF}_{\mathrm{red}}(k)$ of degree equal to the differential degree of $k(\xi)$, i.e. $\mathrm{rank}_{k(\xi)}\, \Omega^{1,\mathrm{sep}}_{k(\xi)/k}$ (cf. prop. 2.1.13). Since $k(x) \to k(\xi)$ is topologically étale and unramified, the same is true of $k(x)^{(p/k)} \to k(\xi)^{(p/k)}$. Comparing degrees one finds that $k(x) \otimes_{k(x)^{(p/k)}} k(\xi)^{(p/k)} \to k(\xi)$ is an isomorphism of clusters of TLFs.

In our situation we get $k(x)^{(p^j/k)} \subset K$ for $j >> 0$ and we define

$$K_\xi := K \otimes_{k(x)^{(p^j/k)}} k(\xi)^{(p^j/k)} , \tag{4.1.10}$$

a cluster of TLFs. The homomorphism $u : K \to K_\xi$ is also topologically étale and unramified, and $k(x) \otimes_K K_\xi \xrightarrow{\simeq} k(\xi)$. Because there exists some continuous k-algebra homomorphism $K_\xi \to \mathcal{O}_{X,\xi}$ (e.g. take $K_\xi \to k(\xi) \xrightarrow{\tau_\xi} \mathcal{O}_{X,\xi}$ arising from some coefficient field $\tau : k(x) \to \mathcal{O}_{X,(x)}$), σ extends uniquely to a homomorphism

$$\sigma_\xi : K_\xi \to \mathcal{O}_{X,\xi} . \tag{4.1.11}$$

In §2.2 we find the notion of finitely ramified base change. It is a universal construction in the category of clusters of TLFs, generalizing the tensor product.

Theorem 4.1.12 *Let $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ be saturated chains in X and let $\sigma : K \to \mathcal{O}_{X,(y)}$ be a pseudo-coefficient field s.t. $\bar{\sigma} : K \to k(y)$ is purely inseparable. Let $u : K \to K_\eta$ be the finitely ramified homomorphism defined above. Then the diagram*

$$\begin{tikzcd} k(\xi) \arrow[r, "\partial"] & k(\xi \vee \mathrm{d}_0\eta) \\ K \arrow[u, "\bar{\sigma}"] \arrow[r, "u"] & K_\eta \arrow[u, "\bar{\sigma}_\eta"'] \end{tikzcd}$$

is a finitely ramified base change.

Proof Let $K_\eta \to B'$ be the morphism gotten by the finitely ramified base change $K \to K_\eta$. By universality there is a finite morphism $B' \to k(\xi \vee \mathrm{d}_0\eta)$ in $\mathsf{CTLF}(k)$. The ring B' is reduced, because $K \to K_\eta$ is topologically étale relative to k (cf. proof of thm. 2.4.23). For each $\mathfrak{n} \in \operatorname{Spec} k(\xi \vee \mathrm{d}_0\eta)$ lying over some $\mathfrak{m}' \in \operatorname{Spec} B'$, the finite morphism of TLFs $B'/\mathfrak{m}' \to k(\xi \vee \mathrm{d}_0\eta)/\mathfrak{n}$ is an isomorphism since $k(x) \to k(\xi \vee \mathrm{d}_0\eta)$ is unramified. Thus it remains to show that the map of sets $\operatorname{Spec} k(\xi \vee \mathrm{d}_0\eta) \to \operatorname{Spec} B'$ is bijective. Taking n-th residue fields, where n is the length of ξ, we reduce to showing that $\kappa_n(B') \to \kappa_n(\, k(\xi \vee \mathrm{d}_0\eta)\,)$ is bijective. It is known (cor. 3.3.4) that

$$\kappa_n(\, k(\xi)\,) \otimes_K K_\eta \cong \kappa_n(\, k(\xi)\,) \otimes_{k(y)} k(\eta) \cong \kappa_n(\, k(\xi)\,)_\eta \to \kappa_n(\, k(\xi \vee \mathrm{d}_0\eta)\,)$$

is bijective. By the following lemma the same holds for B' (using the fact that in characteristic p any topologically étale homomorphism is separable, cf. cor. 2.1.16). □

Lemma 4.1.13 *Let K, K' and L be TLFs over k, let $f : K \to L$ be a morphism of dimension n, and let $u : K \to K'$ be a finitely ramified, separable homomorphism. Let $f' : K' \to L'$ be the morphism gotten by finitely ramified base change, and let $v : L \to L'$ be the corresponding finitely ramified homomorphism. Then the canonical homomorphism*

$$\kappa_n(L) \otimes_K K' \to \kappa_n(L') \tag{4.1.14}$$

is an isomorphism.

Proof Let F be the separable closure of K in $\kappa_n(L)$. Then one can lift F into L and get $f : K \to F \to L$. Correspondingly we get $f' : K' \to F' \to L'$, where $F' := F \otimes_K K'$. But $\kappa_n(L) \otimes_F F' \cong \kappa_n(L) \otimes_K K'$, so we may assume that $K \to \kappa_n(L)$ is purely inseparable. Let $(t_1, \ldots, t_n)$ be an initial system of regular parameters in L. Counting degrees we have

$$\begin{aligned}&[\kappa_n(L) \otimes_K K' : K'] = [\kappa_n(L) : K] = [L : K((t_1, \ldots, t_n))] \\ &= [L' : K'((t_1, \ldots, t_n))] \geq [\kappa_n(L') : K'] ,\end{aligned} \tag{4.1.15}$$

the gap going towards ramification in $K'((t_1, \ldots, t_n)) \to L'$. However in our case $\kappa_n(L) \otimes_K K'$ is a field so (4.1.14) is a bijection (and there is equality in (4.1.15)). □

Corollary 4.1.16 (Transitivity) *Let $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ be saturated chains in X and let σ/τ be compatible coefficient fields for η. Then*

$$\mathrm{Res}_{\xi \vee \mathrm{d}_0\eta, \tau} = \mathrm{Res}_{\eta,\tau} \circ \mathrm{Res}_{\xi,\sigma} : \Omega^*_{k(x)/k} \to \Omega^*_{k(z)/k} .$$

Proof Apply the preceding theorem and thm. 2.4.23 to the diagram

$$\begin{tikzcd}
k(x) \arrow[r] & k(\xi) \arrow[r] & k(\xi \vee \mathrm{d}_0\eta) \\
& k(y) \arrow[u, "\bar{\sigma}"'] \arrow[r] & k(\eta) \arrow[u, "\bar{\sigma}_\eta"'] \\
& & k(z) \arrow[u, "\bar{\tau}"']
\end{tikzcd}$$

□

4.2 Poles of Meromorphic Differential Forms

In this section we consider a high dimensional version of a pole of a differential form. Even though the residue map depends on a choice of coefficient field, the order of pole of a form along a chain is independent of this choice. The key result is:

Lemma 4.2.1 *Let $\xi = (x, \ldots, y)$ be a saturated chain in X and let $M \subset \Omega^{*,\mathrm{sep}}_{k(\xi)/k}$ be a left $\Omega^{*,\mathrm{sep}}_{X/k,y}$-submodule. Then the following conditions on M are equivalent:*

i) *For any pseudo-coefficient field $\sigma : K \to \mathcal{O}_{X,(y)}$, $\mathrm{Res}_{k(\xi)/K}(M) = 0$.*

ii) *For any coefficient field* $\sigma : k(y) \to \mathcal{O}_{X,(y)}$, $\mathrm{Res}_{k(\xi)/k(y),\sigma}(M) = 0$.

iii) *For any saturated chain* $\eta = (y, \dots, z)$ *with* z *a closed point,* $\mathrm{Res}_{k(\xi \vee \mathrm{d}_0\eta)/k}$ $(M) = 0$.

Proof i) $\Rightarrow$ ii): Trivial.

ii) $\Rightarrow$ iii): As in the proof of cor. 4.1.16.

iii) $\Rightarrow$ i): Let L be the separable closure of K in $k(y)$. Then σ factors through L, so we can assume that $K \to k(y)$ is purely inseparable. Choose a chain η as in iii). Now $\mathrm{Res}_{k(\xi \vee \mathrm{d}_0\eta)/k}$ is continuous, k is separated and $\Omega^{*}_{X/k,y} \to \Omega^{*,\mathrm{sep}}_{X/k,\eta}$ is dense. Condition iii) implies that $\mathrm{Res}_{k(\xi \vee \mathrm{d}_0\eta)/k}(\Omega^{*,\mathrm{sep}}_{X/k,\eta} \cdot M) = 0$, so we can assume that M is an $\Omega^{*,\mathrm{sep}}_{X/k,\eta}$-module.

Suppose for some $\alpha \in M$ the form $\beta := \mathrm{Res}_{k(\xi)/K}(\alpha) \in \Omega^{*}_{K/k}$ is non-zero. Define $\sigma_\eta : K_\eta \to \mathcal{O}_{X,\eta}$ like in (4.1.11). Then the image of β in $\Omega^{*,\mathrm{sep}}_{K_\eta/k} \cong K_\eta \otimes_K \Omega^{*}_{K/k}$ is also non-zero. Because the residue pairing $\langle -, - \rangle_{K_\eta/k}$ is perfect (see thm. 2.4.22) there is some $\gamma \in \Omega^{*,\mathrm{sep}}_{K_\eta/k}$ s.t. $\mathrm{Res}_{K_\eta/k}(\gamma \wedge \beta) \neq 0$. But then $\sigma_\eta(\gamma) \in \Omega^{*,\mathrm{sep}}_{X/k,\eta}$, so $\bar{\sigma}_\eta(\gamma) \wedge \alpha \in M$ with

$$\begin{aligned}&\mathrm{Res}_{k(\xi \vee \mathrm{d}_0\eta)/k}(\bar{\sigma}_\eta(\gamma) \wedge \alpha)\\ &\quad = \mathrm{Res}_{K_\eta/k} \circ \mathrm{Res}_{k(\xi \vee \mathrm{d}_0\eta)/K_\eta}(\bar{\sigma}_\eta(\gamma) \wedge \alpha) = \mathrm{Res}_{K_\eta/k}(\gamma \wedge \beta) \neq 0 \ ,\end{aligned}$$

a contradiction. □

Definition 4.2.2 a) *Let* K *be a TLF over* k *of differential degree* d. *Define* $\omega_K := \Omega^{d,\mathrm{sep}}_{K/k}$.

b) *Let* $A = \prod_{\mathfrak{m} \in \operatorname{Spec} A} A/\mathfrak{m}$ *be a reduced cluster of TLFs over* k. *Define* $\omega_A := \bigoplus_{\mathfrak{m} \in \operatorname{Spec} A} \omega_{A/\mathfrak{m}}$, *a free ST* A*-module of rank* 1.

c) *Let* ξ *be a saturated chain in* X. *Define* $\omega(\xi) := \omega_{k(\xi)}$.

For $\xi = (x)$ we shall write $\omega(x)$ instead of $\omega(\,(x)\,)$; thus $\omega(x) = \Omega^{d}_{k(x)/k}$, where $d = \mathrm{tr.deg}_k\, k(x) = \dim \{x\}^{-}$. Recall that given a saturated chain $\xi = (x, \dots)$ the face map $\partial : k(x) \to k(\xi)$ is topologically étale relative to k, so $\omega(\xi) \cong k(\xi) \otimes_{k(x)} \omega(x)$. If X is integral with generic point x then the elements of $\Omega^{*,\mathrm{sep}}_{k(\xi)/k} = \Omega^{*,\mathrm{sep}}_{X/k,\xi}$ are called the meromorphic forms on X along ξ.

Definition 4.2.3 (Holomorphic Forms) *Let* $\xi = (x, \dots, y)$ *be a saturated chain in* X.

a) *A form* $\alpha \in \omega(\xi)$ *is said to be holomorphic if the equivalent conditions of lemma 4.2.1 hold for the module* $\mathcal{O}_{X,y} \cdot \alpha \subset \omega(\xi)$. *Define*

$$\omega(\xi)_{\mathrm{hol}} := \{\alpha \in \omega(\xi) \mid \alpha \text{ is holomorphic } \} .$$

b) *A form* $\alpha \in \omega(x)$ *is said to be holomorphic along* ξ *if its image in* $\omega(\xi)$ *is holomorphic. Define*

$$\omega(x)_{\mathrm{hol}:\xi} := \{\alpha \in \omega(x) \mid \alpha \text{ is holomorphic along } \xi\} = \omega(x) \cap \omega(\xi)_{\mathrm{hol}} .$$

Let $A \to B$ be a morphism in $\mathsf{CTLF}_{\mathrm{red}}(k)$. In §2.4 the residue pairing

$$\langle -,- \rangle_{B/A} : B \times \omega_B \xrightarrow{\mathrm{mult}} \omega_B \xrightarrow{\mathrm{Res}_{B/A}} \omega_A \tag{4.2.4}$$

is defined. It is a perfect pairing of ST A-modules. Now let $\xi = (x, \ldots, y)$ be a saturated chain and let $\sigma : K \to \mathcal{O}_{X,(y)}$ be a pseudo-coefficient field. Then the K-module $\omega(\xi)_{\mathrm{hol}} \subset \omega(\xi)$ is precisely the perpendicular space to $\mathcal{O}_{X,y}$ under the pairing $\langle -,- \rangle_{k(\xi)/K}$.

Lemma 4.2.5 *Given saturated chains* $\xi = (x, \ldots, y)$ *and* $\eta = (y, \ldots, z)$, *the face map* $\omega(\xi) \to \omega(\xi \vee \mathrm{d}_0\eta)$ *sends* $\omega(\xi)_{\mathrm{hol}}$ *into* $\omega(\xi \vee \mathrm{d}_0\eta)_{\mathrm{hol}}$. *Therefore* $\omega(x)_{\mathrm{hol}:\xi} \subset \omega(x)_{\mathrm{hol}:\xi \vee \mathrm{d}_0\eta}$.

Proof Choose compatible coefficient fields σ/τ for η and use lemma 4.2.1. □

Lemma 4.2.6 *Let* $\xi = (x, \ldots, y)$ *be a saturated chain of length* ≥ 1 *and let* $\sigma : K \to \mathcal{O}_{X,(y)}$ *be a pseudo-coefficient field. Then* $\mathrm{Res}_{k(\xi)/K}(\Omega^{*,\mathrm{sep}}_{\mathcal{O}_1(k(\xi))/k}) = 0$. *Therefore the image of the canonical homomorphism* $\Omega^{d,\mathrm{sep}}_{X/k,\mathrm{d}_0\xi} \to \Omega^{d,\mathrm{sep}}_{k(\xi)/k} = \omega(\xi)$ *is inside* $\omega(\xi)_{\mathrm{hol}}$; *here* d *is the differential degree of* $k(\xi)$.

Proof By lemma 4.2.1 we can assume that y is a closed point and that $K = k$. Let $\mathfrak{m} \in \operatorname{Spec} k(\xi)$ and let $L := k(\xi)/\mathfrak{m}$. Then $L \cong F((t_1, \ldots, t_d))$ and $\mathcal{O}_1(L) \cong F((t_2, \ldots, t_d))[[t_1]]$, with $[F : k] < \infty$. Since $F[t_1, \ldots, t_d] \to F((t_2, \ldots, t_d))[[t_1]]$ is topologically étale relative to k we get $\Omega^{d,\mathrm{sep}}_{\mathcal{O}_1(L)/k} \cong F((t_2, \ldots, t_d))[[t_1]] \cdot \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d$, so by definition the residue map vanishes on it. □

Theorem 4.2.7 *Given a saturated chain* ξ *in* X, *the* k*-submodule of holomorphic forms* $\omega(\xi)_{\mathrm{hol}} \subset \omega(\xi)$ *is open.*

Proof The proof is by induction on the length of ξ. For $\xi = (x)$ the module $\omega(x)$ is discrete so $\omega(x)_{\mathrm{hol}} = 0$ is open. Suppose that $\xi = (x, y, \ldots, z)$ is of length ≥ 1 (so possibly $y = z$) and that $\omega(\mathrm{d}_0\xi)_{\mathrm{hol}} \subset \omega(\mathrm{d}_0\xi)$ is open. Choose compatible coefficient fields σ/τ for $\mathrm{d}_0\xi$. Let $\bar{\sigma} = \bar{\sigma}_{\mathrm{d}_0\xi} : k(\mathrm{d}_0\xi) \to k(\xi)$ and $\bar{\tau} : k(z) \to k(\mathrm{d}_0\xi)$ be the induced morphisms in $\mathsf{CTLF}_{\mathrm{red}}(k)$. We claim that

$$\omega(\xi)_{\mathrm{hol}} = \{\alpha \in \omega(\xi) \mid \forall a \in \mathcal{O}_{X,z},\ \mathrm{Res}_{\bar{\sigma}}(a\alpha) \in \omega(\mathrm{d}_0\xi)_{\mathrm{hol}}\}\ . \tag{4.2.8}$$

This follows from condition ii) of lemma 4.2.1, since $\mathrm{Res}_{\bar{\sigma}\circ\bar{\tau}} = \mathrm{Res}_{\bar{\tau}} \circ \mathrm{Res}_{\bar{\sigma}}$.

Choose elements $a_1, \ldots, a_r \in \mathfrak{m}_{(z)} \subset \mathcal{O}_{X,(z)}$ which span $\mathfrak{m}_{(z)}/\mathfrak{m}_{(z)}^2$. Then the continuous $k(z)$-algebra homomorphism $k(z)[[a_1, \ldots, a_r]] \to \mathcal{O}_{X,(z)}$ extending τ is surjective. Let A be the polynomial ring $k(z)[a_1, \ldots, a_r]$. An open subgroup $U \subset \omega(\xi)$ is also closed, so such U is an $\mathcal{O}_{X,z}$-module iff it is an $\mathcal{O}_{X,(z)}$-module, iff it is an A-module. In particular, $\omega(\mathrm{d}_0\xi)_{\mathrm{hol}}$ is an A-submodule of $\omega(\mathrm{d}_0\xi)$. By continuity of the residue map,

$$\omega(\xi)_{\mathrm{hol}} = \{\alpha \in \omega(\xi) \mid \mathrm{Res}_{\bar{\sigma}}(A \cdot \alpha) \subset \omega(\mathrm{d}_0\xi)_{\mathrm{hol}}\}\ .$$

Let $M := \Omega^{d,\mathrm{sep}}_{\mathcal{O}_1(k(\xi))/k} \subset \omega(\xi)$. It is a free ST $\mathcal{O}_1(k(\xi))$-module of rank 1, and $k(\xi) \cdot M = \omega(\xi)$. By lemma 4.2.6 we get $\mathrm{Res}_{\bar{\sigma}}(M) = 0$. Choose a regular parameter t in $\mathcal{O}_1(\,k(\xi)\,)$, so $\omega(\xi) = \bigcup_{j\geq 0} t^{-j-1}M$. Since $\mathfrak{m}_{\mathrm{d}_0\xi} \subset t\mathcal{O}_1(\,k(\xi)\,)$, for every $j \geq 0$ we have $\mathfrak{m}_{\mathrm{d}_0\xi}^{j+1} t^{-j-1} M \subset M$. According to prop. 1.4.4, the $k(\mathrm{d}_0\xi)$-linear homomorphism $\mathrm{Res}_{\bar{\sigma}}|_{t^{-j-1}M}$ is a DO of order $\leq j$ over $\mathcal{O}_{X,\mathrm{d}_0\xi}$. From formula (1.4.2) we see that for any fixed $\alpha \in t^{-j-1}M$,

$$\mathrm{Res}_{\bar{\sigma}}(A\alpha) \subset \sum_{(i_1,\ldots,i_r)\in I(j)} A \cdot \mathrm{Res}_{\bar{\sigma}}(a_1^{i_1} \cdots a_r^{i_r} \alpha)$$

where $I(j)$ is the finite set $\{(i_1, \ldots, i_r) \in \mathbb{N}^r \mid i_1 + \cdots + i_r \leq j\}$. Therefore

$$\omega(\xi)_{\mathrm{hol}} \cap t^{-j-1}M = \{\alpha \in t^{-j-1}M \mid \sum_{(i_1,\ldots,i_r)\in I(j)} \mathrm{Res}_{\bar{\sigma}}(a_1^{i_1} \cdots a_r^{i_r} \alpha) \in \omega(\mathrm{d}_0\xi)_{\mathrm{hol}}\}$$

is open in $\omega(\xi)$. By definition of the direct limit topology, $\omega(\xi)_{\mathrm{hol}} \subset \omega(\xi)$ is open. □

Corollary 4.2.9 *Let $\xi = (x, \ldots, y)$ be a saturated chain. The canonical map*

$$\frac{\omega(x)}{\omega(x)_{\mathrm{hol}:\xi}} \to \frac{\omega(\xi)}{\omega(\xi)_{\mathrm{hol}}}$$

is bijective.

Proof By definition the map is injective and according to cor. 3.2.12 it is dense. But by the theorem the module $\omega(\xi)/\omega(\xi)_{\mathrm{hol}}$ is discrete. □

From prop. 4.1.4 and prop. 1.4.6 it follows that $\omega(x)/\omega(x)_{\mathrm{hol}:\xi}$ is an artinian $\mathcal{O}_{X,y}$-module.

Definition 4.2.10 (Poles of Meromorphic Forms) *Let $\xi = (x, \ldots, y)$ be a saturated chain in X.*

a) *Given a differential form $\alpha \in \omega(x)$, let l be the length of the $\mathcal{O}_{X,y}$-module $(\mathcal{O}_{X,y} \cdot \alpha + \omega(x)_{\mathrm{hol}:\xi})/\omega(x)_{\mathrm{hol}:\xi}$. Then α is said to have a pole of order l along ξ.*

b) *If $l \leq 1$ then α is said to have a simple pole along ξ. Define*

$$\omega(x)_{\mathrm{sim}:\xi} := \{\alpha \in \omega(x) \mid \alpha \text{ has a simple pole along } \xi\} .$$

For any $\mathcal{O}_{X,y}$-module M denote its socle $\mathrm{Hom}_{\mathcal{O}_{X,y}}(k(y), M)$ by $\mathrm{soc}_{\mathcal{O}_{X,y}} M$. Then one has

$$\omega(x)_{\mathrm{sim}:\xi}/\omega(x)_{\mathrm{hol}:\xi} = \mathrm{soc}_{\mathcal{O}_{X,y}} \left(\omega(x)/\omega(x)_{\mathrm{hol}:\xi}\right) . \tag{4.2.11}$$

Proposition 4.2.12 *Let $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ be saturated chains in X, let $\sigma : k(y) \to \mathcal{O}_{X,(y)}$ be a coefficient field and let $\tau : K \to \mathcal{O}_{X,(z)}$ be a pesudo-coefficient field. Then for any form $\alpha \in \omega(x)_{\mathrm{sim}:\xi}$ one has*

$$\mathrm{Res}_{\xi \vee \mathrm{d}_0 \eta, \tau}(\alpha) = \mathrm{Res}_{\eta,\tau} \circ \mathrm{Res}_{\xi,\sigma}(\alpha) .$$

Proof We can assume that $K \to k(z)$ is purely inseparable. Choose a saturated chain $\zeta = (z, \ldots, w)$ with w a closed point and define K_ζ as in thm. 4.1.12. Define $\beta_1 := \mathrm{Res}_{\xi \vee \mathrm{d}_0 \eta, \tau}(\alpha)$ and $\beta_2 := \mathrm{Res}_{\eta,\tau} \circ \mathrm{Res}_{\xi,\sigma}(\alpha)$. If $\beta_1 \neq \beta_2$ there exists some $c \in K$ s.t. $\mathrm{Res}_{K_\zeta/k}(c(\beta_2 - \beta_1)) \neq 0$. Let $\bar{\tau} : K \to k(\eta)$ be the morphism induced by τ, and let $\sigma_\eta : k(\eta) \to \mathcal{O}_{X,\eta}$ be the lifting extending σ. Define $\tilde{c} := \sigma_\eta \circ \bar{\tau}(c) \in \mathcal{O}_{X,\eta}$. Then $\mathrm{Res}_{k(\xi \vee \mathrm{d}_0 \eta)/K, \sigma_\eta \circ \bar{\tau}}(\tilde{c}\alpha) = c\beta_2$. We claim that $\mathrm{Res}_{k(\xi \vee \mathrm{d}_0 \eta)/K, \tau}(\tilde{c}\alpha) = c\beta_1$. This leads to a contradiction, since by theorems 2.4.23 and 4.1.12, one has

$$\mathrm{Res}_{K_\zeta/k}(c\beta_2) = \mathrm{Res}_{k(\xi \vee \mathrm{d}_0 \eta \vee \mathrm{d}_0 \zeta)/k}(\tilde{c}\alpha) = \mathrm{Res}_{K_\zeta/k}(c\beta_1) .$$

In order to prove the claim, note that $\tau(c) - \tilde{c} \in \mathfrak{m}_\eta \subset \mathcal{O}_{X,\eta}$. The submodule $\omega(\xi \vee \mathrm{d}_0 \eta)_{\mathrm{hol}} \subset \omega(\xi \vee \mathrm{d}_0 \eta)$ is closed, and it contains $\omega(\xi)_{\mathrm{hol}}$ (by lemma 4.2.5). On the other hand $\mathfrak{m}_y \subset \mathfrak{m}_\eta$ is dense. Since $\mathfrak{m}_y \cdot \alpha \subset \omega(\xi)_{\mathrm{hol}}$, the continuity of multiplication implies that $\mathfrak{m}_\eta \cdot \alpha \subset \omega(\xi \vee \mathrm{d}_0 \eta)_{\mathrm{hol}}$. Therefore

$$\mathrm{Res}_{k(\xi \vee \mathrm{d}_0 \eta)/K, \tau}(\tilde{c}\alpha) = \mathrm{Res}_{k(\xi \vee \mathrm{d}_0 \eta)/K, \tau}(\tau(c)\alpha) = c\beta_1 .$$

□

Corollary 4.2.13 *Let $\xi = (x, \ldots, y)$ be a saturated chain in X. There is a canonical $\mathcal{O}_{X,y}$-linear homomorphism* $\mathrm{Res}_\xi : \omega(x)_{\mathrm{sim}:\xi} \to \omega(y)$. *If σ is any coefficient field for y, then* $\mathrm{Res}_\xi = \mathrm{Res}_{\xi,\sigma}|_{\omega(x)_{\mathrm{sim}:\xi}}$.

Proof Taking $\eta = (y)$ in the proposition it follows that for any two coefficient fields σ, σ' and all $\alpha \in \omega(x)_{\mathrm{sim}:\xi}$, one has $\mathrm{Res}_{\xi,\sigma}(\alpha) = \mathrm{Res}_{\xi,\sigma'}(\alpha)$. The $\mathcal{O}_{X,y}$-linearity follows from equation (4.2.11). □

Proposition 4.2.14 *Let $\alpha \in \omega(x)$ be a form. Then α is holomorphic along all but finitely many saturated chains $\xi = (x, \ldots)$.*

Proof Because X is quasi-compact we can assume that $X = \operatorname{Spec} A$. The proof is by induction on the length of ξ. First consider chains of length 1, $\xi = (x, y)$. For all but finitely many points $y \in \{x\}^-$ of codimension 1, α is in the image of $\Omega^*_{X/k,y}$; by lemma 4.2.6 α is holomorphic along such (x, y).

Now fix (x, y) and consider chains $\xi = (x, y, \ldots, z)$ of length $n \geq 2$. Write $\xi = (x, y) \vee \mathrm{d}_0\eta$ with $\eta = (y, \ldots, z)$ a chain of length $n - 1$. Choose a coefficient field $\sigma : k(y) \to \mathcal{O}_{X,(y)}$. Since $\mathrm{Res}_{(x,y),\sigma}|_{A\cdot\alpha}$ is a DO over A (prop. 4.1.4), there are forms $\beta_1, \ldots, \beta_r \in \omega(y)$ s.t. $\mathrm{Res}_{(x,y),\sigma}(A \cdot \alpha) \subset \sum_{i=1}^r A \cdot \beta_i$. By induction each β_i is holomorphic along all but finitely many chains η. For each η, $\mathrm{Res}_{(x,y),\sigma}(\mathcal{O}_{X,z} \cdot \alpha) \subset \sum \mathcal{O}_{X,z} \cdot \beta_i$ because $A \to \mathcal{O}_{X,z}$ is formally étale. Using lemma 4.2.1, if all β_i are holomorphic along η, then α is holomorphic along $(x, y) \vee \mathrm{d}_0\eta$. □

The following important theorem is due to Parshin. For surfaces see [Pa1] and for schemes of higher dimensions see [Lo]; cf. also [Be]. By prop. 4.2.14 it makes sense to consider, for fixed $x > y$ and for a pseudo-coefficient field $\sigma : K \to \mathcal{O}_{X,(y)}$, the sum $\sum_{\xi=(x,\ldots,y)} \mathrm{Res}_{\xi,\sigma} : \omega(x) \to \omega_K$.

Theorem 4.2.15 (Parshin-Lomadze) *Let X be a scheme of finite type over a perfect field k.*

a) *Let $\xi = (\ldots, x)$ and $\eta = (y, \ldots, z)$ be saturated chains in X s.t. $x > y$ and* $\mathrm{codim}(\{y\}^-, \{x\}^-) = 2$, *and let $\sigma : K \to \mathcal{O}_{X,(z)}$ be a pseudo-coefficient field. Then*

$$\sum_{w \in X, x > w > y} \mathrm{Res}_{\xi \vee (w) \vee \eta, \sigma} = 0 \ .$$

b) *Suppose X is proper over k, and let $\xi = (\ldots, x)$ be a saturated chain in X s.t.* $\dim \{x\}^- = 1$. *Then*

$$\sum_{w \in X, x > w} \mathrm{Res}_{\xi \vee (w), k} = 0 \ .$$

Proof By lemma 4.2.1 we can assume that in part a), z is a closed point and $K = k$. Then this is an instance of [Lo] thm. 3. □

4.3 The Residue Complex $\mathcal{K}^{\cdot}_X$ - Construction

In [RD] ch. VI §1 we find the following definitions. Let X be a locally noetherian scheme. For a point $x \in X$ let I be an injective hull of $k(x)$ as an $\mathcal{O}_{X,x}$-module, and let $J(x)$ be the skyscraper sheaf which is I on the closed set $\{x\}^-$ and 0 elsewhere. Then $J(x)$ is a quasi-coherent, injective $\mathcal{O}_X$-module.

Definition 4.3.1 *A residual complex on X is a complex $\mathcal{R}^{\cdot}$ of quasi-coherent, injective $\mathcal{O}_X$-modules, bounded below, with coherent cohomology sheaves, and such that there is an isomorphism of $\mathcal{O}_X$-modules*

$$\bigoplus_{p \in \mathbb{Z}} \mathcal{R}^p \cong \bigoplus_{x \in X} J(x) \ .$$

Now suppose X is a reduced scheme of finite type over a perfect field k. In this section we will construct a complex $\mathcal{K}^{\cdot}_X$ on X. We will show that it has all the properties of a residual complex, apart from having coherent cohomology sheaves. This last property shall be verified in §4.5. The complex $\mathcal{K}^{\cdot}_X$ is called the *Grothendieck residue complex* of X (relative to k).

Definition 4.3.2 *Let $x \in X$ be a point and let $\sigma : K \to \mathcal{O}_{X,(x)}$ be a pseudo-coefficient field. Define*

$$\mathcal{K}(\sigma) := \mathrm{Hom}^{\mathrm{cont}}_K(\mathcal{O}_{X,(x)}, \omega_K) \ ,$$

considered as a skyscraper sheaf supported on the closed set $\{x\}^-$. $\mathcal{K}(\sigma)$ is called the dual module of the local ring $\mathcal{O}_{X,x}$ (relative to k) determined by σ.

$\mathcal{K}(\sigma)$ is a quasi-coherent sheaf. By Matlis duality it is an injective hull of $k(x)$ over the local ring $\mathcal{O}_{X,x}$. Thus $\mathcal{K}(\sigma) \cong J(x)$ in the notation used above. In [Gr] these dual modules are the building blocks of the residue complex, and the same is true here. The main effort will be to identify the various $\mathcal{K}(\sigma)$ to a single module $\mathcal{K}(x)$.

Lemma 4.3.3 *Let $\xi = (x, \ldots, y)$ be a saturated chain in X and let σ/τ be compatible coefficient fields for ξ. Denote by* $\operatorname{loc} : \mathcal{O}_{X,y} \to \mathcal{O}_{X,x}$ *the localization homomorphism. Given any $\phi \in \mathcal{K}(\sigma)$ put*

$$\delta(\phi) = \delta_{\xi,\sigma/\tau}(\phi) := \operatorname{Res}_{\xi,\tau} \circ \phi \circ \operatorname{loc} : \mathcal{O}_{X,y} \to \omega(y)$$

(see diagram). Then

a) *The k-linear homomorphism $\delta(\phi)$ is continuous for the $\mathfrak{m}_y$-adic topology.*

b) *The continuous homomorphism $\delta(\phi)_{(y)} : \mathcal{O}_{X,(y)} \to \omega(y)$ extending $\delta(\phi)$ is $k(y)$-linear (via τ).*

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "\phi"] & \omega(x) \arrow[d, "\operatorname{Res}_{\xi,\tau}"] \\ \mathcal{O}_{X,y} \arrow[u, "\operatorname{loc}"] \arrow[r, dotted, "\delta(\phi)"] & \omega(y) \end{tikzcd} \tag{4.3.4}$$

Proof a) Since ϕ is continuous, $\phi(\mathfrak{m}_x^{i+1}) = 0$ for $i >> 0$, so it is a DO over $\mathcal{O}_X$. By prop. 4.1.4, $\operatorname{Res}_{\xi,\tau}$ is a locally DO. Thus the composition $\delta(\phi)$ is a DO over $\mathcal{O}_{X,y}$ (see lemma 3.1.9) and $\delta(\phi)(\mathfrak{m}_y^{j+1}) = 0$ for $j >> 0$.
b) Let $\phi_\xi : \mathcal{O}_{X,\xi} \to \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^{i+1} = (\mathcal{O}_{X,x}/\mathfrak{m}_x^{i+1})_\xi \to \omega(\xi)$ be the $k(\xi)$-linear map obtained by applying the completion $(-)_\xi$ to ϕ. Then by definition of $\operatorname{Res}_{\xi,\tau}$ we get

$$\delta(\phi)_{(y)} = \operatorname{Res}_{k(\xi)/k(y),\tau} \circ \phi_\xi \circ \partial : \mathcal{O}_{X,(y)} \to \omega(y)$$

where $\partial : \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,\xi}$ is the face map. Since σ/τ are compatible for ξ it follows that ϕ_ξ is $k(y)$-linear, and hence so is $\delta(\phi)_{(y)}$. $\square$

Remark 4.3.5 We adopt the following convention: operators denoted by the symbol "δ" are $\mathcal{O}_X$-linear, whereas operators denoted by the symbol "Res" are locally differential operators.

The crucial ingredient of our construction is the coboundary map δ between dual modules.

Definition 4.3.6 *Let $\xi = (x, \ldots, y)$ be a saturated chain and let σ/τ be compatible coefficient fields for ξ. The coboundary map $\delta_{\xi,\sigma/\tau} : \mathcal{K}(\sigma) \to \mathcal{K}(\tau)$ is by definition the $\mathcal{O}_X$-linear homomorphism $\phi \mapsto \delta_{\xi,\sigma/\tau}(\phi)$ of lemma 4.3.3. Also define $\delta_{\xi,\tau} : \omega(x) \to \mathcal{K}(\tau)$ by $\delta_{\xi,\tau}(\alpha)(a) := \operatorname{Res}_{\xi,\tau}(a\alpha)$, $\alpha \in \omega(x)$, $a \in \mathcal{O}_{X,(y)}$.*

The map $\delta_{(x),\sigma}$ is a canonical isomorphism $\omega(x) \xrightarrow{\simeq} \mathrm{soc}_{\mathcal{O}_{X,x}} \mathcal{K}(\sigma)$, and under isomorphism we have

$$\delta_{\xi,\sigma/\tau} \circ \delta_{(x),\sigma} = \delta_{\xi,\tau} : \omega(x) \to \mathcal{K}(\tau) \ . \tag{4.3.7}$$

Note also that $\ker(\delta_{\xi,\tau}) = \omega(x)_{\mathrm{hol}:\xi}$.

Suppose $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ are saturated chains and ρ, σ, τ are coefficient fields for x, y, z respectively, s.t. ρ/σ and σ/τ are compatible for ξ and η respectively. Then by lemma 4.1.6 and cor. 4.1.16 one has

$$\delta_{\xi \vee \mathrm{d}_0 \eta, \tau} = \delta_{\eta,\tau} \circ \delta_{\xi,\sigma} \ . \tag{4.3.8}$$

Recall that a module M over a noetherian local ring $(A, \mathfrak{m})$ is called cofinite if $M \cong \mathrm{Hom}_A(N, I)$ for some finitely generated A-module N and for some injective hull I of $A/\mathfrak{m}$.

Proposition 4.3.9 *Let $\xi = (x, \ldots, y)$ be saturated chain. Then $\omega(x)/\ \omega(x)_{\mathrm{hol}:\xi}$ is a cofinite $\mathcal{O}_{X,y}$-module. As such it can be regarded as a skyscraper quasi-coherent $\mathcal{O}_X$-module, supported on $\{y\}^-$. The map Res_ξ of cor. 4.2.13 induces a canonical isomorphism of $\mathcal{O}_X$-modules $\mathrm{Res}_\xi : \omega(x)_{\mathrm{sim}:\xi}/\omega(x)_{\mathrm{hol}:\xi} \xrightarrow{\simeq} \omega(y)$.*

Proof Choosing a coefficient field τ for y one gets an injection $\delta_{\xi,\tau} : (\omega(x)/\ \omega(x)_{\mathrm{hol}:\xi}) \hookrightarrow \mathcal{K}(\tau)$. By Matlis duality submodules of $\mathcal{K}(\tau)$ are duals of quotients of $\mathcal{O}_{X,(y)}$ with respect to the duality $\mathrm{Hom}_{\mathcal{O}_{X,y}}(-, \mathcal{K}(\tau))$. Since the residue map $\mathrm{Res}_{k(\xi)/k(y),\tau}$ is nonzero, and since the socle of $\mathcal{K}(\tau)$ is simple, it follows that $\delta_{\xi,\tau}$ induces an isomorphism on socles. □

Definition 4.3.10 *A system of residue data on X consists of the data $(\{\mathcal{K}(x)\}, \{\delta_\xi\}, \{\Phi_\sigma\})$, where:*

a) *For every $x \in X$, $\mathcal{K}(x)$ is a quasi-coherent sheaf, called the dual module of the local ring $\mathcal{O}_{X,x}$ (relative to k).*

b) *For every saturated chain $\xi = (x, \ldots, y)$, $\delta_\xi : \mathcal{K}(x) \to \mathcal{K}(y)$ is an $\mathcal{O}_X$-linear homomorphism, called the coboundary map along ξ.*

c) *For every $x \in X$ and every coefficient field $\sigma : k(x) \to \mathcal{O}_{X,(x)}$, $\Phi_\sigma : \mathcal{K}(\sigma) \xrightarrow{\simeq} \mathcal{K}(x)$ is an isomorphism of $\mathcal{O}_X$-modules.*

The following condition must be satisfied:

(†) *For every saturated chain* $\xi = (x, \ldots, y)$ *and all compatible coefficient fields* σ/τ *for* ξ, *the diagram below commutes:*

$$\begin{array}{ccc} \mathcal{K}(\sigma) & \xrightarrow{\Phi_\sigma} & \mathcal{K}(x) \\ \Big\downarrow{\scriptstyle \delta_{\xi,\sigma/\tau}} & & \Big\downarrow{\scriptstyle \delta_\xi} \\ \mathcal{K}(\tau) & \xrightarrow{\Phi_\tau} & \mathcal{K}(y) \end{array}$$

Before stating the next result we have to broaden our definitions regarding differential forms and coboundary maps.

Definition 4.3.11 a) *Let* S *be a finite set of saturated chains in* X. *Define* $k(S) := \prod_{\xi \in S} k(\xi)$ *and* $\omega(S) := \omega_{k(S)} = \bigoplus_{\xi \in S} \omega(\xi)$. *If* $G \subset X$ *is a finite subset, define* $k(G)$ *and* $\omega(G)$ *by replacing* G *with the set of chains* $\{(x) \mid x \in G\}$.

b) *Let* $y \in X$ *be a point and let* $\sigma : K \to \mathcal{O}_{X,(y)}$ *be a pseudo-coefficient field. Suppose* $G \subset X$ *is a finite subset and* $S = \bigcup_{x \in G} S_x$ *is a finite set of chains, s.t. each* $\xi \in S_x$ *begins with* x *and ends with* y. *Define* $\delta_{S,\sigma} : \omega(G) \to \mathcal{K}(\sigma)$ *by* $\delta_{S,\sigma} := \sum_{x \in G} \sum_{\xi \in S_x} \delta_{\xi,\sigma}$.

c) *Let* X_{gen} *be the set of generic points of irreducible components of* X. *If* X *is a reduced scheme define* $\omega(X) := \omega(X_{\mathrm{gen}}) = \bigoplus_{x \in X_{\mathrm{gen}}} \omega(x)$.

Recall that the total ring of fractions of X is denoted by $k(X)$. Thus for X reduced we have $k(X) = k(X_{\mathrm{gen}})$ and $\omega(X)$ is a free $k(X)$-module of rank 1. In part b) of the definition we don't require that $x > y$ for all $x \in G$; if $x \not> y$ then $S_x = \emptyset$ and $\delta_{S,\sigma}$ still makes sense.

Lemma 4.3.12 *Let* S *and* σ *be as in def. 4.3.11 b). Then the* k*-submodule* $\omega(G)_{\mathrm{hol}:S} := \ker(\delta_{S,\sigma}) \subset \omega(G)$ *is independent of* σ.

Proof Copy the proof of lemma 4.2.1. □

The main result of this article is:

Theorem 4.3.13 (Internal Residue Isomorphism) *Let* X *be a reduced scheme of finite type over a perfect field* k. *Let* $y \in X$ *be a point and let* $\sigma : K \to \mathcal{O}_{X,(y)}$ *be any pseudo-coefficient field. Then:*

a) *There exists a finite set of chains* $S = \bigcup_{x \in G} S_x$ *as in definition 4.3.11 b) s.t.*

$$\delta_{S,\sigma} : \omega(G) \to \mathcal{K}(\sigma) \text{ is surjective.} \tag{4.3.14}$$

One may choose $G = X_{\text{gen}}$.

b) *Let* $\sigma' : K' \to \mathcal{O}_{X,(y)}$ *be another pseudo-coefficient field and let* $S = \bigcup_{x \in G} S_x$ *and* $S' = \bigcup_{x' \in G'} S'_{x'}$ *be sets of chains as in def. 4.3.11 b), with* $\delta_{S,\sigma}$ *surjective. Let* $\Phi_{\sigma,\sigma'}$ *be the map which makes the lower triangle in the the diagram*

$$\begin{tikzcd}
\frac{\omega(G')}{\omega(G')_{\mathrm{hol}:S'}} \arrow[r, "\delta_{S',\sigma'}"] \arrow[d, "\delta_{S',\sigma}"'] & \mathcal{K}(\sigma') \\
\mathcal{K}(\sigma) \arrow[ur, "\Phi_{\sigma,\sigma'}"] & \frac{\omega(G)}{\omega(G)_{\mathrm{hol}:S}} \arrow[l, "\delta_{S,\sigma}"] \arrow[u, "\delta_{S,\sigma'}"']
\end{tikzcd}$$

commute. Then $\Phi_{\sigma,\sigma'}$ *is an isomorphism and the upper triangle commutes too.*

Proof a) By thm. 3.3.16 there exists a set of chains $S = \bigcup_{x \in X_{\text{gen}}} S_x$ s.t. the face map $\partial : \mathcal{O}_{X,(y)} \to \prod_{\xi \in S} \mathcal{O}_{X,\xi} = k(S)$ is a strict monomorphism. Since the topology on $k(S)$ is K-linear (prop. 3.2.5), any continuous K-linear homomorphism $\phi : \mathcal{O}_{X,(y)} \to \omega_K$ extends (not uniquely) to a continuous K-linear homomorphism $\hat{\phi} : k(S) \to \omega_K$. The residue pairing (4.2.4) is a perfect pairing of ST K-modules (thm. 2.4.22); there exists a form $\beta \in \omega(S)$ s.t. $\hat{\phi} = \langle -, \beta \rangle_{k(S)/K}$.

Let $\omega(S)_{\text{hol}} \subset \omega(S)$ be the perpendicular space to $\mathcal{O}_{X,y}$ under the pairing $\langle -, - \rangle_{k(S)/K}$. Since $\bigoplus_{\xi \in S} \omega(\xi)_{\text{hol}} \subset \omega(S)_{\text{hol}}$, and by thm. 4.2.7, it follows that $\omega(S)_{\text{hol}}$ is an open submodule of $\omega(S)$. According to cor. 3.2.12, $\omega(X) \subset \omega(S)$ is dense; so we can assume that $\beta \in \omega(X)$. Doing so we get $\phi = \delta_{S,\sigma}(\beta)$.

b) First note that the surjectivity of $\delta_{S,\sigma}$ implies that $\mathcal{O}_{X,(y)} \to k(S)$ is a strict monomorphism. This is because $\mathcal{O}_{X,(y)}$ is a separated ST K-module with a topology generated by K-subspaces of finite codimension. Hence $\delta_{S,\sigma'}$ is surjective too.

To show that the upper triangle is commutative amounts to proving the following statement: if $\alpha = \sum_{x \in G} \alpha_x \in \omega(G)$ and $\alpha' = \sum_{x' \in G'} \alpha'_{x'} \in \omega(G')$ are forms s.t. $\delta_{S,\sigma}(\alpha) = \delta_{S',\sigma}(\alpha')$, then also $\delta_{S,\sigma'}(\alpha) = \delta_{S',\sigma'}(\alpha')$. Now $\delta_{S,\sigma}(\alpha) = \delta_{S',\sigma}(\alpha')$ iff

$$\text{for all } a \in \mathcal{O}_{X,y},\ \sum_{x\in G}\sum_{\xi\in S_x} \operatorname{Res}_{\xi,\sigma}(a\alpha_x) = \sum_{x'\in G'}\sum_{\xi'\in S'_{x'}} \operatorname{Res}_{\xi',\sigma}(a\alpha'_{x'})\ . \quad (4.3.15)$$

But just like in the proof of lemma 4.2.1, if $\eta = (y, \dots, z)$ is any saturated chain with z a closed point, condition (4.3.15) is equivalent to

$$\text{for all } a \in \mathcal{O}_{X,y},\ \sum_{x\in G}\sum_{\xi\in S_x} \operatorname{Res}_{\xi\vee \mathrm{d}_0\eta,k}(a\alpha_x) = \sum_{x'\in G'}\sum_{\xi'\in S'_{x'}} \operatorname{Res}_{\xi'\vee \mathrm{d}_0\eta,k}(a\alpha'_{x'})$$

which is independent of σ. □

Let $(\{\mathcal{K}(x)\}, \{\delta_\xi\}, \{\Phi_\sigma\})$ and $(\{\mathcal{K}'(x)\}, \{\delta'_\xi\}, \{\Phi'_\sigma\})$ be two systems of residue data. An isomorphism between them is a family of isomorphisms $\Psi_x : \mathcal{K}(x) \xrightarrow{\simeq} \mathcal{K}'(x)$ s.t. $\Psi_y \circ \delta_\xi = \delta'_\xi \circ \Psi_x$ and $\Psi_x \circ \Phi_\sigma = \Phi'_\sigma$ for all chains $\xi = (x, \dots, y)$ and all coefficient fields σ for x.

Corollary 4.3.16 *There exists a system of residue data on X, unique up to a unique isomorphism.*

Proof If $x \in X_{\text{gen}}$, set $\mathcal{K}(x) := \omega(x)$. For any $y \in X$ we identify the $\mathcal{O}_X$-modules $\mathcal{K}(\sigma)$, where σ ranges over the coefficient fields for y, via the isomorphisms $\Phi_{\sigma,\sigma'}$. Let $\mathcal{K}(y)$ be this identified module. The coboundary map $\delta_\eta : \mathcal{K}(y) \to \mathcal{K}(z)$ attached to a saturated chain $\eta = (y, \dots, z)$ is represented by $\delta_{\eta,\sigma/\tau} : \mathcal{K}(\sigma) \to \mathcal{K}(\tau)$, where σ/τ are compatible coefficient fields for η. Suppose σ'/τ' are other compatible coefficient fields for η. Let S be a set of chains as in part a) of the theorem, so $\delta_{S,\sigma} : \omega(G) \to \mathcal{K}(\sigma)$ is surjective. Setting $S \vee \mathrm{d}_0\eta := \{\xi \vee \mathrm{d}_0\eta \mid \xi \in S\}$ we get $\delta_{S\vee \mathrm{d}_0\eta,\tau} = \delta_{\eta,\sigma/\tau} \circ \delta_{S,\sigma}$. Let $M := \operatorname{im}(\delta_{S\vee \mathrm{d}_0\eta,\tau}) \subset \mathcal{K}(\tau)$. By the theorem $\Phi_{\tau,\tau'}|_M = \delta_{S\vee \mathrm{d}_0\eta,\tau'} \circ (\delta_{S\vee \mathrm{d}_0\eta,\tau})^{-1}$. Hence

$$\begin{aligned}
\delta_{\eta,\sigma'/\tau'} \circ \Phi_{\sigma,\sigma'} &= \delta_{\eta,\sigma'/\tau'} \circ \delta_{S,\sigma'} \circ (\delta_{S,\sigma})^{-1} \\
&= \delta_{S\vee \mathrm{d}_0\eta,\tau'} \circ (\delta_{S\vee \mathrm{d}_0\eta,\tau})^{-1} \circ (\delta_{\eta,\sigma/\tau} \circ \delta_{S,\sigma}) \circ (\delta_{S,\sigma})^{-1} \\
&= \Phi_{\tau,\tau'} \circ \delta_{\eta,\sigma/\tau}
\end{aligned}$$

so $\delta_\eta : \mathcal{K}(y) \to \mathcal{K}(z)$ is well-defined.

Given another system of residue data $(\{\mathcal{K}'(x)\}, \{\delta'_\xi\}, \{\Phi'_\sigma\})$, for every $y \in X$ and every coefficient field σ the map $\Psi_y : \mathcal{K}(y) \xrightarrow{\Phi_\sigma^{-1}} \mathcal{K}(\sigma) \xrightarrow{\Phi'_\sigma} \mathcal{K}'(y)$ is an isomorphism of $\mathcal{O}_X$-modules. Using compatible coefficient fields one has $\Psi_y \circ \delta_\xi = \delta'_\xi \circ \Psi_x$ for any saturated chain $\xi = (x, \dots, y)$. Thus $\{\Psi_x\}$ is the unique isomorphism between the two systems. □

Remark 4.3.17 Theorem 4.3.13 actually implies more: there is a canonical isomorphism $\Phi_\sigma : \mathcal{K}(\sigma) \xrightarrow{\simeq} \mathcal{K}(x)$ for any pseudo-coefficient field $\sigma : K \to \mathcal{O}_{X,(x)}$.

Corollary 4.3.18 *Given $y \in X$ there is a canonical isomorphism of $\mathcal{O}_X$-modules $\delta_{(y)} : \omega(y) \xrightarrow{\simeq} \mathrm{soc}_{\mathcal{O}_{X,y}} \mathcal{K}(y)$. If σ is a coefficient field for y then one has $\delta_{(y)} = \Phi_\sigma \circ \delta_{(y),\sigma}$.*

Proof We must show that if $\sigma' : k(y) \to \mathcal{O}_{X,(y)}$ is any other coefficient field then $\Phi_{\sigma,\sigma'} \circ \delta_{(y),\sigma} = \delta_{(y),\sigma'}$. Choose any saturated chain $\xi = (x, \ldots, y)$ with $x \in X_{\mathrm{gen}}$. According to the theorem $\Phi_{\sigma,\sigma'}|_{\mathrm{soc}\,\mathcal{K}(\sigma)}$ can be computed using $\delta_{\xi,\sigma}$. Let $\beta \in \omega(y)$; by prop. 4.3.9 we can find $\alpha \in \omega(x)_{\mathrm{sim}:\xi}$ s.t. $\mathrm{Res}_\xi(\alpha) = \beta$. Then using prop. 4.2.12

$$\Phi_{\sigma,\sigma'} \circ \delta_{(y),\sigma}(\beta) = \Phi_{\sigma,\sigma'} \circ \delta_{\xi,\sigma}(\alpha) = \delta_{\xi,\sigma'}(\alpha) = \delta_{(y),\sigma'}(\beta) \ .$$

□

Lemma 4.3.19 a) *Let $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ be saturated chains. Then $\delta_{\xi \vee \mathrm{d}_0 \eta} = \delta_\eta \circ \delta_\xi$.*

b) *Let $y \in X$ and let $\phi \in \mathcal{K}(y)$. Then for all but finitely many saturated chains $\eta = (y, \ldots)$ one has $\delta_\eta(\phi) = 0$.*

c) *Let (x, z) be a chain in X with* $\mathrm{codim}(\{z\}^-, \{x\}^-) = 2$. *Then*

$$\sum_{y \in X, x > y > z} \delta_{(y,z)} \circ \delta_{(x,y)} = 0 \ .$$

Proof a) Choose coefficient fields ρ, σ, τ for x, y, z respectively s.t. ρ/σ and σ/τ are compatible for ξ snd η respectively. Use formula (4.3.8) and the isomorphisms $\Phi_\rho, \Phi_\sigma, \Phi_\tau$.

b) Choose a set of saturated chains $S = \bigcup_{x \in X_{\mathrm{gen}}} S_x$ as in thm. 4.3.13 a), and let $\delta_S := \sum_{\xi \in S} \delta_\xi : \omega(x) \to \mathcal{K}(y)$ be the corresponding surjection. Then $\phi = \delta_S(\alpha)$ for some $\alpha \in \omega(x)$ and $\delta_\eta(\phi) = \delta_{S \vee \mathrm{d}_0 \eta}(\alpha)$ by part a) of the lemma. By prop. 4.2.14, for all but finitely many such chains η, $\alpha \in \bigcap_{\xi \in S} \omega(X)_{\mathrm{hol}:\xi \vee \mathrm{d}_0 \eta}$, and for those η one has $\delta_\eta(\phi) = 0$.

c) By part a) this sum equals $\sum_{y \in X, x > y > z} \delta_{(x,y,z)}$. Choose coefficient fields as in a). For any $\phi \in \mathcal{K}(\rho)$ and any $a \in \mathcal{O}_{X,z}$ we have by definition

$$\sum_{y \in X, x > y > z} \delta_{(x,y,z),\tau}(\phi)(a) = \sum_{y \in X, x > y > z} \mathrm{Res}_{(x,y,z),\tau}(\phi(a))$$

which is zero by theorem 4.2.15. □

The stage is set to present the residue complex. For every natural number q let $X_q \subset X$ be the subset $\{x \in X \mid \dim \{x\}^- = q\}$.

Theorem 4.3.20 *Let X be a reduced scheme of finite type over a perfect field k and let $(\{\mathcal{K}(x)\}, \{\delta_\xi\}, \{\Phi_\sigma\})$ be the unique system of residue data on X. There exists a complex $(\mathcal{K}^{\cdot}_X, \delta_X)$ of $\mathcal{O}_X$-modules, together with homomorphisms of $\mathcal{O}_X$-modules $\Psi_x : \mathcal{K}(x) \to \mathcal{K}^{\cdot}_X$ for all $x \in X$, s.t. for every integer q the homomorphism*

$$\sum_{x \in X_q} \Psi_x : \bigoplus_{x \in X_q} \mathcal{K}(x) \to \mathcal{K}^{-q}_X \tag{4.3.21}$$

is an isomorphism. The coboundary map δ_X satisfies the formula

$$\delta_X \circ \sum_{x \in X} \Psi_x = \sum_{(x,y)} \Psi_y \circ \delta_{(x,y)} : \bigoplus_{x \in X} \mathcal{K}(x) \to \mathcal{K}^{\cdot}_X \ . \tag{4.3.22}$$

The complex $(\mathcal{K}^{\cdot}_X, \delta_X)$ is unique up to a unique isomorphism, and is called the Grothendieck residue complex of X (relative to k).

Proof Use formulas (4.3.21) and (4.3.22) to define the complex $\mathcal{K}^{\cdot}_X$. From lemma 4.3.19 b) it follows that $\sum_{(x,y)} \delta_{(x,y)} : \bigoplus_{x \in X} \mathcal{K}(x) \to \bigoplus_{y \in X} \mathcal{K}(y)$ is well-defined, and from part c) of the same lemma it follows that $\delta_X^2 = 0$. □

4.4 Functorial Properties of the Complex $\mathcal{K}^{\cdot}_X$

Let X be a reduced scheme of finite type over a perfect field k, with structural morphism π. In this section we will examine the behavior of the residue complex $\mathcal{K}^{\cdot}_X$ with respect to finite morphisms and open immersions. We will also show that when π is proper there is a canonical nonzero trace map $\mathrm{Tr}_\pi : \mathrm{H}^0 \pi_* \mathcal{K}^{\cdot}_X \to k$.

Proposition 4.4.1 *Let $f : X \to Y$ be an open immersion of reduced k-schemes of finite type. There is a canonical isomorphism of complexes of $\mathcal{O}_X$-modules $\gamma^*_f : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} f^* \mathcal{K}^{\cdot}_Y$. If $g : Y \to Z$ is another such open immersion then*

$$\gamma^*_{gf} = f^*(\gamma^*_g) \circ \gamma^*_f : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} f^* g^* \mathcal{K}^{\cdot}_Z \cong (gf)^* \mathcal{K}^{\cdot}_Z \ .$$

Proof f induces an isomorphism between the system of residue data on X and the restriction to $f(X)$ of the residue data on Y. Since $\mathcal{K}^{\cdot}_Y$ is a sum of skyscraper sheaves $\mathcal{K}(y)$, $f^*\mathcal{K}^{\cdot}_Y$ is the sum of the sheaves $f^*\mathcal{K}(y)$ for $y \in f(X)$. $\square$

Remark 4.4.2 In prop. 4.4.1, "open immersion" can be replaced with "étale". Indeed, suppose $f : X \to Y$ is étale. Given $y \in Y$, $f^*\mathcal{K}(y) = \bigoplus_{x|y} f^*\mathcal{K}(y)_x$, since f is quasi-finite. For any coefficient field $\sigma : k(y) \to \mathcal{O}_{Y,(y)}$ and any $x|y$, we have an induced coefficient field $\sigma_x : k(x) \to \mathcal{O}_{X,(x)}$, and $\mathcal{O}_{X,(x)} \cong k(x) \otimes_{k(y)} \mathcal{O}_{Y,(y)}$. This induces an isomorphism $f^*\mathcal{K}_Y(\sigma)_x \xrightarrow{\simeq} \mathcal{K}_X(\sigma_x)$ which is compatible with the coboundaries.

Let $f : X \to Y$ be a finite morphism of noetherian schemes. Following [RD] ch. III §6 we let $\bar{f} : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ be the corresponding morphism of ringed spaces, and we denote by $\mathsf{Mod}(Y, f_*\mathcal{O}_X)$ the category of sheaves of $f_*\mathcal{O}_X$-modules on Y. The functor $\bar{f}^* : \mathsf{Mod}(Y, f_*\mathcal{O}_X) \to \mathsf{Mod}(X)$ is exact.

Definition 4.4.3 *Given a finite morphism $f : X \to Y$ define a functor $f^\flat$: $\mathsf{Mod}(Y) \to \mathsf{Mod}(X)$ by*

$$f^\flat := \bar{f}^*\mathcal{H}om_Y(f_*\mathcal{O}_X, -) .$$

If $U = \operatorname{Spec} B$ and $V = \operatorname{Spec} A$ are affine open subsets in X and Y respectively s.t. $U = f^{-1}(V)$, and if $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $\Gamma(U, f^\flat\mathcal{M}) = \operatorname{Hom}_A(B, \Gamma(V, \mathcal{M}))$. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are finite morphisms then $(gf)^\flat \cong f^\flat g^\flat$ naturally.

Remark 4.4.4 In [RD] the functor $f^\flat$ is a derived functor, defined using $\mathrm{R}\mathcal{H}om$ instead of $\mathcal{H}om$. However we shall only apply $f^\flat$ to injective $\mathcal{O}_Y$-modules, making this discrepancy disappear.

Theorem 4.4.5 *Let $f : X \to Y$ be a finite morphism of reduced k-schemes of finite type. There is a canonical isomorphism of complexes of $\mathcal{O}_X$-modules $\gamma^\flat_f : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} f^\flat\mathcal{K}^{\cdot}_Y$. If $g : Y \to Z$ is another such finite morphism then*

$$\gamma^\flat_{gf} = f^\flat(\gamma^\flat_g) \circ \gamma^\flat_f : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} f^\flat g^\flat\mathcal{K}^{\cdot}_Z \cong (gf)^\flat\mathcal{K}^{\cdot}_Z .$$

Before proving the theorem we need to establish some more notation. Suppose $G, H \subset X$ are finite subsets and suppose $S = \bigcup_{w \in G} S_w = \bigcup_{x \in H} S^x$ is a finite set of chains in X s.t. each $\xi \in S_w \cap S^x$ begins with w and ends with x. Suppose also for that every $x \in H$ we are given a pseudo-coefficient field $\sigma_x : K_x \to \mathcal{O}_{X,(x)}$. Define $\sigma := \prod_{x \in H} \sigma_x$ and $\mathcal{K}(\sigma) := \bigoplus_{x \in H} \mathcal{K}(\sigma_x)$. Let $\delta_{S,\sigma} : \omega(G) \to \mathcal{K}(\sigma)$ be the $\mathcal{O}_X$-module homomorphism $\delta_{S,\sigma} := \sum_{w \in G} \sum_{x \in H} \sum_{\xi \in S_w \cap S^x} \delta_{\xi,\sigma_x}$.

Now let $f : X \to Y$ be a finite morphism, let $y \in f(X)$ be a point and let $H \subset f(X)$ be a finite subset. Suppose $T = \bigcup_{w \in H} T_w$ is a finite set of chains in Y, s.t. each $\eta \in T_w$ begins with w and ends with y. Let $G := f^{-1}(H)$ and write $x|y$ for $x \in f^{-1}(y)$. Then $S := f^{-1}(T)$ decomposes into $S = \bigcup_{w \in G} S_w = \bigcup_{x|y} S^x$ as above. Given a pseudo-coefficient field $\tau : K \to \mathcal{O}_{Y,(y)}$, the local homomorphisms $f^* : \mathcal{O}_{Y,(y)} \to \mathcal{O}_{X,(x)}$ induce pseudo-coefficient fields $f^*\tau_x : K \to \mathcal{O}_{X,(x)}$. Set $f^*\tau := \prod_{x|y} f^*\tau_x$. As explained above, there is an $\mathcal{O}_X$-linear homomorphism $\delta_{S,f^*\tau} : \omega(G) \to \mathcal{K}(f^*\tau)$. Define $\omega(G)_{\mathrm{hol}:S} := \ker(\delta_{S,f^*\tau})$.

Let

$$\theta_1 : \omega(G) \xrightarrow{\simeq} f^\flat \omega(H) = \mathrm{Hom}_{k(H)}\left(k(G), \omega(H)\right) \tag{4.4.6}$$

be the isomorphism induced by the trace map $\mathrm{Tr}_{k(G)/k(H)} : \omega(G) \to \omega(H)$, and let

$$\theta_2 : \mathcal{K}(f^*\tau) = \bigoplus_{x|y} \mathcal{K}(f^*\tau_x) \xrightarrow{\simeq} f^\flat \mathcal{K}(\tau) = \bigoplus_{x|y} \mathrm{Hom}_{\mathcal{O}_{Y,y}}\left(\mathcal{O}_{X,x}, \mathcal{K}(\tau)\right) \tag{4.4.7}$$

be the isomorphism of adjunction.

Lemma 4.4.8 *The diagram of $\mathcal{O}_X$-modules below is commutative:*

$$\begin{tikzcd} \omega(G) \arrow[r, "\theta_1"] \arrow[d, "\delta_{S,f^*\tau}"] & f^\flat \omega(H) \arrow[d, "f^\flat(\delta_{T,\tau})"] \\ \mathcal{K}(f^*\tau) \arrow[r, "\theta_2"] & f^\flat \mathcal{K}(\tau) \end{tikzcd} \tag{4.4.9}$$

Proof We can localize at any $x \in f^{-1}(y)$. Choose $\alpha \in \omega(G)_x$, $a \in \mathcal{O}_{X,x}$ and $b \in \mathcal{O}_{Y,y}$. Then

$$(f^\flat(\delta_{T,\tau}) \circ \theta_1)(\alpha)(a)(b) = \sum_{\eta \in T} \mathrm{Res}_{\eta,K} \circ \mathrm{Tr}_{k(G)/k(H)}(ba\alpha) \in \omega_K \ .$$

On the other hand

$$(\theta_2 \circ \delta_{S,f^*\tau})(\alpha)(a)(b) = \sum_{\xi \in S} \mathrm{Res}_{\xi,K}(ba\alpha) \in \omega_K \ .$$

According to prop. 3.2.3, $k(S) \cong k(G) \otimes_{k(H)} k(T)$ as reduced clusters of TLFs. Since $k(H) \to k(T)$ is topologically étale relative to k, we know by thm. 2.4.23 (cf. proof of cor. 4.1.16) that

$$\sum_{\eta \in T} \mathrm{Res}_{\eta,K} \circ \mathrm{Tr}_{k(G)/k(H)} = \sum_{\xi \in S} \mathrm{Res}_{\xi,K} : \omega(G) \to \omega_K \ .$$

□

Proof (of the theorem) Fix $y \in Y$ and $x \in f^{-1}(y)$. It suffices to give an isomorphism $\mathcal{K}(x) \xrightarrow{\simeq} f^{\flat}\mathcal{K}(y)_x$ which is compatible with the coboundaries δ_X and δ_Y. Let $\sigma : k(x) \to \mathcal{O}_{X,(x)}$ and $\tau : k(y) \to \mathcal{O}_{Y,(y)}$ be coefficient fields, and let $f^*\tau_x : k(y) \to \mathcal{O}_{X,(x)}$ be the induced pseudo-coefficient field. The isomorphism $\Phi_{\sigma,f^*\tau_x}$ of thm. 4.3.13 is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{K}(\sigma) \xrightarrow{\simeq} \mathcal{K}(f^*\tau_x)$. By adjunction we get an isomorphism $\mathcal{K}(f^*\tau_x) \xrightarrow{\simeq} f^{\flat}\mathcal{K}(\tau)_x$ (called θ_2 in lemma 4.4.8), and the composition is by definition $\gamma^{\flat}_{f,\sigma,\tau} : \mathcal{K}(\sigma) \xrightarrow{\simeq} f^{\flat}\mathcal{K}(\tau)_x$.

Suppose σ' and τ' are other coefficient fields for x and y respectively. By thm. 4.3.13

$$\Phi_{\sigma',f^*\tau'_x} \circ \Phi_{\sigma,\sigma'} = \Phi_{f^*\tau_x,f^*\tau'_x} \circ \Phi_{\sigma,f^*\tau_x} \ .$$

It remains to show that

$$\theta_2 \circ \Phi_{f^*\tau_x,f^*\tau'_x} = f^{\flat}(\Phi_{\tau,\tau'}) \circ \theta_2 : \mathcal{K}(f^*\tau_x) \xrightarrow{\simeq} f^{\flat}\mathcal{K}(\tau')_x \ . \tag{4.4.10}$$

Then the isomorphism $\gamma^{\flat}_f : \mathcal{K}(x) \xrightarrow{\simeq} f^{\flat}\mathcal{K}(y)_x$ represented by $\gamma^{\flat}_{f,\sigma,\tau}$ is well defined.

Choose a finite set of chains $\tilde{S} = \bigcup_{w \in X_{\mathrm{gen}}} \tilde{S}_w$ in X s.t. each $\xi \in \tilde{S}_w$ begins with w and ends with x, and s.t. the face map $\mathcal{O}_{X,(x)} \to k(\tilde{S})$ is a strict monomorphism. Define $H := f(X_{\mathrm{gen}}) \subset Y$, $T := f(\tilde{S})$, $G := f^{-1}(H) \subset X$ and $S := f^{-1}(T)$. Then $\tilde{S} \subset S$, so $k(S) = k(\tilde{S}) \times k(S - \tilde{S})$ and $\mathcal{O}_{X,(x)} \to k(S)$ is a strict monomorphism. The isomorphism $\Phi_{f^*\tau_x,f^*\tau'_x}$ can be computed using S:

$$\Phi_{f^*\tau_x,f^*\tau'_x} = \delta_{S,f^*\tau'_x} \circ (\delta_{S,f^*\tau_x})^{-1} : \mathcal{K}(f^*\tau_x) \xrightarrow{\simeq} \mathcal{K}(f^*\tau'_x) \ .$$

By thm. 4.3.13 the isomorphism $\Phi_{\tau,\tau'}$ on Y, restricted to $\mathrm{im}(\delta_{T,\tau}) \subset \mathcal{K}(\tau)$, equals $\delta_{T,\tau'} \circ (\delta_{T,\tau})^{-1}$. According to the previous lemma we have

$$\theta_1^{-1} \circ f^{\flat}(\delta_{T,\tau})^{-1} \circ \theta_2 = (\delta_{S,f^*\tau})^{-1} : \mathcal{K}(f^*\tau) \xrightarrow{\simeq} \omega(G)/\omega(G)_{\mathrm{hol}:S}$$

and the same for τ', together yielding formula (4.4.10). Similar arguments show that $\gamma^{\flat}_f$ commutes with the coboundaries. The transitivity of the trace on differential forms implies that $\gamma^{\flat}_{gf} = f^{\flat}(\gamma^{\flat}_g) \circ \gamma^{\flat}_f$. $\square$

Definition 4.4.11 (Traces)

a) *Let $f : X \to Y$ be a finite morphism of reduced k-schemes of finite type. Define a homomorphism of complexes of $\mathcal{O}_X$-modules*

$$\mathrm{Tr}_f : f_*\mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$$

by taking the composition of $f_*(\gamma^{\flat}_f) : f_*\mathcal{K}^{\cdot}_X \xrightarrow{\simeq} f_*f^{\flat}\mathcal{K}^{\cdot}_Y$ with the homomorphism $f_*f^{\flat}\mathcal{K}^{\cdot}_Y \cong \mathcal{H}om_Y(f_*\mathcal{O}_X, \mathcal{K}^{\cdot}_Y) \to \mathcal{K}^{\cdot}_Y$ given locally by $\phi \mapsto \phi(1)$.

b) *Given $x \in X_0$ (a closed point) let $\sigma : k(x) \to \mathcal{O}_{X,(x)}$ be the unique coefficient field. Then there is a canonical isomorphism $\Phi_\sigma : \mathcal{K}(\sigma) = \mathrm{Hom}^{\mathrm{cont}}_{k(x)}(\mathcal{O}_{X,(x)}, k(x)) \xrightarrow{\simeq} \mathcal{K}(x)$. Define*

$$\mathrm{Res}_{(x),k} : \mathcal{K}(x) \to k$$

by $\mathrm{Res}_{(x),k}\ (\phi) := \mathrm{Tr}_{k(x)/k} \circ (\Phi_\sigma^{-1}(\phi))(1)$, $\phi \in \mathcal{K}(x)$.

c) *Let $\pi : X \to \operatorname{Spec} k$ be the structural morphism. Define*

$$\mathrm{Tr}_\pi : \pi_* \mathcal{K}_X^0 = \bigoplus_{x \in X_0} \mathcal{K}(x) \to k$$

by $\mathrm{Tr}_\pi := \sum_{x \in X_0} \mathrm{Res}_{(x),k}$.

Corollary 4.4.12 a) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be finite morphisms of reduced k-schemes of finite type. Then*

$$\mathrm{Tr}_{gf} = \mathrm{Tr}_g \circ g_*(\mathrm{Tr}_f) : (gf)_* \mathcal{K}_X^{\cdot} \to \mathcal{K}_Z^{\cdot}\ .$$

b) *Let $f : X \to Y$ be a finite morphisms of reduced k-schemes of finite type and let $\rho : Y \to \operatorname{Spec} k$ be the structural morphism. Then*

$$\mathrm{Tr}_\pi = \mathrm{Tr}_\rho \circ \rho_*(\mathrm{Tr}_f) : \pi_* \mathcal{K}_X^0 \to k\ .$$

Proof Both assertions are consequences of thm. 4.4.5 and some diagram chasing. □

Corollary 4.4.13 *The homomorphism $\mathrm{Tr}_\pi : \Gamma(X, \mathcal{K}_X^0) \to k$ is nonzero. Moreover, given any nonzero element $a \in \Gamma(X, \mathcal{O}_X)$, there exists some $\alpha \in \Gamma(X, \mathcal{K}_X^0)$ s.t. $\mathrm{Tr}_\pi(a\alpha) \neq 0$.*

Proof Since X is reduced, there is some closed point $x \in X$ s.t. $a(x) \in k(x)$ is nonzero. If $\rho : \operatorname{Spec} k(x) \to \operatorname{Spec} k$ is the structural morphism, then $\mathrm{Tr}_\rho = \mathrm{Tr}_{k(x)/k}$. Consider the finite morphism $f : \operatorname{Spec} k(x) \to X$. By the corollary, $\mathrm{Tr}_\pi \circ \pi_*(\mathrm{Tr}_f) = \mathrm{Tr}_\rho$. Choose any $b \in k(x)$ s.t. $\mathrm{Tr}_{k(x)/k}(ab) \neq 0$ and set $\alpha := \pi_*(\mathrm{Tr}_f)(b)$. □

Theorem 4.4.14 *Suppose $\pi : X \to \operatorname{Spec} k$ is proper. Then $\mathrm{Tr}_\pi \circ \pi_*(\delta_X)(\pi_* \mathcal{K}_X^{-1}) = 0$, so $\mathrm{Tr}_\pi : \pi_* \mathcal{K}_X^{\cdot} \to k$ is a homomophism of complexes of k-modules.*

Proof We have $\pi_* \mathcal{K}_X^{-1} = \bigoplus_{x \in X_1} \mathcal{K}(x)$. Choose $x \in X_1$, let $\sigma : k(x) \to \mathcal{O}_{X,(x)}$ be a coefficient field and let $\phi \in \mathcal{K}(\sigma)$. For any $y \in X_0 \cap \{x\}^-$ we have $\mathrm{Res}_{(y),k} \circ \delta_{(x,y)} \circ \Phi_\sigma(\phi) = \mathrm{Res}_{(x,y),k} \circ \phi(1)$, so

$$\mathrm{Tr}_\pi \circ \pi_*(\delta_X) \circ \Phi_\sigma(\phi) = \sum_{y \in X_0, x > y} \mathrm{Res}_{(x,y),k}(\phi(1)) = 0$$

by the classical residue formula (cf. thm. 4.2.15 b)). □

Definition 4.4.15 *Suppose X is an n-dimensional, equidimensional scheme. Define $\tilde{\omega}_X$ to be the sheaf $\mathrm{H}^{-n}\mathcal{K}_X^{\cdot}$.*

Observe that $\tilde{\omega}_X$ is a subsheaf of $\mathcal{K}_X^{-n} = \omega(X) = \Omega^n_{k(X)/k}$. Now suppose X is integral. In [Ku2] E. Kunz introduced the sheaf of *regular differential forms* $\omega^n_{X/k}$ (cf. [Li1] §0). It is a subsheaf of $\omega(X)$, coherent, and coincides with $\Omega^n_{X/k}$ if X is smooth.

Theorem 4.4.16 *Let X be an integral scheme of finite type over a perfect field k. Then $\tilde{\omega}_X$ is the sheaf of regular differential forms.*

Proof Say X has dimension n and generic point v. We claim that for any open set $U \subset X$, $\Gamma(U, \tilde{\omega}_X) = \bigcap_{x \in X_{n-1} \cap U} \omega(X)_{\mathrm{hol}:(v,x)}$. This is because $\Gamma(U, \mathcal{K}_X^{-n+1}) = \bigoplus_{x \in X_{n-1} \cap U} \mathcal{K}(x)$ and for each such x, $\omega(X)_{\mathrm{hol}:(v,x)} = \ker(\delta_{(v,x)})$. If X is smooth over k then $\omega(X)_{\mathrm{hol}:(v,x)} = \Omega^n_{X/k,x}$ since $\mathcal{O}_{X,x}$ is a DVR formally smooth over k (cf. proof of lemma 4.2.6). Hence for any $y \in X$ we have $\tilde{\omega}_{X,y} = \bigcap_{x \in X_{n-1}, x \geq y} \Omega^n_{X/k,x} = \Omega^n_{X/k,y}$.

It remains to show that given a finite surjective morphism $f : X = \mathrm{Spec}\, B \to Y = \mathrm{Spec}\, A$ with X integral and Y smooth over k, then

$$\Gamma(X, \tilde{\omega}_X) = \{\alpha \in \omega(X) \mid \mathrm{Tr}_f(B \cdot \alpha) \subset \Omega^n_{A/k}\} .$$

Since Tr_f sends $f_* \tilde{\omega}_X$ into $\tilde{\omega}_Y = \Omega^n_{Y/k}$, the inclusion "$\subset$" is trivial. Let us prove the other inclusion. Fix $\alpha \in \omega(X)$ s.t. $\mathrm{Tr}_f(B \cdot \alpha) \subset \Omega^n_{A/k}$. Let v and w be the generic points of X and Y respectively. It suffices to show that for every $x \in X_{n-1}$, $\alpha \in \omega(X)_{\mathrm{hol}:(v,x)}$. Fix such x and let $y := f(x)$.

Define $K := k(Y)_{(y)} = k(\,(w,y)\,)$ and $L := k(X)_{(y)} \cong k(X) \otimes_{k(Y)} K \cong \prod_{x' | y} k(\,(v,x')\,)$. Since Tr_f is $\mathcal{O}_Y$-linear we get upon completion along (y):

$$\mathrm{Tr}_{L/K}((f_* \mathcal{O}_X)_{(y)} \cdot \alpha) \subset \Omega^{n,\mathrm{sep}}_{Y/k,(y)} = \omega(\,(w,y)\,)_{\mathrm{hol}} .$$

Let $e \in (f_* \mathcal{O}_X)_{(y)} \cong \prod_{x'|y} \mathcal{O}_{X,(x')}$ be the idempotent which projects onto $\mathcal{O}_{X,(x)}$. Choose a coefficient field $\sigma : k(y) \to \mathcal{O}_{X,(y)}$. Then

$$\begin{aligned} \operatorname{Res}_{k((v,x))/k(y),\sigma}(\mathcal{O}_{X,(x)} \cdot \alpha) &= \operatorname{Res}_{L/k(y),\sigma}((f_*\mathcal{O}_X)_{(y)} \cdot e\alpha) \\ &= \operatorname{Res}_{K/k(y),\sigma} \circ \operatorname{Tr}_{L/K}((f_*\mathcal{O}_X)_{(y)} \cdot e\alpha) = 0 \end{aligned}$$

so $\alpha \in \omega((v,x))_{\mathrm{hol}}$ (cf. lemma 4.2.1). □

Remark 4.4.17 When X is integral of dimension n and π is proper we get a canonical k-linear homomorphism

$$\tilde{\theta}_X : \mathrm{H}^n(X, \tilde{\omega}_X) \to \mathrm{H}^0(X, \mathcal{K}^{\cdot}_X) \xrightarrow{\mathrm{H}^0(\mathrm{Tr}_\pi)} k \ . \tag{4.4.18}$$

It follows from thm. 1 of the appendix that the pair $(\tilde{\omega}_X, \tilde{\theta}_X)$ is a *dualizing pair* in the sense of [Li1] §0. A separate local calculation is needed to check that $\tilde{\theta}_X$ equals Lipman's map, up to a sign (cf. [Li1] thm. 0.6 (d), and [SY]).

4.5 Exactness for Smooth Schemes; More Functorial Properties

In this section X is a reduced scheme of finite type over a perfect field k. We shall exhibit a canonical quasi-isomorphism $\mathrm{C}_X : \Omega^n_{X/k}[n] \to \mathcal{K}^{\cdot}_X$ for X smooth of dimension n over k. Using the variance of $\mathcal{K}^{\cdot}_X$ with respect to finite morphisms we will prove that it is a residual complex for any X. Finally, we shall show that when $\pi : X \to \operatorname{Spec} k$ is proper (and some extra hypothesis) the pair $(\mathcal{K}^{\cdot}_X, \mathrm{Tr}_\pi)$ represents the functor $\mathcal{F}^{\cdot} \mapsto \operatorname{Hom}_k(\mathrm{R}\pi_*\mathcal{F}^{\cdot}, k)$ on the category $\mathsf{D}^-_{\mathrm{qc}}(X)$.

Suppose X is integral, of dimension n. Following [EZ] ch. III §3.1 we call the canonical homomorphism of $\mathcal{O}_X$-modules $\Omega^n_{X/k} \to \Omega^n_{k(X)/k} = \mathcal{K}^{-n}_X$ the *fundamental class* and denote it by C_X. According to lemma 4.2.6, $\delta_X \circ \mathrm{C}_X = 0$. The *augmented residue complex* on X is the complex

$$\cdots \to 0 \to \Omega^n_{X/k} \xrightarrow{\mathrm{C}_X} \mathcal{K}^{-n}_X \xrightarrow{\delta_X} \mathcal{K}^{-n+1}_X \xrightarrow{\delta_X} \cdots \xrightarrow{\delta_X} \mathcal{K}^0_X \to 0 \to \cdots \ .$$

Remark 4.5.1 The fundamental class C_X is defined on any reduced scheme X. It is a global section of the double complex $\mathcal{K}^{\cdot,*}_X := \mathcal{H}om_X(\Omega^*_{X/k}, \mathcal{K}^{\cdot}_X)$, and $\mathrm{d}'_X(\mathrm{C}_X) = \delta'_X(\mathrm{C}_X) = 0$; see [EZ] ch. III §3.1 and our digression 4.5.13.

Theorem 4.5.2 *Let X be a smooth irreducible scheme over k. Then the augmented residue complex on X is exact.*

First let us set up some notation. Suppose X is an irreducible smooth n-dimensional scheme over k, $z \in X$ is a point and p is an integer in the range $[0, n]$. The stalk of $\mathcal{K}_X^{-p}$ at z, $\mathcal{K}_{X,z}^{-p}$, can be identified with $\bigoplus_{x \in X_p, x \geq z} \mathcal{K}(x)$; it is a direct summand of the group of global sections $\Gamma(X, \mathcal{K}_X^{-p}) = \bigoplus_{x \in X_p} \mathcal{K}(x)$. Any section $\alpha \in \Gamma(X, \mathcal{K}_X^{-p})$ is a sum $\alpha = \sum_{x \in X_p} \alpha_x$, and α_x is identified with the germ of α at x. The module $\mathcal{K}(x)$ is an artinian $\mathcal{O}_{X,x}$-module, so the cyclic submodule $\mathcal{O}_{X,x} \cdot \alpha_x$ has finite length. We shall call a point $x \in X$ bad if it is not contained in any smooth hypersurface $Y \subset X$.

Given $\alpha = \sum_{x \in X_p} \alpha_x \in \Gamma(X, \mathcal{K}_X^{-p})$ define:

$$\begin{aligned}
\mathrm{Ass}_z(\alpha) &:= \{x \in X_p \mid x \geq z,\ \alpha_x \neq 0\} \\
\mathrm{length}_z(\alpha) &:= \textstyle\sum_{x \in \mathrm{Ass}_z(\alpha)} \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x} \cdot \alpha_x) \\
\mathrm{codim}(\alpha) &:= n - p \\
\mathrm{badness}_z(\alpha) &:= \text{number of bad points in } \mathrm{Ass}_z(\alpha) \\
\mathrm{weight}_z(\alpha) &:= (\mathrm{codim}(\alpha), \mathrm{badness}_z(\alpha), \mathrm{length}_z(\alpha)) \in \mathbb{N}^3 .
\end{aligned}$$

We say that $\alpha \in \mathcal{K}_{X,z}^{-p}$ is a cocycle if $\delta_{X,z}(\alpha) = 0$, and that is a coboundary if $\alpha = \delta_{X,z}(\beta)$ for some $\beta \in \mathcal{K}_{X,z}^{-p-1}$ (or $\alpha = \mathrm{C}_{X,z}(\beta)$ for some $\beta \in \Omega^n_{X/k,z}$, if $p = n$). Note that for $\alpha \in \mathcal{K}_{X,z}^{-p} \subset \Gamma(X, \mathcal{K}_X^{-p})$, the support of α is precisely the closure of $\mathrm{Ass}_z(\alpha)$.

Suppose $Y \subset X$ is a smooth hypersurface, with ideal sheaf $\mathcal{I}$, and inclusion morphism i. The canonical isomorphism $\Omega^n_{X/k} \otimes \mathcal{I}^{-1} \otimes \mathcal{O}_Y \cong \Omega^{n-1}_{Y/k}$ gives rise to a surjection of $\mathcal{O}_X$-modules, the Poincaré residue map, $\mathrm{Res}_{(X,Y)} : \Omega^n_{X/k} \otimes \mathcal{I}^{-1} \twoheadrightarrow \Omega^{n-1}_{Y/k}$ (we are omitting the functor i_*). There is also a canonical injection $\Omega^n_{X/k} \otimes \mathcal{I}^{-1} \hookrightarrow \Omega^n_{k(X)/k} = \mathcal{K}_X^{-n}$, which identifies $\Omega^n_{X/k} \otimes \mathcal{I}^{-1}$ with the sheaf of meromorphic forms with simple poles along Y. The trace map $\mathrm{Tr}_i : \mathcal{K}_Y^{\cdot} \to \mathcal{K}_X^{\cdot}$ is an injection of complexes; for any $x \in X$, $\mathrm{im}(\mathrm{Tr}_i)_x$ consists of those germs $\alpha_x \in \mathcal{K}_{X,x}^{\cdot}$ annihilated by $\mathcal{I}_x$.

Lemma 4.5.3 *The diagram of $\mathcal{O}_X$-module homomorphisms below is commutative*

$$\begin{tikzcd}
\Omega^n_{X/k} \otimes \mathcal{I}^{-1} \arrow[r, "\subset"] \arrow[dr, "\mathrm{Res}_{(X,Y)}"] & \mathcal{K}_X^{-n} \arrow[r, "\delta_X"] & \mathcal{K}_X^{-n+1} \\
& \Omega^{n-1}_{Y/k} \arrow[r, "\subset"] & \mathcal{K}_Y^{-n+1} \arrow[u, "\mathrm{Tr}_i"']
\end{tikzcd}$$

Proof Let x and y be the generic points of X and Y respectively. For any $y' \in X_{n-1}$ other than y, both paths are 0, since $(\Omega^n_{X/k} \otimes \mathcal{I}^{-1})_{y'} \subset \omega(x)_{\mathrm{hol}:(x,y')}$. Therefore we can localize at y. Then $\mathcal{I}_y = \mathfrak{m}_y = (t)$ for some $t \in \mathcal{O}_{X,y}$, and $(\Omega^n_{X/k} \otimes \mathcal{I}^{-1})_y = t^{-1} \cdot \Omega^n_{X/k,y}$ which equals $\omega(x)_{\mathrm{sim}:(x,y)}$ (cf. proof of lemma 4.2.6). Since $\mathrm{Res}_{(X,Y)}(\alpha \wedge t^{-1}\mathrm{d}t) = \alpha(y) \in \Omega^{n-1}_{k(y)/k}$ for all $\alpha \in \Omega^{n-1}_{X/k,y}$ we get $\mathrm{Res}_{(X,Y)} = \mathrm{Res}_{(x,y)} : (\Omega^n_{X/k} \otimes \mathcal{I}^{-1})_y \to \omega(y)$ (see prop. 4.3.9). Now $(\mathrm{Tr}_i)_y : \mathcal{K}_Y(y) = \omega(y) \to \mathcal{K}_X(y)$ is given by $(\gamma_i^\flat)_y = \delta_{X,(x,y)} \circ \mathrm{Res}^{-1}_{(x,y)}$ (cf. proof of thm. 4.4.5), so the diagram commutes. □

Proof (of theorem) Using induction on weight in the well-ordered set $(\mathrm{N}^3, \mathrm{lex})$, it suffices to prove the following claim:

(†) Let X be an irreducible smooth scheme over k, let $z \in X$ be a point, let p be an integer in the range $[0, \dim X]$ and let $\alpha \in \mathcal{K}^{-p}_{X,z}$ be a cocycle. Suppose that for all quadruples (X', z', p', α') as above with $\mathrm{weight}_{z'}(\alpha') < \mathrm{weight}_z(\alpha)$, α' is a coboundary. Then α is a coboundary.

The claim is proved case by case. We may assume that $\alpha \neq 0$ and $\dim \{z\}^- \leq p$. Let $n := \dim X$.

<u>case 1</u> $\operatorname{codim}(\alpha) = 0$, so $\alpha \in \mathcal{K}^{-n}_{X,z} = \omega(X)$. Apply thm. 4.4.16.

<u>case 2</u> $\operatorname{codim}(\alpha) \geq 1$ and $\alpha \in \mathrm{Hom}_{\mathcal{O}_{X,z}}(\mathcal{O}_{Y,z}, \mathcal{K}^{-p}_{X,z})$ for some smooth hypersurface $Y \subset X$. Denoting the inclusion morphism of Y by i we have $\alpha \in (i^\flat \mathcal{K}^{-p}_X)_z$, so $\alpha = \mathrm{Tr}_i(\beta)$ for some $\beta \in \mathcal{K}^{-p}_{Y,z}$. Since $\mathrm{Tr}_i : \mathcal{K}^{\cdot}_Y \to \mathcal{K}^{\cdot}_X$ is an injection of complexes, β is a cocycle. We have $\operatorname{codim}(\beta) = \operatorname{codim}(\alpha) - 1$, so by the hypothesis $\beta = \delta_{Y,z}(\gamma)$ for some $\gamma \in \mathcal{K}^{-p-1}_{Y,z}$, or $\beta = \mathrm{C}_{Y,z}(\gamma)$ for some $\gamma \in \Omega^{n-1}_{Y/k,z}$ if $p = n$. If $p < n$ we get $\alpha = \delta_{X,z} \circ (\mathrm{Tr}_i)_z(\gamma)$. If $p = n$ we can lift γ to some $\tilde{\gamma} \in (\Omega^n_{X/k} \otimes \mathcal{I}^{-1})_z$, where $\mathcal{I}$ is the ideal sheaf of Y. According to lemma 4.5.3, $\delta_{X,z}(\tilde{\gamma}) = \alpha$.

<u>case 3</u> $\operatorname{codim}(\alpha) \geq 1$ and there is some $x \in \mathrm{Ass}_z(\alpha)$ which is not bad. So $x \in Y$ where $Y \subset X$ is a smooth hypersurface. Let $U = \operatorname{Spec} A \subset X$ be an open affine neighborhood of z s.t. $U \cap Y = \operatorname{Spec} A/(t)$ for some $t \in A$, and let $i : U \to X$ be the inclusion morphism. Applying the isomorphism $\gamma_i^* : \mathcal{K}^{\cdot}_U \xrightarrow{\simeq} i^*\mathcal{K}^{\cdot}_X$, and observing that weight_z, being defined locally at z, remains unchanged, we see that it is possible to assume that $X = \operatorname{Spec} A$.

Since the class $t(x)$ of t in the residue field $k(x)$ is zero, it follows that $\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x} \cdot t\alpha) < \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x} \cdot \alpha)$, and hence also $\mathrm{weight}_z(t\alpha) < \mathrm{weight}_z(\alpha)$. By hypothesis $t\alpha$ is a coboundary: $t\alpha = \delta_{X,z}(\beta)$ for some $\beta \in \mathcal{K}^{-p-1}_{X,z}$. Since t is a non-zero-divisor on $\mathcal{O}_{X,z}$ and since $\mathcal{K}^{-p-1}_{X,z}$ is an injective

$\mathcal{O}_{X,z}$-module, there is some $\gamma \in \mathcal{K}_{X,z}^{-p-1}$ s.t. $t\gamma = \beta$. Thus $t(\delta_{X,z}(\gamma) - \alpha) = 0$ and we reduce the problem to case 2.

case 4 codim$(\alpha) \geq 1$ and all points in $\mathrm{Ass}_z(\alpha)$ are bad. Choose some $x \in \mathrm{Ass}_z(\alpha)$ and let $f : X \to Y = \mathbb{A}^n_k$ be a finite surjective morphism which linearizes (x). Since X and Y are regular schemes f is a flat morphism (cf. [AK] ch. V cor. 3.6). By choosing a small enough affine neighborhood of $v := f(z)$ in Y we can assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and B is a free A-module of rank $N = \deg f$.

Choose an A-basis $\epsilon_1, \ldots, \epsilon_N$ for $\mathrm{Hom}_A(B, A)$. We get an isomorphism of complexes of $\mathcal{O}_Y$-modules

$$f_* \mathcal{K}_X^{\cdot} \xrightarrow{f_*(\gamma_f^b)} f_* f^b \mathcal{K}_Y^{\cdot} \cong \mathcal{H}om_Y(f_* \mathcal{O}_X, \mathcal{K}_Y^{\cdot}) \cong \bigoplus_{i=1}^{N} \mathcal{O}_Y \cdot \epsilon_i \otimes \mathcal{K}_Y^{\cdot} \tag{4.5.4}$$

s.t. $f_*(\delta_X)(\sum \epsilon_i \otimes \beta_i) = \sum \epsilon_i \otimes \delta_Y(\beta_i)$ for all global sections $\beta_i \in \Gamma(Y, \mathcal{K}_Y^{\cdot})$. There is an exact sequence of complexes of $\mathcal{O}_X$-modules

$$0 \to \mathcal{J}^{\cdot} \to f^b \mathcal{K}_{Y,v}^{\cdot} \xrightarrow{\psi} \mathcal{K}_{X,z}^{\cdot} \to 0 \tag{4.5.5}$$

where ψ is localization at z and $\mathcal{J}^{\cdot} = \bigoplus_{x' \in f^{-1}(v), x' \not\succeq z} \mathcal{K}(x')$. For every $\beta \in \Gamma(X, \mathcal{J}^{\cdot})$ the support of β does not contain z; therefore there is some $t \in B$ s.t. $t\beta = 0$ but $t(z) \neq 0$.

The sequence (4.5.5) is naturally split as a sequence of $\mathcal{O}_X$-modules, although not as complexes. Applying $f_*(-)_v$ to this sequence and recalling that $\delta_{X,z}(\alpha) = 0$ we get $f_*(\delta_X)_v(\alpha) = f_* f^b(\delta_{Y,v})(\alpha) \in f_* \mathcal{J}_v^{\cdot} \cong \Gamma(X, \mathcal{J}^{\cdot})$. Let $t \in B$ be s.t. $t f_*(\delta_X)_v(\alpha) = 0$ but $t(z) \neq 0$. Write $t\alpha = \sum_{i=1}^{N} \epsilon_i \otimes \beta_i$, $\beta_i \in \mathcal{K}_{Y,v}^{-p}$. Then $\sum \epsilon_i \otimes \delta_{Y,v}(\beta_i) = f_*(\delta_X)_v(t\alpha) = 0$, so each β_i is a cocycle. Since $\mathrm{Ass}_v(\beta_i) \subset f(\mathrm{Ass}_z(t\alpha)) \subset f(\mathrm{Ass}_z(\alpha))$ and since $\{f(x)\}^- \subset Y = \mathbb{A}^n_k$ is a linear subspace, we see that $\mathrm{badness}_v(\beta_i) < \mathrm{badness}_z(\alpha) = \#\mathrm{Ass}_z(\alpha)$. The codimension hasn't changed, so $\mathrm{weight}_v(\beta_i) < \mathrm{weight}_z(\alpha)$. By hypothesis $\beta_i = \delta_{Y,v}(\gamma_i)$ for some $\gamma_i \in \mathcal{K}_{Y,v}^{-p-1}$. But $t \in \mathcal{O}_{X,z}^{\times}$ (a unit at z), so we conclude that $\alpha = \delta_{X,z}(t^{-1} \sum \epsilon_i \otimes \gamma_i)$. □

Corollary 4.5.6 *For any reduced scheme X of finite type over k, $\mathcal{K}_X^{\cdot}$ is a residual complex.*

Proof We have to show that $\mathcal{K}_X^{\cdot}$ has coherent cohomology sheaves. Since this is a local question we may assume that X is a closed subscheme of $Y = \mathbb{A}^n_k$ for some n. Then for all p the $\mathcal{O}_X$-module $\mathrm{H}^p \mathcal{K}_X^{\cdot} \cong \mathrm{H}^p \mathcal{H}om_Y^{\cdot}(\mathcal{O}_X, \mathcal{K}_Y^{\cdot}) \cong \mathcal{E}xt_Y^{p+n}(\mathcal{O}_X, \Omega^n_{Y/k})$ is coherent. □

Corollary 4.5.7 *Suppose X is a Cohen-Macaulay, n-dimensional, equidimensional, reduced scheme. Let $\tilde{\omega}_X[n]$ be the complex consisting of the sheaf $\tilde{\omega}_X$ in dimension $-n$. Then the homomorphism of complexes $\tilde{\omega}_X[n] \to \mathcal{K}^{\cdot}_X$ is a quasi-isomorphism.*

Proof The question is local so we may assume X is a closed subscheme of $Y = \mathbb{A}^m_k$. Then $\mathrm{H}^p\mathcal{K}^{\cdot}_X \cong \mathcal{E}xt^{p+m}_Y(\mathcal{O}_X, \Omega^m_{Y/k})$ is 0 for $p \neq -n$ (cf. proof of [Ha] ch. III thm. 7.6). $\square$

Let $\mathsf{D}(X)$ be the derived category of complexes of $\mathcal{O}_X$-modules, localized with respect to quasi-isomorphisms. Let $\mathsf{D}^+_{\mathrm{c}}(X)$ be its full subcategory consisting of bounded below complexes with coherent cohomology sheaves. Consider the category FT/k of schemes of finite type over k and k-morphisms. From [RD] ch. VII cor. 3.4 it follows that there is a contravariant pseudofunctor $!$ on FT/k. To every morphism $f : X \to Y$ in FT/k it assigns a functor $f^! : \mathsf{D}^+_{\mathrm{c}}(Y) \to \mathsf{D}^+_{\mathrm{c}}(X)$, with the following properties:

1) For two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ there is an isomorphism $c^{\mathrm{RD}}_{f,g} : (gf)^! \xrightarrow{\simeq} f^! g^!$.

2) For a finite morphism $f : X \to Y$ there is an isomorphism $d^{\mathrm{RD}}_f : f^! \xrightarrow{\simeq} f^{\flat}$.

3) For a smooth morphism $f : X \to Y$ of relative dimension n, there is an isomorphism $e^{\mathrm{RD}}_f : f^! \xrightarrow{\simeq} \omega_{X/Y}[n] \otimes_{\mathcal{O}_X} f^*$, where $\omega_{X/Y}$ is the invertible sheaf $\Omega^n_{X/Y}$.

4) For a proper morphism $f : X \to Y$ there is a trace morphism $\mathrm{Tr}^{\mathrm{RD}}_f : \mathrm{R}f_* f^! \to 1$ in $\mathsf{D}^+_{\mathrm{c}}(Y)$. It induces a functorial isomorphism

$$\theta^{\mathrm{RD}}_f : \mathrm{R}f_*\mathrm{R}\mathcal{H}om^{\cdot}_X(\mathcal{F}^{\cdot}, f^!\mathcal{G}^{\cdot}) \xrightarrow{\simeq} \mathrm{R}\mathcal{H}om^{\cdot}_Y(Rf_*\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$$

for all $\mathcal{F}^{\cdot} \in \mathsf{D}^-_{\mathrm{qc}}(X)$ and $\mathcal{G}^{\cdot} \in \mathsf{D}^+_{\mathrm{c}}(Y)$.

In particular, taking the structural morphism $\pi : X \to \operatorname{Spec} k$ and the complex $k \in \mathsf{D}^+_{\mathrm{c}}(\operatorname{Spec} k)$, we get an object $\pi^! k \in \mathsf{D}^+_{\mathrm{c}}(X)$. The next corollary says that in many instances there is an isomorphism $\pi^! k \cong \mathcal{K}^{\cdot}_X$ in $\mathsf{D}(X)$ (e.g. when X is quasi-projective).

Corollary 4.5.8 *Suppose the structural morphism $\pi : X \to \operatorname{Spec} k$ factors as $\pi = \rho f$ with $f : X \to Y$ finite and $\rho : Y \to \operatorname{Spec} k$ smooth. Then there is an isomorphism $\zeta : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} \pi^! k$ in $\mathsf{D}(X)$.*

Proof Say Y has dimension n. By thm. 4.5.2 one has isomorphisms

$$\mathrm{C}_Y \circ e_\rho^{\mathrm{RD}} : \rho^! k \xrightarrow{\simeq} \omega_{Y/k}[n] = \omega_Y[n] = \Omega^n_{Y/k}[n] \xrightarrow{\simeq} \mathcal{K}^{\cdot}_Y$$

in $\mathsf{D}^+_{\mathrm{c}}(Y)$, and by thm. 4.4.5 one has

$$(\gamma_f^\flat)^{-1} \circ d_f^{\mathrm{RD}} \circ f^!(\mathrm{C}_Y \circ e_\rho^{\mathrm{RD}}) \circ c_{f,\rho}^{\mathrm{RD}} : \pi^! k \xrightarrow{\simeq} f^! \rho^! k \xrightarrow{\simeq} f^! \mathcal{K}^{\cdot}_Y \xrightarrow{\simeq} f^\flat \mathcal{K}^{\cdot}_Y \xrightarrow{\simeq} \mathcal{K}^{\cdot}_X \; .$$

□

Now assume that π is proper. In thm. 4.4.14 we produced a morphism $\mathrm{Tr}_\pi : \pi_* \mathcal{K}^{\cdot}_X \to k$ in $\mathsf{D}(k)$.

Theorem 4.5.9 *Assume that $\pi : X \to \operatorname{Spec} k$ is proper and that there is some isomorphism $\mathcal{K}^{\cdot}_X \cong \pi^! k$ in $\mathsf{D}(X)$. Then there is a unique isomorphism $\zeta_X : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} \pi^! k$ in $\mathsf{D}(X)$ s.t.*

$$\mathrm{Tr}_\pi = \mathrm{Tr}_\pi^{\mathrm{RD}} \circ \mathrm{R}\pi_*(\zeta_X) : \pi_* \mathcal{K}^{\cdot}_X \to k \; .$$

Proof Say we are given an isomorphism $\zeta : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} \pi^! k$ in $\mathsf{D}^+_{\mathrm{c}}(X)$. Then $\mathrm{Tr}_\pi^{\mathrm{RD}}$ induces an isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules

$$\mathrm{H}^0 \pi_* \mathcal{K}^{\cdot}_X \xrightarrow{\mathrm{H}^0 \mathrm{R}\pi_*(\zeta)} \mathrm{H}^0 \mathrm{R}\pi_* \pi^! k \xrightarrow{\theta_\pi^{\mathrm{RD}}} \mathrm{Hom}_k(\pi_* \mathcal{O}_X, k) \; .$$

Now $\Gamma(X, \mathcal{O}_X)$ is a finite reduced k-algebra, hence a semi-simple artinian ring. It follows that $\mathrm{H}^0(X, \mathcal{K}^{\cdot}_X)$ is a free $\Gamma(X, \mathcal{O}_X)$-module of rank 1. By cor. 4.4.13 the trace $\mathrm{H}^0(\mathrm{Tr}_\pi)$ is nondegenerate, so $\mathrm{H}^0(\mathrm{Tr}_\pi) = \mathrm{H}^0(\mathrm{Tr}_\pi^{\mathrm{RD}}) \circ \mathrm{H}^0 \mathrm{R}\pi_*(\zeta) \circ a$ for some global unit $a \in \Gamma(X, \mathcal{O}_X)^\times$. Then $\zeta_X := \zeta \circ a^{-1}$ is the desired isomorphism. □

Observe that thm. 4.5.9 applies when X is projective over k - this follows from cor. 4.5.8.

Remark 4.5.10 In the appendix (thm. 1) it is shown that there exists a canonical isomorphism $\zeta_X : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} \pi^! k$ in $\mathsf{D}(X)$, as in thm. 4.5.9, on any *proper* reduced scheme X. Moreover, the exercise at the end of the appendix shows that there is a canonical isomorphism of complexes $\zeta_X : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} \pi^\Delta k$ on any reduced scheme X. Here π^Δ is the pseudo-functor of [RD] ch. VI.

Remark 4.5.11 Suppose X is both smooth and proper over k. Are the isomorphisms ζ and ζ_X of cor. 4.5.8 and thm. 4.5.9, respectively, equal? In other words, what is the unit $a \in \Gamma(X, \mathcal{O}_X)$ occurring in the proof of thm. 4.5.9? In [SY] it is proved that $a = \pm 1$.

To conclude the paper, let us indicate some applications of our construction. These shall appear in detail in a future publication [Ye2].

Digression 4.5.12 Let K be a TLF over k. Denote by $\mathcal{D}_K$ the ring $\mathrm{Diff}^{\mathrm{cont}}_{K/k}(K, K)$ of continuous differential operators over K. We can show that there is a canonical right $\mathcal{D}_K$ action on ω_K. This action gives an isomorphism of filtered k-algebras

$$\mathcal{D}_K^{\circ} \xrightarrow{\simeq} \mathrm{Diff}^{\mathrm{cont}}_{K/k}(\omega_K, \omega_K)$$

where $\mathcal{D}_K^{\circ}$ is the opposite ring. A topologically étale homomorphism (relative to k) $K \to K'$ extends to a k-algebra homomorphism $\mathcal{D}_K \to \mathcal{D}_{K'}$, and with respect to it the map $\omega_K \to \omega_{K'}$ becomes a homomorphism of right $\mathcal{D}_K$-modules.

On the other hand, if $k \to K$ is itself a morphism in $\mathsf{TLF}(k)$, the residue pairing $\langle -, - \rangle_{K/k}$ induces an adjoint action of $\mathcal{D}_K$ on ω_K. A calculation shows that this adjoint action coincides with the canonical right action.

Now let $x \in X$ be a point. For any coherent sheaf $\mathcal{M}$ define $\mathcal{M}^{\vee}_{(x)} := \mathcal{H}om_X(\mathcal{M}, \mathcal{K}(x))$, the "canonical Matlis dual" of $\mathcal{M}$ at x. Using the results on $\mathcal{D}$-modules over TLFs mentioned above we prove that any DO $D : \mathcal{M} \to \mathcal{N}$ between $\mathcal{O}_X$-modules induces a DO (of equal order) $D^{\vee}_{(x)} : \mathcal{N}^{\vee}_{(x)} \to \mathcal{M}^{\vee}_{(x)}$. The assignment $D \mapsto D^{\vee}_{(x)}$ is functorial. Moreover, given a saturated chain $\xi = (x, \ldots, y)$, the natural transformation $\delta_{\xi} : (-)^{\vee}_{(x)} \to (-)^{\vee}_{(y)}$ induced by the coboundary $\delta_{\xi} : \mathcal{K}(x) \to \mathcal{K}(y)$ respects DOs.

Consider the sheaf of DOs on X, $\mathcal{D}_X := \mathcal{D}iff_{X/k}(\mathcal{O}_X, \mathcal{O}_X)$. An immediate consequence is that $\mathcal{K}^{\cdot}_X = \bigoplus_{x \in X} (\mathcal{O}_X)^{\vee}_{(x)}$ is a complex of right $\mathcal{D}_X$-modules. One checks that if X is smooth of dimension n and the characteristic is 0, then the induced action on $\Omega^n_{X/k} = \mathrm{H}^{-n}\mathcal{K}^{\cdot}_X$ is by the Lie derivative (cf. [Bo] ch. VI §3.2). For any left (resp. right) $\mathcal{D}_X$-module $\mathcal{M}$, $\mathcal{M}^{\vee}_{(x)}$ is a right (resp. left) $\mathcal{D}_X$-module. So $\mathcal{M}^{\cdot} \mapsto \mathcal{H}om^{\cdot}_X(\mathcal{M}^{\cdot}, \mathcal{K}^{\cdot}_X)$ is a functor $\mathsf{D}(\mathcal{D}_X)^{\circ} \leftrightarrow \mathsf{D}(\mathcal{D}_X^{\circ})$, inducing an equivalence $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X)^{\circ} \leftrightarrow \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{D}_X^{\circ})$ (where "c" means coherent over $\mathcal{O}_X$).

Digression 4.5.13 In [EZ] ch. II §2.1 the bigraded $\mathcal{O}_X$-module $\mathcal{K}^{\cdot,*}_X$ is defined. For any p, q set $\mathcal{K}^{p,q}_X := \mathcal{H}om_X(\Omega^q_{X/k}, \mathcal{K}^p_X)$. Using our construction we get a canonical structure of double complex on $\mathcal{K}^{\cdot,*}_X$ (independent of embedding and for arbitrary characteristic; cf. [EZ] §2.1.3). The first differential δ'_X is simply $\phi \mapsto \delta_X \circ \phi$. The second differential d'_X is a DO of order ≤ 1, defined using the results sketched in digression 4.5.12. We have $\mathcal{K}^{\cdot,*}_X = \bigoplus_{x \in X} (\Omega^*_{X/k})^{\vee}_{(x)}$ and we may set $\mathrm{d}'_X := \sum_{x \in X} \mathrm{d}^{\vee}_{(x)}$. Then $(\mathrm{d}'_X)^2 = 0$ and $\mathrm{d}'_X \circ \delta'_X = \delta'_X \circ \mathrm{d}'_X$. Given a finite morphism $f : X \to Y$ there is a canonical trace map $\mathrm{Tr}_f : f_* \mathcal{K}^{\cdot,*}_X \to \mathcal{K}^{\cdot,*}_Y$. When

X is smooth of dimension n the isomorphism $\mathcal{K}_X^{\cdot,*} \cong \Omega_{X/k}^* \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{K}_X^{\cdot}$ sends $\mathrm{d}_{(x)}^{\vee}$ to $\mathrm{H}_x^{n-p}(\mathrm{d})$ for $x \in X_p$, which is the differential used in [EZ].

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