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Appendix A Pointwise Criterion for Dualizing Pairs

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Appendix

A POINTWISE CRITERION FOR DUALIZING PAIRS

Introduction

Throughout this appendix k will denote a perfect field. The main text of this book will be referred to as the "text". Our only references will be (a) the text, (b) [RD] and (c) [V], and consequently we assume familiarity with the language used in these. For simplicity, we assume that all schemes occurring in this appendix are connected. This does not affect the main result, viz., Theorem 1.

Set $\{p\} = \operatorname{Spec} k$. Let $\pi : X \to \{p\}$ be proper with X reduced, and let $\mathcal{K}_X^\bullet$, $\operatorname{Tr}_\pi : \pi_* \mathcal{K}_X^\bullet \to k$ be as in the text. The main theorem of this appendix is the following :

THEOREM 1. *The map* $\operatorname{Tr}_\pi : \pi_* \mathcal{K}_X^\bullet \to k$ *induces an isomorphism*

$$\mathbb{R}\operatorname{Hom}_X^\bullet(\mathcal{F}^\bullet, \mathcal{K}_X^\bullet) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}_{\{p\}}^\bullet(\mathbb{R}\pi_* \mathcal{F}^\bullet, k)$$

in $\mathbf{D}(\{p\})$ *for every* $\mathcal{F}^\bullet \in \mathbf{D}^+(X)$.

The appendix is organized as follows. The first section gives a quick review of residual complexes – stating facts about them without proofs. The major reference here is [RD]. But we are dealing here with reduced algebraic schemes over k, or a localization of such schemes, and residual complexes on such schemes are special, for the value the "co–dimension function" (associated to the residual complex) takes at a point on such schemes depends only on the dimension of the closure of the point and on a fixed integer constant. Hence we can "normalize" (by shifting) so that the codimension function is zero on closed points. Once we observe this, we deal only with such normalized residual complexes.

The next section reviews the notion of a dualizing pair, defines the notion of a residue pair, and the notion of a pointwise residue pair. Briefly a residue pair is a representative (in the category of residual complexes) of the dualizing pair (which is defined only at the derived category level). To prove Theorem 1 then amounts to showing that the pair $(\mathcal{K}^\bullet_X, \mathrm{Tr}_\pi)$ is a residue pair. Which leads to the question: Is there a local criterion – one that could be checked pointwise – to help decide if a pair $(\mathcal{R}^\bullet, \theta)$, consisting of a residual complex $\mathcal{R}^\bullet$ and a map θ (of complexes) between $\pi_*\mathcal{R}^\bullet$ and k, is a residue pair ? Theorem 2 provides this criterion. But the proof requires us to develop some local theory, which we do in the third section, and the proof itself is carried out in the fourth section. One then checks via Lemma 1 that $(\mathcal{K}^\bullet_X, \mathrm{Tr}_\pi)$ satisfies this criterion.

In the last section we take a sideways glance and address a slightly different question. According to Verdier [V, p. 395, Cor. 1], dualizing pairs are well behaved with respect to open immersions. In other words if an algebraic scheme U over k admits open immersions to two different complete algebraic schemes, then the restrictions of the corresponding dualizing complexes on the two complete schemes to U are canonically isomorphic. This isomorphism arises out of the univeral properties of the two dualizing pairs. Via Theorem 1, this gives an automorphism of $\mathcal{K}^\bullet_U$ (of course, we are also using the fact that the family $\{\mathcal{K}^\bullet_X\}$ (as X varies over reduced algebraic k–schemes) is well–behaved with respect to open immersions (cf. (4.4.1) of the text)). We show that this automorphism is the identity automorphism, using the local theory developed in the third section.

Preliminaries

Let $\mathcal{R}^\bullet$ be a residual complex (cf. (4.3.1) of the text) on a locally noetherian scheme Y. By [RD, p. 304, (1.1)] it follows that $\mathcal{R}^\bullet$ is a pointwise dualizing complex on Y. Hence by [ibid, p. 287, 4] it carries a "codimension function" $d = d_{\mathcal{R}^\bullet} : Y \to \mathbb{Z}$ (i.e. a function such that $d(x) = d(y) + 1$ for any immediate specialization $y \to x$). If $d(x) = p$, then the $\mathcal{O}_X$–module $J(x)$ occurs as a direct summand of $\mathcal{R}^p$.

We say that $\mathcal{R}^\bullet$ is a *normalized residual complex* if $d(y) = -\dim \{y\}^-$ for every $y \in Y$.

Y need not carry any normalized residual complex, even if residual complexes exist on it. However, note that if Y is either:

(a) A scheme of finite type over k, or

(b) the spectrum of $\mathcal{O}_{Z,z}$, where Z is as in (a)

then there exists a normalized residual complex on Y.

NORMALIZATION CONVENTION. *Unless otherwise stated, every residual complex in this appendix is a normalized residual complex.*

We recall some key properties of residual complexes from [RD]. So suppose $\mathcal{R}^\bullet$ and $\mathcal{R}'^\bullet$ are two residual complexes on Y. Then:

P1. If $U \subset Y$ is an open subscheme, then $\mathcal{R}^\bullet|U$ is clearly a residual complex (perhaps not normalized, but certainly normalizable by shifting). If $y \in Y$ is a closed point, then the complex $\mathcal{R}^\bullet_y$ is residual on $\mathrm{Spec}(\mathcal{O}_{Y,y})$.

P2. $\mathcal{R}^\bullet$ is a Cousin complex with respect to the filtration $Z^p = \{x | d(x) \geq p\}$. Hence by [RD, p. 247, (3.2)(a)], any map $\mathcal{R}^\bullet \to \mathcal{R}'^\bullet$ homotopic to zero is actually zero.

P3. Thus ($\mathcal{R}^\bullet$ and $\mathcal{R}'^\bullet$ being injective complexes) any quasi–isomorphism $\mathcal{R}^\bullet \to \mathcal{R}'^\bullet$ is an isomorphism of complexes.

P4. Since $\mathcal{R}^\bullet$ is pointwise dualizing and is bounded (being normalized) therefore it is a dualizing complex. It follows from [ibid, p. 266, (3.1)] that there exists an invertible sheaf $\mathcal{L}$ on Y (unique up to isomorphism) such that

$$\mathcal{R}'^\bullet \xrightarrow{\sim} \mathcal{R}^\bullet \otimes \mathcal{L}$$

– the integer n occurring in loc.cit. being zero because our complexes are normalized. Further, the proof of loc.cit. yields that

$$\mathcal{L} \xrightarrow{\sim} \mathcal{H}om_{\mathbf{C}(Y)}(\mathcal{R}^\bullet, \mathcal{R}'^\bullet)$$

P5. In view of **P4**, there is an isomorphism

$$\mathcal{O}_Y \xrightarrow{\sim} \mathcal{E}\mathrm{nd}_{\mathcal{O}_Y}(\mathcal{R}^\bullet)$$

and an examination of the proof of loc.cit. shows that for an open set U of Y, a section $\alpha \in \Gamma(U, \mathcal{O}_Y)$, corresponds to the endomorphism "multiplication by α" of $\mathcal{R}^\bullet|U$.

Dualizing pairs and residue pairs

Closely associated to residual complexes are "dualizing pairs". For $\pi : X \to \{p\}$ as in Theorem 1, [RD, pp. 383–384, (3.4)] shows that there exists a complex of quasi-coherent sheaves $\pi^! = \pi^! k \in \mathbf{D_c}(X)_{\text{fid}}$ (= complexes in $\mathbf{D}(X)$ of finite injective dimension and coherent cohomology sheaves) and a morphism $\int_\pi : \mathbb{R}\pi_*\pi^! \to k$ in $\mathbf{D}(\{p\})$ inducing an isomorphism

$$\mathbb{R}\operatorname{Hom}^\bullet_X(\mathcal{F}^\bullet, \pi^!) \xrightarrow{\sim} \mathbb{R}\operatorname{Hom}^\bullet_{\{p\}}(\mathbb{R}\pi_*\mathcal{F}^\bullet, k)$$

in $\mathbf{D}(\{p\})$ for every $\mathcal{F}^\bullet$ in $\mathbf{D}^+(X)$.

The pair $(\pi^!, \int_\pi)$ is unique up to unique isomorphism. We call such a pair a *dualizing pair.*

In [RD] a residual complex (normalized because of [ibid, p. 333, (3.4)]) $\pi^\Delta = \pi^\Delta k$ is constructed [ibid, p. 318, (3.1)] along with a trace map $\mathrm{T}_\pi : \pi_*\pi^\Delta \to k$ (denoted Tr_π in loc.cit.). Since π is proper, this is a map of complexes by [ibid, p. 342, (4.2)]. If Q denotes both the localization functors $\mathbf{K}(X) \to \mathbf{D}(X)$ and $\mathbf{K}(\{p\}) \to \mathbf{D}(\{p\})$ then $(Q\pi^\Delta, Q\mathrm{T}_\pi)$ is a dualizing pair [ibid, p. 379, (3.3)] .

DEFINITION. *We call a pair $(\mathcal{R}^\bullet, \theta : \pi_*\mathcal{R}^\bullet \to k)$ a residue pair if $\mathcal{R}^\bullet$ is residual and $(Q\mathcal{R}^\bullet, Q\theta)$ is a dualizing pair.*

Residue pairs exist – for example $(\pi^\Delta, \mathrm{T}_\pi)$.

Let $\mathfrak{C}$ be the category whose objects are pairs $(\mathcal{R}^\bullet, \theta)$ where $\mathcal{R}^\bullet$ is a residual complex and $\theta : \pi_*\mathcal{R}^\bullet \to k$ is a map of complexes. A morphism $(\mathcal{R}^\bullet, \theta) \to (\mathcal{R}'^\bullet, \theta')$ in $\mathfrak{C}$ is a map of complexes $\alpha : \mathcal{R}^\bullet \to \mathcal{R}'^\bullet$ such that $\theta = \theta' \circ \pi_*\alpha$.

PROPOSITION 1. *Residue pairs are final objects in $\mathfrak{C}$, and hence are unique up to unique isomorphism in $\mathfrak{C}$.*

Proof. Let $(\mathcal{F}^\bullet, T)$ be a residue pair. Let $(\mathcal{R}^\bullet, \theta)$ be any object in $\mathfrak{C}$. Since $(Q\mathcal{F}^\bullet, QT)$ is a dualizing pair, therefore there is a map $\bar{\alpha} : Q\mathcal{R}^\bullet \to Q\mathcal{F}^\bullet$ such that $Q\theta = QT \circ \mathbb{R}\pi_*\alpha$. By [RD, p. 304, (1.1)] and [ibid, p. 306, Rmk 1] it follows that $\bar{\alpha} = Q\alpha$ for a unique map $\alpha : \mathcal{R}^\bullet \to \mathcal{F}^\bullet$. Moreover θ and $T \circ \pi_*\alpha$ are homotopic to each other (we are using the fact that $\mathcal{R}^\bullet$ and $\mathcal{F}^\bullet$ are injective complexes and hence $\mathbb{R}\pi_*\mathcal{R}^\bullet = \pi_*\mathcal{R}^\bullet$, $\mathbb{R}\pi_*\mathcal{F}^\bullet = \pi_*\mathcal{F}^\bullet$ and $Q(\pi_*\alpha) = \mathbb{R}\pi_*\alpha)$. However, k (as a complex) lives in degree zero and $\pi_*\mathcal{R}^\bullet$ is zero in positive degrees. Hence $\theta = T \circ \pi_*\alpha$.

The uniqueness of the $\mathfrak{C}$–morphism α is clear. Q.E.D.

Example. Let $X = \mathbb{P}^n$, the projective space of dimension n. Let $(\mathcal{R}^\bullet, \theta)$ be a residue pair on X. Let $\mathcal{R}'^\bullet = \mathcal{R}^\bullet \otimes \mathcal{O}_X(-1)$, and let $\alpha : \mathcal{R}'^\bullet \to \mathcal{R}^\bullet$ be any non-zero map induced by a non-zero global section $s \in \Gamma(X, \mathcal{O}_X(1))$. The map α cannot be an isomorphism (consider its effect on $J(y)$ where y is the generic point of the divisor given by the global section $s \in \Gamma(X, \mathcal{O}_X(1)))$. Let θ' be $\theta \circ \pi_*\alpha$. Then the pair $(\mathcal{R}'^\bullet, \theta')$ is a non-trivial object in $\mathfrak{C}$, and is not a residue pair.

How do we decide if a pair $(\mathcal{R}^\bullet, \theta) \in \mathfrak{C}$ is a residue pair ? One would like local conditions – conditions which can be checked pointwise– which force $(\mathcal{R}^\bullet, \theta)$ to be a residue pair. To that end we define a *pointwise residue pair.* Let $x \in X$ be a *closed point.* Set $\mathcal{R}(x) = \Gamma_x\mathcal{R}^\bullet$ (Γ_x = sections with support in $\{x\}$) and $\theta(x) : \mathcal{R}(x) \to k$ the k– linear map induced by θ. More precisely, $\theta(x)$ is the composition

$$\mathcal{R}(x) \longrightarrow \pi_*\mathcal{R}^\bullet \xrightarrow{\theta} k$$

where the first map arises from the functorial map $\Gamma_x \to \Gamma(X, \ )$.

DEFINITION. $(\mathcal{R}^\bullet, \theta) \in \mathfrak{C}$ *is said to be a pointwise residue pair if for every closed point $x \in X$ and every artinian $\mathcal{O}_{X,x}$– module M, the map*

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(M, \mathcal{R}(x)) \longrightarrow \mathrm{Hom}_k(M, k)$$

(induced by $\theta(x) : \mathcal{R}(x) \to k$) is an isomorphism.

THEOREM 2. *$(\mathcal{R}^\bullet, \theta) \in \mathfrak{C}$ is a residue pair if and only if it is a pointwise residue pair.*

Lemma 1 of the next section says that $(\mathcal{K}_X^\bullet, \mathrm{Tr}_\pi)$ is a pointwise residue pair. Theorem 1 is then an immediate consequence of Theorem 2.

The local theory

Let $(A, \mathfrak{m})$ be a local ring, essentially of finite type over k, such that the residue field $L = A/\mathfrak{m}$ is a finite algebraic extension of k. Let $\mathfrak{A}$ be the category of artinian A–modules. Define a contravariant functor

$$F = F_A : \mathfrak{A} \longrightarrow \mathrm{Mod}(A)$$

by

$$F(M) = \mathrm{Hom}_k(M, k)$$

for $M \in \mathfrak{A}$.

Note that if $M \in \mathfrak{A}$, then M is an $\hat{A}$–module since every element of M is annihilated by some power of $\mathfrak{m}$.

The most important example, for this appendix, of an object in $\mathfrak{A}$ is an injective hull of L over A. One model for this injective hull is the one presented in the text, viz, $\mathcal{K} = \mathrm{Hom}_L^c(\hat{A}, L)$ where $\hat{A}$ is considered an L–module via the unique section $\sigma : L \to \hat{A}$ of the natural surjection $\hat{A} \to L$. With $\mathrm{tr} = \mathrm{tr}_{L/k} : L \to k$ the trace map, and $e : \mathcal{K} \to L$ the map given by "evaluation at 1" we define

$$t : \mathcal{K} \longrightarrow k$$

by the composition $\mathrm{tr} \circ e$.

LEMMA 1. *(Local Duality). The functor F is represented by $(\mathcal{K}, t)$.*

Proof. Since $\mathrm{Hom}_L(M, L)$ is isomorphic to $\mathrm{Hom}_k(M, k)$ via $\mathrm{tr} : L \to k$, therefore we only need to check that the natural map

$$\mathrm{Hom}_A(M, \mathcal{K}) \longrightarrow \mathrm{Hom}_L(M, L)$$

induced by e is an isomorphism. The injectivity of the map is easy to establish. To check surjectivity note that: If $m \in M$ and $\varphi \in \mathrm{Hom}_L(M, L)$, then the map $\hat{A} \to L$ given by $a \mapsto \varphi(am)$ is continuous (since m is annihilated by some power of $\mathfrak{m}$). Q.E.D.

For some aspects of this appendix, we only need to know that F is representable and that the representing object is an injective hull of L over A. Concrete realizations of the representing pair, while useful, can also distract. The next Lemma explores the abstract properties of pairs which represent F.

Let (E, τ) be a pair which represents F. Consider another pair (not necessarily representing F), (J, q), where

- J is an injective hull of L over A
- $q : J \to k$ is a k–linear map.

LEMMA 2. *If $\gamma : E \to J$ is an A–linear map such that $q \circ \gamma = \tau$, then γ is an isomorphism and (J, q) represents the functor F.*

Proof. Since E and J are both injective hulls of L over A, therefore they are isomorphic, and hence it is enough to prove the proposition for $J = E$.

(E, τ) represents F – therefore the map $q : E \to k$ gives rise to a unique A–map $\delta : E \to E$ such that $\tau \circ \delta = q$. Using the universal property of (E, τ) yet again and the equations $\tau \circ (\delta\gamma) = q \circ \gamma = \tau$, we conclude that $\delta\gamma : E \to E$ is the identity map.

By Matlis theory it is known that $\mathrm{Hom}_A(E, E)$ is canonically isomorphic to $\hat{A}$. More precisely, every A–endomorphism of E is given by multiplication by a unique element of $\hat{A}$ (E being artinian is an $\hat{A}$–module). Consequently the ring of A–endomorphisms of E is commutative, and hence $\gamma\delta = \delta\gamma = id$ which implies that γ is an isomorphism. The Lemma follows. Q.E.D.

By a residual complex on A, we mean a complex of A–modules $\mathcal{S}^\bullet$ such that the associated complex of quasi–coherent complexes on $\mathrm{Spec}\, A$ is residual. Let $\Gamma_{\mathfrak{m}} : \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ denote the 0th local cohomolgy functor. Note that if $\mathcal{S}^\bullet$ is residual then $\mathcal{S}(\mathfrak{m}) = \Gamma_\mathfrak{m}(\mathcal{S}^\bullet)$ is an injective hull of L over A.

LEMMA 3. *Let $\mathcal{S}^\bullet$ and $\mathcal{S}'^\bullet$ be residual complexes on A. Then*

(a) *A morphism $\alpha : \mathcal{S}^\bullet \to \mathcal{S}'^\bullet$ is an isomorphism if and only if the map $\Gamma_\mathfrak{m}(\alpha) : \mathcal{S}(\mathfrak{m}) \to \mathcal{S}'(\mathfrak{m})$ is an isomorphism.*

(b) *If $\mathcal{S}'^\bullet$ is equal to $\mathcal{S}^\bullet$ in (a), then α is the identity map if and only if $\Gamma_\mathfrak{m}(\alpha)$ is the identity map.*

Proof. By **P4**, $\mathcal{S}^\bullet$ and $\mathcal{S}'^\bullet$ differ by a rank–one projective module, therefore they are isomorphic via some isomorphism (A being local). Hence it is enough to prove (a) for $\mathcal{S}'^\bullet = \mathcal{S}^\bullet$. By **P5** we have an isomorphism $A \xrightarrow{\sim} \mathrm{End}_A(\mathcal{S}^\bullet)$ (given by $a \mapsto$ "multiplication by a"), and by Matlis theory a similar isomorphism holds between $\hat{A}$ and $\mathrm{End}_A(\mathcal{S}(\mathfrak{m}))$. Consequently, under these identifications, the natural map $A \to \hat{A}$ corresponds to the natural map $\mathrm{End}_A(\mathcal{S}^\bullet) \to \mathrm{End}_A(\mathcal{S}(\mathfrak{m}))$. Part (a) follows from the fact that $a \in A$ is a unit if and only if it is a unit in $\hat{A}$. Next, an element $a \in A$ maps to $1 \in \hat{A}$ if and only if $a = 1$. Part (b) follows.

Proof of Theorem 2

Let $(\mathcal{R}^\bullet, \theta)$ be a pointwise residue pair. Let $(\pi^\Delta, \mathrm{T}_\pi)$ be a residue pair. By Proposition 1, there exists a unique map of complexes $\alpha : \mathcal{R}^\bullet \to \pi^\Delta$ such that $\theta = \mathrm{T}_\pi \circ \pi_*\alpha$. To show α is an isomorphism, it is enough to check that for each closed point $x \in X$, $\alpha_x : \mathcal{R}^\bullet_x \to (\pi^\Delta)_x$ is an isomorphism. By Lemma 3 it suffices to prove that $\Gamma_x(\alpha)$ is an isomorphism. Let $\pi^\Delta(x) = \Gamma_x(\pi^\Delta)$ and $\mathrm{T}_\pi(x) : \pi^\Delta(x) \to k$, the map induced by T_π. Then $\theta(x) = \mathrm{T}_\pi(x) \circ \Gamma_x(\alpha)$ and hence by Lemma 2, $\Gamma_x(\alpha)$ is an isomorphism.

For the converse, since any two residue pairs are isomorphic in $\mathfrak{C}$ therefore, by what we have just proved, it is enough to prove that pointwise residue pairs exist. Lemma 1 shows that $(\mathcal{K}^\bullet_X, \mathrm{Tr}_\pi)$ is a pointwise residue pair.

Compatibility with open immersions

Let $\mathcal{V}$ denote the category of reduced algebraic schemes of finite type over k, and $\mathcal{V}_{\text{zar}}$ the subcategory whose objects are the same as the objects in $\mathcal{V}$, but whose morphisms are open immersions.

Let $U \xrightarrow{i_1} X_1$ and $U \xrightarrow{i_2} X_2$ be morphisms in $\mathcal{V}_{\text{zar}}$ with X_1 and X_2 proper. Let $\pi_1 : X_1 \to \{p\}$ and $\pi_2 : X_2 \to \{p\}$ be the structural morphisms. Then Deligne in [RD, Appendix] and, more explicitly, Verdier in [V, p. 395, Cor. 1], show that there is an isomorphism in $\mathbf{D}(U)$

$$\mu_{12} : i_2^*\pi^!{}_2 \xrightarrow{\sim} i_1^*\pi^!{}_1$$

We do not intend to repeat the arguments of Verdier and Deligne, but we outline here the main thrust of the argument (omitting proofs). First assume there is a map $h : X_2 \to X_1$ in $\mathcal{V}$ such that $h \circ i_1 = i_2$ and $i_1^* h_* = i_2^*$. In such a case we say that X_2 *dominates* X_1. One then has (via the univeral property of $(\pi^!{}_1, \int_{\pi_1})$) a map $\alpha : \mathbb{R}h_*\pi_2^! \to \pi_1^!$ such that $\int_{\pi_2} = \int_{\pi_1} \circ \mathbb{R}\pi_{1*}(\alpha)$. Noting that $i_1^*\mathbb{R}h_*$ is canonically isomorphic to i_2^*, μ_{12} is then simply $i_1^*(\alpha)$ [V, p. 395, Cor. 1].

The general case is reduced to the above as follows : One finds a third morphism $U \xrightarrow{i_3} X_3$ in $\mathcal{V}_{\text{zar}}$ such that X_3 is complete and dominates X_1 and X_2 (for example, as in [V], the closure of U in $X_1 \times X_2$). Then one sets $\mu_{12} = \mu_{13} \circ \mu_{23}{}^{-1}$. In [RD, pp. 414–415], Deligne essentially establishes that μ_{12} so defined is independent of the auxiliary compactification $i_3 : U \to X_3$.

This argument can be repeated (without changes) at the level of residue pairs instead of dualizing pairs [RD, p. 342, (4.2)].

By (4.4.1) of the text, one can canonically identify $i_1^*\mathcal{K}^\bullet_{X_1}$ and $i_2^*\mathcal{K}^\bullet_{X_2}$ with $\mathcal{K}^\bullet_U$. From Theorem 1 and the preceding discussion, we obtain an automorphism $\mu = \mu_{12} : \mathcal{K}^\bullet_U \to \mathcal{K}^\bullet_U$.

THEOREM 3. *The automorphism $\mu : \mathcal{K}^\bullet_U \to \mathcal{K}^\bullet_U$ is the identity.*

Proof. We may reduce to the case where X_2 dominates X_1, say via $h : X_2 \to X_1$. Let $\alpha : \mathbb{R}h_*Q\mathcal{K}^\bullet_{X_2} \to Q\mathcal{K}^\bullet_{X_1}$ be the unique $\mathbf{D}(X_1)$ – morphism arising from the univeral property of the dualizing pair $(Q\mathcal{K}^\bullet_{X_1}, Q\,\mathrm{Tr}_{\pi_1})$. Note that since $\mathcal{K}^\bullet_{X_1}$ is an injective complex, the map α has a (homotpy unique) representative $h_*\mathcal{K}^\bullet_{X_2} \to \mathcal{K}^\bullet_{X_1}$ in the category of complexes. We continue to call this representative α, and note that $\mu = i_1^*(\alpha)$. Also note that Tr_{π_2} and $\mathrm{Tr}_{\pi_1} \circ \pi_{1*}\alpha$ are homotopic and hence equal by arguing as in Proposition 1.

Pick a closed point $x \in U$. Let $\mu(x) = \Gamma_x(\mu) = \Gamma_x(\alpha)$, and let $\mathcal{K}(x)$, $\mathrm{Res}_x : \mathcal{K}(x) \to k$ be as in the text. Since the following diagram commutes

$$\begin{tikzcd}
\mathcal{K}(x) \arrow[r] \arrow[d, "\mu(x)"'] & \pi_{2*}\mathcal{K}^\bullet_{X_2} \arrow[r, "\mathrm{Tr}_{\pi_2}"] \arrow[d, "\alpha"] & k \arrow[d, equal] \\
\mathcal{K}(x) \arrow[r] & \pi_{1*}\mathcal{K}^\bullet_{X_1} \arrow[r, "\mathrm{Tr}_{\pi_1}"] & k
\end{tikzcd}$$

It follows that $\mathrm{Res}_x \circ \mu(x) = \mathrm{Res}_x$, whence by Lemma 1, $\mu(x)$ is the identity map. By part (b) of Lemma 3 it follows that $\mu_x : \mathcal{K}^\bullet_{U,x} \to \mathcal{K}^\bullet_{U,x}$ is the identity map. Q.E.D.

Exercise. Let X be an object in $\mathcal{V}$, not necessarily complete. Show, using Theorem 3, that one has a canonical isomorphism $\gamma : \mathcal{K}^\bullet{}_X \xrightarrow{\sim} \pi^\Delta$ (π = structural morphism of X), where π^Δ is the complex $\pi^\Delta k$ in [RD, p. 318, (3.1)]. [*Hint:* Cover X by open sets $\{U_\alpha\}$ which are compactifiable (e.g. U_α quasi-projective). Deduce isomorphisms $\gamma_\alpha : \mathcal{K}^\bullet_{U_\alpha} \xrightarrow{\sim} \pi^\Delta|U_\alpha$. Use Theorem 3 to show that these isomorphisms patch. Show that γ is independent of the open cover $\{U_\alpha\}$ by using Theorem 3 once again].

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