# *Astérisque*

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**Spectral theory of elliptic operators on non-compact manifolds**

# SPECTRAL THEORY OF ELLIPTIC OPERATORS ON NON-COMPACT MANIFOLDS.

## M.A.SHUBIN

## TABLE OF CONTENTS

# SPECTRAL THEORY OF ELLIPTIC OPERATORS ON NON—COMPACT MANIFOLDS

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## Introduction

This paper contains an enlarged and modified part of my five lectures given in June 1991 at Nantes during the Summer School on Semiclassical Methods. Of course the whole subject as given in the title is inexhaustible since even the "simplest" particular case of the Schrödinger operator on euclidean space can not be exhausted because it contains the whole Quantum Mechanics and hence its complete understanding would provide us with the complete understanding of a considerable part of the Universe. So I did not pretend to be complete in my lectures and I make even less pretensions in this paper. Actually this paper contains only a description of some qualitative results on the spectra of elliptic operators on non–compact manifolds. The lectures contained also a beginning of a quantitative theory, namely integrated density of states and applications of von Neumann algebra techniques to this topic. I hope that these things some day will be described in a second part of this paper but they seemed to me too voluminous and disorderly to include in this paper now.

This paper contains two chapters each having an Appendix. In Chapter 1 we discuss the first question which natually arises when you begin to study a differential operator: what is the natural domain, where this operator is defined? Actually, if the operator is to be considered in a Banach space, one can always take minimal and maximal domain arriving in this way to minimal and maximal operators in this Banach space. We concentrate on the question whether these operators coincide because then they provide

a natural operator in the Banach space associated with the given differential operator. We describe several methods of proving the coincidence based on finite speed propagation for evolution equations, regularity results and estimates of the Green function. The necessary technique concerning manifolds of bounded geometry and behaviour of the Green function is described in Appendix 1 to this chapter. Note that a non–trivial difference between minimal and maximal operator would mean that boundary conditions should be imposed but this certainly goes out of the scope of this paper. The only thing we do about it here is that we explain how to write the unique solution of the hyperbolic Cauchy problem in operator terms in case when the corresponding generating second order operator is symmetric but not essentially self–adjoint due to the behaviour of lower–order terms at infinity (Theorem 3.4).

In Chapter 2 we discuss some general topics concerning elliptic operators on manifolds of bounded geometry. Namely first we apply the general abstract eigenfunction expansion theorem, described in Appendix 2, to provide weighted Sobolev spaces which contain complete orthonormal system of generalized eigenfunctions for any self–adjoint operator. We use the ellipticity to narrow these spaces by use of regularity theorems. Next we discuss Schnol–type theorems giving sufficient conditions for the given complex number $\lambda$ to belong to the spectrum if a non–trivial and non–square –integrable eigenfunction with an appropriate behaviour at infinity is given.

Some parts of this paper are based on methods and technique that were described in [44] and [45], and I felt free to borrow from these papers which were only published in a volume of the PDE seminar in École Polytechnique. But many of the results of [44] are essentially improved here and also some clarifications are added.

Maggie Beucler for her careful work of typing the manuscript.

**Numerational convention.** We numbered all formulas and also Definitions, Theorems etc. separately in every Chapter or Appendix. Inside a Chapter or an Appendix we refer to a formula, Definition, Theorem etc. from the same Chapter or Appendix without any indication of the division where it belongs.

## Chapter 1. Minimal and maximal operators.

### 1.1. Abstract preliminaries

Let $\mathcal{H}$ be a complex Hilbert space, $A$ a densely defined linear operator in $\mathcal{H}$ (the domain of $A$ will be denoted $D(A)$). Suppose that $A$ has a closure $\overline{A}$ or, equivalently, that the adjoint operator $A^*$ is densely defined (see e.g. [32]). We shall denote by $G_A$ the graph of $A$ i.e. the set of pairs $\{u, Au\}$, $u \in D(A)$. Then $G_{\overline{A}} = \overline{G}_A$, i.e. the graph of $\overline{A}$ is the closure of the graph of $A$. Moreover $\overline{A} = A^{**} = (A^*)^*$.

Now let $A^+$ be another densely defined linear operator in $\mathcal{H}$.

DEFINITION 1.1. $A^+$ is called *formally adjoint* to $A$ if

$$(1.1) \qquad (Au, v) = (u, A^+v), \ u \in D(A), \ v \in D(A^+),$$

where $(\cdot, \cdot)$ is the scalar product in $\mathcal{H}$.

If $A = A^+$ then $A$ is called *symmetric* or *formally self–adjoint*.

Note that since $A, A^+$ are densely defined, both $A$ and $A^+$ have closures.

DEFINITION 1.2. Let $A, A^+$ be as in Definition 1.1. Then the *minimal* and the *maximal* operator for $A$ are defined as follows:

$$A_{\min} = \overline{A} = A^{**}, \ A_{\max} = (A^+)^*.$$

Note that both $A_{\min}$ and $A_{\max}$ are closed and $A_{\min} \subset A_{\max}$ i.e. $D(A_{\min}) \subset D(A_{\max})$ and $A_{\max}$ is an extension of $A_{\min}$. The important question that arises in analytic situations and will be discussed later is whether $A_{\min} = A_{\max}$ or not. In an important particular case $A = A^+$ the coincidence $A_{\min} = A_{\max}$ means that $A$ is *essentially self–adjoint* i.e. $A$ is a self–adjoint operator in $\mathcal{H}$.

Now let us consider a more general abstract context. Let $\mathcal{B}, \mathcal{B}'$ be complex Banach spaces and a continuous non–degenerated pairing $\mathcal{B} \times \mathcal{B}' \to \mathbb{C}$ be given which will be denoted $\langle \cdot, \cdot \rangle$. Here continuity may be understood as separate continuity i.e. continuity with respect to each variable. Non–degeneracy means first that if $u \in \mathcal{B}$ and $\langle u, v \rangle = 0$ for all $v \in \mathcal{B}'$ then $u = 0$, and second that if $v \in \mathcal{B}'$ and $\langle u, v \rangle = 0$ for all $u \in \mathcal{B}$ then $v = 0$. Also this pairing may supposed to be bilinear as well as hermitean i.e. linear with respect to the first variable and antilinear with respect to the second variable (in the latter case we shall denote it by $(\cdot, \cdot)$. Now let two pairs $\mathcal{B}_i, \mathcal{B}'_i$, $i = 1, 2$, be given with continuous non–degenerated pairings described as before. Suppose that $A : \mathcal{B}_1 \to \mathcal{B}_2$ and $A^t : \mathcal{B}'_2 \to \mathcal{B}'_1$ are two densely defined linear operators. Then $A^t$ is called a formally transposed operator to $A$ if

$$(1.2) \qquad \langle Au, v \rangle = \langle u, A^t v \rangle, \ u \in D(A), \ v \in D(A^t).$$

If we have hermitean pairings between $\mathcal{B}_i$ and $\mathcal{B}'_i$ and (1.1) is satisfied for two densely defined linear operators $A : \mathcal{B}_1 \to \mathcal{B}_2$ and $A^+ : \mathcal{B}'_2 \to \mathcal{B}'_1$ then $A^+$ is called formally adjoint to $A$. In both situations the following definition is applicable

DEFINITION 1.2'. $A_{\min} = \overline{A}$, $A_{\max} = (A^t)^*$ or $A_{\max} = (A^+)^*$

Here $\overline{A}$ is the operator whose graph is the closure of the graph of $A$ in $\mathcal{B}_1 \times \mathcal{B}_2$ and $(A^t)^*$ and $(A^+)^*$ are naturally defined as the maximal operators such that the following natural identities hold:

$$(1.3) \qquad \langle (A^t)^* u, v \rangle = \langle u, A^t v \rangle, \ u \in D((A^t)^*), \ v \in D(A^t),$$

$$(1.3') \qquad ((A^+)^* u, v) = (u, A^+ v), \ u \in D((A^+)^*), \ v \in D(A^+).$$

It is easy to see that $A_{\min}$ is well defined as for the case of Hilbert space and $A_{\min} \subset A_{\max}$. Now it is natural to ask about the conditions of coincidence $A_{\min}$ and $A_{\max}$.

Sometimes it is useful to pass from a couple $A, A^t$ (or $A, A^+$) to the matrix

$$(1.4) \qquad \mathfrak{a} = \begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix} \left( \text{or} \begin{bmatrix} 0 & A \\ A^+ & 0 \end{bmatrix} \right): \mathcal{B}_2' \oplus \mathcal{B}_1 \to \mathcal{B}_2 \oplus \mathcal{B}_1'$$

Then we naturally have $\mathfrak{a}^t = \mathfrak{a}$ (or $\mathfrak{a}^+ = \mathfrak{a}$).

**Proposition 1.3.** *Equality* $\mathfrak{a}_{\min} = \mathfrak{a}_{\max}$ *is equivalent to the simultaneous fulfilment of two equalities*

$$(1.5)$$
$$A_{\min} = A_{\max} \text{ and } (A^t)_{\min} = (A^t)_{\max} \ (\text{or} \ (A^+)_{\min} = (A^+)_{\max}).$$

(So the trick of passing to the matrix operator $\mathfrak{a}$ allows to reduce the proof of the equalities (1.5) to a similar equality for a "symmetric" operator $\mathfrak{a}$.)

*Proof.* It is easy to check that

$$\bar{\mathfrak{a}} = \begin{bmatrix} 0 & \overline{A} \\ \overline{A}^t & 0 \end{bmatrix} \text{ and } \mathfrak{a}^* = \begin{bmatrix} 0 & (A^t)^* \\ A^* & 0 \end{bmatrix}$$

(and similar equalities for hermitean case are valid too). The Proposition immediately follows. $\square$

Now it it well known that for a symmetric densely defined operator $A$ in a Hilbert space essential self–adjointness is equivalent to two equalities

$$(1.6) \qquad \text{Ker } (A^* - iI) = 0, \ \text{Ker}(A^* + iI) = 0$$

It easily follows that actually they are equivalent to inclusions

$$(1.7) \qquad \text{Ker}(A^* - iI) \subset D(\overline{A}), \ \text{Ker}(A^* + iI) \subset D(\overline{A})$$

(see e.g. [41]). Also the following proposition is sometimes useful.

**Proposition 1.4 ([42]).** *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces,*

$$A : \mathcal{H}_1 \to \mathcal{H}_2, \ A^+ : \mathcal{H}_2 \to \mathcal{H}_1$$

*a pair of densely defined linear operators and (1.1) is fulfilled. Suppose that the operator $A^+A$ is densely defined and essentially self–adjoint. Then $A_{\min} = A_{\max}$ and $(A^+)_{\min} = (A^+)_{\max}$.*

This statement actually means that $A$ and $A^+$ are "essentially adjoint" to each other i.e.

$$\overline{A} = (A^+)^* \text{ and } \overline{A^+} = A^*.$$

So Proposition 1.4 in a sense gives an inverse statement to the well–known fact (first established by von Neumann) that if $A$ is a closed densely defined linear operator in a Hilbert space then the operator $A^*A$ is self–adjoint.

Now we shall recall some facts concerning a connection between self–adjointness and evolution equations (see e.g. [4]). First let us consider the following Cauchy problem for functions of a real variable $t$ with values in a Hilbert space $\mathcal{H}$ where a densely defined symmetric operator $A$ is given:

(1.8) $$\ddot{u} = -A^*u, \ u(0) = u_0, \ \dot{u}(0) = u_1.$$

Here $\dot{u} = \frac{du}{dt}$, $\ddot{u} = \frac{d^2u}{dt^2}$ and the derivatives are understood as the limits in the norm–topology of $\mathcal{H}$ and they may be supposed continuous in this topology. Also the solutions $u$ may supposed to be defined for all real values of $t$. Actually we shall only speak about the uniqueness of the solutions of (1.8) and in the context given all the uniqueness statements are equivalent. So the uniqueness of the solution of (1.8) can be formulated as follows: if $u : \mathbb{R} \to \mathcal{H}$, $u$ is continuous, $\dot{u}$, $\ddot{u}$ exists in the norm sense and are continuous, $u(t) \in D(A^*)$ for every $t \in \mathbb{R}$ and (1.8) are satisfied for all $t$ with $u_0 = u_1 = 0$ then $u \equiv 0$.

**Proposition 1.5 ([4]).** *Suppose that $A$ is semi–bounded from below i.e.*

$$(1.9) \qquad (Au, u) \geq -C(u, u), \ u \in D(A)$$

*with a real constant $C$. Suppose that we have the uniqueness of solutions for the Cauchy problem (1.8). Then $A$ is essentially self–adjoint.*

The idea of the proof is as follows: if $A$ is not essentially self-adjoint then it has at least two different semi–bounded from below self–adjoint extensions. But for any such an extension $\tilde{A}$ we can write the solution of (1.8) in the form

$$(1.10) \qquad u(t) = (\cos \sqrt{\tilde{A}}t)u_0 + \frac{\sin \sqrt{\tilde{A}}t}{\sqrt{\tilde{A}}}u_1$$

(the choice of the branch of the square roots does not matter because both functions

$$\mu \mapsto \cos \mu t, \ \mu \mapsto \frac{\sin \mu t}{\mu}$$

are even). So using two different semi–bounded from below extensions $\tilde{A}_1$ and $\tilde{A}_2$ in (1.10) and taking the difference $u = u^{(1)} - u^{(2)}$ of two solutions $u_1$ and $u_2$ obtained in this way with the same initial values $u_0, u_1 \in D(A)$ we shall come to a non–zero function satisfying (1.8) with vanishing initial values.

Observe that if, vice versa, $A$ is essentially self-adjoint (and semi–bounded from below) than even the uniqueness of the weak solution of (1.8) can be easily proved by the use of the Holmgren principle.

There is a possiblity to use a first–order evolution problem (of heat equation type)

$$(1.11) \qquad \dot{u} = -A^*u, \ u(0) = u_0.$$

Then the statement of Proposition 1.5 is still true if we change (1.8) to (1.11) in this statement (and the proof does not change). But there is also a possibility to avoid the semiboundedness requirement (1.9) by considering a Schrödinger–type evolution equation

$$(1.12) \qquad \dot{u} = iA^*u, \ u(0) = u_0.$$

Let us introduce "deficiency indices"

$$(1.13) \qquad \kappa_\pm = \dim \operatorname{Ker}(A^* \pm iI)$$

(which may be non–negative integers or $+\infty$)

**Proposition 1.6 ([4]).** *Suppose that $\kappa_+ = \kappa_-$ and there is the uniqueness of solutions for the Schrödinger type Cauchy problem (1.12). Then $A$ is essentially self–adjoint.*

Here the uniqueness should be understood in the sense which is similar to that described before Proposition 1.5 for the problem (1.8) (of course only continuity of $u$ and $\dot{u}$ is required). The idea of the proof is also similar to the one of the Proposition 1.5 (the condition $\kappa_+ = \kappa_-$ is necessary and sufficient for self–adjoint extensions to exist and $\kappa_+ = \kappa_- > 0$ implies that there are at least two such extensions).

**1.2. Minimal and maximal operators, essential self–adjointness for differential operators (basic definitions and notations).**
Let us consider a linear differential operator

$$(2.1) \qquad A: \ C^\infty(X, E_1) \to C^\infty(X, E_2),$$

where $X$ is a $C^\infty$–manifold, $E_1, E_2$ are complex $C^\infty$–vector bundles over $X$, $C^\infty(X, E_i)$ is the space of all $C^\infty$–sections of $E_i$ over $X$.

When we want to study such a differential operator, especially spectral properties of this operator, the first thing to be done is to supply it with an appropriate domain so as to make it a reasonable operator in a Hilbert space or, more generally, in a Banach space. So we begin by describing different possibilities to do this.

Let $\Omega = \Omega(X)$ be the vector bundle of (complex) densities (or 1–densities) on $X$. Integration of densities gives a linear map

$$\int : \ C_0^\infty(X, \Omega) \to \mathbb{C}, \ \omega \longmapsto \int_X \omega,$$

where $C_0^\infty(X, E)$ for any vector bundle $E$ over $X$ denotes the space of all compactly supported $C^\infty$–sections of $E$ over $X$. Now for any vector bundle $E$ over $X$ we define (following [3]) the dual bundle $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \Omega)$. Hence we have a natural bilinear pairing of bundles $E \times E^* \to \Omega$, hence applying integration we obtain natural bilinear pairings in sections

(2.2)
$$C_0^\infty(X, E) \times C^\infty(X, E^*) \to \mathbb{C}, \ C^\infty(X, E) \times C_0^\infty(X, E^*) \to \mathbb{C},$$

which we will denote $\langle \cdot, \cdot \rangle$. Now the transposed operator to $A$ is a differential operator

$$A^t : \ C^\infty(X, E_2^*) \to C^\infty(X, E_1^*),$$

defined by the identity

(2.3)     $\langle Au, v \rangle = \langle u, A^t v \rangle, \ u \in C_0^\infty(X, E), \ v \in C_0^\infty(X, F^*).$

Now let $\mathcal{D}'(X, E)$ denote the space of all distributional sections of $E$ over $X$ which is the dual space to $C_0^\infty(X, E^*)$, i.e. the space of all linear forms on $C_0^\infty(X, E^*)$ which are continuous in the usual sense (see e.g. [22], Ch. 2). Then we have a natural inclusion $C^\infty(X, E) \subset \mathcal{D}'(X, E)$ and the identity (2.3) allows then to extend $A$ to a linear operator

(2.4) $$A : \mathcal{D}'(X, E_1) \to \mathcal{D}'(X, E_2)$$

which we denote $A$ again because it does not lead to a confusion.

DEFINITION 2.1. Suppose that we are given Banach spaces $\mathcal{B}_1, \mathcal{B}_2$ such that $C_0^\infty(X, E_i) \subset \mathcal{B}_i \subset \mathcal{D}'(X, E_i)$, $i = 1, 2$, and the inclusions $\mathcal{B}_i \subset \mathcal{D}'(X, E_i)$ are continuous in the weak topology of $\mathcal{D}'(X, E_i)$ (which means that if $\lim_{k \to \infty} u_k = u$ in the norm of $\mathcal{B}_i$ then $\lim_{k \to \infty} \langle u_k, \psi \rangle = \langle u, \psi \rangle$ for every $\psi \in C_0^\infty(X, E_i^*)$).

The *minimal operator* $A_{\min} : \mathcal{B}_1 \to \mathcal{B}_2$ is the closure of $A : C_0^\infty(X, E_1) \to C_0^\infty(X, E_2)$ i.e. a linear operator from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that its graph in $\mathcal{B}_1 \times \mathcal{B}_2$ is the closure of the set of pairs $\{u, Au\}$ with $u \in C_0^\infty(X, E_1)$. The *maximal operator* is a linear operator $A_{\max} : \mathcal{B}_1 \to \mathcal{B}_2$ such that its domain $D(A_{\max}) = \{u | u \in \mathcal{B}_1, Au \in \mathcal{B}_2\}$, where $A$ is applied in the sense of distributions (i.e. as in (2.4)) and $A_{\max}$ is a restriction of the operator (2.4) (i.e. $A_{\max}u = Au$ if $u \in D(A_{\max})$).

It is easy to see that the minimal operator is well defined and $A_{\min} \subset A_{\max}$ i.e. $D(A_{\min}) \subset D(A_{\max})$ and $A_{\max}$ is an extension of $A_{\min}$. The important question we will discuss below is whether $A_{\min}$ and $A_{\max}$ coincide or not.

An example of the spaces $\mathcal{B}_i$ appears if we have an hermitean metric on each bundle $E_i$, $i = 1, 2$, and also a positive $C^\infty$–density $d\mu$ on $X$. Then we can define a space $L^p(X, E_i)$, $1 \le p < \infty$ which is the completion of $C_0^\infty(X, E_i)$ with respect to the norm

$$\|u\|_p = [\int_X |u(x)|^p d\mu(x)]^{1/p},$$

where $|u(x)|$ denotes the norm of $u(x)$ induced by the hermitian metric in the fiber. So we can take $\mathcal{B}_i = L^{p_i}(X, E_i)$, $i = 1, 2$, and speak about the coincidence of $A_{\min}$ and $A_{\max}$ from $L^{p_1}$ to $L^{p_2}$. In case of $p_1 = p_2 = p$ we will just speak about the coincidence of $A_{\min}$ and $A_{\max}$ in $L^p$.

The case when $\mathcal{B}_i = L^{p_i}(X, E_i)$ can be also viewed as a particular case of the setting described in Sect. 1.1 if we take $\mathcal{B}_i' = L^{p_i'}(X, E_i^*)$ with $1/p_i' + 1/p_i = 1$.

Now instead of the usual space $L^\infty(X, E)$ it is often more convenient to use the Banach space $\tilde{C}(X, E)$ of all continuous sections of $E$ vanishing at infinity. We shall also denote this space by $\tilde{L}^\infty(X, E)$. It also has a natural non–degenerated duality with $L^1(X, E)$ but is more convenient than $L^\infty(X, E)$ because $C_0^\infty(X, E)$ is dense in $\tilde{L}^\infty(X, E)$ (but not in $L^\infty(X, E)$). We shall also define $\tilde{L}^p(X, E) = L^p(X, E)$, $1 \le p < \infty$, to be able to use the whole scale $\tilde{L}^p(X, E)$, $1 \le p \le \infty$.

Now instead of linear duality between $E$ and $E^*$ we can also consider an hermitean duality. We will actually use only the case when $E = E^*$ so $E$ is supplied with a fiberwise positive hermitean map $E \times E \to \Omega(X)$. Then we get a Hilbert space $L^2(X, E)$. Suppose that we have $E_1 = E_2 = E$ in (2.1) and $A$ is symmetric. Then the coincidence $A_{\min} = A_{\max}$ means that $A$ is essentially self–adjoint.

## 1.3. Finite speed propagation and essential

## self–adjointness.

Here we describe how the finite speed propagation for hyperbolic equations and systems allows to make use of abstract Propositions 1.5 and 1.6 in order to prove essential self–adjointness of some differential operators. The idea to apply uniqueness for evolution equations to prove essential self–adjointness is due to A. Ja. Povzner ([31]), it was formulated in an abstract form by Ju. M. Berezanskii ([4]) and later rediscovered and applied in geometric situations by P. Chernoff ([9]).

Let $X$ be a Riemannian manifold, $\Delta$ is the scalar Laplacian on $X$. This means that $\Delta = -\delta d$ where $d: C^\infty(X) \to \Lambda^1(X)$ is the standard differential ($\Lambda^1(X)$ is the space of all smooth 1–forms on $X$), $\delta_{\cdot}: \Lambda^1(X) \to C^\infty(X)$ is the formally adjoint operator to $d$. The simplest example of the application of Proposition 1.5 is given by the following

**Theorem 3.1.** *Let $X$ be a complete Riemannian manifold i.e. all geodesics can be extended indefinitely. Let $A: C^\infty(X) \to C^\infty(X)$ be a linear differential operator of the form*

(3.1) $$A = -\Delta + B, \ \text{ord} \ B \leq 1.$$

*Suppose that $A$ is formally self–adjoint and semibounded from below on $C_0^\infty(X)$. Then $A$ is essentially self–adjoint.*

*Proof.* Consider the Cauchy problem

(3.2) $$\frac{\partial^2 u}{\partial t^2} = -Au, \ u|_{t=0} = u_0, \ \frac{\partial u}{\partial t}|_{t=0} = u_1.$$

The equation in (3.2) is strictly hyperbolic, the bicharacteristic flow is essentially the geodesic flow on $X$. Hence due to the finite speed propagation we can always find a solution $u \in C^\infty(\mathbb{R}, C_0^\infty(X))$ provided $u_0, u_1 \in C_0^\infty(X)$. (Here $C^\infty(\mathbb{R}, C_0^\infty(X))$ denotes the space of functions $u : \mathbb{R} \times X \to \mathbb{C}$, such that $t \mapsto u(t, \cdot)$ is a $C^\infty$–function of $t$ with values in $C_0^\infty(X)$; this implies in particular $\text{supp} \, u \cap ([-T, T] \times X)$ is a compact for every $T > 0$). Hence the standard application of the Holmgren principle gives the uniqueness of the Cauchy problem required to apply Proposition 1.5. $\square$

Theorem 3.1 was formulated in a slightly weaker form by P. Chernoff ([9]) (for the case when $\text{ord} \ B = 0$ i.e. when $A$ is the Schrödinger operator) though the reasoning given in [9] works for the operator (3.1) too. The arguments in [9] directly use the evolution equations like (3.2) considering invariance properties of domains of operators i.e. they do not appeal to abstract statements like Propositions 1.5, 1.6. Therefore they allow to prove the self–adjointness for all powers of $A$ as well as for self–adjoint geometric matrix differential operators e.g. Laplacians or signature operator $d + \delta$ on differential forms on complete Riemannian manifolds. Remark that the proof of essential self–adjointness of $d + \delta$ can be done by use of Proposition 1.6 if we use the Friedrichs theory of symmetric hyperbolic systems ([13]). Besides any zero order terms (which do not change formal self–adjointness) can be

added to $d + \delta$ without changing the essential self–adjointness. As we will see below this is in a sharp contrast with the behaviour of the second order operators where lower order terms may be of crucial importance.

Observe that the essential self–adjointness of pure Laplacian $\Delta$ (without lower order terms) on differential forms on a complete Riemannian manifold was first stated and proved by M.P. Gaffney [14–16] with the help of cut–off functions and Friedrichs mollifiers, and independently by W. Roelcke [34]. H.O. Cordes [10] used a beautiful inequality technique to prove essential self–adjointness of the powers of the scalar Laplacian and some Schrödinger operators. The essential self–adjointness of generalized Dirac operators on complete Riemannian manifolds was proved by M. Gromov and H.B. Lawson ([21]).

There exist a lot of results about essential self–adjointness of elliptic operators in $\mathbb{R}^n$ or in open subsets of $\mathbb{R}^n$. We shall mention only a very small part of them which is most closely connected with the results on manifolds that we have discussed here.

The essential self–adjointness of semi–bounded elliptic second–order symmetric operator in $\mathbb{R}^n$ was first proved by E. Wienholtz ([49]; see also a very simple exposition for the Schrödinger operator in the Glazman's book [18]).

Now let us mention the following Titchmarsh–Sears theorem (see [48], [39] and an exposition in [5]).

**Theorem 3.2.** *Let $A = -\Delta + V(x)$ be a Schrödinger operator on $\mathbb{R}^n$ and $V(x) \geq -Q(|x|)$, where $Q$ is a positive non–decreasing function on $[0, \infty)$ such that*

$$(3.3) \qquad \int_0^\infty Q(r)^{-1/2} dr = \infty$$

*Then $A$ is essentially self–adjoint.*

Observe that condition (3.3) is satisfied for $Q(r) = (1 + r)^\alpha$ if and only if $\alpha \leq 2$. On the other hand the Schrödinger operator with the potential $V(x) = -(1 + |x|^2)^{\alpha/2}$ is essentially self–adjoint

if and only if $\alpha \leq 2$ (see [5]) which shows that the condition (3.3) is relatively precise. Note that if we consider the classical Hamiltonian on $\mathbb{R}^{2n}$

$$H(p,q) = |p|^2 - (1 + |q|^2)^{\alpha/2}$$

corresponding to the quantum Hamiltonian $A = -\Delta - (1 + |x|^2)^{\alpha/2}$ then the condition $\alpha \leq 2$ is equivalent to the completeness of the classical dynamics for $H$ (i.e. the existence of solutions for the corresponding Hamiltonian system for all values of $t$–variable). Hence in this example the properties to be well–defined for the corresponding classical and quantum systems are equivalent though no direct connection has been established. Note that the completeness condition for the manifold in Theorem 3.1 (and in other similar more general results mentioned before) are also in fact conditions of completeness of the corresponding classical systems. We refer the reader to P. Chernoff [9] for a beautiful speculation why lower order terms do not matter for the first–order operators from this point of view: first–order operators correspond to relativistic systems and no conditions are needed to infinity because the particle never gets there.

Theorem 3.2 was improved and generalized in many directions. T. Ikebe and T. Kato ([23]) extended it to Schrödinger operators with magnetic field so as to include quantum Hamiltonians of Stark and Zeeman effects. Many improvements and generalizations (e.g. for the cases where no spherically symmetric minorante is required) were made by F.S. Rofe–Beketov and his collaborators (see e.g. [37], review papers [35], [36] and references there).

T. Kato ([24]) used the evolution equation approach by P. Chernoff to prove that if $A = -\Delta + V$ is a Schrödinger operator in $\mathbb{R}^n$ with a real valued $V \in C^\infty(\mathbb{R}^n)$ and $A \geq -a - b|x|^2$ on $C_0^\infty(\mathbb{R}^n)$ with some constants $a$ and $b$ then $A$ is essentially self–adjoint. This means that we can use a minorante like $-a - b|x|^2$ not only for the potential $V$ but also for the operator $A$ itself.

Many other results and references about the essential self–adjointness of Schrödinger operators in $\mathbb{R}^n$ can be found in [32], vol. II.

Recently Igor Oleinik ([30]) proved the following generalization of Theorem 3.2 to manifolds.

**Theorem 3.3.** *Let $X$ be a Riemannian manifold, and aasume that there exists a point $x_0 \in X$ such that the exponential map $\exp_{x_0} : T_{x_0} X \to X$ is a diffeomorphism. Consider the Schrödinger operator $A = -\Delta + V(x)$ on $X$ and suppose that $V(x) \geq -Q(r)$, where $r = dist(x, x_0)$ and $Q$ is a positive non–decreasing function on $[0, \infty)$ satisfying (3.3). Then $A$ is essentially self–adjoint.*

The condition on the exponential map is probably not necessary but let us mention that it is satisfied for all rotationally symmetric manifolds (e.g. for the hyperbolic space).

The proof of Theorem 3.3 may be given along the same lines as for the euclidean case $X = \mathbb{R}^n$ with the standard metric (see e.g. [5]) but with the use of refined Green's formulas and cut–off functions.

Now we turn to the situation when a formally self–adjoint elliptic second–order operator is not essentially self–adjoint due to the lower order terms. What happens with the solution of the corresponding hyperbolic Cauchy problem like (1.8)? Can it be expressed in operator terms by a formula like (1.10)? We shall give now a more precise statement of the problem and the answer in a simplest case.

Let $X$ be a complete Riemannian manifold and $A = \Delta + V$ be the Schrödinger operator with a real–valued potential $V \in C^\infty(X)$. Hence $A$ is formally self–adjoint but not necessarily semibounded. We can consider the Cauchy problem (3.2) which will be a strictly hyperbolic problem, hence well posed in spaces like $C_0^\infty(X)$, $C^\infty(X)$, $L^2_{comp}(X)$, $L^2_{loc}(X)$ etc. due to the finite speed of propagation.

Now suppose that $u_0, u_1 \in C_0^\infty(X)$. Then we can find a unique $u \in C^\infty(\mathbb{R}, C_0^\infty(X))$ which is a solution of (3.2). Obviously $u(t, \cdot) \in D(A) = C_0^\infty(X)$ for all $t \in \mathbb{R}$, in particular $u(t, \cdot) \in D(A_{\min})$ for all $t \in \mathbb{R}^n$. How can this solution be expressed in operator terms?

Note that $A$ is a real operator hence it has equal deficiency indices (complex conjugation interchanges $\mathrm{Ker}(A^* - iI)$ and

$\mathrm{Ker}(A^* + iI))$. Therefore there exists a self–adjoint extension of $A$ which we shall denote $\tilde{A}$ (it may not be unique, namely when the deficiency indices do not vanish).

We shall need cut–offs $\tilde{A}_N$ for the operator $\tilde{A}$ which are defined as $E((-N, \infty); \tilde{A})\tilde{A}$, where $E(I; \tilde{A})$ means the spectral projection of $\tilde{A}$ corresponding to the interval $I$ i.e. $E(I; \tilde{A}) = \chi_I(\tilde{A})$ where $\chi_I : \mathbb{R} \to \{0,1\}$, $\chi_I(\lambda) = 1$ if $\lambda \in I$, $\chi_I(\lambda) = 0$ if $\lambda \notin I$. Hence $\tilde{A}_N \geq -NI$. Now for every $u_0, u_1 \in C_0^\infty(X)$ we can consider

$$(3.4) \qquad \tilde{u}_N(t) = (\cos t\sqrt{\tilde{A}_N})u_0 + \frac{\sin t\sqrt{\tilde{A}_N}}{\sqrt{\tilde{A}_N}}u_1$$

(The choice of the branch for the square root is not important because the functions $\lambda \mapsto \cos t\sqrt{\lambda}$ and $\lambda \mapsto (\sin t\sqrt{\lambda})/\sqrt{\lambda}$ are even; the fraction in the right hand side of (3.4) should be understood as the result of substitution of $\tilde{A}_N$ into the second function.) Now we can state the result.

**Theorem 3.4.** *Let $A$ be a Schrödinger operator on a complete Riemannian manifold $X$ with the real potential $V \in C^\infty(X)$. Let $u$ be the solution of (3.2) with initial values $u_0, u_1 \in C_0^\infty(X)$, $\tilde{A}$ a self–adjoint extension of $A$, $\tilde{u}_N$ are defined by (3.4). Then*

$$(3.5) \qquad \tilde{u}_N \in C^\infty(\mathbb{R}, L^2(X)) \cap C^\infty(\mathbb{R} \times X)$$

*and*

$$(3.6) \qquad \lim_{N \to \infty} \tilde{u}_N = u \text{ in } C^\infty(\mathbb{R} \times X).$$

*In particular the limit in (3.6) does not depend on the choice of the self–adjoint extension $\tilde{A}$.*

*Proof.* The inclusion $\tilde{u}_N \in C^\infty(\mathbb{R}, L^2(X))$ is obvious since $u_0, u_1$ belong to the domain $D(\tilde{A}^k)$ hence to $D(\tilde{A}_N^k)$ for every $k \in \mathbb{Z}_+$.

The operator inclusion $\tilde{A} \subset A^*$ and the ellipticity of $A$ imply now that $\tilde{u}_N \in C^\infty(\mathbb{R} \times X)$.

Let us decompose $u_j, j = 0, 1$, as follows

$$u_j = u'_{j,N} + u''_{j,N},$$

$$u'_{j,N} = E((-N, \infty); \tilde{A})u_j, \quad u''_{j,N} = E((-\infty, -N]; \tilde{A})u_j.$$

Then (3.4) can be rewritten as

$$\tilde{u}_N(t) = \tilde{u}'_N(t) + \tilde{u}''_N(t),$$

where

$$\tilde{u}'_N(t) = \cos t\sqrt{\tilde{A}_N}u'_{0,N} + \frac{\sin t\sqrt{\tilde{A}_N}}{\sqrt{\tilde{A}_N}}u'_{1,N}; \quad \tilde{u}''_N(t) = u''_{0,N} + tu''_{1,N}.$$

Now note that $\tilde{u}'_N$ is the solution of the Cauchy problem (3.2) with $u_0, u_1$ replaced by $u'_{0,N}, u'_{1,N}$ because $\tilde{A}^k_N u'_{j,N} = \tilde{A}^k u'_{j,N}$ for every $k \in \mathbb{Z}_+$, $j = 0, 1$. Since $\lim_{N \to \infty} \tilde{A}^k u''_{j,N} = 0$ in $L^2(X)$ for every $k \in \mathbb{Z}$, it follows due to the ellipticity of $A$ that $\lim_{N \to \infty} u''_{j,N} = 0$ in $C^\infty(X)$, $j = 0, 1$, hence $\lim_{N \to \infty} \tilde{u}''_N = 0$ in $C^\infty(\mathbb{R} \times X)$ and $\lim_{N \to \infty} u'_{j,N} = u_j$ in $C^\infty(X)$, $j = 0, 1$. It remains to notice that then $\lim_{N \to \infty} \tilde{u}'_N = u$ in $C^\infty(\mathbb{R} \times X)$ due to the well known local energy estimates for the Cauchy problem (3.2). $\square$

## 1.4. Minimal and maximal operators on manifolds of bounded geometry.

We shall use definitions, notations and facts about manifolds of bounded geometry, which are collected in Appendix 1 to this Chapter.

Let $X$ be a manifold of bounded geometry, $E, F$ are vector bundles of bounded geometry on $X$ and

(4.1) $$A : C_0^\infty(X, E) \to C_0^\infty(X, F)$$

is a $C^\infty$-bounded uniformly elliptic differential operator of order $m$. Recall that $A$ can be extended to a bounded linear operator

(4.2) $$A : W_p^m(X, E) \to L^p(X, F), \ 1 \leq p \leq \infty$$

Lemma 1.4 from Appendix 1 easily implies that $A_{\min} = A_{\max}$ in $L^p(X, E)$ if $1 < p < \infty$. More exactly

**Proposition 4.1.** *If $1 < p < \infty$ and $A$ is a uniformly elliptic operator (4.1) then $A_{\min} = A_{\max}$ in $L^p(X, E)$ and*

(4.3) $$D(A_{\min}) = D(A_{\max}) = W_p^m(X, E).$$

*Proof.* Clearly due to the continuity of $A$ in (4.2)

$$W_p^m(X, E) \subset D(A_{\min}) \subset D(A_{\max})$$

But Lemma 1.3 from Appendix 1 implies that $D(A_{\max}) \subset W_p^m(X, E)$, hence $D(A_{\min}) = D(A_{\max}) = W_p^m(X, E)$.
$\square$

**Corollary 4.2.** *Let $A$ be as in Proposition 4.1 with $E = F$ and let $E$ have a hermitean $C^\infty$-bounded scalar product on fibers, $(\cdot, \cdot)$ is the scalar product on $L^2(X, E)$ induced by the scalar product on fibers and the Riemannian density on $X$. Let $A$ be formally self–adjoint with respect to this scalar product. Then $A$ is essentially self–adjoint in $L^2(X, E)$.*

Proposition 4.1 does not cover exceptional values $p = 1$ and $\infty$ but actually $A_{\min} = A_{\max}$ also for the case $p = 1$. As to the case $p = \infty$, a natural modification is necessary: we have to consider $\tilde{L}^\infty = \tilde{C}$ instead of $L^\infty$ (see notations in Sect. 1.2). So we have

**Theorem 4.3 ([45]).** *Let $A$ be a $C^\infty$-bounded uniformly elliptic operator acting as in (4.1). Then $A_{\min} = A_{\max}$ in $\tilde{L}^p(X, E)$ for all $p \in [1, \infty]$.*

Following [45] we shall give a proof that uses the theory of operators with a parameter. A much more complicated parabolic operation approach was suggested by Yu. A. Kordyukov [27],[28] who proved the same statement in the case where $E = F$ and $A$ has a positive–hermitian principal symbol. Many authors have obtained the equality $A_{\min} = A_{\max}$ (in $L^1$ or $\tilde{C}$) or results which imply this in various special cases. E.B. Davies [12] obtains such results for second order operators on homogeneous spaces, Lie groups and on some more general manifolds. The work of R.S. Strichartz [47] also treats the second order case on manifolds. T. Kato [25] studies the Schrödinger operator on $\mathbb{R}^n$ with non smooth potential. H.B. Stewart [46] studies strongly elliptic operators in the Euclidean case and obtains resolvent estimates in the case $p = 1, \infty$. He also refers to some unpublished seminar notes of Masuda.

First we shall suppose that the following Agmon–Agranovich–Vishik condition is satisfied:

(H) $E = F$ and there exist constants $\rho \in \mathbb{C}$ and $C > 0$ with

$|\rho| = 1$ such that $\|(a_m(\nu) - \rho\lambda)^{-1}\| \leq C$ for all $\nu \in T^*X$ with $|\nu| = 1, \lambda > 0$.

Here $a_m$ is the principal symbol of $A$, $|\nu|$ means the norm of the cotangent vector $\nu$ with respect to the given Riemannian metric and $\|\cdot\|$ is the operator norm in fibers of $E$ which is taken in local trivializations of $E$ making it a vector bundle of bounded geometry (see Appendix 1).

The following Lemma summarizes the necessary part of the Agmon–Agranovich–Vishik theory of the operators satisfying (H) (see e.g. [1], [2], [7], [40], or [41]):

**Lemma 4.4.** *There exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ the operator $A - \lambda\rho I : W_2^{m+\ell}(X, E) \to W_2^\ell(X, E)$ is bijective for*

every $\ell \in \mathbb{R}$ with a bounded inverse $(A - \lambda\rho)^{-1} : W_2^\ell(X, E) \to W_2^{m+\ell}(X, E)$ satisfying the estimate

$$
(4.4) \quad \begin{aligned} &\|(A - \lambda\rho)^{-1}u\|_{m+\ell} + \lambda^{1/m}\|(A - \lambda\rho)^{-1}u\|_{m+\ell-1} + \dots \\ &+ \lambda\|(A - \lambda\rho)^{-1}u\|_\ell \le C\|u\|_\ell \end{aligned}
$$

for every $u \in W_2^\ell(X, E)$. Here $\|\cdot\|_s$ denotes the norm in $W_2^s(X, E)$ and $C > 0$ is a constant which is independent of $u$ and of $\lambda$.

*Proof.* We first notice that it is enough to prove the result with $A$ replaced by $\rho^{-1}A$, which satisfies $\|(\rho^{-1}a_m(x, \xi) - \lambda)^{-1}\| \le C$, $x \in X$, $|\xi| = 1$. This is the usual uniform Agmon condition so we can apply the Seeley construction of a local parametrix of $(\rho^{-1}A - \lambda)$ which will satisfy uniform estimates. (See [40].) We then get a global parametrix by using the uniform partition of unity of Lemma 1.3 in Appendix 1. (Making use of the fact that $A$ is a differential operator, one can give simpler proofs, see for instance [41].) $\quad\square$

Later on we shall abbreviate $W_2^s(X, E)$ to $W_2^s$, $L^p(X, E)$ to $L^p$ etc.

Let $f \in C^\infty(X)$ have the property that $\nu(x, \partial_x)f$ is a $C^\infty$-bounded function for every $C^\infty$-bounded vector field $\nu$. Then:

$$
(4.5) \qquad e^f \circ A \circ e^{-f} = A + B_f,
$$

where $B_f$ is a $C^\infty$-bounded differential operator of order $m - 1$. We then have:

$$
e^f \circ (A - \lambda\rho) \circ e^{-f} = (A - \lambda\rho) + B_f,
$$

and if we choose $\lambda > \lambda_0$, where $\lambda_0$ is given in Lemma 4.1, then in the sense of bounded operators from $W_2^{m+\ell}$ to $W_2^\ell$, we can write

$$
(4.6) \quad e^f \circ (A - \lambda\rho) \circ e^{-f} = (1 + B_f(A - \lambda\rho)^{-1}) \circ (A - \lambda\rho).
$$

If $\lambda > 0$ is large enough (depending only on the bounds of $\partial^\alpha f$ for $1 \leq |\alpha| \leq m$ in canonical coordinates), the norm of $B_f(A - \lambda\rho)^{-1}$ : $L^2 \to L^2$ is smaller than $\frac{1}{2}$. We conclude that the right hand side of (4.6), viewed as an operator $W_2^m \to L^2$, is bijective with a uniformly bounded inverse when $\lambda > \lambda_1$, and $\lambda_1 > 0$ is large enough. The identity (4.6) is of course to be understood in the sense of distributions, but we have:

**Lemma 4.5.** *Let $f$ be as above. Then there exists a constant $\lambda_1 > 0$ depending only on the bounds of $\partial^\alpha f$ for $1 \leq |\alpha| \leq m$ (in canonical coordinates) such that for $\lambda > \lambda_1$ the uniformly bounded inverse, $G_\lambda$ of the operator $A - \lambda\rho : W_2^m \to L^2$ (which exists according to Lemma 4.4) has the following property: The operator $e^f \circ G_\lambda \circ e^{-f}$ (which a priori maps $L^2 \cap \mathcal{E}'$ into $W_{2,\mathrm{loc}}^m$) has a bounded extension $L^2 \to W_2^m$, and the norm can be bounded by a constant which is independent of $\lambda$ and of $f$.*

*Proof.* If $f$ is a bounded function, then multiplication by $e^{\pm f}$ is a bounded operator on all the spaces $W_2^s$, and we see that $e^f \circ G_\lambda \circ e^{-f}$ is the inverse of the operator (4.6), and the proposition follows in that case. If $f$ is not a bounded function, we let $\psi(s)$ be a smooth increasing real valued function with $\psi(s) = s$ for $-1 \leq s \leq 1$, $\psi(s) = 2$ for $s \geq 3$, $\psi(s) = -2$ for $s \leq -3$ and put $\psi_\epsilon(s) = \epsilon^{-1}\psi(\epsilon s)$, for $0 < \epsilon \leq 1$. Notice that $|\partial_s^k \psi_\epsilon(s)| \leq C_k$ for $k = 1, 2, \ldots$, where $C_k$ are independent of $s$ and of $\epsilon$, so that the functions $f_\epsilon = \psi_\epsilon \circ f$ satisfy $|\partial^\alpha f_\epsilon(x)| \leq \tilde{C}_\alpha$ for $1 \leq |\alpha| \leq m$, with $\tilde{C}_\alpha$ independent of $\epsilon$. We can then apply Lemma 4.5 with $f$ replaced by $f_\epsilon$. We conclude that $e^{f_\epsilon} \circ G_\lambda \circ e^{-f_\epsilon}$ is bounded as an operator $L^2 \to W_2^m$, uniformly with respect to $\lambda$ and $\epsilon$. If $u \in L^2 \cap \mathcal{E}'$, then for $\epsilon > 0$ small enough, we have $f_\epsilon = f$ on the support of $u$, and if $K$ is an arbitrary compact subset in $X$, then for $\epsilon > 0$ small enough, we have $e^f G_\lambda e^{-f} u = e^{f_\epsilon} G_\lambda e^{-f_\epsilon} u$ on $K$, hence $\|e^f G_\lambda e^{-f} u\|_{m,K} \leq C\|u\|_0$, with a constant $C > 0$ which is independent of $u$ and $K$. Here $\|\cdot\|_{m,K}$ denotes the $W_2^m$-norm over $K$. Since $K$ is arbitrary, we conclude that $e^f G_\lambda e^{-f} u$ belongs to $W_2^m$ and $\|e^f G_\lambda e^{-f} u\|_m \leq C\|u\|_0$. It is then clear that

$e^f \circ G_\lambda \circ e^{-f}$ extends to a bounded operator $L^2 \to W_2^m$. $\quad\square$

Notice that the distribution kernel of $e^f \circ G_\lambda \circ e^{-f}$ is of the form $e^{f(x)-f(y)}K_{G_\lambda}(x,y)$, if we denote the distribution kernel of $G_\lambda$ by $K_{G_\lambda}(x,y)$. Also notice that $K_{G_\lambda}$ is $C^\infty$ outside the diagonal. We shall apply the above result with $f = f_x(y) = (t+1)\tilde{d}(x,y)$, where $\tilde{d}$ is the function constructed by Kordyukov (see Lemma 2.1 in Appendix 1). Here $x$ may be an arbitrary point of $X$, and $t > 0$ may be arbitrary but fixed. Then the hypotheses of Lemma 4.5 are satisfied uniformly when $x$ varies in $X$ and as in Theorem 2.2 of Appendix 1 we obtain:

**Lemma 4.6.** *Let $t > 0$. Then there exists $\lambda(t) > 0$ such that for $\lambda \geq \lambda(t)$ we have the following: For every $\delta > 0$ and all multiindices $\alpha, \beta$ there exists $C_{\alpha,\beta,\delta} > 0$ such that*

(4.7)
$$|\partial_x^\alpha \partial_y^\beta G_\lambda(x,y)| \leq C_{\alpha,\beta,\delta} e^{-td(x,y)} \text{ for all } x,y \in X \text{ with } d(x,y) > \delta.$$

The study of $K_{G_\lambda}$ in the region $d(x,y) < \delta$ goes through exactly as in section 3 of Appendix 1, and we obtain the following analogue of Theorem 3.7 of Appendix 1:

**Theorem 4.7.** *Let $t > 0$. Then there exists $\lambda(t) > 0$ such that for $\lambda \geq \lambda(t)$ we have the following: For all multiindices $\alpha, \beta$ there exists a constant $C_{\alpha,\beta} > 0$ such that when $m < n$ and $x \neq y$:*

(4.8) $\qquad |\partial_x^\alpha \partial_y^\beta G_\lambda(x,y)| \leq C_{\alpha,\beta} d(x,y)^{m-n-|\alpha|-|\beta|} e^{-td(x,y)},$

*and when $m \geq n$ and $x \neq y$:*

(4.9)
$$|\partial_x^\alpha \partial_y^\beta G_\lambda(x,y)|$$
$$\leq C_{\alpha,\beta}(1 + d(x,y)^{m-n-|\alpha|-|\beta|}|\log(d(x,y))|)e^{-td(x,y)}.$$

We here also notice that it is well known that the kernel is locally integrable in $y$ for every fixed $x$ and in $x$ for every fixed $y$.

We have the following result where the only assumption is that $X$ is of bounded geometry:

**Lemma 4.8.** *Let $B(x, r) = \{y \in X; d(y, x) < r\}$. There exists a constant $C = C(X)$ such that for all $x \in X$ and $r \geq 0$:*

$$(4.10) \qquad Vol(B(x, r)) \leq e^{Cr}.$$

*Proof.* We supply a simple proof for the sake of completeness. A more general result due to Bishop, can be found in the book of M. Gromov [20]. We shall use reasoning as in the proof of Lemma 1.2 of Appendix 1. Let us take a maximal system of points $\{x_j | j = 1, 2, \dots, N\} \subset B(x, r)$ such that the balls $B(x_i, \varepsilon)$ and $B(x_j, \varepsilon)$ do not intersect if $i \neq j$. Then $B(x, r)$ will be covered by the balls $B(x_i, 2\varepsilon)$, $i = 1, 2, \dots, N$. Now evidently

$$N \leq C_1(\varepsilon) \mathrm{Vol} B(x, r), \text{ where } C_1(\varepsilon) = [\inf_{x \in X} \mathrm{Vol} B(x, \varepsilon)].$$

Since the ball $B(x, r + \varepsilon)$ is covered by the balls $B(x_i, 3\varepsilon)$, $i = 1, \dots, N$, we have

$$\mathrm{Vol} \, V(x, r + \varepsilon) \leq C(\varepsilon) \, \mathrm{Vol} \, V(x, r)$$

where $C(\varepsilon) = C_1(\varepsilon) \sup_{x \in X} \mathrm{Vol} \, B(x, 3\varepsilon)$. Now (4.10) evidently follows. $\square$

Using the lemma one obtains the following corollary of Theorem 4.7.

**Corollary 4.9.** *There exists $\lambda_0 > 0$, such that if $\lambda > \lambda_0$, then:*

$$(4.11)$$
$$\sup_{x \in X} \int |K_{G_\lambda}(x, y)| dy < +\infty, \quad \sup_{y \in X} \int |K_{G_\lambda}(x, y)| dx < +\infty.$$

*Proof.* Using (4.8), (4.9), it is easy to see that

$$\sup_{x \in X, |x-y| \leq \delta} \int |K_{G_\lambda}(x,y)| \, dy < +\infty,$$

$$\sup_{y \in X |x-y| \leq \delta} \int |K_{G_\lambda}(x,y)| \, dx < +\infty,$$

so we only have to estimate the corresponding integrals over the domain $|x-y| > \delta$, and here we may use (4.7): We get for $\lambda \geq \lambda(t)$

$$\int_{|x-y|>\delta} |K_{G_\lambda}(x,y)| \, dy \leq C_0 \int_0^{+\infty} e^{-td(x,y)} \, dy = C_0 \int_0^{+\infty} e^{-tr} \, dV(r),$$

where $V(r) = \text{Vol}(B(x,r))$. We choose $t$ strictly larger than the constant "$C$" which appears in Lemma 4.8. Then the last integral is convergent and an integration by parts gives:

$$\int_0^\infty e^{-tr} \, dV(r) = \int_0^\infty t e^{-tr} V(r) \, dr \leq \int_0^\infty t e^{(C-t)r} \, dr = t/(t-C).$$

The same estimate is valid for the $x$–integrals and the corollary follows. $\square$

From now on we take $\lambda > 0$ sufficiently large so that Corollary 4.9 applies. By Schur's lemma (see e.g. Lemma 18.1.12 in [22], vol. 3) we then know that the restriction of $G_\lambda$ to $C_0^\infty$ has a unique bounded extension $\tilde{L}^p \to \tilde{L}^p$, when $1 \leq p < \infty$. It is also easy to see (using also (4.11)), that $G_\lambda$ has a unique bounded extension, $\tilde{L}^\infty \to \tilde{L}^\infty$. Working with some fixed $p$, we denote this extension $\tilde{G}_\lambda$. For $u \in C_0^\infty$ we have $(A - \lambda\rho)G_\lambda u = u$, and using the continuity of $\tilde{G}_\lambda$ in $\tilde{L}^p$ and the continuity of $A - \lambda\rho$ for the weak topology of distributions, we get:

$$(4.12) \qquad (A - \lambda\rho)\tilde{G}_\lambda = I \text{ on } \tilde{L}^p$$

Let $u \in D(A_{\max})$ so that $u$ and $Au$ belong to $\tilde{L}^p$. Then if $\varphi \in C_0^\infty$, we get formally:

(4.13)
$$(\tilde{G}_\lambda(A - \lambda\rho)u, \varphi) = ((A - \lambda\rho)u, G_\lambda^*\varphi) = (u, (A - \lambda\rho)^*G_\lambda^*\varphi),$$

where the scalar products are taken either in $L^2$ and $*$ indicates that we take the formal complex adjoint in the sense of distributions. To justify these manipulations we may use the cut-off functions constructed as follows:

(4.14)
$$\chi_N(x) = \sum_{1 \le i \le N} \varphi_i(x),$$

where $\{\varphi_i | i = 1, 2, \dots\}$ is the partition of unity described in Lemma 1.3 of Appendix 1. Clearly $\chi_N \in C_0^\infty(X)$, $0 \le \chi_N \le 1$ and for every compact $K \subset X$ there exists $N$ such that $\chi_N = 1$ in a neighbourhood of $K$. Moreover $|\partial^\alpha \chi_N| \le C_\alpha$ in canonical coordinates uniformly with respect to $N$. Now we can begin with the obvious equality

$$(\tilde{G}_\lambda\chi_N(A - \lambda\rho)u, \varphi) = (u, (A - \lambda\rho)^*\chi_N G_\lambda^*\varphi)$$

and then take limit as $N \to \infty$. Using the boundedness $\tilde{G}_\lambda : \tilde{L}^p \to \tilde{L}^p$ in the left-hand side and the estimates (4.8), (4.9) in the right-hand side we shall conclude that the limits exist and (4.13) is fulfilled. Now $(A - \lambda\rho)^*G_\lambda^*\varphi = \varphi$ as can be seen by replacing $u$ by a $C_0^\infty$-section $\psi$ in (4.13) and using that $\tilde{G}_\lambda(A - \lambda\rho)\psi = G_\lambda(A - \lambda\rho)\psi = \psi$. Thus (4.13) reduces to:

(4.15)
$$(\tilde{G}_\lambda(A - \lambda\rho)u, \varphi) = (u, \varphi).$$

and varying $\varphi$ we conclude that:

(4.16)
$$\tilde{G}_\lambda(A - \lambda\rho) = I \text{ on } D(A_{\max}).$$

Thus we have proved that for $\lambda$ sufficiently large, $(A - \lambda\rho)$ is bijective from $D(A_{\max})$ onto $\tilde{L}^p$ and that the inverse is $\tilde{G}_\lambda$.

We can now end Proof of Theorem 4.3. First suppose that (H) is satisfied. Let $u \in D(A_{\max})$ and $v = Au$. Let $w_j$, $j = 1, 2, \ldots$ be a sequence of $C_0^\infty$–sections converging to $v - \lambda\rho u$ in $\tilde{L}^p$, and put $u_j = \tilde{G}_\lambda w_j \in \tilde{L}^p \cap C^\infty$. Then $u_j \to u$ in $\tilde{L}^p$ and $Au_j = w_j + \lambda\rho u_j \to u$ in $\tilde{L}^p$. It only remains to prove that $u_j$ belongs to $D(A_{\min})$. We note that if $\Omega_j = \mathrm{supp}(w_j)$ then

$\sup_x \int_{\Omega_j} (1 - \chi_N(x)) |K_{G_\lambda}(x, y)| dy$ and

$\sup_{y \in \Omega_j} \int (1 - \chi_N(x)) |K_{G_\lambda}(x, y)| dx$ tend to zero when $N$ tends to infinity, and similarly when $(1 - \chi_N(x)) K_{G_\lambda}$ is replaced by some $x$–derivative of the same function. (Indeed, this is proved in the same way as Corollary 4.9.) Hence (still with $j$ fixed) $\chi_N u_j \to u_j$ and $A(\chi_N u_j) \to Au_j$ in $\tilde{L}^p$ when $N \to \infty$, and the proof is complete provided (H) is satisfied.

Now consider the general case. Here we just have to apply Proposition 1.3. We may assume that $E$ and $F$ are uniformly $C^\infty$–bounded hermitean vector bundles. Let $A^+$ denote the formal complex adjoint of $A$, and consider the uniformly elliptic $C^\infty$–bounded formally self adjoint operator: $\mathfrak{a} : C^\infty(M; F \oplus E) \to C^\infty(M; F \oplus E)$ given by the matrix

$$\mathfrak{a} = \begin{pmatrix} 0 & A \\ A^+ & 0 \end{pmatrix}$$

We notice that $\mathfrak{a}$ satisfies (H) with $\rho = \sqrt{-1}$, so we know that $\mathfrak{a}_{\max} = \mathfrak{a}_{\min}$. It follows due to Proposition 1.3 that $A_{\max} = A_{\min}$ q.e.d. $\square$

## Appendix 1. Analysis on manifolds of bounded geometry.

In this Appendix we mostly follow [44].

**A1.1. Preliminaries.** Let $X$ be a Riemannian manifold $n = \dim X$. In what follows we shall always suppose for the sake of simplicity that $X$ is connected. Then the Riemannian distance

$d : X \times X \to [0, +\infty)$ is well defined; namely $d(x, y)$ is the infinium of Riemannian lengths of all arcs connecting $x$ and $y$.

Denote by $T_x X$ the tangent space of $X$ at a point $x \in X$ and let $\exp_x : T_x X \to X$ be the usual exponential geoddesic map: $\exp_x v = \gamma(1)$, where $\gamma(t)$ is the geodesic (with a canonical parameter which is proportional to the arc length) starting at $x$ with the initial speed $v \in T_x X$, i.e. $\gamma(0) = x$, $\dot{\gamma}(0) = v$. We shall always suppose that $X$ is complete or equivalently that $\exp_x$ is defined everywhere i.e. for every $x \in X$ and $v \in T_x X$ the corresponding geodesic $\gamma(t)$ can be defined for all $t \in \mathbb{R}$. The exponential map $\exp_x : T_x X \to X$ is a diffeomorphism of a ball $B_x(0, r) \subset T_x X$ of radius $r > 0$ with the center $0$ on a neighborhood $U_{x,r}$ of $x$ in $X$. (Actually for a fixed $x$ this neighborhood $U_{x,r}$ will be the ball $B(x, r)$ of the radius $r$ centered at $x$ on the manifold $X$ with respect to the distance $d$ induced by the given Riemannian metric, provided $r$ is sufficiently small). Denoting by $r_x$ the supremum of possible radii of such balls we can define the injectivity radius of $X$ as $r_{inj} = \inf_{x \in X} r_x$. If $r_{inj} > 0$ then taking $r \in (0, r_{inj})$ we see that $\exp_x : B_x(0, r) \to U_{x,r}$ will be a diffeomorphism for every $x \in X$. Euclidean coordinates in $T_x X$ (associated with an orthonormal frame in $T_x X$) define coordinates on $U_{x,r}$ (by means of $\exp_x$) which are called canonical.

DEFINITION 1.1 (see e.g. [8] or [33]) $X$ is called a *manifold of bounded geometry* if the following two conditions are satisfied:

a) $r_{inj} > 0$

b) $|\nabla^k R| \le C_k$, $k = 0, 1, 2, \ldots$ (i.e. every covariant derivative of the Riemann curvature tensor is bounded).

Note that a) implies that $X$ is complete i.e. all geodesics can be extended indefinitely. It follows that $X$ is complete as a metric space with the metric given by the Riemannian distance $d$, and every ball $\{x | d(x, x_0) \le r\}$ is compact whatever $x_0 \in X$, $r > 0$.

The property b) can be replaced by the following equivalent property which will be more convenient for the use here

b') let us fix any $r \in (0, r_{inj})$ and let $U_{x,r}, U_{x',r}$ be two domains of canonical coordinates $y : U_{x,r} \to \mathbb{R}^n, y' : U_{x',r} \to \mathbb{R}^n$ such that $U_{x,r} \cap U_{x',r} \ne \emptyset$: consider the vector function $y' \circ y^{-1} :$

$y(U_{x,r} \cap U_{x',r}) \to \mathbb{R}^n$; then

$$|\partial_y^\alpha (y' \circ y^{-1})| \le C_{\alpha,r}$$

for every multiindex $\alpha$.

Examples of manifolds of bounded geometry are Lie groups or more general homogeneous manifolds (with invariant metrics), covering manifolds of compact manifolds (with a Riemannian metric which is lifted from the base manifold), leaves of a foliation on a compact manifold (with a Riemannian metric which is induced by a Riemannian metric of the compact manifold).

Below we shall always use only canonical coordinates with a fixed $r \in (0, r_{inj})$. Then all the change of coordinate functions have bounded derivatives of all orders. This property allows to formulate a correct notion of $C^k$–boundedness ($k = 0, 1, 2, \cdots$) or $C^\infty$–boundedness for functions, vector fields, exterior forms and other tensor fields on $X$. Namely a function $f : X \to \mathbb{C}$ is called $C^k$–bounded if $f \in C^k(X)$ and $|\partial_y^\alpha f(y)| \le C_\alpha$ for every multiindex $\alpha$ with $|\alpha| \le k$ and for any choice of canonical coordinates. A function $f : X \to \mathbb{C}$ is called $C^\infty$–bounded if $f \in C^\infty(X)$ and $f$ is $C^k$–bounded for every $k = 0, 1, 2, \cdots$. Let $C_b^k(X)$ be the space of all $C^k$–bounded complex–valued functions on $X$ (here $k = 0, 1, 2, \cdots$ or $k = \infty$). Of course $C^k$–boundedness of a function $f \in C^k(X)$ is equivalent to the estimate $|\nabla^k f(x)| \le C$ but the formulation in local coordinates is sometimes more convenient.

Similarly a vector field, an exterior form on any general tensor field on $X$ is called $C^k$–bounded ($k = 0, 1, 2, \cdots$ or $k = \infty$) if all components of the field in any canonical coordinate system are $C^k$–bounded as $C^k$–functions of corresponding coordinates (with bounds depending only on the order of the differentiation but not on the chosen coordinate neighbourhood).

Let $A : C^\infty(X) \to C^\infty(X)$ be a differential operator of order $m$ with $C^\infty$–coefficients. We shall call it $C^\infty$–bounded if in any canonical coordinate system $A$ is written in the form

$$(1.1) \qquad A = \sum_{|\alpha| \leq m} a_\alpha(y) \partial_y^\alpha$$

where the coefficients $a_\alpha$ are (complex–valued) functions satisfying the estimates $|\partial_y^\beta a_\alpha(y)| \leq C_\beta$ for any multiindex $\beta$ (with a constant $C_\beta$ which does not depend on the chosen canonical neighbourhood). A $C^\infty$–bounded vector field defines a $C^\infty$–bounded differential operator of order 1.

Let $E$ be a complex vector bundle on $X$. We shall say that $E$ is a **bundle of bounded geometry** if it is supplied by an additional structure: trivializations of $E$ on every canonical coordinate neighbourhood $U$ such that the corresponding matrix transition functions $g_{UU'}$ on all intersections $U \cap U'$ of such neighbourhoods are $C^\infty$–bounded i.e. all their derivatives $\partial_y^\alpha g_{UU'}(y)$ with respect to canonical coordinates are bounded with bounds $C_\alpha$ which do not depend on the chosen pair $U, U'$. Examples of vector bundles of bounded geometry are: trivial bundle $X \times \mathbb{C}$, complexified tangent and cotangent bundles $TX \otimes \mathbb{C}$ and $T^*X \otimes \mathbb{C}$, complexified exterior powers $\Lambda^\ell T^*X \otimes \mathbb{C}$ of the cotangent bundle ($C^\infty$–sections of $\Lambda^\ell T^*X \otimes \mathbb{C}$ are exterior complex–valued $\ell$–forms on $X$), complexified tensor bundles etc. The definition of $C^\infty$–bounded differential operator is easily generalized to the case of operators

$$(1.2) \qquad A: \; C^\infty(X, E) \to C^\infty(X, F)$$

acting between spaces of $C^\infty$–sections of vector bundles of bounded geometry $E, F$ (the definition is the same as for scalar operators but with the use of the representation (1.1) in canonical coordinates and chosen trivializations. Examples of $C^\infty$–bounded differential operators in this more general context are the exterior differentiation de Rham operator $d: \; \Lambda^\ell(X) \to \Lambda^{\ell+1}(X)$ where $\Lambda^\ell(X) = C^\infty(X, \Lambda^\ell T^*X \otimes \mathbb{C})$, operators of covariant differentiation of tensors, Laplace–Beltrami operators on functions or forms etc.

If $E$ is a vector bundle of bounded geometry on $X$ then the notion of $C^\ell$–boundedness and the corresponding spaces $C^\ell_b(X, E)$ of $C^\ell$–bounded sections are also defined for $\ell = 0, 1, 2, \cdots$ or $\ell = \infty$. Also the space $L^p(X, E)$ of the sections with the integrable $p$–th power of a fiber norm $(1 \le p < \infty)$ is naturally defined as well as the spaces $\tilde{L}^p(X, E)$, $1 \le p \le \infty$.

The following Lemma is essentially due to M. Gromov [19].

**Lemma 1.2.** *There exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ then there exists a countable covering of $X$ by balls of the radius $\varepsilon$ : $X = \cup B(x_i, \varepsilon)$ such that the covering of $X$ by the balls $B(x_i, 2\varepsilon)$ with the double radius and the same centers has a finite multiplicity.*

Here the multiplicity (or index in the terminology of [19]) of the covering by balls is the maximal number of the balls with non–empty intersection in this covering.

*Proof.* Let us choose $\varepsilon_0 > 0$ so that $3\varepsilon_0 < r_{inj}$, hence the canonical coordinates are defined on the ball $B(x, 3\varepsilon)$ for every $x \in X$ and the transition functions from one set of canonical coordinates to another have bounded derivatives of every order (see Definition 1.1). Also the components $g_{ij}$ and $g^{ij}$ of the Riemannian metric have bounded derivatives of every order in chosen canonical coordinates. It follows in particular that there exists $C > 0$ such that

$$C^{-1} \le \frac{V(x, r)}{V(y, r)} \le C, \ x, y \in X, \ r \in (0, 3\varepsilon_0),$$

where $V(x, r) = \text{Vol } B(x, r)$ (here Vol means volume with respect to the standard Riemannian density).

Let us choose a maximal set of disjoint balls $B(x_1, \varepsilon/2)$, $B(x_2, \varepsilon/2), \ldots$ (such a set exists due to Zorn Lemma and is obviously countable). For every $x \in X$ there exists $i$ such that $d(x, x_i) < \varepsilon$ (otherwise we could add $B(x, \varepsilon/2)$ to the chosen balls). Hence $X = \cup B(x_i, \varepsilon)$.

Now if $y \in B(x_i, 2\varepsilon)$ then $B(x_i, \varepsilon/2) \subset B(y, 3\varepsilon)$. Hence if $y$ is covered by each of different balls $B(x_{i_k}, 2\varepsilon)$, $k = 1, \ldots, N$, then

$\Sigma_{1 \leq k \leq N} V(x_{i_k}, \varepsilon/2) \leq V(y, 3\varepsilon)$ and we get the required estimate of multiplicity

$$N \leq (\sup_{y \in X} V(y, 3\varepsilon))(\inf_{x \in X} V(x, \varepsilon/2)). \quad \square$$

Lemma 1.1 implies the existence of "uniform" partition of unity which is subordinate to a covering by balls from Lemma 1.1. Let us choose $\varepsilon < r/2$ where $r \in (0, r_{inj})$ is fixed as before.

**Lemma 1.3.** *For every $\varepsilon > 0$ there exists a partition of unity $1 = \Sigma_{i=1}^{\infty} \varphi_i$ on $X$ such that*
*1) $\varphi_i \geq 0$, $\varphi_i \in C_0^{\infty}(X)$, supp $\varphi_i \subset B(x_i, 2\varepsilon)$,*
*where $\{x_i\}$ is the sequence of points from Lemma 1.2;*
*2) $|\partial_y^{\alpha} \varphi_i(y)| \leq C_{\alpha}$*
*for every multiindex $\alpha$ in canonical coordinates uniformly with respect to $i$ (i.e. with the constant $C_{\alpha}$ which does not depend on $i$).*

This Lemma is a useful tool to construct global objects on $X$ from their local prerequisites. One of the important examples is the uniform Sobolev or Besov spaces $W_p^s(X)$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$ (see e.g. [33] in case $p = 2$). First introduce the Sobolev norm $\|.\|_{s,p}$ on $C_0^{\infty}(X)$ by the formula

$$(1.3) \qquad \|u\|_{s,p}^p = \Sigma_{i=1}^{\infty} \|\varphi_i u\|_{s,p;B(x_i,2\varepsilon)}^p,$$

where $\|.\|_{s,p;B(x_i,2\varepsilon)}$ means the usual Sobolev (Besov or Bessel potential) norm of order $s$ in canonical coordinates on $B(x_i, 2\varepsilon)$. Actually we only need the case $s \in \mathbb{Z}_+$; then the local Sobolev norm can be written for every open set $\Omega \subset \mathbb{R}^n$ as

$$\|v\|_{s,p;\Omega} = (\Sigma_{|\alpha| \leq s} \int_{\Omega} |\partial^{\alpha} v(y)|^p dy)^{1/p}, \ 1 \leq p < \infty,$$

$$\|v\|_{s,\infty;\Omega} = \Sigma_{|\alpha| \leq s} \operatorname*{ess\,sup}_{\Omega} |\partial^{\alpha} v(y)|$$

Also if we choose a system $Y_1, \cdots, Y_N$ of $C^\infty$–bounded vector fields on $X$ such that $Y_1(x), \cdots, Y_N(x)$ generate $T_x X$ for every $x \in X$ then we can introduce the following norm which is equivalent to (1.3)

(1.3')
$$\|u\|_{s,p}^p = \sum_{k=0}^{s} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} \int_X |Y_{i_1} \cdots Y_{i_k} u(x)|^p dx, \ 1 \le p < \infty,$$

where $dx$ is the standard Riemannian density on $X$,

$$\|u\|_{s,\infty} = \sum_{k=0}^{s} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} \operatorname*{ess\,sup}_X |Y_{i_1} \cdots Y_{i_k} u(x)|.$$

Another equivalent norm for $s \in \mathbb{Z}_+$ is given by

$$\|u\|_{s,p}^p = \Sigma_{k=0}^s \int_X |\nabla^k u(x)|^p dx, \ 1 \le p < \infty,$$

$$\|u\|_{s,\infty} = \sum_{k=0}^{s} \operatorname*{ess\,sup}_X |\nabla^k u(x)|$$

(here $|\cdot|$ is understood as the norm induced by the Riemannian metric on tensors).

Now we can introduce the uniform Sobolev space $W_p^s(X)$ as the completion of $C_0^\infty(X)$ with respect to the norm (1.3). The spaces $W_p^s(X)$ have the same properties as the corresponding spaces in the case $X = \mathbb{R}^n$. All of them are naturally included in the space of distributions $\mathcal{D}'(X)$. The space $W_2^s(X)$ has a natural Hilbert structure and will be also denoted $H^s(X)$. The usual embedding theorems are true, e.g. $W_p^0(X) = L^p(X)$ if $1 \le p < \infty$, $W_p^s(X) \subset C_b^k(X)$ if $s > k + n/p$. If $E$ is a vector bundle of bounded geometry then the Sobolev norms of sections and the

corresponding Sobolev spaces of sections $W_p^s(X, E)$ are defined in the same way.

Denote $W_p^{-\infty}(X) = \cup_{s \in \mathbb{R}} W_p^s(X)$, $W_p^\infty(X) = \cap_{s \in \mathbb{R}} W_p^s(X)$ and the similar meaning have the notations $W_p^{-\infty}(X, E)$, $W_p^\infty(X, E)$.

Let $A$ be a differential operator of order $m$ acting as in (1.2) between spaces of sections of vector bundles of bounded geometry. The principal symbol of $A$ gives a family of linear maps

$$a_m(x, \xi) : \ E_x \to F_x$$

where $x \in X$, $(x, \xi) \in T_x^* X$ is a cotangent vector based at $x$, $E_x$ and $F_x$ are fibers of bundles $E$ and $F$ over $x$. Let us choose admissible trivializations of $E$ and $F$ over a neighbourhood of $x$. Then $a_m(x, \xi)$ becomes a (complex) matrix. The operator $A$ is called elliptic if this matrix is invertible for every $(x, \xi)$ with $\xi \neq 0$. It is called **uniformly elliptic** if there exists $C > 0$ such that

$$(1.4) \qquad |a_m^{-1}(x, \xi)| \leq C|\xi|^{-m}, \ (x, \xi) \in T^* X, \xi \neq 0.$$

Here $|\xi|$ is the length of $(x, \xi)$ with respect to the given Riemannian metric, $|a_m^{-1}(x, \xi)|$ is the operator norm of the matrix $a_m^{-1}(x, \xi)$ in the above mentioned trivializations.

Let $A$ be a $C^\infty$–bounded differential operator of order $m$ on $M$. Then $A$ defines a bounded linear operator $A : \ W_p^s(X) \to W_p^{s-m}(X)$ for every $s \in \mathbb{R}$, $1 \leq p \leq \infty$ (if $A$ acts as in (1.2) then it defines a bounded linear operator $A : \ W_p^s(X, E) \to W_p^{s-m}(X, F)$). Now we shall formulate regularity properties and a priori estimates which follow from uniform ellipticity.

**Lemma 1.4.** *Let $A$ be a $C^\infty$–bounded uniformly elliptic differential operator acting as in (1.2) between spaces of sections of vector bundles of bounded geometry. Then for every $s, t \in \mathbb{R}$, $p \in (1, +\infty)$ there exists $C > 0$ such that*

$$(1.5) \qquad \|u\|_{s,p} \leq C(\|Au\|_{s-m,p} + \|u\|_{t,p}), \ u \in C_0^\infty(X, E).$$

*Moreover if* $u \in W_p^{-\infty}(X, E)$ *and* $Au \in W_p^{s-m}(X, F)$ *then* $u \in W_p^s(X, E)$.

*Proof.* Let us choose the points $x_1, x_2, \ldots$ and $\varepsilon > 0$ as in Lemma 1.1. We have the usual local a priori estimate

$$(1.6) \quad \|u\|_{s,p;B(x_i,\varepsilon)}^p \leq C_1 (\|Au\|_{s-m,p;B(x_i,2\varepsilon)}^p + \|u\|_{t,p;B(x_i,2\varepsilon)}^p)$$

with a constant $C_1$ which does not depend on $i$. Summing over all $i$ we evidently obtain an estimate which is equivalent to (1.5). The last statement also follows from the corresponding local regularity result and the estimate (1.6). $\square$

## A1.2. Weight estimates and decay of the Green function.

We begin with a construction which gives a substitute with natural smoothness properties for the distance $d = d(x, y)$ on a connected Riemannian manifold $X$ of bounded geometry. Such a substitute will be a function which we shall denote by $\tilde{d} = \tilde{d}(x, y)$. For the case of Lie groups it can be constructed as a convolution of $d(x, .)$ with a $C_0^\infty$–function ([29]). The general case requires a more complicated procedure which we shall give now ([27],[28]).

**Lemma 2.1.** *(Yu.A. Kordyukov). There exists a function* $\tilde{d}$ : $X \times X \to [0, +\infty)$ *satisfying the following conditions:*

(i) *there exists* $\rho > 0$ *such that*

$$|\tilde{d}(x, y) - d(x, y)| < \rho$$

*for every* $x, y \in X$;

(ii) *for every multiindex* $\alpha$ *with* $|\alpha| > 0$ *there exists a constant* $C_\alpha > 0$ *such that*

$$|\partial_y^\alpha \tilde{d}(x, y)| \leq C_\alpha, \quad x, y \in X,$$

*where the derivative* $\partial_y^\alpha$ *is taken with respect to canonical coordinates.*

Moreover for every $\varepsilon > 0$ the exists a function $\tilde{d} : X \times X \rightarrow [0, \infty)$ satisfying (i) with $\rho < \varepsilon$.

*Proof.* Let us choose a covering $X = \cup B(x_i, 2\varepsilon)$ and a partition of unity $1 = \Sigma \varphi_i$ described in Lemmas 1.2 and 1.3. We shall suppose that an orthonormal frame is chosen in every tangent space $T_{x_i} X$, $i = 1, 2, \cdots$, so $T_{x_i} X$ is identified with $\mathbb{R}^n$ and the exponential maps at the points $x_i$ can be considered as the maps $\exp_{x_i} : \mathbb{R}^n \rightarrow X$.

Let us choose a function $\theta_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\theta_1 \geq 0$, supp $\theta_1 \subset \{x \| x | < 1\}$, $\int_{\mathbb{R}^n} \theta_1(x) \mathrm{d}x = 1$ and define $\theta_\delta(x) = \delta^{-n} \theta_1(x/\delta)$ for any $\delta > 0$. Now choosing $\delta$ sufficiently small we can define

$$(2.1) \quad \tilde{d}(x,y) = \Sigma_{i=1}^\infty \varphi_i(y) \int_{\mathbb{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(x, \exp_{x_i}(z)) dz.$$

Subtracting the evident identity

$$d(x,y) = \Sigma_{i=1}^\infty \varphi_i(y) \int_{\mathbb{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(x, y) dz$$

from (2.1) and using the triangle inequality we obtain the estimate

$$|\tilde{d}(x,y) - d(x,y)| \leq \Sigma_{i=1}^\infty \varphi_i(y) \int_{\mathbb{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z) d(\exp_{x_i}(z), y) dz.$$

It follows from the bounded geometry conditions that there exists $C > 0$ such that $d(\exp_{x_i}(z), y) < C\delta$ if $y \in$ supp $\varphi_i$ and $|\exp_{x_i}^{-1}(\dot{y}) - z| < \delta$, so we obtain

$$|\tilde{d}(x,y) - d(x,y)| < C\delta$$

which proves (i) with small $\rho$ provided $\delta$ is chosen sufficiently small.

To prove (ii) let us consider first the case $|\alpha| = 1$.

Using the notation $\partial_j = \partial/\partial y_j$ in some canonical coordinates we obtain

(2.2)

$$\partial_j \tilde{d}(x,y) = \Sigma_{i=1}^{\infty}[\partial_j \varphi_i(y)] \int_{\mathbb{R}^n} \theta_\delta(\exp_{x_i}^{-1}(y) - z)d(x, \exp_{x_i}(z))dz +$$

$$\Sigma_{i=1}^{\infty}\varphi_i(y)\Sigma_{k=1}^{n} \int_{\mathbb{R}^n} b_{ijk}(y)[\frac{\partial}{\partial z_k}\theta_\delta(\exp_{x_i}^{-1}(y) - z)]d(x, \exp_{x_i}(z))dz$$

where $b_{ijk}$ are some functions (in the chosen canonical coordinates) which are $C^\infty$–bounded uniformly with respect to $i, j, k$ and the chosen coordinates. The same arguments as we used in proving (i) show that the first term in the right hand side of (2.2) is estimated by a constant. To estimate the second term we can subtract from it a similar term which is obtained by changing $d(x, \exp_{x_i}(z))$ to $d(x, y)$ (this modified term evidently vanishes). Following then the reasoning used for the proof of (i) we obtain that the second term is estimated by a constant.

Further inductive reasoning shows that (ii) is true for every $\alpha$ q.e.d. $\square$

Now we can introduce exponential weights $f_{\varepsilon,y} \in C^\infty(X)$ by

(2.3) $$f_{\varepsilon,y}(x) = \exp(\varepsilon\tilde{d}(y,x)), \ x,y \in X,$$

where $\varepsilon \in \mathbb{R}$ (usually $\varepsilon$ will be sufficiently small).

Let us introduce a weight Sobolev space

$$W_{p,\varepsilon}^s(X) = \{u | u \in \mathcal{D}'(X), f_{\varepsilon,y}u \in W_p^s(X)\}$$

where $s \in \mathbb{R}$, $p \in [1, \infty]$ and $y$ is any fixed point in $X$. It is easy to check that

$$f_{\varepsilon,y_1}^{-1}f_{\varepsilon,y_2} \in C_b^\infty(X)$$

for any fixed points $y_1, y_2 \in X$. It follows that the space $W_{p,\varepsilon}^s(X)$ does not depend on the chosen point $y$. The space $W_{p,\varepsilon}^s(X)$ is a Banach space with the norm

(2.4) $$\|u\|_{s,p;\varepsilon,y} = \|f_{\varepsilon,y}u\|_{s,p}.$$

These norms obtained by use of different points $y$ are equivalent but the dependence on $y$ is sometimes essential.

Now we shall consider a $C^\infty$–bounded uniformly elliptic operator $A : C^\infty(X,E) \to C^\infty(X,E)$ where $E$ is a vector bundle of bounded geometry. Then $A_{\min} = A_{\max}$ in $L^p(X,E)$, $1 < p < \infty$ (see Sect. 1.4 in Ch. 1) and we denote $\sigma_p(A)$ the spectrum of $A_{\min}$ (or $A_{\max}$) in $L^p(X,A)$. Let us suppose that $\lambda \in \mathbb{C}\backslash\sigma_p(A)$ for $p \in (1,+\infty)$. Then there is a bounded everywhere defined inverse operator

$$(A - \lambda I)^{-1} : L^p(X,E) \to L^p(X,E).$$

The L. Schwartz kernel of this inverse operator will be denoted $G = G(x,y)$ and will be called **the Green function** ($p$ and $\lambda$ are fixed). We are ready to prove estimates of decay of the Green function off the diagonal $\Delta = \{(x,x)|x \in X\} \subset X \times X$. Note that $G$ is a distributional section of the bundle $E \otimes E^*$ on $X \times X$ (the fiber of $E \otimes E^*$ over a point $(x,y) \in X \times X$ is $E_x \otimes E_y^*$, where $E_y^*$ is the dual linear space to $E_y$). We identify the density bundle over $M$ with a trivial bundle by use of the standard Riemannian density.

**Theorem 2.2.** *Let $A : C^\infty(X,E) \to C^\infty(X,E)$ be a $C^\infty$-bounded uniformly elliptic differential operator. Let $p \in (1,+\infty)$ and $\lambda \in \mathbb{C}\backslash\sigma_p(A)$ be fixed, $G = G(x,y)$ the Green function. Then $G \in C^\infty(X \times X\backslash\Delta)$ and there exists $\varepsilon > 0$ such that for every $\delta > 0$ and for every multiindices $\alpha, \beta$ there exists $C_{\alpha\beta\delta} > 0$ such that*

(2.5) $$|\partial_x^\alpha \partial_y^\beta G(x,y)| \le C_{\alpha\beta\delta} \exp(-\varepsilon d(x,y)) \text{ if } d(x,y) \ge \delta.$$

*Here the derivatives $\partial_x^\alpha$ and $\partial_y^\beta$ are taken with respect to canonical coordinates and absolute value in the left hand side is taken in the corresponding fibers.*

*Proof.* Without loss of generality we can suppose that $\lambda = 0$. For the sake of simplicity of notations we shall only consider the scalar case i.e. the case of trivial $E = X \times \mathbb{C}$. Let us for every $\varepsilon \in \mathbb{R}$, $y \in X$ consider a differential operator $A_{\varepsilon,y} = F_{\varepsilon,y} A F_{\varepsilon,y}^{-1}$ where $F_{\varepsilon,y}$ is the multiplication operator $(F_{\varepsilon,y}u)(x) = f_{\varepsilon,y}(x)u(x)$ with $f_{\varepsilon,y}$ defined by (2.3). Choosing any $s \in R$ we obtain a commutative diagram

$$
(2.6) \qquad
\begin{array}{ccc}
W_p^s(X) & \xrightarrow{\;A_{\varepsilon,y}\;} & W_p^{s-m}(X) \\[2pt]
\Big\uparrow{\scriptstyle F_{\varepsilon,y}} & & \Big\uparrow{\scriptstyle F_{\varepsilon,y}} \\[2pt]
W_{p,\varepsilon}^s(X) & \xrightarrow{\;A\;} & W_{p,\varepsilon}^{s-m}(X)
\end{array}
$$

where the vertical arrows are linear topological isomorphisms and even isometries if we use the norm (2.4) in $W_{p,\varepsilon}^s(X)$ and the corresponding norm in $W_{p,\varepsilon}^{s-m}(X)$. It follows from the properties of $\tilde{d}$ described in Lemma 2.1 that

$$
(2.7) \qquad\qquad A_{\varepsilon,y} = A + \varepsilon B_{\varepsilon,y},
$$

where $\{B_{\varepsilon,y} | y \in X, |\varepsilon| < 1\}$ is a family of uniformly $C^\infty$–bounded differential operators of order $m - 1$. It follows that the operator norm

$$
\|A_{\varepsilon,y} - A : \; W_p^s(X) \to W_p^{s-m}(W)\|
$$

tends to 0 as $\varepsilon \to 0$. The required invertibility of $A$ implies now that $A$ defines a linear topological isomorphism of Banach spaces

$$
A : \; W_p^s(X) \to W_p^{s-m}(X),
$$

so $A_{\varepsilon,y}$ in the diagram (2.6) also defines a linear topological isomorphism if $|\varepsilon| < \varepsilon_0$ where $\varepsilon_0 > 0$ is sufficiently small. Besides all

norm estimates are uniform with respect to $y \in X$. Hence $A$ in the diagram is also uniformly topologically invertible if $|\varepsilon| < \varepsilon_0$.

Now notice that

$$(2.8) \qquad G(x,y) = [A^{-1}\delta_y(\cdot)](x),$$

where $\delta_y$ is the standard Dirac $\delta$–measure on $X$ supported at $y \in X$. The Sobolev embedding theorem implies that if $s < -n/p$ then $\delta_y \in \cap_{\varepsilon \in \mathbb{R}} W^s_{p,\varepsilon}(X)$ and $\|\delta_y\|_{s,p;\varepsilon y} \leq C_{s,p}$ uniformly over $y \in X$ and $\varepsilon$ with $|\varepsilon| < 1$. It follows from (2.8) that

$$(2.9) \qquad \|G(\cdot,y)\|_{s+m,p;\varepsilon,y} \leq C_{s,p}$$

if $|\varepsilon| < \varepsilon_0$.

Now note that

$$A_x G(x,y) = 0 \text{ if } x \neq y.$$

It follows from (2.9) and the uniform local a priori estimate like (1.6) that for every $\delta > 0$, $s \in \mathbb{R}$, $p \in (1,+\infty)$, $y \in X$ and $x \in X$ with $d(x,y) > \delta$

$$\|G(\cdot,y)\|_{s,p,B(x,\delta/2)} \leq C_{s,p,\delta} \exp(-\varepsilon d(x,y)).$$

The Sobolev embedding theorem implies now that the required estimate (2.5) is satisfied if $\beta = 0$. Now the same reasoning can be applied with respect to $y$ because we can use the uniformly elliptic equation

$$A^t_y G(x,y) = 0 \text{ if } x \neq y$$

where $A^t$ is the formally transposed operator to $A$ defined by the equality

$$\langle Au,v \rangle = \langle u, A^t v \rangle, \ u,v \in C^\infty_0(X),$$

where

$$\langle f, g \rangle = \int_X f(x) g(x) dx,$$

$dx$ is the Riemannian density on $X$. This immediately leads to the estimates (2.5). $\square$

Actually estimates (2.5) prove to be adequate only in case of subexponential growth of the volume of the balls on $X$. For the case of exponential growth stronger estimates in terms of $L^p$-norms are available.

**Theorem 2.3.** *Let $p \in (1, +\infty)$ and $\lambda \in \mathbb{C} \backslash \sigma_p(A)$ be fixed, $G = G(x, y)$ the Green function. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ and for every multiindices $\alpha, \beta$ there exists $C_{\alpha\beta\delta} > 0$ such that*

$$(2.10) \qquad \int_{x : d(x,y) \geq \delta} |\partial_x^\alpha \partial_y^\beta G(x,y)|^p \exp(\varepsilon d(x,y)) dx \leq C_{\alpha\beta\delta}$$

$$(2.11) \qquad \int_{y : d(x,y) \geq \delta} |\partial_x^\alpha \partial_y^\beta G(x,y)|^{p'} \exp(\varepsilon d(x,y)) dy \leq C_{\alpha\beta\delta},$$

*where $1/p' + 1/p = 1$, the derivatives and absolute values are understood as in Theorem 2.2.*

*Proof.* We should just return to (2.9) but use it a little differently. Namely, using the same reasoning as in the proof of Theorem 2.2 we can evidently conclude from (2.9) that for every $s \in \mathbb{R}$

$$\sum_{j=1}^\infty \|G(\cdot, y)\|_{s,p,B(x_j,\delta/2)}^p \exp(\varepsilon d(x_j, y)) < \infty$$

where $x_j$ are chosen as in Lemma 1.2 (with $\varepsilon$ replaced by $\delta$ there). Then (2.10) obviously follows. To prove (2.11) we should apply (2.10) to the transposed operator $A^t$. $\square$

We need also uniform local estimates of the Green function near the diagonal but the simplest way to obtain them is in a use of pseudo–differential operators. This will be done in the next Section.

## A1.3. Uniform properly supported pseudo–differential operators and structure of inverse operators.

We shall introduce here classes of uniform properly supported pseudo–differential operators on a manifold $X$ of bounded geometry which coincide locally with well–known Hörmander classes $\Psi^m$ and $\Psi^m_{phg}$ ([22], vol. 3). Such classes were inroduced first on Lie groups in [29] and later in the general case in [28]

DEFINITION 3.1. $U\Psi^{-\infty}(X)$ is a class of all operators $R$ with a L. Schwartz kernel $K_R \in C^\infty(X \times X)$ satisfying the following conditions

  (i) there exists $C_R > 0$ such that $K_R(x,y) = 0$ if $d(x,y) > C_R$;
  (ii) $|\partial_x^\alpha \partial_y^\beta K_R(x,y)| \le C_{\alpha\beta}$, $x, y \in X$, where the derivatives are taken in canonical coordinates.

The class $U\Psi^{-\infty}(X)$ will serve as a class of negligible operators in our context. Notice that an operator $R \in U\Psi^{-\infty}(X)$ is not necessarily compact e.g. in $L^2(X)$.

In the next definition we fix $r \in (0, r_{inj})$ as was already done before.

DEFINITION 3.2. $U\Psi^m(X)$ is a class of all operators $A : C_0^\infty(X) \to C_0^\infty(X)$ satisfying the following conditions:

  (i) there exists $C_A > 0$ such that $K_A(x,y) = 0$ if $d(x,y) > C_A$ (here $K_A$ is the L. Schwartz kernel of $A$);
  (ii) let $B(x_0, r)$ be a ball on $X$, then in canonical coordinates on $B(x_0, r)$ the operator

$$A_{x_0} = A|_{C_0^\infty(B(x_0,r))} : \ C_0^\infty(B(x_0,r)) \to C^\infty(B(x_0,r)),$$

$$u \mapsto Au|_{B(x_0,r)}$$

can be written as

(3.1)
$$A_{x_0} = a_{x_0}(x, D_x) + R_{x_0}$$

where $a_{x_0} \in S^m$ uniformly with respect to $x_0$, i.e.

$$|\partial_\xi^\alpha \partial_x^\beta a_{x_0}(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}$$

with $C_{\alpha\beta}$ which do not depend on $x_0$, and $R_{x_0}$ is an operator with a L. Schwartz kernel $K_{R_{x_0}} \in C^\infty(B(x_0,r) \times B(x_0,r))$ satisfying the following estimates

$$|\partial_x^\alpha \partial_y^\beta K_{R_{x_0}}(x,y)| \le C'_{\alpha\beta}$$

with constants $C'_{\alpha\beta}$ which do not depend on $x_0$.

DEFINITION 3.3. $U\Psi_{phg}^m(X)$ is a class of operators $A \in U\Psi^m(X)$ which have polyhomogeneous local symbols $a_{x_0}(x,\xi)$ with uniform estimates of homogeneous terms in local representations (3.1). More exactly it is required that there exist $a_{x_0,j} = a_{x_0,j}(x,\xi)$, $j = 0,1,2,\ldots$, such that the following conditions are satisfied:

(i) $a_{x_0,j}(x,\xi)$ is defined when $x \in B(x_0,r), \xi \ne 0$ and is homogeneous of degree $m - j$ with respect to $\xi$, i.e.

$$a_{x_0,j}(x,t\xi) = t^{m-j}a_{x_0,j}(x,\xi), \quad x \in B(x_0,r), \quad \xi \in \mathbb{R}^n \backslash 0, t > 0;$$

(ii) $a_{x_0,j} \in C^\infty$ when $\xi \ne 0$ and $|\partial_\xi^\alpha \partial_x^\alpha a_{x_0,j}(x,\xi)| \le C_{\alpha\beta j}$ when $x \in B(x_0,r)$ and $|\xi| = 1$ with the constants $C_{\alpha\beta j}$ which do not depend on $x_0$;

(iii) let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(\xi) = 1$ when $\xi$ is close to 0, and $\chi$ is fixed, then for every $N, \alpha, \beta, x_0$

$$|\partial_\xi^\alpha \partial_x^\beta [a_{x_0}(x,\xi) - \Sigma_{j=0}^{N-1}(1-\chi(\xi))a_{x_0,j}(x,\xi)]| \le C_{\alpha\beta N}(1+|\xi|)^{m-N}$$

with $C_{\alpha\beta N}$ which do not depend on $x_0$.

So the classes $U\Psi^m, U\Psi_{phg}^m$ are just usual Hörmander classes of properly supported pseudo–differential operators but with appropriate uniformity conditions.

The classes $U\Psi^m, U\Psi^m_{phg}$ are defined for all $m \in \mathbb{R}$. The class $U\Psi^m_{phg}(X)$ can be defined also for $m \in \mathbb{C}$ as a class of operators $A \in U\Psi^{\mathrm{Re}\ m}(X)$ such that the conditions (i), (ii), (iii) of Definition 3.3 are satisfied if we replace $m$ by Re $m$ in (iii).

The usual algebraic and continuity properties are satisfied for the classes $U\Psi^m(X)$, $U\Psi^m_{phg}(X)$.

In particular the following statements are easily checked:

(a) if $A_j \in U\Psi^{m_j}(X)$, $j = 1,2$, then $A_1 A_2 \in U\Psi^{m_1+m_2}(X)$; the same is true for the classes $U\Psi^m_{phg}(X)$;

(b) if $A \in U\Psi^m(X)$ (or $U\Psi^m_{phg}(X)$) then $A^* \in U\Psi^m(X)$ (resp. $U\Psi^{\bar{m}}_{pgh}(X)$ where $\bar{m}$ is complex conjugate to $m$).

(c) if $A \in U\Psi^m(X)$ then $A$ defines for every $s \in \mathbb{R}$, $p \in (1, +\infty)$ a continuous linear operator

$$A : W^s_p(X) \to W^{s-m}_p(X)$$

**Proposition 3.4.** *Let $A$ be a $C^\infty$-bounded uniformly elliptic differential operator of order $m$ on $X$. Then there exists $B \in U\Psi^{-m}_{phg}(X)$ such that $I - AB, I - BA \in U\Psi^{-\infty}(X)$.*

*Proof.* The operator $B$ with required properties is easily constructed by use of inform local parametrices $B_i$ for $A$ in the balls $B(x_i, \varepsilon)$ from Lemma 1.2 and then patching them up by the formula

$$B = \Sigma_i \Psi_i B_i \Phi_i,$$

where $\Phi_i, \Psi_i$ are multiplication operators $\Phi_i u(x) = \varphi_i(x) u(x)$, $\Psi_i u(x) = \psi_i(x) u(x)$, $\varphi_i$ is taken from the partition of unity of Lemma 1.3, $\psi_i \in C^\infty_0(B(x_i, 2\varepsilon))$ are chosen to be uniformly $C^\infty$-bounded and such that $\psi_i(x) = 1$ in a neighbourhood of supp $\varphi_i$. $\square$

REMARK 3.5. Choosing $\varepsilon > 0$ sufficiently small we can obtain the parametrix $B$ with a L. Schwartz kernel $K_B$ with

$$\mathrm{supp}\ K_B \subset \{(x,y)|d(x,y) < \varepsilon_1\}$$

where $\varepsilon_1 = \varepsilon_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now we can describe the structure of the operator $(A - \lambda I)^{-1}$ in case $\lambda \notin \sigma_p(A)$ more precisely.

First note that all the definitions and statements of this Section can be easily generalized to operators acting in spaces of sections of vector bundles of bounded geometry on $X$. The corresponding classes of operators $A : C_0^\infty(X, E) \to C_0^\infty(X, F)$ will be denoted $U\Psi^{-\infty}(X; E, F)$, $U\Psi^m(X; E, F)$, $U\Psi_{phg}^m(X; E, F)$ or $U\Psi^{-\infty}(X, E)$ etc. in case $E = F$.

**Theorem 3.6.** *Let $A : C_0^\infty(X, E) \to C_0^\infty(X, F)$ be a uniformly elliptic $C^\infty$–bounded differential operator of order $m$. Let the closure of $A$ in $L^p(X, E)$ have an everywhere defined bounded inverse $A^{-1}$. Then there exists $\varepsilon > 0$ and a representation:*

$$(3.2) \qquad A^{-1} = B + T,$$

*where $B \in U\Psi_{phg}^{-m}(X; F, E)$, $T$ has a L. Schwartz kernel $K_T \in C^\infty$ satisfying the following estimates*

$$(3.3) \qquad |\partial_x^\alpha \partial_y^\beta K_T(x, y)| \le C_{\alpha\beta} \exp(-\varepsilon d(x, y)).$$

*Also*

$$(3.3') \qquad \int_X |\partial_x^\alpha \partial_y^\beta K_T(x, y)|^p \exp(\varepsilon d(x, y)) dx \le C_{\alpha\beta}$$

$$(3.3'') \qquad \int_X |\partial_x^\alpha \partial_y^\beta K_T(x, y)|^{p'} \exp(\varepsilon d(x, y)) dy \le C_{\alpha\beta}$$

*where $1/p' + 1/p = 1$. Here the derivatives and the norm in the left–hand side are taken with respect to the canonical coordinates and canonical trivializations of $E$ and $F$.*

*Proof.* For the sake of simplicity of notations we shall consider the case of trivial $E = F = X \times \mathbb{R}$. It follows from Proposition 3.4 that there exists $B \in U\Psi_{phg}^{-m}(X)$ such that

$$AB = I - R,$$

where $R \in U\Psi^{-\infty}(X)$. Multiplying by $A^{-1}$ from the left we obtain (3.2) with $T = A^{-1}R$. Now it is clear that

$$(3.4) \qquad K_T(x, y) = [A^{-1}K_R(\cdot, y)](x).$$

Notice that $K_R(\cdot, y) \in C_0^{\infty}(X)$ and supp $K_R(\cdot, y) \subset B(y, r_0)$ for some $r_0 > 0$ which does not depend on $y$. Hence it follows from (3.4) and Theorem 2.1 that the estimates (3.3) are fulfilled if $d(x, y) \geq r_0$ with $r_0 > 0$ arbitrarily small so the estimates (3.3) are proved outside $\delta$–neighbourhood of the diagonal for every $\delta > 0$.

Now we have to prove (3.3) in the set

$$\{(x, y)|d(x, y) < \delta\}$$

where $\delta > 0$ can be chosen arbitrarily small. But then (3.3) reduces to the boundedness of all derivatives which follows from the Sobolev embedding theorem and the boundedness of the operator

$$A^{-1}: \ W_p^s(X) \to W_p^{s+m}(X)$$

for every $s \in \mathbb{R}$ which is due to the regularity properties (Lemma 1.4) and the closed graph theorem.

Now to prove (3.3$'$) we use (3.4) again but apply the boundedness of $A_{\varepsilon,y}^{-1}$ instead of Theorem 2.2 itself. The estimate (3.3$''$) is proved by applying the same arguments to $A^t$ instead of $A$. $\square$

Now we can prove estimates of the Green function near the diagonal.

**Theorem 3.7.** *Let $A, p, \lambda$ satisfy the conditions of Theorem 2.2, $G$ be the Green function (the L. Schwartz kernel of $(A - \lambda I)^{-1}$). Then there exists $\varepsilon > 0$ such that*

$$(3.5) \quad |\partial_x^\alpha \partial_y^\beta G(x, y)| \leq C_{\alpha\beta} d(x, y)^{m-n-|\alpha|-|\beta|} \exp(-\varepsilon d(x, y))$$

provided $m < n$;

$$(3.6) \qquad \begin{aligned} &|\partial_x^\alpha \partial_y^\beta G(x,y)| \leq \\ &C_{\alpha\beta}[1 + d(x,y)^{m-n-|\alpha|-|\beta|}|\log d(x,y)|]\exp(-\varepsilon d(x,y)) \end{aligned}$$

provided $m \geq n$.

*Proof.* As usual we shall consider the scalar case. Due to Theorem 2.2 it is sufficient to prove (3.5) and (3.6) for $x, y \in X$ such that $d(x,y) \leq \delta$ with some fixed $\delta > 0$. Let us consider the representation (3.2). Clearly the L. Schwartz kernel $K_T$ satisfies the required estimates due to (3.3). Now we have to consider $K_B$ and to do this let us present $B$ locally in $B(x_0, r)$ in the form (3.1)

$$B_{x_0} = b_{x_0}(x, D_x) + R_{x_0}$$

where the L. Schwartz kernel of $R_{x_0}$ satisfies the required estimates and $b_{x_0} = b_{x_0}(x, \xi)$ is a polyhomogeneous symbol with uniform estimates. The L. Schwartz kernel of $b_{x_0}(x, D_x)$ in local canonical coordinates near $x_0$ is equal to

$$K_{x_0}(x, y) = F_{\xi \to x-y} b_{x_0}(x, \xi) = (2\pi)^{-n} \int b_{x_0}(x, \xi) e^{i\langle x-y, \xi \rangle} d\xi$$

so to prove the necessary estimates it is sufficient to use the well known properties of the Fourier transform of homogeneous functions or their appropriate distributional regularizations (see e.g. [22], vol. 1). $\square$

REMARK 3.8. Most part of the results described here can be generalized to pseudo–differential operators. Namely, Theorem 3.6 is true for uniformly elliptic pseudo–differential operators $A \in U\Psi_{phg}^m(X; E, F)$ if $m > 0$. Also if $A \in U\Psi^m(X; E, F)$ is uniformly elliptic in appropriate sense (see [29] for the case of Lie

groups) then the statement of Theorem 3.6 is true with $B \in U\Psi^{-m}(X; F, E)$. So Theorem 3.7 is also true in the case $A \in U\Psi^m_{phg}(X; E, F)$ if $m > 0$ (the estimate (3.5) will be true when $m < n$ or $m - n \notin \mathbb{Z}$).

In fact it is not necessary to consider only pseudo–differential operators which are properly supported. Everything is true e.g. for the operators like the right–hand side in (3.2) i.e. for the operators of the form $A = A_0 + T$, where $A_0 \in U\Psi^m_{phg}(X; E, F)$ and $T$ satisfies some decay conditions as in the formulation of Theorem 3.6. Moreover the requirement of exponential decay of the kernel off the diagonal can also be relaxed if the volume of balls on $X$ grows even more slowly. The corresponding machinery was developed in [29] for Lie groups and is perfectly suitable for general manifolds of bounded geometry so we omit the details.

## Chapter 2. Eigenfunctions and spectra.

### 2.1. Generalized eigenfunctions.

Let $X$ be a manifold of bounded geometry which we shall suppose to be connected for the sake of simplicity, $\dim X = n$, and $E$ a complex vector bundle of bounded geometry on $X$. We shall always suppose that $E$ is provided with an hermitian scalar product of bounded geometry on fibers. In particular the Hilbert space of sections $L^2(X, E)$ is well defined. We shall construct a special Hilbert–Schmidt rigging of this space, hence its negative space will contain a complete orthonormal system of generalized eigenfunctions of any self–adjoint operator (see Appendix 2 after this Chapter). In the elliptic case additional regularity properties of these generalized eigenfunctions will be proved.

Denote $V_x(r) = \mathrm{Vol}B(x, r)$, $V(r) = \sup_{x \in X} V_x(r)$. Lemma 4.4 from Chapter 1 immediately implies that there exists $a > 0$ such that

$$(1.1) \qquad V(r) \leq e^{ar}.$$

Also both $V_x(r), V(r)$ are increasing functions on $[0, \infty)$ with values in $[0, \infty)$, positive on $(0, \infty)$. The reasoning in the proof of Lemma 4.4, Ch.1 shows that there exists $C > 0$ such that

$$(1.2) \qquad V_x(r+1) \leq C V_x(r), \ r \geq 1, \ x \in X.$$

Taking supremum over $x \in X$ on both sides we obtain

$$(1.2') \qquad V(r+1) \leq C V(r), \ r \geq 1.$$

with the same constant $C$. Hence (again with the same constant $C > 0$) we obtain

$$(1.3) \qquad C^{-1} V_x(r) \leq V_x(\rho) \leq C V_x(r) \text{ if } \rho \in [r-1, r+1],$$

$$(1.3') \qquad C^{-1} V(r) \leq V(\rho) \leq C V(r) \text{ if } \rho \in [r-1, r+1].$$

REMARK 1.1. It is not always possible to estimate $V_x(r)$ from below by $C^{-1} V(r)$ whatever $C > 0$. For example, if we take a manifold $X$ which is diffeomorphic to $\mathbb{R}^n$ with coordinates $x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n)$ with the hyperbolic metric $x_n^{-2}(dx'^2 + dx_n^2)$ in $\{x | x_n \geq 1\}$ and the euclidean metric $dx'^2 + dx_n^2$ in $\{x | x_n \leq -1\}$ with a smooth transition in $\{x | -1 \leq x_n \leq 1\}$ making $X$ a manifold of bounded geometry then $V_x(r)$ for a fixed $r$ varies at least between volumes of the euclidean and the hyperbolic ball of radius $r$ (the first one being $0(r^n)$ and the second growing exponentially as $r \to +\infty$).

**Lemma 1.2.** *There exist increasing $C^\infty$ functions $\tilde{V} : [0, \infty) \to (0, \infty)$, $\tilde{V}_x : [0, \infty) \to (0, \infty)$ such that*

$$(1.4) \qquad C^{-1} V_x(r) \leq \tilde{V}_x(r) \leq C V_x(r), \ r \geq 1,$$

(1.4')     $$C^{-1}V(r) \leq \tilde{V}(r) \leq CV(r), \ r \geq 1,$$

*with the same constant C as in (1.2), (1.2'), (1.3), (1.3'). Besides*

(1.5)     $$|\partial_r^k \tilde{V}_x(r)| \leq C_k \tilde{V}_x(r), \ |\partial_r^k \tilde{V}(r)| \leq C_k \tilde{V}(r)$$

*for every $k = 0, 1, 2, \ldots$.*

*Proof.* Let us extend $V_x(r), V(r)$ by 0 on $(-\infty, 0)$ and then take

$$\tilde{V}_x(r) = \int V_x(r+s)\varphi(s)ds, \ \tilde{V}(r) = \int V(r+s)\varphi(s)ds,$$

where $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geq 0$, $\int \varphi(s)ds = 1$ and supp $\varphi \subset [-1/4, 1/4]$. The estimates (1.4), (1.4'), (1.5) now obviously follow from (1.3), (1.3'). Also $\tilde{V}_x, \tilde{V}$ are increasing due to the same property of $V_x, V$. $\square$

Now let us define positive weight $C^\infty$ functions

(1.6)     $$f_{x_0}(x) = \tilde{V}_{x_0}(\tilde{d}(x_0, x)), \ f(x) = \tilde{V}(\tilde{d}(x_0, x)),$$

where $\tilde{d}$ is the smoothed distance–function constructed in Lemma 2.1 of Appendix 1.

**Lemma 1.3.** *In canonical coordinates*

(1.7)     $$|\partial^\alpha f_{x_0}(x)| \leq C_\alpha f_{x_0}(x), \ |\partial^\alpha f(x)| \leq C_\alpha f(x), \ x \in X$$

*with constants $C_\alpha$ which do not depend on x.*

*Proof.* The estimates (1.7) obviously follow from (1.5), the "derivative of composition formula", e.g.

$$(1.8) \quad \partial^\alpha f(x) =$$

$$\sum_{\substack{\alpha_1+\ldots+\alpha_k=\alpha \\ |\alpha_j|>0}} c_{\alpha_1,\ldots,\alpha_k}(\partial_r^k \tilde{V})(\tilde{d}(x_0,x))\partial_x^{\alpha_1}\tilde{d}(x_0,x)\ldots\partial_x^{\alpha_k}\tilde{d}(x_0,x),$$

and boundedness of the derivatives $\partial^\alpha \tilde{d}(x_0,x)$ for $|\alpha| > 0$ (see Lemma 2.1 in Appendix 1). $\square$

Now change $f_{x_0}, f$ to real powers of these functions.

**Lemma 1.4.** *For any $t \in \mathbb{R}$ in canonical coordinates*

$$(1.9) \qquad |\partial^\alpha f_{x_0}^t(x)| \leq C_{\alpha,t} f_{x_0}^t(x), \; |\partial^\alpha f^t(x)| \leq C_{\alpha,t} f^t(x).$$

*Proof.* Using "derivative of composition formula" like (1.8) we obtain e.g.

$$(1.10)$$
$$\partial^\alpha f^t(x) = \sum_{\substack{\alpha_1+\ldots+\alpha_k=\alpha \\ |\alpha_j|>0}} c_{\alpha_1,\ldots,\alpha_k} f^{t-k}(x)\partial^{\alpha_1}f(x)\ldots\partial^{\alpha_k}f(x)$$

and (1.9) follows from Lemma 1.3. $\square$

**Lemma 1.5.** *If $t > 1/2$ then $f_{x_0}^{-t}, f^{-t} \in L^2(X)$. Also*

$$f_{x_0}^{-1/2}(\log f_{x_0})^{-1/2-\varepsilon}, f^{-1/2}(\log f)^{-1/2-\varepsilon} \in L^2(X)$$

*for every $\varepsilon > 0$.*

*Proof.* Let us fix $t > 1/2$. We clearly have due to Lemma 1.2

$$\int_{x:d(x,x_0)\geq 1} f_{x_0}^{-2t}(x)dx \leq C_1 \int_1^\infty V_{x_0}^{-2t}(r)dV_{x_0}(r) =$$

$$C_1 \int_{V_{x_0}(1)}^\infty \lambda^{-2s}d\lambda < \infty.$$

with a consant $C_1 > 0$. Hence $f_{x_0}^{-t} \in L^2$. Now $V_x(r) \leq V(r)$, therefore $V^{-t}(r) \leq V_{x_0}^{-t}(r)$ and $f^{-t}(x) \leq C_2 f_{x_0}^{-t}(x)$. Hence $f^{-t} \in L^2(X)$. Other inclusions are checked similarly. $\square$

Now let $g : X \to (0, \infty)$ be a positive $C^\infty$–function such that

$$(1.10) \qquad |\partial^\alpha g(x)| \leq C_\alpha g(x), \ x \in X.$$

Examples of such functions are $f_{x_0}^t, f^t$ due to Lemma 1.4. We could also take $f_{x_0}^t (\log f_{x_0})^{t_1}$ or $f^t (\log f)^{t_1}$ with $t, t_1 \in \mathbb{R}$.

Now let us define the weighted Sobolev space $H_g^s = H_g^s(X, E)$ with $s \in \mathbb{R}$ as follows

$$H_g^s(X, E) = \{u | u \in \mathcal{D}'(X, E), \ gu \in H^s(X, E)\},$$

where

$$H^s(X, E) = W_2^s(X, E)$$

is the uniform Sobolev space defined in Sect. A1 of Appendix 1.

Clearly $H_g^s(X, E) \supset \overset{\circ}{C}_0^\infty(X, E)$, hence $H_g^s(X, E)$ continuously included and dense in $L^2(X, E)$ provided $s \geq 0$ and $g(x) \geq g_0 > 0$. Therefore in this case we can use $H_g^s$ as a positive space to construct a rigging of $L^2(X, E)$.

**Lemma 1.6.** *If we use $H_g^s(X, E)$ with $s \geq 0$ and $g(x) \geq g_0 > 0$ as a positive space to construct a rigging of $L^2(X, E)$ then the corresponding negative (dual) space will be equal to $H_{g^{-1}}^{-s}(X, E)$*

*Proof.* Denote $\mathcal{H}_+ = H_g^s(X, E)$. Then in the notations of Appendix 2 we obviously have:

$$\mathcal{H}_- = \{u | u \in \mathcal{D}'(X, E), \ g^{-1}u \in H^{-s}(X, E)\} = H_{g^{-1}}^{-s}(X, E)$$

due to the standard duality by $H^s(X, E)$ and $H^{-s}(X, E)$. $\square$

**Proposition 1.7.** *Suppose that $s > n/2$, $g \in C^\infty(X)$ satisfies (1.10), $g(x) \geq g_0 > 0$ and $g^{-1} \in L^2(X)$. Then the rigging of $\mathcal{H} = L^2(X, E)$ with the positive space $\mathcal{H}_+ = H^s_g(X, E)$ is a Hilbert–Schmidt rigging.*

*Proof.* Choosing an elliptic pseudo–differential operator $B \in U\Psi^{s/2}_{phg}(X, E)$ we may take $A = I + B^*B \in U\Psi^s_{phg}(X, E)$ which will be elliptic invertible self–adjoint operator of order $s$. Hence $u \in H^s(X, E)$ if and only if $u \in L^2(X, E)$ and $Au \in L^2(X, E)$. Now obviously $H^s(X, E) = \mathrm{Im}(\bar{A}^{-1})$, where $\bar{A}$ is the self–adjoint operator defined by $A$ on $L^2(X, E)$ with the domain $D(\bar{A}) = H^s(X, E)$. Hence

$$H^s_g(X, E) = \{g^{-1}\bar{A}^{-1}u \,|\, u \in L^2(X, E)\} = g^{-1}\bar{A}^{-1}L^2(X, E).$$

Therefore it is sufficient to establish that $g^{-1}\bar{A}^{-1}$ is a Hilbert–Schmidt operator. But his Schwartz kernel is given by

$$K(x, y) = g^{-1}(x)G(x, y)$$

where $G(\cdot, \cdot)$ is the Schwartz kernel of $\bar{A}^{-1}$ (or the Green function of $A$). Now we can use Theorems 2.3 and 3.7 from Appendix 1 to conclude that

$$\int_X |G(x, y)|^2 dy \leq C < \infty$$

It follows that

$$\int_{X \times X} |K(x, y)|^2 dx dy \leq C \int_X g^{-2}(x) dx < \infty$$

hence $g^{-1}\bar{A}^{-1}$ is a Hilbert–Schmidt operator. $\square$

Now applying Theorem 2.3 from Appendix 2 we immediately obtain

**Theorem 1.8.** *Suppose that $s > n/2$ and $g$ satisfies the conditions in Proposition 1.7. Then for any self–adjoint operator $A$ in $L^2(X, E)$ the space $H_{g^{-1}}^{-s}(X, E)$ contains complete orthonormal system of generalized eigenfunctions of $A$ in the sense of Definition 2.2 of Appendix 2.*

**Corollary 1.9.** *For any $\varepsilon > 0, \delta > 0$ both spaces $H_{f^{-1/2-\delta}}^{-n/2-\varepsilon}(X, E)$,*

*$H_{f^{-1/2}(\log f)^{-1/2-\delta}}^{-n/2-\varepsilon}$ contain complete orthonormal system of generalized eigenfunctions of any self–adjoint operator $A$ in $L^2(X, E)$.*

REMARK 1.10. Using the composition formula for pseudo–differential operators of classes $U\Psi^m$ we can describe the space $H_g^s(X, E)$ also in a dual way as the space of all $u \in \mathcal{D}'(X, E)$ such that $gBu \in L^2(X, E)$ for every $B \in U\Psi^s(X, E)$. If $s \in \mathbb{Z}_+$ then we can equivalently write $g\partial^\alpha u \in L^2(X, E)$ for every multiindex $\alpha$ with $|\alpha| \leq s$ (here $\partial^\alpha u$ can be taken in canonical coordinates for any piecewise constant choice of such coordinates induced by coverings described in Lemma 1.2 of Appendix 1). Using this description we can skip the requirement of smoothness of $g$ and estimates (1.10) defining e.g. $H_{f^t}^s$ for $s \in \mathbb{Z}_+$ as the space of sections $u \in L^2(X, E)$ such that $[1 + V(d(\cdot, x_0))]^t \partial^\alpha u \in L^2(X, E)$ for every $\alpha$ with $|\alpha| \leq s$. Hence the dual space $H_{f^{-t}}^{-s}$ consists of distributions which have the form

$$v = \sum_{\substack{k \leq s \\ X_1 \dots X_k}} X_1 \dots X_k [(1 + V(d(\cdot, x_0)))^t v_\alpha], \quad v_\alpha \in L^2(X, E),$$

where $X_1, \dots, X_s$ are first–order uniformily $C^\infty$–bounded differential operators in $C^\infty(X, E)$, the sum is taken over a finite set of such tuples $X_1, \dots, X_k$ with $k \leq s$. Similarly for general $s > 0$ the space $H_{f^{-t}}^{-s}$ consists of sections $u \in \mathcal{D}'(X, E)$ of the form

$$u = \sum_{j=1}^{N} B_j [(1 + V(d(\cdot, x_0))^t v_j], \quad v_j \in L^2(X, E), \quad B_j \in U\Psi^s(X, E)$$

where $N$ and the set $B_1, \ldots, B_N$ depend on $u$.

**EXAMPLE 1.11.** If $X = \mathbb{R}^n$ with the standard euclidean metric then $V(d(x,0)) = c_n|x|^n$ and for any $\varepsilon > 0$, $\delta > 0$ we can take the space $H^{-n/2-\varepsilon}_{(1+|x|^2)^{-n/4-\delta}}(\mathbb{R}^n)$ as the negative space containing a complete orthonormal space of generalized eigenfunctions of any self–adjoint operator in $L^2(\mathbb{R}^n)$.

**EXAMPLE 1.12.** If $X = \mathbb{H}^n$ is the hyperbolic space with the curvature $-1$ then $V_x(r) = V(r) \sim c_n e^{(n-1)r}$ as $r \to \infty$. Let us denote $|x| = d(x,0)$, where $0$ is a fixed point in $\mathbb{H}^n$, and choose a positive $C^\infty$–function $x \mapsto \langle x \rangle$ coinciding with $|x|$ if $|x| \geq 1$. Then for any $\varepsilon > 0$, $\delta > 0$ we can take one of the spaces $H^{-n/2-\varepsilon}_{\exp(-(n-1+\delta)\langle x \rangle/2)}(\mathbb{H}^n)$ or $H^{-n/2-\varepsilon}_{\exp(-(n-1)\langle x \rangle/2)\langle x \rangle^{-1/2-\delta}}(\mathbb{H}^n)$ as the desired negative space for any self–adjoint operator in $L^2(\mathbb{H}^n)$.

Now suppose that we consider not a general self–adjoint operator but a uniformly elliptic $C^\infty$–bounded self–adjoint differential operator $A : C^\infty(X, E) \to C^\infty(X, E)$. Then we can use local a priori estimates to increase $-n/2 - \varepsilon$ up to any $s$. Actually any generalized eigenfunction will be a solution of a uniformly elliptic equation, hence it should be a $C^\infty$–function (or rather $C^\infty$–section). Hence we arrive to the following

**Theorem 1.13.** *Let* $A : C^\infty(X, E) \to C^\infty(X, E)$ *be a* $C^\infty$–*bounded uniformly elliptic self–adjoint operator. Let* $g$ *be a positive* $C^\infty$–*function on* $X$, *satisfying (1.10), such that* $g^{-1} \in L^2(X)$. *Then there exist a complete orthonormal system of eigenfunctions for* $A$, *such that any eigenfunction* $\psi$ *in this system satisfies the following estimates*

$$(1.11) \qquad \int_X |\partial^\alpha \psi(x)|^2 g^{-2}(x) < \infty, \ x \in X,$$

*for any multiindex* $\alpha$.

Now using locally (on balls of a fixed radius) the Sobolev imbedding theorem we obtain

**Corollary 1.14.** *Under the conditions of Theorem 1.13 there exists a complete orthonormal system of eigenfunctions such that any eigenfunction $\psi$ in this system satisfies estimates*

$$(1.12) \qquad |\partial^\alpha \psi(x)| \leq C_\alpha g(x).$$

REMARK 1.15. Clearly $g$ here can be replaced by a positive function $g_1$ such that

$$C^{-1} g(x) \leq g_1(x) \leq C g(x)$$

with a constant $C > 0$. In particular both Theorem 1.13 and Corollary 1.14 remain true if we replace $g$ by one of the following functions:

$$[1 + V(d(\cdot, x_0))]^{1/2+\varepsilon}, \quad [1 + V(d(\cdot, x_0))]^{1/2} \ \log[2 + V(d(\cdot, x_0))]^{1+\varepsilon},$$

where $\varepsilon > 0$.

## 2.2. Schnol–type theorems.

In the previous section we gave a sufficient condition for a space to contain a complete orthonormal system of generalized eigenfunctions for a self–adjoint operator. The corresponding eigenvalues then will be in the spectrum of this operator (at least almost everywhere) and actually the closure of the set of these eigenvalues constitutes the spectrum in $L^2$. In this section we will consider an opposite question: assume that for some $\lambda \in \mathbb{C}$ we know a solution $\psi$ of the equation $A\psi = \lambda\psi$ satisfying some estimates at infinity; when can we conclude that $\lambda$ is in the spectrum $\sigma(A)$ of the operator $A$ in $L^2$?

An example of the sort is the well known Schnol theorem ([38], [11]) which (with some simplifying restrictions) states that if $A = -\Delta + q(x)$ is a Schrödinger operator in $L^2(\mathbb{R}^n)$ with the potential $q \in L^\infty_{loc}(\mathbb{R}^n)$ such that $q(x) \geq -C$ for all $x \in \mathbb{R}^n$ and there exists a non–trivial solution $\psi$ of the equation $A\psi = \lambda\psi$ such that for every $\varepsilon > 0$

$$\psi(x) = O(\exp(\varepsilon|x|))$$

then $\lambda \in \sigma(A)$. Another Schnol theorem ([38]) also concerning the Schrödinger operator states that if the negative part $q_-(x) = \min(0, q(x))$ satisfies the estimate

$$q_-(x) = o(|x|^2)$$

then the existence of a non–trivial polynomially bounded solution (i.e. a solution $\psi$ such that $\psi(x) = 0((1+|x|)^N)$ with some $N > 0$) for the equation $A\psi = \lambda\psi$ implies that $\lambda \in \sigma(A)$.

T. Kobayashi, K. Ono and T. Sunada ([26]) introduced.

DEFINITION 2.1. An operator $A$ satisfies the **weak Bloch property** (WBP) if the following implication is true:

{there exists a bounded $\psi \not\equiv 0$ such that $A\psi = \lambda\psi$} $\Longrightarrow \lambda \in \sigma(A)$

So each of the mentioned Schnol theorems implies that the Schrödinger operator on $\mathbb{R}^n$ with a locally bounded and semi–bounded below potential satisfies WBP.

On the other hand the Laplacian $\Delta$ of the standard Riemannian metric on the hyperbolic space $\mathbb{H}^n$ does not satisfy WBP because $\Delta 1 = 0$ but $0 \notin \sigma(\Delta)$.

It is natural to investigate the following WBP–problem: **describe classes of manifolds and operators which satisfy WBP.**

It is easy to notice that the WBP–problem is closely connected with the problem of coincidence of spectra of an operator in spaces $L^p(X)$ for different $p$: if all these spectra for $1 \leq p \leq \infty$ coincide then WBP evidently holds because if $\sigma_p(A)$ means the spectrum of $A$ in $L^p(X)$ then the existence of a non–trivial bounded solution $\psi$ of $A\psi = \lambda\psi$ implies that $\lambda \in \sigma_\infty(A)$ so $\lambda \in \sigma_2(A) = \sigma(A)$. The problem of the coincidence of spectra was considered on discrete metric spaces in [43] where it was pointed out that the coincidence follows from the exponential decay of the Green function off the diagonal provided the space has a subexponential growth

of the number of points lying in a ball of the radius $r$ as $r \to +\infty$. The exponential decay of the Green function off the diagonal was proved in [43] for some operators which were called pseudodifference operators, e.g. difference operators with a finite radius of action and bounded coefficients on discrete groups etc.

The same reasoning works also for continuous objects when the appropriate estimates of the Green function hold. Such estimates were obtained in [29] for uniformly elliptic operators on unimodular Lie groups and in [27],[28] on general manifolds of bounded geometry. It follows (though it was not noticed in [29] or [27],[28]) that the spectra of corresponding operators in $L^p(X)$ coincide for all $p \in (1, +\infty)$ provided the volumes of balls of radius $r$ grow subexponentially as $r \to +\infty$, and also that WBP is satisfied in this situation. The main ideas of this approach will be explained here in detail. The important point here is a use of some weighted Sobolev spaces with exponential weights. In [26] the authors used an entirely different method which is quite close to the original Schnol method (see also [11]). The WBP was proved in [26] for the Schrödinger operators with periodic potentials on Riemannian manifolds $X$ with a subexponential growth of volumes of balls and with a discrete group of isometries $\Gamma$ such that the orbit space $X/\Gamma$ is compact.

Now let $X$ be a complete connected Riemannian manifold, $d(x, y)$ be the Riemannian distance between $x$ and $y$, $x, y \in X$. Let $A$ be a differential operator on $X$. Denote by $\sigma(A)$ its spectrum in $L^2(X)$.

DEFINITION 2.2.

    **i)** The operator $A$ satisfies the **weak Schnol property** (WSP) if the existence of a non–trivial solution $\psi$ of the equation $A\psi = \lambda\psi$ satisfying an estimate of the form

$$|\psi(x)| = O(1 + d(x, x_0)^N)$$

    (with some $N > 0$ and a fixed $x_0$) implies that $\lambda \in \sigma(A)$.

    **ii)** The operator $A$ satisfies the **strong Schnol property** (SSP) if the following implication is true: if there exists

a non–trivial solution $\psi$ of the equation $A\psi = \lambda\psi$ such that for every $\varepsilon > 0$

$$|\psi(x)| = O(\exp(\varepsilon d(x, x_0)))$$

(with a fixed $x_0$) then $\lambda \in \sigma(A)$.

Clearly SSP implies WSP, and WSP implies WBP. We shall prove that if $X$ is a manifold of bounded geometry with a subexponential growth of volumes of balls and $A$ is a uniformly elliptic differential operator with $C^\infty$–bounded coefficients on $X$ then $A$ satisfies (SSP) and even stronger property: if for every $\varepsilon > 0$ there exists a non–trivial solution $\psi_\varepsilon$ of $A\psi_\varepsilon = \lambda\psi_\varepsilon$ with

(2.1) $$|\psi_\varepsilon(x)| = O(\exp(\varepsilon d(x, x_0))$$

(with a fixed $x_0$) then $\lambda \in \sigma(A)$. We even prove the following Theorem which does not require any subexponential growth conditions

**Theorem 2.3.** *Let $X$ be a manifold of bounded geometry, $E$ a vector bundle of bounded geometry on $X$,*

$$A: \ C^\infty(X, E) \to C^\infty(X, E)$$

*a uniformly elliptic $C^\infty$–bounded differential operator. Let $p \in (1, \infty)$, $\lambda \in \mathbb{C}$ and for every $\varepsilon > 0$ there exists $\psi_\varepsilon \in C^\infty(X, E)$ such that $A\psi_\varepsilon = \lambda\psi_\varepsilon$, $\psi_\varepsilon \not\equiv 0$ and*

(2.2) $$\psi_\varepsilon \exp(-\varepsilon d(\cdot, x_0)) \in L^p(X, E).$$

*Then $\lambda \in \sigma_p(A)$.*

Here $\sigma_p(A)$ means the spectrum of $A_{\min} = A_{\max}$ in $\tilde{L}^p(X, E)$ (see Sect. 1.4 in Ch. 1), $1 \leq p \leq \infty$.

Before proving Theorem 2.3 we will give its corollaries and particular cases.

DEFINITION 2.4. Let $X$ be a manifold of bounded geometry. We shall say that $X$ has a *subexponential growth* (or is a *manifold of subexponential growth*) if for every $\varepsilon > 0$

$$(2.3) \qquad V(r) = O(e^{\varepsilon r}), \; r \to \infty$$

where $V(\cdot)$ is introduced in Sect. 1.

**Corollary 2.5.** *Suppose that $X$ is a manifold of subexponential growth, $A$ is a uniformly elliptic $C^{\infty}$–bounded differential operator on $X$ and $\lambda \in \mathbb{C}$. Suppose that for every $\varepsilon > 0$ there exists $\psi_\varepsilon \not\equiv 0$ satisfying $A\psi_\varepsilon = \lambda\psi_\varepsilon$ and the estimate (2.1). Then $\lambda \in \sigma_p(A)$, $1 < p < \infty$. In particular (WBP), (WSP) and (SSP) are satisfied for $A$ in this case.*

*Proof.* To apply Theorem 2.3 we have to check that $\exp(-\varepsilon d(\cdot, x_0)) \in L^p(X)$ for any $\varepsilon > 0$, $1 \le p \le \infty$. This can be proved if we notice that (2.3) implies for any $\varepsilon > 0$, $\delta > 0$

$$\exp(-\varepsilon d(\cdot, x_0)) \le V^{-1-\delta}(d(\cdot, x_0)) \le V_{x_0}^{-1-\delta}(d(\cdot, x_0))$$

and then use the same reasoning as in the proof of Lemma 1.5. $\square$

Corollary 2.5 gives the same sufficient condition for $\lambda \in \sigma_p(A)$ to be true whatever $p \in (1, \infty)$. So we may expect that $\sigma_p(A)$ does not depend on $p$ in the case of subexponential growth. We shall prove this and even give some information about extremal cases $p = 1$ and $p = \infty$.

**Proposition 2.6.** *Let $X, A$ be as in Corollary 2.5 (in particular $X$ has a subexponential growth). Then the spectrum $\sigma_p(A)$ does not depend on $p \in (1, \infty)$. Moreover denoting this spectrum by $\sigma(A)$ we have*

$$(2.4) \qquad \sigma_1(A) \subset \sigma(A), \; \sigma_\infty(A) \subset \sigma(A).$$

*Proof.* For the sake of simplicity of notations let us consider the case of trivial bundle $E$ with the fiber $\mathbb{C}$. We have to prove that if $\lambda \in \mathbb{C} - \sigma_{p_0}(A)$ for some $p_0 \in (1, \infty)$ then $\lambda \notin \sigma_p(A)$ for all $p \in [1, \infty]$. Now we may also suppose that $\lambda = 0$.

Due to Theorem 3.7 of Appendix 1 we obtain for the Green function $G(\cdot, \cdot)$ (the L. Schwartz kernel of $A^{-1}$) that

$$\sup_y \int |G(x, y)| dx < \infty, \quad \sup_x \int |G(x, y)| dy < \infty.$$

Hence due to the well known Schur lemma (see e.g. Lemma 18.1.12 in [22], vol. 3) we obtain that the integral operator $G$ with the Schwartz kernel $G(\cdot, \cdot)$ can be extended to a linear bounded operator

$$G: \ L^p(M) \to L^p(M)$$

for every $p \in [1, \infty]$. Let us introduce for any $\varepsilon > 0$ a space $W_\varepsilon$ which contains functions $\varphi \in C^\infty(X)$ such that

$$|\partial^\alpha \varphi(x)| = O(\exp(-\varepsilon d(x, x_0)))$$

for every multiindex $\alpha$ (with the derivative $\partial^\alpha$ in canonical co-ordinates) and a chosen fixed $x_0 \in X$ (the condition does not depend on $x_0$). The subexponentiality condition clearly implies that $W_\varepsilon \subset L^p(X)$ for all $\varepsilon > 0$, $p \in [1, \infty]$ and moreover

$$(2.5) \qquad W_\varepsilon \subset \bigcap_{p \in [1, \infty]} \bigcap_{s \in \mathbb{R}} W_p^s(X), \ \varepsilon > 0.$$

Now it follows from Theorem 3.6 of Appendix 1 that $G$ maps $C_0^\infty(X)$ into $W_\varepsilon$ with some $\varepsilon > 0$. Evidently $AG = GA = I$ on $C_0^\infty(X)$. Note that the first equality implies that $A_x G(x, y) = \delta_y(x)$ and the second implies that $A^t G^t = I$ on $C_0^\infty(X)$, hence $A_y^t G(x, y) = \delta_x(y)$. Another important algebraic corollary is that $G^t A^t = I$ on $C_0^\infty(X)$.

Now it is easy to check that $AG = I$ on $L^p(X)$ for every $p \in [1, \infty]$ if $A$ is applied in the sense of distributions. In fact if $u \in L^p(X)$, $v \in C_0^\infty(X)$ then

$$\langle AGu, v \rangle = \langle Gu, A^t v \rangle = \langle u, G^t A^t v \rangle = \langle u, v \rangle,$$

hence $AGu = u$. It follows that $Gu \in D_p(A)$ where $D_p(A)$ is the domain of $A$ in $\tilde{L}^p(X)$. Hence $A : D_p(A) \to L^p(X)$ is surjective.

Let us prove that $GA = I$ on $D_p(A)$, $p \in [1, \infty]$. If $u \in D_p(A)$, $v \in C_0^\infty(X)$ then

$$\langle GAu, v \rangle = \langle Au, G^t v \rangle$$

due to the Fubini theorem. Note that $G^t v \in W_\varepsilon$ for some $\varepsilon > 0$. So it is enough to prove that

(2.6)          $\langle Au, \varphi \rangle = \langle u, A^t \varphi \rangle$, $u \in D_p(A)$, $\varphi \in W_\varepsilon$

Let us define a cut–off function

$$\chi_N(x) = \Sigma_{i=1}^N \varphi_i(x)$$

where $\varphi_i$ are the functions from the partition of unity of Lemma 1.3 in Appendix 1. It is clear that $\chi_N \in C_0^\infty(X)$, $0 \le \chi_N \le 1$ and for every compact $K \subset X$ there exists $N$ such that $\chi_N = 1$ in a neighbourhood of $K$. Moreover $|\partial^\alpha \chi_N| \le C_\alpha$ in canonical coordinates uniformly with respect to $N$.

Now we can begin with the equality

(2.7)          $\langle Au, \chi_N \varphi \rangle = \langle u, A(\chi_N \varphi) \rangle$, $u \in D_p(A)$, $\varphi \in W_\varepsilon$,

and try to take limit as $N \to \infty$ to obtain (2.6). Note that $(Au)\varphi \in L^1(X)$ due to (2.5), therefore $\lim_{N \to \infty} \langle Au, \chi_N \varphi \rangle = \langle Au, \varphi \rangle$ due to the dominated convergence theorem. The same reasoning can be applied to the right–hand side of (2.7) due to the estimates of derivatives of $\chi_N$, so we obtain (2.6).

We have proved that the operators $A : D_p(A) \to L^p(X)$ and $G : L^p(X) \to D_p(A)$ are mutually inverse as required. $\square$

Proposition 2.6 immediately implies that WBP holds under its conditions, i.e. if $X$ has a subexponential growth, $A$ is uniformly elliptic $C^\infty$–bounded operator on $X$, $\lambda \in \mathbb{C}$ and there exists $u \in L^\infty$, $u \not\equiv 0$ such that $Au = \lambda u$, then $\lambda \in \sigma_p(A)$, $1 < p < \infty$, because $\sigma_\infty(A) \subset \sigma_p(A)$. But Theorem 2.3 will give us a stronger result as mentioned in Corollary 2.5.

Corollary 2.5 and Proposition 2.6 were proved in the paper [44] which was inspired by the beautiful paper [26], though the paper [44] relied heavily on ideas contained in [43], [29] and [28]. Theorem 2.3 improves the results of [44] extending it to general manifolds of bounded geometry.

Now we are ready for the proof of the main theorem.

*Proof of Theorem 2.3.* Let us consider the scalar case and suppose that $\lambda = 0$. We should repeat arguments given in the proof of Proposition 2.6. Let us suppose that $0 \notin \sigma_p(A)$. Then we can construct the Green operator $G = A^{-1}$ which has a Schwartz kernel $G(\cdot, \cdot)$ satisfying estimates (2.10), (2.11) in Theorem 2.3 of Appendix 1.

Using the local a priori estimates it is easy to prove that (2.2) implies the same inclusion for derivatives of $\psi_\varepsilon$:

$$(2.8) \qquad |\partial^\alpha \psi_\varepsilon(\cdot)| \exp(-\varepsilon d(\cdot, x_0)) \in L^p(X)$$

for every multiindex $\alpha$ (with the derivatives taken in local coordinates). But (2.8) and the estimate (2.11) in Appendix 1 imply now that $GA\psi_\varepsilon$ makes sense due to the Hölder inequality if $\varepsilon > 0$ is sufficiently small. Moreover $GA\psi_\varepsilon = \psi_\varepsilon$. Indeed for every $v \in C_0^\infty(X)$ we obtain using the Fubini theorem and estimates (2.10), (2.11) from Appendix 1:

$$\langle GA\psi_\varepsilon, v \rangle = \langle A\psi_\varepsilon, G^t v \rangle = \langle \psi_\varepsilon, A^t G^t v \rangle = \langle \psi_\varepsilon, v \rangle$$

(the middle equality is obtained by a limit procedure with the same use of the cut–off functions $\chi_N$ as in the proof of Proposition

2.6. On the other hand $A\psi_\varepsilon = 0$ implies $GA\psi_\varepsilon = 0$, hence $\psi_\varepsilon = 0$, so we get a contradiction which proves the theorem. $\square$

REMARK 2.7. Suppose that $X$ has a free isometric action of a discrete group $\Gamma$ such that $X/\Gamma$ is compact. Let $\Delta$ be the scalar Laplacian on $X$. Then R. Brooks [6] proved that $0 \in \sigma(\Delta)$ if and only if $\Gamma$ is amenable. Note that we always have $\Delta 1 = 0$, hence $0 \in \sigma_\infty(\Delta)$ and WBP does not hold on $X$ if $\Gamma$ is not amenable. However it is not clear whether something like this is true for more general operators (e.g. Schrödinger operator with a $\Gamma$–invariant potential, which is the case where WBP was proved in [26] for the case of subexponential growth).

Now the amenability of $\Gamma$ is equivalent to the amenability of $X$ which means the existence of compacts $K_j \subset X$, $j = 1, 2, \ldots$ , such that

$$\lim_{j \to \infty} \frac{\mathrm{Vol}[(K_j)_1 - K_j]}{\mathrm{Vol}\, K_j} = 0$$

where $(K_j)_1 = \{x | \mathrm{dist}(x, K_j) \le 1\}$. This makes sense for general manifolds of bounded geometry. So it is natural to ask whether WBP is true for general $C^\infty$–bounded uniformly elliptic operators on amenable manifolds. The positive answer for the Schrödinger operator in the $\Gamma$–periodic case was conjectured in [26].

Similar questions may be asked for WSP and SSP (for SSP the natural question is whether the subexponential growth condition can be weakened or not).

REMARK 2.8. There is an essential gap between Theorems 1.8 (or 1.13) and 2.3. Namely Theorems 1.8 and 1.13 do not allow to exclude $C^\infty$–bounded functions from negative spaces where we are trying to find a complete orthogonal system of generalized eigenfunctions. On the other hand the condition (2.2) in Theorem 2.3 (in case $p = 2$) is not satisfied for $\psi_\varepsilon \equiv 1$ unless $X$ has a subexponential growth. The gap disappears for the manifolds of subexponential growth but it is natural to try to fill it in case of manifolds of exponential growth (like $\mathbb{H}^n$). No considerable improvement can be expected in Theorem 2.3 because its growth

conditions come close to those which exist in examples like $\mathbb{H}^n$. On the other hand the abstract Theorem 2.3 from Appendix 2 can not be improved in the sense that the condition on the rigging to be a Hilbert–Schmidt rigging is necessary if we want the negative space to contain a complete orthonormal system of generalized eigenvectors for any self–adjoint operator. So possible improvement can be made here only if we switch from abstract operators e.g. to $C^\infty$–bounded uniformly elliptic ones.

REMARK 2.9. It is sufficient to have only a sequence $\varepsilon_j \to 0$, and it is not necessary to keep $\lambda$ fixed when we change $\varepsilon$. For instance in Theorem 2.3 we can only require that there exist a sequence $\varepsilon_j > 0$, $\varepsilon_j \to 0$ as $j \to +\infty$, and sections $\psi_j \not\equiv 0$, such that $\psi_j \exp(-\varepsilon_j d(\cdot, x_0)) \in L^p(X, E)$, $A\psi_j = \lambda_j \psi_j$ with $\lambda_j \to \lambda$ as $j \to \infty$. Then we can easily prove by the same reasoning that $\lambda \in \sigma_p(A)$. (Here $1 < p < \infty$.)

In case of subexponential growth it is easy to prove, using Theorem 1.8, that for the self-adjoint operators satisfying the conditions of Theorem 2.3 this condition is also necessary (hence necessary and sufficient) for the inclusion $\lambda \in \sigma(A)$, as well as the existence of sequences $\lambda_j \to \lambda$, $\psi_j \in C^\infty(X, E)$, $\psi_j \not\equiv 0$, such that $A\psi_j = \lambda_j \psi_j$ and $|\psi_j(x)| = 0(\exp \varepsilon_j d(x, x_0))$ where $\varepsilon_j \to 0$ as $j \to 0$.

## Appendix 2. Rigged spaces and generalized eigenvectors of self–adjoint operators.

In this Appendix we shall briefly describe some well–known results about rigged spaces and generalized eigenvectors of abstract self–adjoint operators. We will mainly follow [5] referring the reader to the book for proofs and more details. An alternative approach can be found in [4].

### A2.1. Rigged Hilbert spaces.

Usually Hilbert spaces arise in Analysis as spaces of square–integrable functions, sections of a vector bundle etc. But in this case usually additional restrictions of smoothness or (and) decay may be imposed to form a smaller Hilbert space. Also then the dual to this smaller space can be defined as a Hilbert space which naturally includes the basic Hilbert space. This situation

is described in the following

DEFINITION 1.1. A *rigged Hilbert space* is a triple

$$(1.1) \qquad \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$$

where $\mathcal{H}, \mathcal{H}_+, \mathcal{H}_-$ are (complex) Hilbert spaces (with the scalar products and norms denoted by $(\cdot, \cdot)$, $(\cdot, \cdot)_+$, $(\cdot, \cdot)_-$, $\| \cdot \|$, $\| \cdot \|_+$, $\| \cdot \|_-$ respectively) and the following conditions are satisfied:
  i) Both inclusions $\mathcal{H}_+ \subset \mathcal{H}$ and $\mathcal{H} \subset \mathcal{H}_-$ are linear continuous operators with dense image.
  ii) The scalar product $(\cdot, \cdot)$ in $\mathcal{H}$ can be extended to a continuous hermitian form $(\cdot, \cdot) : \mathcal{H}_+ \times \mathcal{H}_- \to \mathbb{C}$ which is non–degenerate in the following strong sense: every linear continuous functional $\ell : \mathcal{H}_+ \to \mathbb{C}$ can be uniquely represented in the form $\ell(\cdot) = (\cdot, f)$ where $f \in \mathcal{H}_-$ and $\|f\|_- = \|\ell\|$ where $\|\ell\|$ is the usual (operator) norm of $\ell$; similarly, every anti–linear continuous functional $\ell' : \mathcal{H}_- \to \mathbb{C}$ can be uniquely represented in the form $\ell'(\cdot) = (g, \cdot)$ with $g \in \mathcal{H}_+$ and $\|g\|_+ = \|\ell'\|$.
  The triple (1.1) is called then a rigging for the Hilbert space $\mathcal{H}$. Spaces $\mathcal{H}_+, \mathcal{H}_-$ (and norms $\| \cdot \|_+, \| \cdot \|_-$) are usually called positive and negative spaces (and norms) respectively. Actually the negative space $\mathcal{H}_-$ can be obviously reconstructed if only the couple $\mathcal{H}_+ \subset \mathcal{H}$ is given with the continuous imbedding operator having dense image.
  A convenient general procedure of constructing a rigging for a given Hilbert space $\mathcal{H}$ is to use a continuous linear operator $K : \mathcal{H} \to \mathcal{H}$ such that Ker $K = 0$ and Ker $K^* = 0$ (hence with a dense image $K\mathcal{H}$). Having such an operator we can put

$$(1.2) \qquad \mathcal{H}_+ = K\mathcal{H}, \ (Ku, Kv)_+ = (u, v), \ u, v \in \mathcal{H}.$$

Then $\mathcal{H}_-$ can be reconstructed as the dual space to $\mathcal{H}_+$ or as the completion of $\mathcal{H}$ with respect to the norm $\|h\|_- = \|K^*h\|$.
  Actually without loss of generality $K$ can be chosen self–adjoint because replacing $K$ by $|K| = \sqrt{K^*K}$ does not change the space $\mathcal{H}_+$ (and its norm).

DEFINITION 1.2. A *Hilbert–Schmidt rigging* is a rigging constructed with the help of a Hilbert–Schmidt operator $K$ in (1.2).

Hilbert–Schmidt riggings play a special role in spectral theory as we shall see in the next section.

Supposing that $K^* = K$ we may consider $A = K^{-1}$ as a self-adjoint operator in $\mathcal{H}$; besides Ker $A = 0$. If such an operator is given then we can construct the rigging by putting $\mathcal{H}_+ = D(A)$ and $(u,v)_+ = (Au, Av)$. This will be a Hilbert–Schmidt rigging if and only if $A$ has a discrete spectrum and its eigenvalues $\{\lambda_j | j = 1, 2, \ldots\}$ satisfy

$$(1.3) \qquad \sum_{j=1}^{\infty} \lambda_j^{-2} < \infty$$

Note that only separable Hilbert space $\mathcal{H}$ may have a Hilbert–Schmidt rigging in the sense described here. But this is the only case which we need in applications.

**A2.2. Generalized eigenvectors.**

First recall a general formulation of the spectral theorem for self–adjoint operators (see e.g. [5] or [32]).

**Theorem 2.1.** *Let $A$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then there exists a measure space $(M, \mu)$, a unitary operator $U : \mathcal{H} \to L^2(M, d\mu)$ and a real-valued measurable function $a$ on $M$ which is defined and finite almost everywhere such that*

*(i) $\psi \in D(A)$ if and only if $a(\cdot)(U\psi)(\cdot) \in L^2(M, d\mu)$*
*(ii) If $\varphi \in U(D(A))$ then $(UAU^{-1}\varphi)(m) = a(m)\varphi(m)$.*

In other words $A$ can be represented as a multiplication operator $M_a$ given by $(M_a\varphi)(m) = a(m)\varphi(m)$ in $L^2(M, d\mu)$ with a real-valued measurable and almost everywhere finite function $a$. More exactly $A = U^{-1}M_aU$ with a unitary $U$. Let us recall that under the given conditions the operator $M_a$ with the natural domain

$$D(M_a) = \{\varphi | \varphi \in L^2(M, d\mu), \ a\varphi \in L^2(M, d\mu)\}$$

is self–adjoint.

Now let us consider a rigging (1.1) of $\mathcal{H}$. Let $A$ be a self–adjoint operator in $\mathcal{H}$. Suppose further that we are given a measure space $(M, \mu)$ and a vector–valued function $\Phi : M \to \mathcal{H}_-$ (which may be actually defined almost everywhere) with values in the negative space of the rigging.

DEFINITION 2.2. A vector–valued function $\Phi : M \to \mathcal{H}_-$ is called a *complete orthonormal system of generalized eigenvectors* (or *eigenfunctions*) of the operator $A$ if the following conditions are fulfilled:

(i) for any $h_+ \in \mathcal{H}_+$ the function $m \mapsto (h_+, \Phi(m))$ on $M$ belongs to $L^2(M, d\mu)$;

(ii) the map $h_+ \mapsto (h_+, \Phi(\cdot))$ can be extended to a unitary operator $U : \mathcal{H} \to L^2(M, d\mu)$ which gives a spectral representation of $A$ as in Theorem 2.1.

The reader can find motivations and explanations of this definition in [5]. Let us remark only that $\Phi(m)$ is really a generalized eigenfunction of $A$ with an eigenvalue $a(m)$ in a reasonable sense. For example if we take any complex–valued Borel function $f : \mathbb{R} \to \mathbb{C}$ then

$$(2.1) \qquad (\Phi(m), \bar{f}(A)g) = f(a(m))(\Phi(m), g)$$

for any $g \in \mathcal{H}_+ \cap \bar{f}(A)^{-1}\mathcal{H}_+$ (i.e. $g \in \mathcal{H}_+ \cap D(\bar{f}(A))$ and $\bar{f}(A)f \in \mathcal{H}_+$) and for almost every $m \in M$. In particular

$$(\Phi(m), Ag) = a(m)(\Phi(m), g)$$

for any $g$ such that $g \in \mathcal{H}_+$ and $Ag \in \mathcal{H}_+$, and for almost every $m \in M$.

Actually the set $M_0 \subset M$ where all relations (2.1) are true (and such that $\mu(M - M_0) = 0$) may be choosen independent of $g$ provided $\mathcal{H}_+$ is separable (see [5], Proposition 2.7 in Supplement 1).

Now we shall remind the main result about riggings and generalized eigenfunctions. It is due to Ju. M. Berezanskii but in a weaker form it was proved earlier by I.M. Gelfand and A.G. Kostyuchenko [17].

**Theorem 2.3.** *Given a Hilbert–Schmidt rigging (1.1) of $\mathcal{H}$ and a self–adjoint operator $A$ in $\mathcal{H}$, there exists in (1.1) a complete orthonormal system of generalized eigenvectors for the operator $A$.*

A simple proof can be found in [5]. Remark that the condition on the rigging (1.1) to be Hilbert–Schmidt is necessary in a natural sense (see [4]).

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