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0. Introduction

The main purpose of these lecture notes, which are a revised and expanded version of the survey paper [S-U V], is to give an overview of the mathematical developments in the last few years in inverse boundary value problems. In these problems one attempts to discover internal properties of a body by making measurements at the boundary. We concentrate mainly in the problem of determining the conductivity of a body from measurements of voltage potentials and corresponding current fluxes at the boundary. This problem which is often referred to as *Electrical Impedance Tomography* arose in geophysics from attempts to determine the composition of the earth. More recently it has been proposed as a potentially valuable diagnostic tool for the medical sciences. The methods developed to study this problem have lead to new results in inverse scattering and inverse spectral problems. We also give an account of some of these developments in these notes.

1. Electrical impedance tomography; the isotropic case.

In this section we formulate the inverse conductivity problem and a similar problem for the Schrödinger equation at zero energy.

Let $\Omega \subseteq \mathbf{R}^n$ $n \geq 2$, be a smooth bounded domain. If the conductivity of Ω is independent of direction (isotropic case) it is represented by a positive function, which we assume in $C^{1,1}(\overline{\Omega})$, with a positive lower bound. If we assume that there are no sources or sinks of current in Ω, the conductivity equation for the potential u in Ω is

$$(1.1) \qquad L_\gamma u = \operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega.$$

If f represents the induced potential on the boundary (assume $f \in H^{\frac{1}{2}}(\partial\Omega)$), $u \in H^1(\Omega)$ solves the Dirichlet problem

$$(1.2) \qquad \begin{aligned} L_\gamma u &= 0 \qquad \text{in } \Omega \\ u|_{\partial\Omega} &= f. \end{aligned}$$

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The Dirichlet to Neumann map is then defined by

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu} \tag{1.3}$$

where u is the solution of (1.2) and ν is the unit outer normal to the boundary. The map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is selfadjoint and is often called the voltage to current map because $\gamma\frac{\partial u}{\partial \nu}$ measures the current flux at the boundary.

The inverse conductivity problem consists of the study of various properties of the map

$$\gamma \xrightarrow{\Phi} \Lambda_\gamma. \tag{1.4}$$

These properties include the injectivity, range, and continuity of the map and its inverse (when an inverse exists). From the point of view of applications, an even more important problem is to give a method to reconstruct γ(or at least to deduce as much information as possible about γ) from Λ_γ.

A closely related problem is to consider instead of the conductivity equation, the Schrödinger equation at zero energy

$$L_q = \Delta - q \tag{1.5}$$

where $q \in L^\infty(\Omega)$.

If 0 is not an eigenvalue of L_q, we can solve the Dirichlet problem

$$\begin{aligned} L_q u &= 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} &= f \end{aligned} \tag{1.6}$$

and define the Dirichlet to Neumann map by

$$\Lambda_q(f) = \frac{\partial u}{\partial \nu} \tag{1.7}$$

where u is the solution of (1.6). We want to study the map

$$q \xrightarrow{\tilde{\Phi}} \Lambda_q. \tag{1.8}$$

Λ_γ and Λ_q are related in the following way: If u is a solution of (1.1) then

$$w = \gamma^{\frac{1}{2}} u$$

is a solution of $L_q w = 0$ with $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$. It is a straightforward computation to see that

$$\Lambda_q = \gamma^{-\frac{1}{2}}\Lambda_\gamma\gamma^{-\frac{1}{2}} + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial\nu}. \tag{1.9}$$

Thus if we know $\Lambda_\gamma, \gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}$ we can determine Λ_q. In the next section we shall see that Λ_γ determines $\gamma|_{\partial\Omega}$ and $\frac{\partial\gamma}{\partial\nu}|_{\partial\Omega}$, so that knowledge of Λ_γ determines Λ_q.

2. Results at the boundary

Kohn and Vogelius ([K-V, I]) proved that if $\gamma \in C^\infty(\overline{\Omega})$ one can determine $\frac{\partial^j\gamma}{\partial\nu^j}|_{\partial\Omega} \quad \forall j$.

Theorem 2.1. *Let $\gamma_i (i = 1,2)$ be in $L^\infty(\Omega)$ with a positive lower bound. Let $x_0 \in \partial\Omega$ and let B be a neighborhood of x_0 relative to $\overline{\Omega}$. Suppose that*

$$\gamma_i \in C^\infty(B), \quad i = 1,2$$

and

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \quad \forall f \in H^{\frac{1}{2}}(\partial\Omega) \qquad \textit{with}$$

supp $f \subset B \cap \partial\Omega$, then

$$\left(\frac{\partial}{\partial x}\right)^\alpha \gamma_1(x_0) = \left(\frac{\partial}{\partial x}\right)^\alpha \gamma_2(x_0)$$

where

$$\left(\frac{\partial}{\partial x}\right)^\alpha \textit{ denotes } \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Sketch of proof.

Kohn and Vogelius proved this result by cleverly choosing boundary data. We outline here a different approach taken in [S-U, I] which makes use of the fact that Λ_γ is a pseudodifferential operator of order 1. This means that, in local coordinates near $x_0 \in \partial\Omega$ which we denote by x', and for f supported near x_0,

$$\Lambda_\gamma f(x') = \int e^{ix'\cdot\xi'}\lambda_\gamma(x',\xi')\hat{f}(\xi')d\xi'. \tag{2.2}$$

$\lambda_\gamma(x',\xi')$ is the full symbol of Λ_γ and has an asymptotic expansion for large $|\xi'|$

$$\lambda_\gamma(x',\xi') \sim \sum_{j\le 1} \lambda_\gamma^{(j)}(x',\xi') \tag{2.3}$$

with $\lambda_\gamma^{(j)}$ homogeneous of degree j in ξ'. We have $\lambda_\gamma^{(1)}(x',\xi') = \gamma|_{\partial\Omega}(x')|\xi'|$ and it was proven in [S-U, I] that $\lambda_\gamma^{(j)}(x',\xi')$ determines inductively $\frac{\partial^{j-1}}{\partial\nu^{j-1}}\gamma|_{\partial\Omega}$ (For a simpler proof of this see the paper [L-U] and also the sketch in section 9 of this paper.) $\square$

The previous result implies the injectivity of Φ at real-analytic conductivities. Kohn and Vogelius extended this result further to cover piecewise real-analytic conductivities ([K-V, II]).

Sylvester and Uhlmann ([S-U I]) used the proof of Theorem 2.1 outlined above to give continuous dependence estimates at the boundary.

Theorem 2.4. *Let γ_i, $i = 1,2$ be in $L^\infty(\Omega)$ with a positive lower bound. Then*

(a)
$$\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}} \leq C\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)}.$$

If γ_1, γ_2 are continuous, then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C_1\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}}.$$

(b) If γ_1, γ_2 are Lipschitz continuous then

$$B_i = \Lambda_{\gamma_i} - \gamma_i\Lambda_1 \quad \textit{satisfy}$$

$$\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} \leq C_2\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\Omega)}$$

and $\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(\partial\Omega)} + \|\frac{\partial}{\partial\nu}(\gamma_1 - \gamma_2)\|_{L^\infty(\partial\Omega)}$

$$\leq C_3(\|B_1 - B_2\|_{\frac{1}{2},\frac{1}{2}} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}})$$

On the operators we use the operator norm. C_1 depends only on Ω and the lower bound of the γ_i's. C_2, C_3 depends only on Ω and the γ_i's are normalized to have Lipschitz norm less than or equal to one.

3. Linearization at constants; Calderón's approach

Calderón formulated the inverse conductivity problem in a different way. He considered the Dirichlet integral associated to the solution of (1.2)

$$Q_\gamma(f) = \int_\Omega \gamma|\nabla u|^2, \tag{3.1}$$

$Q_\gamma(f)$ measures the power necessary to maintain the potential f on the boundary.

Polarizing the quadratic form Q_γ we obtain the bilinear form

$$Q_\gamma(f,g) = \int_\Omega \gamma \nabla u \cdot \nabla v \tag{3.2}$$

where u is a solution of (1.2) and v solves

$$\begin{aligned} L_\gamma v &= 0 \quad \text{in } \Omega \\ v|_{\partial\Omega} &= g\,. \end{aligned} \tag{3.3}$$

The divergence theorem gives

$$Q_\gamma(f,g) = \int_{\partial\Omega} g\Lambda_\gamma f \tag{3.4}$$

In other words $\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is the unique selfadjoint operator associated to the quadratic form Q_γ with domain $H^{\frac{1}{2}}(\partial\Omega)$. The inverse conductivity problem can then be reformulated as the study of the map

$$\gamma \xrightarrow{Q} Q_\gamma. \tag{3.5}$$

For the Schrödinger equation L_q we look at the Dirichlet form

$$Q_q(f,g) = \int_\Omega \nabla u \cdot \nabla v + quv \tag{3.6}$$

where u, v solve

$$\begin{aligned} L_q u = L_q v &= 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} = f; \quad v|_{\partial\Omega} &= g \end{aligned} \tag{3.7}$$

and we can consider the map

$$q \xrightarrow{\widetilde{Q}} Q_q\,. \tag{3.8}$$

Calderón computed the formal linearization of Q near γ. He obtained

$$\lim_{\varepsilon \to 0} \frac{(Q_{\gamma+\varepsilon\varphi} - Q_\gamma)}{\varepsilon}(f,g) = \int_\Omega \varphi \nabla u \cdot \nabla v \tag{3.9}$$

with u, v as in (1.2) and (3.3).

An analogous computation shows that

$$\lim_{\varepsilon\to 0}\frac{(\widetilde{Q}_{q+\varepsilon\varphi}-\widetilde{Q}_q)}{\varepsilon}(f,g)=\int_\Omega \varphi uv \tag{3.10}$$

with u, v as in (3.7).

Formulas (3.9) and (3.10) imply that the formal linearization of Q (resp. $\widetilde{Q}$) at γ (resp. q) is injective iff the linear span of the inner products of gradients of solutions to (3.3)(resp. products of solutions to (3.7)) is dense in $L^2(\Omega)$; or equivalently that any function orthogonal to all such inner products (resp. products) is identically zero.

Calderón exploited this by proving:

Theorem 3.11. *The linear span of the inner products of gradients of solutions of harmonic functions (or the product of harmonic functions) is dense in $L^2(\Omega)$.*

Proof. Calderón chose the complex exponential harmonic functions

$$\begin{aligned} u &= e^{x\cdot\rho} \\ v &= e^{-x\cdot\overline{\rho}} \end{aligned} \tag{3.12}$$

where $\rho \in \mathbf{C}^n$. These functions are harmonic iff

$$\rho\cdot\rho = 0\,. \tag{3.13}$$

For $\rho = \eta + ik$, with $\eta, k \in \mathbf{R}^n$, (3.13) is satisfied iff $\eta\cdot k = 0$, $|\eta| = |k|$. Inserting (3.12) into (3.9)(resp. (3.10)) yields

$$\int_\Omega \varphi\nabla u\cdot\nabla v = -2|k|^2\int_\Omega e^{2ix\cdot k}\varphi(x)dx$$

and

$$\int_\Omega \varphi uv = \int_\Omega e^{2ix\cdot k}\varphi(x)dx\,.$$

In both cases we conclude by the Fourier inversion formula that $\varphi = 0$ in Ω. $\square$

4. Special solutions

Motivated by Calderón's approach, Sylvester and Uhlmann constructed an analog for the elliptic equations (3.3) (or (3.7)) of the geometrical optics solutions for hyperbolic equations. These solutions behave like the complex exponentials $e^{x\cdot\rho}$, $\rho\cdot\rho = 0$ for large complex frequencies ρ.

Theorem 4.1. *Let $q \in L^\infty(\Omega)$ so that $q = 0$ in Ω^c.*

Let $\rho \in \mathbf{C}^n$, $n \geq 2$ be such that

$$\rho \cdot \rho = 0 \tag{4.2}$$

and

$$|\rho| > \|(1+|x|^2)^{\frac{1}{2}} q\|_{L^\infty}, \tag{4.3}$$

then there exists a unique solution to

$$L_q u = 0 \quad \text{in } \mathbf{R}^n$$

of the form

$$u(x,\rho) = e^{x\cdot\rho}(1 + \psi_q(x,\rho)) \tag{4.4}$$

where $\psi_q(\cdot,\rho) \in L^2_\delta(\mathbf{R}^n)$, $-1 < \delta < 0$.

Furthermore

$$\|\psi_q\|_{H^m_\delta} \leq \frac{C}{|\rho|}\|q\|_{H^m_{\delta+1}}, \quad m \geq 0. \tag{4.5}$$

$L^2_\delta(\mathbf{R}^n)$ is the weighted L^2-space

$$L^2_\delta(\mathbf{R}^n) = \{f; \int |f|^2(1+|x|^2)^\delta dx < \infty\},$$

$H^m_\delta(\mathbf{R}^n)$ is the corresponding Sobolev space.

An analogous statement is valid for the conductivity equation. Extend $\gamma \in C^{1,1}(\Omega)$ to $\gamma \in C^{1,1}(\mathbf{R}^n)$ with $\gamma = 1$ outside a ball. Then the solution (4.4) is replaced by

$$u(x,\rho) = e^{x\cdot\rho}\gamma^{-\frac{1}{2}}(1 + \psi_\gamma(x,\rho)) \tag{4.6}$$

ψ_q (resp. ψ_γ) in (4.4) (resp. (4.6)) satisfy the "transport" equation

$$\Delta\psi_q + 2\rho\cdot\nabla\psi_q - q\psi_q = q \tag{4.7}$$

(resp. $\Delta\psi_\gamma + 2\rho\cdot\nabla\psi_\gamma - q_\gamma\psi_q = q_\gamma$ with $q_\gamma = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$). The solution of the singular perturbation problem (4.7) with growth condition at infinity is easily seen to be a regular perturbation of the following proposition (see [S-U, II], Prop. 2.1)

Proposition 4.8. *Suppose*

$$\rho \cdot \rho = 0, |\rho| > B > 0, -1 < \delta < 0$$

and $f \in L^2_{\delta+1}$. *Then there exists a unique* $\phi \in L^2_\delta$ *solving*

$$\Delta\phi + \rho \cdot \nabla\phi = f.$$

Moreover,

$$\|\phi\|_{H^m_\delta} \leq \frac{C(B,\delta)}{|\rho|}\|f\|_{H^m_{\delta+1}}, \quad m \geq 0.$$

Theorem 4.1 has been extended to more singular potentials (see for instance [Ch]). Isakov [Is I] has given a different construction of special solutions which also applies to other equations with constant coefficient principal part. However, he doesn't obtain weighted estimates for the solutions.

5. Uniqueness and continuous dependence, $n \geq 3$

Sylvester and Uhlmann [S-U, II] proved that the map Φ (resp. $\widetilde{\Phi}$) is injective for smooth conductivities (potentials). The smoothness assumptions were relaxed to $\gamma \in C^{1,1}(\overline{\Omega})$ $(q \in L^\infty(\Omega))$ in [N-S-U].

Theorem 5.1. *(a) Let* $n \geq 3$, $\gamma_1, \gamma_2 \in C^{1,1}(\overline{\Omega})$ *with a positive lower bound and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2 \,.$$

(b) Let $n \geq 3$, $q_1, q_2 \in L^\infty(\Omega)$ *and*

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$q_1 = q_2.$$

Proof. We first prove (b). An easy application of Green's theorem gives

$$\int_\Omega (q_1 - q_2)u_1u_2 = \int_{\partial\Omega} f_1\Lambda_{q_1}f_2 - f_2\Lambda_{q_2}f_1 \tag{5.2}$$

where u_i is solution of $L_{q_i}u_i = 0$ and $f_i = u_i|_{\partial\Omega}$, $i = 1, 2$. Since Λ_q is a self adjoint map we obtain the identity proven by Alessandrini ([A])

$$\int_\Omega (q_1 - q_2)u_1u_2 = \int_{\partial\Omega} f_1(\Lambda_{q_1} - \Lambda_{q_2})f_2 \,. \tag{5.3}$$

If $\Lambda_{q_1} = \Lambda_{q_2}$ we have

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0 \tag{5.4}$$

for all u_i which solve $L_{q_i} u_i = 0$, $i = 1, 2$. We let

$$u_i = e^{x\cdot\rho_i}(1 + \psi_{q_i}(x, \rho_i)) \tag{5.5}$$

with ρ_i as in Theorem 4.1 and choose (in order to guarantee (4.2))

$$\begin{aligned} \rho_1 &= \frac{\eta}{2} + \frac{i(r\omega + k)}{2} \\ \rho_2 &= -\frac{\eta}{2} + \frac{i(-r\omega + k)}{2} \end{aligned} \tag{5.6}$$

where $\eta, \omega, k \in \mathbf{R}^n$, $|\omega| = 1$, $r \in \mathbf{R}$ with

$$\eta \cdot k = \eta \cdot \omega = \omega \cdot k = 0$$

and

$$|\eta|^2 = r^2 + k^2 .$$

Substituting (5.5) into (5.4) gives

$$\int_\Omega e^{ix\cdot k}(q_1 - q_2) = -\int_\Omega e^{ix\cdot k}(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2})(q_1 - q_2) . \tag{5.7}$$

However, the estimate (4.5) implies that $\psi_{q_i} \to 0$ in $\overline{\Omega}$ as $r \to \infty$. Therefore

$$\widehat{q_1}(k) = \widehat{q_2}(k)$$

and thus

$$q_1 = q_2 .$$

A proof of part (a) follows from the fact that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\Lambda_{q_{\gamma_1}} = \Lambda_{q_{\gamma_2}}$ with $q_{\gamma_i} = \frac{\Delta\sqrt{\gamma_i}}{\sqrt{\gamma_i}}$ because of (1.9) and Theorem (2.1). Now it is easy to check (see for example [S-U II]) that $q_{\gamma_1} = q_{\gamma_2}$ implies $\gamma_1 = \gamma_2$. $\square$

A very interesting problem is to extend the uniqueness result above to the case of piecewise continuous conductivities. Isakov [Is II] has proven such a result for conductivities with jump type singularities across the boundary of an open bounded subset of Ω.

Alessandrini ([Al]) used the identity (5.3), the special solutions of Theorem (4.1) and the continuous dependence estimate at the boundary (Theorem 2.4) to prove a stability estimate (i.e. a logarithmic continuous dependence result, which depends on an a-priori bound in a high Sobolev norm) for the conductivity.

Theorem 5.8. *Let $s > \frac{n}{2}, n \geq 3$, $\gamma_i \in H^{s+2}(\Omega)$ with*

$$0 < \alpha \leq \gamma_i(x) \quad x \in \overline{\Omega}$$

and

$$\|\gamma_i\|_{H^{s+2}(\Omega)} \leq \frac{1}{\alpha}, \quad i = 1, 2.$$

Then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C_\alpha w(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},-\frac{1}{2}})$$

where

$$w(t) = (\frac{1}{-\log t})^\delta, \quad 0 < t < 1$$

and δ, $0 < \delta < 1$ depends only on n and s.

It is not known whether this is the best possible continuous dependence result. However, for conductivities having special features better continuous dependence results are known. Friedman and Vogelius ([F-V]) have shown that if one seeks to find spheres of zero or infinite conductivity inside a medium with ambient constant conductivity, then the radii and diameters are Lipschitz continuous functions of the measurements in two dimensions. It would be useful to understand the mechanism of ill-posedness in the general problem in order to better study special problems where the dependence could be better.

6. Complex frequency Born approximation, $n \geq 3$

In this section we discuss briefly the relationship between the Dirichlet to Neumann map Λ_q and the function T defined in the $\overline{\partial}$-approach to multidimensional inverse scattering theory by Ablowitz and Nachman ([N-A]) and Beals and Coifman ([B-C I]). In one dimension it had been developed earlier in [B-C II]. For more details the reader should look at those papers and the more recent ones like [N-H], [N] and [No] and the references indicated there.

Let us assume $q \in L^\infty(\Omega)$ with $q = 0$ outside Ω. The scattering amplitude can then be written in terms of the outgoing eigenfunction (see for example [Ag])

$$a(\lambda, \theta, \omega) = c_n \int e^{-i\lambda x \cdot \theta} q(x) \psi_+(\lambda, x, \omega) dx, \tag{6.1}$$

where $\lambda \in \mathbf{R}, \theta, \omega \in S^{n-1}$ and $\psi_+(\lambda, x, \omega)$ is the outgoing eigenfunction of $-\Delta + q$ i.e. ψ_+ is the solution of the Lippmann-Schwinger equation

$$\psi_+(\lambda, x, \omega) = e^{i\lambda x\cdot\omega} - \int G_\lambda^+(x-y)q(y)\psi_+(\lambda, y, \omega)dy \tag{6.2}$$

where G_+^λ is the outgoing Green's kernel

$$G_\lambda^+(x) = (2\pi)^{-n}\int \frac{e^{ix\cdot k}}{k^2 - \lambda^2 - i0}dk\,. \tag{6.3}$$

The outgoing eigenfunction ψ_+ has the asymptotic expression for large $|x|$ (see [Ag])

$$\psi_+(\lambda, x, \omega) = e^{i\lambda x\cdot\omega} + \frac{a(\lambda, \theta, \omega)}{|x|^{\frac{n-1}{2}}}e^{i\lambda|x|} + O(|x|^{\frac{-(n+1)}{2}}), \tag{6.3'}$$

where $\theta = \frac{x}{|x|}$.
Moreover the following estimate holds (see [Ag])

$$\|\psi_+ - e^{i\lambda x\cdot\omega}\|_{L^2_\delta} \le \frac{C}{\sqrt{\lambda}}\|q\|_{L^2_\delta}, \quad \delta < -\frac{1}{2}. \tag{6.4}$$

¿From (6.4) and (6.1) it is easy to derive the Born approximation for the scattering amplitude.

Faddeev [F] proposed to construct exponentially growing eigenfunctions of

$$(-\Delta + q)u(x,\zeta) = \zeta^2 u(x,\zeta) \tag{6.5}$$

where $\zeta \in \mathbf{C}^n$ is arbitrary but non-real, by solving the integral equation

$$u(x,\zeta) = e^{x\cdot\zeta} - \int G_\zeta(x-y)q(y)u(y,\zeta)dy \tag{6.6}$$

where $G_\zeta(x)$ is a new Green's kernel for $\Delta - \zeta^2$:

$$G_\zeta(x) = \frac{1}{(2\pi)^n}e^{x\cdot\zeta}\int \frac{e^{ix\cdot k}}{-|k|^2 + 2i\zeta\cdot k}\,dk\,. \tag{6.7}$$

Notice that G_ζ satisfies formally

$$(\Delta - \zeta^2)G_\zeta = \delta(x)\,. \tag{6.8}$$

Faddeev proposed using these generalized eigenfunctions for complex parameters ζ with imaginary part tending to zero as a generalization to 3 dimensions of the Gelfand-Levitan approach to inverse scattering in one dimension.

Notice that $h_\zeta(x) = e^{-x\cdot\zeta}G_\zeta(x)$ is the solution of

$$(\Delta + 2\zeta\cdot\nabla)h_\zeta = \delta(x)\,. \tag{6.9}$$

Proposition (4.8) implies, for $\zeta\cdot\zeta = 0$ and $|\zeta|$ large, the integral equation (6.6) has a unique solution. These generalized eigenfunctions were also considered by Ablowitz and Nachman ([N-A]) and Beals and Coifman [B-C I,II] in their $\overline{\partial}$-approach to the study of the scattering amplitude. In particular, in analogy with (6.1) they considered the function

$$T_q(k,\zeta) = \int e^{-ix\cdot k}q(x)u(x,\zeta)dx \tag{6.10}$$

with u solution of (6.6). The point is that the compatibility conditions for the $\overline{\partial}$-equation leads to compatibility conditions for the range of the map

$$q \xrightarrow{T} T_q. \tag{6.11}$$

Henkin and Novikov ([N-H]) gave a characterization of T for sufficiently smooth potentials (the derivation in [N-A] is formal and Beals and Coifman [B-C] gave proofs for small potentials and $\zeta\cdot\zeta = 0$). The relationship between $T(k,\zeta)$ and the physical scattering amplitude has been studied ([L-N]) and [N-H] but there is still not complete understanding of this. We want to point out here the relation between $T(k,\zeta)$ (or rather a closely related function; see below) and Λ_q. For this we shall give yet another proof of Theorem 4.1 which appeared in [N-S-U]. We define

$$t(k,\rho) = \int e^{-ix\cdot k}e^{-x\cdot\rho}q(x)u(x,\rho)dx \tag{6.12}$$

where $u(x,\rho)$ is the solution of $L_q u = 0$ in Theorem 4.1 and we require, for $k \in \mathbf{R}^n$, that $\rho \in \mathbf{C}^n$ satisfy:

$$\rho\cdot\rho = 0, \quad (ik+\rho)\cdot(ik+\rho) = 0, \quad |\rho| > \|(1+|x|^2)^{\frac{1}{2}}q(x)\|_{L^\infty}\,. \tag{6.13}$$

We then have

$$\Delta e^{ix\cdot k + x\cdot\rho} = 0, \quad \Delta u = qu. \tag{6.14}$$

Using (6.14) and Green's theorem we see that

$$t(k,\rho) = \int_{\partial\Omega} e^{-ix\cdot k}e^{-x\cdot\rho}[\Lambda_q u|_{\partial\Omega} + (ik+\rho)\cdot \nu u|_{\partial\Omega}]\, dS \tag{6.15}$$

with dS euclidean surface measure on $\partial\Omega$.

Hence we can compute $t(k,\rho)$ for (k,ρ) satisfying (6.13)) if we know Λ_q and the boundary values of the special solution $u(x,\rho)$. moreover, we prove next (see [S-U, I]) that $u|_{\partial\Omega}$ is actually determined uniquely by Λ_q.

Proposition 6.16. *Let $q_1, q_2 \in L^\infty(\Omega)$ such that*

$$\Lambda_{q_1} = \Lambda_{q_2}.$$

Let u_1, u_2 be solution of

$$L_{q_i} u_i = 0, \;\; i = 1,2$$

as in Theorem 4.1. Then

$$u_1|_{\Omega^c} = u_2|_{\Omega^c}$$

Proof. Let us consider the solution of

$$L_{q_1} w = 0 \tag{6.17}$$

$$w|_{\partial\Omega} = u_2.$$

Let us define

$$z = \begin{cases} w & \text{in } \Omega \\ u_2 & \text{in } \Omega^c. \end{cases} \tag{6.18}$$

Now, z obviously satisfies (6.17) in $\mathbf{R}^n \backslash \partial\Omega$; in addition,

$$\frac{\partial z}{\partial \nu}|_{\partial\Omega} = \Lambda_{q_1} z|_{\partial\Omega} = \Lambda_{q_1}(u_2|_{\partial\Omega}) = \Lambda_{q_2}(u_2|_{\partial\Omega}) = \frac{\partial u_2}{\partial \nu}|_{\partial\Omega}. \tag{6.19}$$

Hence $z \in C^{1,1}(\overline{\Omega})$ and solves (6.17) in all of $\mathbf{R}^n$. Because z satisfies the required growth conditions at ∞, the uniqueness part of Theorem 4.1 implies that $w = u_1$, concluding the proof. □

Proposition (6.16) implies that, if $\Lambda_{q_1} = \Lambda_{q_2}$, then $t_1(k,\rho) = t_2(k,\rho)$ with (k,ρ) as in (6.13).

Now for these (k, ρ)

$$\lim_{|\rho| \to \infty} t(k, \rho) = \int e^{-ix \cdot k} q(x) dx = \widehat{q}(k) . \tag{6.20}$$

Proposition (6.16) and (6.20) provide another proof of Theorem 5.1. Equation (6.20) may be thought of as an analog of the Born-approximation for complex-frequencies. Nachman ([N]) observed that $u(\cdot, \rho)|_{\partial\Omega}$ as in (4.4) satisfies a Fredholm integral equation on the boundary. Because $q = 0$ in Ω^c, $u(x, \rho)$ must satisfy

$$\Delta u = 0 \text{ in } \Omega^c \tag{6.21}$$

$$\frac{\partial u}{\partial \nu}|_{\partial\Omega} = \Lambda_q u|_{\partial\Omega}.$$

Because u has the same asymptotics as G_ρ in (6.7), it must be a combination of the single and double layer potentials

$$S_\rho f(x) = \int_{\partial\Omega} G_\rho(x - y) f(y) dS_y \tag{6.22}$$

$$B_\rho f(x) = \int_{\partial\Omega} \frac{\partial G_\rho}{\partial \nu}(x - y) f(y) dS_y. \tag{6.23}$$

Nachman showed that $u(x, \rho)|_{\partial\Omega}$ was the unique solution to

$$f(x, \rho) = e^{x \cdot \rho} - (S_\rho \Lambda_q - B_\rho - \frac{1}{2}) f(x, \rho) \tag{6.24}$$

for every $x \in \partial\Omega$.

The point is that equation (6.24) does not depend on q and therefore provides a direct method for finding $u(x, \rho)|_{\partial\Omega}$ without a priori knowledge of q. Novikov [No] studied similar integral equations.

7. The two dimensional case

The Schwartz kernel of the Dirichlet to Neumann map is a distribution of $(n - 1) + (n - 1) = 2n - 2$ variables, while the conductivity itself is a function of n variables. Hence the inverse conductivity problem is formally overdetermined in dimension $n \geq 3$ and formally determined in dimension 2. This is reflected in the lack of freedom to choose enough exponential solutions as in the proof of Theorem 4.1. The first result in this case was proven by Sylvester and Uhlmann [S-U III] for conductivities (resp. potentials) close to constant (resp. zero).

Theorem 7.1.

(a) Let $\gamma_i \in W^{3,\infty}(\Omega)$, $i = 1,2$ with positive lower bound. There exists $\varepsilon > 0$ such that if

$$\|\gamma_i - 1\|_{W^{3,\infty}(\Omega)} < \varepsilon, \quad i = 1,2$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

(b) Let $q_i \in W^{1,\infty}(\Omega)$ such that L_{q_i}, $i = 1,2$ does not have zero as an eigenvalue. There exists $\varepsilon > 0$ such that if

$$\|q_i\|_{W^{1,\infty}(\Omega)} < \varepsilon, \quad i = 1,2$$

and

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$q_1 = q_2.$$

Brief Sketch of proof. Again we only indicate how to prove (b). As in the proof of Theorem 5.1, we substitute the special solutions (5.5) into the identity (5.4). However, in two dimensions we may not choose ρ_i as in (5.6), but must be content with

$$\rho_1 = \frac{l+ik}{2} \qquad (7.2)$$
$$\rho_2 = \frac{-l+ik}{2}$$

where $l \cdot k = 0$ and $|l|^2 = |k|^2 = \frac{1}{2}|\rho|^2$ is sufficiently large. This yields estimates for the Fourier transform of $q_1 - q_2$ for all sufficiently large frequencies.

We may estimate the Fourier transform of $q_1 - q_2$ at sufficiently low frequencies by inserting into (5.4) solutions of $L_{q_i}\widetilde{u}_i = 0$ of the form

$$\widetilde{u}_1 = e^{x\cdot\rho} + \delta\widetilde{u}_1, \quad \delta\widetilde{u}_1|_{\partial\Omega} = 0, \qquad (7.3)$$
$$\widetilde{u}_2 = e^{-x\cdot\widetilde{\rho}} + \delta\widetilde{u}_2, \quad \delta\widetilde{u}_2|_{\partial\Omega} = 0.$$

If q_1 q_2 are small enough, both estimates combine to produce an inequality which can be satisfied only when $q_1 - q_2$ is identically zero. □

The uniqueness question for the inverse conductivity problem for smooth conductivity remains open. We report in this section on the progress obtained. The "transport" equation (4.7) has special features in two dimensions.

Let $\rho \in \mathbf{C}^2$ be such that

$$\rho \cdot \rho = 0, |\rho| > \|(1+|x|^2)^{\frac{1}{2}} q\|_{L^\infty}. \tag{7.4}$$

We write such a ρ in the form

$$\rho = \frac{\eta + ik}{2}, \eta \cdot k = 0, |\eta| = |k|; \eta, k \in \mathbf{R}^2, \quad k = (k_1, k_2).$$

Then the equation for ψ in two dimensions can be written in the form

$$\overline{\partial}\partial\psi + (k_2 + ik_1)\partial\psi - q\psi = q \tag{7.5}$$

where

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \quad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right). \tag{7.6}$$

In [S-U, III] it was proven that ψ can be written in the form

$$\psi(x,k) = \frac{a(x)}{k_2 + ik_1} + \frac{b(x,k)}{(k_2 + ik_1)^2} \tag{7.7}$$

with

$$\|a\|_{H^1_\delta}, \|a\|_{L^\infty(\Omega)}, \|b\|_{H^1_\delta} \leq C\|q\|_{W^{1,\infty}(\Omega)}. \tag{7.8}$$

Moreover a solves

$$\overline{\partial}a = q \tag{7.9}$$

¿From Proposition (6.16) and the expansion (7.7) we conclude

Proposition 7.10. *Suppose $q_i \in L^\infty(\Omega)$, $i = 1, 2$, $q_i = 0$ in Ω^c and L_{q_i} has not zero as eigenvalue. Suppose*

$$\Lambda_{q_1} = \Lambda_{q_2}$$

then

$$a_1 = a_2 \text{ in } \Omega^c$$

where a_i are as in (7.7).

We can write a_i in terms of q_i using the Cauchy integral representation

$$(7.11) \qquad a_i(x) = \frac{1}{2\pi i}\int \frac{q_i(w)}{x-w}dw \wedge d\overline{w}\,.$$

For $|x|$ sufficiently large, we can write

$$(7.12) \qquad a_i(x) = \frac{1}{2\pi i x}\sum_{n=0}^{\infty}\int q_i(w)\frac{w^n}{x^n}dw \wedge d\overline{w}$$

Therefore we conclude from Proposition 7.10 the following result proven in [S-U, IV] and [Su, I]

Theorem 7.13. *(a) Let γ_i, $i = 1,2$ be in $W^{3,\infty}(\Omega)$ with a positive lower bound. Assume Ω simply connected and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\int_\Omega (q_{\gamma_1} - q_{\gamma_2})h = 0$$

for all h harmonic in $\overline{\Omega}$.

(b) Let q_i be in $W^{1,\infty}(\Omega)$ so that L_{q_i} has no eigenvalue 0, $i = 1,2$. Assume Ω simply connected and

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$\int_\Omega (q_1 - q_2)h = 0$$

for all h harmonic in $\overline{\Omega}$.

In particular one can prove the global uniqueness result

Corollary 7.14. *Let $\gamma_1 \in W^{3,\infty}(\Omega)$ with a positive lower bound. Suppose $\gamma_2 =$ constant > 0 and*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2 = \text{ constant.}$$

Sun ([Su, II]) has observed that Theorem 7.13 gives the following global uniqueness result for conductivities:

Theorem 7.15. *Let $\gamma_i \in W^{3,\infty}(\Omega)$, $i = 1,2$ with positive lower bound. Assume γ_2^α is harmonic for some $\alpha \in \mathbf{R}$ or $\log \gamma_2$ is harmonic. If*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

Sun also gave a logarithmic continuous dependence result for conductivities γ_1, γ_2 as in the hypothesis of Theorem 7.15 under an priori $C^4(\overline{\Omega})$ bound on γ_i's, $i = 1,2$. For local uniqueness Sun ([Su, II]) improved on the local result, Theorem 7.1, to prove

Theorem 7.16. *Let $\gamma_i \in W^{3,\infty}(\Omega)$, $i = 1,2$ with positive lower bound. Let $\gamma_0 \in C^3(\overline{\Omega})$ be such that either (a) γ_0^α is harmonic for some $\alpha \in \mathbf{R}$ or (b) $\gamma_0 = e^{\mathrm{Re}\,\phi}$ where ϕ is an injective conformal map in $\overline{\Omega}$. Then there is $\varepsilon > 0$ such that if*

$$\|\gamma_i - \gamma_0\|_{W^{3,\infty}(\Omega)} < \varepsilon, \quad i = 1,2$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

All the results assume some a priori restriction on the conductivities or potentials besides smoothness. Recently Sun and Uhlmann [Su-U I] proved that for almost all conductivities or potentials injectivity and local injectivity for the map Φ and $\widetilde{\Phi}$ holds. More precisely:

Theorem 7.17. *(a) There exists an open and dense set $\mathcal{O}$ in $W^{3,\infty}_{pos}(\Omega)$*. If $\gamma \in \mathcal{O}$ there exists an $\varepsilon > 0$ such that if*

$$\|\gamma_i - \gamma\|_{W^{3,\infty}(\Omega)} < \varepsilon, \quad i = 1,2$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then

$$\gamma_1 = \gamma_2.$$

---

* *$W^{3,\infty}_{pos}(\Omega)$ denotes the set of positive functions in $W^{3,\infty}(\Omega)$.*

(b) There exists an open and dense set $\mathcal{O}$ in $W^{1,\infty}(\Omega)$. If $q \in \mathcal{O}$ there exists $\varepsilon > 0$ such that if

$$\|q_i - q\|_{W^{1,\infty}(\Omega)} < \varepsilon\,, i = 1, 2$$

and

$$\Lambda_{q_1} = \Lambda_{q_2}$$

then

$$q_1 = q_2.$$

For global uniqueness it was proven in [Su-U I]:

Theorem 7.18.

(a) There exists an open dense set $\mathcal{O}$ in $W^{3,\infty}_{pos}(\Omega) \times W^{3,\infty}_{pos}(\Omega)$, such that if $(\gamma_1, \gamma_2) \in \mathcal{O}$ and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then

$$\gamma_1 = \gamma_2\,.$$

(b) There exists an open dense set $\mathcal{O}$ in $W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$ such that if $(q_1, q_2) \in \mathcal{O}$ and

$$\Lambda_{q_1} = \Lambda_{q_2}$$

then

$$q_1 = q_2.$$

Sketch of proof.

We indicate how to prove part (b) of Theorems 7.17 and 7.18. Part (a) follows in a similar way to the proof of part (a) of Theorem 5.1 from part (b).

The proof of Theorem (7.17) is reduced to show that

Lemma 7.19. *Let $q \in L^\infty(\Omega)$. Then*

$$D_q = \{uv; u, v \text{ are solutions of } L_q u = L_q v = 0 \text{ in } \Omega\}$$

is complete in $L^2(\Omega)$ for $q \in \mathcal{O}$ where $\mathcal{O}$ is an open and dense set in $W^{1,\infty}(\Omega)$.

Sketch of proof of Lemma 7.19

Consider the q's in $W^{1,\infty}(\Omega), q = 0$ in Ω^c with $\|q\|_{W^{1,\infty}(\Omega)} < R$. By Theorem 4.1 there exists $L_R > 0$ and solutions u, v of $L_q u = L_q v = 0$ in $\mathbf{R}^2$ of the form

$$\begin{aligned} u_1(x,k) &= e^{x\cdot\rho}(1 + \psi(x,k)) \text{ for } |k| > L_R \\ u_2(x,k) &= e^{-x\cdot\overline{\rho}}(1 + \widetilde{\psi}(x,k)) \text{ for } |k| > L_R\,. \end{aligned} \tag{7.20}$$

We require further that L_q does not have zero as an eigenvalue (this set of q's is easily seen to be open and dense in $W^{1,\infty}(\Omega)$) and denote by $\widetilde{u}_i$ the solutions of the Dirichlet problem (7.3).

Next we define the operator

$$A_q f(k) = \begin{cases} \int f e^{ix\cdot k} + \int_\Omega e^{ix\cdot k} f(\psi + \widetilde{\psi} + \psi \cdot \widetilde{\psi}) & \text{for } |k| > L_R \\ \int f e^{ixk} + \int_\Omega f(\widetilde{u}_1 \widetilde{u}_2 - e^{ix\cdot k}) & \text{for } |k| \leq L_R \end{cases} \tag{7.22}$$

The operator M_q is defined by

$$A_q f = \widehat{f} + M_q f \tag{7.23}$$

and K_q is defined by taking the inverse Fourier transform

$$(A_q f)^\vee = f + K_q f. \tag{7.24}$$

It is easy to see that D_q is complete in $L^2(\Omega)$ if A_q is injective in $L^2(\mathbf{R}^2)$.

The next two propositions are the main technical points of the proof. Because of the decay in $|k|$ of the lower order terms ψ and $\widetilde{\psi}$ and the representation (7.7) one can prove:

Proposition (7.25). *$K_q : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is compact .*

Moreover the explicit construction of ψ as in (7.7) allows to prove

Proposition (7.26). *K_q depends analytically on q, that is, $K_{q_0+\lambda q_1} f$ has a convergent power series in $L^2(\mathbf{R}^2)$ for those λ's so that $\|q_0+\lambda q_1\|_{W^{1,\infty}(\Omega)} < R$ and $L_{q_0+\lambda q_1}$ does not have zero as an eigenvalue.*

Then for $\lambda \in \mathbf{C}$

$$(A_{\lambda q} f)^\vee = (Id + K_{\lambda q}) f\,,$$

$K_{\lambda q}$ is an analytic function of λ for λ's so that $|\lambda| \|q\|_{W^{1,\infty}(\Omega)} < R$ and $L_{\lambda q}$ does not have zero as eigenvalue. By the analytic Fredholm theorem then $(A_{\lambda q} f)^\vee$ is an isomorphism except for a discrete set of λ's. This sketches the proof of Theorem 7.17. For more details see [Su-U]. □

The proof of the global result Theorem 7.18 proceeds along similar lines. If $\Lambda_{q_1} = \Lambda_{q_2}$ relation (5.4) motivates the definition of a similar operator to K_q above. Let $q_1, q_2 \in W^{1,\infty}(\Omega)$ so that $\|q_i\|_{W^{1,\infty}(\Omega)} < R$ and L_{q_i}, $i = 1, 2$ does not have zero as eigenvalue. We define

$$(A_{q_1,q_2} f)^\vee = f + K_{q_1,q_2} f \tag{7.27}$$

where

$$(K_{q_1,q_2}f)^\wedge = \begin{cases} \int_\Omega e^{ix\cdot k} f(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2}), |k| > L_R \\ \int_\Omega f(\widetilde{u}_{q_1}\widetilde{u}_{q_2} - e^{ix\cdot k}), |k| \le L_R \end{cases},$$

$$u_i = e^{x\cdot\rho}(1+\psi_{q_i}), \quad |k| > L_R \tag{7.28}$$

is a solution of $L_{q_i}u_i = 0$ $i = 1,2$ as in theorem 4.1, and $\widetilde{u}_{q_i}$ solves

$$L_{q_i}\widetilde{u}_{q_i} = 0; \quad \widetilde{u}_{q_1}|_{\partial\Omega} = e^{x\cdot\rho}|_{\partial\Omega}; \quad \widetilde{u}_{q_2}|_{\partial\Omega} = e^{-x\cdot\overline{\rho}}|_{\partial\Omega}. \tag{7.29}$$

Again, $K_{q_1,q_2} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is compact and $K_{\lambda q_1,\lambda q_2}$ depends analytically on λ for λ such that $\|\lambda q_i\|_{W^{1,\infty}} < R$, $i = 1,2$. Then by the analytic Fredholm theorem $A_{\lambda q_1,\lambda q_2}$ is an isomorphism except for a discrete set of λ's.

Now if $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 - q_2$ is in the kernel of A_{q_1,q_2} (see (5.4)). Then for an open dense set $\mathcal{O}$ in $W^{1,\infty} \times W^{1,\infty}$ if $(q_1,q_2) \in \mathcal{O}$ and $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$. This finishes the sketch of proof of Theorem 7.17. □

8. Determining Lamé parameters by boundary measurements

Another inverse boundary value problem which arises in applications is to determine the elastic properties of a material by measuring the stress energy to maintain it in a prescribed shape. We formulate below more precisely the mathematical problem.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary which will be considered in this paper as a linear, inhomogeneous, isotropic, elastic medium. The elastic properties of Ω are determined by the pair of Lamé parameters $\gamma = (\lambda,\mu) \in L^\infty(\Omega)$. Moreover we assume the strong convexity assumption

$$\mu > 0, n\lambda + 2\mu > 0 \text{ on } \Omega. \tag{8.1}$$

Under the assumption (8.1) we can solve uniquely, with $\vec{u} \in H^1(\Omega)$, the displacement boundary value problem:

$$\begin{cases} (L_\gamma \overrightarrow{u})_i = \sum_{j,k,\ell=1}^n \partial_{x_j}(c_{ijk\ell}\partial_{x_k}u_\ell) = 0 \quad \text{in } \Omega, i = 1,\dots,n, \\ \overrightarrow{u}|_{\partial\Omega} = \overrightarrow{\phi} \in H^{\frac{1}{2}}(\partial\Omega), \end{cases} \tag{8.2}$$

where the elasticity tensor is given by

$$c_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \tag{8.3}$$

the displacement vector is denoted by $\overrightarrow{u} = (u_1, \dots, u_n)$, and δ_{ij} denotes the Kronecker delta.

Associated to the displacement vector $\overrightarrow{u}$, there are two tensor fields

$$\epsilon(\overrightarrow{u}) = \frac{1}{2}(\nabla \overrightarrow{u} +^t \nabla \overrightarrow{u}) \tag{8.4}$$

and

$$\tau(\overrightarrow{u}) = \lambda(\text{trace } \epsilon(\overrightarrow{u}))I + 2\mu\epsilon(\overrightarrow{u}) \tag{8.5}$$

which are called the strain tensor and stress tensor respectively. Here $\overrightarrow{u} = (u_1, \dots, u_n)$, $\nabla \overrightarrow{u} = (\partial_{x_j} u_i; \begin{smallmatrix} i \downarrow 1, \dots, n \\ j \rightarrow 1, \dots, n \end{smallmatrix})$, trace $\epsilon(\overrightarrow{u}) = \Sigma_{j=1}^n \partial_{x_j} u_j$. The equation in (8.2) simply means that the stress tensor is divergence free (i.e. there no source or sinks of stress):

$$L_\gamma \overrightarrow{u} = \nabla \cdot \tau(\overrightarrow{u}) = 0 \text{ in } \Omega. \tag{8.6}$$

The energy associated to a solution $\overrightarrow{u}$ of (8.6) is given by

$$\begin{aligned} Q_\gamma(\overrightarrow{\phi}) &= \inf_{\overrightarrow{v} \in H^1(\Omega)} \int_\Omega \text{trace}(\tau(\overrightarrow{u})\overline{\epsilon(\overrightarrow{v})})dx \\ &= \int_\Omega \sum_{i,j,k,\ell} c_{ijk\ell} \partial_{x_j} u_i \overline{\partial_{x_\ell} u_k} dx \\ &= \int_\Omega \{\lambda | \text{ div } \overrightarrow{u}|^2 + 2\mu|\epsilon(\overrightarrow{u})|^2\} dx. \end{aligned} \tag{8.7}$$

The stress energy form obtained by polarization of (8.7) is given by

$$Q_\gamma(\overrightarrow{\phi}, \overrightarrow{\psi}) = \int_\Omega (\lambda \text{ div } \overrightarrow{u} \cdot \overline{\text{div } \overrightarrow{u}} + 2\mu\epsilon(\overrightarrow{u}) \cdot \overline{\epsilon(\overrightarrow{v})} dx \tag{8.8}$$

where $\overrightarrow{u}$, $\overrightarrow{v}$ are solutions of

$$L_\gamma \overrightarrow{u} = L_\gamma \overrightarrow{v} = 0 \quad \text{in } \Omega, \overrightarrow{u}|_{\partial\Omega} = \overrightarrow{\phi}, \overrightarrow{v}|_{\partial\Omega} = \overrightarrow{\psi}. \tag{8.9}$$

By using Green's theorem one can easily prove that

$$Q_\gamma(\overrightarrow{\phi}, \overrightarrow{\psi}) = \int_{\partial\Omega} \tau(\overrightarrow{u})\nu \cdot \overline{\overrightarrow{v}} dS = \int_{\partial\Omega} \Lambda_\gamma \overrightarrow{\phi} \cdot \overline{\overrightarrow{\psi}} dS \tag{8.10}$$

where ν denotes the unit outer normal to $\partial\Omega$ and dS denotes surface measure on $\partial\Omega$. The Dirichlet to Neumann map is defined by

$$(8.11)\qquad (\Lambda_\gamma \overrightarrow{\phi})_i = (\tau(\overrightarrow{u})|_{\partial\Omega} \cdot \nu)_i = \sum_{j,k,\ell=1}^{n} \nu_j c_{ijk\ell} \partial_{x_k} u_\ell|_{\partial\Omega}.$$

Physically, $\Lambda_\gamma \overrightarrow{f} = T\overrightarrow{u}$ where T measures traction on the boundary.

The inverse problem is whether knowledge of $\Lambda_\gamma \overrightarrow{\phi}$ for any $\overrightarrow{\phi} \in H^{\frac{1}{2}}(\partial\Omega)$, which involves only boundary measurements, determines the Lamé parameters λ and μ in $\overline{\Omega}$. That is we want to determine the injectivity of the map

$$L^\infty(\Omega) \times L^\infty(\Omega) \ni \gamma = (\lambda, \mu) \xrightarrow{\Lambda} \Lambda_\gamma.$$

Because of (8.10) knowledge of the selfadjoint map

$$\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is equivalent to knowledge of $Q_\gamma(\overrightarrow{\phi}, \overrightarrow{\psi})$ for any $\overrightarrow{\phi}, \overrightarrow{\psi} \in H^{\frac{1}{2}}(\partial\Omega)$.

In [N-U] it was proven, in two dimensions, local injectivity of Λ in a $W^{31,\infty}(\Omega)$ neighborhood of constant λ, μ.

Let $\gamma_* = (\lambda_*, \mu_*)$ denote a pair of constant Lamé parameters in Ω satisfying (8.1). Then we have

(8.12) Theorem. *Let $n = 2$. There exists $\epsilon > 0$ such that if $\gamma_j = (\lambda_j, \mu_j)$ satisfy (8.1),*

$$\|\lambda_j - \lambda_*\|_{W^{31,\infty}(\Omega)} + \|\mu_j - \mu_*\|_{W^{31,\infty}(\Omega)} < \epsilon$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2},$$

then $\gamma_1 = \gamma_2$ on $\overline{\Omega}$.

There are several new difficulties in extending the method used in [S-U II] for the conductivity problem to this case. First of all L_γ is an elliptic system and second we have to determine two functions λ, μ of $\gamma = (\lambda, \mu)$. To underscore these difficulties let us look at the linearized problem. The Fréchet derivative of Λ at a constant pair $\gamma_* = (\lambda_*, \mu_*)$ in a direction $h = (h_1, h_2)$ is given by

$$(8.13)\qquad \begin{aligned} &(d\Lambda_{\gamma_*}(h)(\overrightarrow{u}_*|_{\partial\Omega}), (\overrightarrow{v}_*|_{\partial\Omega})) = \\ &\int_\Omega \{h_1 \operatorname{div} \overrightarrow{u}_* \cdot \overline{\operatorname{div} \overrightarrow{v}_*} + 2h_2(\epsilon(\overrightarrow{u}_*) \cdot \overline{\epsilon(\overrightarrow{v}_*)})\}dx \end{aligned}$$

where $\overrightarrow{u}_*, \overrightarrow{v}_*$ are solutions of

$$L_{\gamma_*}(\overrightarrow{u}_*) = L_{\gamma_*}(\overrightarrow{v}_*) = 0 \quad \text{in } \mathbf{R}^2. \tag{8.14}$$

We first construct analogous solutions of (8.14) to the ones considered by Calderón for the linearized problem for the inverse conductivity problem. Namely we take

$$\overrightarrow{u}_* = \nabla e^{x\cdot\zeta}, \overrightarrow{v}_* = \nabla e^{-x\cdot\overline{\zeta}} \text{ with } \zeta \in \mathbf{C}^2, \zeta\cdot\zeta = 0. \tag{8.15}$$

Notice that $\overrightarrow{u}_*, \overrightarrow{v}_*$ are vector-valued harmonic functions. Substituting (8.15) in (8.13) we find that

$$(d\Lambda_*(h)(\overrightarrow{u}_*|_{\partial\Omega})\overline{(\overrightarrow{v}_*|_{\partial\Omega})}) = |k|^2 \int_\Omega 2h_2 e^{ix\cdot k} dx, \tag{8.16}$$

where

$$\Lambda_* = \Lambda_\gamma \text{ with } \gamma = \gamma_*$$

and

$$\zeta = \frac{1}{2}(Jk + ik), J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = (k_1, k_2) \in \mathbf{R}^2. \tag{8.17}$$

If $d\Lambda_*(h) = 0$, then we get by the Fourier inversion formula $h_2 = 0$ in Ω.

So we need different solutions of (8.14) to get information about h_1. Ikehata [I] used a different set of solutions of (8.14) other than (8.15) that allowed him to prove injectivity of the linearized map (8.16) at the constant pair γ_*.

Ikehata found these by constructing new solutions of the biharmonic operator Then he used the so called Boussineq-Somigliana- Garlekin method to construct solutions of the elasticity system at a constant pair. Namely if g solves

$$\Delta^2 g = 0 \text{ in } \Omega \tag{8.18}$$

then

$$u = (\lambda_* + 2\mu_*)\Delta g - (\lambda_* + \mu_*)\nabla(\nabla\cdot g) = F(g), \tag{8.19}$$

solves

$$L_{\gamma_*}(u) = 0 \quad \text{in } \Omega. \tag{8.20}$$

Ikehata considered

$$(8.21) \qquad g_1 = -\frac{1}{2}|\zeta|^{-2}(x\cdot\overline{\zeta})e^{-x\cdot\zeta}$$

$$g_2 = -\frac{1}{2}|\overline{\zeta}|^{-2}(x\cdot -\zeta)e^{x\cdot\overline{\zeta}},$$

with ζ as in (8.17), as solutions of (8.18) and $u_* = F(g_1), v_* = F(g_2)$ as solutions of (8.14). Plugging these in (8.13) we find that
(8.22)

$$d\Lambda_{\gamma_*}(h)(u_*|_{\partial\Omega}, v_*|_{\partial\Omega}) = \mu_*^2\frac{|k|^2}{4}\int_\Omega e^{ix\cdot k}h_1(x)dx + \mu_*^2\frac{|k|^2}{4}\int_\Omega e^{ix\cdot k}h_2(x)dx.$$

$$+(\lambda_* + \mu_*)^2\frac{|k|^4}{8}\int_\Omega |x|^2 h_2(x)e^{ix\cdot k}dx$$

We already know that $h_2 = 0$ if $d\Lambda_{\gamma *}(h) = 0$, therefore we conclude that $h_1 = 0$ concluding the proof that the linearized problem is injective at constant Lamé parameters.

The main difficulty in the non-linear case is to construct for high frequencies the analog of the solutions (8.21). This was done in [N-U]. We outline some of the ideas.

Akamatsu, Nakamura and Steinberg [A-N-S] proved the analog of the Kohn-Vogelius result in this case. We have

(8.23) Theorem. *(Akamatsu, Nakamura, Steinberg) Let* $\gamma_j \in C^2(\overline{\Omega})(j = 1,2)$ *satisfying (8.1). Assume*

$$\Lambda_1 = \Lambda_2 \text{ where } \Lambda_j = \Lambda_\gamma \text{ with } \Lambda = \Lambda_j (j = 1,2).$$

Then

$$\partial^\alpha\gamma_1|_{\partial\Omega} = \partial^\alpha\gamma_2|_{\partial\Omega} \quad (|\alpha| \le 2).$$

Hence we may assume $\gamma_1 - \gamma_2 \in C_0^2(\overline{\Omega}), \gamma_j - \gamma_* \in C_0^2(B(0,r_0))(j = 1,2)$, where $B(0,r_0) = \{x \in \mathbf{R}^2; |x| < r_0\} \supset \overline{\Omega}$.

Another important fact is that in two dimensions one can diagonalize the elasticity system to a system whose principal part is the biharmonic operator Δ^2.

(8.24) Proposition. *Let* $n = 2$ *and* $\gamma = (\lambda,\mu) \in C^2(\overline{\Omega})$ *satisfying (8.1). Moreover let*

$$T(D) = \begin{bmatrix} D_1 & D_2 \\ D_2 & -D_1 \end{bmatrix}$$

where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. *Then*
(8.25)

$$T(D)L_\gamma(x,D)T(D) = \begin{bmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{bmatrix}(-\Delta^2)I + M_\gamma^3(x,D) + M_\gamma^2(x,D),$$

where
(8.26)

$$M_\gamma^3(x,\xi) = 2i|\xi|^2 \begin{bmatrix} \nabla(\lambda+2\mu)\cdot\xi & -(\xi\wedge\nabla)\mu \\ (\xi\wedge\nabla)\mu & \nabla\mu\cdot\xi] \end{bmatrix}, \xi\wedge\nabla = \xi_1\frac{\partial}{\partial x_2} - \xi_2\frac{\partial}{\partial x_1}$$

and $M_\gamma^2(x,D)$ *is a system of second order differential operators whose coefficients consist of second order derivatives of* γ.

Let

$$M_\gamma(x,D) = -\Delta^2 I + \begin{bmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{bmatrix}^{-1}(M_\gamma^3(x,D)) + M_\gamma^2(x,D)). \tag{8.27}$$

Factorizing $-\Delta$ in (1.25) we get

$$M_\gamma(x,D) = (-\Delta)\{\Delta I + \widetilde{M_\gamma^1}(x,D) + \widetilde{M_\gamma^0}(x,D)\} \tag{8.28}$$

where

$$\widetilde{M_\gamma^1}(x,\xi) = \begin{bmatrix} \lambda+2\mu & 0 \\ 0 & \mu \end{bmatrix}^{-1} M_\gamma^3(x,\xi)|\xi|^{-2} \tag{8.29}$$

and $\widetilde{M_\gamma^0}(x,D)$ is a pseudodifferential operator of order 0 such that

$$M_\gamma(x,D) = (-\Delta)\widetilde{M_\gamma^0}(x,D) \tag{8.30}$$

is a system of second order differential operator whose coefficients are p-th $(2 \le p \le 4)$ order derivatives of γ.

For each compact set high frequency solutions of

$$(\Delta I + \widetilde{M_\gamma^1}(x,D) + \widetilde{M_\gamma^0}(x,D)\overrightarrow{w} = 0 \text{ or a constant vector in } \mathbf{R}^2 \tag{8.31}$$

are constructed of the form

$$\overrightarrow{w} = e^{x\cdot\zeta}(A_0(x,\zeta) + A_{-1}(x,\zeta)), \zeta \in \mathbf{C}^2, \zeta\cdot\zeta = 0, \tag{8.32}$$

where $A_0(x,\zeta), |\zeta|A_{-1}(x,\zeta)$ are uniformly bounded. Here we remark that although $\overrightarrow{w}$ is constructed on a compact set, $\Delta\overrightarrow{w}$ has a natural extension to $\mathbf{R}^2$ so that it satisfies (8.31).

One difference with the conductivity equation is that in that case $A_0(x,\zeta)$ is independent of ζ (in fact $A_0(x,\zeta) = \gamma^{-\frac{1}{2}}$ where γ is the conductivity). Moreover one does not solve the transport equation for A_{-1} in a unique fashion in an appropriate weighted class. However, it is solved in every compact set. Everything works out since one can check that

$$\overline{D^1}\{L\gamma(x,D)T(D)\overrightarrow{w}\} \in L^p_\delta(\mathbf{R}^2) = L^p(\mathbf{R}^2;(1+|x|^2)^{\frac{p\delta}{2}}dx);$$
$$1 < p < \infty, -\frac{2}{p} < \delta < 1 - \frac{2}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

where $\overline{D^k} = (\partial^\alpha/\partial x^\alpha; |\alpha| \leq k)$ for $k \in N$. Since $T(D)^2 = -\Delta I$ one gets by standard estimates that in fact

$$L_\gamma(x,D)T(D)\overrightarrow{w} = 0 \quad \text{in } \mathbf{R}^2.$$

One must also match the two types of low frequency solutions that are constructed (as in [I], but slightly different) with the high frequency solutions. Full details are in the paper [N-U].

9. Electrical impedence tomography; the anisotropic case

If the conductivity of Ω depends on direction then it is represented by a positive definite symmetric matrix $\gamma = (\gamma^{ij})$ in $\overline{\Omega}$ which we assume to be smooth. Kohn and Vogelius ([K-V III]) suggested a constructive approach to the isotropic case based on an algorithm developed by Wexler et al ([W-F-N]). This consists of minimizing an appropriate functional. The functional is not quasi-convex and, therefore, a minimizing sequence will not in general converge to a solution, but will in general have limit points which are solutions to the "relaxed problem". Kohn and Vogelius computed the relaxation of one such problem which turned out to be a variational problem for an anisotropic conductivity. Numerical performance of this method has been recently studied by Kohn and McKenney ([K-M]). Thus the anisotropic problem occurs naturally even when considering isotropic conductivities. We now formulate more precisely the inverse conductivity problem in the anisotropic case.

The conductivity equation is

$$L_\gamma u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}(\gamma^{ij}\frac{\partial}{\partial x_j}u) = 0 \quad \text{in } \Omega. \tag{9.1}$$

The Dirichlet to Neumann map is defined by

$$\Lambda_\gamma f = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}|_{\partial\Omega} dS \tag{9.2}$$

where ν^i is the i-th component of the unit euclidean conormal, dS represents the $(n-1)$ dimensional euclidean surface measure on $\partial\Omega$ and u is the solution of the Dirichlet problem

$$L_\gamma u = 0 \quad \text{in } \Omega \tag{9.3}$$

$$u|_{\partial\Omega} = f,$$

It is convenient to define the Dirichlet to Neumann map as a $(n-1)$ form since in actual measurements one integrates the current flux rather than measure it pointwise. Moreover it helps to understand the invariance properties of Λ_γ under the action of diffeomorphisms. We have, again, using the divergence theorem

$$\int_{\partial\Omega} g\Lambda_\gamma(f) = \int_\Omega \sum_{i,j=1}^{n} \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dV \tag{9.4}$$

where dV is the euclidean volume element in Ω, u as in (8.3) and v solves

$$L_\gamma v = 0 \quad \text{in } \Omega \tag{9.5}$$

$$v|_{\partial\Omega} = g.$$

Again, instead of considering the map

$$\gamma \xrightarrow{\Phi} \Lambda_\gamma \tag{9.6}$$

we can consider the map

$$\gamma \xrightarrow{Q} Q_\gamma \tag{9.7}$$

where Q_γ is the quadratic form

$$Q_\gamma(f) = \int_\Omega \sum_{i,j=1}^{n} \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dV \tag{9.8}$$

with u solution of (9.3).

Unfortunately injectivity of Φ (or Q) is not valid in the anisotropic case. The following observation can be found in [K-V IV]: Let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism so that $\Psi|_{\partial\Omega} = Id$. Let

$$\widetilde{\gamma} = \frac{(D\Psi)^T \cdot \gamma \cdot D\Psi) \circ \Psi^{-1}}{|detD\Psi|} \tag{9.9}$$

where $D\Psi$ denotes the differential of Ψ and $(D\Psi)^T$ its transpose. The relevant point is that

$$\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma \ (Q_{\widetilde{\gamma}} = Q_\gamma). \tag{9.10}$$

This is a consequence of the following observation:

Proposition 9.11. *Let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism so that $\Psi|_{\partial\Omega} = Id$. Then if u solves*

$$L_\gamma u = 0 \quad \text{in } \Omega$$

$$u|_{\partial\Omega} = f,$$

then $u \circ \psi^{-1} = \widetilde{u}$ solves

$$L_{\widetilde{\gamma}} \widetilde{u} = 0 \quad \text{in } \Omega$$

$$\widetilde{u}|_{\partial\Omega} = f$$

with $\widetilde{\gamma}$ as in (9.9).

More generally, let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism so that $\Psi|_{\partial\Omega} = \psi$. Then

$$Q_\gamma(f) = Q_{\widetilde{\gamma}}(f \circ \psi^{-1}) \tag{9.12}$$

with $\widetilde{\gamma}$ as in (9.9).

We disgress a little to discuss the corresponding relation for Λ_γ given by (9.4). It is convenient to give an invariant interpretation of (9.9), (9.10) and (9.12). For more details see the discussion in the introduction in [S]. Ohm's law (or rather its differential version) in a wire is given by

$$i(x) = \gamma(x) du(x)$$

where $u(x)$ is the voltage potential, $i(x)$ the current flowing through x and $\gamma(x) = 1/\rho(x)$ where $\rho(x)$ is the resistivity.

In higher dimensions the current i is represented by an $(n-1)$ form. Then it is natural to interpret the conductivity as a map from 1 forms $(du(x))$ to $(n-1)$ forms $(i(x))$. The conductivity is then a map

$$\gamma : \Lambda^1(\overline{\Omega}) \to \Lambda^{n-1}(\overline{\Omega}) \tag{9.13}$$

which is symmetric and positive definite as explained below.

In standard Euclidean coordinates $x^1, \ldots, x^n$ and ω_k the $(n-1)$ forms

$$\omega_k = (-1)^{k-1} dx^1 \wedge \cdot \cdot \wedge dx^{k-1} \wedge dx^{k+1} \wedge \ldots \wedge dx^n \tag{9.14}$$

then the components γ^{ij} of γ are given by

$$\gamma(dx^i) = \sum_{i=1}^{n} \gamma^{ij} \omega_j \, ,$$

with γ^{ij} a symmetric, positive definite matrix in $\overline{\Omega}$.

If $\Psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism then the push forward of γ as in (9.13) is given by

$$(\Psi_* \gamma)\alpha = \Psi_*(\gamma(\Psi^* \alpha)) \tag{9.15}$$

where $\Psi^* \alpha$ denotes the pull back of the 1-form α and $\Psi_* = (\Psi^{-1})^*$ denotes the pull back by Ψ^{-1} acting on the $(n-1)$ form $\gamma(\Psi^* \alpha)$. In coordinates (9.15) reads

$$(\Psi_* \gamma(y))^{\ell m} = \frac{\frac{\partial \Psi^\ell}{\partial x^i} \gamma^{ij} \frac{\partial \Psi^m}{\partial x^j}}{|\det(\frac{\partial \Psi}{\partial x})|} \tag{9.16}$$

which is exactly the relation (9.9). Thus we may rewrite the relation (9.9) in an invariant way as

$$\widetilde{\gamma} = \Psi_* \gamma \tag{9.17}$$

Now we define the Dirichlet to Neumann map by

$$\Lambda_\gamma f = \gamma du|_{\partial\Omega}$$

which, in coordinates, is just (9.2).

If $\Psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism with $\Psi|_{\partial\Omega} = \psi$ we can define the push forward $\Psi_* \Lambda_\gamma$ by

$$(\psi_* \Lambda_\gamma) f = \psi_*(\Lambda_\gamma(\psi^* f))$$

where $\psi^* f = f \circ \psi^{-1}$. Then the relation (9.12) can be rewritten as

$$\Lambda_{\Psi_* \gamma} = \psi_* \Lambda_\gamma \tag{9.18}$$

Of course, if $\Psi|_{\partial\Omega} = Id$ we obtain

$$\Lambda_{\Psi_* \gamma} = \Lambda_\gamma \tag{9.19}$$

which is (9.10).

The natural conjecture is that (9.19) is the only obstruction to uniqueness.

Conjecture 9.20. Let γ_1, γ_2 be smooth anisotropic symmetric and positive definite conductivities in $\overline{\Omega}$. Suppose

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}.$$

Then there exists a diffeomorphism $\Psi : \overline{\Omega} \to \overline{\Omega}$ diffeomorphism such that $\Psi|_{\partial\Omega} = Id$ so that

$$\Psi_* \gamma_1 = \gamma_2 .$$

Progress has been made in proving the conjecture even though the general case remains unsolved. In the two dimensional case Sylvester ([S]) proved

Theorem 9.21. *($n = 2$) Let γ_i be C^3 anisotropic conductivities with*

$$\|\gamma_i\|_{C^3(\overline{\Omega})} \leq M, \quad i = 1, 2.$$

Then there exists $\varepsilon(\Omega, M)$ such that if

$$\|\log(\det \gamma_i)\|_{C^3(\overline{\Omega})} < \varepsilon \quad i = 1, 2$$

and

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then there exists a C^3 diffeomorphism Ψ with

$$\gamma_1 = \Psi_* \gamma_2, \quad \Psi|_{\partial\Omega} = Id.$$

Sketch of Proof of Theorem 9.21. The first step in the proof to use the existence of isothermal coordinates (see for instance [A]) to reduce the proof to a new isotropic problem.

Proposition (9.22). *(Isothermal coordinates) Given a C^3 anisotropic conductivity γ with*

$$\|\gamma\|_{C^3(\overline{\Omega})} \leq M,$$

we can find a constant $k = k(M)$, and a C^3 diffeomorphism Ψ such that

$$\Psi : \overline{\Omega} \to \overline{D} = \{x \in \mathbf{R}^2; |x| \leq 1\},$$

$$\|\Psi\|_{C^3} \leq k$$

and

$$\Psi_* \gamma \quad \text{is isotropic.} \tag{9.23}$$

Let γ_i belong to $C^3(\overline{\Omega})$. Then there exists a diffeomorphism $\Psi_i : \overline{\Omega} \to \overline{D}$ so that

$$(9.24) \qquad (\Psi_i)_* \gamma_i = \alpha_i e \quad i = 1, 2$$

where $\alpha_i \in C^3(\overline{\Omega})$ has a positive lower bound, $i = 1, 2$ and e is the euclidean conductivity.

Let $\Psi_i|_{\partial\Omega} = \psi_i$ for $i = 1, 2$. Then using (9.24) and (9.18) we obtain

$$(9.25) \qquad \Lambda_{\alpha_1 e} = \Lambda_{(\Psi_1)_* \gamma_1} = (\psi_1)_* \Lambda_{\gamma_1} .$$

Using the hypothesis $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$,(9.18), and (9.24) we have

$$(\psi_1)_* \Lambda_{\gamma_1} = (\psi_1)_* (\psi_2^{-1})_* (\psi_2)_* \Lambda_{\gamma_2} = (\phi)_* \Lambda_{(\Psi_2)_* \gamma_2} = \phi_* \Lambda_{\alpha_2 e}$$

where

$$(9.26). \qquad \phi = \psi_1 \psi_2^{-1}$$

We conclude that

$$(9.27) \qquad (\phi)_* \Lambda_{\alpha_2 e} = \Lambda_{\alpha_1 e}$$

which is a relation between two isotropic conductivities. The main technical result in [S] is

Lemma (9.28). *Let α_i be C^3 isotropic conductivity, $i = 1, 2$ such that (9.27) is satisfied. Then there exists a C^3 conformal map $\Phi : \overline{D} \to \overline{D}$, such that*

$$(9.29) \qquad \Phi|_{\partial D} = \phi.$$

Assuming the lemma for a moment we complete the proof of Theorem 9.21. Let Φ be as in (9.29). Then by (9.18) and (9.27)

$$(9.30) \qquad \Lambda_{\alpha_1 e} = (\phi)_* \Lambda_{\alpha_2 e} = \Lambda_{\Phi_* (\alpha_2 e)}.$$

Since α_2 is isotropic and Φ is conformal, then $(\Phi)_*(\alpha_2 e)$ is also isotropic. The smallness hypothesis in Theorem 9.21 and the bounds for Ψ_i imply that α_i is close to 1, $i = 1, 2$. Using now the local result of [SU-II] (see Theorem (7.1)) for the isotropic case in dimension 2 we conclude

$$\alpha_1 = \Phi_*(\alpha_2 e)$$

and by (9.9) since Φ is conformal

$$\alpha_1 = \alpha_2 \circ \Phi.$$

Unwinding the definitions then yields

$$\gamma_1 = (\Psi_1^{-1}\Phi\Psi_2)_*\gamma_2.$$

It follows from (9.26) that, on the boundary,

$$\psi_1^{-1}\phi\psi_2 = Id$$

which proves the theorem. □

The proof of the lemma begins by constructing a C^2-diffeomorphism

$$\Phi : \overline{D^c} \to \overline{D^c}, \Phi = Id \text{ for } |x| > R \,, \Phi|_{\partial D} = \phi$$

such that the (anisotropic) conductivity given by

$$\gamma_{12} = \begin{cases} \alpha_2 & \text{for } |x| < 1 \\ \Phi_*\alpha_1 & \text{for } |x| > 1 \end{cases} \tag{9.31}$$

is in $C^{1,1}(\mathbf{R}^2)$ where α_2 has been extended as a $C^{1,1}$ function to $\mathbf{R}^2$. To see that such a Φ exists involves the formal solution to a Beltrami equation as well as the computation of the two first two terms in the expansion of the full symbol of $\phi_*\Lambda_{\alpha_1}$ and Λ_{α_2} (see [S], Prop. 3.1).

A more precise version of the existence of isothermal coordinates allows the construction of a unique C^2-diffeomorphism $F^{12} : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$(F^{12})_*\gamma_{12} = (\det \gamma_{12} \circ (F^{1,2})^{-1})^{\frac{1}{2}} e \tag{9.32}$$

where e is the euclidean conductivity. If we consider F^{12} as a complex valued function, it is the unique solution to the Beltrami equation

$$\overline{\partial} F^{12} = \mu_{12} F^{12}. \tag{9.33}$$

which is asymptotic to z at infinity (see [S], Prop 2.1) for a more precise description). In (9.33), μ_{12} is a rational function in the coefficients of γ which is called the complex dilitation. In particular,

$$\mu_{12} = 0 \Longleftrightarrow \gamma_{12} \text{ is isotropic }. \tag{9.34}$$

Therefore,

$$F^{12} = Id \Longleftrightarrow \gamma_{12} \quad \text{is isotropic }. \tag{9.35}$$

This version of isothermal coordinates can be used to prove that the special solutions of Theorem (4.1) exist in the anisotropic case; that is, there exist unique solutions $u(z,k)$ which are asymptotic to e^{kz} at infinity (it is convenient to use complex notation, $k, z \in \mathbf{C}$) which solve

$$L_{\gamma_{12}} u = 0 \quad \text{in } \mathbf{R}^2 . \tag{9.36}$$

Moreover, one can show that

$$\lim_{|k|\to\infty} \frac{\log u(z,k)}{k} = F^{12} \tag{9.37}$$

uniformly on compact sets.

Arguments similar to those used in the proof of Proposition 6.16 can be used to show that

$$u = \begin{cases} v & \text{for } |x| < 1 \\ u_1 \circ \Phi^{-1} & \text{for } |x| \geq 1 \end{cases} \tag{9.38}$$

where Φ is as in (9.31), v solves the Dirichlet problem

$$L_{\alpha_2} v = 0 \text{ in } D \tag{9.39}$$

$$v|_{\partial\Omega} = u_1|_{\partial\Omega} \circ \phi^{-1} = u_1 \circ \Phi^{-1}|_{\partial\Omega} \quad ,$$

and $u_1(z,k)$ is the special solution of

$$L_{\alpha_1} u_1 = 0 \text{ in } \mathbf{R}^2 \tag{9.40}$$

which is asymptotic to e^{kz} at infinity. Now since γ_{12} is isotropic in D we have

$$\overline{\partial} F^{12} = 0 \text{ in } D. \tag{9.41}$$

For points on the boundary of D, (9.37) implies that (recall that u is smooth across ∂D)

$$F^{12} = \lim_{|k|\to\infty} \frac{\log u}{k} = \lim_{|k|\to\infty} \log \frac{u_1(z,k) \circ \Phi^{-1}}{k} = F_1 \circ \phi^{-1} . \tag{9.42}$$

where F_1 is the solution to the Beltrami equation associated to the conductivity α_1. Since α_1 is isotropic (9.35) implies that $F_1(z) = z$ and hence that

$$F^{12}|_{\partial D} = \phi^{-1}\,. \tag{9.43}$$

¿From (9.41) and (9.43) we conclude that F^{12} is the conformal map with boundary value ϕ^{-1}. Therefore $(F^{12})^{-1}$ is the desired conformal map. □

The proof above relied heavily on the construction of isothermal coordinates This is not available in dimension $n \geq 3$.

J. Lee and G. Uhlmann ([L-U]) have proved conjecture 9.20 in dimension $n \geq 3$ in the real-analytic category under certain restrictions.

First we note that in dimensions $n \geq 3$ we can identify Riemannian metrics and anisotropic conductivities.

Let g be a smooth Riemannian metric in $\overline{\Omega}$. We denote by Δ_g the Laplace-Beltrami operator associated to g. In local coordinates

$$\Delta_g u = \sum_{i,j=1}^{n} (\det g_{k\ell})^{-1/2} \frac{\partial}{\partial x_i}((\det g_{k\ell})^{\frac{1}{2}} g^{ij} \frac{\partial u}{\partial x_j}) \tag{9.45}$$

where g^{ij} is the inverse of the metric g_{ij}.

We can solve the Dirichlet problem

$$\Delta_g u = 0 \quad \text{in } \Omega \tag{9.46}$$

$$u|_{\partial\Omega} = f$$

and define the Dirichlet to Neumann map as map from functions on the boundary to $(n-1)$ forms in the boundary, by

$$\Lambda_g f = \nabla_g u \lrcorner\, dV_g|_{\partial\Omega}, \tag{9.47}$$

where ∇_g denotes the gradient with respect to the metric g, dV_g is the Riemannian volume element and $\lrcorner$ denotes interior differentiation. (We recommend the book by Spivak [Sp] for the reader unfamiliar with the differential geometric terms used.)

Let γ be an anisotropic conductivity given in local coordinates by γ^{ij}. Then if $n \geq 3$

$$g_{ij} = (\det \gamma^{k\ell})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \tag{9.48}$$

is a Riemannian metric with

$$\Lambda_g = \Lambda_\gamma. \tag{9.49}$$

Conversely, if g is Riemannian metric given in local coordinates by g_{ij}, then

$$\gamma^{ij} = (\det g_{k\ell})^{1/2}(g_{ij})^{-1} \tag{9.50}$$

is an anisotropic conductivity satisfying (9.49). We shall identify in the rest of this section conductivities and Riemannian metrics.

We first compute the full symbol of Λ_g if g is a smooth Riemannian metric. For this it is convenient to use boundary normal coordinates. For each $x_0 \in \partial\Omega$, let γ_{x_0} be the unit speed geodesic starting at x_0 and normal to $\partial\Omega$. If $\{x^1, \ldots, x^{n-1}\}$ are local coordinates for $\partial\Omega$ near $p \in \partial\Omega$, we can extend them smoothly to functions in a neighborhood of p in $\overline{\Omega}$ by letting them be constant along each normal geodesic γ_{x_0}. If we then define x^n to be the parameter along each γ_{x_0}, it follows that $\{x^1, \ldots, x^n\}$ are coordinates in $\overline{\Omega}$, which we call *boundary normal coordinates* determined by $\{x^1, \ldots, x^{n-1}\}$. In these coordinates $x^n > 0$ in Ω and $\partial\Omega$ is locally characterized by $x^n = 0$. The metric g takes the form

$$g = \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}(x)dx^\alpha dx^\beta + (dx^n)^2, \tag{9.51}$$

and the Laplace-Beltrami operator is given by

$$-\Delta_g = D_{x^n}^2 + iE(x)D_{x^n} + Q(x, D_{x'}) \tag{9.52}$$

where

$$E(x) = -\frac{1}{2}\sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(x)\partial_{x^n} g_{\alpha\beta}(x),$$

$$Q(x, D_{x'}) = \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}(x)D_{x^\alpha}D_{x^\beta} - i\sum_{\alpha,\beta=1}^{n-1}\left(\frac{1}{2}g^{\alpha\beta}(x)\partial_{x^\alpha}\log r(x) + \partial_{x^\alpha}g^{\alpha\beta}(x)\right)D_{x^\beta}$$

and $x = (x', x^n)$. Moreover

$$r(x) = \det(g_{ij})\,.$$

We use (9.52) to factorize Δ_g and give an easy way to compute the full symbol of the Dirichlet to Neumann map (see [L-U] Proposition 1.1).

Proposition 9.53. *There exists a pseudodifferential operator $A(x, D_{x'})$ of order one in x' depending smoothly on x^n such that*

$$-\Delta_g = (D_{x^n} + iE(x) - iA(x, D_{x'}))(D_{x^n} + iA(x, D_{x'})) \tag{9.54}$$

modulo a smoothing operator.

We can actually write the full symbol, $a(x,\xi')$, of $A(x,D_{x'})$

$$(9.55) \qquad a(x,\xi') \sim \sum_{j\leq 1} a_j(x,\xi'), \quad \xi' \in \mathbf{R}^{n-1},$$

with a_j homogeneous of degree j in ξ' and

$$(9.56) \qquad a_1(x,\xi') = -\sqrt{q_2}$$

with q_2 the principal symbol of Q as in (9.52)

$$a_{m-1}(x',\xi') = \frac{1}{2\sqrt{q_2}} \sum_{\substack{j,k,K \\ m\leq j,k\leq 1 \\ |K|=j+k-m}} \frac{1}{K!}(\partial_{\xi'}^K(a_j)D_{x'}^K(a_k) + \partial_{x^n}a_m - Ea_m)$$

The main point is that

Proposition 9.57. *$\Lambda_g f = (-1)^{n-1}r^{1/2}Afdx^1\wedge\ldots\wedge dx^{n-1}|_{\partial\Omega}$ modulo a smoothing operator.*

Sketch of Proof. This follows from the factorization (9.54). Let u satiusfies (9.46). Then using the factorization (9.54) we get that

$$(9.58) \qquad (D_{x^n} + iA)u = v$$

$$u|_{x^n=0} = f$$

with

$$(9.59) \qquad (D_{x^n} + iE - iA)v = h \in C^\infty([0,T]\times\mathbf{R}^{n-1}) \text{ for } T>0.$$

It follows, since (9.59) can be viewed as a backwards generalized heat equation (make the substitution $t = T - x_n$), that v is also smooth (see [T]). Therefore from (9.58) and elliptic regularity we conclude $(D_{x^n}u) = -iAu$ modulo a smooth function and $\Lambda_g f = \frac{1}{i}D_{x^n}u|_{x^n=0}$ in boundary coordinates. $\square$

The computation (9.56), together with Proposition (9.57) shows (see [L-U], Prop. 1.3) that one can determine from a_j the full Taylor series of g in boundary normal coordinates. This is the analog in the anisotropic case of the Kohn-Vogelius result theorem in the isotropic case.

Theorem 9.60. *Let $n\geq 3$. Let $\{x^1,\ldots,x^{n-1}\}$ be any local coordinates for an open set $U\subset\partial M$ and let $\{a_j, j\leq 1\}$ denote the full symbol of A in* *these coordinates. For any $p \in U$, the full Taylor series of g at p in boundary normal coordinates is given by an explicit formula in terms of the functions $\{r^{1/2}a_j\}$ and their tangential derivatives at p.*

Now in case that $\partial\Omega$, g_1 and g_2 are real-analytic and $\Lambda_{g_1} = \Lambda_{g_2}$ we can use the last result to easily find a collar neighborhood of $\partial\Omega$ and a real-analytic diffeomorphism $\Psi_0 : U \to \overline{\Omega}$, $\Psi_0|_{\partial\Omega} = Id$ so that (see [L-U], Lemma 2.1)

$$\Psi_0^* g_1 = g_2.$$

One needs to extend the diffeomorphism Ψ_0 to Ω. In [L-U] this was done by analytic continuation along geodesics. We mention one of the results obtained (For a more general statement see Prop. 2.2 in [L-U]).

Theorem 9.61. *Let g_i, $i = 1, 2$ be real-analytic Riemannian metrics so that $\Lambda_{g_1} = \Lambda_{g_2}$. Assume Ω is simply connected and $\overline{\Omega}$ is strongly convex with respect to the metrics g_1, g_2. Then $\exists \Psi : \overline{\Omega} \to \overline{\Omega}$ real-analytic diffeomorphism so that*

$$\Psi^* g_1 = g_2, \Psi|_{\partial\Omega} = Id.$$

Theorems (9.21) and (9.61) use special features. In two dimensions isothermal coordinates are used to break the diffeomorphism invariance. In dimension $n \geq 3$, in the real-analytic case, geodesic flow is used to break the diffeomorphism invariance.

Jack Lee has suggested the use of harmonic maps to break this invariance. We discuss this idea in more detail. The material that follows is taken from [S-U VI].

For a general reference on harmonic maps see [Ha]. We shall only consider the case where the domain and range of a map is $\overline{\Omega}$, with Ω a smooth bounded domain in $\mathbf{R}^n$.

Let $f : (\overline{\Omega}, g) \to (\overline{\Omega}, h)$ be a smooth map where g and h are Riemannian metrics in $\overline{\Omega}$. The *energy* associated to the map f is given in local coordinates by

$$E(f) = \sum_{\alpha,\beta,i,j=1}^{n} \int_\Omega g^{ij}(x) h_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\beta}{\partial x_j} \sqrt{\det g}\, dx. \tag{9.62}$$

The Euler-Lagrange equation associated to the quadratic form (9.62) is given by the non-linear elliptic system

$$\frac{-2}{\sqrt{\det g}} \sum_{\alpha,i,j=1}^{n} \frac{\partial}{\partial x_j} \quad ((\sqrt{\det g}) g^{ij} h_{\alpha\beta} \frac{\partial f^\alpha}{\partial x_i}) + \sum_{\alpha,\gamma,i,j=1}^{n} g^{ij} \frac{\partial h_{\alpha\gamma}}{\partial f_\beta} \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\gamma}{\partial x_j} = 0 \quad \forall \beta. \tag{9.63}$$

Definition 9.64. A C^∞ map $f : (\overline{\Omega}, g) \to (\overline{\Omega}, h)$ is called *harmonic* if it is a critical point of (9.62) (i.e., it is a solution of (9.63)).

Note that if h is the Euclidean metric, then (1.13) simply states that the components of f are harmonic functions with respect to the metric g.

We are going to reduce conjecture 9.20 to the proof of a uniqueness theorem by means of the following Proposition, which follows readily from the definition of a harmonic map.

(9.65) Proposition. *Let $(\overline{\Omega}, g)$ and $(\overline{\Omega}, h)$ be two smooth bounded domains with Riemannian metrics g and h. Suppose there is a harmonic map*

(9.66)
$$\psi : (\overline{\Omega}, g) \to (\overline{\Omega}, h) \text{ such that } \psi|_{\partial\Omega} = \text{Identity and } \psi \text{ a diffeomorphism.}$$

Then the Identity: $(\overline{\Omega}, g) \to (\overline{\Omega}, \psi^* h)$ is harmonic.

We shall show that conjecture 9.20 is reduced to prove

Conjecture 9.67. Suppose g and h are Riemannian metrics on $\overline{\Omega}$ and that Identity: $(\overline{\Omega}, g) \to (\overline{\Omega}, h)$ is harmonic and $\Lambda_g = \Lambda_h$.

Then $g = h$.

(9.68) Proposition. *Conjecture 9.67 $\Longrightarrow$ 9.20 if there exists harmonic ψ satisfying (9.66).*

Proof. If $\Lambda_g = \Lambda_h$, and there is a ψ with $\psi|_{\partial\Omega} =$ Identity and $\Lambda_g = \Lambda_{\psi^* h} = \Lambda_h$. Then using Proposition 9.65 and Conjecture 9.67 we conclude that $h = \psi^* g$.

The solvability of the harmonic Dirichlet problem (9.66) is known if h has nonpositive sectional curvature ([H]) or if g and h are sufficiently close in the C^3 topology to the euclidean metric ([L-M-S-U]).

Thus, we have reduced the proof of Conjecture (9.20) to the uniqueness statement in Conjecture (9.67), under the additional assumption of the existence of an harmonic diffeomorphism which is the identity on the boundary.

In [S-U VI] it was proven that the linearization at the identity of conjecture (9.67) holds. We sketch the proof. In analogy with (9.8) the quadratic form associated to Λ_g is given by

$$Q_g(f,g) = \sum_{i,j=1}^{n} \int_\Omega g^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \sqrt{\det g}\, dx \tag{9.69}$$

with u, v solution of $\Delta_g u = \Delta_g v = 0$ in Ω; $u|_{\partial\Omega} = g$. We consider the linearization of Q at the euclidean metric in the direction of the quadratic form $m \in C_0^\infty(\Omega)$

$$dQ_m(f,g) = \lim_{\epsilon \to 0} \frac{Q_{e+\epsilon m}(f,g) - Q_e(f,g)}{\epsilon}. \tag{9.70}$$

A computation yields:

$$dQ_m(f,g) = \sum_{i,j=1}^{n} \int_{\Omega} (m_{ij} - \frac{1}{2} trm) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \tag{9.71}$$

where $\Delta u = \Delta v = 0$ in Ω; $u|_{\partial\Omega} = f, v|_{\partial\Omega} = g$ and $trm = \sum_{i=1}^{n} m_{ii}$.

We assume that $dQ_m = 0$. As Calderón did for the isotropic case, we take

$$u = e^{x\cdot\xi}, v = e^{-x\cdot\bar{\xi}} \tag{9.72}$$

where $\xi \in \mathbf{C}^n$, $\xi = \eta + ik$ with $\eta, k \in \mathbf{R}^n$ and $\langle \eta, k \rangle = 0$, $|\eta| = |k|$. Substituting (9.72) in (9.71) we obtain

$$\sum_{i,j=1}^{n} \int_{\Omega} (m_{ij} - \frac{1}{2} trm) e^{i\langle x,k \rangle} (\eta_i \eta_j + k_i k_j) = 0.$$

We rewrite (9.72) in the form

$$k^t(\widehat{m} - \frac{1}{2} tr\widehat{m})k + \eta^t(\widehat{m} - \frac{1}{2} tr\widehat{m})\eta = 0 \tag{9.73}$$

where t denotes transpose and $\widehat{}$ the Fourier transform.

Now the fact that the identity is a harmonic map implies the following system of n first order linear partial differential equations for $m = g - h$ (g is the euclidean metric in this computation):

$$-2 \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(m_{j\beta}) + \frac{\partial}{\partial x_\beta} trm = 0 \text{ in } \Omega, \quad \beta = 1, \cdots, n. \tag{9.74}$$

Taking the Fourier transform of (9.74) we obtain

$$-2 \sum_{j=1}^{n} k_j \widehat{m}_{j\beta}(k) + k_\beta tr\widehat{m}(k) = 0, \quad \beta = 1, \cdots, n. \tag{9.75}$$

Let us take $k = (1, 0, \ldots, 0)$, $\eta \in k^\perp$ with $|\eta| = |k| = 1$. Using (9.75) we get

$$\begin{aligned} \widehat{m}_{1\beta}(k) &= 0, \quad \beta = 2, \cdots, n \\ \widehat{m}_{11}(k) &= \frac{1}{2} tr\widehat{m}(k). \end{aligned} \tag{9.76}$$

Using (9.73) we obtain

(9.77) $\widehat{m}_{\beta\gamma} - tr\widehat{m}(k) = -(\widehat{m}_{11} - tr\widehat{m})(k), \quad \beta = 2,\dots,n, \quad \gamma = 2,\dots,n.$

Combining (9.76) and (9.77) we conclude

$$tr\widehat{m}(k) = 0$$

Using (9.76) again we see that $\widehat{m}_{ij}(k) = 0 \quad i,j = 1,\dots,n$. Rotating coordinates shows that $\widehat{m}(k) = 0 \ \forall\ k$ and therefore $m = 0$.

10. The Borg-Levinson theorem. We consider in this section an application of the methods developed for the inverse conductivity problem to study an inverse spectral problem. This involves, in an essential way, the study of the Dirichlet to Neumann map for the equation $\Delta - q + \lambda$.

We consider the equation

$$L_{q-\lambda} = \Delta - q + \lambda \tag{10.1}$$

with $q \in L^\infty(\Omega)$ and $\lambda \in \mathbf{C}$.

The following theorem appears in [N-S-U] :

Theorem 10.2. *Let $n \geq 2$ and $q_i \in L^\infty(\Omega)$, $\quad i = 1,2$. Suppose that, as meromorphic operator valued functions of λ,*

$$\Lambda_{q_1-\lambda} = \Lambda_{q_2-\lambda} \quad \forall \lambda \in \mathbf{R}. \tag{10.3}$$

Then

$$q_1 = q_2 .$$

Remark 10.4. For $n \leq 3$ it is enough as a consequence of Theorem (5.1) to assume $\Lambda_{q_1-\lambda_0} = \Lambda_{q_2-\lambda_0}$ for λ_0 not an eigenvalue of L_{q_1} or L_{q_2}.

Sketch of proof of Theorem 10.2.

Because we know the Dirichlet to Neumann map $\Lambda_{q-\lambda}$ for all λ (except for a discrete set) we may use the scattering solutions (6.2) instead of the exponentially growing solutions from theorem (4.1). Let us take

$$\psi_+^i = e^{ix\cdot\xi_i} + \Omega_i \quad i = 1,2 \tag{10.5}$$

where $\xi_i \in \mathbf{R}^n$ and (assume $\lambda > 0$)

$$\xi_i \cdot \xi_i = \lambda \tag{10.6}$$

and

$$\|\Omega_i\|_{L^2_\delta} \le \frac{C}{\sqrt{\lambda}}\|q_i\|_{L^2_\delta}\,, \quad -\frac{1}{2} < \delta\,. \tag{10.7}$$

Using the hypothesis of the theorem we conclude, as in the proof of theorem (5.1),

$$\int_\Omega (q_1 - q_2)u_1u_2 = 0 \tag{10.8}$$

for all u_i solution of $L_{q_i-\lambda}u_i = 0$, $i = 1, 2$.

We fix $k \in \mathbf{R}^n$ and choose

$$\begin{aligned} \xi_1 &= \frac{1}{2}(k+\ell), \quad k\cdot\ell = 0, |k|^2 + |\ell|^2 = \lambda \\ \xi_2 &= \frac{1}{2}(k-\ell)\,. \end{aligned} \tag{10.9}$$

Now replacing (10.5), with ξ as in (10.9), in (10.8) and letting l and $\lambda \to \infty$ we conclude

$$\widehat{q}_1(k) = \widehat{q}_2(k)$$

which proves the theorem. □

The Dirichlet to Neumann map $\Lambda_{q-\lambda}$ can be related to the eigenvalues and eigenfunctions of the Schrödinger operator $\Delta - q$. We give only a formal argument here. The reader is referred to [N-S-U] for complete proofs.

Let $q \in L^\infty(\Omega)$ be real-valued and let $\{\lambda_i\}_{i=1}^\infty$ denote the Dirichlet eigenvalues of L_q. Let $G(\lambda, x, y), \lambda \neq \lambda_i$, be the Green's kernel for the Dirichlet problem

$$(\Delta - q + \lambda)G = \delta(x-y), \quad G(\lambda, \cdot, y)|_{\partial\Omega} = 0, \quad \forall y \in \Omega\,.$$

The solution of

$$\begin{aligned} L_{q-\lambda}u &= 0 \\ u|_{\partial\Omega} &= f \end{aligned} \tag{10.10}$$

has the representation

$$u(x) = \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(\lambda, x, y) f(y)\, dS_y \tag{10.11}$$

where dS_y is the euclidean surface measure on $\partial\Omega$.

G can be written in term of the λ_i's and the corresponding set of orthonormal eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$

$$G(\lambda,x,y)=\sum_{i=1}^{\infty}\frac{\varphi_i(x)\varphi_i(y)}{\lambda-\lambda_i}. \tag{10.12}$$

Inserting (10.12) into (10.11) we obtain

$$u(x)=\int_{\partial\Omega}\sum_{i=1}^{\infty}\varphi_i(x)\frac{\partial\varphi_i}{\partial\nu}(y)f(y)dS_y \tag{10.13}$$

and therefore

$$\Lambda_{q-\lambda}f=\frac{\partial u}{\partial\nu}=\int_{\partial\Omega}\sum_{i=1}^{\infty}\frac{\frac{\partial\varphi_i}{\partial\nu}(x)\frac{\partial\varphi_i}{\partial\nu}(y)f(y)dS_y}{\lambda-\lambda_i}. \tag{10.14}$$

Formula (10.14) and Theorem 10.2 lead directly to the following result ([N-S-U]; Novikov [No] proved this result independently) which states that the Dirichlet eigenvalues and normal derivatives at the boundary of an orthonormal set of eigenfunctions uniquely determine the potential.

Theorem 10.15. *Let $q_i\in L^\infty(\Omega)$ $i=1,2$ be real-valued. Let $\lambda_j(q_i)$, $j=1,2,\dots$ denote the Dirichlet eigenvalues of L_{q_i}, $i=1,2$ with $\lambda_j\geq\lambda_{j+1}$ and eigenvalues repeated according to their multiplicity. Assume*

$$\lambda_j(q_i)=\lambda_j(q_2)\quad\forall j. \tag{10.16}$$

For q_i, $i=1,2$ we choose orthonormal sets of eigenfunctions $\{\varphi_j(\cdot,q_i)\}_{i=1}^{\infty}$ with

$$\frac{\partial\varphi_j}{\partial\nu}(x,q_1)=\frac{\partial\varphi_j}{\partial\nu}(x,q_2). \tag{10.17}$$

Then

$$q_1=q_2.$$

Remark 10.18. Theorem (10.15) can be thought of as an n-dimensional analog of the one-dimensional Borg-Levinson theorem, which states that the Dirichlet eigenvalues and the norming constants determine the potential uniquely. Alessandrini and Sylvester ([A-S]) have given stability estimates for the result Theorem 10.2. Roughly speaking, they showed that if q is a-priori bounded in some Sobolev norm, then, in some lower Sobolev norm, q

depends continuously on its Dirichlet eigenvalues and the normal derivatives of an orthonormal set of Dirichlet eigenfunctions.

11. The hyperbolic Dirichlet to Neumann map

We consider the mixed problem for the wave equation associated to the Schrödinger equation

$$(11.1) \qquad ((\frac{\partial^2}{\partial t^2} - \Delta) + q)u = 0 \quad \text{in} \quad \Omega \times (0,T)$$

$$u|_{t=0} = \varphi, \frac{\partial u}{\partial t}|_{t=0} = \psi ,$$

$$u|_{\partial\Omega\times(0,T)} = f$$

where $q \in L^\infty(\Omega)$.

The (hyperbolic) Dirichlet to Neumann map is then defined by

$$(11.2) \qquad \Lambda_q^h(f) = \frac{\partial u}{\partial \nu}.$$

with u solution of (11.1). Notice that φ, ψ are fixed throughout. As shown in [Ra-S] the choice of φ, ψ is inmaterial. Rakesh and Symes ([Ra-S]) proved

Theorem 11.3. *($n \geq 2$) Let $q_1, q_2 \in L^\infty(\Omega)$. Assume $\Lambda_{q_1}^h = \Lambda_{q_2}^h$ for $t \in [0,T]$ with $T >$ diam(Ω). Then*

$$q_1 = q_2.$$

Remark 11.4

If one knows $\Lambda_q^h(f)$ for all t, then taking Fourier transform in the time variable, one obtains the Dirichlet to Neumann map $\Lambda_{q-\lambda^2}$ considered in Theorem 10.2. In Theorem 11.3, we require only knowledge of Λ_q^h in the interval $[0,T]$.

Sketch of proof.

Rakesh and Symes use geometrical optics solutions concentrated near lines with direction $\omega \in S^{n-1}$ and an identity similar to (5.4) to prove that one can recover the X-ray transform of q knowing Λ_q^h.

We indicate here another way of obtaining this information from the hyperbolic Dirichlet to Neumann map. We consider for simplicity the case $q \in C_0^\infty(\Omega)$. Let $q_1, q_2 \in C_0^\infty(\Omega)$ such that

$$(11.5) \qquad \Lambda_{q_1}^h = \Lambda_{q_2}^h, \quad 0 \leq t \leq T, \text{ with } T > \text{diam}\,(\Omega)\,.$$

Let u_i, $i = 1, 2$ be the solution of

$$(11.6) \qquad ((\frac{\partial^2}{\partial t^2} - \Delta) + q_i)u_i = 0$$

$$u_i = \delta(t - x \cdot \omega), t \ll 0$$

where $\omega \in S^{n-1}$ is the direction of the plane wave $\delta(t - x \cdot \omega)$. We proceed now to show that the information (11.5) implies $u_1 = u_2$ in $\Omega^c \times [0, T]$. We proceed as in (6.18). Let

$$(11.7) \qquad z = \begin{cases} w & \text{in } \Omega \times [0, T] \\ u_2 & \text{in } \Omega^c \times [0, T] \end{cases}$$

where w solves the initial boundary value problem

$$(11.8) \qquad (\frac{\partial^2}{\partial t^2} - \Delta + q_1)w = 0$$

$$w = \delta(t - x \cdot \omega), t \ll 0$$

$$w|_{\partial\Omega \times (0,T)} = u_2|_{\partial\Omega \times (0,T)}.$$

Now

$$\frac{\partial w}{\partial \nu} = \Lambda_{q_1}^h(w|_{\partial\Omega \times [0,T]}) = \Lambda_{q_2}^h(u_2|_{\partial\Omega \times [0,T]}) = \frac{\partial u_2}{\partial \nu}.$$

Therefore z solves

$$(11.9) \qquad (\frac{\partial^2}{\partial t^2} - \Delta + q_1)z = 0 \quad \text{in} \quad \mathbf{R}^n \times \mathbf{R}$$

$$z = \delta(t - x \cdot \omega), \quad t \ll 0.$$

By the uniqueness of the solution of (11.9) we obtain

$$z = u_1$$

proving that

$$(11.10) \qquad u_1 = u_2 \text{ in } \Omega^c \times [0, T].$$

Now one can use the progressive wave expansion of Courant-Lax ([C-L]) to conclude for u_i as in (11.6)

$$(11.12) \qquad u_i = \delta(t - x \cdot \omega) + a_i(x, \omega)H(t - x \cdot \omega) + b_i(t, x, \omega)$$

where $b_i \in C^0(\mathbf{R} \times \mathbf{R}^n \times S^{n-1})$ $i = 1,2$, $H(x)$ is the Heaviside function and

$$\nabla a_i \cdot w = \frac{-q_i(x)}{2}, \quad i = 1,2,$$

$$a_i = 0 \text{ for } x \cdot \omega \ll 0.$$

Since $u_1 = u_2$ in $\Omega^c \times [0,T]$ we conclude

$$a_1 = a_2 \text{ in } \Omega^c.$$

But a_i, $i = 1,2$, can be obtained as integration of the potential q_i in the direction ω, therefore implying that the X-ray transform of q_1 and q_2 coincide

$$\int_{-\infty}^{\infty} q_i(x + t\omega)dt = \int_{-\infty}^{\infty} q_2(x + t\omega)dt \quad \forall \omega, x.$$

Now by the inversion of the X-ray transform ([H]) we conclude

$$q_1 = q_2.$$

Stefanov [St] and Ramm and Sjöstrand [R-Sj] have extended Theorem 11.3 result to the case of potentials depending on time. Isakov [Is III] has considered the case of wave equation plus first order perturbations. In all these works geometrical optics solutions and the relationship between the hyperbolic Dirichlet to Neumann and the X-ray transform play a crucial role.

We now consider the hyperbolic Dirichlet to Neumann map in the anisotropic case. In particular we would like to describe the relationship of this map and the inverse kinematic problem in seismology. The material that follows is taken from [S-U VI] and is part of work in progress of the author with Jack Lee, Gerardo Mendoza and John Sylvester [L-M-S-U].

Let Ω be a smooth bounded domain in $\mathbf{R}^n$ and g a smooth Riemannian metric on $\overline{\Omega}$. We consider the initial boundary value problem

$$\text{(11.13)} \qquad \left(\frac{\partial^2}{\partial t^2} - \Delta_g\right) u = 0 \text{ in } \Omega \times (0,T), \quad T > 0$$

$$u|_{t=0} = \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \text{ in } \Omega$$

$$u|_{\Omega \times (0,T)} = f.$$

We define the (hyperbolic) Dirichlet to Neumann map by

$$\text{(11.14)} \qquad \Lambda_g^h(f) = \sum_{i,j=1}^{n} \left(\nu_i g^{ij} \frac{\partial u}{\partial x_j}\right)\Big|_{\partial\Omega}$$

where u is a solution of (11.13).

As in the elliptic case, it is easy to see that the map

$$g \xrightarrow{\Lambda^h} \Lambda_g^h \tag{11.15}$$

is not injective since $\Lambda^h_{\psi^* g} = \Lambda^h_g$ for any diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ such that $\psi|_{\partial\Omega} =$ Identity. One can show,as in the elliptic case, that knowledge of Λ^h_g determines the Taylor series of g at $\partial\Omega$ in boundary normal coordinates.

If $\Lambda_{g_0} = \Lambda_{g_1}$ one can extend $g_0 = g_1$ to Ω^c such that both are smooth and both are euclidean outside a ball. Using similar arguments to the ones used in the proof of Proposition 6.16 we have

(11.16) Proposition. *Let g_0, g_1 the smooth Riemannian metrics on $\overline{\Omega}$. Assume $\Lambda^h_{g_0} = \Lambda^h_{g_1}$. Let $(u_0, u_1) \in \mathcal{E}'(\mathbf{R}^n) \times \mathcal{E}'(\mathbf{R}^n)$, supp $u_k \subseteq \Omega^c$, $k = 0, 1$. The solution v_k of the initial value problem*

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_{g_k}\right) v_k = 0 \ \text{in } \mathbf{R}^n \times (0,T)$$

$$v_k|_{t=0} = u_0$$

$$\frac{\partial v_k}{\partial t}\Big|_{t=0} = u_1$$

satisfies $v_0 = v_1$ in $\Omega^c \times (0, T)$.

One can use the proposition above and the geometrical optics construction (2.7) to solve the wave equation with data supported outside Ω^c (say $u_0 = \delta_y, y \in \Omega^c, u_1 = 0$) to conclude that the geodesic distance function for points $y, x \in \Omega^c$ is the same. We are going to use an alternative method which is the Hadamard parametrix construction (see Hörmander [Hö], section 12.4).

Let $F_k(t, x, y)$ be the solution of

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_{g_k}\right) F_k = 0, \quad k = 0, 1$$

$$F_k(0, x, y) = \delta(x - y), y \in \Omega^c$$

$$\frac{\partial F_k}{\partial t}(0, x, y) = 0.$$

Then, assuming that the exponential map for each of the metrics g_k is a global diffeomorphism near $\overline{\Omega}$ (i.e., no caustics in a neighborhood of Ω), we may write

$$F_k(t, x, y) = \sum_{j=0}^{N} A_j^k(x, y)\left(t^2 - (s_k(x, y))^2\right)_+^{-j+\frac{1}{2}(n-1)} + F_N^k \tag{11.17}$$

where $F_N^k \in C^{N+1-[\frac{1}{2}(n-1)]}(\mathbf{R}_t \times \mathbf{R}_x^n \times \mathbf{R}_y^n)$ and $A_j^k \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n), k = 0, 1$. Here $s_k(x, y)$ denotes the geodesic distance between x and y in the metric g_k, $k = 0, 1$. The distributions

$$\left(t^2 - (s(x,y)^2\right)_+^\lambda = \begin{cases} \frac{\left(t^2 - (s(x,y))^2\right)^{-\lambda}}{\Gamma(1-\lambda)} & \text{for } t^2 > \left(s(x,y)\right) \\ 0 & t^2 < \left(s(x,y)\right) \end{cases}$$

are defined for Re $\lambda \ll 0$ and have an analytic continuation to $\lambda \in \mathbf{C}$.

Now from proposition (11.16) we know that if $\Lambda_{g_0}^h = \Lambda_{g_1}^h$, then $F_0(t,x,y) = F_1(t,x,y)$ in Ω^c, for $t > 0$. Therefore, comparing the most singular terms in (2.20) we conclude that

$$\left(t^2 - (s_0(x,y))^2\right)_+^{\frac{1}{2}(n-1)} = \left(t^2 - (s_1(x,y))^2\right)_+^{\frac{1}{2}(n-1)}.$$

Thus we have proved

Theorem 11.18. *Let g_0 and g_1 be Riemannian metrics with $\Lambda_{g_0}^h = \Lambda_{g_1}^h$. Then if the exponential map is a global diffeomorphism in $\overline{\Omega}$ for g_k, $k = 0, 1$ and $s_k(x,y)$ denotes the geodesic distance from x to y in the metric g_k, we have*

$$s_0(x,y) = s_1(x,y) \quad \forall x, y \in \partial\Omega.$$

The inverse kinematic problem in seismology is to recover g from $s_g(x,y), x, y \in \partial\Omega$. Again this is not possible since if $\psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism such that $\psi|_{\partial\Omega} =$ Identity, then $s_{\psi^* g} = s_g$. As in conjecture 1, the question is whether this is the only obstruction to uniqueness. It is proven in [S-U V] that the linearized version at the euclidean metric of this conjecture is valid using again the harmonic map equation.

Let g_ϵ be a family of Riemannian metrics in Ω, $g_\epsilon = e + \epsilon h$, where e is the euclidean metric. We also assume that $g_\epsilon = e$ in Ω^c and

$$s_{g_\epsilon}(x,y) = s_e(x,y) \ \forall \epsilon. \tag{11.19}$$

An easy computation shows that

$$\int_{\gamma(x,t,v)} (h_{ij})(v,v)dt = 0 \tag{11.20}$$

where $\gamma(x,t,v)$ denotes a straight line through x with direction v at time t. Formula (11.20) means that the X-ray transform of the quadratic form h_{ij} vanishes in the direction v.

We recall that the linearization at the identity of the harmonic map equation (in the direction h) is

$$-2\sum_{i=1}^n \frac{\partial}{\partial x_i}(h_{i\beta}) + \frac{\partial}{\partial x_\beta} trh = 0, \quad \beta = 1, \dots, n. \tag{11.21}$$

Integrating (11.21) along the lines with direction v yields

$$\int_{\gamma(x,t,v)} v_j(h_{ij})w_\beta = 0 \ \forall w \in \mathbf{R}^n \text{ with } \langle w, v\rangle = 0.$$

Arguments similar to those at the end of section 9 show that

$$\int_{\gamma(x,t,v)} (h_{ij}(w,w) = 0 \quad \forall w \in \mathbf{R}^n$$

proving that the X-ray transform of $h_{ij}(w,w)$ is zero for all w and therefore that $h = 0$.

12. The scattering amplitude at fixed energy

In the previous section 10 we related the Dirichlet to Neumann map $\Lambda_{q-\lambda}$ to spectral information about q. One can also relate this to scattering information, now fixing the frequency λ (this is more or less implicit in the hyperbolic Dirichlet to Neumann map and in the analog to formula (6.15) for the scattering amplitude).

In the same way that we obtained (6.15), it is possible to show that the scattering amplitude satisfies

$$a(\lambda,\theta,\omega) = \int_{\partial\Omega} e^{i\lambda x\cdot\omega}(\Lambda_{q-\lambda^2}\psi_+ + i\lambda\nu\cdot\omega\psi_+)dS \tag{12.1}$$

where ψ_+ is the outgoing eigenfunction.

(In [N] and [No] an integral equation was derived for $\psi_+|_{\partial\Omega}$ in terms of $\Lambda_{q-\lambda^2}$ similar to (6.24). See also the nice exposition of Colton and Kress [C-K] on integral equation methods in scattering theory).

Arguments analogous to those in proof of Proposition 6.16 show that if

$$\Lambda_{q_1-\lambda_0^2} = \Lambda_{q_2-\lambda_0^2} \tag{12.2}$$

for $q_1, q_2 \in L^\infty(\Omega)$ then

$$\psi_+^{(1)} = \psi_+^{(2)} \quad \text{in } \Omega^c. \tag{12.3}$$

with $\psi_+^{(i)}, i = 1,2$ the outgoing eigenfunction associated to q_i. Then using (12.1) we conclude that if (12.3) is satisfied,

$$a_1(\lambda_0,\theta,\omega) = a_2(\lambda_0,\theta,\omega) \quad \forall\theta,\omega \in S^{n-1}$$

with a_i $i = 1,2$ the scattering amplitude associated to q_i. One can prove the converse

Theorem 12.4. $(n \geq 3)$. *Let* $q_i \in L^\infty(\mathbf{R}^n)$, supp $q_i \subseteq \Omega = \{x; |x| < R\}$, $i = 1, 2$ *such that*

$$a_1(\lambda_0, \theta, \omega) = a_2(\lambda_0, \theta, \omega) \tag{12.5}$$

for some $\lambda_0 \neq 0$, $\forall \theta, \omega \in S^{n-1}$. *Then if* λ_0^2 *is not an eigenvalue of* L_{q_1} *or* L_{q_2} *(in* Ω *with Dirichlet boundary conditions),*

$$\Lambda_{q_1 - \lambda_0^2} = \Lambda_{q_2 - \lambda_0^2}$$

and therefore

$$q_1 = q_2 \, .$$

Sketch of proof

Let $G_q(x, y, \lambda_0)$ be the outgoing Green's kernel for $-\Delta + q - \lambda_0^2$. The single-layer operator, which is an invertible operator from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega)$, is defined by

$$\mathcal{S}_{\lambda_0} f(x) = \int_{\partial B(0,R)} G_q(x, y, \lambda_0) f(y) dS \tag{12.6}$$

where dS denotes surface measure.

It was proven in [N] (see Theorem 1.6; the proof is also valid in two dimensions) that

$$\Lambda_{q - \lambda_0^2} \to \mathcal{S}_{\lambda_0} \tag{12.7}$$

is injective. More precisely (see (1.40) in [N])

$$\Lambda_{q - \lambda_0^2} = \Lambda_{-\lambda_0^2} + \mathcal{S}_{\lambda_0}^{-1} - (\mathcal{S}_{\lambda_0}^{+})^{-1} \tag{12.8}$$

where $\mathcal{S}_{\lambda_0}^{+}$ is as in (12.6) with $q = 0$. Next we sketch how to prove that the map

$$\mathcal{S}_{\lambda_0} \to \mathcal{A}_{\lambda_0} \tag{12.9}$$

is injective, where $\mathcal{A}_{\lambda_0}(q) = a(\lambda_0, \theta, \omega)$.

This is an old result of Berezanskii ([B]) who showed how to go from the far field $(\mathcal{A}_{\lambda_0})$ to the near field $(\mathcal{S}_{\lambda_0})$ in a quite explicit fashion. One can see the injectivity of (12.9) using the asymptotic expansion of the outgoing Green's kernel, namely

$$G_q(x, y, \lambda_0) = \frac{e^{i\lambda_0 |x|}}{|x|^{\frac{n-1}{2}}} \psi(\lambda_0, y, \theta) + 0(|x|^{-\frac{(n-1)}{2} - 1}) \tag{12.10}$$

with $\theta = -\frac{x}{|x|}$ and ψ_+ the outgoing eigenfunction. Now if $\mathcal{A}_{\lambda_0}(q_1) = \mathcal{A}_{\lambda_0}(q_2)$, by (12.10) and (6.3') we get

$$G_{q_1}(x,y,\lambda_0) - G_{q_2}(x,y,\lambda_0) = 0(|x|^{-\frac{(n-1)}{2}-1}|y|^{-\frac{(n-1)}{2}-1}). \tag{12.11}$$

Now

$$\varphi(x,y) = G_{q_1}(x,y,\lambda_0) - G_{q_2}(x,y,\lambda_0)$$

solves

$$(-\Delta_x - \lambda_0^2)\varphi = 0 \quad \text{for } |x| \geq R, |y| \geq R.$$

Therefore by Rellich's lemma we obtain that

$$G_{q_1}(x,y,\lambda_0) = G_{q_2}(x,y,\lambda_0) \quad \text{for } |x|, |y| \geq R$$

proving the injectivity of the map (12.9). □

In two dimensions Novikov [N II] proved injectivity of the map

$$q \to \mathcal{A}_{\lambda_0}(q) \tag{12.12}$$

for q close to 0. This result can be also proven using the method outlined in the proof of Theorem 12.4 and the local result in [S-U II] stated in section 7. Sun and Uhlmann [S-U II] used the generic results in [Su-U I] to prove generic injectivity of the map (12.2). More recently in [Su-U III] it was proven that in two dimensions for a singular potential having jump type discontinuities across a subdomain, knowledge of the map (12.2) determines both the location of the singularity and the jump at the singularity. This result follows from a corresponding one for the Dirichlet to Neumann map.

Remark 12.13. Ramm stated Theorem 12.4 in several papers. However some of his proofs, as indicated by Novikov ([No]), are incorrect (for instance [R I]). A corrected proof appears in [R II].The proof sketched above was communicated to us by A. Nachman. Stefanov [St II] has used similar ideas to obtain continuous dependence results for the map (12.12). Henkin and Novikov ([N-H]) had proved Theorem 12.4 earlier in the case of small potentials. Novikov ([No]) sketched a proof of Theorem 12.4 without the smallness assumption using the results in [N-H].

13. An analogous discrete problem

A discrete version of the inverse conductivity problem described in section 1 is to consider a network of resistors. The problem is to determine the resistances in the network by making voltage and current measurements at the boundary of the network. Of course the geometry of the network is important for uniquely determining the resistors. For instance it is easy to see that two resistances wired in series cannot be determined by making voltage and current measurements at the boundary.

We consider rectangular network Ω of resistors in the plane. We follow here the approach of [Cu-M I]. The nodes of Ω are the lattice points $p = (i, j)$ for which $a \leq i \leq b$ and $c \leq j \leq d$ with the four corner points (a, c), (b, c), (a, d) and (b, d) excluded. The set of nodes is denoted by Ω_0. The interior int Ω_0 consists of those nodes in Ω_0 all of whose four adjoint points are in Ω_0. The *edges* of Ω, Ω_1 are the horizontal and vertical line segments which connect each pair of adjacent points in Ω_0. The *conductivity* is a function

$$\gamma : \Omega_1 \to \mathbf{R}^+$$

where $\mathbf{R}^+$ is the set of positive numbers and $\frac{1}{\gamma(\sigma)}$ is the resistance of the edge.

The conductivity equation is easily obtained used Kirkhoff's law: The sum of all currents at an interior node is zero

$$(13.1) \qquad L_\gamma u(p) = \sum_{q \sim p} \gamma(p, q)(u(q) - u(p)) = 0, \quad p \in \text{int } \Omega_0$$

where $q \sim p$ means that q and p are nodes connected by a resistance; $\gamma(p, q)$ represents the conductivity associated to the edge joining p and q.

The discrete Dirichlet to Neumann is then defined by

$$(13.2) \qquad \Lambda_\gamma^d f(p) = \gamma(p, q)(u(q) - u(p)), p \in \partial\Omega_0$$

where q is the unique node in Ω_0 connected to p by an edge and u is the solution to the Dirichlet problem

$$(13.3) \qquad \begin{aligned} L_\gamma u &= 0 \text{ in } \text{ int } \Omega_0 \\ u|_{\partial\Omega_0} &= f. \end{aligned}$$

Again $\Lambda_\gamma^d f(p)$ is the induced current at p by the potential u induced by the voltage f.

In analogy with the continuous case it is easy to see that if we consider the total power to maintain the potential f on the boundary, with u solution of (13.3)

$$(13.4) \qquad Q_\gamma^d(f) = \sum_{q \sim p} \gamma(p, q)(u(q) - u(p))^2,$$

then

$$(13.5) \qquad Q_\gamma^d(f) = \sum_{p \in \partial\Omega_0} f(p)\Lambda_\gamma^d f(p).$$

The inverse conductivity problem for the network of resistances can then be reduced to study the map

$$\gamma \xrightarrow{\Phi} \Lambda_\gamma^d \tag{13.6}$$

with Λ_γ^d as in (13.2) or equivalently the map

$$\gamma \xrightarrow{Q} Q_\gamma^d \tag{13.7}$$

with Q_γ^d as in (13.4).

Lawler and Sylvester ([L-S]) proved the injectivity of the map Φ (or Q) for conductivities which are a small deviation of constant conductivities. They used the analog of the growing exponential solutions of Calderón (section 2). In [Cu-M I] completely different solutions of (13.1) are constructed which don't have an analog in the continuous case. This allows to prove not only injectivity for Φ (or Q) and to give a reconstruction method to get γ from Λ_γ^d but also to give a characterization of all possible Λ_γ^d which arise ([Cu-M II]). We first state

Theorem 13.8. *Let Ω_0 be a network of resistors in the plane with edges Ω_1. Let γ_i, $i = 1, 2$ be two conductivities $\gamma_i : \Omega_1 \to \mathbf{R}^+$. Assume*

$$\Lambda_{\gamma_1}^d = \Lambda_{\gamma_2}^d,$$

then

$$\gamma_1 = \gamma_2 \,.$$

Sketch of proof.

Similar to the approach taken in the continuous case, we look at Q_γ^d. Polarizing the quadratic form (13.4) we obtain the bilinear form

$$Q_\gamma^d(f, g) = \sum_{q \sim p} \gamma(p, q)(u(q) - u(p))(v(q) - v(p)) \tag{13.9}$$

where u is a solution of (13.1) and v solves the Dirichlet problem.

$$L_\gamma v = 0 \text{ in } \operatorname{int} \Omega_0 \tag{13.10}$$

$$v|_{\partial\Omega_0} = g.$$

One can easily prove the corresponding identity to (5.4) in the case of a network of resistors. Namely if $\gamma_i, i = 1,2$, are conductivities so that $\Lambda^d_{\gamma_1} = \Lambda^d_{\gamma_2}$, then

$$\sum_{q\sim p}(\gamma_1(p,q) - \gamma_2(p,q))(u_1(p) - u_1(q))(u_2(p) - u_2(q)) = 0 \tag{13.11}$$

where u_i is solution of

$$L_{\gamma_i}u_i = 0, \quad i = 1,2.$$

The main technique used in [Cu-M I] is "harmonic continuation". More precisely given a conductivity γ one can show that there exist solutions of

$$L_\gamma u = 0 \quad \text{in } \Omega_0$$

so that $u = 0$ below any line of slope plus or minus one (of course there is no analog of these solutions in the continuous case). By choosing u_1 in (13.11) to be zero below the appropiate line of slope one, and u_2 to be zero below a line of slope minus one, Curtis and Morrow proved

Proposition 13.12. *Given an edge joining p_0 and q_0 and two conductivities γ_1, γ_2 in a network, one can construct solutions*

$$L_{\gamma_i}u_i = 0 \textit{ in } \operatorname{int}\Omega_0$$

so that for $q \sim p$

$$(u_1(q) - u_1(p))(u_2(q) - u_2(p)) = \delta_{q_0p_0} \tag{13.13}$$

where

$$\delta_{q_0p_0} = \begin{cases} 1 & q = q_0, p = p_0 \\ 0 & \textit{otherwise.} \end{cases}$$

The theorem follows immediately from Proposition (13.12) since we may insert u_1 and u_2 as in (13.13) into (13.11) to get

$$\gamma_1(p_0, q_0) = \gamma_2(p_0, q_0),$$

which proves the theorem. □

This method of proof allowed Curtis and Morrow to give a reconstructive procedure to get γ from Λ^d_γ and moreover to formulate necessary conditions for a matrix Λ_{ij} to be the Dirichlet to Neumann map associated to a conductivity. They have recently proved that these conditions are also sufficient ([Cu-M, II]).

Let Ω_0 be a square network of side $n \times n$ and $\gamma : \Omega_1 \to \mathbf{R}^+$ a conductivity. The Dirichlet to Neumann map Λ^d_γ is represented by the matrix $\Lambda_{i,j}$ (if we number the boundary nodes clockwise, then the functions which are one at the j'th node and zero elsewhere form a basis for functions on the boundary). Curtis and Morrow proved

Theorem 13.14. *Let $\Lambda_{i,j}$ be a $4n$ by $4n$ matrix representing the linear map Λ. Then there is a unique conductivity function γ on Ω_1, such that $\Lambda = \Lambda_\gamma^d$ iff $\Lambda_{i,j}$ satisfies the four properties listed below.*

(R1) Let k be an integer with $1 \leq k \leq n$, and take $m = 4n - k + 1$. Then there is a unique set of numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that for each i with $k < i < m$,

$$\Lambda_{i,m} + \sum_{i=1}^{k} \Lambda_{ij}\alpha_j = 0$$

A similar relation holds for any node in any face, and columns from faces either clockwise or anti-clockwise from that node.

(R2) Λ_γ is symmetric: $\Lambda_{i,j} = \Lambda_{j,i}$.Thus, there are relations similar to (R1) involving the rows of Λ_γ.

(R3) For each $i = 1, 2, \ldots, 4n$,

$$\sum_{j=1}^{4n} \Lambda_{i,j} = 0$$

(DP) Each of the six $n \times n$ blocks which lie entirely above the diagonal, and each of their transposes has the Determinant Property –A matrix has the determinant property if any k by k submatrix M satisfies: det $M < 0$ if $k \equiv 1$ or 2 mod 4; det $M > 0$ if $k \equiv 3$ or 4 mod 4 .

An interesting open question is to analyze the relationship between the discrete and continuous Dirichlet to Neumann map.

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