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ON RECURRENCES FOR SOME HYPERGEOMETRIC TYPE POLYNOMIALS

by

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1. Introduction

A well-known result for Legendre type polynomials $P_n = {}_2F_1\left(\begin{smallmatrix}-n,-n\\1\end{smallmatrix}\,\middle|\,t\right)$ is the recurrence relation $nP_n - (2n-1)(1+t)P_{n-1} + (n-1)(1-t)^2P_{n-2} = 0$. When t=-1 this implies the known sum formula $\sum_{k=0}^{2m}\binom{2m}{k}^2(-1)^k = (-1)^m\binom{2m}{m}$ $(m = 0,1,2,...)$.

There are numerous extensions of this recurrence for special values of t. ASKEY and WILSON [2] achieved three term recurrences for the sums $\sum_{k=0}^{n}\binom{n}{k}\binom{n+a+d}{k+d}\binom{n+k+b+e}{k+e}\binom{n+k+c+f}{k+f}$ $(a+d=b+c)$ by contiguous relations for hypergeometric series. PERLSTADT [6] obtained recurrences for sums $\sum_{k=0}^{n}\binom{n}{k}^r$ $(r = 2,...,6)$ by the method of telescoping series.

In the following we want to apply so called 'SISTER CELINE's method' to the following kind of hypergeometric polynomials - say ${}_3F_2\left(\begin{smallmatrix}-n,A,B\\1,\,1\end{smallmatrix}\,\middle|\,t\right)$, where $(A,B) = (-n,-n),(-n,1/2)$ or $(-n,n+1)$ and ${}_4F_3\left(\begin{smallmatrix}-n,A,B,C\\1,\,1,\,1\end{smallmatrix}\,\middle|\,t\right)$, where $(A,B,C) = (-n,-n,-n)$ or $(-n,n+1,n+1)$. For the hypergeometric notation we refer to RAINVILLE [8]. These polynomials satisfy four or five order recurrences and with special values of the parameter t we shall obtain some three term recurrences like the APÉRY recurrences [1] considered in detail by VAN DEER POORTEN [7]. Also we shall get DIXON's sum formula $\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k = (-1)^m\binom{2m}{m}\binom{3m}{m}$ $(m = 0,1,2,...)$ (MACMAHON [5]). More references can be found from ASKEY and WILSON [2] and PERLSTADT [6].

2. Sister Celine's method

From the remarks of BAILEY [3] it is clear that ${}_3F_2$-polynomials satisfy at most four term recurrences and ${}_4F_3$-polynomials satisfy at most five term recurrences. To obtain the recurrence

$$(1) \qquad a_0(n)p_n + (a_1(n) + a_2(n)t)p_{n-1} + (a_3(n) + a_4(n)t + a_5(n)t^2)p_{n-2} + ... = 0$$

for the polynomial $p_n = p_n(t)$ of degree n we shall use SISTER CELINE's method, see FASENMYER [4] or RAINVILLE [8]. Note that some of the coefficient polynomials of $p_n, p_{n-1}, ...$

may be zero e.g. in (4), (13) and (14). In these cases we shall rise the index n so that in all cases our results have the form

$$A(n,t)\, p_n \;+\; B(n,t)\, p_{n-1} \;+\; C(n,t)p_{n-2} \;+ ... = 0.$$

Let us set $p_n(t) = \sum_{k=0}^{n} \epsilon(k,n)t^k$. In the case of the above mentioned polynomials it is easy to write polynomials $p_{n-1}, tp_{n-1}, p_{n-2}, tp_{n-2}, t^2 p_{n-2}, ...$ in the form

$$\sum_{k=0}^{n} r_i(k,n)\epsilon(k,n)t^k \qquad (i = 1, 2, ...)$$

(see the proof of Theorem 1), where the $r_i(k,n)$'s are certain rational expressions of k and n. Now from (1) shall we obtain the identity

$$a_0(n) + a_1(n)r_1(k,n) + a_2(n)r_2(k,n) + a_3(n)r_3(k,n) + ... \;=\; 0. \tag{2}$$

Thus it is straightforward to get enough equations to solve the unknowns $a_i(n)$ $(i = 0, 1, ...)$. Let the number of a_i's be m. In all our cases it sufficied to set $k = 1, ..., m$ to achieve a solvable system of equations. In some cases due the symmetry it was possible to reduce significantly the number of equations (see Theorem 1).

3. Theorems

In the following theorem we shall state recurrences for

$$J_n(t) \;=\; {}_3F_2\left(\begin{matrix} -n, -n, -n \\ 1,\ 1 \end{matrix}\,\middle|\, t\right) \quad \text{and} \quad H_n(t) \;=\; {}_4F_3\left(\begin{matrix} -n, -n, -n, -n \\ 1,\ 1,\ 1 \end{matrix}\,\middle|\, t\right).$$

Now due the relations $J_n(1/t) \;=\; (-1/t)^n J_n(t)$ and $H_n(1/t) \;=\; (1/t)^n H_n(t)$ there are only six unknowns $a_i(n)$ in the first case and 14 in the second case.

THEOREM 1. *The recurrences for J_n and H_n are given by*

$$\begin{aligned} &(3n-5)n^2 J_n \;-\; (9n^3 - 24n^2 + 17n - 4)(1-t)J_{n-1} \\ &+ (3n-4)(3n^2 - 7n + 3 + (21n^2 - 49n + 24)t + (3n^2 - 7n + 3)t^2)J_{n-2} \\ &- (3n-2)(n-2)^2(1-t)^3 J_{n-3} \;=\; 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} &(b_0(n) + b_1(n)t + b_0(n)t^2)H_n \;+\; (b_2(n) + b_3(n)t + b_3(n)t^2 + b_2(n)t^3)H_{n-1} \;+ \\ &(b_4(n) + b_5(n)t + b_6(n)t^2 + b_5(n)t^3 + b_4(n)t^4)H_{n-2} \;+ \\ &(b_7(n) + b_8(n)t + b_9(n)t^2 + b_9(n)t^3 + b_8(n)t^4 + b_7(n)t^5)H_{n-3} \;+ \end{aligned} \tag{4}$$

$$(b_{10}(n) + b_{11}(n)t + b_{12}(n)t^2 + b_{13}(n)t^3 + b_{12}(n)t^4 + b_{11}(n)t^5 + b_{10}(n)t^6)H_{n-4} \;=\; 0,$$

where

$$\begin{aligned}
b_0(n) &= (n-1)n^3(64n^4 - 544n^3 + 1697n^2 - 2303n + 1151),\\
b_1(n) &= (n-1)n^3(272n^4 - 2312n^3 + 7231n^2 - 9859n + 4958),\\
b_2(n) &= -(n-1)(256n^7 - 2560n^6 + 10212n^5 - 20844n^4 + 23317n^3 -\\
&\quad 14341n^2 + 4655n - 630),\\
b_3(n) &= -(n-1)(1344n^7 - 13440n^6 + 53688n^5 - 109816n^4 + 123098n^3 -\\
&\quad 75804n^2 + 24615n - 3330),\\
b_4(n) &= 384n^8 - 4800n^7 + 25318n^6 - 73416n^5 + 127613n^4 - 135624n^3 +\\
&\quad 85700n^2 - 29385n + 4230,\\
b_5(n) &= -6304n^8 + 78800n^7 - 418938n^6 + 1237876n^5 - 2224383n^4 +\\
&\quad 2489364n^3 - 1694065n^2 + 640890n - 103320,\\
b_6(n) &= -10(3296n^8 - 41200n^7 + 218996n^6 - 646196n^5 + 1157126n^4 -\\
&\quad 1286706n^3 + 867069n^2 - 323775n + 51462),\\
b_7(n) &= -(n-2)(256n^7 - 3072n^6 + 15140n^5 - 39448n^4 + 58199n^3 -\\
&\quad 48180n^2 + 20600n - 3555),\\
b_8(n) &= -(n-2)(9024n^7 - 108288n^6 + 536320n^5 - 1412352n^4 + 2119776n^3 -\\
&\quad 1798290n^2 + 793925n - 142335),\\
b_9(n) &= -5(n-2)(8384n^7 - 100608n^6 + 498988n^5 - 1318024n^4 + 1987917n^3 -\\
&\quad 1698342n^2 + 756763n - 137142),\\
b_{10}(n) &= (n-2)(n-3)^3(64n^4 - 288n^3 + 449n^2 - 285n + 65),\\
b_{11}(n) &= (n-1)(n-2)(n-3)^3(16n^3 - 56n^2 + 75n - 30),\\
b_{12}(n) &= -5(n-2)(n-3)^3(128n^4 - 576n^3 + 913n^2 - 597n + 141),\\
b_{13}(n) &= 10(n-2)(n-3)^3(112n^4 - 504n^3 + 797n^2 - 519n + 122),
\end{aligned}$$

respectively.

PROOF: We shall shortly describe the proof of the recurrence (4). For technical reasons we shall lower the index n to $n-2$ in the formula (4). When we denote

$$H_n = \sum_{k=0}^{n} \epsilon(k,n)t^k,$$

where $\epsilon(k,n) = (-n)_k^4/(k!)^4$, then

$$t^i H_{n-j} = \sum_{k=i}^{n+i-j} \left(\frac{(k-i+1)_i(-n+k)_{j-i}}{(-n)_j}\right)^4 \epsilon(k,n)t^k \quad (i \le j)$$

so that $r_3 = ((k-n)(k-n+1)/n(n-1))^4$, $r_4 = (k(k-n)/n(n-1))^4, \ldots$. Hence the left hand side of (4) can be combined into one sum

$$\sum_{k=0}^{n} \frac{N(k,n)}{(n(n-1)(n-2)(n-3)(n-4)(n-5))^4} \epsilon(k,n) t^k,$$

where

$$N(k,n) =$$

$$(n-2)^4(n-3)^4(n-4)^4(n-5)^4(((k-n)^4(k-n+1)^4+(k-1)^4k^4)b_0(n-2)+(k-n)^4k^4b_1(n-2))+\ldots$$

$$+(k-n)^4(k-n+1)^4(k-n+2)^4(k-2)^4(k-1)^4k^4b_{13}(n-2) \;=\; 0$$

identically on k and n. Counting this could have been quite tedious without the assistance of some symbolic mathematical program system like Macsyma, Musimp or Reduce. ∎

When $t = 1$ we get from (3) $\quad J_n = \dfrac{-3(3n-4)(3n-2)}{n^2} J_{n-2}$. So we have DIXON's result $\sum_{k=0}^{2m} \binom{2m}{k}^3 (-1)^k = (-1)^m \binom{2m}{m}\binom{3m}{m}$ $\quad (m = 0,1,2,\ldots)$ (MACMAHON [5]).

In the following corollary we shall state three term recurrences for $j_n = \sum_{k=0}^{n} \binom{n}{k}^3$, $h_n = \sum_{k=0}^{n} \binom{n}{k}^4$ and $g_n = \sum_{k=0}^{n} \binom{n}{k}^4 (-1)^k$. The recurrences for numbers j_n and h_n are results of FRANEL (PERLSTADT [6]), while the recurrence (7) for g_n is perhaps new.

COROLLARY 1. *The recurrences for j_n, h_n and g_n are given by*

$$n^2 j_n \;-\; (7n^2-7n+2)j_{n-1} \;-\; 8(n-1)^2 j_{n-2} \;=\; 0, \tag{5}$$

$$n^3 h_n \;-\; 2(2n-1)(3n^2-3n+1)h_{n-1} \;-4(n-1)(4n-3)(4n-5)h_{n-2} \;=\; 0 \tag{6}$$

and

$$(n-1)(12n^2-63n+83)n^3 g_n \;+ \tag{7}$$

$$4(408n^6-3774n^5+13760n^4-25203n^3+24465n^2-11970n+2340)g_{n-2}+$$

$$16(n-2)(12n^2-15n+5)(n-3)^3 g_{n-4} \;=\; 0$$

respectively.

PROOF: We shall prove the first formula, the proof of second formula goes similarly. The third formula is an immediate consequence of (4) with $t = -1$.

Let us define the operator E by $Er_n = r_{n+1}$. As a consequence of formula (3) with $t=-1$ we get

$$(8) \qquad (3n-5)n^2 j_n - 2(9n^3-24n^2+17n-4)j_{n-1} - (3n-4)(15n^2-35n+18)j_{n-2} - 8(3n-2)(n-2)^2 j_{n-3} = 0.$$

Then (8) is equivalent to

$$(9) \qquad P(n)j_{n-3} = ((3n-5)n^2E^3 - 2(9n^3-24n^2+17n-4)E^2 - (3n-4)(15n^2-35n+18)E - 8(3n-2)(n-2)^2)j_{n-3} = 0,$$

where the operator $P(n)$ factors in the following way

$$P(n) = ((3n-5)E+3n-2)((n-1)^2E^2-(7(n-1)(n-2)+2)E-8(n-2)^2).$$

Let us denote

$$(10) \qquad s_{n-3} = ((n-1)^2E^2-(7(n-1)(n-2)+2)E-8(n-2)^2)j_{n-3}.$$

Thus (9) is equivalent to

$$(11) \qquad ((3n-5)E+3n-2)s_{n-3} = 0$$

i.e. $(3n-5)s_{n-2} = -(3n-2)s_{n-3}$. From (10) one gets $s_0 = 2^2j_2-(7\cdot 2\cdot 1+2)j_1-8j_0 = 0$, so $s_k = 0$ $(k=0,1,2,\ldots)$ and thus (10) implies (5).

In the second case one sets $t=1$ in (4) and gets $P(n)h_{n-4}=0$, where the operator $P(n)$ factors in the following way

$$P(n) = ((n-1)(20n^3-115n^2+215n-132)E + 2(2n-5)(20n^3-55n^2+45n-12)) ((n-1)^3E^2 - 2(2n-3)(3n^2-9n+7)E - 4(n-2)(4n-7)(4n-9)).$$

Analogously to the first case one obtains the recurrence (6). ∎

By similar method like in Theorem 1 we can achieve recurrences for

$$F_n(t) = {}_3F_2\left(\begin{matrix}-n,-n,1/2\\ 1,\ 1\end{matrix}\,\middle|\, t\right), \quad B_n(t) = {}_3F_2\left(\begin{matrix}-n,-n,n+1\\ 1,\ 1\end{matrix}\,\middle|\, t\right)$$

and

$$A_n(t) = {}_4F_3\left(\begin{matrix}-n,-n,n+1,n+1\\ 1,\ 1,\ 1\end{matrix}\,\middle|\, t\right).$$

THEOREM 2. *The recurrences for F_n, B_n and A_n are given by*

$$(12) \qquad 4(4n-7)n^2F_n - 2(6(4n^3-11n^2+8n-2)+(16n^3-44n^2+34n-9)t)F_{n-1} +$$

$$(4(12n^3-45n^2+52n-18)+2(2n-3)t+(4n-3)(2n-3)^2t^2)F_{n-2}$$
$$-\ 4(4n-3)(n-2)^2(1-t)^2F_{n-3} \ = \ 0$$

and

$$(b_0(n)+b_1(n)t)B_n \ + \ (b_2(n)+b_3(n)t+b_4(n)t^2)B_{n-1} \ +$$
$$(b_5(n)+b_6(n)t+b_7(n)t^2+b_8(n)t^3)B_{n-2} \ + \ (b_9(n)+b_{10}(n)t+b_{11}(n)t^2)B_{n-3} \ = \ 0, \tag{13}$$

where

$$\begin{aligned}
b_0(n) &= 3(9n-14)n^2, & b_6(n) &= -4(159n^3-567n^2+626n-200),\\
b_1(n) &= -4(8n-13)n^2, & b_7(n) &= 4(67n-40)(2n-3)^2,\\
b_2(n) &= -3(27n^3-69n^2+47n-10), & b_8(n) &= -16(8n-5)(2n-3)^2,\\
b_3(n) &= -10(12n^3-30n^2+23n-6), & b_9(n) &= -3(9n-5)(n-2)^2,\\
b_4(n) &= 8(32n^3-84n^2+62n-15), & b_{10}(n) &= (59n-35)(n-2)^2,\\
b_5(n) &= 3(3n-5)(9n^2-17n+6), & b_{11}(n) &= -4(8n-5)(n-2)^2,
\end{aligned}$$

and

$$(c_0(n)+c_1(n)t)A_n \ + \ (c_2(n)+c_3(n)t+c_4(n)t^2)A_{n-1} \ +$$
$$(c_5(n)+c_6(n)t+c_7(n)t^2+c_8(n)t^3)A_{n-2} \ + \ (c_9(n)+c_{10}(n)t+c_{11}(n)t^2)A_{n-3} \ +$$
$$(c_{12}(n)+c_{13}(n)t)A_{n-4} \ = \ 0, \tag{14}$$

where

$$\begin{aligned}
c_0(n) &= (n-1)(2n-3)(2n-5)^2n^3,\\
c_1(n) &= -(n-1)(2n-5)(4n-7)(4n-9)n^3,\\
c_2(n) &= -(n-1)(2n-1)(2n-5)(8n^4-40n^3+62n^2-33n+6),\\
c_3(n) &= -(n-1)(2n-1)(2n-5)(4n-3)(8n^3-34n^2+41n-18),\\
c_4(n) &= 4(n-1)(2n-1)(2n-5)(4n-9)(16n^3-44n^2+34n-9),\\
c_5(n) &= (2n-3)(24n^6-216n^5+754n^4-1284n^3+1101n^2-441n+66),\\
c_6(n) &= -(2n-3)(864n^6-7776n^5+27514n^4-48444n^3+44169n^2-\\
&\qquad 19485n+3270),\\
c_7(n) &= 4(2n-1)(2n-3)^3(2n-5)(64n^2-192n+99),\\
c_8(n) &= -16(2n-1)(2n-3)^3(2n-5)(4n-3)(4n-9),\\
c_9(n) &= -(n-2)(2n-1)(2n-5)(8n^4-56n^3+134n^2-123n+33),\\
c_{10}(n) &= -(n-2)(2n-1)(2n-5)(4n-9)(8n^3-38n^2+53n-15),\\
c_{11}(n) &= 4(n-2)(2n-1)(2n-5)(4n-3)(16n^3-100n^2+202n-129),\\
c_{12}(n) &= (n-2)(n-3)^3(2n-1)^2(2n-3),\\
c_{13}(n) &= -(n-2)(n-3)^3(2n-1)(4n-3)(4n-5),
\end{aligned}$$

respectively.

In the following corollary we shall state three term recurrences for $f_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$, $e_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \left(\frac{1}{4}\right)^k$, $b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ and $a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$. The recurrence for the numbers f_n is proved in STIENSTRA and BEUKERS [9], the recurrences for the APÉRY numbers b_n and a_n can be found in APÉRY [1] and POORTEN VAN DEER A. [7].

COROLLARY 2. *The recurrences for f_n, e_n, b_n and a_n are given by*

$$n^2 f_n - (10n^2 - 10n + 3) f_{n-1} + 9(n-1)^2 f_{n-2} = 0, \tag{15}$$

$$4n^2 e_n - 2(10n^2 - 10n + 3) e_{n-1} + (4n-3)(4n-5) e_{n-2} = 0, \tag{16}$$

$$n^2 b_n - (11n^2 - 11n + 3) b_{n-1} - (n-1)^2 b_{n-2} = 0 \tag{17}$$

and

$$n^3 a_n - (2n-1)(17n^2 - 17n + 5) a_{n-1} + (n-1)^3 a_{n-2} = 0 \tag{18}$$

respectively.

For example let $a_n = A_n(1) = {}_4F_3\left(\begin{matrix}-n,-n,n+1,n+1\\ 1,\ 1,\ 1\end{matrix}\,\middle|\, 1\right)$ then the formula (14) gives $P(n)a_{n-4} = 0$, where

$$\begin{aligned} P(n) &= ((n-2)(2n-5)E^2 - (2n-1)(2n-5)E + (n-1)(2n-1)) \\ &\quad ((n-2)^3 E^3 - (2n-5)(17(n-2)(n-3)+5)E + (n-3)^3). \end{aligned}$$

Again like in the proof of Corollary 1 we see that a_n satisfies (18). ∎

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