Astérisque

EDGAR H. BROWN
ROBERT H. SZCZARBA

Continuous cohomology and real homotopy type II

Astérisque, tome 191 (1990), p. 45-70
<http://www.numdam.org/item?id=AST_1990__191__45_0>

CONTINUOUS COHOMOLOGY AND REAL HOMOTOPY TYPE II

EDGAR H. BROWN AND ROBERT H. SZCZARBA

Introduction.

In our earlier paper "Continuous Cohomology and Real Homotopy Type" [**3**], we studied localization of simplicial spaces at the reals and established an equivalence between the category of free nilpotent differential graded commutative algebras of finite type over the reals and nilpotent simplicial spaces of finite type localized at the reals. In this paper, we extend these results by eliminating the nilpotent condition on the algebraic side, thus proving a conjecture of Sullivan [**8**]. (See Theorem 1.2, Part (iv), below.) The main technical work consists in introducing local coefficients into continuous cohomology, continuous de Rham cohomology, the Serre Spectral Sequence, and the constructions involved in real homotopy type.

We also obtain information about secondary characteristic classes of G foliations in the sense of Haefliger [**1,3,4,6**], namely that when G is compact, the continuous cohomology of the appropriate classifying space injects into the ordinary cohomology. This result is stated and proved at the end of Section 2. (See Proposition 2.5).

Our main results are stated in Section 1. The remainder of the paper is devoted to proving these results.

1. Statements of Results.

We begin by recalling some of the notation and definitions from [**3**].

Let $\mathcal{CA}$ denote the category of differential (degree $+1$), graded, commutative (in the graded sense), locally convex topological algebras with unit over R and $\Delta\mathcal{T}$ the category of compactly generated simplicial spaces. Let Ω_q^p denote the space of C^∞ differential p-forms on the standard q-simplex Δ^q in the C^∞ topology. Then $\Omega^p = \{\Omega_q^p\}$ is in $\Delta\mathcal{T}$, $\Omega_q = \{\Omega_q^p\}$ is in $\mathcal{CA}$, and $\Omega = \{\Omega_q^p\}$ is in $\Delta\mathcal{CA}$. Define contravariant functors $\Delta : \mathcal{CA} \longrightarrow \Delta\mathcal{T}$ and $\mathcal{A} : \Delta\mathcal{T} \longrightarrow \mathcal{CA}$ by

---

[1]The first author was partially supported by the National Science Foundation.

Astérisque 191 (1990)
S.M.F

$\Delta(A)_q = (A, \Omega_q) =$ the simplicial space of algebra mappings $A \to \Omega_q$,
$\mathcal{A}(X)^p = (X, \Omega^p) =$ the vector spaces of continuous simplicial mappings $X \to \Omega^p$.

The simplicial structure on Ω gives $\Delta(A)$ a simplicial structure and the algebra structure on Ω gives one on $\mathcal{A}(X)$. We view $\Delta(A)$ as the *simplicial realization* of A and $\mathcal{A}(X)$ as the *algebra of differential forms* on X.

For $X \in \Delta\mathcal{T}$ and any topological abelian group G, let $C^q(X;G)$ be the space of continuous mappings $u : X_q \to G$ with $u \circ s_i = 0, 0 \le i \le q-1$, and define $\delta : C^q(X;G) \to C^{q+1}(X;G)$ by

$$\delta u = \sum_{j=0}^{q+1} (-1)^j u \circ \partial_j$$

Here, s_i, ∂_j denotes the face and degeneracy mappings of X. *The continuous cohomology of X with coefficients in G* is defined by

$$H^*(X;G) = H_*(C^*(X;G);\delta).$$

The usual deRham mapping defines an isomorphism

$$\psi : H^*(\mathcal{A}(X);d) \longrightarrow H^*(X;R) = H^*(X).$$

(See Theorem 2.4 of [**3**].)

We next describe homology of $A \in \mathcal{CA}$ with local coefficients. Suppose L is a finite dimensional Lie algebra which acts on a finite dimensional vector space V via a Lie algebra homomorphism $\gamma : L \to g\ell(V) = \mathrm{Hom}(V,V)$. Let $C^*(L)$ denote the usual cochain algebra on L, that is, $C^p(L)$ is the space of alternating, multilinear functions

$$u : L^p = L \times L \times \cdots \times L \to R$$

with $d : C^p(L) \to C^{p+1}(L)$ given by

$$du(\ell_1, \dots, \ell_{p+1}) = \sum_{i<j} (-1)^{i+j} u([\ell_i, \ell_j], \ell_1, \dots, \hat{\ell}_i, \dots, \hat{\ell}_j, \dots, \ell_{p+1})$$

For $A \in \mathcal{CA}$, we define L-local coefficients on A as follows. Let $\ell_1, \dots, \ell_n$ be a basis for L, $\ell_1^*, \dots, \ell_n^*$ the dual basis for L^*, and suppose $\lambda : C^*(L) \to A$ is a $\mathcal{CA}$ mapping. Define $d_\lambda : A \otimes V \to A \otimes V$ by

$$d_\lambda(a \otimes v) = da \otimes v + (-1)^p \sum_{i=1}^{n} a\lambda(\ell_i^*) \otimes \ell_i v.$$

where $\ell_i v = \gamma(\ell_i)(v)$. It is easy to check that d_λ is independent of the choice of basis, that $d_\lambda^2 = 0$, and that d_λ is functorial in both A and V. Let $H_*(A; V_\lambda) = H_*(A \otimes V, d_\lambda)$.

Remark 1.1. If $A = C^*(L), \lambda =$ identity, $\gamma : L \to g\ell(V)$, and $J : C^*(L) \otimes V \to C^*(L; V)$ is the standard isomorphism, then $Jd_\lambda = d_\gamma J$ where $d_\gamma : C^p(L; V) \to C^{p+1}(L; V)$ is given by

$$d_\gamma \omega = d\omega + \gamma \wedge \omega.$$

Here, γ is considered as a $g\ell(V)$-valued 1-form on L and the wedge product $\gamma \wedge \omega$ is defined using the action of $g\ell(V)$ on V.

Suppose now that $A \in \mathcal{CA}$ is free and of finite type; that is, A is the tensor product of a polynomial algebra on even dimensional generators with an exterior algebra on odd dimensional generators and each A^j is a finite dimensional vector space, $j \geq 0$. According to Proposition 7.11 of [**2**], we can find a basis $t_1, \ldots, t_n$ for A^1 such that, for $1 \leq i \leq m$,

$$dt_i = \sum_{\substack{1 \leq i < j \leq m \\ j < k}} a_i^{jk} t_j t_k$$

and for $m < i \leq n, dt_i$ is a polynormal generator for A. One easily sees that, if A and B are free and of finite type, then $\Delta(A \otimes B) = \Delta(A) \times \Delta(B)$ and if $A = R[x, y]$ with $dx = y$, then ΔA is contractible in $\Delta\mathcal{T}$. Hence, up to homotopy type, $\Delta(A)$ is unchanged by dividing A by the ideal generated by $\{t_i, dt_i \mid i > m\}$. Henceforth, we include the condition $n = m$ in the notion of free and of finite type.

Given A as above, let L be the dual vector space to A^1 and let $\alpha_1, \ldots, \alpha_m$ be the basis for L dual to $t_1, \ldots, t_m$. Then L is a Lie algebra with

$$[\alpha_j, \alpha_k] = 2 \sum_{i=1}^{m} a_i^{jk} \alpha_i.$$

The inclusion

$$\lambda : C^*(L) \simeq R[t_1, \ldots, t_m] \subset A$$

defines L-local coefficients on A. As in [**3**], we define $i : A \to \mathcal{A}(\Delta(A))$ by $i(a)(u) = u(a)$. Then $i\lambda : C^*(L) \to \mathcal{A}(\Delta(A))$ defines L-local coefficients on $\mathcal{A}(\Delta(A))$. Finally, if $A^{(1)}$ denotes the subalgebra of A generated by $t_1, \ldots, t_m$, then $C^*(L)$ is naturally isomorphic to $A^{(1)}$.

The following result is stated in [**8**] as "Theorem" 8.1.

THEOREM 1.2. *Suppose $A \in \mathcal{CA}$ is free of finite type, and that $A^{(1)} = C^*(L)$ as above.*

(i) Let $G = G_A$ be the connected, simply connected Lie group with $L(G) = L$. Then

$$\begin{aligned} \pi_i(\Delta A^{(1)}) &\simeq G \quad \textit{for } i = 1, \\ &\simeq \pi_i(G) \quad \textit{for } i > 1. \end{aligned}$$

(ii) Let V be a finite dimensional vector space on which L acts and $\lambda : C^*(L) \to A$ the inclusion map. Then the mapping $i : A^{(1)} \to \mathcal{A}(\Delta(A^{(1)}))$ induces an isomorphism

$$i_* : H_*(A^{(1)}; V_\lambda) \to H_*(\mathcal{A}(\Delta(A^{(1)})); V_{i\lambda}) \simeq H^*(\Delta A^{(1)}; V_{i\lambda}).$$

*(iii) Let $\tilde{A}$ be the quotient algebra of A by the ideal generated by $A^{(1)}$. Then $A^{(1)} \subset A$ induces a fibre map $\Delta(A) \to \Delta(A^{(1)})$ with fibre $\Delta(\tilde{A}) \subset \Delta(A)$. (Note that $\tilde{A}$ is free, nilpotent, and of finite type and hence the homotopy type of $\Delta(\tilde{A})$ is described in [**3**].)*

(iv) For any action of L on V, the mapping $i : A \to \mathcal{A}(\Delta(A))$ induces isomorphisms

$$i_* : H_*(A; V_\lambda) \longrightarrow H_*(\mathcal{A}(\Delta(A)); V_{i\lambda}) \simeq H^*(\Delta(A); V_{i\lambda}).$$

In [8], Sullivan gives a very brief sketch of (i) and (iii) and, asserts that (ii) is "a reformulation of the theorem of Van Est". No proof is given for (iv). We give a detailed proof of (iv) in general and of (ii) when $G = G_A$ in the universal cover of a compact group. Actually (i) follows from Proposition 2.4 and (iii) follows from results of Section 5. In Section 2, we give an analysis of $\Delta(C^*(L))$ (see Theorem 2.3 and Proposition 2.4). Section 3 deals with local coefficients and the de Rham theorem. Proposition 2.4, the de Rham theorem with local coefficients, and an unpublished result of Graeme Segal are used in Section 4 to prove (ii) when G is the universal cover of a compact group. The result of Segal is that the continuous cohomology and the ordinary cohomology of the singular complex of a CW complex are isomorphic. We give Segals proof in Section 7.

The development of the proof of (iv) is as follows: Suppose $A \in \mathcal{CA}$ is free and of finite type. Then $A = UA^{(n)}$ where $A^{(0)} = R$ and $A^{(n)} = A^{(n-1)}[x_1^{(n)}, \ldots, x_k^{(n)}]$ and the $x_i^{(n)}$ have dimension n. We compute $H^*(\Delta(A); V_{i\lambda})$ by computing $H^*(\Delta(A^{(n)}); V_{i\lambda})$ using induction on n. In Section 6, we use Proposition 2.4 to prove Theorem 5.3, namely that

$$\Delta(A[x_1, \ldots, x_k]) \to \Delta(A)$$

is a fibration with fibre $\Delta(R[x_1, \ldots, x_k])$. In Section 6 we develop the Serre Spectral Sequence for continuous cohomology with local coefficients and apply it to the above fibration to prove the inductive step in the proof of (iv).

Recall that, for $A, B \in \mathcal{CA}$, a function complex $\mathcal{F}(A, B) \in \Delta\mathcal{T}$ was defined in [3] (following [2]) by $\mathcal{F}(A, B)_q = (A, \Omega_q \otimes B)$, the space of continuous differential graded algebra mappings from A to $\Omega_q \otimes B$. If A and $B \in \mathcal{CA}$ are free and of finite type and $h : A^{(1)} \to B^{(1)}$ is a map in $\mathcal{CA}$, define $\mathcal{F}(A, B; h)$ to be the simplicial subspace of $\mathcal{F}(A, B)$ whose q simplicies are maps $u : A \to \Omega_q \otimes B$ which give a commutative diagram

$$\begin{array}{ccc} A^{(1)} & \xrightarrow{h} & B^{(1)} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow{u} & \Omega_q \otimes B \end{array}$$

where $i : A^{(1)} \to A$ is the inclusion and $j(b) = 1 \otimes b$.

Similarly, for $X, Y \in \Delta\mathcal{T}$, $\mathcal{F}(X,Y) \in \Delta\mathcal{T}$ is given by $\mathcal{F}(X,Y)_q = (X \times \Delta[q], Y)$, the space of simplicial mappings from $X \times \Delta[q]$ to Y where $\Delta[q]$ is the simplicial model for the standard q-simplex. If $h : A^{(1)} \to B^{(1)}$ is as above, let $\mathcal{F}(\Delta B, \Delta A; \Delta(h))$ be the simplicial subspace of $\mathcal{F}(\Delta B, \Delta A)$ whose q-simplicies are mappings $f : \Delta[B] \times \Delta[q] \to \Delta(A)$ for which the diagram

$$\begin{array}{ccc} \Delta[B] \times \Delta[q] & \xrightarrow{f} & \Delta(A) \\ {\scriptstyle \bar{j}}\downarrow & & \downarrow{\scriptstyle \Delta(i)} \\ \Delta(B^{(1)}) & \xrightarrow{\Delta(h)} & \Delta(A^{(1)}), \end{array}$$

is commutative, where $\bar{j}(s,u) = u \mid \Delta(B^{(1)})$. Just as in [3], Theorem 1.20, we prove

THEOREM 1.3. *Suppose $A, B \in \mathcal{CA}$ are free and of finite type and $h : A^{(1)} \to B^{(1)}$ is a mapping in $\mathcal{CA}$. Then $\Delta : \mathcal{F}(A,B) \to \mathcal{F}(\Delta B, \Delta A)$ defines a weak equivalence*

$$\Delta : \mathcal{F}(A, B; h) \to \mathcal{F}(\Delta B, \Delta A; \Delta h).$$

The proof of this result is given at the end of Section 5.

2. The Simplicial Space $\Delta(C^\star(L))$.

We give here an analysis of the simplicial space $\Delta(C^\star(L))$ and prove an independence result for characteristic classes of G-foliations. Although the results of this section are stated for finite dimensional Lie groups, they do hold more generally for infinite dimensional Lie groups which are regular (in the sense of Milnor [7]) and for which the Lie algebra $L(G)$ is reflexive. In particular, they hold for $G = \mathrm{Diff}(M)$, M compact, where $L(G)$ is the Lie algebra of vector fields on M.

Let X be a manifold, G a Lie group, and let (X, G) be the space of C^∞ mappings $f : X \to G$. Let G act on (X,G) by $(gf)(x) = gf(x)$ and let $\hat{\Omega}^1(X;L) \subset \Omega^1(X,L)$ be given by

$$\hat{\Omega}^1(X;L) = \{w \in \Omega^1(X;L) \mid dw - w \wedge w = 0\}.$$

Define $\hat{\rho} : (X,G) \to \Omega^1(X;L)$ by $\hat{\rho}(f) = -f^{-1}df$ where

$$(f^{-1}df)(v) = dL^{-1}_{f(x)} df(v) \in TG_e = L(G)$$

for $v \in TX_x$. It is easily checked that $\hat{\rho}(g \cdot f) = \hat{\rho}(f)$ for any $f \in (X, G), g \in G$, and that $d(\hat{\rho}(f)) - \hat{\rho}(f) \wedge \hat{\rho}(f) = 0$. Thus, $\hat{\rho}$ defines

$$\rho : (X, G)/G \to \hat{\Omega}^1(X; L).$$

Similarly, we define $\tilde{\sigma} : (X, G) \to (C^*(L), \Omega^*(X))$ by $\tilde{\sigma}(f)(\alpha) = f^*(\bar{\alpha})$ for $\alpha \in C^1(L) = L^*$ where $\bar{\alpha} \in \Omega^1(G)$ is the invariant 1-form defined by α. Then $\tilde{\sigma}(g \cdot f) = \tilde{\sigma}(f)$ so $\tilde{\sigma}$ defines

$$\sigma : (X, G)/G \to (C^*(L), \Omega^*(X)).$$

Finally, we define $\psi : \Omega^1(X, L) \to \mathrm{Hom}(L^*; \Omega^1(X))$ by $\psi(w)(\alpha)(v) = \alpha(w(v))$. One easily checks (see [**5**]) that ψ defines a bijection

$$\psi : \tilde{\Omega}^1(X; L) \to (C^*(L), \Omega^*(X)).$$

THEOREM 2.1. *Suppose X is simply connected. Then each of the mappings ρ and σ defined above are bijections and the diagram*

$$\begin{tikzcd}
(X,G)/G \arrow[r, "\rho"] \arrow[dr, "\sigma"'] & \hat{\Omega}^1(X;L) \arrow[d, "\zeta"] \\
& (C^*(L), \Omega^*(X))
\end{tikzcd}$$

is commutative where $\zeta(w) = \psi(-w)$.

Remark. If the spaces above are given the C^∞ topologies, then each of the mappings in the diagram is a homeomorphism.

PROOF: The fact that the diagram of Theorem 2.1 commutes is an immediate consequence of the definitions. Since $\psi : \tilde{\Omega}(X; L) \to (C^*(L), \Omega^*(X))$ is a bijection, it follows that $\zeta : \tilde{\Omega}(X; L) \to (C^*(L), \Omega^*(X))$ is a bijection so Theorem 2.1 will be proved if we can show that $\rho : (X, G)/G \to \hat{\Omega}^1(X; L)$ is a bijection. This is an immediate consequence of the following.

LEMMA 2.2. *Let U be a neighborhood of $x \in X$ and suppose $w \in \Omega^1(U, L)$ satisfies $dw = w \wedge w$. Then there is a neighborhood $U_0 \subset U$ of x and a unique C^∞ function $f : U_0 \to G$ such that $f(x) = e$ and $w = f^{-1}df$.*

For a proof of this lemma, see [**9**].

Let ΔG denote the simplicial space of C^∞ singular simplices of G. Setting $X = \Delta^q, q = 0, 1, 2, \ldots$, in the previous discussion yields simplicial mappings

$$\tilde{\sigma} : \Delta G \to \Delta C^*(L)$$
$$\sigma : \Delta G/G \to \Delta C^*(L)$$

As an immediate consequence of Theorem 2.1, we have

THEOREM 2.3. *The mapping $\sigma : \Delta G/G \to \Delta C^*(L)$ is a simplicial homeomorphism.*

The next result proves part (i) of Theorem 1.2.

PROPOSITION 2.4. *The mapping $\tilde{\sigma} : \Delta G \to \Delta C^*(L)$ is a twisted cartesian product with fibre and group $\tilde{G}$ where $\tilde{G}$ is the simplicial group with $\tilde{G}_q = G$ for all q and with the identity as face and degeneracy mappings.*

By Theorem 2.3, it is sufficient to show that the natural mapping $\pi : \Delta G \to \Delta G/G$ is a twisted cartesian product with fibre and group $\tilde{G}$. To accomplish this, define $\tau : \Delta G/G \to \tilde{G}$ by $\tau([T]) = T(v_1)T(v_0)^{-1}$ and $h : \Delta G \to (\Delta G/G) \times_\tau \tilde{G}$ by $h(T) = ([T], T(v_0))$ where $T \in (\Delta G/G)_q$ and $v_0, v_1, \ldots, v_q$ are the vertices of Δ^q. Then τ is a twisting function and h is a simplicial homeomorphism such that the diagram

$$\begin{tikzcd} \Delta G \arrow[r, "h"] \arrow[d, "\pi"'] & (\Delta G/G) \times_\tau \tilde{G} \arrow[d, "\pi_1"] \\ \Delta G/G \arrow[r, "id"] & \Delta G/G \end{tikzcd}$$

is commutative.

We conclude this section with a result concerning characteristic classes of G foliations (in the sense of [5]). According to Haefliger [5], if L is the Lie algebra of a Lie group G, then $\Delta C^*(L)$ is a classifying space for G-foliations transverse to fibres of a product. The following can be interpreted as an independence statement for the continuous cohomology characteristic classes of these foliations.

PROPOSITION 2.5. *Let G be a compact Lie group with Lie algebra L. Then the homomorphism $H^*(\Delta C^*(L)) \to H^*(\Delta C^*(L)^\delta)$ is injective.*

Here, $\Delta C^*(L)^\delta$ is the simplicial space $\Delta C^*(L)$ in the discrete topology.

PROOF: By Theorem 2.3, it is enough to prove that $i^* : H^*(\Delta G/G) \to H^*((\Delta G/G)^\delta)$ is injective. To do this, we consider the commutative diagram

$$\begin{tikzcd} C^*(\Delta G/G) \arrow[r, "j"] \arrow[d, "i^*"'] & C^*(\Delta G)^G \arrow[r, "\ell"] \arrow[d] & C^*(\Delta G) \arrow[d, "k"] \\ C^*((\Delta G/G)^\delta) \arrow[r] & C^*(\Delta G^\delta)^G \arrow[r] & C^*(\Delta G^\delta). \end{tikzcd}$$

The mapping j is an isomorphism and k is a homology isomorphism by Theorem 4.9. Using the Haar integral, we can construct a cochain mapping $r : C^*(\Delta G) \to C^*(\Delta G)^G$ with $r\ell = id$. (See Proposition 4.4.) It follows that the composite $k\ell j$ is injective on homology so i^* is injective on homology.

REMARK: The analogue of Proposition 2.5 with local coefficients can be proved using the techniques developed in Section 4.

3. Local Coefficients and the de Rham Theorem.

In this section, we describe local coefficients systems in several different ways. We also prove a local coefficient version of the continuous cohomology de Rham theorem.

Let G be a Lie group with Lie algebra L and let V be a finite dimensional vector space. Suppose G acts on V via a representation $\Gamma : G \to GL(V)$ so that L acts on V via a representation $\gamma : L \to g\ell(V)$. In Section 1, we defined a local L system on $A \in \mathcal{CA}$ to be a $\mathcal{CA}$ map $\lambda : C^*(L) \to A$ and a differential

$$d_\lambda : A^p \otimes V \to A^{p+1} \otimes V$$

given by the formula

$$d_\lambda(a \otimes v) = (da) \otimes v + (-1)^p \sum_i a\lambda(\ell_i^*) \otimes \ell_i v$$

where $\{\ell_i\}$ is a basis for L, $\{\ell_i^*\}$ the dual basis for $L^*, a \in A^p$ and $\ell_i v = \gamma(\ell_i)(v)$. We now translate this into a more familiar form.

Let V be as above and define $\Omega(V)$ to be the simplicial topological differential graded vector space given by

$$\Omega_q^p(V) = \Omega^p(\Delta^q; V),$$

the smooth differential p-forms on Δ^q with values in V. For $X \in \Delta\mathcal{T}$, let $\mathcal{A}(X, V)$ be the differential topological graded vector space with

$$\mathcal{A}^p(X; V) = (X, \Omega^p(V)),$$

the space of simplicial mappings from X to $\Omega^p(V)$. It is easy to see that $\mathcal{A}(X, V) = \mathcal{A}(X) \otimes V$. Note that $\Delta C^*(L)$ can be considered to be contained in $\Omega^1(L)$.

Suppose $\lambda : C^*(L) \to \mathcal{A}(X)$ is an $\mathcal{CA}$ map and let $\phi = \phi_\lambda$ be the composite

$$X \xrightarrow{j} \Delta(\mathcal{A}(X)) \xrightarrow{\Delta(\lambda)} \Delta(C^*(L)) \subset \Omega^1(L)$$

where $j(x)(u) = u(x)$. Then $\phi \in \mathcal{A}^1(X; L)$ and one easily checks that $d\phi(x) + \phi(x)_\wedge \phi(x) = 0$ for all $x \in X_q$. We define $d_\phi : \mathcal{A}(X; V) \to \mathcal{A}(X, V)$ by

$$d_\phi \omega = d\omega + \phi_\wedge \omega.$$

Then $d_\phi^2 = 0$ and we have

PROPOSITION 3.1. *Let $\iota : \mathcal{A}(X) \otimes V \to \mathcal{A}(X,V)$ be the isomorphism defined by $\iota(\omega \otimes v)(x) = \omega(x)v$. Then $\iota d_\lambda = d_\phi \iota$.*

PROOF: If $\omega \in \mathcal{A}^p(X), v \in V$, and $x \in X_q$, then

$$\begin{aligned}
\iota d_\lambda(\omega \otimes v)(x) &= \iota((d\omega) \otimes v + (-1)^p \sum_i \omega\lambda(\ell_i^*) \otimes \ell_i v)(x) \\
&= (d\omega)(x)v + (-1)^p \sum_i \omega(x)\lambda(\ell_i^*)(x)\ell_i v \\
&= (d\omega)(x)v + (-1)^p \omega(x)\phi(x) \\
&= d_\phi \iota(\omega \otimes v)(x).
\end{aligned}$$

since $\phi(x) = \sum_i \lambda(\ell_i^*)(x)\ell_i$.

We next reformulate these notions into an equivariant setting. Let $\tilde{\lambda}$ be the composite

$$X \xrightarrow{j} \Delta(\mathcal{A}(X)) \xrightarrow{\Delta(\lambda)} \Delta C^*(L)$$

and let $\tilde{X}$ be the pullback

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d, "\tilde{p}"'] & \Delta G \arrow[d, "\tilde{\sigma}"] \\
X \arrow[r, "\tilde{\lambda}"] & \Delta C^*(L)
\end{tikzcd}$$

where $\tilde{\sigma}$ is defined in Section 2. Let G act on $\mathcal{A}(\tilde{X}, V)$ by

$$(g\omega)(x,T) = g\omega(x, g^{-1}T)$$

where $x \in X_q, g \in G$, and $T \in (\Delta G)_q$ and let $H : \mathcal{A}(X,V) \to \mathcal{A}(\tilde{X}, V)$ be given by

$$H(\omega)(x,T) = T \cdot \omega(x)$$

for $\omega \in \mathcal{A}(X,V)$ and $(x,T) \in \tilde{X}$. Here

$$T \cdot \omega(x)(w_1, \dots, w_p) = (\Gamma T)(y) \cdot (\omega(x)(w_1, \dots, w_p))$$

where $\Gamma : G \to GL(V)$ is the homomorphism defined by the action of G on V, $y \in \Delta^q, w_1, \dots, w_p \in T\Delta^q_y$, and $(\Gamma T)(y)$ acts on $\omega(x)(w_1, \dots, w_p) \in V$.

PROPOSITION 3.2. *The map H defines an isomorphism of $\mathcal{A}(X,V)$ onto $\mathcal{A}(\tilde{X}, V)^G$ with $Hd_\phi = dH$.*

PROOF: The verification that $gH(\omega) = H(\omega)$ is straightforward. To see that H is an isomorphism, let $\mathcal{O} : \Delta C^*(L) \to \Delta G$ be the composite

$$\Delta C^*(L) \xrightarrow{\sigma^{-1}} \Delta G/G \xrightarrow{\beta} \Delta G$$

where $\sigma : \Delta G/G \to \Delta C^*(L)$ is the simplicial homeomorphism defined in Section 2, $\beta[T] = T(v_0)^{-1}T$, and v_0 is the initial vertex of Δ^q. Then $H^{-1} : \mathcal{A}(\tilde{X}, V)^G \to \mathcal{A}(X, V)$ is given by

$$H^{-1}(\omega)(x) = \mathcal{O}(\tilde{\lambda}(x))^{-1}\omega(x, \mathcal{O}(\tilde{\lambda}(x))).$$

In order to prove that $Hd_\phi = dH$, we need the following, which will also be useful in the next section.

LEMMA 3.3. *Let M be a manifold, V a finite dimensional vector space, $f \in \Omega^0(M; GL(V))$*
$\subset \Omega^0(M; g\ell(V))$, and $\omega \in \Omega^1(M; V)$. Define $\Delta f \in \Omega^1(M; g\ell(V))$ by $\Delta f = f^{-1}df$. Then

$$d(fw) = f(dw + \Delta f \wedge w).$$

The proof is straightforward.

To prove $Hd_\phi = dH$, consider $\omega \in \mathcal{A}(X, V), (x, T) \in \tilde{X}_q \subset X_q \times \Delta G_q$, and let $f : \Delta^q \to GL(V)$ be the composite

$$\Delta^q \xrightarrow{T} G \xrightarrow{\Gamma} GL(V).$$

Then $\phi(x) = \Delta f$ since $\tilde{\lambda}(x) = \sigma(T) = T^{-1}dT$ and we have

$$\begin{aligned}(Hd_\phi\omega)(x, T) &= (\beta T) \cdot d_\phi\omega(x)\\ &= f \cdot (\cdot\omega(x) + \Delta f \wedge \omega(x))\\ &= d(f \cdot \omega(x)) = dH(\omega)(x, T).\end{aligned}$$

by Lemma 3.3.

We conclude this section by reviewing the usual definition of local coefficients.

Let $t : X_1 \to G$ be a continuous function satisfying $t(\partial_1 x) = t(\partial_2 x)t(\partial_0 x)$ for $x \in X_2$. Define $\delta_t \cdot C^p(X; V) \to C^{p+1}(X; V)$ by

$$(\delta_t u)(x) = t(\partial_2^{p-1} x)u(\partial_0 x) + \sum_{i=1}^{p+1}(-1)^i u(\partial_j x)$$

for $x \in X_{p+1}$. Then $\delta_t^2 = 0$ and we define

$$H^*(X; V_t) = H_*(C^*(X; V); \delta_t).$$

Suppose $\lambda : C^*(L) \to \mathcal{A}(X)$ with $\mathcal{O}, \phi$, and $\tilde{X}$ as above. Define $t = t_\lambda$ by

$$t_\lambda(x) = \mathcal{O}(\phi(x))(v_1)$$

where $v_0, v_1, \ldots, v_q$ are the vertices of Δ^q and let G act on $C^*(\tilde{X}; V)$ by

$$(gu)(x, T) = gu(x, g^{-1}T).$$

Define $K : C^*(X; V) \to C^*(\tilde{X}; V)$ by $K(u)(x, T) = T(v_0)u(x)$.

PROPOSITION 3.4. *The function K maps $C^*(X, V)$ isomorphically onto $C^*(\tilde{X}, V)^G$ with $Kd_t = dK$ where $t = t_\lambda$.*

PROOF: It is easy to see that K^{-1} is given by

$$K^{-1}(u)(x) = u(x, \mathcal{O}(\tilde{\lambda}(x))).$$

The remainder of the proof is similar to the proof of Proposition 3.2 and we omit it.

Let $\tilde{\Psi} = \Psi \otimes \mathrm{id} : \mathcal{A}(X, V) \to C^*(X, V)$ where $\Psi : \mathcal{A}(X) \to C^*(X)$ is defined in [3]. We now have the following local coefficient version of the de Rham Theorem.

THEOREM 3.5. *The map $\tilde{\psi}$ induces an isomorphism*

$$\tilde{\psi}_* : H_*(\mathcal{A}(\tilde{X}; V)^G) \to H_*(C^*(\tilde{X}; V)^G)$$

and hence an isomorphism

$$(K^{-1}\tilde{\psi}H)_* : H_*(\mathcal{A}(X, V), d_\phi) \to H_*(C^*(X, V), d_t)$$

where $\phi = \phi_\lambda$ and $t = t_\lambda$.

PROOF: In the proof of Theorem 2.4 of [3], natural mappings $\phi : C^*(X) \to \mathcal{A}(X)$ and $\gamma : \mathcal{A}^p(X) \to \mathcal{A}^{p-1}(X)$ were constructed satisfying $\psi\phi = \mathrm{id}$ and $d\gamma + \gamma d = \phi\psi - \mathrm{id}$. Tensoring everything in sight with V gives the desired result.

4. The Proof of Theorem 1.2 (ii).

We now prove part (ii) of Theorem 1.2 in the case where $G = G_A$ is compact.

Let G be a connected, simply connected Lie group G with Lie algebra L. Suppose L acts on a finite dimensional vector space V via a homomorphism $\gamma : L \to g\ell(V)$. Viewing γ in $C^1(L; g\ell V)$, define a differential d_γ on $C^*(L; V)$ by

$$d_\gamma(\alpha) = d\alpha + \gamma_\wedge \alpha$$

as in Remark 1.1. Similarly, we define a differential $d_{i\gamma}$ on $\mathcal{A}(\Delta C^*(L); V)$ by

$$d_{i\gamma}(\omega) = d\omega + (i\gamma) \wedge \omega$$

where $i : C^*(L; g\ell(V)) \to \mathcal{A}(\Delta(C^*(L)); g\ell(V))$ is the canonical map. The wedge product $(i\gamma) \wedge \omega$ is defined using the pairing

$$\mathcal{A}(\Delta C^*(L); g\ell(V)) \otimes \mathcal{A}(\Delta C^*(L); V) \to \mathcal{A}(\Delta C^*(L); V).$$

According to Remark 1.1, part (ii) of Theorem 1.2 is a consequence of the following.

THEOREM 4.1. *If G is compact, then the mapping*

$$i : (C^*(L;V); d_\gamma) \to (\mathcal{A}(\Delta C^*(L);V); d_{i\gamma})$$

induces an isomorphism on homology.

The remainder of this section will be devoted to proving this result.

Define $\tilde{F} : C^*(L;V) \longrightarrow \mathcal{A}(\Delta G/G;V)$ by $\tilde{F}(\alpha \otimes v)(T) = (T^*\bar{\alpha}) \otimes v$ where $v \in V$, $T \in \Delta G, \alpha \in C^p(L)$, and $\bar{\alpha} \in \Omega^p(G)$ is the left invariant p-form defined by α. Define $\tilde{\gamma} = \tilde{F}\gamma \in \mathcal{A}^1(\Delta G/G; g\ell(V))$ where $\tilde{F}$ is defined as above with V replaced by $g\ell(V)$. Let

$$d_{\tilde{\gamma}} : \mathcal{A}^p(\Delta G/G;V) \to \mathcal{A}^{p+1}(\Delta G/G;V)$$

be given by $d_{\tilde{\gamma}}(u) = du + \tilde{\gamma} \wedge u$ where d is the usual differential on $\mathcal{A}^*(\Delta G/G)$ and $\tilde{\gamma} \wedge v$ is the wedge product defined using the pairing

$$\Omega^*(\Delta^q; g\ell(V)) \otimes \Omega^*(\Delta^q;V) \to \Omega^*(\Delta^q;V).$$

LEMMA 4.2. *For any $\alpha \in C^*(L;V), \tilde{F}d_\gamma(\alpha) = d_{\tilde{\gamma}}\tilde{F}(\alpha)$ and the diagram*

$$\begin{tikzcd} & \mathcal{A}(\Delta C^*(L);V) \arrow[dr, "\sigma^*"] & \\ C^*(L;V) \arrow[ur, "i"] \arrow[rr, "\tilde{F}"] & & \mathcal{A}(\Delta G/G;V) \end{tikzcd}$$

commutes.

PROOF: We first verify that the diagram commutes. It is enough to do this when $V = R$ in which case each of the mappings is an algebra homomorphism. Since $C^*(L)$ is generated by one dimensional elements, we need only show that $\sigma^* i(\alpha) = \tilde{F}(\alpha)$ for $\alpha \in L^* = C^1(L)$. If $T : \Delta^q \to G$, we have

$$\begin{aligned} \sigma^* i(\alpha)(T) &= i(\alpha)(\sigma(T)) \\ &= \sigma(T)(\alpha) = \alpha(T^{-1}dT) \end{aligned}$$

Now $T^{-1}dT$ is the L-valued 1-form on Δ^q given by $(T^{-1}dT)(u) = dL_{T(t)^{-1}}dT(u)$ where u is a tangent vector to Δ^q at $t \in \Delta^q$ and $L_{T(t)^{-1}} : G \to G$ is left translation by $T(t)^{-1}$. Thus, $\alpha(dL_{T(t)^{-1}}dT(u)) = \bar{\alpha}(dT(u))$ so that $\alpha(T^{-1}dT) = T^*(\bar{\alpha})$ and the diagram commutes.

To prove $\tilde{F}d_\gamma\alpha = d_{\tilde{\gamma}}\tilde{F}(\alpha)$, we note that $d\tilde{F} = \tilde{F}d$ so it is enough to show that $\tilde{F}(\gamma \wedge \alpha) = \tilde{\gamma} \wedge \tilde{F}(\alpha)$. If $\gamma = \beta \otimes A$ as an element of $C^1(L) \otimes g\ell(V) \simeq C^1(L; g\ell(V))$ and $\alpha = \alpha_1 \otimes v$ as an element of $C * (L) \otimes V \simeq C^*(L;V)$, then

$$\begin{aligned} \tilde{F}(\gamma \wedge \alpha)(T) &= \tilde{F}(\beta \wedge \alpha_1 \otimes A(v))(T) \\ &= T^*(\bar{\beta} \wedge \bar{\alpha}_1) \otimes A(v) \\ &= (\tilde{F}(\gamma) \wedge \tilde{F}(\alpha))(T) = \tilde{\gamma} \wedge \tilde{F}(\alpha)(T). \end{aligned}$$

The general case now follows from the fact that any $\gamma \in C^1(L; g\ell(V))$ and $\alpha \in C^*(L;V)$ are sums of elements of the form considered above.

Since σ is a simplicial homeomorphism, σ^* is an isomorphism of differential graded vector spaces and i will be a homology isomorphism if and only if $\tilde{F}$ is a homology isomorphism. To prove $\tilde{F}$ a homology isomorphism, we will define mappings which give the following commutative diagram of differential graded vector spaces:

$$
\begin{tikzcd}
(C^*(L;V);d_\gamma) \arrow[r, "\tilde{F}"] \arrow[d, "\mu"'] & (\mathcal{A}(\Delta G/G;V);d_{\tilde\gamma}) \arrow[d, "\tilde\mu"] \\
(\Omega^*(G;V)^G;d) \arrow[r, "\bar F"] \arrow[d, shift left, "j"] & (\mathcal{A}(\Delta G;V)^G;d) \arrow[d, shift left, "\tilde j"] \\
(\Omega^*(G;V);d) \arrow[r, "F"] \arrow[u, shift left, "r"] & (\mathcal{A}(\Delta G;V);d) \arrow[u, shift left, "\tilde r"]
\end{tikzcd} \tag{4.3}
$$

Here, j and $\tilde{j}$ are inclusion mappings.

PROPOSITION 4.4. *Suppose that, in the commutative diagram (4.3), μ and $\tilde\mu$ are isomorphisms, F is a homology isomorphism, $rj = id$, and $\tilde r \tilde j = id$. Then $\tilde F$ is a homology isomorphism.*

PROOF: It is clearly enough to prove $\bar F$ a homology isomorphism. Now, $rj = id$ implies that j_* is injective. Thus $(Fj)_*^{-1}(\tilde j \bar F)_*$ is injective and it follows that $\bar F_*$ is injective. To prove $\bar F_*$ surjective, consider $u \in H_*(\mathcal{A}(\Delta G;V)^G;d)$ and let $v = r_* F_*^{-1} \tilde j_* u$. Then $\bar F_*(v) = u$ so $\bar F_*$ is surjective and thus an isomorphism.

We now proceed to define the mappings in diagram (4.3) and prove that the hypotheses of Proposition 4.4 are satisfied. We begin by defining an action of G on $\Omega^*(G;V)$ and on $\mathcal{A}(\Delta G;V)$ which give the middle row of (4.3).

The Lie algebra homomorphism $\gamma : L \to g\ell(V)$ determines a unique Lie group homomorphism $\Gamma : G \to GL(V)$ with $\gamma = d\Gamma : L = TG_e \to TGL(V)_e = g\ell(V)$. (Recall that G is assumed simply connected.) Define actions of G on $\Omega^*(G;V)$ and on $\mathcal{A}(\Delta G;V)$ as follows. For $g \in G, w \in \Omega^*(G;V)$, let $gw \in \Omega^p(G;V)$ be given by $gw = \Gamma(g)(L^*_{g^{-1}}w)$. Similarly, for $u : \Delta G \to \Omega^p(\Delta^q;V) \in \mathcal{A}^p(\Delta G;V), g \in G$, and $T \in (\Delta G)_q$, let gu be the element of $\mathcal{A}^p(\Delta G;V)$ given by $(gu)(T) = \Gamma(g)u(g^{-1}T)$. Then $\Omega^*(G;V)^G$ and $\mathcal{A}(\Delta G;L)^G$ denote the cochain complexes of elements fixed under the action of G with the standard differential d. It is straightforward to show that $d(gu) = g(du)$ for $u \in \Omega^*(G;V)$ or $u \in \mathcal{A}(\Delta G;V)$.

Define $F : \Omega^*(G;V) \to \mathcal{A}(\Delta G;V)$ by $F(w)(T) = T^*w$ for $w \in \Omega^*(G;V), T \in \Delta G$. Then $F(dw) = dF(w)$ is immediate and

$$\begin{aligned}
F(gw)(T) &= T^*(gw) \\
&= T^*\Gamma(g)L^*_{g^{-1}}w \\
&= (id \otimes \Gamma(g))(T^* \otimes id)L^*_{g^{-1}}w \\
&= \Gamma(g)(L_{g^{-1}}T)^*w = (gF(w))(T).
\end{aligned}$$

Thus F induces $\bar{F} : (\Omega^*(G,V)^G, d) \to (\mathcal{A}(\Delta G;V)^G, d)$.

Define mappings

$$r : \Omega^*(G;V) \to \Omega^*(G;V) \quad \tilde{r} : \mathcal{A}(\Delta G;V) \to \mathcal{A}(\Delta G;V)$$

by

$$r(w) = \int_G gw, \quad \tilde{r}(u)(T) = \int_G (gu)(T)$$

for $w \in \Omega^*(G;V), u \in \mathcal{A}(\Delta G;V), T \in \Delta G$, and the integral is the Haar integral on G normalized so that the volume of G is one.

PROPOSITION 4.5. *For any $g \in G, w \in \Omega^*(G;V), u \in \mathcal{A}^1(\Delta G;V)$, we have $g(r(w)) = r(w)$ and $g(\tilde{r}(u)) = \tilde{r}(u)$. Furthermore, $dr = rd, d\tilde{r} = \tilde{r}d$, and $\tilde{r}F = \bar{F}r$.*

The proof of this proposition is straightforward. For example, to prove that $\bar{F}r = \tilde{r}F$, we simply use the definitions:

$$\begin{aligned}
(\bar{F}r(w))(T) &= T^*r(w) \\
&= T^* \int_G \Gamma(g)L^*_{g^{-1}}w \\
&= \int_G \Gamma(g)T^*L^*_{g^{-1}}w \\
&= \int_G \Gamma(g)(L_{g^{-1}}T)^*w \\
&= \int \Gamma(g)F(w)(L_{g^{-1}}T) = (\tilde{r}F(w))(T).
\end{aligned}$$

It follows from Proposition 4.5 that

$$r(\Omega^*(G;V)) \subset \Omega^*(G;V)^G, \quad \tilde{r}(\mathcal{A}(\Delta G;V)) \subset \mathcal{A}(\Delta G;V)^G.$$

Thus, we have established the existence of the lower rectangle of mappings in (4.3).

To obtain the upper rectangle in (4.3), we need first of all to define a second action of G on $\Omega^*(G;V)$ and $\mathcal{A}(\Delta G;V)$. For $w \in \Omega^*(G,V)$ and $g \in G$, let $g * w$ be the element of $\Omega^*(G;V)$ given by

$$g * w = L^*_{g^{-1}} w.$$

Similarly, for $u \in (\Delta G;V)$, let $g * u \in \mathcal{A}(\Delta G;V)$ be defined by

$$(g * u)(T) = u(g^{-1}T)$$

for $T \in \Delta G$. Then $d(g * w) = g * dw$ for $w \in \Omega^*(G;V)$ or $w \in \mathcal{A}(\Delta G;V)$ and we let $\Omega^*(G;V)^{G*}$ and $\mathcal{A}(\Delta G;V)^{G*}$ denote the subspaces of elements fixed under these actions of G. Of course, $C^*(L;V)$ can be identified with $\Omega^*(G;V)^{G*}$. The next result gives the corresponding identification for $\mathcal{A}(\Delta G;V)$.

LEMMA 4.6. *The natural mapping $p : \Delta G \to \Delta G/G$ induces an isomorphism*

$$p^* : \mathcal{A}(\Delta G/G;V), d_{\tilde{\gamma}}) \to (\mathcal{A}(\Delta G;V)^{G*}, d_{\tilde{\gamma}})$$

of differential graded vector spaces.

The proof is trivial.

In order to define the mappings μ and $\tilde{\mu}$ of diagram (4.3), we first define related mappings. Let

$$\eta : \Omega^*(G;V) \to \Omega^*(G;V), \quad \tilde{\eta} : \mathcal{A}(\Delta G;V) \to \mathcal{A}(\Delta G;V)$$

be defined by $\eta(\omega) = \Gamma \cdot \omega$, $\tilde{\eta}(u)(T) = (\Gamma \circ T) \cdot u(T)$.

LEMMA 4.7. *The mappings η and $\tilde{\eta}$ are bijective. Furthermore, we have $\eta(g * \omega) = g\eta(\omega)$ and $\tilde{\eta}(g * u) = g\tilde{\eta}(u)$ for any $g \in G$, $\omega \in \Omega^*(G;V)$, and $u \in \mathcal{A}(\Delta G;V)$.*

PROOF: The inverses to η and $\tilde{\eta}$ are defined just as η and $\tilde{\eta}$ are defined using $\Gamma^{-1} : G \to GL(V), \Gamma^{-1}(g) = \Gamma(g)^{-1}$, in place of Γ. The verification of the two equations of Lemma 4.7 are similar; we do only the second, leaving the first to the reader. Thus, for $g \in G, u \in \mathcal{A}(\Delta G;V)$, and $T \in \Delta G$, we have

$$\begin{aligned}(g\tilde{\eta}(u))(T) &= \Gamma(g)\tilde{\eta}(u)(g^{-1}T)\\ &= \Gamma(g)(\Gamma \circ g^{-1}T)u(g^{-1}T)\\ &= (\Gamma(g)\Gamma(g^{-1})\Gamma \circ T)u(g^{-1}T)\\ &= (\Gamma \circ T)u(g^{-1}T) = \tilde{\eta}(g * u)(T).\end{aligned}$$

As indicated above, any element $\alpha \in C^*(L;V)$ can be considered as a left invariant form $\bar{\alpha} \in \Omega^*(G;V)$. In particular, $\gamma \in C^1(L;g\ell(V))$ determines $\bar{\gamma} \in \Omega^1(G;g\ell(V))$ where $\gamma(X) = \gamma(dL_{g^{-1}}X)$ for $X \in TG_g$. Define

$$d_{\bar{\gamma}} : \Omega^*(G;V) \to \Omega^*(G;V)$$

by $d_{\bar\gamma} w = dw + \bar\gamma \wedge w$. Define $\tilde\gamma \in \mathcal{A}(\Delta G; g\ell(V))$ by $\tilde\gamma = F\bar\gamma$ where $F : \Omega^*(G;V) \to \mathcal{A}(\Delta G; V)$ is given by $F(w)(T) = T^*w$, (see Proposition 4.5), and let

$$d_{\tilde\gamma} : \mathcal{A}(\Delta G; V) \to \mathcal{A}(\Delta G; V)$$

be defined by $d_{\tilde\gamma} u = du + \tilde\gamma \wedge u$. Then $d^2_{\bar\gamma} = 0$ and $d^2_{\tilde\gamma} = 0$ and we have

Lemma 4.8. *For $w \in \Omega^*(G;V)$ and $u \in \mathcal{A}(\Delta G; V)$, we have $d\eta(w) = \eta(d_{\bar\gamma} w)$ and $d\tilde\eta(u) = \tilde\eta(d_{\tilde\gamma} u)$*

Proof: Again, the verifications of the two equations are similar so we carry out only the proof of the second equation. Thus,

$$\begin{aligned}(d\tilde\eta(u))(T) &= d(\tilde\eta(u)(T))\\ &= d((\Gamma \circ T)u(T))\\ &= \Gamma \circ T(du(T) + \Delta(\Gamma \circ T) \wedge u(T))\end{aligned}$$

by Lemma 3.3. Now, if $X \in T\Delta^q_t$, we have $\Delta(\Gamma\circ T)(X) = dL_{\Gamma(T(t)^{-1})} d\Gamma dT(X)$. Identifying $\gamma : L \to g\ell(V)$ with $d\Gamma : TG_e \to TGL(V)_e$, we have $d\Gamma = dL_{\Gamma(T(t))}\gamma dL_{T(t)^{-1}}$ (since Γ is a homomorphism) and

$$\begin{aligned}\Delta(\Gamma \circ T)(X) &= \gamma dL_{T(t)^{-1}} dT(X)\\ &= \bar\gamma(dT(X))\\ &= F(\bar\gamma)(X) = \tilde\gamma(X).\end{aligned}$$

Then

$$\begin{aligned}d(\tilde\eta(u))(T) &= (\Gamma \circ T)(du(T) + \tilde\gamma \wedge u(T))\\ &= (\tilde\eta(d_{\tilde\gamma})u)(T).\end{aligned}$$

According to Lemmas 4.7 and 4.8, η and $\tilde\eta$ induce isomorphisms

$$\begin{aligned}\eta &: (\Omega^*(G;V)^{G*}; d_{\bar\gamma}) \to (\Omega^*(G;V)^G; d)\\ \tilde\eta &: (\mathcal{A}(\Delta G; V)^{G*}; d_{\tilde\gamma}) \to (\mathcal{A}(\Delta G; V)^G; d)\end{aligned}$$

of differential graded vector spaces. Let μ and $\tilde\mu$ be the composites

$$\begin{aligned}\mu &: (C^*(L;V); d_\gamma) \simeq (\Omega^*(G;V)^{G*}; d_{\bar\gamma}) \xrightarrow{\eta} (\Omega^*(G;V)^G; d)\\ \tilde\mu &: (\mathcal{A}(\Delta G/G; V); d_{\tilde\gamma}) \xrightarrow{p^*} (\mathcal{A}(\Delta G; V)^{G*}; d_{\tilde\gamma}) \xrightarrow{\tilde\eta} (\mathcal{A}(\Delta G; V)^G; d)\end{aligned}$$

If $\bar F : \Omega^*(G;V)^G \to \mathcal{A}(\Delta G; V)^G$ is the restriction of $F : \Omega^*(G;V) \to \mathcal{A}(\Delta G; V)$, then diagram (4.3) is easily seen to be commutative.

We have now established the commutative diagram (4.3) of differential, graded vector spaces. Moreover, the mappings μ and $\tilde\mu$ are isomorphisms. Thus, according to Proposition 4.4, the proof of Theorem 1.2, part (ii), will be complete if we can show that F is a homology isomorphism. For this we need the following.

THEOREM 4.9. *Let X be a CW complex, ΔX the singular complex of X in the compact open topology, and ΔX^δ the singular complex of X in the discrete topology. Then the natural mapping $j : \Delta X^\delta \to \Delta X$ induces an isomorphism*

$$H^*(\Delta X) \xrightarrow{j^*} H^*(\Delta X^\delta) = H^*(X).$$

We are indebted to Graeme Segal for the proof of this result which we give in Section 7.

To see that $F : \Omega^*(G;V) \to \mathcal{A}(\Delta G;V)$ induces an isomorphism on homology, consider the following commutative diagram

$$\begin{array}{ccc} \Omega^*(G;V) & \xrightarrow{F} & \mathcal{A}(\Delta G;V) \\ {\scriptstyle\tilde\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ C^*(\Delta G^\delta;V) & \xrightarrow{j^*} & C^*(\Delta G;V) \end{array}$$

Here all differentials are the ordinary untwisted differentials, ψ is defined in [**3**], Section 5, and $\tilde\psi$ is the usual deRham mapping. Now, ψ is a homology isomorphism by Theorem 2.4 of [**3**], j^* is a homology isomorphism by Theorem 4.9 above, and $\tilde\psi$ is well known to be a homology isomorphism. Thus F is a homology isomorphism and Theorem 1.2, part (ii) follows from Proposition 4.4.

5. Fibrations.

Suppose G is a connected, simply connected Lie group with Lie algebra L, $A \in \mathcal{CA}$, and $\lambda : C^*(L) \to A$ is a map in $\mathcal{CA}$ Let X be a graded vector space with basis $x_1, \dots, x_k$, deg $x_j = n, j = 1, \dots, n$, and let $\{\ell_i\}$ be a basis for L. Let $A[X] = A[x_1, \dots, x_k]$ be the free algebra over A on $x_1, \dots, x_k$ and suppose $A[X]$ has a differential d with $dA \subset A$ and such that

$$dx_i = \sum b_i^{jm} \lambda(\ell_j^*) x_m + c_i \tag{5.1}$$

where $b_i^{jm} \in R$, $c_i \in A^{n+1}$ and $\{\ell_i^*\}$ is the basis for L^* dual to $\{\ell_i\}$. The relation $d^2 x_i = 0$ yields

$$dc_i = \sum b_i^{jm} \lambda(\ell_j^*) c_m \tag{5.2}$$

Let X^* be the dual space of X, $\{x_i^*\}$ the basis for X^* dual to $\{x_i\}$, and define $\mu : L \otimes X^* \to X^*$ by

$$\mu(\ell_j \otimes x_m^*) = \sum b_i^{jm} x_i^*.$$

The equation $d^2 = 0$ implies that μ defines an action of L on X^* *as a Lie algebra.* Therefore, we have a corresponding action of G on X^*.

Let $\tilde{G} \in \Delta\mathcal{T}$ be given by $\tilde{G}_q = G$, $\delta_i = s_i = id$ for all q, i and set $P = \Delta R[X] \times \tilde{G}$. (Here, $R[X] \in \mathcal{CA}$ with $dX = 0$.) We make P into a simplicial topological group and define an action of P on $\Delta(R[X])$ by

$$(v, g)(v', g') = (v + g \cdot v', gg')$$
$$(v, g)\omega = v + g\omega$$

for $g, g' \in \tilde{G}_q = G$, $v, v', \omega \in \Delta(R[X])_q \subset \Omega^n(\Delta^q; X^*)$.

In [2], Section 5, we defined a map $\mu_0 : \Omega^p(\Delta^q) \to \Omega^{p-1}(\Delta^q)$ satisfying $d\mu_0 + \mu_0 d = id$, $\mu_0 s_j = s_j \mu_0$ for $j \geq 0$, and $\mu_0 \partial_i = \partial_i \mu_0$ for $i > 0$. We extend this map to a mapping

$$\mu_0 : \Omega^p(\Delta^q; X^*) \to \Omega^{p-1}(\Delta^q; X^*)$$

with these same properties by $\mu_0(\omega \otimes x) = \mu_0(\omega) \otimes x$ and define $c : X \to A^{n+1}$ by $c(x_i) = c_i$.

THEOREM 5.3. *The simplicial space $\Delta(A[X])$ is a twisted cartesian product $\Delta(A) \times_\tau \Delta(R[X])$ with group P and twisting function $\tau : \Delta A_q \to P_{q-1}$ given by*

$$\tau(u) = (\mathcal{O}(u \circ \lambda)(v_1)^{-1}((\partial_0\mu_0 - \mu_0\partial_0)(\mathcal{O}(u \circ \lambda)u \circ c),\ \mathcal{O}(u \circ \lambda)(v_1)^{-1})$$

for $u \in \Delta A_q = (A, \Omega_q)$ and v_1 is the second vertex of Δ^q. Here $\mathcal{O}$ is defined in Section 3 to be the composite $\Delta C^*(L) \xrightarrow{\sigma^{-1}} \Delta G/G \xrightarrow{\beta} \Delta G$ where $\beta(T) = T(v_0)^{-1}T$.

PROOF: We identify (X, Ω_q^n), the space of linear mappings from X to Ω_q^n, with $\Omega^n(\Delta^q; X^*)$ by $v \longmapsto \sum v(x_i)x_i^*$, $v : X \to \Omega_q^n$. If $u \in \Delta(A)_q$, then $u \circ c \in (X, \Omega_q^{n+1}) = \Omega^{n+1}(\Delta^q; X^*)$ and $u \circ \lambda \in \Delta(C^*(L)) \subset \Omega^1(\Delta^q; L)$. Thus

$$\begin{aligned}
\Delta(A[X])_q &= (A[X], \Omega_q) \\
&= \{(u, v) \in \Delta(A)_q \times (X, \Omega_q^n) \mid dv(x_i) = \Sigma b_i^{jm}(u \circ \lambda(\ell_j^*))v(x_m) + u(c_i)\} \\
&= \{(u, v) \in \Delta(A)_q \times \Omega^n(\Delta^q; X^*) \mid dv = (u \circ \lambda) \wedge v + u \circ c\}
\end{aligned}$$

Note that

$$\begin{aligned}
\Delta(R[X])_q &= (R[X], \Omega_q) \\
&= \{v : X \to \Omega_q \mid dv = 0\} \\
&= \{v \in \Omega^n(\Delta^q; X^*) \mid dv = 0\}.
\end{aligned}$$

Define $f : \Delta(A)_q \times \Delta(R[X])_q \to \Delta(A[X])_q$ by $f(u, v) = (u, v')$ where

$$v' = \mathcal{O}(u \circ \lambda)^{-1}(\mu_0(\mathcal{O}(u \circ \lambda)u \circ c) + v)$$

where $\beta([T]) = T(v_0)^{-1}T$. In order to insure that $(u, v') \in \Delta(A[X])_q$, we need to show that $dv' = (u \circ \lambda) \wedge v' + u \circ c$. If $\mathcal{O}_0 = \mathcal{O}(u \circ \lambda) \in (\Delta G)_q$, then

$$\begin{aligned} dv' &= d\mathcal{O}_0^{-1} \cdot (\mu_0(\mathcal{O}_0 \cdot u \circ c) + v) + \mathcal{O}_0^{-1} \cdot (d\mu_0(\mathcal{O}_0 \cdot u \circ c) + dv) \\ &= -\mathcal{O}_0^{-1} \cdot d\mathcal{O}_0 \cdot \mathcal{O}_0^{-1}(\mu_0(\mathcal{O} \cdot u \circ c) + v) + \mathcal{O}_0^{-1} \cdot (\mathcal{O}_0 \cdot u \cdot c - \mu_0(d(\mathcal{O}_0 \cdot u \circ c))) \\ &= -\mathcal{O}_0^{-1} d\mathcal{O}_0 \wedge v' + u \circ c - \mathcal{O}_0^{-1} \cdot \mu_0(d(\mathcal{O}_0 \cdot u \circ c)) \end{aligned}$$

since $d\mathcal{O}_0^{-1} = -\mathcal{O}_0^{-1} d\mathcal{O}_0 \mathcal{O}_0^{-1}$.

We now need the following results.

If $u \circ \lambda \in \Delta C^*(L)_q$ is considered an element of $\Omega^1_q(L)$ (as in the discussion following Theorem 2.10), we have

LEMMA 5.4. $d\mathcal{O}_0 = -\mathcal{O}_0 \cdot u \circ \lambda$.

PROOF: Identifying $\Delta C^*(L)_q$ with $\hat{\Omega}^1(\Delta^q; L)$, we have

$$\begin{aligned} -\mathcal{O}_0^{-1} d\mathcal{O} &= \rho(\mathcal{O}_0) \\ &= \rho(\mathcal{O}(u \circ \lambda)) = u \circ \lambda \end{aligned}$$

by the definition of the mappings involved.

COROLLARY. $\mathcal{O}_0^{-1} d\mathcal{O}_0 = -u \circ \lambda$.

LEMMA 5.5. $d(u \circ c) = u \circ \lambda \wedge u \circ c$.

PROOF: The element in $\Omega^{n+1}(\Delta^q; X^*)$ corresponding to $u \circ c$ is $\Sigma(u \circ c)(x_i)x_i^* = \Sigma u(c_i)x_i^*$. Thus

$$\begin{aligned} d(u \circ c) &= d\Sigma u(c_i)x_i^* \\ &= \Sigma u(dc_i)x_i^* \\ &= \Sigma b_i^{jm} u \circ \lambda(\ell_j^*)u(c_m)x_i^* \\ &= (\Sigma u \circ \lambda(\ell_j^*)\ell_j)(\Sigma u \circ c(x_m)x_m^*) \\ &= u \circ \lambda \wedge u \circ c. \end{aligned}$$

COROLLARY. $d(\mathcal{O}_0 \cdot u \circ c) = 0$.

PROOF:

$$\begin{aligned} d(\mathcal{O}_0 \cdot u \circ c) &= (d\mathcal{O}_0) \wedge u \circ c + \mathcal{O}_0 \cdot du \circ c \\ &= -\mathcal{O}_0 \cdot u \circ \lambda \wedge u \circ c + \mathcal{O}_0 \cdot u \circ \lambda \wedge u \circ c = 0. \end{aligned}$$

It follows from the two corollaries above that dv' has the required form.

Define $f^{-1} : \Delta(A[X])_q \to \Delta(A)_q \times \Delta R[X]_q$ by $f^{-1}(u, v') = (u, v)$ where

$$v = \mathcal{O}(u \circ \lambda)v' - \mu_0(\mathcal{O}(u \circ \lambda) \cdot u \circ c).$$

It is easy to check that f^{-1} is actually an inverse for f. We now determine the twisting function τ.

For $(u, v) \in \Delta A_q \times \Delta R[X]_q$, set $\partial_0(u, v) = (\partial_0 u, \bar{v})$. Then $f(\partial_0(u, v)) = (\partial_0 u, \bar{v}')$ where

$$\bar{v}' = \mathcal{O}(\partial_0(u \circ \lambda))^{-1}(\mu_0(\mathcal{O}(\partial_0(u \circ \lambda)) \cdot \partial_0(u \circ c)) + \bar{v})$$

Furthermore, $\partial_0 f(u, v) = (\partial_0 u, \partial_0 v')$ where

$$\partial_0 v' = \partial_0 \mathcal{O}(u \circ \lambda)^{-1}(\partial_0 \mu_0(\mathcal{O}(u \circ \lambda) \cdot u \circ c) + \partial_0 v)$$

Thus, if $\partial_0 v' = \bar{v}$, we have

$$\bar{v} = \mathcal{O}(\partial_0(u \circ \lambda)) \cdot \partial_0 \mathcal{O}(u \circ \lambda)^{-1}(\partial_0 \mu_0(\mathcal{O}(u \circ \lambda) \cdot u \circ c) + \partial_0 v) - \mu_0(\mathcal{O}(\partial_0(u \circ \lambda)) \cdot \partial_0(u \circ c))$$

It is easy to see that $g \cdot \mu_0(\omega) = \mu_0(g \cdot \omega)$ and $\mathcal{O}(\partial_0 \alpha) = \mathcal{O}(\alpha)(v_1)^{-1}\partial_0 \mathcal{O}(\alpha)$ for $g \in G$, $\omega \in \Omega^*(\Delta^q; X^*)$, and $\alpha \in \Delta C^*(L)$. It follows that

$$\bar{v} = \mathcal{O}(u \circ \lambda)(v_1)^{-1}((\partial_0 \mu_0 - \mu_0 \partial_0)(\mathcal{O}(u \circ \lambda) \cdot u \circ c) + \mathcal{O}(u \circ \lambda)(v_1)^{-1}\partial_0 v)$$

so that

$$\tau(u) = (\mathcal{O}(u \circ \lambda)(v_1)^{-1}((\partial_0 \mu_0 - \mu_0 \partial_0)(\mathcal{O}(u \circ \lambda) \cdot u \circ c), \mathcal{O}(u \circ \lambda)(v_1)^{-1}).$$

The verification that $\partial_i f = f \partial_i$ for $i > 0$ and $s_i f = f s_i$ is routine and left to the reader.

We conclude this section with a proof of Theorem 1.3. The proof of Theorem 5.3 of this paper can easily be extended to show that

$$\mathcal{F}(R[X], B) \to \mathcal{F}(A[X], B) \to \mathcal{F}(A, B) \tag{5.6}$$

is a twisted cartesian product and hence a fibration in $\Delta\mathcal{T}$. (Theorem 5.3 corresponds to the case $B = R$.) For example, identifying $(X, \Omega_q \otimes B)$ with $\Omega(\Delta^q; X^*) \otimes B)^n$, we have

$$\mathcal{F}(A[X, B)_q = \{(u, v) \in \mathcal{F}(A[X], B) \times (\Omega(\Delta^q; X^*) \otimes B)^n \mid dv = u\lambda v + uc\}.$$

The pullback of (5.6) to $\mathcal{F}(A, B; h) \subset \mathcal{F}(A, B)$ yields a fibration and a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(R[X], B) \arrow[r] \arrow[d, "\Delta_1"] & \mathcal{F}(A[X], B; h) \arrow[r] \arrow[d] & \mathcal{F}(A, B; h) \arrow[d] \\
\mathcal{F}(\Delta B, \Delta R[X]) \arrow[r] & \mathcal{F}(\Delta B, \Delta(A[X]); \Delta h) \arrow[r] & \mathcal{F}(\Delta B, \Delta A; \Delta h)
\end{tikzcd}$$

If we write $A = \cup A_n$, then induction, the diagram above, and the fact that Δ_1 is a weak equivalence gives the theorem for $A = A_n$. A limit argument as in the proof of Theorem 2.20 of [**3**] yields the desired result.

6. The Continuous Cohomology Serre Spectral Sequence with Local Coefficients

Suppose that $E = B \times_\tau F$ is a twisted cartesian product in $\Delta\mathcal{T}$ with group P and suppose $\pi_0(P) = P_0$. In [**3**], Section 8, we constructed a local system $\tau = \tau \mid B_1 \to P_0$, on action of P_0 on $C^*(F; R)$, and a map

$$\Delta^* : C^*(B; C^*(F; R)) \to C^*(B \times_\tau F; R)$$

which was filtration preserving with respect to the obvious filtrations. In general, Δ^* is not a cochain mapping (relative to the usual differential δ on $C^*(B \times_\tau F; R)$ and the twisted differential δ_τ on $C^*(B; C^*(F; R))$) but it does in fact induce an isomorphism on $E_r^{p,q}$ for $r \leq 2$. (See [**3**], Section 8.) Furthermore, if F is splittable, this map gives an isomorphism

$$E_2^{p,q}(B \times_\tau F) \simeq H^*(B; H^*(F; R)).$$

Suppose now that L, G, and V are as in Section 3 and $t : B_1 \to G$ is a local system. If, in the above paragraph, one replaces R by V, δ on $C^*(E; V)$ by $\delta_{tp}, p : E \to B$, and δ_τ by $\delta_{\bar{\tau}}, \bar{\tau} : B_1 \to P_0 \times G, \bar{\tau}(b) = (\tau(b), t(b))$, then the statements remain true with the same proofs as in Section 8 of [**3**]. Hence we have

THEOREM 6.1. *If F is splittable, then the Serre spectral sequence for $H^*(B \times_\tau F; V_{tp})$ converges in the usual way and Δ^* induces an isomorphism*

$$E_2^{p,q} \simeq H^p(B; H^q(F; V)_{\bar{\tau}}).$$

We next apply Theorem 6.1 to $\Delta(A[X]) = \Delta(A) \times_\tau \Delta(R[X])$. Let $\lambda : C^*(L) \to A$ be a map in $\mathcal{CA}$ and recall that

$$i = i_A : A \otimes V \to (\Delta(A)) \otimes V$$

is given by $i(a \otimes v) = i(a) \otimes v$ where $i(a)(f) = f(a)$.

LEMMA 6.2. *If i_A induces an isomorphism*

$$i_A : H_*(A \otimes V, d_\lambda) \xrightarrow{\simeq} H_*(\mathcal{A}(\Delta(A)), d_\phi);$$

then the same is true for $i_{A[X]} : A[X] \otimes V \to \mathcal{A}^*(\Delta(A[X])) \otimes V$.

PROOF: Let $\bar{\psi} = K^{-1}\tilde{\psi}H : \mathcal{A}(\Delta(A); V) \to C^*(\Delta A; V)$ be the mapping defined in Section 3 (see Theorem 3.5) and let

$$\bar{i} = \bar{\psi} i : A \otimes V \to C^*(\Delta A; V).$$

It is sufficient to prove Lemma 6.2 with i replaced by $\bar{i}$ and d_ϕ by d_t. Define a filtration on $A[X] \otimes V = A \otimes R[X] \otimes V$ by

$$F^p = \{a \otimes w \otimes v \mid \dim a \geq p\}.$$

Exactly as in [**3**], Lemma 9.4, one checks that $\bar{i}$ is filtration preserving and hence induces a mapping of the corresponding spectral sequences.

We prove Lemma 6.2 by showing that $\bar{i}$ for $A[X]$ induces an isomorphism at the E_2 level. As in [**3**], Section 9, the $\hat{E}_1$ term for $A \otimes R[X] \otimes V$ is

$$\hat{E}_1^{p,q} = A^p \otimes R[X]^q \otimes V$$

and for $\Delta(A[X])$,

$$E_1^{p,q} = C^p(\Delta(A), H^q(\Delta(R[X]; V))).$$

In both cases, d_2 is the appropriate local coefficient differential. Furthermore,

$$\bar{i} : R[X] \otimes V \to C^*(\Delta(R[X]); V)$$

induces an isomorphism on homology, this being the untwisted version of our theorem which we proved as Proposition 2.8 in [**3**]. By hypothesis

$$\bar{i}_{R[X]} \otimes \bar{i}_A : \hat{E}_2 \to E_2$$

induces an isomorphism. Thus we must show that $\bar{i}_{A[X]}$ induces this map. The map induced by $\bar{i}_{A[X]}$ is the composition

$$\begin{aligned} A[X] \otimes V \xrightarrow{\bar{i}} C^*(\Delta(A[X]), V) &\xrightarrow{f^*} C^*(\Delta(A) \times_\tau \Delta(R[X])) \\ &\xrightarrow{\eta} \sum C^p(\Delta(A), C^{*-p}(\Delta(R[X]), V)) \end{aligned}$$

where η is induced by the usual Eilenberg-Zilber map $C_p(X) \otimes C_q(X) \to C_{p+q}(X)$ involving shuffles of degeneracy maps.

For any $X, x \in X$, and $w \in \Omega^n(X;V)$,

$$\begin{aligned}\bar{\psi}\omega(x) &= K^{-1}\mathcal{O}\psi H\omega(x)\\ &= \tilde{\psi}H\omega(x,\mathcal{O}(\phi(x)))\\ &= \int_{\Delta^n} H\omega(x,\mathcal{O}(\phi(x)))\\ &= \int_{\Delta^n} \mathcal{O}(\phi(x)\omega(x)).\end{aligned}$$

For $u \in \Delta^n(A), v \in \Delta_q(R[X])$, and (α,β) a (p,q) shuffle,

$$f(s_\alpha u, s_\beta v) = (s_\alpha u, s_\alpha(\mathcal{O}(u\lambda)^{-1}\mu_0\mathcal{O}(u\lambda)uw) + s_\alpha\mathcal{O}(u\lambda)^{-1}s_\beta v$$

The first terms in each factor are s_α degenerate, thus will be s_α degenerate when evaluated on an element of $A[X]\otimes V$ and hence will drop out when we integrate. Suppose $a \in A^p, e \in R[X]^q$ and $z \in V$. Then

$$\bar{i}f^*\eta(a\otimes e\otimes z)(u)(v) = \sum_{\alpha,\beta}\pm\int_{\Delta^{p+q}} s_\alpha\mathcal{O}(u\lambda)(s_\alpha u(a)(s_\alpha\mathcal{O}(u\lambda)^{-1}s_\beta\hat{v}(e))z$$

$$= \int_{\Delta^p}\mathcal{O}(u\lambda)(z)u(a)\mathcal{O}(u\lambda)^{-1}\int_{\Delta^q}\hat{v}(e)$$

where $\hat{v} \in (R[X],\Omega)$ is defined by $v \in \mathcal{A}(\Delta_q;X)$. Similarly

$$i\bar{\psi}i_A : A\otimes R[X]\otimes V \to C^*(\Delta(A), C^*(\Delta R[X],V))$$

is given as follows: Note first that the group in question is $\pi_0(G\times P) = G\times G$ which acts on $R[X]\otimes V$ by

$$(g_1,g_2)(e\otimes z) = (eg_2, g_1 z).$$

Since G acts on the left of X^*, it acts on the right of $R[X]$ and on the left of $\Delta(R[X])$. For $u \in \Delta(A)$ the $\mathcal{O}$ in this case is $(\mathcal{O}(u\lambda),\mathcal{O}(u\lambda)^{-1})$ (see Section 5). Hence

$$\bar{\psi}i_A(a\otimes e\otimes z)(u) = \int_{\Delta^p} e\mathcal{O}(u\lambda)^{-1}\otimes\mathcal{O}(u\lambda)u(a)z$$

and

$$i\bar{\psi}i_A(a\otimes e\otimes z)(u)(v) = \int_{\Delta^q}\int_{\Delta^p}(\mathcal{O}(u\lambda)^{-1}v)(e)\mathcal{O}(u\lambda)u(a)z$$

where $\mathcal{O}(u\lambda) \in \Omega^0(\Delta^p;G), u(a) \in \Omega^p(\Delta^p), v(e) \in \Omega^q(\Delta^q)$. Comparing this with
$\bar{i}f^*\eta(a\otimes e\otimes z)(u)(v)$ we see they are equal and the lemma is proved.

We conclude this section with a proof of Theorem 1.2, (iv). Let $A \in \mathcal{CA}$ be free and finite type so that $A = \cup A^{(n)}, A^{(n)} = A^{(n-1)}[X_n], A_0 = R$. By Theorem 1.2, (ii), i induces an isomorphism

$$H_*(A^{(1)}; V_\lambda) \simeq H_*(\mathcal{A}(\Delta(A^{(1)}) \otimes V_{i\lambda}).$$

By Lemma 6.2 and induction on n, i induces an isomorphism

$$H_*(A^{(n)} \otimes V, d_\lambda) \approx H_*(\mathcal{A}(\Delta(A^{(n)})) \otimes V, d_\lambda)$$

and hence the same holds for A.

7. The Proof of Theorem 4.9.

We give here the proof of Theorem 4.9 which was communicated to us by G. Segal.

Let X be a paracompact space and $\Delta(X)$ the singular complex of X in the compact open topology. Thus, $H^*(\Delta X)$ is the continuous cohomology of the simplicial space $\Delta(X)$ and $H^*(\Delta(X)^\delta)$ is the singular cohomology of X. Let $\mathcal{U}$ be a covering of X which has a partition of unity $\{\lambda_U; U \in \mathcal{U}\}$ subordinate to it. Let $\Delta(X,\mathcal{U}) \subset \Delta(X)$ be the simplicial subspace consisting of those $T : \Delta^q \to X$ with $T(\Delta^q) \subset U$ for some $U \in \mathcal{U}$.

LEMMA 7.1. *The inclusion mapping $\Delta(X,\mathcal{U}) \subset \Delta(X)$ induces an isomorphism*

$$H^*(\Delta(X,\mathcal{U})) \xrightarrow{\simeq} H^*(\Delta(X))$$

on continuous cohomology.

PROOF: Let $C_q(X)$ be the singular chains on X with integer coefficients, $sd : C_q(X) \to C_q(X)$ be the usual subdivision mapping, and $D : C_q(X) \to C_{q+1}(X)$ the chain homotopy with $\partial D + D\partial = sd - id$. The maps sd and D are natural and obtained from the first barycentric subdivision of Δ^q. Let sd^n be the n^{th} iterate of $sd, id_q : \Delta^q \to \Delta^q$ the identity map considered as an element of $\Delta_q(\Delta^q)$, and write

$$sd^n(id_q) = \sum_i \pm\tau_i^n$$

where $\tau_i^n \in \Delta_q(\Delta^q)$. Note that the diameter of $\tau_i^n(\Delta^q)$ approaches 0 as n approaches infinity.

Let $\rho : R \to R$ be a smooth non decreasing function with $\rho(x) = 0$ for $x \leq 0$ and $\rho(x) = 1$ for $x \geq \frac{1}{2}$. Define continuous functions $\varphi^n : \Delta(X) \to R$ by

$$\varphi^n(T) = \rho(\min_i \sum_{U\in\mathcal{U}} \min_{t\in\Delta^q}(\lambda_U(T(\tau_i^n(t)))))$$

The following may be verified by inspection.

LEMMA 7.2. *The functions $\varphi^n : \Delta(X) \to R$ satisfy the following.*

(i) *φ^n is continuous.*

(ii) *For each $T \in \Delta(X)$, there is an integer $N = N_T$ with $\varphi^n(T) = 1$ for $n \geq N$.*

(iii) *If $\varphi^n(T) \neq 0$, then there is a $U \in \mathcal{U}$ with*

$$T(\tau_i^n(\Delta^q)) \subset supp\ \lambda_U \subset U$$

for all i.

Remark. The functions $\psi_n = \varphi_i - \varphi_{n-1}, \psi_0 = \varphi_0$, define a partition of unity subordinate to the covering $\{\Delta^n(X, \mathcal{U}), n \geq 0\}$ where

$$\Delta^n(X, \mathcal{U}) = \{T \in \Delta(X) \mid sd^n T \in C_q(\Delta(X, \mathcal{U}))\}.$$

Let $sd^* : C^q(\Delta(X)) \to C^q(\Delta(X))$ and $D^* : C^q(\Delta(X)) \to C^{q-1}(\Delta(X))$ be induced by sd and D. We show $H^q(\Delta(X)) \approx H^q(\Delta(X;\mathcal{U}))$. Suppose $u \in C^q(\Delta(X)), \delta u = 0$ and $v \in C^{q-1}(\Delta(X;\mathcal{U}))$ with $\delta v = u$ on $\Delta(X;\mathcal{U})$. We define $v_n \in C^{q-1}(\Delta^n(X;\mathcal{U}))$ with $\delta v_n = u$ on $\Delta^n(X;\mathcal{U})$ by induction on n. Let $v_0 = v$ and define

$$v_{n+1} = sd^* v_n - D^* u.$$

Then, if $\delta v_n = u$,

$$\begin{aligned} \delta v_{n+1} &= sd^* u - (sd^* u - u - D^* \delta u) \\ &= u \end{aligned}$$

We modify the v_n so that they fit together to give an element of $C^{q-1}(\Delta(X))$. If $w \in C^q(\Delta^n(X;\mathcal{U}))$, define $\varphi^n w$ by $(\varphi^n w)(T) = \varphi^n(T) w(T)$. Since sup $\varphi^n \subset \Delta^n(X, \mathcal{U})$, we have $\varphi^n w \in C^q(\Delta(X))$. Let $t \in C^{q-1}(\Delta(X))$ be defined by

$$t \mid \Delta^n(X;\mathcal{U}) = t_n = v_n - \delta(\sum_{i<n} \varphi^i D^* v_i - \sum_{i \geq n} (1 - \varphi^i) D^* v_i)$$

Since $\varphi^i(T) = 1$ for i large, the above sum makes sense. Note that $\delta t_n = u$ and that

$$\begin{aligned} \delta D^* v_i &= sdv_i - v_i - D^* U \\ &= v_{i+1} - v_i \end{aligned}$$

Hence, for $T \in \Delta^n(X;\mathcal{U})$ and N large,

$$t_n(T) = (v_{N+1} + \delta \sum_{i \leq N} \varphi^i D^* v_i)(T).$$

It follows that $t_{n+1}(T) = t_n(T)$, t is well defined and $\delta t = 0$.

Suppose $v \in C^q(\Delta(X;\mathcal{U}))$ and $\delta v = 0$. Let $U \in C^q(\Delta(X))$ be defined by

$$u \mid \Delta^n(X,\mathcal{U}) = u_n = sd^n v - \delta(\sum_{i<n} \varphi^i D^* sd^i v - \sum_{i\geq n}(1-\varphi^i) D sd^i v)$$

Then for $T \in \Delta^n(X,\mathcal{U})$ and N large

$$u_n(T) = sd^{N+1} v - \delta(\sum_{i\leq N} \varphi^i D^* sd^i v)$$

Thus u is well defined and $u = u_0 = v - \delta z$ on $\Delta(X;\mathcal{U})$. This completes the proof of Lemma 7.1.

PROOF OF THEOREM 4.9: We verify that $H^*(\Delta(X))$ satisfies additivity and the

Eilenberg-Steenrod axioms on pairs of CW complexes. Homotopy, additivity and the dimension axiom are obvious. Excision follows from Lemma 7.1. To verify exactness one needs to show that if (X, A) is a CW pair, then $u \in C^q(\Delta(A))$ can be extended to $v \in C^q(\Delta(A))$. Let U be a neighborhood of A, $r : U \to A$ a retraction, and $\sigma : X \to [0,1]$ a mapping with support in U and with $f(a) = 1$ for $a \in A$. Define $v \in C^q(\Delta(X))$ by

$$v(T) = (\min_{y\in\Delta^q} \sigma(T(s)))u(r \otimes T)$$

This completes the proof of Theorem 4.9.

REFERENCES

1. Bott, R., *Lectures on characteristic classes and poliations*, Lecture Notes in Math, Springer **279** (1973), 1–74.
2. Bousfield, A.K. and Gugenheim, V.K.A.M., *On PL de Rham theory and rational homotopy type*, Memoirs of A.M.S. 179, Providence, 1976.
3. Brown, E.H. and Szczarba, R.H., *Continuous cohomology and real homotopy type*, Trans. Amer. Math. Soc. **311** (1989), 57–106.
4. Godbillon, C. and Vey, J., *An invariant des fenilletages de codimension 1*, CR Acad. Sci. Paris, Ser. A **273** (1971), 92.
5. Haefliger, A., *Cohomology of Lie algebras and foliations, Symposium of Differential and Algebraic Topology, Rio de Janeiro*, Lecture Notes in Mathematics 652, Springer-Verlag, 1–12.
6. Kamber, F.W. and Tondeur, Ph., *Algebres de Weil semi-simpliciales*, CR Acad. Sci. Paris, Ser. A **276** (1973), 1177–1179.
7. Milnor, J., *Remarks on infinite-dimensional Lie groups*, Relativity, Groups, and Topology II, Les Houches, North Holland, 1009–1057.
8. Sullivan, D., *Infinitesimal computations in topology*, Publ. Math. I.H.E.S. **47** (1977), 269–331.
9. Palais, R. and Terng, C., *Critical point theory and submanifold geometry*, Lecture Notes in Math 1353, Springer Verlag, p. 6.

Edgar H. Brown and Robert H. Szczarba

-Brandeis University, Department of Mathematics, Waltham, MA, 06254, USA
-Yale University, Department of Mathematics, Box 2155 Yale Station, New Haven, CT., 06520, USA