SAID ZARATI

Derived functors of the destabilization and the Adams spectral sequence

Astérisque, tome 191 (1990), p. 285-298

<http://www.numdam.org/item?id=AST_1990__191__285_0>

DERIVED FUNCTORS OF THE DESTABILIZATION and THE ADAMS SPECTRAL SEQUENCE

by Said ZARATI

Introduction

Let A be the modulo 2 Steenrod algebra, $\mathcal{M}$ the category of graded A-modules and A-linear maps of degree zero, and $\mathcal{U}$ the full sub-category of $\mathcal{M}$ whose objects are unstable A-modules. We denote by D : $\mathcal{M}$ ---> $\mathcal{U}$ the destabilization functor and by D_s, $s \geq 0$, its derived functors. We have a natural transformation : D_s ---> $\Sigma D_s \Sigma^{-1}$, $s \geq 0$, induced by the adjoint of the identity $\Omega D = D \Sigma^{-1}$ where Σ^m, $\mathcal{M}$ ---> $\mathcal{M}$, $m \in \mathbb{Z}$, is the m^{th} suspension functor and Ω is the left adjoint of Σ : $\mathcal{U}$ ----> $\mathcal{U}$.

In this note we prove the following theorem wich will be more precise in section 2.3.

Theorem 1.1. Let M be a nil-closed unstable A-module. Then the natural map $\Omega D_s \Sigma^{-s} M$ ---> $D_s \Sigma^{-s-1} M$ is an isomorphism for every $s \geq 0$.

Using the higher Hopf invariants introduced in [7] we prove the following property of the Adams spectral sequence, in the modulo 2 cohomology, for the group {X,Y} of homotopy classes of stable maps from X to Y, in certain cases.

Theorem 1.2. : Let X and Y two pointed CW-complexes such that

(i) $\bar{H}^*(X,\mathbb{F}_2) \simeq \Sigma^2 I$ where ΣI is an injective unstable A-module.

(ii) $\bar{H}^*(Y;\mathbb{F}_2)$ is gradually finite and nil-closed.

Then, the Adams spectral sequence for the group {X,Y} degenerate at the E_2-term : $E_2^{s,s} \approx E_r^{s,s}$ for every $r \geq 2$ and $s \geq 0$.

S.M.F.
Astérisque 191 (1990)

The infinite real projective space $\mathbb{R}P^\infty$ is an example of a space Y satisfying the hypotheses of theorem 1.2.

The organization of the rest of this note is as follows. In section 2 we give a characterization of nil-closed A-modules which allows us to prove the theorem 1.1 (see theorem 2.3.3). Section 3 gives the proof of theorem 1.2 and an application. We finish this note by a remark concerning the case $p > 2$.

All cohomology is taken with $\mathbb{F}_2$ coefficients. We write $H^*(\ )$ for $H^*(\ ;\mathbb{F}_2)$ and we denote by $\bar{H}^*(\ )$ the reduced modulo 2 cohomology.

2. Derived functors of the destabilization

2.1. Let A be the modulo 2 Steenrod algebra. We denote by $\mathcal{M}$ the category whose objects are graded A-modules ($M = \{M^n,\ n \in \mathbb{Z}\}$) and whose morphisms are A-linear maps of degree zero. We denote by $\mathcal{U}$ the full sub-category of $\mathcal{M}$ whose objects are unstable A-modules (an A-module M is called unstable if $Sq^i x = 0$ for every x in M^n and every $i > n$; in particular $M^n = 0$ if $n < 0$).

The forgetful functor $\mathcal{U} \dashrightarrow \mathcal{M}$ has a left adjoint functor $D : \mathcal{M} \dashrightarrow \mathcal{U}$, called the destabilization functor, which satisfies : $\mathrm{Hom}_{\mathcal{M}}(M,N) = \mathrm{Hom}_{\mathcal{U}}(DM,N)$ for every A-module M and every unstable A-module N. The functor $D : \mathcal{M} \dashrightarrow \mathcal{U}$ is right exact, we denote $D_s : \mathcal{M} \dashrightarrow \mathcal{U}$, $s \geq 0$, its derived functors. One of the motivations for the study of the derived functors of the destabilization is the following isomorphism :

$$(2.1) \qquad \mathrm{Ext}^s_{\mathcal{M}}(M,I) \simeq \mathrm{Hom}_{\mathcal{U}}(D_s M,I)$$

for every A-module M and every unstable injective A-module I.

Let $\Sigma^m : \mathcal{M} \dashrightarrow \mathcal{M}$, $m \in \mathbb{Z}$, the m^{th} suspension functor

which associates to a module $M = \{M^n, n \in \mathbb{Z}\}$ the module $\Sigma^m M = \{M^{n-m}, n \in \mathbb{Z}\}$. The A-module structure on $\Sigma^m M$is given by $Sq^i(\Sigma^m x) = \Sigma^m Sq^i x$, x in M. The computation of $D_s\Sigma^{-t}M$, where M is an unstable A-module, is done by Lannes and Zarati [5] for $t \leq s$.
In this paragraph we will compute $D_s\Sigma^{-(s+1)}M$ for a particular unstable A-modules called nil-closed. First let us recall the definition and some properties of nil-closed unstable A-modules.

2.2. Nil-closed unstable A-modules [1], [6]

Definition 2.2.1 An unstable A-module M is called reduced if the cup-square $Sq^n : M^n \text{---}> M^{2n}$, $x \text{---}> Sq^n x$, is injective for every $n \geq 0$.

Remark 2.2.2 We can verify easily that an unstable A-module is reduced if and only if it does not contain a non trivial nilpotent sub-A-module. An unstable A-module N is called nilpotent if for every x in M^n, there exist $r \geq 0$ such that $Sq^{2^r n} \ldots\ldots Sq^n x = 0$.

Definition 2.2.3. An unstable A-module M is called nil-closed if
(i) M is reduced
(ii) An element x in M of even degree is in the image of the cup-square if and only if $Q_i x = 0$, for all $i \geq 0$, where Q_i is the i^{th} Milnor primitive in A.

Example 2.2.4 Let $B\mathbb{Z}/2$ denote a classifying space of the group $\mathbb{Z}/2$. The unstable A-module $H^*(B\mathbb{Z}/2)$ is nil-closed indeed, as a graded $\mathbb{F}_2$-algebra $H^*(B\mathbb{Z}/2)$ is freely generated by one generator of degree one.

2.3.Computation of $D_s\Sigma^{-(s+1)}M$, M nil-closed and $s \geq 0$.

2.3.1 To state our result we use the functor $R_s : \mathcal{U} \text{---}> \mathcal{U}$, $s \geq 0$, introduction in [5] page 29 (see also [9]) whose main properties are:

(i) The module R_sM is a sub-A-module of $H^*(B(\mathbb{Z}/2)^s) \otimes M$. In particular R_sM is an unstable A-module.

(ii) Let $H^*(B\mathbb{Z}/2) = \mathbb{F}_2[u]$ where u is of degree one. We denote by $L_s = H^*(B(\mathbb{Z}/2)^s)^{GL_s(\mathbb{Z}/2)}$ the Dickson algebra, that is the sub-algebra of $H^*(B(\mathbb{Z}/2)^s)$ of invariants under the natural action of the general linear group $GL_s(\mathbb{Z}/2) = GL((\mathbb{Z}/2)^s)$. The module R_sM is the L_s-module generated by the elements $St_s(x)$, x in M. These elements $St_s(x)$ are defined inductively by :

$$St_0(x) = x \quad , \qquad x \in M.$$

$$St_1(x) = \sum_{i=0}^{n} u^{n-i} \otimes Sq^i x \quad , \; x \in M^n.$$

$$St_s(x) = St_1(St_{s-1}(x)) \quad , \; s \geq 1, \; x \in M$$

iii) Let $E_+\mathfrak{S}_{2^s}$ be the disjoint union of a base point and a contractible space on which the symmetric group $\mathfrak{S}_{2^s}$ acts freely. For any pointed space X, we denote by $\mathfrak{S}_{2^s}X$ the quotient of the space $E_+\mathfrak{S}_{2^s} \wedge (X \wedge .... \wedge X)$, X is smashed with itself 2^s times, by the diagonal action of $\mathfrak{S}_{2^s}$ ($\mathfrak{S}_{2^s}$ acts on $X \wedge ...... \wedge X$ by permutation of the factors). Let $\Delta_s : B_+(\mathbb{Z}/2)^s \wedge X \text{ -----> } \mathfrak{S}_{2^s} X$ be a "Steenrod diagonal" determined by a bijection between $(\mathbb{Z}/2)^s$and $\{1,2,....,2^s\}$. The unstable A-module R_sH^*X is the image of $\Delta_s{}^*$ in the modulo 2 cohomology.

2.3.2 Let $\Omega : \mathcal{U} \text{ ----> } \mathcal{U}$ be the left adjoint functor of $\Sigma : \mathcal{U} \text{ ---> } \mathcal{U}$, that is :

$$Hom_{\mathcal{U}}(M,\Sigma N) = Hom_{\mathcal{U}}(\Omega M,N)$$

for every unstable A-modules M and N.

We are now ready to state the main result of this paragraph which will be proved in 2.6

Theorem 2.3.3 : Let M be a nil-closed unstable A-module. There exist a natural isomorphism :

$$D_s \Sigma^{-(s+1)} M \simeq \Omega R_s M \quad , \quad s \geq 0$$

2.4. Some properties of nil-closed unstable A modules

In this paragraph we give two characterizations of nil-closed unstable A-modules which allow us to prove theorem 2.3.3

2.4.1. The first characterization of nil-closed unstable A-modules is given in [6] page 314.

Proposition 2.4.1.1. Let M be an unstable A-module. The following conditions are equivalent.

(i) M is nil-closed

(ii) $Ext_{\mathcal{U}}^{i}(N,M) = 0$ for every nilpotent N in $\mathcal{U}$ and i = 0,1.

(iii) There exist an injective resolution of M starting

$$0 \dashrightarrow M \dashrightarrow K^0 \dashrightarrow K^1$$

where K^0 and K^1 are reduced injective unstable A-modules.

Remark 2.4.1.2. The condition (iii) of the proposition 2.4.1.1 can be replaced by the following (see [4] page 163)

(iii)' There exist an injective resolution of M starting

$$0 \dashrightarrow M \dashrightarrow \prod_{\alpha} H^*(BV_\alpha) \dashrightarrow \prod_{\beta} H^*(BV_\beta)$$

where V_α and V_β are elementary abelian 2-groups. We have the following easy corollary.

Corollary 2.4.1.3. Let M be an unstable A-module. The following conditions are equivalent.

(i) M is nil-closed.

(ii) There exist a nil-closed unstable A-module L containing M such that the quotient L/M is reduced.

2.4.2. Another characterization of nil-closed.

Proposition 2.4.2.1 Let M be an unstable A-module. The following properties are equivalent.
(i) M is nil-closed
(ii) M and ΩM are reduced

The proof of this proposition is based on the following technical lemma. Let Q_i, $i \geq 0$, the i^{th} Milnor primitive in A and Sq_k the cohomology operation defined by $Sq_k\, x = Sq^{n-k}x$ where x is an element of degree n of an A-module ($Sq^{n-k} = 0$ if $n < k$).

Lemma 2.4.2.2 Let M be an unstable A-module. We have the following formula :

$$(Q_{i+1} \circ Sq_1)(x) = (Sq_0 \circ Q_i)(x)$$

for every x in M and every $i \geq 0$.

Proof. The proof is done by induction on i using Adem's relations. Recall that the elements Q_i, $i \geq 0$, are defined by

$$Q_0 = Sq^1$$

$$Q_i = Q_{i-1}\, Sq^{2^i} + Sq^{2^i}\, Q_{i-1}\ , \quad i \geq 1$$

The case i = 0. Let x be an element of degree n of an unstable A-module, we have :

$$Sq^1 Sq_1(x) = Sq^1 Sq^{n-1}(x) = \begin{cases} 0 & \text{if } n \equiv 0(2). \\ Sq_0 x & \text{if } n \equiv 1(2). \end{cases}$$

$$Sq^2 Sq_1(x) = Sq^2 Sq^{n-1}(x) = \begin{cases} 0 & \text{if } 2 > 2n-1. \\ Sq^2 Sq^1 x & \text{if } 2 = 2n-2. \\ \displaystyle\sum_{c=0}^{1} C^{2-2c}_{n-2-c}\, Sq^{n+1-c}\, Sq^c x & \text{if } 2 < 2n-2. \end{cases}$$

$$= \begin{cases} 0 & \text{if } n = 1 \\ Sq^n Sq^1 x & \text{if } n \geq 2. \end{cases}$$

These formulas imply the case i = 0 because we have :

$$\begin{aligned} Q_1 Sq_1 x &= Sq^3 Sq_1 x + Sq^2 Sq^1 Sq_1 x \\ &= Sq_0 Sq^1 x \\ &= Sq_0 Q_1 x. \end{aligned}$$

Suppose $Q_i Sq_1 x = Sq_0 Q_{i-1} x$ for evry $i : 0 \leq i \leq j-1$ and for every element x (of degree n) of an unstable A-module. To prove this formula for i = j we consider :

$$Q_j Sq_1(x) = Sq^{2^j} Q_{j-1} Sq_1(x) + Q_{j-1} Sq^{2^j} Sq_1(x)$$

$$= Sq^{2^j} Sq_0 Q_{j-2}(x) + Q_{j-1} Sq^{2^j} Sq_1(x) \text{ , (inductive assumption)}$$

$$= Sq_0 Sq^{2^{j-1}} Q_{j-2}(x) + Q_{j-1} Sq^{2^j} Sq_1(x).$$

In the last equality we have used the following easy formula :

$$Sq^k Sq_0 = \begin{cases} 0 & \text{if } k \equiv 1(2) \ . \\ Sq_0 Sq^{\frac{k}{2}} & \text{if } k \equiv 0(2) \ . \end{cases}$$

If remains to show :

$$Q_{j-1} Sq^{2^j} Sq_1(x) = Sq_0 Q_{j-2} Sq^{2^{j-1}}(x).$$

Using the unstability of M and Adem's relations we prove :

$$Sq^{2^j}Sq_1(x) = \begin{cases} 0 & \text{if } n \le 2^{j-1}. \\ Sq_1Sq^{2^{j-1}}(x) & \text{if } n \ge 2^{j-1}+1. \end{cases}$$

This formula gives :

$$Q_{j-1}Sq^{2^j}Sq_1(x) = \begin{cases} 0 & \text{if } n \le 2^{j-1}. \\ Q_{j-1}Sq_1Sq^{2^{j-1}}(x) & \text{if } n \ge 2^{j-1}+1. \end{cases}$$

$$= \begin{cases} 0 & \text{if } n \le 2^{j-1}. \\ Sq_0\, Q_{j-2}Sq^{2^{j-1}}(x) & \text{if } n \ge 2^{j-1}+1 \text{, (inductive assumption).} \end{cases}$$

$$= Sq_0Q_{j-2}Sq^{2^{j-1}}(x)$$

2.4.3. Functor R_S and nil-closed A-modules.

Proposition 2.4.3.1. Let M be an unstable A-module. If M is nil-closed then R_SM is nil-closed.

Proof : Let (*) $0 \dashrightarrow M \dashrightarrow \prod_\alpha H^*(V_\alpha) \dashrightarrow \prod_\beta H^* V_\beta$ be the beginning of an injective resolution of the nil-closed unstable A-module M (see remark 2.4.1.2). The functor R_S is exact and comutes with products (see [6]) ; then, when we apply it to the exact sequence (*) we get the following exact sequence :

$$0 \dashrightarrow R_sM \dashrightarrow \prod_\alpha R_sH^*V_\alpha \longrightarrow \prod_\beta R_sH^*V_\beta \;.$$

The computation of R_sH^*V, where V is an elementary abelian 2-group, is done by induction on s (see [6] page 321). Let $V_s = (\mathbb{Z}/2)^s \oplus V, R_sH^*V$ is the sub-module of $H^*(V_s)$ of invariants under the action of the sub-group of $GL(V_s)$, denoted $GL(V_s,V)$, of automorphisms of V_s which induces the identity on V. The proposition 2.4.3.1 is now a consequence of the corollary 2.4.1.3 and of the fact that the sub-A-module $H^*(V)^G$, $G < GL(V)$, of $H^*(V)$ is nil-closed (see [6] page 314).

Remark 2.4.3.2. A different proof of the proposition 2.4.3.1 for s = 1 is given in [3]

2.5.Proof of the proposition 2.4.2.1.

2.5.1. First let us recall some properties of the functor Ω introduced in 2.3.2. Let $\Phi : \mathcal{U} \dashrightarrow \mathcal{U}$ be the functor which associates to each unstable A-module A-module M, the "double of M", denoted ΦM, defined by :

$$(\Phi M)^n = \begin{cases} 0 & \text{if } n \equiv 1(2). \\ M^{n/2} & \text{if } n \equiv 0(2). \end{cases} \quad \text{and } Sq^i(\Phi x) = \begin{cases} 0 & \text{if } i \equiv 1(2). \\ \Phi Sq^{i/2}x & \text{if } i \equiv 0(2). \end{cases}$$

we verify that the map $Sq_0 : \Phi M \dashrightarrow M$, $\Phi x \mapsto Sq_0 x$, is A-linear and that the kernel and the cokernel of Sq_0 are respectively $\Sigma\Omega_1 M$ and $\Sigma\Omega M$ where Ω_1 is the first and unique derived functor of Ω (see [5] page 30). We remark that an unstable A-module M is reduced if and only if $\Omega_1 M = 0$.

2.5.2. Proof the proposition 2.4.2.1. (i) ==> (ii). It suffices to prove that ΩM is reduced. Let y be an element of $(\Omega M)^k$ such that $Sq_0 y = 0$. To prove that y = 0 we envision two cases :

(*) The case $k \equiv 0(2)$. in this case $(\Omega M)^k = (\Sigma^{-1}M/\mathrm{Im}Sq_0)^k = M^{k+1}$ then $y = \Sigma^{-1}x$ where x is an element of M^{k+1}. $Sq_0 y =$

$\Sigma^{-1}Sq_1x = 0$. This implies that $Sq^1Sq_1x = Sq_0x = 0$ and then $x = o$ since M is reduced. This shows that $y = \Sigma^{-1}x = 0$.

(**) The case $k \equiv 1(2)$. In this case $(\Omega M)^k = (\Sigma^{-1}M/\mathrm{Im}Sq_0)^k = (M/\mathrm{Im}Sq_0)^{k+1}$ then $y = \Sigma^{-1}[x]$ where x is an element of M^{k+1}. $Sq_0y = \Sigma^{-1}[Sq_1x] = \Sigma^{-1}Sq_1x = 0$ (Sq_1x is an element of M of odd degree) ; then, $Sq_1x = 0$. This implies that $Q_{i+1}Sq_1x = 0$ for every $i \geq 0$. Using the lemma 2.4.2.2 we get : $Sq_0Q_ix = 0$ for every $i \geq 0$ and then $Q_i(x) = 0$, $i \geq 0$, since M is reduced. Now x is an element of even degree of a nil-closed A-module M annulated by all the Q_i, $i \geq 0$ then x is in the image of Sq_0 and then $y = \Sigma^{-1}[x] = 0$.

(ii) ==> (i). Since M is reduced then M embeds in a reduced injective unstable A-module K (see [6] page 313). To prove M nil-closed it suffices to prove that the quotient K/M is reduced and to use the corollary 2.4.1.3. If we apply the functor Ω to the exact sequence 0 ---> M ---> K ---> K/M ---> 0 we get the following exact sequence : 0 ---> Ω_1(K/M) ---> ΩM ---> ΩK ---> Ω(K/M) ---> 0. The module Ω_1(K/M) is trivial because it is a nilpotent sub-A-module of the reduced unstable A-module ΩM. Ω_1(K/M) is nilpotent because, by definition, it is concentrated in odd degree. This shows that K/M is reduced and then M is nil-closed

2.6. Proof of the theorem 2.3.3

Let M be an unstable A-module. Consider the following exact sequence introduced in [5] page 32 :

(*) $0 \longrightarrow \Omega D_s\Sigma^{-s}M \longrightarrow D_s\Sigma^{-(s+1)}M \longrightarrow \Omega_1 D_{s-1}\Sigma^{-s}M \longrightarrow 0$

When M is reduced, the module $D_s\Sigma^{-s}M$ is naturally isomorphic to R_sM ([5] proposition 4.6.2). The exact sequence becomes :

(**) $0 \longrightarrow \Omega R_s M \longrightarrow D_s\Sigma^{-(s+1)}M \longrightarrow \Omega_1 D_{s-1}\Sigma^{-s}M \longrightarrow 0$

The proof of the theorem 2.3.3 is done by induction on s.For s = 0 it is the identity $D\Sigma^{-1} = \Omega D$. Suppose that : (H_k) $D_k\Sigma^{-(k+1)}M \simeq \Omega R_kM$ for every $k : 0 \leq k \leq s-1$ and every nil-closed A-module M.

To prove (H_s) it suffices to remark that since M is nil-closed then, by the proposition 2.4.3.1, $R_{s-1}M$ is nil-closed. This implies that $\Omega R_{s-1}M$ is reduced (proposition 2.4.2.1), that is : $\Omega_1 \Omega R_{s-1}M = 0$. The exact sequence (**) and the inductive assumption give, for M nil-closed, the following natural isomorphism : $D_s\Sigma^{-(s+1)}M \simeq \Omega R_s M$.

3. Applications.

The topological applications of this note are based on the higher Hopf invariants introduced by Lannes and Zarati in [7]. Let X and Y be two pointed CW-complexes. We donote by {X,Y} the group of homotopy classes of stable maps from X to Y. The Adams spectral sequencee, in the modulo 2 cohomology, for the group {X,Y} is denoted $\{E_r^{s,s} = E_r^{s,s}(X,Y),\ s \geq 0,\ d_r\}_{r\geq 2}$; $d_r : E_r^{s,s} \dashrightarrow E_r^{s+r,s+r-1}$ is the differential. We have the following theorem which will be proved in the section 3.4

Theorem 3.1 Let X and Y be two pointed CW-complexes such that :
(i) $\bar{H}^*(X) \simeq \Sigma^2 I$ where ΣI is an injective unstable A-module.
(ii) $\bar{H}^*(Y)$ is gradually finite ($\dim_{\mathbb{F}_2} H^n(Y) < +\infty$, $n \geq 0$) and nil-closed.
Then, the Adams spectral sequence, in the modulo 2 cohomology, for the group {X,Y} degenerate at the E_2-term $E_2^{s,s} \approx E_r^{s,s}$ for every $r \geq 2$ and $s \geq 0$.

Remark 3.2 In [8] (see also [7]) there exist an analogous property of the Adams spectral sequence as in theorem 3.1 in the following two cases :
(3.2.1) (i) $\bar{H}^*(X)$ is a reduced injective unstable A-module.
(ii) $\bar{H}^*(Y)$ is gradually finite.

(3.2.2) (i) $\Sigma\bar{H}^*(X)$ is an injective unstable A-module.
(ii) $\bar{H}^*(Y)$ is a reduced gradually finite unstable A-module.

Corollary 3.3. Let X and Y be two pointed CW complexes which verify the hypothesis (i) and (ii) of theorem 3.1 and such that the Adams spectral sequence for the group $\{X,Y\}$ converges.
Then, the natural map :

$$h : \{S^1X,Y\} \dashrightarrow \mathrm{Hom}_{\mathcal{U}}(\bar{H}^*Y, \Sigma\bar{H}^*X)$$

is surjective.

Proof. Theorem 3.1 shows that the term $E_2^{0,1} \simeq \mathrm{Hom}_{\mathcal{U}}(\bar{H}^*Y, \Sigma\bar{H}^*X)$ persists at the infinity. Since the Adams spectral sequences for $\{X,Y\}$ converges, then the natural map $h : \{S^1X,Y\} \dashrightarrow \mathrm{Hom}_{\mathcal{U}}(\bar{H}^*Y,\Sigma\bar{H}^*X)$ is surjective.

3.4. Proof of the theorem 3.1

Consider the following diagram whose commutativity is proved in [7], [8].

$$\begin{array}{ccccc}
 & & E_\infty^{s,s} & \xrightarrow{\mathcal{H}_\infty^{s,s}} & \mathrm{Hom}_{\mathcal{U}}(R_s\bar{H}^*Y, \Sigma^2 I) \\
 & \nearrow & & & \wr \\
Z_{2,\infty}^{s,s} & & & & \mathrm{Hom}_{\mathcal{U}}(\Omega R_s\bar{H}^*Y, \Sigma I) \\
\downarrow & & & & \uparrow \wr \ ③ \\
E_2^{s,s} \overset{①}{\approx} \mathrm{Ext}_{\mathcal{M}}^{s}(\Sigma^{-s-1}\bar{H}^*Y, \Sigma I) & & \xrightarrow[②]{\mathcal{H}_2^{s,s}\ \sim} & & \mathrm{Hom}_{\mathcal{U}}(D_s\Sigma^{-s-1}\bar{H}^*Y, \Sigma I)
\end{array}$$

($Z_{2,\infty}^{s,s}$ is the inverse image of $E_\infty^{s,s}$ in E_2, $\mathcal{H}_\infty^{s,s}$ and $\mathcal{H}_2^{s,s}$ are the Hopf invariants at the E_∞-level and the E_2-level respectively). The isomorphism 1 is clear since $E_2^{s,s} = \mathrm{Ext}^s_{\mathcal{M}}(\Sigma^{-s}\bar{H}^*Y, \Sigma^2 I) \simeq \mathrm{Ext}_{\mathcal{M}}^{s}(\Sigma^{-s-1}\bar{H}^*Y, \Sigma I)$. The isomorphism 2 follows from the fact that ΣI is an injective unstable A-module. The isomorphism 3 is a consequence of the theorem 2.3.3.

By definition of the differential $d_r : E_r^{s-r,s-r+1} \dashrightarrow E_r^{s,s}$ we have : $\mathrm{Im}\, d_r \subset Z^{s,s}_{r,\infty}$ and $Z^{s,s}_{r+1,\infty} = Z^{s,s}_{r,\infty}/\mathrm{Im}\ d_r$ (see, for example [2]). It follows from the commutativity of the previous diagram that $Z^{s,s}_{2,\infty} \xrightarrow{\simeq} E^{s,s}_{\infty}$ and then the differential $d_r : E_r^{s-r,s-r+1} \dashrightarrow E_r^{s,s}$, $r \geq 2$, is trivial. To prove that the differential $d_r : E_r^{s,s} \dashrightarrow E_r^{s+r,s+r-1}$, $r \geq 2$, is trivial we use the following isomorphism $E_2^{s,t}(X,Y) \approx E_2^{s,t+1}(X,SY)$ which allows us to use the results of [8] (see remark ?.2).

4. The case p > 2

In this note we can't replace 2 by an odd prime p since the proposition 2.4.2.1, which is the main algebraic result of this note, is false for $p > 2$. Here is an example ; the unstable A-module $H = H^*(B(\mathbb{Z}/p)\ ;\ \mathbb{F}_p)$ is the tensor product, $E(u) \otimes \mathbb{F}_p[v]$ of an exterior algebra on one generator u of degree one and of a polynomial algebra generated by v the Bockstein of u. We know that H is nil-closed (see [6]) but ΩH is not λ-projective (λ is the analog of Sq_0 for $p > 2$) ; the element $\Sigma^{-1} v^2$ of degree three of ΩH is such that : $\lambda(\Sigma^{-1}v^2) = \Sigma^{-1}\beta P^1\ v^2 = 0$.

REFERENCES

[1] **C.BROTO and S.ZARATI** : Nil-localization of unstable algebras over the Steenrod algebra, Math Z. 199, 525-537 (1988).

[2] **H.CARTAN and S.EILENBERG** : Homological algebra, Princeton Univ. Press 1956.

[3] **J.H.GUNAWARDENA, J.LANNES et S.ZARATI** : Cohomologie des groupes symétriques et application de Quillen, preprint 1986.

[4] **J.LANNES and L.SCHWARTZ** : Sur la structure des A-modules instables injectifs ; Topology vol. 28, N°2, pp 153-169, 1989.

[5] **J.LANNES et S.ZARATI** : Sur les foncteurs dérivés de la déstabilisation, Math. Z. 194, 25-59 (1987).

[6] **J.LANNES et S.ZARATI** : Sur les $\mathcal{U}$-injectifs, Ann. Ec. Norm. Sup 4 e série, t. 19, 1986, p. 303 - 333.

[7] **J.LANNES et S.ZARATI** : Invariants de Hopf d'ordre supérieur et suite spectrale d'Adams, C.R.A.S t. 296 (1983) p. 695-698.

[8] **J.LANNES et S.ZARATI** : Same title, to appear.

[9] **W.M.SINGER** : The construction of certain algebras over the steenrod algebra, J. of pure and applied algebra, 11 (1977), 53-59.

S.ZARATI
Universite de Tunis
Faculté des Sciences
Département de Mathématiques
1060 TUNIS - TUNISIA -