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# On the André-Quillen Cohomology of Commutative $\mathbf{F}_2$-algebras

by Paul G. Goerss*

In the late 1960's and early 1970's the authors Michel André and Daniel Quillen developed a notion of homology and cohomology for commutative rings that, in many respects, behaves much like the ordinary homology and cohomology for topological spaces. For example, one can construct long exact sequences such as Quillen's transitivity sequence [21] or a product in cohomology [2]. They further noticed that this homology could say much about the commutative ring at hand. Again, for example, Quillen conjectured that the vanishing of homology groups implied that the ring was of a particularly simple type and, recently, his conjecture has been born out by Luchezar Avramov [3].

One can approach the subject of the homology and cohomology of commutative rings from two points of view. The first is from the point of view of commutative algebra. Many authors have been interested in the following situation: let $\Lambda$ be a commutative, local ring with residue field $\mathbf{k}$. Then the quotient map $\Lambda \to \mathbf{k}$ allows one to define the André-Quillen homology $H_*(\Lambda, \mathbf{k})$. In this case $H_*(\Lambda, \mathbf{k})$ is of concern to local ring theorists, and squarely in the province of commutative ring theory. This is a traditional point of view. Certainly it was adopted by André and Quillen — who, if $\mathbf{k}$ was of characteristic 0, could effectively compute $H_*(\Lambda, \mathbf{k})$ in terms of $Tor_*^{\Lambda}(\mathbf{k}, \mathbf{k})$ — and more recently, by such authors as Avramov and Stephen Halperin [4].

In this work, however, we take another viewpoint — that of homotopy

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theory. The starting point is other work of Quillen [20] on non-abelian homological algebra and homotopical algebra. One of the many advances of this work of Quillen's was to isolate exactly what was required of a category $\mathcal{C}$ so that one could make all of the familiar constructions of homotopy theory in $\mathcal{C}$. If $\mathcal{C}$ satisfies the resulting list of axioms, then $\mathcal{C}$ is called a closed model category. If, in addition, $\mathcal{C}$ has a sub-category of abelian objects $\mathbf{AB}(\mathcal{C})$ and the inclusion functor $\mathbf{AB}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint, then one can define the homology of objects in $\mathcal{C}$.

The model for this sort of set-up is the category of spaces; that is, the category of simplicial sets. The abelian objects are the simplicial abelian groups, and one obtains the usual homology with integer coefficients.

In this paper, we will consider the category $s\mathcal{A}$ of simplicial, supplemented, commutative $\mathbf{F}_2$-algebras. A commutative $\mathbf{F}_2$-algebra $\Lambda$ is supplemented if there is an augmentation $\epsilon : \Lambda \to \mathbf{F}_2$ so that the composite

$$\mathbf{F}_2 \xrightarrow{\eta} \Lambda \xrightarrow{\epsilon} \mathbf{F}_2$$

is the identity. Here $\eta$ is, of course, the unit map. An object in $s\mathcal{A}$ is then a sequence of supplemented algebras $A_n$, $n \geq 0$, linked by face and degeneracy maps that satisfy the simplicial identities. The category $s\mathcal{A}$ is a closed model category, with a notion of homotopy and homology. In fact, the notion of homology is exactly that of André and Quillen. We will explore homotopy and homology together and use them to illuminate each other. Indeed, the work of A.K. Bousfield [5] and the work of William Dwyer [11] imply that we know much about homotopy in $s\mathcal{A}$ and we can use their results as a foundation for our study of homology and cohomology.

Since André and Quillen define homology and cohomology using simplicial resolutions and the like, it is a natural step to studying the category $s\mathcal{A}$.

The efficacy of this approach is this: by studying simplicial objects $A \in s\mathcal{A}$ and the homology $H_*^{\mathcal{Q}}A$ and cohomology $H_{\mathcal{Q}}^*A$, we can not only take advice from the commutative algebra, but also from homotopy theory — and two angles are better than one. For example, homotopy theory tell us that cohomology should support a product, and because we are working in characteristic 2, something like Steenrod operations. This is indeed the case and it is exactly these operations that explain why Quillen's fundamental spectral sequence — the main computational tool of [21] — does not collapse in characteristic 2, as it did in characteristic 0. By studying the product and operations, we systematize this difficulty and then can proceed to compute.

In the end we find that we will have a situation very unusual in homotopy theory: we will understand all primary homotopy operations in $s\mathcal{A}$, and all primary cohomology operations in $s\mathcal{A}$, but neither homotopy or homology will be in any sense trivial. That is to say, we will have a category rich in structure, but we can understand, appreciate, and exploit that structure. I hope you will find the answer pleasant.

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## Notation and Conventions

1.) $\mathbf{F}_2$ is the field with two elements, and $v\mathbf{F}_2$ is the category of vector spaces over $\mathbf{F}_2$

2.) If $\mathcal{C}$ is any category, we let

$$n\mathcal{C} = \text{the category of graded objects in } \mathcal{C}$$

and

$$s\mathcal{C} = \text{the category of simplicial objects in } \mathcal{C}.$$

The exceptions are $n\mathbf{F}_2$ and $s\mathbf{F}_2$, which are the categories of graded and simplicial objects in the category of $\mathbf{F}_2$-vector spaces, respectively.

3.) If $V$ is a simplicial $\mathbf{F}_2$ vector space, we define the homotopy of $V$ to be the homotopy of $V$ regarded as a simplicial set. This is the same as the following familiar calculation. Let $V = \{V_n\}$ and

$$\partial = \sum_{i=0}^{n} d_i : V_n \to V_{n-1}$$

be the sum of the face operators. (Working over $\mathbf{F}_2$ means that we do not need the alternating sum.) Then $\partial^2 = 0$ and

$$\pi_* V \cong H_*(V, \partial).$$

We can, and will, write $C(V)$ for the chain complex associated to $V$ with the differential $\partial$.

4.) Following the conventions in 2.), $ss\mathbf{F}_2$ is the category of simplicial simplicial vector spaces, which we will call bisimplicial vector spaces. Thus an object $V \in ss\mathbf{F}_2$ is a sequence of vector spaces $V_{p,q}$, one for each pair of non-negative integers $(p, q)$, equipped with horizontal face and degeneracy operators

$$d_i^h : V_{p,q} \to V_{p-1,q} \qquad \text{and} \qquad s_i^h : V_{p,q} \to V_{p+1,q}$$

and vertical face and degeneracy operators

$$d_i^v : V_{p,q} \to V_{p,q-1} \qquad \text{and} \qquad s_i^v : V_{p,q} \to V_{p,q+1}.$$

This specifies vertical and horizontal direction in $V$ — we will reserve the first variable "$p$" for the horizontal direction and the second variable for the vertical direction. The vertical and horizontal maps commute.

For fixed $p$ we have vertical homotopy

$$\pi_q V_{p,\cdot} \cong H_q(V_{p,\cdot}, \partial^v)$$

and, for fixed $q$, horizontal homotopy

$$\pi_p V_{\cdot,q} \cong H_p(V_{\cdot,q}, \partial^h).$$

5.) If $V \in ss\mathbf{F}_2$, we have the total chain complex of $V$ given by

$$C(V)_n = \oplus_{p+q=n} V_{p,q}$$

and

$$\partial = \partial^h + \partial^v : C(V)_n \to C(V)_{n-1}.$$

We can filter $C(V)$ by degree in $p$ and obtain a spectral sequence

$$\pi_p \pi_q V \Rightarrow H_{p+q} C(V).$$

There is also the diagonal simplicial vector space $diag(V)$ with

$$diag(V)_n = V_{n,n}$$

and the Eilenberg-Zilber-Cartier Theorem implies that the existence of a chain equivalence $C(V) \to C(diag(V))$. Thus we obtain one of our most useful tools, a spectral sequence

$$\pi_p \pi_q V \Rightarrow \pi_{p+q} diag(V).$$

If we filter $C(V)$ by degree in $q$ we obtain the other spectral sequence of a bisimplicial vector space

$$\pi_q \pi_p \Rightarrow \pi_{p+q} diag(V).$$

6.) A graded vector space $W$ is of finite type if $W$ is finite in each degree.

# Chapter I: Overview and Statement of Results

In this chapter, we draw the outlines for the homotopy and homology theory of simplicial algebras. Like an architect's sketch, this section is intended to make clear the form of things to come, but without smothering ideas in a welter of details. We leave these details to the later sections. The justification for this approach is that amidst the dust and smoke of the actual construction — say, section 7 or 15 — the architect's dream may become obscured, and it will be well to have it firmly in mind beforehand.

We end the chapter with some historical notes, with credits to other authors.

To begin, let $\mathcal{A}$ be the category of commutative, supplemented algebras over the field $\mathbb{F}_2$. Then let $s\mathcal{A}$ be the category of simplicial objects in $\mathcal{A}$; that is, $A \in s\mathcal{A}$ is a simplicial, commutative, supplemented $\mathbb{F}_2$-algbera — simplicial algebra, for short. Quillen [20,II.4] points out that $s\mathcal{A}$ is a closed model category. The details of this observation are contained in section 1, but, for now, it is sufficient to know that for an object $A \in s\mathcal{A}$, we have a notion of homotopy groups $\pi_*A$ — indeed, $\pi_*A$ is the homotopy of $A$ regarded as a simplicial vector space — and a we have a notion of what it means for two morphisms $f, g : A \to B$ in $s\mathcal{A}$ to be homotopic. Furthermore, we can make all the usual constructions of homotopy theory in $s\mathcal{A}$ and, hence, we know what we mean by the homotopy classes of morphisms from one object $A$ in $s\mathcal{A}$ to another object $B$. We will call the set of homotopy classes of maps by the name $[A, B]_{s\mathcal{A}}$.

An advantage of the category $s\mathcal{A}$ is that homotopy groups have rich, but understood, structure. This is in contrast to the homotopy groups of spaces which, although laden with structure, represent virtually uncharted territory to the homotopy theoretic explorer.

To be specific, if $A \in s\mathcal{A}$, then $\pi_*A$ is a graded, commutative, supplemented $\mathbb{F}_2$-algebra, equipped with an augmentation

$$\epsilon : \pi_*A \to \mathbb{F}_2$$

where we regard $\mathbb{F}_2$ as concentrated in degree 0. Of course, $\pi_0A$ becomes a supplemented algebra and we could recover much of the concerns of André and Quillen by concentrating on the case where $\pi_*A \cong \pi_0 A$; that is, $\pi_*A$ is concentrated in degree 0. However, as we shall see, this would be severely limiting.

Homotopy in $s\mathcal{A}$ is a representable functor: there is an object $S(n) \in s\mathcal{A}$ for $n \geq 0$ and a class $\iota_n \in \pi_n S(n)$ so that for all $A \in s\mathcal{A}$, the map

$$[S(n), A]_{s\mathcal{A}} \longrightarrow \pi_n A$$

given by

$$f \longmapsto \pi_* f(\iota_n)$$

is an isomorphism for $n > 0$ and defines an isomorphism

$$[S(0), A]_{s\mathcal{A}} \cong I\pi_0 A$$

where $I\pi_0 A = ker(\epsilon : \pi_0 A \to \mathbf{F}_2)$ is the augmentation ideal. This is analogous to the situation for topological spaces; indeed, these isomorphisms virtually demand that we refer to $S(n)$ as the $n$-sphere in $s\mathcal{A}$.

A computation of $\pi_* S(n)$ for all $n$ would be a computation of the homotopy groups of spheres and a calculation of all primary homotopy operations. This is possible. Many authors have worked on this, of which, perhaps, the most notable include Cartan [8], Bousfield [5], Dwyer [11], and Tom Lada (unpublished) — and, of course, Quillen [21,22]. The work of these authors combines to prove the following result.

**Theorem A:** Suppose $A \in s\mathcal{A}$. Then there are natural operations

$$\delta_i : \pi_n A \to \pi_{n+i} A, \qquad 2 \leq i \leq n$$

so that

1.) $\delta_i$ is a homomorphism $2 \leq i < n$ and $\delta_n = \gamma_2$ — the divided square — so that

$$\delta_n(x + y) = \delta_n(x) + \delta_n(y) + xy;$$

2.) the operation $\delta_i$ acts on products as follows:

$$\begin{aligned} \delta_i(xy) &= x^2 \delta_i(y) \qquad \text{if } x \in \pi_0 A \\ &= y^2 \delta_i(x) \qquad \text{if } y \in \pi_0 A \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

3.) if $i < 2j$, then

$$\delta_i\delta_j(x) = \sum_{i+1/2\leq s\leq i+j/3} \binom{j-i+s-1}{j-s} \delta_{i+j-s}\delta_s(x).$$

Because the top operation $\delta_n : \pi_n A \to \pi_{2n} A$ is the divided square, these operations are called higher divided squares. We define, for a $k$-tuple of integers $I = (i_1, \ldots, i_k)$ with $i_i \geq 2$ for all $t$,

$$\delta_I = \delta_{i_1} \cdots \delta_{i_k}$$

to be the composition, when defined, of the operations $\delta_{i_t}$. Define $\delta_i$ to be *admissible* if $i_t \geq 2i_{t+1}$ for all $t$. Theorem A.3. implies that we may write any composition of the operations $\delta_i$ as a sum of admissible operations. Define the *excess* of $I$ by the formula

$$e(I) = i_1 - i_2 - \cdots - i_k.$$

Then we have the following computation of the homotopy groups of spheres. Let $\Lambda(\cdot)$ denote the exterior algebra on the indicated elements.

**Theorem B:** Let $\iota_n \in \pi_n S(n)$ be the universal homotopy class. Then, if $n > 0$, there is an isomorphism of graded algebras

$$\pi_* S(n) \cong \Lambda(\delta_I(\iota_n) \ : \ e(I) \leq n)$$

where $\delta_I$ must be admissible. $\pi_* S(0) \cong \mathbb{F}_2[\iota_0]$, a polynomial algebra on one generator concentrated in degree 0.

It is the purpose of section 1 through 3 to define and understand the structure of homotopy in $s\mathcal{A}$. There we will discuss Theorems A and B, and more. This structure will be extremely useful in later sections.

The primary concern of this work, however, is not homotopy, but homology and cohomology. In section 4, we will define, for every $A \in s\mathcal{A}$, the André-Quillen homology $H_*^Q A$ and cohomology $H_Q^* A$. These will be, initially, graded $\mathbb{F}_2$-vector spaces, but it is our larger purpose to uncover as much structure in $H_*^Q A$ and $H_Q^* A$ as we can. We will immediately see that $H_*^Q(\cdot)$ has some of the good properties that homology should have; for example, we can construct cofibration sequences in $s\mathcal{A}$ and a cofibration sequence yields a long exact sequence in homology. Also, there is a suspension functor

$$\Sigma : s\mathcal{A} \to s\mathcal{A}$$

with the properties that for every $A, B \in s\mathcal{A}$, $[\Sigma A, B]_{s\mathcal{A}}$ is a group and $H_n^Q \Sigma A \cong H_{n-1}^Q A$ and $H_0^Q \Sigma A = 0$.

It is worth pointing out that $\pi_* \Sigma A$ is almost never concentrated in degree 0, so that to restrict ourselves to simplicial algebras with that property would be to deny ourselves the flexibility of this vital tool.

We begin almost immediately to impose further structure on $H_Q^*(\cdot)$. It is the purpose of sections 5 and 7 to prove the following results.

**Theorem C:** For $A \in s\mathcal{A}$, there is a natural commutative, bilinear product

$$[\ ,\ ] : H_Q^m A \otimes H_Q^n A \to H_Q^{m+n+1} A$$

and natural homomorphisms

$$P^i : H_Q^n A \to H_Q^{n+i+1} A$$

so that

1.) $[\ ,\ ]$ satisfies the Jacobi identity: for all $x, y, z \in H_Q^* A$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0;$$

2.) $P^i = 0$ if $i > n$ and $P^n(x) = [x, x]$;

3.) for all $x, y \in H_Q^* A$ and all integers $i$, $[x, P^i y] = 0$.

The "+1" will not seem unnatural to homotopy theorists used to dealing with the Whitehead product. Indeed, the construction of this product owes a great deal to the work of Bousfield and Daniel Kan [7] who, in that paper, are concerned exactly with the Whitehead product. The operations $P^i$ are "Steenrod operations" for $H_Q^*$ and Theorem C.2 is an unstable condition on these operations.

After we have defined the product and operations, we become concerned with their structure. For this, like Quillen, we must study a spectral sequence that passes from $H_*^Q A$ to $\pi_* A$. This is an Adams-type spectral sequence quite analogous to the lower central series spectral sequence in the homotopy theory of spaces that was studied by Bousfield, Curtis, Kan, Rector, Quillen, and others. We devote section 6 to fitting the operations $\delta_i$ of Theorem A and the operations and product of Theorem C into Quillen's spectral sequence. Then, in section 8 we supply some applications — including the facts that the product [ , ] and operations $P^i$ are non-trivial (which is good!) and that $P^i = 0$ if $i < 2$. Then in section 9 we prove the existence of Adem relations.

**Theorem D:** For $A \in s\mathcal{A}$, $x \in H_Q^* A$, and $i \geq 2j$ there is an equation

$$P^i P^j(x) = \sum_{s=i-j+1}^{i+j-2} \binom{2s-i-1}{s-j} P^{i+j-s} P^s(x).$$

Then, as a last application of Quillen's spectral sequence, we discover and examine a final operation defined only on $H_Q^0 A$.

**Theorem E:** Let $A \in s\mathcal{A}$. Then there is a natural quadratic operation

$$\beta : H_Q^0 A \to H_Q^1 A$$

so that

1.) $\beta(x+y) = \beta(x) + \beta(y) + [x, y]$;

2.) for all $x \in H_Q^0 A$ and $y \in H_Q^* A$,

$$[\beta(x), y] = [x, [x, y]].$$

This result is one of the purposes of section 10.

Now, in calling $\beta$ the "final" operations, we are implicitly stating that Theorems C, D, and E encode all the structure of $H_Q^*(\cdot)$; or, put another way, that there are no universally defined operations except those implied by these results, and that there are no further relations except those implied by these results. This is, in fact, the case, and the final five sections are devoted to proving this result. The method is a time-honored one: compute the cohomology of the universal examples. For spaces, we would be computing the cohomology of Eilenberg-MacLane spaces; for simplicial algebras, we wish to compute the André-Quillen cohomology of abelian objects in $s\mathcal{A}$.

If $V$ is a simplicial $\mathbf{F}_2$-vector space, we can define a simplicial algebra $V_+$ by setting, as a vector space, $V_+ = V \oplus \mathbf{F}_2$, requiring that $\mathbf{F}_2$ be the unit, $V$ the augmentation ideal, and $V^2 = 0$. Thus, $V_+$ could be called a trivial simplicial algebra. For $A \in s\mathcal{A}$,

$$Hom_{s\mathcal{A}}(A, V_+) \cong Hom_{s\mathbf{F}_2}(QA, V_+)$$

where $QA = IA/(IA)^2$ is the vector space of indecomposables of $A$. Since $Hom_{s\mathcal{A}}(A, V_+)$ is a group, $V_+$ is an abelian object in $s\mathcal{A}$. We will compute that for $A \in s\mathcal{A}$,

$$[A, V_+]_{s\mathcal{A}} \cong Hom_{n\mathbf{F}_2}(H_*^Q A, \pi_* V).$$

Thus, if we choose a simplicial vector space $K(n)$ so that

$$\pi_* K(n) \cong \mathbf{F}_2$$

concentrated in degree $n$, we will have

$$[A, K(n)_+]_{s\mathcal{A}} \cong [H_n^Q A]^* = H_Q^n A.$$

Here $[\cdot]^*$ denotes the $\mathbf{F}_2$-dual vector space and the last equality will be, in fact, the definition of the André-Quillen cohomology group. The exact form of this isomorphism is the usual one: there is a universal cohomology class $\iota_n \in H_Q^n K(n)_+$ so that the map

$$[K(n)_+, A]_{s\mathcal{A}} \longrightarrow H_Q^n A$$

given by

$$f \longmapsto H_Q^* f(\iota_n)$$

is an isomorphism.

Thus $K(n)_+$ represents cohomology and computing $H^*_Q K(n)_+$ for all $n$ would yield all natural cohomology operations in one variable. This we can do. For a sequence of integers $I = (i_1, \ldots, i_s)$ of integers, let

$$P^I = P^{i_1} \ldots P^{i_s}$$

be the appropriate composition of the operations $P^{i_t}$. Call this composition *allowable* if $i_t < 2i_{t+1}$ for all $t$ and define the *length* of $I$ by $f(I) = s$. The length can be 0, in which case $I$ is empty and $P^I(x) = x$ for all $x$. The reader is invited to contrast these definitions and the following result with the definitions and conclusions surrounding Theorem A.

**Theorem F:** Let $n \geq 1$ and let $\iota_n \in H^n_Q K(n)_+$ be the universal cohomology class of degree $n$. Then a basis for $H^*_Q K(n)_+$ is given by all allowable compositions

$$P^I(\iota_n) = P^{i_1} \cdots P^{i_s}(\iota_n)$$

with $s \geq 0$, $i_t \geq 2$ for all $t$, and $i_s \leq n$. For $n = 0$ we have

$$H^n_Q K(0)_+ \cong \begin{cases} \mathbb{F}_2, & \text{if } n = 0, \text{ generated by } \iota_0; \\ \mathbb{F}_2, & \text{if } n = 1, \text{ generated by } \beta\iota_0; \\ 0, & \text{otherwise.} \end{cases}$$

The computation of $H^*_Q K(0)_+$ is done in section 9. The computation of $H^*_Q K(n)_+$, for $n > 0$, is this work's most lengthy project, consuming section 12 through 15. The core idea is a spectral sequence due to Haynes Miller [16, Section 4] that passes from $\pi_* A$ to $H^*_Q A$ — a reverse Adams spectral sequence, if you will. Since $\pi_* K(n)_+$ is an exterior algebra on a single generator of degree $n$, we have the input computed and reaching the output is a theoretically simple matter. Practically, however, there is a smothering welter of details and, hence, the numerous sections.

Theorem F, it turns out, is the crucial calculation, and we can parlay that result into other computations. To see how, we use the Hilton-Milnor Theorem of [12]. If we have a sequence of non-negative integers $n_1, n_2, \ldots, n_s$, we may consider the abelian simplicial algebra

$$[K(n_1) \times \cdots K(n_s)]_+ \in s\mathcal{A}$$

obtained by taking the product of the relevant simplicial vector spaces $K(n_i)$. Then, for $A \in s\mathcal{A}$

$$[A, [K(n_1) \times \cdots \times K(n_s)]_+]_{s\mathcal{A}} \cong H_Q^{n_1} A \times \cdots \times H_Q^{n_s} A.$$

Therefore a computation of $H_Q^*[K(n_1) \times \cdots \times K(n_s)]_+$ would compute all natural cohomology operations in $s$ variables. We make this computation by considering the homotopy type, in $s\mathcal{A}$, of $\Sigma[K(n_1) \times \cdots K(n_s)]_+$, where $\Sigma$ is the suspension functor above. The projections

$$[K(n_1) \times \cdots \times K(n_s)]_+ \to K(n_k)_+$$

with $1 \leq k \leq s$ and the universal classes $\iota_k \in H_Q^{n_k} K(n_k)_+$, define classes

$$\iota_k \in H_Q^{n_k}[K(n_1 \times \cdots \times K(n_s)]_+.$$

Let $L$ be the free Lie algebra on the $s$ generators $\iota_k$ and let $B \subseteq L$ be a basis of monomials in the $\iota_k$. If $b \in B$, the $b$ is some iterated bracket in the generators $\iota_k$; let $j_k(b)$ be the number of appearances of $\iota_k$ in $b$ and let $\ell(b) = \sum j_k(b)$. Then, for $b \in B$, the elements $\iota_k$ and the product of Theorem C define a map in the homotopy category

$$f_b : [K(n_1) \times \cdots \times K(n_s)]_+ \to K(n_b)_+$$

where

$$n_b = \sum j_k(b) n_k + \ell(b) - 1.$$

Then, by suspension, we get a map

$$\Sigma f_b : \Sigma[K(n_1) \times \cdots \times K(n_s)]_+ \to \Sigma K(n_b)_+.$$

And then, using the fact that $[\Sigma A, B]_+$ is a group for all $A, B \in s\mathcal{A}$, we get a map

$$f : \Sigma[K(n_1) \times \cdots \times K(n_s)]_+ \to \otimes_{b \in B} \Sigma K(n_b)_+.$$

The Hilton-Milnor Theorem says that this map is a weak equivalence in $s\mathcal{A}$. Thus, using the fact that suspension commutes with cohomology and that cohomology takes tensor products to products, we may conclude that there is an isomorphism

$$H^*_Q[K(n_1) \times \cdots \times K(n_s)]_+ \cong \times_{b \in B} H^*_Q K(n_b)_+$$

and we can finish the computation using Theorem F. The Hilton-Milnor Theorem is explored more thoroughly in section 11.

We can now write down the two main structure theorems of this paper, including their proofs — the only proofs of this introduction.

We now define a category $\mathcal{W}$ that will be the target category for $H^*_Q$.

**Definition G:** An object $W \in \mathcal{W}$ is a graded vector space $W = \{W^n\}$ equipped with

1.) a commutative, bilinear product

$$[\ ,\ ] : W^m \otimes W^n \to W^{m+n+1}$$

satisfying the Jacobi identity;

2.) homomorphisms

$$P^i : W^n \to W^{n+i+1}$$

so that i.) $P^i = 0$ if $i < 2$ or $i > n$, and $P^n(x) = [x, x]$;

ii.) $[x, P^i(y)] = 0$ for all $x, y$ and $i$;

iii.) if $i \geq 2j$, there is an equality

$$P^i P^j = \sum_{s=i-j+1}^{i+j-2} \binom{2s-i-1}{s-j} P^{i+j-s} P^s;$$

3.) a quadratic operation $\beta : W^0 \to W^1$ so that

i.) $\beta(x + y) = \beta(x) + \beta(y) + [x, y]$; and

ii.) $[\beta(x), y] = [x, [x, y]]$.

Morphisms in $\mathcal{W}$ commute with the product and operations.

Then we have

**Theorem H:** André-Quillen cohomology defines a functor

$$H^*_Q(\cdot) : s\mathcal{A} \to \mathcal{W}.$$

**Proof:** This follows from Theorems C, D, and E.

Such a result, by itself, has no teeth — after all, $H^*_Q$ also defines a functor to graded vector spaces. Theorem F gives this theorem some force, but the next result puts real weight behind Theorem H by stating that, in some sense, $\mathcal{W}$ is the best possible category.

The forgetful functor $\mathcal{W} \to n\mathbf{F}_2$ from $\mathcal{W}$ to graded vector spaces has an evident left adjoint $U$. If $W \in n\mathbf{F}_2$ has a basis $\{w_\alpha\}$, then $U(W)$ has a basis given by the union of

1.) a basis of monomials $B$ for the free Lie algebra on $\{w_\alpha\}$ graded by requirements of Theorem C;

2.) $P^{i_1} \cdots P^{i_k}(b)$ where $b \in B$, $k \geq 1$, $i_k \leq deg(b)$, and, for all $t$, $2 \leq i_t < 2i_{t+1}$; and

3.) $\beta(w_\alpha)$ if $w_\alpha$ is of degree 0.

The structure of $U(W)$ as an object in $\mathcal{W}$ is determined by Theorems C, D, and E.

**Theorem I:** Let $V \in s\mathbf{F}_2$ be a simplicial vectors space so that $\pi_* V = W$ is of finite type — that is, $\pi_n V$ is a finite vector space for each $n$. Then there is a natural isomorphism

$$H^*_Q V_+ \cong U(W^*).$$

$W^*$ is dual to $W$.

**Proof:** If $W$ is finite there is a weak equivalence

$$V_+ \simeq [K(n_1) \times \cdots \times K(n_s)]_+$$

for some set of non-negative integers $\{n_1, \ldots, n_s\}$. In this case the result follows from the Hilton-Milnor Theorem and Theorem F. If $W$ is not finite, we write $V$ as a filtered colimit of simplicial vector spaces $V^m \subseteq V$ so that

$$\pi_n V^m \cong \begin{cases} \pi_n V, & \text{if } n \leq m; \\ 0, & \text{if } n > m. \end{cases}$$

Then we use the fact that $H^*_Q V_+ \cong \lim H^*_Q V^m_+$.

The finite type hypothesis is a familiar one: the ordinary cohomology of topological Eilenberg-Maclane spaces becomes troublesome if the homotopy groups of the Eilenberg-MacLane space are not finitely generated in each degree.

We close this chapter by explaining the relationship between this work to the work of others.

**Historical notes:** André-Quillen cohomology was defined — in its full strength — by, of course, André and Quillen, but several authors foreshadowed them. Most notable among these are, perhaps, Lichtenbaum and Schlesinger. Other authors have picked up the thread of studying André-Quillen cohomology as a sub-field of local algebra – Avramov and Halperin, for example. And, of course, there is the work of Luc Illusie [25] where many of the definitions of chapter 1 of this work appear, and which constitutes a globalization of the work of André and Quillen.

André would probably prefer to regard our algebras as rings augmented to $\mathbf{F}_2$; that is, for $A \in s\mathcal{A}$, he would emphasize the augmentation $\epsilon : A \to \mathbf{F}_2$ and define $H^*_Q(A, \mathbf{F}_2)$, using derivations. We will show in section 4 that, in fact,

$$H^*_Q(A, \mathbf{F}_2) \cong H^*_Q \Sigma A$$

so that

$$H^n_Q(A, \mathbf{F}_2) \cong H^{n-1}_Q A.$$

In a paper I studied often for inspiration [2], André defines a product

$$\langle\ ,\ \rangle : H^m_Q(A, \mathbf{F}_2) \otimes H^m_Q(A, \mathbf{F}_2) \to H^{n+m}_Q(A, \mathbf{F}_2)$$

and, using his construction, one could define (although he does not) Steenrod operations in $H^*_Q(A, \mathbf{F}_2)$. We will show in Appendix A, that if $\partial : H^{n-1}_Q A \to H^n_Q(A, \mathbf{F}_2)$ is the isomorphism above, then

$$\partial[x, y] = \langle x, y \rangle.$$

Thus our product could be computed by his and visa versa. André's product is more pleasant in its definition — and thus has the advantage of aesthetics over ours — but it is not clear to me how to put his operations into Quillen's spectral sequence and, hence, to prove the Adem relations of Theorem D.

Finally, I learned of the importance of André-Quillen cohomology from Haynes Miller's Sullivan Conjecture paper [16], especially sections 3, 4, and 5 — this paper is a gold mine, liberally studded with glittering ideas. The reverse Adams spectral sequence I use here first appears in section 4 of that paper, the suspension appears in section 5, and the key technical device of almost-free algebras appears in section 3. I cannot overemphasize my debt to that paper, or the importance of several highly productive conversations with Haynes Miller.

# Chapter II: The Homotopy Theory of Simplicial Algebras

## 1. Preliminaries on simplicial algebras

In this section we describe a closed model category structure on the category of simplicial algebras and give Illusie's canonical factorization of any map as a cofibration followed by an acyclic fibration. At the end of the section, we discuss the homotopy category associated with this closed model category structure. This section details much of the language that we will use in the rest of this paper. There is little new in this section; indeed, we are compiling results of Quillen [20], [21], Illusie [25], and Miller [16].

First, if $V$ is a simplicial $\mathbf{F}_2$ vector space, we define the homotopy of $V$ to be the homotopy of $V$ regarded as a simplicial set. This is the same as the following familiar calculation. Let $V = \{V_n\}$ and

$$\partial = \sum_{i=0}^{n} d_i : V_n \to V_{n-1}$$

be the sum of the face operators. (Working over $\mathbf{F}_2$ means that we do not need the alternating sum.) Then $\partial^2 = 0$ and

$$\pi_* V \cong H_*(V, \partial).$$

We now turn to the closed model category structure on the category of simplicial algebras. Recall that $\mathcal{A}$ is the category of commutative supplemented $\mathbf{F}_2$ algebras and $s\mathcal{A}$ is the category of simplicial objects in this category. Then $s\mathcal{A}$ has a structure of a closed model category in the sense of Quillen. There are weak equivalences, fibrations and cofibrations satisfying the axioms CM1–CM5 of [20]. We now supply the definitions. Notice that for $A \in s\mathcal{A}$, we have that $\pi_* A$ is a graded, commutative, supplemented $\mathbf{F}_2$-algebra, and that that $\pi_0 A \in \mathcal{A}$. Furthermore, $\pi_0 A$ is a quotient of $A_0$ and the quotient map

$$A_0 \to \pi_0 A$$

defines a map of simplicial algebras

$$\epsilon : A \to \pi_0 A$$

where $\pi_0 A$ is regraded as a constant simplicial algebra. This construction is natural in $A$, so that if $f : A \to B$ is a morphism in $s\mathcal{A}$, we obtain a diagram

$$\begin{array}{ccc} A & \xrightarrow{\epsilon} & \pi_0 A \\ \downarrow f & & \downarrow \pi_0 f \\ B & \xrightarrow{\epsilon} & \pi_0 B \end{array}$$

and hence a canonical map in $s\mathcal{A}$

$$(f, \epsilon) : A \to B \times_{\pi_0 B} \pi_0 A$$

where the target is the evident pullback. The morphism $f$ will be called *surjective on components* if this map is a surjection.

**Definition 1.1:**1.) A morphism $f : A \to B$ in $s\mathcal{A}$ is a *weak equivalence* if

$$\pi_* f : \pi_* A \to \pi_* B$$

is an isomorphism.

2.) $f : A \to B$ is a *fibration* if it is a surjection on components; $f$ is an *acyclic fibration* if it is a fibration and a weak equivalence.

3.) $f : A \to B$ is a *cofibration* if for every acyclic fibration $p : X \to Y$ in $s\mathcal{A}$, there is a morphism $B \to X$ so that is the following diagram both triangles commute:

$$\begin{array}{ccc} A & \to & X \\ \downarrow f & \nearrow & \downarrow p \\ B & \to & Y \end{array}$$

As specializations of these ideas we have *fibrant* and *cofibrant* objects. We write $\mathsf{F}_2$ for the terminal and the initial object of $s\mathcal{A}$. Then we say that $A \in s\mathcal{A}$ is cofibrant if the unit map $\eta : \mathsf{F}_2 \to A$ is a cofibration. Similarly, we say that $A$ is fibrant if the augmentation $\epsilon : A \to \mathsf{F}_2$ is a fibration. Every object in $s\mathcal{A}$ is fibrant, so we say no more about this concept.

The following now follows from Theorem 4, pII.4.1 of [20].

**Proposition 1.2:** With the notions of weak equivalence, fibration, and cofibration defined above, $s\mathcal{A}$ is a closed model catgory.

Of course, cofibrations are somewhat mysterious objects and difficult to recognize at this point. We will now be more concrete.

Let $S : v\mathbf{F}_2 \to \mathcal{A}$ be left adjoint to the augmentation ideal functor. $S$ is, of course, the symmetric algebra functor. We will call a morphism $f : A \to B$ in $s\mathcal{A}$ *almost-free* if, for every n $\geq 0$, there is a sub-vector space $V_n \subseteq IB_n$ and maps of vector spaces

$$\delta_i : V_n \to V_{n-1}, \qquad 1 \leq i \leq n$$

$$\sigma_i : V_n \to V_{n+1}, \qquad 0 \leq i \leq n$$

so that the evident extension

$$A_n \otimes S(V_n) \to B_n$$

is an isomorphism for every $n$ and there are commutive diagrams, with the horizontal maps isomorphisms:

$$\begin{array}{ccc} A_n \otimes S(V_n) & \xrightarrow{\cong} & B_n \\ \downarrow {\scriptstyle d_i \otimes S\delta_i} & & \downarrow {\scriptstyle d_i} \\ A_{n-1} \otimes S(V_{n-1}) & \xrightarrow{\cong} & B_{n-1} \end{array}$$

for i $\geq 1$ and

$$\begin{array}{ccc} A_n \otimes S(V_n) & \xrightarrow{\cong} & B_n \\ \downarrow {\scriptstyle s_i \otimes S\sigma_i} & & \downarrow {\scriptstyle s_i} \\ A_{n+1} \otimes S(V_{n+1}) & \xrightarrow{\cong} & B_{n+1} \end{array}$$

for i $\geq 0$. Only $d_0$ is *not* induced up from $n\mathbf{F}_2$. The following results (which are implicit in Quillen, section II.4) can be proved exactly as the corresponding result in section 3 of [16,17].

**Theorem 1.3:**1.) Almost free morphisms are cofibrations.

2.) Any cofibration is a retract of an almost free morphism.

**Proof:** The first statement is proved by Miller, using a "skeletal filtration" of the morphism. The second follows from Definition 1.1 and the next result.

**Proposition 1.4:** Any morphism $f : A \to B$ in $s\mathcal{A}$ may be factored canonically as

$$A \xrightarrow{i} X \xrightarrow{p} B$$

with $i$ almost free and $p$ an acyclic fibration.

We will give the proof of Proposition 1.4, as the construction will prove useful in the later discussion. To begin, we say a word about cotriples. The composite functor

$$\bar{S} = S \circ I : \mathcal{A} \to \mathcal{A}$$

has the structure of a cotriple on $\mathcal{A}$. That is, for $\Lambda \in \mathcal{A}$ there are natural transformations

$$\epsilon_\Lambda : \overline{S}(\Lambda) \to \Lambda$$

$$\eta_\Lambda : \overline{S}(\Lambda) \to \overline{S}^2(\Lambda)$$

and these are related in such a manner that we may form the simplicial object $\overline{S}.(\Lambda) \in s\mathcal{A}$. To be specific,

$$\overline{S}_n(\Lambda) = \overline{S}^{n+1}(\Lambda)$$

and

$$d_i : \overline{S}_n(\Lambda) \to \overline{S}_{n-1}(\Lambda)$$

is defined by

$$d_i = \overline{S}^i \epsilon \overline{S}^{n-i}, \qquad 0 \le i \le n$$

and

$$s_i : \overline{S}_n(\Lambda) \to \overline{S}_{n+1}(\Lambda)$$

is given by

$$s_i = \overline{S}^i \eta \overline{S}^{n-i}, \qquad 0 \le i \le n.$$

Now $\overline{S}.(\Lambda)$ is an *augmented* simplicial object in the sense that $\epsilon$ induces map

$$\epsilon_\Lambda : \overline{S}_0(\Lambda) \to \Lambda$$

such that $\epsilon d_0 = \epsilon d_1$. More than this, $\epsilon$ induces an isomorphism

$$\pi_* \overline{S}.(\Lambda) \cong \Lambda$$

concentrated in degree 0. The retraction that guarantees these isomorphisms is given by the inclusions in $v\mathbb{F}_2$

$$I\Lambda \to I\overline{S}(\Lambda)$$

adjoint to the identity.

This idea can be greatly generalized. For example, let $\Lambda \in \mathcal{A}$. Then we may define the category $\Lambda/\mathcal{A}$ to be the category of objects under $\Lambda$; that is, objects $\Gamma \in \mathcal{A}$ equipped with a morphism $\Lambda \to \Gamma$ in $\mathcal{A}$ making $\Gamma$ into an $\Lambda$-algebra. The augmentation ideal functor $I : \Lambda/\mathcal{A} \to v\mathbb{F}_2$ has a left adjoint

$$S^{\Lambda}(V) = \Lambda \otimes S(V).$$

This pair of adjoint functors yields a cotriple $\bar{S}^{\Lambda}:\Lambda/\mathcal{A} \to \Lambda/\mathcal{A}$ and, as above, this yields an augmented simplicial object

$$\bar{S}^{\Lambda}_{\cdot}\Gamma \to \Gamma$$

for any object $\Gamma \in \Lambda/\mathcal{A}$. If $\Lambda = \mathbb{F}_2$, this is exactly the situation above.

**Proof of 1.4:** Let $f : A \to B$ be a morphism in $s\mathcal{A}$. Then the last paragraph yields an augmented bisimplicial algebra

$$\bar{S}^{A}_{\cdot,\cdot}B \to B \tag{1.5}$$

with

$$\bar{S}^{A}_{p,q}B = (\bar{S}^{A_q})^{p+1}B_q.$$

Let

$$\bar{S}^{A}B = diag(\bar{S}^{A}_{\cdot,\cdot}B)$$

be the resulting diagonal simplicial algebra. Thus, we have factored $f : A \to B$ as

$$A \to \bar{S}^{A}B \to B. \tag{1.6}$$

The first map is almost free, the second map is a fibration, and the construction is canonical and functorial in $f$. We need only show that $\bar{S}^{A}B \to B$

is an acyclic fibration. But, since $\bar{S}^A B$ is the diagonal simplicial algebra of $\bar{S}^A_{\cdot,\cdot} B$, we may filter $\bar{S}^A_{\cdot,\cdot} B$ by degree in q to obtain a spectral sequence converging to $\pi_* \bar{S}^A B$. But since $\pi_* \bar{S}^{A_q}_{\cdot} B_q \cong B_q$, and the isomorphism is induced by the augmentation, the result follows.

The great strength of the construction of (1.6) is precisely that $\bar{S}^A B$ is the diagonal of a bisimplicial algebra. This allows the construction of many spectral sequences.

As a bit of notation, if $f = \eta : \mathbf{F}_2 \to B$ we abbreviate $\bar{S}^{\mathbf{F}_2} B$ as $\bar{S}_{\cdot} B$.

Next we come to the notion of homotopy. Notice that in $s\mathcal{A}$, tensor product is the coproduct and if $A \in s\mathcal{A}$, then the algebra multiplication

$$\mu : A \otimes A \to A$$

is the "fold" map; that is, multiplication supplies the canonical map from the coproduct from $A$ to itself. Factor $\mu$ as a cofibration followed by an acyclic fibration

$$A \otimes A \xrightarrow{i} Cy(A) \xrightarrow{p} A.$$

By Proposition 2.4 this may be done functorially in A. $Cy(A)$ is a cylinder object on $A$. Then two morphisms $f, g : A \to B$ in $s\mathcal{A}$ are homotopic if there is a morphism $H$ making the following diagram commute

$$\begin{array}{ccc} A \otimes A & \xrightarrow{i} & Cy(A) \\ \downarrow f \vee g & & \downarrow H \\ B & \xrightarrow{=} & B \end{array}$$

where $f \vee g = \mu(f \otimes g)$. If $f = g$ and we let $H$ be the composite

$$Cy(A) \xrightarrow{p} A \xrightarrow{f} B$$

we obtain the constant homotopy from $f$ to itself. The reader is invited to prove that homotopy defines an equivalence relation on the set of maps from an object $A$ to an object $B$.

We can specialize these notions somewhat. If $h : C \to A$ is another morphism in $s\mathcal{A}$ and $f, g : A \to B$ are two maps, then we say that $f$ and $g$ are *homotopic under* $C$ if $fh = gh$ and there is some homotopy from $f$ to

$g$ which restricts to the constant homotopy on $fh$. If $q : B \to D$ is a map, then there is a corresponding notion of a *homotopy over* $D$.

The following, then, is the lemma that we need to show that many of our definitions are well-defined.The proof is in [21] as Proposition 1.3.

**Lemma 1.7:** Let $f : A \to B$ be a cofibration and $p : X \to Y$ be an acyclic fibration. Then any two solutions $B \to X$ in the diagram

$$\begin{array}{ccc} A & \to & X \\ \downarrow f & \nearrow & \downarrow p \\ B & \to & Y \end{array}$$

are homotopic under $A$ and over $Y$.

**1.8: The homotopy category.** Associated to $s\mathcal{A}$ and the closed model category structure we have on $s\mathcal{A}$ there is an associated homotopy category. This category has the same objects as $s\mathcal{A}$ and morphisms

$$[A, B]_{s\mathcal{A}} = Hom_{s\mathcal{A}}(X, B)/\sim$$

where $\sim$ denotes the equivalence relation generated by homotopy and $p : X \to A$ is an acyclic fibration with $X$ cofibrant. Lemma 1.7 implies that $[A, B]_{s\mathcal{A}}$ is well-defined. A morphism in the homotopy category may be represented by a diagram

$$A \xleftarrow{p} X \xrightarrow{f} B$$

and an *isomorphism* in the homotopy category is such a diagram where $f$ is a weak equivalence. This homotopy category is relatively simple because every object in $s\mathcal{A}$ is fibrant.

## 2. Homotopy operations and the structure of homotopy

In subsequent sections we will use detailed information about the structure of $\pi_* A$, where $A \in s\mathcal{A}$. Of course, $\pi_* A$ is a graded commutative, supplemented $\mathbf{F}_2$ algebra, but it turns out that it supports much more structure than this. We first work theoretically — using the language of triples, then more concretely; that is, we choose bases to continue the discussion. For triples, see [14, Chap VI], where a triple is called a monad.

The next section is devoted to the interior details of much of the discussion of this section.

To begin, it might help to give a definition of the category $\mathcal{A}$. Consider the symmetric algebra functor $S : v\mathbf{F}_2 \to \mathcal{A}$ left adjoint to the augmentation ideal functor. By composing with the forgetful functor $\mathcal{A} \to v\mathbf{F}_2$ and abusing notation, we may regard $S$ as a functor $S : v\mathbf{F}_2 \to v\mathbf{F}_2$. Then $S$ has the structure of a triple; that is, there are natural transformations $\mu : S^2 \to S$ and $\eta : 1 \to S$ so that certain diagrams commute. Then an object $\Lambda \in \mathcal{A}$ is a $S$-algebra in the sense that there is a map of graded vector spaces $\epsilon : S\Lambda \to \Lambda$ which behaves correctly with respect to $\mu$ and $\eta$. A morphism in $\mathcal{A}$ commutes with the structure maps $\epsilon$; thus, an object $A \in s\mathcal{A}$ comes equipped with a morphism of simplicial vector spaces $\epsilon : SA \to A$.

To apply this language and the functor $S$, we recall a result of Dold's [9].

If $\mathbf{B}$ is a category, let $\mathbf{F}(\mathbf{B})$ be the category of "endo-functors" of $\mathbf{B}$; that is, the objects of $\mathbf{F}(\mathbf{B})$ are functors

$$F : \mathbf{B} \to \mathbf{B}$$

and morphisms are natural transformations. The category $\mathbf{F}(\mathbf{B})$ has a composition functor

$$\circ : \mathbf{F}(\mathbf{B}) \times \mathbf{F}(\mathbf{B}) \to \mathbf{F}(\mathbf{B})$$

with

$$G \times F \longmapsto G \circ F : \mathbf{B} \to \mathbf{B}.$$

The results of Dold's paper, especially section 5, can be used to prove the following result. Let $n\mathbf{F}_2$ be the category of graded vector spaces.

**Proposition 2.1:** There is a functor

$$\psi : \mathbf{F}(v\mathbf{F}_2) \to \mathbf{F}(n\mathbf{F}_2)$$

so that

1.) $\psi(G \circ F) = \psi(G) \circ \psi(F)$;

2.) $\psi(1) = 1$;

3.) if $V \in s\mathbf{F}_2$ is a simplicial vector space, then there is a natural isomorphism

$$\pi_* F(V) \cong \psi(F)(\pi_* V)$$

We will abbreviate $\psi(F)$ to $\mathcal{F}$ to shorten notation. Then we have, for 2.1.3

$$\pi_* F(V) \cong \mathcal{F}(\pi_* V).$$

Property 2.1.3 determines the functor $\mathcal{F}$, because, for any graded vector space $W \in n\mathbf{F}_2$, there is a simplicial vector space $V \in s\mathbf{F}_2$ so that $\pi_* V \cong W$. This can be proved by using the normalization functor on simplicial vector spaces; see [15, Section 22]. Indeed, the last two sentences constitute an outline of the proof of 2.1.

We now apply 2.1 to the functor $S$. Thus there is a functor $\mathfrak{S} : n\mathbf{F}_2 \to n\mathbf{F}_2$ so that for $V \in s\mathbf{F}_2$

$$\mathfrak{S}(\pi_* V) \cong \pi_* S(V).$$

The fact that $S$ is a triple on $v\mathbf{F}_2$ and 2.1.1 and 2.1.2 show that $\mathfrak{S}$ is a triple on $n\mathbf{F}_2$. Let $\mathcal{AD}$ — the notation to be explained below — be the category of $\mathfrak{S}$-algebras. An object in $\mathcal{AD}$ is a graded vector space $W$ with a structure map $\epsilon : \mathfrak{S}W \to W$; morphisms in $\mathcal{AD}$ commute with structure maps. The following is nearly obvious.

**Proposition 2.2:** If $A \in s\mathcal{A}$, then $\pi_* A \in \mathcal{AD}$.

**Proof:** The structure map is given by

$$\pi_* \epsilon : \pi_* SA \cong \mathfrak{S}(\pi_* A) \to \pi_* A,$$

where $\epsilon : SA \to A$ is the structure map for $A \in s\mathcal{A}$.

However theoretically pleasing this result may be, we can make few computations until we have more detailed information about $\mathfrak{S}$ – just as we can do little in $\mathcal{A}$ until we understand $S$, and stipulate the existence of a commutative, associative ring multiplication. The next step, then, is to draw on the work of Cartan, Bousfield, and Dwyer to make some observations about $\pi_*A$, with $A \in s\mathcal{A}$.

The first observation is the existence of divided power operations. These are due, principally, to Cartan.

**Proposition 2.3:** [8] Let $A \in s\mathcal{A}$. Then there exist divided power operations

$$\gamma_i : \pi_n A \to \pi_{ni} A$$

for $i \geq 0$ and $n \geq 2$ so that

1.) $\gamma_0(x) = 1 \in \pi_0 A$ and $\gamma_1(x) = x$;
2.) $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$ where $\binom{a}{b}$ is the binomial coefficient;
3.) $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(x)$;
4.) $\gamma_i(xy) = x^2\gamma_i(y) = \gamma_i(x)y^2$; and
5.) $\gamma_i(\gamma_j(x)) = \frac{(ij)!}{i!(j!)^i}\gamma_{ij}(x)$.

These are easy to define, and give a taste of the next section. If $V$ is a simplicial vector space and $C(V)$ is the chain complex with $C(V)_n = V_n$ and $\partial = \sum d_i$ — so that $H_*C(V) \cong \pi_*V$ — then there is a choice of Eilenberg-Zilber chain equivalence

$$\Delta : C(V) \otimes \cdots \otimes C(V) \to C(V \otimes \cdots \otimes V)$$

where the tensor product is taken $i$ times in both domain and range and so that in degrees bigger than zero

$$\Delta = \sum_{g \in \Sigma_i} g\Delta(i)g^{-1}$$

for some homomorphism of graded vector spaces

$$\Delta(i) : C(V)^{\otimes i} \to C(V^{\otimes i})$$

and where the symmetric group $\Sigma_i$ acts on both the domain and the range by permuting coordinates. The map $\Delta(i)$ is not a chain map, and the deviation of $\Delta(i)$ from being a chain map is important (it is one of the topics of the next section), but we can say this: if $A \in s\mathcal{A}$ and

$$\mu : A \otimes \cdots \otimes A \to A$$

is the multiplication, and if $x \in \pi_n A$ is represented by a cycle $\alpha \in A_n = C(A)_n$, then

$$\mu\Delta(i)(\alpha \otimes \cdots \otimes \alpha) \in A_{ni}$$

is a cycle and the residue class of this cycle is $\gamma_i(x)$. Notice that if $x \in \pi_n A$ is represented by $\alpha \in A_n$, then $x^i$ is represented by

$$\begin{aligned}\mu\Delta(\alpha \otimes \cdots \otimes \alpha) &= \sum_{g \in \Sigma_i} \mu\, g\Delta(i)(\alpha \otimes \cdots \otimes \alpha)\\ &= i!\Delta(i)(\alpha \otimes \cdots \otimes \alpha).\end{aligned}$$

Thus $x^i = i!\gamma_i(x)$. Hence the name "divided power." Also if $i \geq 2$, $i! \equiv 0 \bmod 2$, so $x^i = 0$. This argument works for $x \in \pi_n A$, $n \geq 2$, however the conclusion is also true if $n = 1$, as was known to many people. We record this as a proposition.

**Proposition 2.4:** Let $A \in s\mathcal{A}$ and $x \in \pi_n A$, $n \geq 1$. Then $x^2 = 0$. If $i > 0$ and $\gamma_i$ is the divided power operation, and if $i \geq 2$, then

$$\gamma_i(xy) = \begin{cases} x^i\gamma_i(y), & \text{if } x \in \pi_0 A\\ y^i\gamma_i(x), & \text{if } y \in \pi_0 A\\ 0, & \text{otherwise.}\end{cases}$$

**Proof:** This is a consequence of the above remarks and 2.3.4.

Now, a moment's thought will show that if $k = 2^j + i$, $0 \leq i < 2^j$, then

$$\gamma_k(x) = \gamma_{2^j}(x)\gamma_i(x).$$

This follows from 2.3.2. Further, 2.3.5 implies that

$$\gamma_2(\gamma_{2^j}(x)) = \gamma_{2^{j+1}}(x).$$

Thus the action of the divided powers on $\pi_* A$ is determined by the action of $\gamma_2$ and the algebra structure of $\pi_* A$. Thus this operation is paramount. The next result details operations that generalize $\gamma_2$. Because of 2.5.1 we will call them "higher divided powers."

**Theorem 2.5:** ([5],[11], and Lada) Suppose $A \in s\mathcal{A}$. Then there are natural higher divided power operations

$$\delta_i : \pi_n A \longrightarrow \pi_{n+i} A, \qquad 2 \leq i \leq n$$

so that

1.) $\delta_i$ is a homomorphism $2 \leq i < n$ and $\delta_n = \gamma_2$ – the divided square – so that

$$\delta_n(x+y) = \delta_n(x) + \delta_n(y) + xy;$$

2.) the operation $\delta_i$ acts on products as follows:

$$\begin{aligned} \delta_i(xy) &= x^2\delta_i(y) \qquad \text{if } x \in \pi_0 A \\ &= y^2\delta_i(x) \qquad \text{if } y \in \pi_0 A \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

3.) if $i < 2j$, then

$$\delta_i\delta_j(x) = \sum_{i+1/2 \leq s \leq i+j/3} \binom{j-i+s-1}{j-s} \delta_{i+j-s}\delta_s(x).$$

**Remark 2.6:**1.) Bousfield's work extends to odd primes, but the relations analogous to 2.5.3 have not been worked out in admissible form; hence, we choose to concentrate on $p = 2$.

2.) We define a composition

$$\delta_I = \delta_{i_1} \dots \delta_{i_k}$$

to be admissible if $i_t \geq 2i_{t+1}$ for all $t$. Then the relations of 5.3.3 imply that if $\delta_i\delta_j$ is not admissible, then we may rewrite this composition as a sum of admissible operations. The usual argument then shows that any composition of higher divided powers may be rewritten, if necessary, as a sum of admissible operations.

3.) The range of summation in 2.5.3) differs from Dwyer's in that it always returns an admissible answer. The fact that the sum can be written as we say was first proved by Tom Lada (unpublished).

We will discuss how the operations $\delta_i$ are defined in the next section.

Now, the functor $\mathfrak{S}$ above must reflect all this structure. In fact, if $W \in n\mathbf{F}_2$, then $\mathfrak{S}W$ will be the algebra on generators $\delta_I(w)$, $w \in W$ subject to the relations implied by 2.5. Two specific ingredients go into this calculation.

First, as remarked above, for every $W \in n\mathbf{F}_2$, there is a $V \in s\mathbf{F}_2$, so that $\pi_* V \cong W$. Thus $\mathfrak{S}W \cong \pi_* SV$ and, hence, $SW$ supports an action of the higher divided powers, subject to the axioms suggested by 2.5.

Second, it is sufficient to compute $\mathfrak{S}W$ when $W$ is one-dimensional over $\mathbf{F}_2$. This is because if $W$ is arbitrary, then $W$ is the filtered colimit of its finite dimensional subspaces and if $W$ is finite dimensional, then $W$ is the direct sum of one-dimensional vector spaces. Then one notices that since $S$ commutes with colimits and since

$$S(V_1 \oplus V_2) \cong S(V_1) \otimes S(V_2),$$

the naturality of Dold's result implies that $\mathfrak{S}$ commutes with colimits and

$$\mathfrak{S}(W_1 \oplus W_2) \cong \mathfrak{S}(W_1) \otimes \mathfrak{S}(W_2).$$

Now, we have already seen (2.6) what it means for a composition $\delta_I$ to be admissible and that any composition may be rewritten as a sum of admissible operations. Define the excess of $I = (i_1, \ldots, i_k)$ by the formula

$$e(I) = i_1 - i_2 - \cdots - i_k.$$

Let $\Lambda(\ )$ and $\Gamma(\ )$ denote exterior and divided power algebra respectively and let $F(n) \in n\mathbf{F}_2$ be the graded vector space that is one-dimensional over $\mathbf{F}_2$, concentrated in degree $n$.

**Proposition 2.7:** Let $\iota \in F(n)$ be the non-zero element. Then, if $n > 0$, there is an isomorphism of graded algebras

$$\begin{aligned}\mathfrak{S}(F(n)) &\cong \Lambda(\delta_I(\iota) \ : \ e(I) \leq n) \\ &\cong \Gamma(\delta_I(\iota) \ : \ e(I) < n)\end{aligned}$$

where $\delta_I$ must be admissible. $\mathfrak{S}(F(0)) \cong \mathbf{F}_2[\iota]$, a polynomial algebra on one generator concentrated in degree 0.

**Proof:** This is in [5], Section 7, or [11], Remark 2.3.

The action of the higher divided powers is the obvious one suggested by the notation of 2.7 and the axioms of 2.5. Proposition 2.7 allows us to compute $\mathfrak{S}W$ for all $W \in n\mathbf{F}_2$.

We can now give a more concrete description of the category $\mathcal{AD}$. The notation is meant to be suggestive: the $\mathcal{A}$ stands for algebra, and the $\mathcal{D}$ for higher divided powers. By combining the definition of $\mathcal{AD}$ as the category of $\mathfrak{S}$-algebras and Corollary 2.7, we have that $\Lambda \in \mathcal{AD}$ is a graded algebra with an action of the higher divided power operations

$$\delta_i : \Lambda_n \to \Lambda_{n+i}$$

so that the axioms suggested by 2.5 hold. A morphism in $\mathcal{AD}$ is an algebra map that commutes with these operations.

Notice that, in a sense, Proposition 2.7 gives a calculation of the homotopy groups of spheres in $s\mathcal{A}$. To see this, Let $K(n) \in s\mathbf{F}_2$ be a simplicial vector space so that $\pi_* K(n) \cong \mathbf{F}_2$ concentrated in degree $n$. Then, in the homotopy category of simplicial vector spaces

$$[K(n), V]_{s\mathbf{F}_2} \cong \pi_n V.$$

Here we use that a map of simplicial vector spaces is determined up to homotopy by the map on homotopy. This is proved in May's book [15]. Then, using the adjointness between the augmentation ideal functor and the symmetric algebra functor, we have for $A \in s\mathcal{A}$

$$\begin{aligned}[S(K(n)), A]_{s\mathcal{A}} &\cong [K(n), IA]_{s\mathbf{F}_2} \\ &\cong \pi_n IA.\end{aligned}$$

But $\pi_* IA \cong I\pi_* A$ , so

$$\pi_n IA \cong \begin{cases} \pi_n A, & \text{if } n > 0; \\ I\pi_0 A, & \text{if } n = 0. \end{cases}$$

Thus $S(K(n))$ represents homotopy and deserves to be called the $n$-sphere. And, of course,

$$\pi_* S(K(n)) \cong \mathfrak{S}(\pi_* K((n)) \cong \mathfrak{S}(F(n))$$

is calculated by Proposition 2.7.

## 3. Homotopy operations and cohomology operations

For future applications, we need to know how the homotopy operations $\delta_i$ of the previous section are defined. Following Dwyer, this is handled by investigating the symmetries inherent in one of the Eilenberg-Zilber chain equivalences. This contrasts with the usual definition of cohomology operations, which investigates the deviation of the other chain equivalence from being commutative on the chain level. We will compare these two definitions at the end of the section.

We now give Dwyer's definition [11], reserving Bousfield's definition until later. If $V$ is a simplicial vector space, let $C(V)$ be the associated chain complex obtained by taking the differential to be the sum of the face operators. If $V$ and $W$ are simplicial vector spaces, define

$$\phi_n : C(V) \otimes C(W) \to C(V \otimes W), \qquad n \geq 0$$

to be the map of degree $(-n)$ that is zero on $[C(V) \otimes C(W)]_m$ for $m \neq 2n$, and given in degree $2n$ by the projection onto one factor:

$$[C(V) \otimes C(W)]_{2n} = \oplus_{p+q=2n} V_p \otimes W_q \to V_n \otimes W_n.$$

Let $T$ denote any map that switches factors.

**Lemma 3.1:** [11] Let $V, W$ be simplicial vector spaces. Then there exist natural homomorphisms

$$\Delta^k : [C(V) \otimes C(W)]_m \to [C(V \otimes W)]_{m-k}$$

for $k \geq 0$ and defined when $m \geq 2k$ so that

1.) $\Delta^0 + T\Delta^0 T + \phi_0 = \Delta$ is a chain equivalence;

2.) $\partial\Delta^{k-1} + \Delta^{k-1}\partial = \Delta^k + T\Delta^k T + \phi_k$ for $k > 0$, whenever both sides of the equations are defined.

Dwyer proves that the maps $\Delta^k$ are essentially unique, as we will see below. Also, $\Delta^0$ was called $\Delta(2)$ in section 2, where we were discussing the definition of the divided powers.

If $A \in s\mathcal{A}$, then $A$ has commutative multiplication $\mu : A \otimes A \to A$, and we can define a map

$$\Theta_i : C(A)_n \to C(A)_{n+i}, \qquad 1 \leq i \leq n$$

by, for $n - i > 0$

$$\Theta_i(\alpha) = \mu\Delta^{n-i}(\alpha \otimes \alpha) + \mu\Delta^{n-i-1}(\alpha \otimes \partial\alpha) \tag{3.2.1}$$

and, for $i = n$,

$$\Theta_n(\alpha) = \mu\Delta^0(\alpha \otimes \alpha). \tag{3.2.2}$$

$\Theta_i$ is defined for $n - i - 1 \geq 2n$ or $1 \leq i \leq n$. Taking boundaries we have

$$\partial\Theta_i(\alpha) = \Theta_i(\partial\alpha), \qquad 2 \leq i < n \tag{3.3.1}$$

$$\partial\Theta_n(\alpha) = \mu\Delta(\alpha \otimes \partial\alpha) \tag{3.3.2}$$

and

$$\partial\Theta_1(\alpha) = \Theta_1(\partial\alpha) + \alpha^2, \qquad \text{for } n \geq 2. \tag{3.3.3}$$

Now the $\Theta_i$ are not homomorphisms, but are quadratic:

$$\Theta_i(x + y) = \Theta_i(x) + \Theta_i(y) + \partial\mu\Delta^{n-i-1}(x \otimes y), \qquad 2 \leq i < n \tag{3.3.4}$$

and

$$\Theta_n(x + y) = \Theta_n(x) + \Theta_n(y) + \mu\Delta(x \otimes y). \tag{3.3.5}$$

Thus, if $\alpha$ is a cycle representing $x \in \pi_n A$, then the residue class of $\Theta_i(\alpha)$ defines an element

$$\delta_i(x) \in \pi_{n+i} A, \qquad 2 \le i \le n.$$

This defines the operation $\delta_i$. Notice that if $i$ is not between 2 and $n$, then $\delta_i$ is not defined on $\pi_n A$. The formulas of (3.3) will be useful later. Indeed, (3.3.4) and (3.3.5) immediately imply 2.5.1.

Bousfield's definition is more functorial. Let $V$ be a simplicial vector space and let $S_2 V \in s\mathbf{F}_2$ be the quotient of $V \otimes V$ by the action of $\Sigma_2$ that permutes factors. $S_2 V$ is the vector space of coinvariants of this action. Then, notice that the homomorphisms $\Delta^k$ of 3.1 define maps

$$\bar{\delta}_i : \pi_n V \to \pi_{n+i} S_2 V, \qquad 2 \le i \le n$$

by sending $x$, represented by $\alpha \in V_n$, to $\bar{\delta}_i(x)$ represented by the residue class of

$$\Delta^{n-i}(\alpha \otimes \alpha)$$

in $S_2 V$. If $A \in s\mathcal{A}$, then the commutative algebra multiplication $A \otimes A \to A$ defines a map

$$\mu : S_2 A \to A$$

and the operations $\delta_i$ is the composite

$$\pi_n A \xrightarrow{\bar{\delta}_i} \pi_{n+i} S_2 A \xrightarrow{\pi_* \mu} \pi_{n+i} A. \tag{3.4}$$

Thus, to define the operations $\delta_i$, it would be sufficient to compute $\pi_* S_2 A$. This is what Bousfield does. In fact, we can read his calculations off of Proposition 2.7. To see this, let $V \in s\mathbf{F}_2$ and let $S_n V$ denote the coinvariants of $V^{\otimes n}$ under the action of $\Sigma_n$ that permutes the factors. Then, by Dold's result (2.1)

$$\pi_* S_n V \cong \mathfrak{S}_n(\pi_* V)$$

for some functor $\mathfrak{S}_n : n\mathbf{F}_2 \to n\mathbf{F}_2$. Furthermore, since the symmetric algebra on $V$ can be decomposed

$$S(V) \cong \oplus_{n \ge 0} S_n V$$

where $S_0V = \mathbf{F}_2$ regarded as a constant simplicial vector space, we have

$$\mathfrak{S}(\pi_*V) \cong \oplus_{n\geq 0}\mathfrak{S}_n(\pi_*V).$$

A spanning set for $\mathfrak{S}(\pi_*V)$ as a graded vector space is given by products

$$x_{j_1}\ldots x_{j_n}$$

where $x_{j_t}$ is of the form $x_{j_t} = \delta_I(v)$ and $v \in \pi_*V$. We define a weight function on this spanning set by

$$\begin{aligned} wt(v) &= 1 \text{ if } v \in \pi_*V \\ wt(xy) &= wt(x) + wt(y) \\ wt(\delta_i(x)) &= 2wt(x) \end{aligned}$$

Then Bousfield proves the following [5]:

**Proposition 3.5:** $\mathfrak{S}_n(\pi_*V)$ is spanned by the elements of weight $n$.

In particular, $\mathfrak{S}_2(\pi_*V)$ is spanned by elements of the form $vw$ where $v, w \in \pi_*V$ and if $v \in \pi_nV$ with $n > 0$ then $v \neq w$, and by elements $\delta_i(v)$, $v \in \pi_*V$. As a further specialization, if $K(n) \in s\mathbf{F}_2$ is the simplicial vector space so that

$$\pi_*K(n) \cong \mathbf{F}_2$$

in degree $n$ generated by $\iota$, then $\pi_*S_2K(n) \cong \mathfrak{S}_2(F(n))$ where $F(n)$ is as in 2.7, and $\mathfrak{S}_2(F(n))$ is generated by $\delta_i(\iota)$, $2 \leq i \leq n$. If $V \in s\mathbf{F}_2$, there is an isomorphism

$$[K(n), V]_{s\mathbf{F}_2} \xrightarrow{\cong} Hom_{n\mathbf{F}_2}(\pi_*K(n), \pi_*V) \cong \pi_nV$$

given by

$$f \longmapsto \pi_*f(\iota)$$

where $[\ ,\ ]_{s\mathbf{F}_2}$ is the homotopy classes of maps in $s\mathbf{F}_2$. If $A \in s\mathcal{A}$ and $x \in \pi_nA$ choose a map $f : K(n) \to A$ so that $\pi_*f(\iota) = x$ and consider the composition

$$\pi_*S_2K(n) \xrightarrow{S_2f} \pi_*S_2A \xrightarrow{\pi_*\mu} \pi_*A.$$

The image of $\delta_i(\iota)$ under this composition is $\delta_i(x)$. This definition agrees with the previous one, as Dwyer proves that $\bar{\delta}_i(\iota)$ (as in (3.4)) is non-zero, and hence

$$\bar{\delta}_i(\iota) = \delta_i(\iota) \in \pi_* S_2 K(n).$$

For this reason we will drop the notation "$\bar{\delta}_i$". Additionally, we see that Dwyer's definition of the $\delta_i$ is independent of the choice of maps $\Delta^k$ of 3.1. Finally, notice that $\delta_i$ defines an operation

$$\delta_i : \mathfrak{S}_n(V) \to \mathfrak{S}_{2n}(V).$$

We now contrast the definition of the operations $\delta_i$ with the construction of Steenrod operations, stressing the general situation. In this case, we have simplicial vector space $C$ equipped with a cocommutative coproduct

$$\psi : C \to C \otimes C$$

in the category of simplicial vector spaces. Then we use the following result.

**Lemma 3.6:** For simplicial vector spaces $V$ and $W$, there are natural homomorphisms of degree $k$

$$D_i : C(V \otimes W) \to C(V) \otimes C(W)$$

so that

1.) $D_0$ is a chain map, a chain equivalence and the identity in degree 0; and

2.) $\partial D_{i+1} + D_{i+1}\partial = D_i + TD_iT$.

Of course, $T$ is the switch map. This is proved in Dold's paper [10]. If $(\ )^*$ denotes the $\mathbf{F}_2$-dual, then we can define natural operations

$$\mathrm{Sq}^i : \pi^n C^* \to \pi^{n+i} C^*$$

when $C$ is a simplicial vector space equipped with a cocommutative coproduct.

$$\pi^* C^* = H^*(C, \partial^*)$$

where $\partial^* = \sum d_i^* : C_n^* \to C_{n+1}^*$. If $x \in \pi^n C^*$ is represented by the cocycle $\alpha \in C_n^*$, then $\mathrm{Sq}^i(x)$ is represented by the cocycle

$$\psi^* D_{n-i}^*(\alpha \otimes \alpha).$$

Of course, $\mathrm{Sq}^n(x) = x^2$ in the graded algebra $\pi^* C^*$.

The operations $\mathrm{Sq}^i$ are always homomorphisms; therefore, when $\pi_* C$ is of finite type, we obtain dual operations

$$(\cdot)\mathrm{Sq}^i : \pi_n C \to \pi_{n-i} C$$

which we write on the right. Thus if $x \in \pi_n C$, we have $x\mathrm{Sq}^i \in \pi_{n-i} C$. These operations can be defined directly — without the use of the double duals — as follows. Since $\psi : C \to C \otimes C$ is cocommutative, we get an induced map

$$\psi : C \to S^2 C$$

where

$$S^2 C = (C \otimes C)^{\Sigma_2}$$

is the vector space of invariants under the action that permutes the coordinates. Thus it makes sense to compute $\pi_* S^2 C$ and then to examine

$$\pi_* \psi : \pi_* C \to \pi_* S^2 C.$$

So let $V$ be a simplicial vector space. If $V = K(n)$, then we know that

$$\pi_{n+i} S^2 K(n) \cong \mathbf{F}_2, \qquad \text{if } 0 \leq i \leq n$$

generated by a class $\sigma_i(\iota)$. The method of universal examples used above now defines, for all simplicial vector spaces $V$, a natural operations

$$\sigma_i : \pi_n V \to \pi_{n+i} S^2 V.$$

In addition, if we define

$$(1 + T) : V \otimes V \to V \otimes V$$

by

$$(1+T)(x \otimes y) = x \otimes y + y \otimes x$$

then we have a map

$$\pi_*(1+T) : \pi_* V \otimes \pi_* V \to \pi_* S^2 V.$$

**Proposition 3.7:** Let V be a simplicial vector space. Then $\pi_* S^2 V$ is spanned by the classes

$$\sigma_i(x), \qquad 0 \leq i \leq n$$

where $x \in \pi_n V$ and

$$\pi_*(1+T)(x \otimes y)$$

where $x, y \in \pi_* V$ and $x \neq y$. All these classes are non-zero if $x \neq 0$ and $y \neq 0$. If

$$\rho : \pi_* S^2 V \to \pi_* V \otimes \pi_* V$$

is the map induced by the inclusion $S^2 V \subseteq V \otimes V$, then

$$\begin{aligned} \rho \pi_*(1+T)(x \otimes y) &= x \otimes y + y \otimes x \\ \rho \sigma_n(x) &= x \otimes x \\ \rho \sigma_i(x) &= 0, \qquad 0 \leq i < n \end{aligned}$$

for $x \in \pi_n V$.

**Proof:** This is a consequence of Adem's work or a modification of Dwyer's techniques [11, Section 5].

Thus, if $\psi : C \to C \otimes C$ is a cocommutative coproduct, then under

$$\pi_* \psi : \pi_* C \to \pi_* S^2 C$$

we have

(3.8) $$\psi_*(x) = \sum_j \pi_*(1+T)(y_j \otimes z_j) + \sum_i \sigma_i(x \mathrm{Sq}^i).$$

Immediately we see that $x\mathrm{Sq}^i = 0$ if $2i > n$ — the usual unstable condition equivalent to the condition that

$$0 = \mathrm{Sq}^i : \pi^n C^* \longrightarrow \pi^{n+i} C^*$$

if $i > n$. We also note that we can recover coproduct

$$\pi_* \psi : \pi_* C \longrightarrow \pi_* C \otimes \pi_* C$$

using Proposition 3.7. Applying $\rho$ we obtain, for $x$ in degree $n$,

$$\pi_* \psi(x) = x\mathrm{Sq}^{n/2} \otimes x\mathrm{Sq}^{n/2} + \sum_j y_j \otimes z_j + z_j \otimes y_j \in \pi_* C \otimes \pi_* C$$

where $x \in \pi_n C$ and $\mathrm{Sq}^{n/2} = 0$ if $n$ is odd.

A final technical note is this: if we consider

$$(1+T) : V \otimes V \longrightarrow S^2 V$$

we have

$$(1+T)(x \otimes y) = (1+T)(y \otimes x).$$

Therefore, we get an induced map

$$tr : S_2 V \longrightarrow S^2 V.$$

**Lemma 3.9:** Under the map

$$tr_* : \pi_* S_2 V \longrightarrow \pi_* S^2 V$$

we have, for $x \in \pi_n V$

$$\begin{aligned} tr_*(\delta_i(x)) &= \sigma_i(x), \qquad 2 \le i \le n \\ tr_*(xy) &= \pi_*(1+\tau)(x \otimes y). \end{aligned}$$

**Proof:** The statement about $tr_*(\delta_i(x))$ follows from the universal example $V = K(n)$, once we know that

$$tr_* : \pi_* S_2 K(n) \longrightarrow \pi_* S^2 K(n)$$

is an injection. But this follows from Dwyer's work [11, Section 5] and Adem's calculations. The statement about $tr_*(xy)$ is a simple calculation.

Notice that Lemma 3.9 implies that $tr_*$ is an injection in positive degrees and that there is is a commutative diagram.

$$\begin{array}{ccc} \pi_n V & \xrightarrow{\delta_i} & \pi_{n+i} S_2 V \\ \downarrow = & & \downarrow tr_* \\ \pi_n V & \xrightarrow{\sigma_i} & \pi_{n+i} S^2 V \end{array}$$

whenever it makes sense; that is, for $2 \leq i \leq n$.

## Chapter III: Homology and Cohomology

### 4. Homology, cohomology, cofibrations, and the suspension

In this section we define the André-Quillen homology and cohomology of our algebras and discuss when there is a long exact sequence in homology. Then we define one of the most important tools of this paper – the suspension of a simplicial algebra. Finally, since our definition is different than the one that André and Quillen give we show that ours yields the same groups as theirs, with a degree shift.

Homology is, for Quillen, the derived functors of abelianization. Thus we wish to determine the abelian objects in $s\mathcal{A}$. So we turn to the study of group objects in $s\mathcal{A}$; that is, we examine objects $B \in s\mathcal{A}$ so that $Hom_{s\mathcal{A}}(A, B)$ is a group for all $A \in s\mathcal{A}$. For this we need the categorical product in $s\mathcal{A}$. If $A, B \in s\mathcal{A}$, define $A \times_{\mathbf{F}_2} B$ by the pull-back diagram of simplicial vector spaces

$$\begin{array}{ccc} A \times_{\mathbf{F}_2} B & \longrightarrow & B \\ \downarrow & & \downarrow \epsilon \\ A & \xrightarrow{\epsilon} & \mathbf{F}_2 \end{array}$$

where $\epsilon$ is an augmentation. $A \times_{\mathbf{F}_2} B$ is easily seen to be the product of $A$ and $B$ in $s\mathcal{A}$. There is a canonical map

$$\rho : A \otimes B \rightarrow A \times_{\mathbf{F}_2} B$$

given by

$$\rho(a \otimes b) = (a\eta\epsilon(b), \eta\epsilon(a)b)$$

where $\eta$ is the unit map.

Therefore, if $B$ is a group object in $s\mathcal{A}$, then there is a multiplication

$$m : B \times_{\mathbf{F}_2} B \rightarrow B$$

and a commutative diagram

$$(4.1) \qquad \begin{array}{ccc} B \otimes B & \xrightarrow{\rho} & B \times_{\mathbf{F}_2} B \\ \downarrow \mu & & \downarrow m \\ B & \xrightarrow{=} & B \end{array}$$

Here $\mu$ is the algebra multiplication. Hence, if $IB$ is the augmentation ideal of $B$, we have $IB^2 = 0$. Thus

$$B \cong (IB)_+$$

where $(\ )_+ : s\mathbf{F}_2 \to s\mathcal{A}$ is the functor from the category of simplicial vector spaces which sets

$$V_+ = V \oplus \mathbf{F}_2$$

with $\mathbf{F}_2$ the unit, $V$ the augmentation ideal, and $V^2 = 0$. In other words, $V_+$ is a trivial algebra in the sense that all non-trivial products are zero. In particular, it is now obvious that $B$ is is an abelian group object in $s\mathcal{A}$. Therefore, we have proved the following.

**Lemma 4.2:** Let $\mathbf{AB}(s\mathcal{A})$ be the sub-category of $s\mathcal{A}$ whose objects are abelian group objects and whose morphisms preserve the groups multiplication $m$ of (4.1). Then

1.) $(\ )_+ : s\mathbf{F}_2 \to \mathbf{AB}(s\mathcal{A})$ is an isomorphism of categories; and
2.) every group object of $s\mathcal{A}$ is an abelian group object.

There is, for $A \in s\mathcal{A}$ and $V \in s\mathbf{F}_2$, an obvious isomorphism

$$Hom_{s\mathcal{A}}(A, V_+) \cong Hom_{s\mathbf{F}_2}(QA, V)$$

where $QA = IA/IA^2$ is the indecomposables functor. Or, more succinctly, the indecomposables functor $Q$ is left adjoint to $(\ )_+$ and deserves to be called the abelianization functor on $s\mathcal{A}$. Since, for Quillen [20, Section II.5], homology is the derived functors of abelianization, we have the next definition.

**Definition 4.3:** Let $A \in s\mathcal{A}$. Define the André-Quillen homology of $A$ as follows. Choose an acyclic fibration $p : X \to A$ with $X$ cofibrant in $s\mathcal{A}$ and set

$$H_*^Q A = \pi_* QX.$$

The cohomology of $A$ is given by

$$H_Q^* A = (H_*^Q A)^* \cong \pi^*(QX)^*.$$

**Proposition 4.3:**1.) $H_*^Q A$ and $H_Q^* A$ are well-defined and functorial in $A$;

2.) if $f : A \to B$ is a weak equivalence in $s\mathcal{A}$, then

$$H_*^Q f : H_*^Q A \to H_*^Q B$$

is an isomorphism; and

3.) if $V \in s\mathbf{F}_2$, then

$$[A, V_+]_{s\mathcal{A}} \cong Hom_{n\mathbf{F}_2}(H_*^Q A, \pi_* V)$$

where $n\mathbf{F}_2$ is the category of graded vector spaces over $\mathbf{F}_2$.

**Proof:** Parts 1 and 2 follow from the properties of cofibrant objects and Lemma 1.7. Part 3 follows from the isomorphisms

$$\begin{aligned} [A, V_+]_{s\mathcal{A}} &\cong Hom_{s\mathcal{A}}(X, V_+)/\sim \\ &\cong Hom_{s\mathbf{F}_2}(QX, V)/\sim \\ &\cong Hom_{n\mathbf{F}_2}(\pi_* QX, \pi_* V) \end{aligned}$$

where $X \to A$ is an acyclic fibration with $X$ cofibrant, $\sim$ denotes homotopy, and we use the fact that a map between simplicial vector spaces is determined up to homotopy by its effect on homotopy groups. This follows from Proposition 2.1.1. Homotopy in $s\mathbf{F}_2$ is defined in a manner similar to homotopy in $s\mathcal{A}$. The model category structure on $s\mathbf{F}_2$ is explored in [20, Section II.4].

**Example 4.4:**1.) Let $K(n) \in s\mathbf{F}_2$ be a simplicial vector space so that $\pi_* K(n) \cong \mathbf{F}_2$ concentrated in degree $n$. Then, by 4.3.3,

$$\begin{aligned} [A, K(n)_+]_{s\mathcal{A}} &\cong Hom_{n\mathbf{F}_2}(H_*^Q A, \pi_* K(n)) \\ &\cong H_Q^n A. \end{aligned}$$

Thus $K(n)_+ \in s\mathcal{A}$ represents cohomology, similar to the way that Eilenberg-MacLane spaces represent the cohomology of spaces. Hence $H_Q^* K(n)_+$, as $n$ varies, gives all cohomology operations of one variable. Furthermore, since

$$[A, (K(n) \times K(m))_+]_{s\mathcal{A}} \cong H_Q^n A \times H_Q^m A$$

we have that $H^*_Q(K(n) \times K(m))_+$ , as $n$ and $m$ vary, gives all cohomology operations of two variables, such as products. We will show how to compute the cohomology groups $H^*_Q K(n)_+$ and $H^*_Q(K(n) \times K(m))_+$ in later sections.

2.) Let $\Lambda \in \mathcal{A}$ be regarded as the constant simplicial algebra that is $\Lambda$ in each degree and with every face and degeneracy map the identity. If we perform the construction of 1.6 on $\Lambda$ we obtain a simplicial resolution

$$\epsilon : \bar{S}.\Lambda \longrightarrow \Lambda$$

with $\pi_* \bar{S}.\Lambda \cong \Lambda$ concentrated in degree 0. Of course, $\bar{S}.\Lambda$ is almost free and, hence, cofibrant. Thus

$$H^Q_* \Lambda \cong \pi_* Q \bar{S}.\Lambda.$$

But $\pi_n Q \bar{S}.\Lambda$ is often called the $n$th derived functor of the indecomposables functor with respect to the cotriple obtained from the symmetric algebra functor $S$; hence we write

$$H^Q_n \Lambda \cong L^S_* Q(\Lambda).$$

We now give an example of the flexibility that general objects in $s\mathcal{A}$ supply. This is the long exact sequence of a cofibration in $s\mathcal{A}$ — a long exact sequence related to Quillen's transitivity sequence [21]. Let $f : A \rightarrow B$ be a morphism in $s\mathcal{A}$. Using the construction of (1.6), form the commutative square

$$\begin{array}{ccc} \bar{S}.A & \xrightarrow{\bar{S}.f} & \bar{S}.B \\ \downarrow p_A & & \downarrow p_B \\ A & \xrightarrow{f} & B \end{array}$$

and factor $\bar{S}.f$ as an almost free map followed by an acyclic fibration

$$\bar{S}.A \xrightarrow{i} X \xrightarrow{p} \bar{S}.B.$$

Then define the *mapping cone* of the morphism $f$ by the equation

$$M(f) = \mathbf{F}_2 \otimes_{\bar{S}.A} X. \tag{4.5}$$

$M(f)$ is almost free and, hence, cofibrant. Lemma 1.7 implies that $M(f)$ is well-defined up to homotopy equivalence and functorial in $f$ in the homotopy category. (A homotopy equivalence is a weak equivalence with a homotopy inverse.) We could use the construction of (1.6) to make $M(f)$ strictly functorial.

**Proposition 4.6:** There is a long exact sequence in homology

$$\cdots \to H_n^Q A \xrightarrow{H_*^Q f} H_n^Q B \to H_n^Q M(f) \to H_{n-1}^Q A \to \cdots$$

and a long exact sequence in cohomology

$$\cdots \to H_Q^{n-1} A \to H_Q^n M(f) \to H_Q^n B \xrightarrow{H_Q^* f} H_Q^n A \to \cdots.$$

**Proof:** The cohomology result is obtained from the homology result by dualizing. To prove the homology result, notice that since $\bar{S}_.A$ is almost free and $i$ is an almost free morphism, the sequence of simplicial algebras

$$\bar{S}_.A \xrightarrow{i} X \to \mathbb{F}_2 \otimes_{\bar{S}_.A} X$$

yields a short exact sequence of simplicial vector spaces

$$0 \to Q\bar{S}_.A \to QX \to Q(\mathbb{F}_2 \otimes_{\bar{S}_.A} X) \to 0$$

Since $p : X \to \bar{S}_.B$ is an acyclic fibration and the composition of cofibrations is a cofibration, we have that

$$\pi_* QX \cong H_*^Q B$$

and the result follows.

The higher homotopy of $M(f)$ is often non-trivial, even if $\pi_* A$ and $\pi_* B$ are concentrated in degree 0. For computational purposes, we have the following result, from [20, Theorem II.6.b)]. Let $f : A \to B$ be a morphism in $s\mathcal{A}$.

**Proposition 4.7:** There is a first quadrant spectral sequence of algebras

$$Tor_p^{\pi_* A}(\mathbb{F}_2, \pi_* B)_q \Rightarrow \pi_{p+q} M(f).$$

Notice that if $f : \Lambda \to \Gamma$ is a map of constant simplicial objects in $s\mathcal{A}$, then this result implies that

$$\pi_* M(f) \cong Tor_*^{\Lambda}(\mathbf{F}_2, \Gamma).$$

Of particular interest is the case where $B = \mathbf{F}_2$ is the terminal object in $s\mathcal{A}$ and $f = \epsilon : A \to \mathbf{F}_2$ is the augmentation. Because the cofiber of a the map to the terminal object deserves to be called a suspension, we define the *suspension* of A by the equation

$$\Sigma A = M(\epsilon).$$

Since $H_*^Q \mathbf{F}_2 = 0$, 4.6 says that there are isomorphisms

$$\text{(4.8)} \qquad \begin{array}{ll} H_n^Q \Sigma A \cong H_{n-1}^Q A & n \geq 1 \\ H_Q^n \Sigma A \cong H_Q^{n-1} A & n \geq 1 \end{array}$$

and

$$H_0^Q \Sigma A = 0 = H_Q^0 \Sigma A.$$

The suspension has other properties that are worth recording here. For example, from [12] we have that there is a homotopy associative coproduct

$$\psi : \Sigma A \to \Sigma A \otimes \Sigma A$$

that gives $\pi_* \Sigma A$ the structure of a Hopf algebra that is connected in the sense that $\pi_0 \Sigma A = \mathbf{F}_2$. This coproduct can be used to turn the spectral sequence, obtained as a corollary to Proposition 2.12

$$\text{(4.9)} \qquad Tor_*^{\pi_* A}(\mathbf{F}_2, \mathbf{F}_2) \Rightarrow \pi_* \Sigma A$$

into a spectral sequence of Hopf algebras.

To specialize even further, if we regard $\Lambda \in \mathcal{A}$ as a constant simplicial algebra, then the spectral sequence of (4.9) collapses and we obtain an isomorphism of Hopf algebras

$$\pi_* \Sigma \Lambda \cong Tor_*^{\Lambda}(\mathbf{F}_2, \mathbf{F}_2).$$

Finally, the work of Miller [16,Section 5;17] implies that if $\bar{B}(\Lambda)$ is the bar construction, then there is a weak equivalence is $s\mathcal{A}$

$$\Sigma\Lambda \to \bar{B}(\Lambda).$$

Thus the suspension is not so unfamiliar after all.

The following result records some initial observations about $H_*^Q A$. Recall that if $A \in s\mathcal{A}$, then $\pi_0 A \in \mathcal{A}$.

**Lemma 4.10:** Let $A, B \in s\mathcal{A}$. Then

1.) $H_0^Q A \cong Q\pi_0 A$; and

2.) $H_*^Q$ preserves coproducts: $H_*^Q(A \otimes B) \cong H_*^Q A \oplus H_*^Q B$.

3.) Let $V \in s\mathbf{F}_2$ and $S(V) \in s\mathcal{A}$ the resulting simplicial symmetric algebra. The $H_*^Q S(V) \cong \pi_* V$.

**Proof:** The first part follows from the fact that the indecomposables functor is right exact; that is, $Q$ preserves surjections. For the second, if $X \to A$ and $Y \to B$ are acyclic fibrations with $X$ and $Y$ cofibrant, then the induced map

$$X \otimes Y \to A \otimes B$$

is an acyclic fibration. Since the coproduct of cofibrant objects is cofibrant

$$\begin{aligned} H_*^Q(A \otimes B) &\cong \pi_* Q(X \otimes Y) \\ &\cong \pi_* QX \oplus \pi_* QY \cong H_*^Q A \oplus H_*^Q B. \end{aligned}$$

For the third statement, notice that $S(V)$ is almost-free (in fact, "free") and that $QS(V) \cong V$. Hence $H_*^Q S(V) \cong \pi_* QS(V) \cong \pi_* V$.

**Remark:** If $K(n) \in s\mathbf{F}_2$ is so that $\pi_* K(n) \cong \mathbf{F}_2$ concentrated in degree $n$, then

$$H_*^Q S(K(n)) \cong \begin{cases} \mathbf{F}_2, & i = n; \\ 0, & \text{otherwise.} \end{cases}$$

Since $S(K(n))$ is the $n$-sphere in $s\mathcal{A}$, this homology is not surprising.

Quillen and André, taking a more general viewpoint, define cohomology slightly differently. We end this section by explaining that our definition agrees with theirs, with a dimension shift.

Let $\Lambda$ be any commutative ring, $\Gamma$ a $\Lambda$-algebra, and $M$ a $\Gamma$-module. Then define the module of derivations

$$Der_{\Lambda}(\Gamma, M)$$

to be the $\Lambda$-module of $\Lambda$-module homomorphisms

$$\partial : \Gamma \to M$$

so that

$$\partial(xy) = y\partial(x) + x\partial(y).$$

Now let $A \in s\mathcal{A}$. Factor $\epsilon : A \to \mathbf{F}_2$ as a cofibration followed by an acyclic fibration:

$$A \to X \xrightarrow{\epsilon} \mathbf{F}_2.$$

Then André and Quillen define

$$H^*(A, \mathbf{F}_2) = \pi^* Der_A(X, \mathbf{F}_2) \tag{4.11}$$

where $\mathbf{F}_2$ is regarded as an $X$-module via the augmentation. However, one easily checks that

$$Der_A(X, \mathbf{F}_2) \cong Hom_{s\mathbf{F}_2}(Q(\mathbf{F}_2 \otimes_A X), \mathbf{F}_2)$$

so that

$$H^n(A, \mathbf{F}_2) \cong H^n_Q \Sigma A \cong H^{n-1}_Q A. \tag{4.12}$$

## 5. Products and operations in cohomology

This section is devoted to constructing a commutative product

$$[\ ,\ ] : H^n_Q A \otimes H^m_Q A \to H^{n+m+1}_Q A$$

and operations

$$P^i : H^n_Q A \to H^{n+i+1}_Q A$$

for $A \in s\mathcal{A}$. We will show that $P^n(x) = [x, x]$ for $x \in H^n_Q A$, but will leave the proofs of the other properties of the product and operations until later sections.

In the homotopy category associated to our model category structure on $s\mathcal{A}$, any simplicial algebra $A$ is isomorphic to an almost free simplicial algebra. Since homology and cohomology are functors on the homotopy category, by 4.3.2, we may as well assume that $A$ is almost free in $s\mathcal{A}$. Then for all $s \geq 0$,

$$A_s = S(V_s)$$

for some graded vector space $V_s$. Of course, $S : v\mathbf{F}_2 \to \mathcal{A}$ is the symmetric algebra functor left adjoint to the augmentation ideal functor. The vector space diagonal

$$\Delta : V_s \to V_s \oplus V_s$$

yields, after applying $S$, a coproduct

$$\psi_s = S\Delta : A_s = S(V_s) \to S(V_s) \otimes S(V_s) = A_s \otimes A_s$$

that gives $A_s$ the structure of a commutative, cocommutative Hopf algebra with conjugation in $\mathcal{A}$. In particular, for any $\Lambda \in \mathcal{A}$

$$Hom_{\mathcal{A}}(A_s, \Lambda)$$

is a group; indeed

$$\begin{aligned} Hom_{\mathcal{A}}(A_s, \Lambda) &\cong Hom_{\mathcal{A}}(S(V_s), \Lambda) \\ &\cong Hom_{v\mathbf{F}_2}(V_s, I\Lambda) \end{aligned}$$

and all isomorphisms are group isomorphisms. Hence $Hom_{\mathcal{A}}(A_s, \Lambda)$ is an $\mathbf{F}_2$ vector space. Now, because A is almost free,

$$d_i : A_s \to A_{s-1}, \qquad 1 \leq i \leq s$$

and

$$s_i : A_s \to A_{s+1}, \qquad 0 \leq i \leq s$$

are maps of Hopf algebras. Only $d_0$ is not necessarily a map of Hopf algebras; hence, it makes sense to measure the deviation of $d_0$ from being a Hopf algebra map. Define

$$\bar{\xi} : A_s \to A_{s-1} \otimes A_{s-1} \tag{5.1}$$

to be the product (which is the same as difference), in the group

$$Hom_{\mathcal{A}}(A_s, A_{s-1} \otimes A_{s-1}),$$

of

$$(d_0 \otimes d_0)\psi_s : A_s \to A_{s-1} \otimes A_{s-1}$$

and

$$\psi_{s-1} d_0 : A_s \to A_{s-1} \otimes A_{s-1}.$$

The morphism $\bar{\xi}$ actually factors through a subalgebra of $A_{s-1} \otimes A_{s-1}$. If $B, C \in s\mathcal{A}$, let $B \times_{\mathbf{F}_2} C$ be the product defined in the previous section and $\rho : B \otimes C \to B \times_{\mathbf{F}_2} C$ the canonical map from the coproduct to the product. Define $B \wedge C$ by the pull-back diagram (of simplicial vector spaces)

$$\begin{array}{ccc} B \wedge C & \rightarrow & B \otimes C \\ \downarrow \epsilon & & \downarrow \rho \\ \mathbf{F}_2 & \xrightarrow{\eta} & B \times_{\mathbf{F}_2} C \end{array}$$

Then $B \wedge C \in s\mathcal{A}$. As an aside, notice that if $X$ and $Y$ are pointed spaces with smash product $X \wedge Y$, then

$$H^*(X \wedge Y, \mathbf{F}_2) \cong H^*(X, \mathbf{F}_2) \wedge H^*(Y, \mathbf{F}_2).$$

Then, for $A \in s\mathcal{A}$ almost free, there is a factoring

$$
\begin{array}{ccc}
A_s & \xrightarrow{\xi} & A_{s-1} \wedge A_{s-1} \\
\downarrow = & & \downarrow \\
A_s & \xrightarrow{\bar{\xi}} & A_{s-1} \otimes A_{s-1}
\end{array}
\tag{5.2}
$$

To see this, one need only check that the two composites

$$A_s \xrightarrow{\bar{\xi}} A_{s-1} \otimes A_{s-1} \xrightarrow{\epsilon\otimes 1} A_{s-1}$$

and

$$A_s \xrightarrow{\bar{\xi}} A_{s-1} \otimes A_{s-1} \xrightarrow{1\otimes\epsilon} A_{s-1}$$

are the trivial map

$$\eta\epsilon : A_s \to A_{s-1}.$$

For the morphism $\epsilon \otimes 1$, say, this is equivalent to showing that

$$(\epsilon \otimes 1)(d_0 \otimes d_0)\psi_s = (\epsilon \otimes 1)\psi_{s-1}d_0 : A_s \to A_{s-1}.$$

But this is obvious. A similar argument can be given in the other case and that completes the definition of the map $\xi$ of (5.2).

To define the product on cohomology of the simplicial algebra $A \in s\mathcal{A}$, we need the following lemmas. Let $Q(\ \ )$ denote the indecomposables functor.

**Lemma 5.3:** For $B, C \in s\mathcal{A}$, there is a natural map

$$Q(B \wedge C) \to QB \otimes QC$$

**Proof:** The map $B \wedge C \to B \otimes C$ induces a map

$$I(B \wedge C) \to IB \otimes IC$$

where $I(\ \ )$ is the augmentation ideal functor. The result follows by investigating this map.

For the next lemma, we need some notation. If $f, g : A_s \to \Lambda$ with $A \in s\mathcal{A}$ almost free, let $f * g$ denote the product of $f$ and $g$ in the group $Hom_{\mathcal{A}}(A_s, \Lambda)$; that is, $f * g$ is the composite

$$A_s \xrightarrow{\psi_s} A_s \otimes A_s \xrightarrow{f \otimes g} \Lambda \otimes \Lambda \to \Lambda$$

where the last map is multiplication. Notice that

$$Q(f * g) = Qf + Qg : QA_s \to Q\Lambda \tag{5.4.}$$

Thus, the next result will allow us to compute boundary homomorphisms in various chain complexes.

**Lemma 5.5:** Let $A \in s\mathcal{A}$ be almost free. Then if

$$\xi : A_s \to A_{s-1} \wedge A_{s-1}$$

is the map of (5.2), we have

1.) $(d_i \wedge d_i)\xi = \xi d_{i+1}, \qquad i \geq 1$; and

2.) $(d_0 \wedge d_0)\xi = [\xi d_0] * [\xi d_1]$.

**Proof:** These are simple consequences of the simplicial identities; we will do 2.)

It is sufficient to show that for

$$\bar{\xi} : A_s \to A_{s-1} \otimes A_{s-1}$$

we have the equation

$$(d_0 \otimes d_0)\bar{\xi} = [\bar{\xi} d_0] * [\bar{\xi} d_1].$$

This is because the map

$$Hom_{\mathcal{A}}(A_s, A_{s-2} \wedge A_{s-2}) \to Hom_{\mathcal{A}}(A_s, A_{s-2} \otimes A_{s-2})$$

is an injection. However,

$$\bar{\xi} = [(d_0 \otimes d_0)\psi] * [\psi d_0]$$

where the coproducts $\psi_s$ and $\psi_{s-1}$ are abbreviated to $\psi$. Now, since A is almost free, the coproduct $\psi$ commutes with $d_i$ for $i \geq 1$:

$$(d_i \otimes d_i)\psi = \psi d_i, \qquad i \geq 1.$$

Thus we may compute, using the facts that $Hom_{\mathcal{A}}(A_s, \Lambda)$ is an $\mathbf{F}_2$-vector space and that $d_0 d_1 = d_0 d_0$:

$$\begin{aligned}
&[(d_0 \otimes d_0)\bar{\xi}] * [\bar{\xi} d_1] \\
&\qquad = [(d_0 \otimes d_0)^2\psi] * [(d_0 \otimes d_0)\psi d_0] * [(d_0 \otimes d_0)\psi d_1] * [\psi d_0 d_1] \\
&\qquad = [(d_0 \otimes d_0)^2\psi] * [(d_0 \otimes d_0 \psi d_0] * [(d_0 \otimes d_0)^2\psi] * [\psi d_0 d_0] \\
&\qquad = [(d_0 \otimes d_0)\psi d_0] * [\psi d_0 d_0] \\
&\qquad = \bar{\xi} d_0.
\end{aligned}$$

The result follows.

If $V$ is a simplicial vector space let $C(V)$ be the associated chain complex. The following is now an immediate consequence of the previous result.

**Corollary 5.6:** If $\xi : A_s \to A_{s-1} \wedge A_{s-1}$ is the map of (5.2), then $\xi$ induces a chain map of degree $-1$

$$Q\xi : C(QA) \to C(Q(A \wedge A)).$$

We now use $\xi$ to define a product in the cohomology of a simplicial algebra. Let $A \in s\mathcal{A}$. Since $A$ is weakly equivalent to an almost free object, we may assume that $A$ is almost free. Applying Corollary 5.6, we know that $\xi$ induces a map of degree $-1$:

$$QA \xrightarrow{Q\xi} Q(A \wedge A) \to QA \otimes QA. \tag{5.7}$$

Let us call this map $\psi_A$. By (5.3) and 5.6, we know that these are maps of chain complexes. We now define a product

$$[\ ,\ ] : H^n_Q A \oslash H^m_Q A \to H^{n+m+1}_Q A$$

as the map induces by the map of cochain complexes

$$QA^* \otimes QA^* \to (QA \otimes QA)^* \xrightarrow{\psi_A^*} QA^*.$$

The first map is the canonical homomorphism from $V^* \otimes W^*$ to $(V \otimes W)^*$ and we use the Eilenberg-Zilber Theorem to give a natural isomorphism

$$\pi^*(QA^* \otimes QA^*) \cong H_Q^* A \otimes H_Q^* A.$$

The product is commutative on the chain level. We record this fact in the following result. If $V$ is any vector space, let $T : V \otimes V \to V \otimes V$ be the switch map $T(u \otimes v) = v \otimes u$. The next result follows from the definitions.

**Lemma 5.8:** We have equality between the following morphisms:

$$\psi_A = T\psi_A : QA \to QA \otimes QA$$

**Proof:** For $A \in s\mathcal{A}$ almost free the Hopf algebra diagonal map

$$\psi_s : A_s \to A_s \otimes A_s$$

is cocommutative; that is, $\psi_s = T\psi_s$. The result now follows from the definition of $\xi$.

**Corollary 5.9:** The product

$$[\ ,\ ] : H_Q^p A \otimes H_Q^q A \to H_Q^{p+q+1} A$$

is bilinear and commutative.

We now use Lemma 5.8 to define the operations. Let $A \in s\mathcal{A}$ be almost free, and $Q\xi$ as in (5.7). Let $\{D_k\}$ be a collection of higher Eilenberg-Zilber maps as guaranteed by Lemma 3.6. Define a function

$$\Theta^i : QA^* \to QA^*$$

of degree $i+1$ by setting, for $\alpha$ of degree $n$

$$\text{(5.10)} \qquad \Theta^i(\alpha) = \psi_A^* D_{n-i}^*(\alpha \otimes \alpha) + \psi_A^* D_{n-i+1}^*(\alpha \otimes \partial\alpha).$$

Here we let $D_k = 0$ if $k < 0$. Then one easily checks, using Lemma 5.8, that

$$\partial\Theta^i(\alpha) = \psi_A^* D_{n+1-i}^*(\partial\alpha \otimes \partial\alpha) = \Theta^i(\partial\alpha). \tag{5.10.1}$$

Let

$$P^i = \pi^*\Theta^i : H_Q^* A \to H_Q^* A. \tag{5.10.2}$$

If $A \in s\mathcal{A}$ is not almost free, choose an acyclic fibration $X \to A$ and define the operations in $H_Q^* X \cong H_Q^* A$.

**Lemma 5.11:** If $x \in H_Q^n A$, then $P^i(x) = 0$ if $i > n$ and

$$P^n(x) = [x, x].$$

**Proof:** $D_0$ is the Eilenberg-Zilber chain equivalence and $D_k = 0$ if $k < 0$.

As a first application, let $f : A \to B$ be morphism in $s\mathcal{A}$ and

$$\partial : H_Q^s A \to H_Q^{s+1} M(f)$$

the boundary map in the long exact sequence of the resulting cofibration sequence, as in Proposition 4.6. We would like to know how this map behaves with respect to the product and operations.

**Lemma 5.12:**1.) Let $x \in H_Q^* A$. Then, for all $i$

$$\partial P^i(x) = P^i(\partial x).$$

2.) For all $x \in H_Q^* A$ and $y \in H_Q^* M(f)$

$$[\partial x, y] = 0.$$

**Proof:** The map $\partial$ is the connecting homomorphism obtained from a short exact sequence of cochain complexes. See 4.6. Part 1.) follows from investigating formula (5.10.1) and part 2.) follows from the naturality of the homomorphism $D_0$ of Lemma 3.6.

**Corollary 5.13:** Let $A \in s\mathcal{A}$ and let $\Sigma A$ be the suspension of $A$. Then for all $x, y \in H_Q^*\Sigma A$

$$P^i(x) = 0 \qquad \text{for } i \geq deg(x)$$

and

$$[x, y] = 0.$$

**Proof:** This follows from the fact that

$$\partial : H_Q^s A \to H_Q^{s+1}\Sigma A$$

is an isomorphism and the previous lemma.

## Chapter IV: Quillen's Fundamental Spectral Sequence

### 6. Quillen's spectral sequence

In his seminal work on the homology of commutative algebras, Quillen ([21] and [22]) described an Adams-type spectral sequence passing from the homology of a simplicial commutative algebra to its homotopy. Then he used this spectral sequence to study homology — for example, in characteristic zero, there is a simple situation under which the spectral sequence collapses and computes homology. The situation is different in characteristic 2, mostly because of the existence of the operations of the previous section. However, we still study this spectral sequence and gain useful information. For example, in section 8 we show that the product and operations of the previous section are non-trivial, and we can prove the Adem relations among the operations.

Before defining the spectral sequence, we make some computations. Let $\Lambda \in \mathcal{A}$ be an algebra with augmentation ideal $I\Lambda$. Then we can filter $\Lambda$ by powers of the augmentation ideal:

$$F_s\Lambda = (I\Lambda)^s \subseteq \Lambda.$$

In particular, if $\Lambda = S(V)$ — the symmetric algebra on a vector space $V$ — then

$$S(V) = \bigoplus_{n \geq 0} S_n(V)$$

where $S_n(V)$ is the vector space of coinvariants of $V^{\otimes n}$ under the action of the permutation group $\Sigma_n$ that permutes the factors. Then

$$F_sS(V) = (IS(V))^s = \oplus_{n \geq s} S_n(V)$$

so that the filtration quotient is

$$E^0_sS(V) = F_sS(V)/F_{s+1}S(V) \cong S_s(V).$$

Or, because the associated graded vector space of a filtration of an algebra given by an ideal is a graded algebra, we can say that $E^0S(V)$ is the graded symmetric algebra on the vector space $V$ concentrated in degree 1.

Now suppose that $A \in s\mathcal{A}$ is almost-free; that is, for each $t$, $A_t \cong S(V_t)$ for some vector space $V_t$. Then the composite

$$V_t \xrightarrow{\subseteq} IS(V_t) \cong IA_t \to QA_t$$

is an isomorphism. Then, if we filter $A$ by the powers of the simplicial augmentation ideal, we have that the associated graded algebra $E^0 A$ is the simplicial symmetric algebra on $QA$; that is, $E^0_s A \cong S_s(QA)$. In particular, $E^0_1 A \cong QA$. If we apply homotopy, we obtain a spectral sequence

$$E^1_{s,t} A \cong \pi_t E^0_s A \cong \pi_t S_s(QA) \Rightarrow \pi_t A \tag{6.1}$$

with differentials

$$d_r : E^r_{s,t} A \to E^r_{s+r,t-1} A.$$

This is Quillen's spectral sequence.

In section 3, we gave a description of $\pi_* S_s(QA)$. There is a functor $\mathfrak{S}_s : n\mathbf{F}_2 \to n\mathbf{F}_2$, so that

$$\pi_* S_s(QA) \cong \mathfrak{S}_s(H^Q_* A).$$

Here we use the fact that since $A$ is almost-free, $\pi_* QA \cong H^Q_* A$. We will write

$$\mathfrak{S}_s(H^Q_* A)_t$$

for the elements of degree $t$. Then we have

$$\mathfrak{S}_s(H^Q_* A)_t \cong \pi_t S_s(QA).$$

The pairing

$$S_s(QA) \otimes S_{s'}(QA) \to S_{s+s'}(QA)$$

that induces the algebra product $S(QA)$ induces a pairing

$$\begin{aligned} E^1_{s,t} A \otimes E^1_{s',t'} A \cong \mathfrak{S}_s(H^Q_* A)_t \otimes \mathfrak{S}_{s'}(H^Q_* A)_{t'} \\ \to \mathfrak{S}_{s+s'}(H^Q_* A)_{t+t'} \cong E^1_{s+s',t+t'} A \end{aligned}$$

that turns $E^1 A$ into a bigraded algebra and (6.1) into spectral sequence of algebras. Again $E^1 A$ has a succinct description: if

$$\mathfrak{S} : n\mathbf{F}_2 \to n\mathbf{F}_2$$

is the functor so that $\pi_* S(V) \cong \mathfrak{S}(\pi_* V)$ for $V \in s\mathbf{F}_2$, then $E^1 A$ is the associated graded algebra obtained from $\mathfrak{S}(H_*^Q A)$ by filtering by powers of the graded augmentation ideal.

**Theorem 6.2:** (Quillen's spectral sequence) For $A \in s\mathcal{A}$, there is a natural spectral sequence of algebras

$$E^1_{s,t} A \cong \mathfrak{S}_s(H_*^Q A)_t \Rightarrow \pi_t A.$$

This spectral sequence converges if $H_0^Q A = 0$.

**Proof:** If $A$ is almost-free, the existence of the spectral sequence is the content of the previous paragraphs. For a general $A$, choose an weak equivalence $X \to A$ with $X$ almost free and define the spectral sequence for $A$ to be the spectral sequence for $X$. This is well-defined by 1.7. For the convergence statement, Proposition 3.7 implies that if $H_0^Q A = 0$, then $\mathfrak{S}_s(H_*^Q A)_t = 0$ for $t < s$. Thus

$$E^r_{s,t} A \cong E^\infty_{s,t} A$$

if $r$ is sufficiently large. The result follows.

**Remark 6.3:** If $H_0^Q A \neq 0$, one can compute the spectral sequence for $\Sigma A$ instead. Then $H_n^Q \Sigma A \cong H_{n-1}^Q A$ and $H_0^Q \Sigma A = 0$. For example, if $A = \Lambda$ is a constant simplicial algebra, then

$$H_n^Q \Sigma \Lambda \cong H_n(\Lambda, \mathbf{F}_2)$$

by the dual of (4.12) and

$$\pi_* \Sigma \Lambda \cong Tor_*^{\Lambda}(\mathbf{F}_2, \mathbf{F}_2).$$

Making these substitutions into (6.2) yields Quillen's spectral sequence, as he wrote it down.

The next step in understanding this spectral sequence is to put in the higher divided power operations of section 2. The operations $\delta_i$ of sections 2 and 3 induce operations

$$\delta_i : \mathfrak{S}_s(H_*^Q A)_t \to \mathfrak{S}_{2s}(H_*^Q A)_{t+i}. \tag{6.4}$$

We claim that these operations extend to the spectral sequence and abut to the operations on $\pi_* A$. Unfortunately, the operations on $E^r A$ are only defined up to some indeterminacy, which we now define. Let

$$B^q_{s,t}A \subseteq E^r_{s,t}A, \qquad q \geq r$$

be the vector space of elements that survive to $E^q_{s,t}A$ but have zero residue class in $E^q_{s,t}A$. An element $y \in E^r_{s,t}A$ is defined up to indeterminacy $q$ if $y$ is a coset representative for a particular element in $E^r_{s,t}A/B^q_{s,t}A$.

We now define operations

$$\delta_i : E^r_{s,t}A \to E^r_{2s,t+i}A$$

of indeterminacy $2r-1$. These will agree with the operations of (6.4) where $r = 1$. Notice that when $r = 1$ there will be no indeterminacy.

Let $\Theta_i$ be the quadratic functions of (3.2). Then $\Theta_i$ restricts to a function

$$\Theta_i : (IA)^s \to (IA)^{2s}.$$

Now let $y \in E^r_{s,t}A$. Then, modulo $(IA)^{s+1}$, $y$ is represented by an element $\alpha \in (IA)^s$ so that $\partial\alpha \in (IA)^{s+r}$. The class of $\alpha$ in not unique, even modulo $(IA)^{s+1}$. It may be altered by adding elements

$$\partial\beta \in (IA)^s$$

so that $\beta \in (IA)^{s-r+1}$.

Define $\delta_i(y) \in E^r_{2s,t+i}A$ to be the residue class of $\Theta_i(\alpha)$. Since

$$\partial\Theta_i(\alpha) = \Theta_i(\alpha) \in (IA)^{s+r}, \qquad 2 \leq i < t$$

and

$$\partial\Theta_t(\alpha) = \mu\Delta(\alpha \otimes \partial\alpha)$$

$\delta_i(y)$ is an element in $E^r_{s,t}A$; indeed, it survives to $E^{2r}_{s,t}A$ if $2 \leq i < t$. It is not well-defined, however, because of the possible choices of $\alpha$. But an easy calculation using the formulas of (3.3) shows that the indeterminacy of $\delta_i(y)$ is $2r - 1$.

The next result shows that how the operations behave with respect to the differentials in the spectral sequence. It is to be understood that the various formulas that we write down are true modulo appropriate indeterminacy.

**Proposition 6.5:** For $1 \leq r \leq \infty$ there are operations

$$\delta_i : E^r_{s,t}A \to E^r_{2s,t+i}A, \qquad 2 \leq i \leq t$$

of indeterminacy $2r - 1$, satisfying the following properties:

1.) if $r = 1$, then $\delta_i$ is as in (6.4);

2.) if $x \in E^r A$ and $2 \leq i < t$, then $\delta_i(x)$ survives to $E^{2r}A$ and

$$\begin{aligned} d_{2r}\delta_i(x) &= \delta_i(d_r x), \qquad 2 \leq i < t \\ d_r\delta_t(x) &= x\partial x \end{aligned}$$

modulo indeterminacy;

3.) the operations on $E^r A$, $r \geq 2$ are induced by the operations on $E^{r-1}A$. The operations on $E^\infty A$ are induced by the operations on $E^r A$ with $r < \infty$; and

4.) the operations on $E^\infty A$ are also induced by the operations on $\pi_* A$.

**Proof:** This follows from the definition of $\delta_i$ as given above, and the formulas of (3.3). For example, if $x \in E^r_{s,t}A$, then there is an $\alpha \in (IA)^s$ so that $\partial\alpha \in (IA)^{s+r}$ and the residue class of $\alpha$ is $x$. The residue class of $\partial\alpha$ is, of course, $d_r x$. Then (3.3.2) says that

$$\partial\Theta_t(\alpha) = \mu\Delta(\alpha \otimes \partial\alpha)$$

so that the residue class of $\partial\Theta_t(\alpha)$ is $x d_r x$. This proves one of the assertions of part 3. The rest are left to the reader.

**Remark 6.6:** Notice that there is not indeterminacy for the operations at $E^\infty A$. Therefore, the operations are well-defined there.

**Proposition 6.7:** Up to indeterminacy, the operations

$$\delta_i : E^r_{s,t}A \to E^r_{2s,t+i}A, \qquad 2 \leq i \leq t$$

satisfy the properties of Theorem 2.5. In particular, if $i < 2j$, then

$$\delta_i\delta_j(x) = \sum_{i+1/2 \leq s \leq i+j/3} \binom{j-i+s-1}{j-s} \delta_{i+j-s}\delta_s(x).$$

**Proof:** This is true at $r = 1$, by 6.5.1. So the result follows from 6.5.3.

Because of Proposition 3.7, $E^1A$ is generated as an algebra by compositions

$$\delta_I(x) = \delta_{i_1} \cdots \delta_{i_s}(x)$$

with $x \in H^Q_*A$. Therefore, we could go a long way towards computing this spectral sequence by computing

$$d_1 : E^1_1 A \cong H^Q_* A \to E^1_2 A \cong \mathfrak{S}_2(H^Q_* A)$$

and then applying Proposition 6.5. This we now do.

As in any Adams-type spectral sequence, this differential depends only on the product and operations in cohomology, or, in our case, on the coproduct and operations in $H^Q_*A$. At the end of section 3 we explained how the product and operations in cohomology could be defined using homology. We recapitulate this idea in our new setting.

Let $A \in s\mathcal{A}$ be almost-free and let

$$\psi_A : QA \to QA \otimes QA$$

be the chain map of degree $-1$ of (5.7) used to define the product and operations. Because of Lemma 5.8, $\psi_A$ induces a map

$$\psi_A : QA \to S^2 QA = (QA \otimes QA)^{\Sigma_2}$$

where the target is the vector space of invariants under the action the permutes coordinates. Thus, we get a map of degree $-1$

$$(\psi_A)_* : H^Q_* A \to \pi_* S^2(QA).$$

By Dold's Theorem (2.1), $\pi_* S^2(QA)$ is a functor of $\pi_* QA \cong H_*^Q A$. Furthermore, as in (3.8), we have that for $x \in H_*^Q A$

$$(6.8) \qquad (\psi_A)_*(x) = \sum_j \pi_*(1+T)(y_j \otimes z_j) + \sum_i \sigma_i(xP^i)$$

where $1+T : QA \otimes QA \to S^2 QA$ is the averaging map. This equations defines operations

$$(\cdot)P^i : H_n^Q A \to H_{n-i-1}^Q A$$

which we write on the right and which are dual to the operations of the previous section. By composing with the map $\rho : \pi_* S^2 QA \to \pi_* QA \otimes \pi_* QA$ induced by the inclusion, we recover the coproduct

$$(\psi_A)_* : H_*^Q A \to H_*^Q A \otimes H_*^Q A.$$

Namely, for $x \in H_n^Q A$

$$(\psi_A)_*(x) = xP^{\frac{n-1}{2}} \otimes xP^{\frac{n-1}{2}} + \sum_j y_j \otimes z_j + z_j \otimes y_j$$

where $xP^{\frac{n-1}{2}} = 0$ if $n$ is even.

**Proposition 6.9:** Let $d_1 : E_1^1 A \cong H_*^Q A \to E_2^1 A \cong \mathfrak{S}_2(H_*^Q A)$ be the differential in Quillen's spectral sequence. Then, for $x \in H_n^Q A$ with $n \geq 1$,

$$d_1(x) = \sum_j y_j z_j + \sum_{i \geq 2} \delta_i(xP^i)$$

where

$$(\psi_A)_*(x) = \sum_j \pi_*(1+\tau)(y_j \otimes z_j) + \sum_i \sigma_i(xP^i).$$

The rest of this section will be spent in proving this result. Or, more exactly, we will spend the time proving the following result, which implies 6.9. Let

$$tr_* : \pi_* S_2(QA) \to \pi_* S^2(QA)$$

be the map of Lemma 3.9.

**Proposition 6.10:** There is an equality of homomorphisms

$$tr_* d_1 = (\psi_A)_* : H_*^Q A \to \pi_* S^2(QA).$$

**Proof of 6.9:** If $x \in H_*^Q A$, then we apply (6.8) to compute $(\psi_A)_*(x)$ and then use 6.10 and and Lemma 3.9 to compute $d_1(x)$. It is crucial that $tr_*$ is an injection in positive degrees.

We begin the proof of 6.10 with a technical lemma about $\psi_A : QA \to QA \otimes QA$. Let $A \in s\mathcal{A}$ be almost-free and, for each $t$

$$\psi : A_t \to A_t \otimes A_t$$

the resulting Hopf algebra coproduct. $\psi$ commutes with $d_i$, $i > 0$, so that if $\partial$ is the sum of the face operators $d_i$, then

$$\partial\psi + \psi\partial = (d_0 \otimes d_0)\psi + \psi d_0 : A_t \to A_{t-1} \otimes A_{t-1}. \tag{6.11.1}$$

If $x \in IA_t$, then $(\partial\psi + \psi\partial)(x) \in IA_{t-1} \otimes IA_{t-1}$ and we have a map $H$ that completes the commutative diagram

$$\begin{array}{ccc} IA_t & \xrightarrow{H} & IA_{t-1} \otimes IA_{t-1} \\ \downarrow & & \downarrow \\ A_t & \xrightarrow{\partial\psi+\psi\partial} & A_{t-1} \otimes A_{t-1} \end{array} \tag{6.11.2}$$

**Lemma 6.12:** There is a commutative diagram

$$\begin{array}{ccc} IA_t & \xrightarrow{H} & IA_{t-1} \otimes IA_{t-1} \\ \downarrow & & \downarrow \\ QA_t & \xrightarrow{\psi_A} & QA_{t-1} \otimes QA_{t-1}. \end{array}$$

where the vertical maps are the projections.

**Proof:** In (5.1), (5.2), and (5.6) we constructed a diagram

$$\begin{array}{ccccc} IA_t & \xrightarrow{\xi} & I(A_{t-1} \wedge A_{t-1}) & \rightarrow & IA_{t-1} \otimes IA_{t-1} \\ \downarrow & & \downarrow \subseteq & & \\ IA_t & \xrightarrow{\bar{\xi}} & I(A_{t-1} \otimes A_{t-1}) & & \end{array}$$

and the top row was used to compute $\psi_A$. To compute $\xi$, it is sufficient to compute $\bar{\xi}$. Since $A \in s\mathcal{A}$ is almost-free, $A = S(V_t)$ for some vector space $V_t$; indeed, the composite

$$V_t \xrightarrow{\subseteq} IS(V_t) \rightarrow QA_t$$

is an isomorphism. This to compute $\psi_A$ it is sufficient to examine $\bar{\xi}(v)$ for $v \in V_t$. However $v \in V_t$ is primitive in the Hopf algebra $A_t = S(V_t)$; therefore,

$$\bar{\xi}(v) = (d_0 \otimes d_0)\psi(v) + \psi d_0(v)$$

by the definition of $\bar{\xi}$. The result follows.

Now let $x \in H_t^Q A$ be represented by $\alpha \in QA_t$. Let $v \in V_t \subseteq IS(V_t) \cong IA_t$ be the unique element that passes to $\alpha$ under the isomorphism $V_t \cong QA_t$. Then, if $H$ is as in 6.12

$$H(v) = (\partial\psi + \psi\partial)(v) \in IA_{t-1} \otimes IA_{t-1}.$$

Since $\alpha \in QA_t$ is a cycle, we may write $\partial(v) \in IA_{t-1}$ as a unique sum

$$\partial(v) = w_1 + w_2 \tag{6.13}$$

where $w_1 \in S_2V_{t-1}$ and $w_2 \in \oplus_{n>2} S_n V_{t-1}$. Let

$$\bar{tr} : S_2V_{t-1} \rightarrow V_{t-1} \otimes V_{t-1} = S_1V_{t-1} \oplus S_1V_{t-1}$$

induced by the averaging map $(1 + T) : V_{t-1} \oplus V_{t-1} \rightarrow V_{t-1} \otimes V_{t-1}$.

**Lemma 6.14:** $H(v) = \bar{tr}(w_1) + y$ where

$$y \in \bigoplus_{\substack{p+q>2 \\ p \neq 0 \neq q}} S_pV_{t-1} \otimes S_qV_{t-1}.$$

**Proof:** First of all,

$$\partial\psi(v) = \partial(v) \otimes 1 + 1 \otimes \partial(v)$$

because $v$ is primitive. On the other hand,

$$\begin{aligned}\psi\partial(v) &= \psi(w_1) + \psi(w_2) \\ &= \partial(v) \otimes 1 + 1 \otimes \partial(v) + \bar{tr}(w_1) + \bar{\psi}(w_2)\end{aligned}$$

where

$$\bar{\psi}(w_2) = w_2 \otimes 1 + 1 \otimes w_2 + \psi(w_2).$$

Here we use the fact that if $z \in S_2(V_{t-1})$, then

$$\psi(z) = z \otimes 1 + \bar{tr}(z) + 1 \otimes z.$$

Since $w_2 \in \oplus_{n>2} S_n V_{t-1}$, $\bar{\psi}(w_2)$ has the property required of $y$.

**Proof of 6.10:** If $w_1$ is as in (6.13), then $w_1$ is a cycle in $S_2(QA)$ whose residue class is $d_1 x$. Since restricting the range of

$$\bar{tr} : S_2(QA) \to QA \otimes QA$$

to $S^2 QA$ yields $tr$, Lemma 6.14 says that we need only identify the projection to $QA \otimes QA$ of $\bar{H}(v)$. However, the diagram of Lemma 6.12 implies that this projection is $\psi_A(v)$. The result follows.

## 7. Ramifications of the Jacobi identity

In section 11 we will prove that the product $[\ ,\ ]$ on $H^*_Q$ satisfies the Jacobi identity; that is, if $A \in s\mathcal{A}$ and $x, y, z \in H^*_Q A$, then

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0.$$

The proof uses the method of the universal example. In this section we give a chain level proof of this fact. We do not embark on this exercise gratuitously, but because we can use the constructions to provide a proof of the following fact: if $x, y \in H^*_Q A$, then

$$[x, P^i y] = 0$$

for integers $i$. If $i = deg(y)$, then $[x, P^i y] = [x,[y,y]]$ and this claim follows from the Jacobi identity. The other cases are more problematic and necessitate the approach taken here. The crucial technical input is provided by the lemmas of the previous section, which explains the location of these arguments.

The product $[\ ,\ ]$ on $H^*_Q A$ is determined by the cocommutative coproduct

$$\psi_A : QA \to QA \otimes QA$$

of section 5, and we use the notation and ideas of that section freely. Of course, we are assuming that $A \in s\mathcal{A}$ is almost-free. To actually compute $[\ ,\ ]$ we must use an Eilenberg-Zilber chain equivalence

$$D_0 : C(QA \otimes QA) \to C(QA) \otimes C(QA).$$

Here we are writing $C(V)$ for the chain complex associated to a simplicial vector space $V$, and $C(QA) \otimes C(QA)$ is the tensor product of chain complexes with the usual Leibniz differential. Then, of course, the product $[\ ,\ ]$ is defined by dualizing the composition

$$C(QA) \xrightarrow{\psi_A} C(QA \otimes QA) \xrightarrow{D_0} C(QA) \otimes C(QA).$$

It will be useful to make a standard choice for $D_0$. If $V$ is a simplicial vector space, define

$$\tilde{d}^{s-i} : V_s \to V_i$$

to be the composition

$$\tilde{d}^{s-i} = d_{i+1} \circ \cdots \circ d_s$$

where the $d_j$ are the face operators in $V$. Define $d_0^i : V_s \to V_{s-i}$ to be the composition of the respective $d_0$ face operators. Then, for any simplicial vectors spaces $V$ and $W$ we can define

$$D_0 : C(V \otimes W)_s \to [C(V) \otimes C(W)]_s$$

by the formula

$$D_0(v \otimes w) = \sum_{i=0}^{s} \tilde{d}^{s-i} v \otimes d_0^i w. \tag{7.1}$$

This is called the Alexander-Whitney chain equivalence. It is useful to fix this choice for $D_0$, and we do so.

Now in discussing the Jacobi identity and related matters, we are confronted with the following composition

$$\begin{aligned} C(QA) &\xrightarrow{\psi_A} C(QA \otimes QA) \\ &\xrightarrow{D_0} C(QA) \otimes C(QA) \\ &\xrightarrow{\psi_A \otimes 1} C(QA \otimes QA) \otimes C(QA) \\ &\xrightarrow{D_0 \otimes 1} C(QA) \otimes C(QA) \otimes C(QA). \end{aligned}$$

This may be written as

$$(D_0 \otimes 1)(\psi_A \otimes 1) D_0 \psi_A. \tag{7.2}$$

To simplify our calculations, we claim that $\psi_A \otimes 1$ and $D_0$ commute as follows.

**Lemma 7.3:** $(\psi_A \otimes 1) D_0 = D_0(\psi_A \otimes d_0)$.

**Proof:** We refer freely to Lemma 6.12 and diagram 6.11, including the notation established there. To prove the result at hand, we proceed as follows: since $IA \to QA$ is a surjection, it is sufficient to prove that

$$(H \otimes 1) D_0 = D_0(H \otimes d_0) : C(IA \otimes IA) \to C(IA \otimes IA) \otimes C(IA).$$

Since $IA \to A$ is an injection it is sufficient to prove that

$$[(\partial\psi + \psi\partial) \otimes 1]D_0 = D_0[(\partial\psi + \psi\partial) \otimes d_0] : C(A \otimes A) \to C(A \otimes A) \otimes C(A).$$

Since, for $i \geq 1$, $d_i$ commutes with $\psi : A \to A \otimes A$, we have that

$$(\partial\psi + \psi\partial) = (d_0 \otimes d_0)\psi + \psi d_0.$$

The result now follows from a routine calculation with the simplicial identities, using the formula of 7.1.

To exploit this lemma, we proceed in the following manner. For a simplicial vector spaces $U$, $V$, and $W$, let

$$\tau : U \otimes V \otimes W \to W \otimes U \otimes V$$

be the permutation of factors given by

$$\tau(u \otimes v \otimes w) = w \otimes u \otimes v.$$

An acyclic models argument shows that there is a chain homotopy

$$\tau(D_0 \otimes 1)D_0 \simeq (D_0 \otimes 1)D_0\tau.$$

Combing this fact with Lemma 7.3, we see that the Jacobi identity will hold if we can prove that

$$(1 + \tau + \tau^2)(\psi_A \otimes d_0)\psi_A : C(QA) \to C(QA \otimes QA \otimes QA)$$

is chain null-homotopic.

We can actually build an explicit homotopy.

**Lemma 7.4:** Let $A \in s\mathcal{A}$ be an almost-free simplicial algebra. Then there is a homomorphism

$$F : C(QA) \to C(QA \otimes QA \otimes QA)$$

of degree $-1$ so that

$$\partial F + F\partial = (1+\tau+\tau^2)(\psi_A \otimes d_0)\psi_A.$$

**Proof:** Since $A$ is almost-free, we have a group

$$Hom_{\mathcal{A}}(A_{t+1}, A_t \otimes A_t \otimes A_t).$$

In this group, let $\bar{\mathcal{E}}$ be the product of

$$\begin{gathered}(\psi \otimes 1)(d_0 \otimes d_0)\psi \\ \tau(\psi \otimes 1)(d_0 \otimes d_0)\psi \\ \tau^2(\psi \otimes 1)(d_0 \otimes d_0)\psi\end{gathered}$$

and

$$(\psi \otimes 1)\psi d_0 = (1{+}\tau + \tau^2)(\psi \otimes 1)\psi d_0.$$

The last equality is a result of the fact that the coproducts $\psi$ are cocommutative and coassociative.

Now one easily argues that there is a factoring

$$\begin{array}{ccc} A_{t+1} & \xrightarrow{\mathcal{E}} & A_t \wedge A_t \wedge A_t \\ \downarrow = & & \downarrow \\ A_{t+1} & \xrightarrow{\bar{\mathcal{E}}} & A_t \otimes A_t \otimes A_t. \end{array}$$

Therefore, using 5.3, we obtain a map

$$F : QA_{t+1} \to QA_t \otimes QA_t \otimes QA_t.$$

We would like to compute $\partial F + F\partial$. Arguing as in the proof of 6.12, we obtain a diagram

$$\begin{array}{ccccc} QA_{t+1} & \leftarrow & IA_{t+1} & \rightarrow & A_{t+1} \\ \downarrow F & & \downarrow \mathcal{H} & & \downarrow G \\ QA_t \otimes QA_t \otimes QA_t & \leftarrow & IA_t \otimes IA_t \otimes IA_t & \rightarrow & A_t \otimes A_t \otimes A_t \end{array} \tag{7.6}$$

where

$$G = (1+\tau+\tau^2)[(\psi \otimes 1)(d_0 \otimes d_0)\psi + (\psi \otimes 1)\psi d_0].$$

To show the result, it is sufficient to show that

$$\partial G + G\partial = (1+\tau+\tau^2)[(\partial\psi + \psi\partial)\otimes d_0](\partial\psi + \psi\partial).$$

However,

$$\begin{aligned}\partial(\psi\otimes 1)(d_0\otimes d_0)\psi + (\psi\otimes 1)(d_0\otimes d_0)\psi\partial \\ &= (d_0\otimes d_0\otimes d_0)(\psi\otimes 1)(d_0\otimes d_0)\psi \\ &\quad + (\psi\otimes 1)(d_0\otimes d_0)(\partial\psi + \psi\partial)\end{aligned}$$

and

$$\partial(\psi\otimes 1)\psi d_0 + (\psi\otimes 1)\psi d_0\partial = (d_0\otimes d_0\otimes d_0)(\psi\otimes 1)\psi d_0.$$

The result now follows from the fact that

$$\partial\psi + \psi\partial = (d_0\otimes d_0)\psi + \psi d_0.$$

The Jacobi identity written down at the beginning of this section follows from the next result, using the commutativity of the product [ , ].

**Corollary 7.7:** If $A\in s\mathcal{A}$ and $x, y, z\in H_Q^*A$, then

$$[[x,y],z] + [[z,x],y] + [[y,z],x] = 0.$$

**Proof:** We may assume that $A$ is almost-free. Then the result follows from the sequence of chain equivalences

$$\begin{aligned}(1+\tau+\tau^2)(D_0\otimes 1)(\psi_A\otimes 1)D_0\psi_A \\ &= (1+\tau+\tau^2)(D_0\otimes 1)D_0(\psi_A\otimes d_0)\psi_A \\ &\simeq (D_0\otimes 1)D_0(1+\tau+\tau^2)(\psi_A\otimes d_0)\psi_A \\ &\simeq 0.\end{aligned}$$

Before proceeding, we remark that the maps constructed in the proof of 7.4 actually have more properties than we first demanded of them. Since

$$\psi_A : QA \to QA\otimes QA$$

is cocommutative, there is a diagram

$$\begin{array}{ccccc}
QA & \xrightarrow{\psi_A} & QA \otimes QA & \xrightarrow{\psi_A \otimes d_0} & (QA \otimes QA)^{\Sigma_2} \otimes QA \\
\downarrow = & & \downarrow = & & \downarrow \subseteq \\
QA & \xrightarrow{\psi_A} & QA \otimes QA & \xrightarrow{\psi_A \otimes d_0} & QA \otimes QA \otimes QA
\end{array}$$

and

$$\begin{array}{ccc}
(QA \otimes QA)^{\Sigma_2} \otimes QA & \xrightarrow{1+\tau+\tau^2} & (QA \otimes QA \otimes QA)^{\Sigma_3} \\
\downarrow \subseteq & & \downarrow \subseteq \\
QA \otimes QA \otimes QA & \xrightarrow{1+\tau+\tau^2} & QA \otimes QA \otimes QA
\end{array}$$

where $V^G$ are the invariants. The map

$$F : QA \to QA \otimes QA \otimes QA$$

of Lemma 7.4 actually restricts to a map

$$\bar{F} : QA \to (QA \otimes QA \otimes QA)^{\Sigma_3}. \tag{7.8.1}$$

so that

$$\partial \bar{F} + \bar{F} \partial = (1 + \tau + \tau^2)(\psi_A \otimes d_0)\psi_A. \tag{7.8.2}$$

To see this, consider the diagram (7.6). There

$$G = (1 + \tau + \tau^2)[(\psi \otimes 1)(d_0 \otimes d_0)\psi + (\psi \otimes 1)\psi d_0] : A_t \to A_t \otimes A_t \otimes A_t.$$

Since $\psi$ is cocommutative, $(\psi \otimes 1)(d_0 \otimes d_0)\psi$ and $(\psi \otimes 1)\psi d_0$ define maps

$$A_t \to (A_t \otimes A_t)^{\Sigma_2} \otimes A_t.$$

Hence, $G$ restricts to a map

$$\bar{G} : A_{t+1} \to (A_t \otimes A_t \otimes A_t)^{\Sigma_3}.$$

Therefore $\mathcal{H}$ (see 7.6) restricts to a map

$$\bar{\mathcal{H}} : IA_t \to (IA_t \otimes IA_t \otimes IA_t)^{\Sigma_3}.$$

And so $F$ restricts as required by (7.8.1). Since the invariants from a sub-vector space of $QA_t^{\otimes 3}$, (7.8.2) follows from 7.4.

We now approach the computation of

$$[x, P^i y] = [P^i y, x] \in H_Q^* A$$

for $x, y \in H_Q^* A$ and $i$ an integer. For this we must consider the composite chain map

$$\begin{aligned} C(QA) \xrightarrow{\psi_A} C(QA \otimes QA) \xrightarrow{D_0} & C(QA) \otimes C(QA) \\ & \xrightarrow{\psi_A \otimes 1} C(QA \otimes QA)^{\Sigma_2} \otimes C(QA) \\ & \xrightarrow{D_k \otimes 1} (C(QA) \otimes C(QA))_{\Sigma_2} \otimes C(QA) \end{aligned}$$

where $D_k$ is an appropriate higher Eilenberg-Zilber map and $V_G$ is the vector space of coinvariants. This composition, by 7.3, can be written as

$$(D_k \otimes 1) D_0 (\psi_A \otimes d_0) \psi_A : C(QA) \to (C(QA) \otimes C(QA))_{\Sigma_2} \otimes C(QA).$$

We can write this out as

$$\begin{aligned} C(QA) \xrightarrow{\psi_A} C(QA \otimes QA) \xrightarrow{\psi_A \otimes d_0} & C((QA \otimes QA)^{\Sigma_2} \otimes QA) \\ & \xrightarrow{D_0} C(QA \otimes QA)^{\Sigma_2} \otimes C(QA) \\ & \xrightarrow{D_k \otimes 1} (C(QA) \otimes C(QA))_{\Sigma_2} \otimes C(QA). \end{aligned}$$

To deal with this, we first state some generalities about simplicial vector spaces. Let $V$ be a simplicial vector space. Define, for every non-negative integer $k$,

$$D_k' = (1 + \tau + \tau^2)(D_k \otimes 1) D_0 : C(V \otimes V \otimes V) \to C(V) \otimes C(V) \otimes C(V)$$

and

$$D_k'' = (D_k \otimes 1) D_0 (1 + \tau + \tau^2) : C(V \otimes V \otimes V) \to C(V) \otimes C(V) \otimes C(V).$$

If $T : V \otimes V \to V \otimes V$ or $T : C(V) \otimes C(V) \to C(V) \otimes C(V)$ is the switch map, then using the Alexander-Whitney $D_0$ of (7.1), we compute that

$$(T \otimes 1) D_0 = D_0 (T \otimes 1) : C(V \otimes V \otimes V) \to C(V \otimes V) \otimes C(V)$$

and, hence, $D_0'$ and $D_0''$ are chain maps and for $k > 0$,

$$\partial D_k' + D_k'\partial = D_{k-1}' + (T \otimes 1)D_{k-1}'(T \otimes 1)$$

and

$$\partial D_k'' + D_k''\partial = D_{k-1}'' + (T \otimes 1)D_{k-1}''(T \otimes 1).$$

The method of acyclic models now demonstrates the existence of a map $E_k$ so that $E_0 = 0$ and for $k > 0$,

$$\partial E_{k+1} + E_{k+1}\partial = E_k + (T \otimes 1)E_k(T \otimes 1) + D_k' + D_k''.$$

Therefore, there is a chain homotopy between the chain maps

$$D_k', D_k'' : C((V \otimes V)^{\Sigma_2} \otimes V) \to (C(V) \otimes C(V))_{\Sigma_2} \otimes C(V). \tag{7.9}$$

**Theorem 7.10:** Let $A \in s\mathcal{A}$. For all integers $i$ and all $x, y \in H_Q^* A$,

$$[P^i y, x] = 0.$$

**Proof:** We may assume that $A$ is almost-free. Then it is sufficient to show that

$$(D_k \otimes 1)D_0(\psi_A \otimes 1)\psi_A : C(QA) \to (C(QA) \otimes C(QA))_{\Sigma_2} \otimes C(QA)$$

is null-homotopic for all $k$. Let $1 + \tau + \tau^2$ stand for the composite

$$C((QA{\otimes}QA)^{\Sigma_2}{\otimes}QA) \xrightarrow{1+\tau+\tau^2} C(QA{\otimes}QA{\otimes}QA)^{\Sigma_3} \xrightarrow{\subseteq} C((QA{\otimes}QA)^{\Sigma_2}{\otimes}QA).$$

By 7.8.2,

$$D_k''(\psi_A \otimes d_0)\psi_A = (D_k \otimes 1)D_0(1 + \tau + \tau^2)(\psi_A \otimes d_0)\psi_A$$

as a map from $C(QA)$ to $(C(QA) \otimes C(QA))_{\Sigma_2} \otimes C(QA)$ is null-homotopic. Hence, by 7.9,

$$D_k'(\psi_A \otimes d_0)\psi_A = (1 + \tau + \tau^2)(D_k \otimes 1)D_0(\psi_A \otimes d_0)\psi_A$$

is null-homotopic. Thus we have a chain homotopy

$$(D_k \otimes 1)D_0(\psi_A \otimes d_0)\psi_A \simeq (\tau + \tau^2)(D_k \otimes 1)D_0(\psi_A \otimes d_0)\psi_A$$

of maps from $C(QA)$ to $(C(QA) \otimes C(QA))_{\Sigma_2} \otimes C(QA)$. Now, for any simplicial vector space $V$, there is a diagram

$$\begin{array}{ccc}
(C(V) \otimes C(V))_{\Sigma_2} \otimes C(V) & \xrightarrow{=} & (C(V) \otimes C(V))_{\Sigma_2} \otimes C(V) \\
\downarrow {\scriptstyle (1+T)\otimes 1} & & \downarrow {\scriptstyle \tau+\tau^2} \\
C(V) \otimes C(V) \otimes C(V) & \xrightarrow{\tau} & (C(V) \otimes C(V))_{\Sigma_2} \otimes C(V).
\end{array}$$

Hence

$$\begin{aligned}
(D_k \otimes 1)D_0(\psi_A \otimes d_0)\psi_A &\simeq (\tau + \tau^2)(D_k \otimes 1)D_0(\psi_A \otimes d_0)\psi_A \\
&= \tau[(1 + T) \otimes 1](D_k \otimes 1)D_0(\psi_A \otimes d_0)\psi_A \\
&= \tau[(D_k + TD_k) \otimes 1]D_0(\psi_A \otimes d_0)\psi_A \\
&= \tau[(\partial D_{k+1} + D_{k+1}\partial) \otimes 1]D_0(\psi_A \otimes d_0)\psi_A.
\end{aligned}$$

Thus $(D_k \otimes 1)D_0(\psi_A \otimes d_0)\psi_A$ is null-homotopic.

## 8. Applications of Quillen's spectral sequence

We give a number of results that depend either on the spectral sequence of the previous section, or, perhaps, on the technical result 6.12. We postpone the proof of the Adem relations among the operations $P^i$ until the next section.

We begin with a Whitehead Theorem for the category $s\mathcal{A}$. Call a simplicial algebra $A$ *connected* if $\pi_0 A \cong \mathbf{F}_2$ generated by the unit in the algebra $\pi_* A$.

**Proposition 8.1:** Let $f : A \to B$ be a morphism between connected objects in $s\mathcal{A}$. Then $f$ is a weak equivalence if and only if

$$H_*^Q f : H_*^Q A \to H_*^Q B$$

is an isomorphism.

**Proof:** If $f$ is a weak equivalence, the $H_*^Q f$ is an isomorphism by 4.3.2. On the other hand, if $H_*^Q f$ is an isomorphism, then the spectral sequence of 6.2 implies that $f$ is a weak equivalence.

The next result is a Hurewicz theorem. But first, some notation. Let $A \in s\mathcal{A}$. Choose and acyclic fibration $X \to A$ with $X$ cofibrant. Then $\pi_* X \cong \pi_* A$. The projection from the augmentation ideal to the indecomposables

$$IX \to QX$$

induces, for every $A \in s\mathcal{A}$, a natural Hurewicz homomorphism

$$h_* : I\pi_* A \cong \pi_* IA \to H_*^Q A.$$

Since $\pi_n IA \cong \pi_n A$ for $n > 0$, we get maps for $n \geq 1$

(8.2.1) $$h_* : \pi_n A \to H_n^Q A$$

and, dually,

(8.2.2) $$h^* : H_Q^n A \to (\pi_n A)^* \cong \pi^n A^*.$$

It is a simple exercise with the definition of Quillen's spectral sequence to see that the edge homomorphism

$$I\pi_*A \to E_1^\infty A \subseteq E_1^1 A \cong H_*^Q A$$

is the Hurewicz homomorphisms $h_*$.

**Proposition 8.3:** Let $A \in s\mathcal{A}$. Then

1.) $h_*$ induces an isomorphism $Q\pi_0 A \cong H_0^Q A$, and

2.) if $A$ is connected and $\pi_k A = 0$ for $1 \leq k < n$, $h_*$ induces an isomorphism

$$h_* : \pi_n A \xrightarrow{\cong} H_n^Q A.$$

**Proof:** The first result is 4.10.1. The second follows immediately from Quillen's spectral sequence.

**Remark on universal examples:** Because of the naturality of the product and operations in $H_Q^*$, we can often prove general properties by considering specific examples. In Example 4.4.1 we produced an object $K(n)_+ \in s\mathcal{A}$ so that for all $A \in s\mathcal{A}$

$$[A, K(n)_+]_{s\mathcal{A}} \cong H_Q^n A.$$

Indeed, there must be a universal cohomology class $\iota_n \in H_Q^n K(n)_+$ so that this isomorphism is defined by

$$f \longmapsto H_Q^* f(\iota_n).$$

Furthermore $\pi_* K(n)_+ \cong \Lambda(x_n)$ — the exterior algebra on one generator of degree $n$.

Similarly, we can consider the object

$$K(m)_+ \times_{\mathbf{F}_2} K(n)_+ \cong (K(m) \times K(n))_+$$

in $s\mathcal{A}$. Then

$$\begin{aligned}[A, (K(m) \times K(n))_+]_{s\mathcal{A}} &\cong [A, K(m)_+]_{s\mathcal{A}} \times [A, K(n)_+]_{s\mathcal{A}} \\ &\cong H_Q^m A \times H_Q^n A.\end{aligned}$$

This isomorphism, too, can be made explicit. The two projections on factors

$$p_1 : (K(m) \times K(n))_+ \to K(m)_+$$

and

$$p_2 : (K(m) \times K(n))_+ \to K(n)_+$$

define classes $\iota_m \in H^m_Q(K(m) \times K(n)_+$ and $\iota_n \in H^n_Q(K(m) \times K(n)_+$ respectively and the isomorphism is given by

$$f \longmapsto (H^*_Q f(\iota_m), H^*_Q f(\iota_n)).$$

In addition, operations applied to the pair $(\iota_m, \iota_n)$ are universal operations in two variables; for example, $[\iota_n, \iota_m]$ is the universal product in $H^*_Q$.

The projections $p_1$ and $p_2$ have sections that are often of useful. For example, define, in $s\mathbf{F}_2$

$$\bar{f}_1 : K(m) \to K(m) \times K(n)$$

to be the inclusion on the first factor:

$$\bar{f}_1(x) = (x, 0).$$

Then $\bar{f}_1$ induces a map

$$f_1 : K(m)_+ \to (K(m) \times K(n))_+$$

so that $p_1 f_1$ is the identity and $H^*_Q f_1(\iota_n) = 0$. Similarly, there is a map

$$f_2 : K(n)_+ \to (K(m) \times K(n))_+$$

so that $p_2 f_2$ is the identity and $H^*_Q f_2(\iota_m) = 0$.

For contrast to 8.3, we note that the product and operations in $H^*_Q$ are not detected by cohomotopy. Let $h^*$ be as in 8.2.2.

**Proposition 8.4:** Let $A \in s\mathcal{A}$. For all $x, y \in H^*_Q A$

$$h^*[x, y] = 0$$

and

$$h^*P^i(x) = 0.$$

**Proof:** It is sufficient to show this for the universal examples. For the product, let $x \in H^m_Q A$ and $y \in H^n_Q A$. Let $K(n)_+$ be the universal example of 4.4.1. and $\iota_n \in H^n_Q K(n)_+$ the universal cohomology class of degree $n$. Then there are unique maps

$$f : A \to K(m)_+$$

and

$$g : A \to K(n)_+$$

in the homotopy category associated to $s\mathcal{A}$ so that

$$H^*_Q f(\iota_m) = x \qquad \text{and} \qquad H^*_Q g(\iota_n) = y.$$

The maps $f$ and $g$ induce a map

$$f \times g : A \to K(m)_+ \times_{\mathbf{F}_2} K(n)_+ \cong (K(n) \times K(m))_+$$

and $[x, y]$ can be computed by the equations

$$H^Q_*(f \times g)[\iota_m, \iota_n] = [x, y]$$

where $[\iota_m, \iota_n] \in H^Q_*(K(m) \times K(n))_+$ is the product of $\iota_m$ and $\iota_m$ under the inclusions

$$H^*_Q K(m)_+ \to H^*_Q (K(m) \times K(n))_+$$

and

$$H^*_Q K(n)_+ \to H^*_Q (K(m) \times K(n))_+.$$

The naturality of the Hurewicz map shows that it is only necessary to demonstrate that $h^*[\iota_m, \iota_n] = 0$. However

$$[\iota_m, \iota_n] \in H^Q_{n+m+1} K(m) \times K(n))_+$$

and

$$\pi_{n+m+1}(K(m) \times K(n))_+ = 0.$$

The assertion about the operation is proved in the same manner.

This does not mean that the product and operations are identically zero. In fact, we have the following result. Let $A = (K(m) \times K(n))_+$ be the universal example of 4.4.1. Furthermore, let $\iota_m \in H_Q^m A$ and $\iota_n \in H_Q^m A$ be the image of the universal cohomology classes under the maps induced in cohomology by the projections

$$A = (K(m) \times K(n))_+ \longrightarrow K(m)_+$$

and

$$A = (K(m) \times K(n))_+ \longrightarrow K(n)_+.$$

**Proposition 8.5:** Let $m, n \geq 0$. Then

$$[\iota_m, \iota_n] \neq 0 \in H_Q^{m+n+1}(K(n) \times K(m))_+.$$

**Proof:** We first assume that $n > 0$ or $m > 0$. If $n = 0$ or $m = 0$, we have not proved that Quillen's spectral sequence converges for this example; however, formulas 6.9 holds and this is all we will use. Let $A = (K(n) \times K(m))_+$. If $u \in \pi_* K(m)$ and $v \in \pi_* K(n)$ are the non-zero classes, then $\pi_* A \cong \Lambda(u,v)/(uv)$, where $\Lambda(\ )$ denotes the exterior algebra. The inclusion

$$K(m)_+ \longrightarrow (K(m) \times K(n))_+$$

provides a section for the projection above and show that if

$$h_* : I\pi_* A \longrightarrow H_*^Q A$$

is the Hurewicz map, then $h_* u \neq 0$. Indeed $\langle \iota_m, h_* u\rangle \neq 0$ in the pairing between $H_Q^* A$ and $H_*^Q A$. Similarly, $\langle \iota_n, h_* v\rangle \neq 0$. Thus in Quillen's spectral sequence

$$E^1 A \cong \mathfrak{S}(H_*^Q A) \Rightarrow \pi_* A$$

the product class $0 \neq h_* u \cdot h_* v \in E_2^1 A$ survives to $E^\infty A$. This is because we have a spectral sequence of algebras. However, $uv = 0$ in $\pi_* A$. So there must be a class

$$z \in E^1_{1,m+n+1} \cong H^Q_{m+n+1} A$$

so that

$$d_1 z = h_* u \cdot h_* v.$$

Since $n + m > 0$, 6.9 implies that under the coproduct

$$(\psi_A)_* : H_*^Q A \to H_*^Q A \otimes H_*^Q A$$

we have that

$$(\psi_A)_*(z) = h_* u \otimes h_* v + h_* v \otimes h_* u.$$

Since the coproduct is non-trivial, the product (which is dual) must also be non-trivial. In particular, because of the pairings above, $[\iota_m, \iota_n] \neq 0$.

In the case $n = m = 0$, we must appeal directly to Lemma 6.10, instead of 6.9. However, that is sufficient to complete the argument along the same lines above.

Next we decide that some of the operations are identically 0.

**Proposition 8.6:** Let $x \in H_Q^n A$ for $n \geq 0$ and let $i \leq 0$. Then $P^i(x) = 0$.

**Proof:** We may assume that $A$ is almost-free. If $x$ is represented by a cocycle $\alpha \in QA^*$, then $P^i(x)$ is the residue class of

$$(\psi_A)_* D_{n-i}^*(\alpha \otimes \alpha).$$

With a good choice of the higher Eilenberg-Zilber maps $D_k$, such as the choice given by Singer in [S] (see before 3.10), then we know that

$$D_k(\alpha \otimes \alpha) = 0, \qquad k > n.$$

So we have the result for $i < 0$. For $i = 0$, we note that this choice of Eilenberg-Zilber maps also has

$$D_n(\alpha \otimes \alpha) = \alpha \otimes \alpha.$$

So we can show that $(\psi_A)_*(\alpha \otimes \alpha) = 0$ to complete the result. Combining 6.11 and 6.12 and dualizing, we get a diagram

$$\begin{array}{ccc} QA^* \otimes QA^* & \xrightarrow{(\psi_A)^*} & QA^* \\ \downarrow & & \downarrow p^* \\ IA^* \otimes IA^* & \longrightarrow & IA^* \\ \uparrow & & \uparrow \\ A^* \otimes A^* & \xrightarrow{(\partial\psi + \psi\partial)^*} & A^*. \end{array}$$

Since $A^* \longrightarrow IA^*$ is a split surjection and $p^*$ is an injection, it is sufficient to show that

$$\partial^*\psi^*(v \otimes v) + \psi^*\partial^*(v \otimes v) = 0$$

for all $v \in A^*$. Because we are assuming that $A$ is almost-free, we have that $A_t \cong S(V_t)$ for some vector space $V_t$. Indeed, $A_t$ is a primitively generated Hopf algebra. Thus, $A_t^*$ is an exterior algebra with multiplication given by $\psi^*$. Thus $\psi^*(v \otimes v) = 0$. Also

$$\psi^*\partial^*(v \otimes v) = \sum \psi^*(d_i^* v \otimes d_i^* v) = 0.$$

This proves the result.

The previous result combined with the next result shows that we can claim that $P^i = 0$ for all $i < 2$.

**Proposition 8.7:** Let $A \in s\mathcal{A}$ and $x \in H^n_Q A$. Then $P^1(x) = 0$.

**Proof:** For $n = 0$ this follows from 5.11. So assume that $n \geq 1$. It is enough to consider the universal class

$$x = \iota_n \in H^n_Q K(n)_+$$

where $K(n)_+$ is the universal example of 4.4.1. Then $\pi_* K(n)_+ \cong \Lambda(u)$ where $u \in \pi_n K(n)_+$. In Quillen's spectral sequence

$$E^1 K(n)_+ \cong \mathfrak{S}(H^Q_* K(n)_+) \Rightarrow \pi_* K(n)_+$$

one easily see that

$$E^1_{s,t} K(n)_+ \cong \mathfrak{S}_s(H^Q_* K(n)_+)_t = 0$$

of $s > 1$ and $t < n + 2$. For this one can use 3.7. If $P^1(x) \neq 0$, then there would be a non-zero class $z \in H^Q_{n+2} K(n)_+ \cong E^1_{1,n+2} K(n)_+$. Since $\pi_{n+2} K(n)_+ = 0$, there would have to be some $r$ so that $d_r z \neq 0$ in Quillen's spectral sequence. Since this cannot happen, we must have $P^1(x) = 0$.

We can also demonstrate that, often, the operations $P^i$ are non-trivial.

**Proposition 8.7:** Let $\iota_n \in H^n_Q K(n)_+$ be the universal class. If $2 \leq i \leq n$ then

$$P^i(\iota_n) \neq 0$$

in $H^*_Q K(n)_+$.

**Proof:** If $n < 2$ then the statement is vacuous. So we assume that $n \geq 2$. Let $u \in E^1_{1,n} K(n)_+ \cong H^Q_n K(n)_+$ be dual to $\iota_n$. The class $u$ survives to $E^\infty K(n)_+$ in Quillen's spectral sequence; therefore, Proposition 6.5 now implies that

$$\delta_i(u) \in E^1_{2,n+i} K(n)_+ \cong \mathfrak{S}_2(H^Q_* K(n)_+)$$

also survives to $E^\infty K(n)_+$. In addition, $\delta_i(u) \neq 0$ in $E^1 K(n)_+$. Since $\pi_* K(n)_+ \cong \Lambda(u)$, there must exist a class $y \in E^1_{1,n+i+1} K(n)_+$ so that $d_1 y = \delta_i(u)$. But

$$E^1_{1,n+i+1} K(n)_+ \cong H^Q_{n+i+1} K(n)_+$$

and Proposition 6.9 implies that $yP^i = u$. Dualizing, we obtain that

$$P^i(\iota_n) \neq 0.$$

## 9. The proof of the Adem relations.

We now come to the proof of the Adem relations among the operations $P^i$. This is a further application of Quillen's spectral sequence.

**Theorem 9.1:** For $A \in s\mathcal{A}$, $x \in H_Q^* A$, and $i \geq 2j$ there is an equation

$$P^i P^j(x) = \sum_{s=i-j+1}^{i+j-2} \binom{2s-i-1}{s-j} P^{i+j-s} P^s(x).$$

Before starting the proof, we make some observations about Quillen's spectral sequence.

The first is that the following is a special case of Proposition 6.5. If $A \in s\mathcal{A}$ and $x \in E^1 A$ has the property that $d_1 x = 0$, let $[x]$ be the residue class of $x$ in $E^2 A$.

**Lemma 9.2:** Let $x \in E^1_{s,t} A$ and $d_1 x = y \in E^1_{s+1,t-1}$. Then

$$d_1 \delta_t(x) = xy$$

and for $2 \leq i < t$, $d_1 \delta_i(x) = 0$ and

$$d_2[\delta_i(x)] = [\delta_i(y)].$$

**Proof:** This follows from the definition of the $\delta_i$ in the spectral sequence (see before 6.5) and the formulas of 3.3.

The second observation is a remark on compositions

$$\delta_i \delta_j : E^1_1 A \to E^1_4 A.$$

The computations of 2.7 and 3.5 imply the following. If we have $x \in E^1_{s,t} A$, let the degree be given by the formula $deg(x) = t$.

**Lemma 9.3:**1.) Let $\{\delta_i \delta_j(w_{ij})\} \subseteq E^1_{4,t} A$ be a set of elements so that $i \geq 2j$ and $i - j \leq deg(w_{i,j})$. Then this is a linearly independent set; that is, and equation

$$\sum \delta_i \delta_j(w_{ij}) = 0$$

implies that $\delta_i\delta_j(w_{ij}) = 0$ for all pairs $(i,j)$ under consideration.

2.) $\delta_i\delta_j : E^1_{1,t} \to E^1_{4,t+i+j}A$ with $i \geq 2j$ and $i - j < t$ is an injective homomorphism.

**Remark 9.4:** A consequence of this result is that if we have an equation

$$\sum \delta_i\delta_j(w_{ij}) = 0$$

where the sum is over a set of pairs $(i,j)$ so that $i \geq 2j$ and $i-j < deg(w_{ij})$, then $w_{ij} = 0$ for all pairs $(i,j)$ under considerations. It will be our business to produce such an equation.

We can do a little better. Since the product

$$E^1_1A \otimes E^1_2A \cong \mathfrak{S}_1(H^Q_*A) \otimes \mathfrak{S}_2(H^Q_*A) \to \mathfrak{S}_3(H^Q_*A) \cong E^1_3A$$

is onto, and since Quillen's spectral sequence is a spectral sequence of algebras, we can conclude that if $y \in E^1_3A$, then $d_1y \in E^1_4A$ is decomposable in the algebra $E^1A$. However, $\delta_i\delta_j(x)$ is always indecomposable in $E^1_4A$. Combining this observation, 9.2, and 9.4, we have the following result.

**Lemma 9.5:** Given an equation in $E^2_4A$

$$\sum[\delta_i\delta_j(w_{ij})] = 0$$

where the sum is over pairs $(i,j)$ so that $i \geq 2j$ and $i - j < deg(w_{ij})$, then $w_{ij} = 0$ for all pairs $(i,j)$ under consideration.

The third observation is a reduction. It is not necessary to prove Theorem 9.1 for all $x \in H^*_QA$ and $A \in s\mathcal{A}$, but merely for the universal example $x = \iota_n \in H^n_QK(n)_+$. Define, for $i \geq 2j$ and $A \in s\mathcal{A}$, a homomorphism

$$g_{ij} : H^n_QA \to H^{n+i+j+2}_QA$$

by

$$g_{ij} = P^iP^j + \sum_{s=i-j+1}^{i+j-2} \binom{2s-i-1}{s-j} P^{i+j-s}P^s.$$

**Lemma 9.6:** For all $n$ and $i \geq 2j$

$$g_{ij} = 0 : H_Q^n K(n)_+ \to H_Q^{n+i+j+2} K(n)_+.$$

We give the proof after we supply the following.

**Proof of 9.1:** If $x \in H_Q^n A$ with $n \leq 1$, then the conclusion of Theorem 9.1 holds since both sides of the equation are zero. This follows from 5.11, 9.6 and 9.7. So assume that $n \geq 2$. Then there is a unique map in the homotopy category associated to $s\mathcal{A}$

$$f : A \to K(n)_+$$

so that $H_Q^* f(\iota_n) = x$. Since $g_{ij}(\iota_n) = 0$, we have

$$(g_{ij})(x) = H_Q^* f(g_{ij}(\iota_n)) = 0.$$

The result follows.

**Proof of 9.6:** To avoid intricacies involving the top operations $\delta_i$ and to eliminate some of the summands of the formula given in 6.9, we use the suspension. Combining 4.8 and 5.13, we see that it is sufficient to prove that

$$0 = g_{ij} : H_Q^{n+1} \Sigma K(n)_+ \to H_Q^{n+i+j+3} \Sigma K(n)_+.$$

There is a dual homomorphism, we we will write of the right

$$(\cdot) f_{ij} : H^Q_{n+i+j+3} \Sigma K(n)_+ \to H^Q_{n+1} \Sigma K(n)_+$$

given by, for $i \geq 2j$,

$$(\cdot) f_{ij} = (\cdot) P^i P^j + \sum_{s=i-j+1}^{i+j-2} \binom{2s-i-1}{s-j} (\cdot) P^{i+j-s} P^s$$

and it is sufficient to show that $f_{ij} = 0$. Consider Quillen's spectral sequence and let

$$x \in H^Q_{n+i+j+3} \Sigma K(n)_+ \cong E^1_{1,n+i+j+3} \Sigma K(n)_+.$$

We compute, using 5.13 and 6.9, that

$$d_1 x = \sum_{i \geq 2} \delta_i (xP^i).$$

Here we use that all products vanish in $H_Q^* \Sigma K(n)_+$ so that the coproduct must be trivial in $H_*^Q \Sigma K(n)_+$. Using 9.2 and the fact that the residue class of $d_1 x$ in $E^2 \Sigma K(n)_+$ is zero, we have

$$\begin{aligned} 0 = d_2(\sum_{i \geq 2} [\delta_i (xP^i)]) &= \sum_{i \geq 2} [\delta_i (d_1 (xP^i))] \\ &= \sum_{i \geq 2} \sum_{j \geq 2} [\delta_i \delta_j (xP^i P^j)]. \end{aligned}$$

Here we use that the top operation $P^i$ vanishes in $H_Q^* \Sigma K(n)_+$. Thus, using Proposition 6.7, we can write

$$\begin{aligned} 0 = &\sum_{i \geq 2j \geq 4} [\delta_i \delta_j (xP^i P^j)] \\ &+ \sum_{2 \leq i < 2j} \; \sum_{i+1/2 \leq k \leq i+j/3} \binom{j-i+s-1}{j-s} [\delta_{i+j-k} \delta_k (xP^i P^j)]. \end{aligned}$$

Substituting $i = a$, and $j = b$ into the first summand and $j = s$, $k = b$, and $i = a + b - s$ into the second summand, we find that

$$0 = \sum_{a \geq 2b \geq 4} [\delta_a \delta_b (x f_{ab})].$$

Here we use that in $H_Q^* \Sigma K(n)_+$, if $xf_{ab} \neq 0$, then $a - b < deg(xf_{ab})$. This follows from the fact that the top operation $P^i$ vanishes in the cohomology of suspension. Finally Lemma 9.5 implies that for $a \geq 2b$

$$x f_{ab} = 0$$

in $H_*^Q \Sigma K(n)_+$. In particular $xf_{ij} = 0$ and we have proved the result.

## 10. Projective extension sequences and a quadratic operation

In this section we define and discuss a natural quadratic operation

$$\beta : H^0_Q A \to H^1_Q A.$$

This operation will have the property that

$$\beta(x+y) = \beta(x) + \beta(y) + [x, y] \tag{10.1.1}$$

where [ , ] is the product on $H^*_Q A$ and that

$$[\beta(y), x] = [y, [y, x]] \tag{10.1.2}$$

for all $x \in H^*_Q A$ and $y \in H^0_Q A$.

The best way to define this operations is first to consider the André-Quillen cohomology of certain algebras, and to use that calculation in the definition. So we begin with a general computation. Let $V_0, V_1$ be vector spaces — not simplicial vector spaces — and

$$f : S(V_0) \to S(V_1)$$

a map of algebras so that

10.2) $S(V_1)$ is a projective $S(V_0)$ module.

Note that 10.2 implies that the morphism $f$ is an injection.

We are interested in computing the cohomology of the algebra

$$\Lambda = \mathbf{F}_2 \otimes_{S(V_0)} S(V_1).$$

That is, we regard $\Lambda$ as a constant simplicial algebra and compute $H^Q_* \Lambda$. We will say that $\Lambda$ is defined by a projective extension sequence and we will often write that there is a projective extension sequence of algebras

$$\mathbf{F}_2 \to S(V_0) \to S(V_1) \to \Lambda \to \mathbf{F}_2$$

to indicate that this is the case.

To begin the computation, regard $S(V_0)$ and $S(V_1)$ as constant simplicial algebras. Then, since $S(V_0)$ and $S(V_1)$ are almost-free, we have that

$$H^Q_0 S(V_0) \cong V_0 \qquad \text{and} \qquad H^Q_0 S(V_1) \cong V_1$$

and

$$H_i^Q S(V_0) = 0 = H_i^Q S(V_1)$$

for $i > 0$. Compare 4.10.3. If we factor $f : S(V_0) \to S(V_1)$ as a cofibration followed by an acyclic fibration

$$S(V_0) \to X \to S(V_1),$$

then we get a diagram

$$\begin{array}{ccccccc} S(V_0) & \to & X & \to & \mathbf{F}_2 \otimes_{S(V_0)} X & & \\ \downarrow = & & \downarrow & & \downarrow p & & \\ S(V_0) & \xrightarrow{f} & S(V_1) & \to & \mathbf{F}_2 \otimes_{S(V_0)} S(V_1) & = & \Lambda. \end{array}$$

Proposition 4.7 and the hypothesis of 10.2 imply that $p$ is a weak equivalence. Since $\mathbf{F}_2 \otimes_{S(V_0)} X$ is the homotopy cofiber of $f$, Proposition 4.6 yields an exact sequence

$$0 \to H_1^Q \Lambda \xrightarrow{\partial} V_0 \xrightarrow{H_*^Q f} V_1 \to H_0^Q \Lambda \to 0. \tag{10.3}$$

For example, if $\Lambda(x)$ is the exterior algebra on a single generator $x$, there is a projective extension sequence

$$\mathbb{F}_2 \to \mathbb{F}_2[y] \xrightarrow{f} \mathbb{F}_2[x] \to \Lambda(x) \to \mathbb{F}_2$$

where $f(y) = x^2$. Hence

$$H_0^Q \Lambda(x) \cong \mathbb{F}_2 \cong H_1^Q \Lambda(x) \tag{10.4}$$

and

$$H_i^Q \Lambda(x) = 0$$

for $i > 1$.

It is instructive to follow in Quillen's [21] and Bousfield's [6] footsteps and to compute Quillen's spectral sequence for an algebra $\Lambda$ defined by a projective extension sequence. To be precise, notice that if we use the suspension functor $\Sigma$ defined in section 4, then (4.9) implies that

$$\pi_* \Sigma \Lambda \cong Tor_*^{\Lambda}(\mathbb{F}_2, \mathbb{F}_2).$$

For $A \in s\mathcal{A}$, let $Q(\pi_* A)_n$ denote the indecomposables of degree $n$ in the graded algebra $\pi_* A$.

**Proposition 10.5:** Let $\Lambda$ be defined by a projective extension sequence. Then Quillen's spectral sequence

$$\mathfrak{S}(H_*^Q \Sigma\Lambda) \Rightarrow Tor_*^\Lambda(\mathbf{F}_2, \mathbf{F}_2)$$

collapses and

$$H_0^Q \Lambda \cong Tor_1^\Lambda(\mathbf{F}_2, \mathbf{F}_2)$$

and

$$H_1^Q \Lambda \cong Q(Tor_*^\Lambda(\mathbf{F}_2, \mathbf{F}_2))_2.$$

**Proof:** This follows from the isomorphism $H_i^Q \Lambda \cong H_{i+1}^Q \Sigma\Lambda$ and Proposition 6.5. In fact, we can write down an isomorphism of divided power algebras

$$Tor_*^\Lambda(\mathbf{F}_2, \mathbf{F}_2) \cong E(H_0^Q \Lambda) \otimes \Gamma(H_1^Q \Lambda)$$

where $E$ and $\Gamma$ denote the graded exterior and divided power algebra respectively.

The proof of 10.5 suggests the following generalization, due essentially to André [1]. Let $A \in s\mathcal{A}$ and consider the composition

$$(I\pi_* \Sigma A)_n \cong \pi_n I \Sigma A \xrightarrow{h_*} H_{n+1}^Q \Sigma A \xrightarrow{\cong} H_n^Q A.$$

Call this composite $g_*$. The following is proved by examining Quillen's spectral sequence.

**Proposition 10.6:** $g_*$ induces natural isomorphisms

$$\pi_1 \Sigma A \xrightarrow{\cong} H_0^Q A$$

and

$$Q(\pi_* \Sigma A)_2 \xrightarrow{\cong} H_1^Q A.$$

We now define the operation $\beta$, using the method of the universal example. If $x \in H^0_Q A$, then there is a unique morphism in the homotopy category associated to $s\mathcal{A}$

$$f : A \to K(0)_+$$

so that if $\iota_0 \in H^0_Q K(0)_+$ is the universal class, then $H^*_Q f(\iota_0) = x$. Thus, to define a natural operations

$$\beta : H^0_Q A \to H^1_Q A$$

it is sufficient to stipulate $\beta(\iota_0) \in H^1_Q K(0)_+$. However $K(0) \in s\mathbf{F}_2$ is the constant simplicial vector space that is $\mathbf{F}_2$ in each simplicial degree; hence, $K(0)_+ \cong \Lambda(x)$, where the exterior algebra on the generator x is regarded as a constant simplicial algebra. Thus (10.4) implies that $H^0_Q K(0)_+ \cong \mathbf{F}_2 \cong H^1_Q K(0)_+$ and $H^i_Q K(0)_+ = 0$ for $i > 1$. The non-zero class

$$\iota_0 \in H^0_Q K(0)_+$$

is the universal cohomology class of degree 0; let

$$\beta(\iota_0) \in H^1_Q K(0)_+ \tag{10.7}$$

be the non-zero class of degree 1. This defines the operation $\beta$.

To discuss the operations $\beta$, we note that it can be detected in homotopy. Or, to be precise, consider the dual Hurewicz homomorphism

$$h^* : H^*_Q \Sigma A \to (I\pi_*\Sigma A)^*.$$

Since $\pi_*\Sigma A$ is a Hopf algebra (see before 4.9), we have that $\pi^*\Sigma A^*$ is a Hopf algebra, at least when $\pi_* A$ is of finite type. Also, since $H^0_Q \Sigma A = 0$, we have a factoring of $h^*$ as

$$H^*_Q \Sigma A \xrightarrow{\bar{h}^*} \pi^*\Sigma A^* \to (I\pi_*\Sigma A)^*.$$

**Proposition 10.8:** Let $A \in s\mathcal{A}$ be so that $\pi_* A$ is of finite type. Let $x \in H^0_Q A \cong H^1_Q \Sigma A$ and $\beta(x) \in H^1_Q A \cong H^2_Q \Sigma A$. Then

$$\bar{h}^*\beta(x) = (\bar{h}^* x)^2$$

in $\pi^*\Sigma A^*$.

**Remark:** Since $\bar{h}^* : H^1_Q\Sigma A \to \pi^1\Sigma A^*$ is an isomorphism and $\bar{h}^* : H^2_Q\Sigma A \to \pi^2\Sigma A^*$ is an injection (both because of 10.6), this is an effective method for computing $\beta$.

**Proof:** Since a map $f : A \to B$ in $s\mathcal{A}$ induces a map of Hopf algebras

$$\pi^*\Sigma f : \pi^*\Sigma A \to \pi^*\Sigma B$$

it is only necessary to show this result for the universal example $A = K(0)_+$. However,

$$\begin{aligned} \pi^*\Sigma K(0)_+ &\cong [Tor_*^{\Lambda(x)}(\mathbf{F}_2, \mathbf{F}_2)]^* \\ &\cong Ext^*_{\Lambda(x)}(\mathbf{F}_2, \mathbf{F}_2) \cong \mathbf{F}_2[y] \end{aligned}$$

where $y \in Ext^1_{\Lambda(x)}(\mathbf{F}_2, \mathbf{F}_2)$. We apply 10.5. Since $\bar{h}^*\iota_0 = y$ and $\bar{h}^*\beta(\iota_0) \neq 0$, we must have that

$$\bar{h}^*\beta(\iota_0) = y^2.$$

We can now prove the following.

**Proposition 10.9:** Let $A \in s\mathcal{A}$ and $x, y \in H^0_QA$. Then

$$\beta(x+y) = \beta(x) + \beta(y) + [x, y].$$

**Proof:** It is sufficient to do this for the universal example

$$A = (K(0) \times K(0))_+$$

and $x = \iota^1$ and $y = \iota^2$ where these classes are the two cohomology classes induced from the universal cohomology class by the two under the two projections

$$(K(0)_+ \times K(0))_+ \to K(0)_+.$$

We claim that $H^1_QA$ is of dimension 3 over $\mathbf{F}_2$ and that a basis for $H^1_QA$ is given by the three elements $\beta(i^1), \beta(\iota^2)$, and $[\iota^1, \iota^2]$. To see this and to prepare for the rest of the argument, we argue as follows. $\pi^*A \cong \Lambda(u, v)/(uv)$ and, using this fact, it was proved in [12] that there is an isomorphism of Hopf algebras

$$\pi^*\Sigma A \cong Ext^*_{\pi^*A}(\mathbf{F}_2, \mathbf{F}_2) \cong T(y_1, y_2)$$

where $T(\ )$ denotes the primitively generated tensor algebra. Furthermore,

$$\bar{h}^* : H^0_Q A \to Ext^1_{\pi_* A}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism and $\bar{h}^* \iota^1 = y_1$ and $\bar{h}^* \iota^2 = y_2$. Dualizing 10.6 and letting $P(\ )$ denote the primitives we have that

$$\bar{h}^* : H^1_Q A \to P(Ext^*_{\pi_* A}(\mathbb{F}_2, \mathbb{F}_2))^2$$

is an isomorphism. Therefore, $H^1_Q A$ is of dimension 3. The previous result now implies that

$$\bar{h}^* \beta(\iota^1) = y_1^2 \qquad \text{and} \qquad \bar{h}^* \beta(\iota^2) = y_2^2.$$

Next consider the two projections

$$H^*_Q(K(0) \times K(0))_+ \to H^*_Q K(0)_+$$

induced by the two inclusions onto factors

$$K(0) \to K(0) \times K(0).$$

The element $[\iota^1, \iota^2] \in H^*_Q A$ goes to zero under both these projections. On the other hand, $[\iota^1, \iota^2]$ in non-zero, by 10.5. Thus we have

$$\bar{h}^*[\iota^1, \iota^2] = y_1 y_2 + y_2 y_1$$

in $Ext^*_{\pi_* A}(\mathbb{F}_2, \mathbb{F}_2)$. Now we compute using 10.8:

$$\begin{aligned} \bar{h}^* \beta(\iota^1 + \iota^2) &= (y_1 + y_2)^2 \\ &= y_1^2 + y_2^2 + y_1 y_2 + y_2 y_1 \\ &= \bar{h}^*(\beta(\iota^1) + \beta(\iota^2) + [\iota^1, \iota^2]). \end{aligned}$$

Since $\bar{h}^*$ is an injection on $H^1_Q A$, the result follows.

Finally, we append here the following result, although, in its proof, we will reference results of later sections.

**Proposition 10.10:** Let $A \in s\mathcal{A}$, $x \in H^n_Q A$, $y \in H^0_Q A$. Then

$$[\beta(y), x] = [y, [y, x]].$$

**Proof:** Again, it is sufficient to examine the universal example

$$A = (K(0) \times K(n))_+$$

with $y = \iota_0$ and $x = \iota_n$. In [12] it was proved that there is an isomorphism of Hopf algebras

$$\pi^*\Sigma A \cong T(y_1, y_2)$$

where $T(\ )$ is the primitively generated Hopf algebra and

$$\begin{aligned} y_1 &= \bar{h}^*\iota_0 \in \pi^1\Sigma A \\ y_2 &= \bar{h}^*\iota_n \in \pi^{n+1}\Sigma A \end{aligned}$$

We are still using the Hurewicz map $\bar{h}^*$ given by the composition

$$H^m_Q A \cong H^{m+1}_Q A \longrightarrow \pi^{m+1}\Sigma A^*.$$

Now we will prove in 10.8 that

$$\bar{h}^*[x, y] = \bar{h}^*(x)\bar{h}^*(y) + \bar{h}^*(y)\bar{h}^*(x) \tag{10.11}$$

in $\pi^*\Sigma A^*$. Furthermore in 10.11 we will prove that for $A = (K(0) \times K(n))_+$

$$H^{n+3}_Q \Sigma A \cong \mathbf{F}_2.$$

Thus to prove the result it is sufficient to prove that

$$\begin{aligned} \bar{h}^*[\iota_0[\iota_0, \iota_n]] &\neq 0 \\ \bar{h}^*[\beta\iota_0, \iota_n] &\neq 0. \end{aligned}$$

But we compute, using 10.5 and 10.11, that

$$\begin{aligned} \bar{h}^*[\beta\iota_0, \iota_n] &= \bar{h}^*(\beta\iota_0)\bar{h}^*(\iota_n) + \bar{h}^*(\iota_n)\bar{h}^*(\beta\iota_0) \\ &= y_1^2 y_2 + y_2 y_1^2 \\ &= \bar{h}^*[\iota_0, [\iota_0, \iota_n]]. \end{aligned}$$

The result follows.

# Chapter V: The Cohomology of Abelian Objects

In section 4 we determined that all abelian simplicial algebras in our category $s\mathcal{A}$ were of the form $V_+$ where $V$ is a simplicial $\mathbf{F}_2$-vector space and $(\ )_+$ is the trivial algebra functor. In addition, we computed that for $A \in s\mathcal{A}$,

$$[A, V_+]_{s\mathcal{A}} \cong Hom_{n\mathbf{F}_2}(H_*^Q A, \pi_* V).$$

Thus the objects $V_+ \in s\mathcal{A}$ behave exactly as Eilenberg-MacLane spaces do in the category of spaces. Of particular interest are the cases where $V = K(n)$ and $V = K(m) \times K(n)$. Recall that $\pi_* K(n) \cong \mathbf{F}_2$ concentrated in degree $n$. Then

$$\begin{aligned} [A, K(n)_+]_{s\mathcal{A}} &\cong H_Q^n A \\ [A, (K(m) \times K(n))_+]_{s\mathcal{A}} &\cong H_Q^m A \times H_Q^n A. \end{aligned}$$

Thus $H_Q^* K(n)_+$ and $H_Q^*(K(m) \times K(n))_+$ represent, respectively, all cohomology operations of one and two variables.

Our computation of $H_Q^* V_+$ will proceed in stages. In section 11 we will compute $H_Q^* V_+$ for for a finite product of $K(n)$'s:

$$V = K(n_1) \times K(n_2) \times \cdots \times K(n_k)$$

assuming the computation of $H_Q^* K(n)_+$ for all $n$. It will be the purpose of the remaining sections to do the latter computation.

## 11. Applications of a Hilton-Milnor Theorem

The purpose of this section is to give some computations of $H_Q^* V_+$, assuming that we have computed $H_Q^* K(n)_+$ for all $n$. Along the way we will draw some corollaries about the structure of $H_Q^* A$ for general $A \in s\mathcal{A}$.

To be up front with what will be postponed, let $A \in s\mathcal{A}$ and $x \in H^n_Q A$. Then we know that the cohomology operations $P^i$ have the property that if $i > n$, then $P^i(x) = 0$. In addition, consider the composition of operations

$$P^I = P^{i_1} P^{i_2} \ldots P^{i_k}$$

defined for some sequence of integers $I = (i_1, i_2, \ldots, i_k)$. $I$ might be empty, in which case $P^I$ is the identity. Call such a composition *allowable* if $i_t < 2i_{t+1}$ for all $t$. The empty composition is allowable. If $I$ is not allowable then the relations of Theorem 9.1 permit us to rewrite $P^I$ as a sum of allowable compositions. Finally, if $P^I$ is allowable and $i_s \leq n$, then

$$i_t \leq i_{t+1} + \cdots + i_s + n$$

for all $t$. Thus, if $x \in H^n_Q A$, the set of elements

$$P^I(x) = P^{i_1} \cdots P^{i_s}(x)$$

with $P^I$ allowable, $i_t \geq 2$, and $i_k \leq n$ span the sub-vector space of $H^*_Q A$ generated by $P^J(x)$, with $P^J$ running over all possible compositions of the operations $P^i$. Notice that since $I$ may be empty, $x$ itself is in this sub-vector space.

One of the principal results of this paper is that the allowable $P^I$ with $i_k \leq n$ can be linearly independent as well.

**Theorem 11.1:** Let $n \geq 1$ and let $\iota_n \in H^n_Q K(n)_+$ be the universal cohomology class of degree $n$. Then a basis for $H^*_Q K(n)_+$ is given by all allowable compositions

$$P^I(\iota_n) = P^{i_1} \cdots P^{i_k}(\iota_n)$$

with $k \geq 0$, $i_t \geq 2$ for all $t$, and $i_s \leq n$.

This will be proved in later sections. See 14.3.

**Remark 11.2.**1.) This computes $H^*_Q K(n)_+$ for all $n \geq 0$, as $H^*_Q K(0)_+$ was computed in section 10. In fact, $H^*_Q K(0)_+$ has basis $\iota_0$ and $\beta\iota_0 \in H^1_Q K(n)_+$.

2.) One consequence of this result is that there are no universal relations among the operations $P^i$ except those that are implied by Theorem 9.1.

3.) Notice that $H^*_{\mathcal{Q}}K(1)_+ \cong \mathbf{F}_2$ in degree 1, generated by $\iota_1$. We knew this anyway, as the unique non-zero map $SK(1) \to K(1)_+$ is a weak equivalence.

To extend the computation of $H^*_{\mathcal{Q}}K(n)_+$ to a computation of $H^*_{\mathcal{Q}}V_+$ for other simplicial vector spaces $V$, we need the Hilton-Milnor Theorem. Let

$$\Sigma : s\mathcal{A} \to s\mathcal{A}$$

be the suspension functor of section 4. And let $V_1, V_2 \in s\mathbf{F}_2$. Then the Hilton-Milnor Theorem discusses the homotopy type of

$$\Sigma[(V_1)_+ \times_{\mathbf{F}_2} (V_2)_+] \cong \Sigma[(V_1 \times V_2)_+]$$

in $s\mathcal{A}$.

We need some further notation. The category $s\mathbf{F}_2$ is a category of modules and, as such, is equivalent to a category of chain complexes [15, p. 96]. Therefore, it is easy to construct a suspension functor

$$\sigma : s\mathbf{F}_2 \to s\mathbf{F}_2$$

so that there is a natural isomorphism for $V \in s\mathbf{F}_2$

$$\pi_n \sigma V = \begin{cases} \pi_{n-1}V; & \text{if } n \geq 1; \\ 0; & \text{if } n = 0. \end{cases}$$

The functor $\sigma$ has an adjoint $\omega$ so that there is a natural isomorphism $V \cong \omega\sigma V$ for all $V \in s\mathbf{F}_2$ and if $\pi_0 V = 0$, then $\pi_n \omega V \cong \pi_{n+1} V$.

Now let $L$ be the free algebra on two elements $x_1, x_2$. Let $B$ be the Hall basis for $L$. Then $b \in B$ is an iterated Lie product in the elements $x_1$ and $x_2$. Let

$$\begin{aligned} j_1(b) &= \text{the number of appearances of } x_1 \text{ in } b \\ j_2(b) &= \text{the number of appearances of } x_2 \text{ in } b \\ \ell(b) &= j_1(b) + j_2(b) \end{aligned}$$

and if $V_1, V_2 \in s\mathbf{F}_2$, define $V(b) \in s\mathbf{F}_2$ by

$$V(b) = \omega[(\sigma V_1)^{\otimes i(b)} \otimes (\sigma V_2)^{\otimes j(b)}] \tag{11.3}$$

where $W^{\otimes k}$ means the tensor product of $W$ with itself $k$ times.

**Theorem 11.4** (Hilton-Milnor)[12]: Let $V_1$ and $V_2$ be objects in $s\mathbf{F}_2$. Then there is a weak equivalence in $s\mathcal{A}$

$$\Sigma[(V_1)_+ \times_{\mathbf{F}_2} (V_2)_+] \to \otimes_{b \in B} \Sigma[V(b)_+].$$

This equivalence is natural with respect to maps $V_1 \to W_1$ and $V_2 \to W_2$ in $s\mathbf{F}_2$.

**Remark 11.5:** This theorem is useful for computations as follows: notice that if $A, B \in s\mathcal{A}$, then there is a natural isomorphism

$$Q(A \otimes B) \cong QA \oplus QB$$

and, hence, a natural isomorphism

$$H^*_Q(A \otimes B) \cong H^*_Q A \times H^*_Q B.$$

Thus there is a sequence of isomorphisms

$$H^*_Q[(V_1)_+ \times_{\mathbf{F}_2} (V_2)_+] \cong H^{*+1}_Q \Sigma[(V_1)_+ \times_{\mathbf{F}_2} (V_2)_+]$$

and

$$H^*_Q \Sigma[(V_1)_+ \times_{\mathbf{F}_2} (V_2)_+] \cong \times_{b \in B} H^*_Q \Sigma V(b)_+$$

and

$$H^{*+1}_Q \Sigma V(b)_+ \cong H^*_Q V(b)_+.$$

So that

$$H^*_Q (V_1 \times V_2)_+ \cong \times_{b \in B} H^*_Q V(b)_+.$$

As an example and application, we consider the Hilton-Milnor Theorem in the case where $V_1 = K(m)$ and $V_2 = K(n)$, where $\pi_* K(m) \cong \mathbf{F}_2$ and

$\pi_* K(n) \cong \mathbf{F}_2$ concentrated in degrees $m$ and $n$, respectively. If $b \in B \subseteq L$ is a word in the Hall basis, let

$$n_b = mj_1(b) + nj_2(b) + \ell(b) - 1.$$

Then, using the fact that there are weak equivalences in $s\mathbf{F}_2$

$$K(i) \otimes K(j) \simeq K(i+j)$$

and

$$\sigma K(i) \simeq K(i+1)$$

we see that the Hilton-Milnor Theorem yields a weak equivalence

$$\Sigma(K(m) \times K(n))_+ \xrightarrow{\simeq} \otimes_{b \in B} \Sigma K(n_b)_+. \tag{11.6.1}$$

Therefore

$$\begin{aligned} H_Q^* \Sigma(K(m) \times K(n))_+ &\cong \times_{b \in B} H_Q^* \Sigma K(n_b)_+ \\ &\cong H_Q^{*-1} K(n_b)_+ \end{aligned}$$

and, using 11.1, we have effectively computed $H_Q^*(K(m) \times K(n))_+$. To be more concrete in our description, we must develop the relationship between the elements of $B \subseteq L$ and the product $[\ ,\ ]$ of section 5. Let

$$\iota_m, \iota_n \in H_Q^*(K(m) \times K(n))_+$$

be the universal classes. Consider the composition, which we call $g^*$:

$$\begin{aligned} H_Q^* K(m+n+1)_+ &\xrightarrow{\cong} H_Q^{*+1} \Sigma K(m+n+1)_+ \\ &\longrightarrow H_Q^{*+1} \Sigma(K(m) \times K(n))_+ \\ &\xrightarrow{\cong} H_Q^*(K(m) \times K(n))_+ \end{aligned}$$

obtained by considering the word $b = [x_1, x_2] \in B$. Let $\iota_{m+n+1} \in H_Q^* K(m+n+1)_+$ be the universal class and let $[\ ,\ ]$ be the product on $H_Q^*$.

**Lemma 11.7:** $g^*(\iota_{m+n+1}) = [\iota_m, \iota_n]$.
**Proof:** $g^*(\iota_{m+n+1}) \neq 0$ because $g^*$ is an injection. But if

$$f_1 : K(m)_+ \to (K(m) \times K(n))_+$$

and

$$f_2 : K(n)_+ \to (K(m) \times K(n))_+$$

are the inclusion into the first and second factors respectively. Then the naturality clause in the Hilton-Milnor Theorem implies that

$$f_1^* g^*(\iota_{m+n+1}) = 0 = f_2^* g^*(\iota_{m+n+1}).$$

Similarly, 8.5 implies that $[\iota_m, \iota_n] \neq 0$ and

$$f_1^*[\iota_m, \iota_n] = 0 = f_2^*[\iota_m, \iota_n].$$

Now the calculation of 11.6.2 implies that there is a unique class

$$x \in H_Q^{m+n+1}(K(m) \times K(n))_+$$

with these properties: that $x \neq 0$, but $f_1^* x = 0 = f_2^* x$. The result follows.

The following corollary to this result is useful for calculation. Let $A \in s\mathcal{A}$ be so that $\pi_* A$ is of finite type. Consider the degree shifting Hurewicz homomorphism

$$\bar{h}^* : H_Q^* A \cong H_Q^* \Sigma A \to \pi^* \Sigma A^*.$$

Let $x, y \in H_Q^* A$ and $[x, y] \in H_Q^* A$ their product. Also, let

$$\langle\ ,\ \rangle : \pi^* \Sigma A^* \otimes \pi^* \Sigma A^* \to \pi^* \Sigma A^*$$

be the commutator product

$$\langle a, b \rangle = ab + ba$$

in the possibly non-commutative Hopf algebra $\pi^* \Sigma A^*$.

**Proposition 11.8:** $\bar{h}^*[x, y] = \langle \bar{h}^*(x), \bar{h}^*(y) \rangle$.

**Proof:** It is only necessary to consider the universal example $A = (K(m) \times K(n))_+$, and $x = \iota_m$ and $y = \iota_n$. But then it is clear from the decomposition of 11.6.1 that $\bar{h}^*g^*(\iota_{m+n+1}) = \langle \bar{h}^*(x), \bar{h}^*(y)\rangle$, where $g^*(\iota_{m+n+1})$ is as in 11.7. Since

$$[\iota_m, \iota_n] = g^*(\iota_{m+n+1})$$

, the result follows.

We next consider an arbitrary finite product, which we label $K_+$:

$$K_+ = (K(n_1) \times K(n_2) \times \cdots K(n_k))_+.$$

We can use the Hilton-Milnor Theorem and induction to study $K_+$. Let $L_k$ be the free Lie algebra on symbols $x_1, x_2, \ldots, x_k$ and let $B_k \subseteq L_k$ be the Hall basis. If $b \in b_k$, then $b$ is an iterated Lie bracket in the symbols $x_i$, and we define, for $1 \leq i \leq k$,

$$j_i(b) = \text{number of appearances of } x_i \text{ in } b$$

and

$$\ell(b) = j_1(b) + \cdots + j_k(b).$$

Then we let

$$n_b = j_1(b)n_1 + \cdots + j_k(b)n_k + \ell(b) - 1.$$

Then, using Theorem 11.4, 10.6.1, and induction, we have a weak equivalence

$$\Sigma K_+ \xrightarrow{\simeq} \otimes_{b \in B_k} \Sigma K(n_b)_+.$$

In fact, using 11.8, we can give an explicit description of this weak equivalence. Let $b \in B_k$; then, we may write $b$ as a Lie product

$$b = [x_{i_1}[x_{i_2} \cdots [x_{i_{\ell-1}}, x_{i_\ell}] \cdots]$$

where $\ell = \ell(b)$. Let $\iota_{n_i} \in H^*_Q K_+$ be the universal classes. define, in the homotopy category associated to $s\mathcal{A}$,

$$g_b : K_+ \to K(n_b)_+$$

by the requirement that

$$g^*\iota_{n_b} = [\iota_{i_1}[\iota_{i_2}\cdots[\iota_{i_{\ell-1}},\iota_{i_\ell}]\cdots]. \tag{11.9}$$

Then, applying the suspension functor, we obtain a map

$$\Sigma g_b : \Sigma K_+ \to \Sigma K(n_b)_+$$

and then, using the coproduct $\Sigma K_+ \to \Sigma K_+ \otimes \Sigma K_+$, we obtain a map

$$g : \Sigma K_+ \to \otimes_{b\in B_k}\Sigma K(n_b)_+.$$

**Proposition 11.10:** The morphism $g$ is a weak equivalence.

**Proof:** Let $T(\ )$ denote the primitively generated tensor Hopf algebra. In [12] it was proved that

$$\pi^*\Sigma K_+^* \cong T(\pi^*\sigma K^*)$$

where $\sigma : s\mathbf{F}_2 \to s\mathbf{F}_2$ is the shift functor defined before 11.4. Therefore, 11.8 implies that $\pi^*g^*$ is an isomorphism. Since $\pi_*\Sigma K_+$ is of finite type, the result follows.

**Remark 11.11:** It follows from 11.1, 11.9, and 11.10 that a basis for $H_Q^*K_+$ is given by the elements

$$P^{j_1}\cdots P^{j_s}[\iota_{i_1}[\iota_{i_2}\cdots[\iota_{i_{\ell-1}},\iota_{i_\ell}]\cdots]$$

where

$$b = [x_{i_1}[x_{i_2}\cdots[x_{i_{\ell-1}},x_{i_\ell}]\cdots]\in B_k,$$

$P^J$ is allowable, $j_t \geq 2$ for all $t$ and $j_s \leq n_b$, and the elements

$$\beta(\iota_{n_i})$$

when $n_i = 0$.

We state some further corollaries of 11.10. The first was used in the proof of 10.10.

**Corollary 11.12:** Let $n \geq 0$. Then

$$H_Q^{n+3}(K(0) \times K(n))_+ \cong \mathbf{F}_2$$

generated by $[\iota_0, [\iota_0, \iota_n]]$.

**Proof:** This follows immediately from Remark 11.11.

We can also prove that the product $[\ ,\ ]$ on $H_Q^*$ satisfies the Jacobi identity.

**Theorem 11.13:** Let $A \in s\mathcal{A}$ and $x, y, z \in H_Q^* A$. Then

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

**Proof:** We need only prove this for the universal example

$$A = [K(n_1 \times K(n_2) \times K(n_3)]_+$$

and $x = \iota_{n_1}, y = \iota_{n_2}$, and $z = \iota_{n_3}$. Let

$$f_{ij} : (K(n_i) \times K(n_j))_+ \to A$$

with $1 \leq i < j \leq 3$ be the three "monotonic" inclusions and

$$M = ker(f_{12}^*) \cap ker(f_{13}^*) \cap ker(f_{23}^*) \subseteq H_Q^{n_1+n_2+n_3+2} A.$$

Then, using 11.8, 11.10, and 11.11, we have that the composition

$$M \xrightarrow{\subseteq} H_Q^* A \xrightarrow{\bar{h}^*} \pi^* \Sigma A^*$$

is an injection. Since

$$\alpha = [\iota_{n_1}, [\iota_{n_2}, \iota_{n_3}]] + [\iota_{n_3}, [\iota_{n_1}, \iota_{n_2}]] + [\iota_{n_2}, [\iota_{n_3}, \iota_{n_1}]] \in M$$

and $\bar{h}^*(\alpha) = 0$, by 11.8, we have that $\alpha = 0$.

We finish with a uniqueness result about the operations $P^i$.

**Proposition 11.14:** Suppose that for all $A \in s\mathcal{A}$ and all non-negative integers $j$ and $n$ there is a natural operations

$$\bar{P}^i : H^n_Q A \to H^{n+j+1}_Q A$$

so that 1.) $\bar{P}^n(x) = [x, x]$, and

2.) $\bar{P}^j$ commutes with the suspension isomorphism

$$\partial : H^*_Q A \xrightarrow{\cong} H^{*+1}_Q \Sigma A.$$

Then for all $j$ and $n$, $\bar{P}^j = P^j$.

**Proof:** Because both $\bar{P}^j$ and $P^j$ are natural, it is only necessary to prove that

$$\bar{P}^j(\iota_n) = P^j(\iota_n)$$

where $\iota_n \in H^n_Q K(n)_+$ is the universal cohomology class. By Theorem 11.1, there are elements $\alpha_I \in \mathbf{F}_2$ so that

$$\bar{P}^j(\iota_n) = \alpha_j P^j(\iota_n) + \sum \alpha_I P^I(\iota_n)$$

where the sum ranges over all allowable monomials

$$P^I = P^{i_1} \cdots P^{i_s}$$

where $s > 1$ and $i_1 + \cdots + i_s + s = j + 1$. This last condition implies that $i_s < j$.

Consider the map in the homotopy category associated to $s\mathcal{A}$

$$f : \Sigma^{n-j} K(j)_+ \to K(n)_+$$

so that $H^*_Q f(\iota_n) \neq 0$. Then, under the composite

$$H^*_Q K(n)_+ \xrightarrow{H^*_Q f} H^*_Q \Sigma^{n-j} K(j)_+ \xrightarrow{\partial^{n-j}} H^*_Q K(j)_+$$

we have that $\iota_n$ maps to $\iota_j$ and, then,

$$\bar{P}^j(\iota_j) = \alpha_j P^j(\iota_j) + \sum \alpha_I P^I(\iota_j).$$

Furthermore, as a consequence of 11.1, the induced map

$$H^{n+j+1}_Q K(n)_+ \to H^{2j+1}_Q K(j)_+$$

is an injection. Thus, we have

$$[\iota_j, \iota_j] = \alpha_j [\iota_j, \iota_j] + \sum \alpha_I P^I(\iota_j).$$

Hence we may conclude that $\alpha_j = 1$ and $\alpha_I = 0$ for all other $I$.

## 12. A reverse Adams spectral sequence for computing $H^*_Q A$

In this section we begin the computation of $H^*_Q K(n)_+$ by producing a spectral sequence that passes from $\pi_* A$ to $H^*_Q A$ for $A \in s\mathcal{A}$. This spectral sequence is a reasonable tool to consider as we often regard $\pi_* A$ as computable. The spectral sequence is due to Miller [16, Section 4] and [18].

As a look ahead, we remark that this spectral sequence will have a particularly nice form when $\pi_* A$ is the graded algebra underlying some graded Hopf algebra. This will be the case when, for example, $A = \Sigma B$ is the suspension for some object $B \in s\mathcal{A}$ or if $A = K(n)_+$.

To begin, let $\mathcal{AD}$ be the category of graded algebras over the higher divided square operations $\delta_i$, as defined in section 2. Then homotopy defines a functor

$$\pi_* : s\mathcal{A} \to \mathcal{AD}$$

and

$$\bar{\mathfrak{S}} : \mathcal{AD} \to \mathcal{AD}$$

is the functor underlying a cotriple on $\mathcal{AD}$.

Thus, if $\Lambda \in \mathcal{AD}$, we have a simplicial object $\bar{\mathfrak{S}}.\Lambda$ in $\mathcal{AD}$ and, for $\Gamma \in \mathcal{AD}$, we may define

$$Ext^s_{\mathcal{AD}}(\Lambda, \Gamma)$$

by the equation

$$Ext^s_{\mathcal{AD}}(\Lambda, \Gamma) = \pi^s Hom_{\mathcal{AD}}(\bar{\mathfrak{S}}.\Lambda, \Gamma). \tag{12.1}$$

Now let $\Lambda_n \in \mathcal{AD}$ be the exterior algebra on a single generator $x_n$ of degree $n$, with the evident action of the operations $\delta_i$. Notice that

$$\pi_* K(n)_+ \cong \Lambda_n.$$

Here is the reverse Adams spectral sequence.

**Theorem 12.2:** Let $A \in s\mathcal{A}$ be a simplicial algebra. Then there is a spectral sequence

$$Ext^p_{\mathcal{AD}}(\pi_* A, \Lambda_q) \Rightarrow H^{p+q}_Q A.$$

Before proving this we remark that because of the complexity of $\pi_* A$ as an object in $\mathcal{AD}$ and the complexity of the functor $\mathfrak{S}$, this spectral sequence doesn't necessarily appear as a step forward. It is the purpose of some the auxiliary results of this section and of the next section to explore various ways one might compute with this spectral sequence.

**Proof of 12.2:** Let $\bar{S}A = \bar{S}^{\mathbf{F}_2}A \to A$ be the acyclic fibration with $\bar{S}A$ almost-free produced in the proof of 1.4. Then

$$H_Q^* A \cong \pi^* Q\bar{S}A^*.$$

Now,

$$\bar{S}A = diag(\bar{S}_{\cdot,\cdot}A)$$

where $\bar{S}_{\cdot,\cdot}A$ is the bisimplicial algebra produced in section 1: $\bar{S}_{p,q}A = \bar{S}^{p+1}A_q$. The observation that makes this proof go is that if we fix $p$, then

$$\pi_* \bar{S}^{p+1} A \cong \mathfrak{S}^{p+1}(\pi_* A)$$

In other words, by taking homotopy in the simplicial degree $q$ – thereby leaving the simplicial degree $p$ – we obtain the canonical acyclic almost-free resolution of $\pi_* A$ as an object in $\mathcal{AD}$:

$$\bar{\mathfrak{S}}.(\pi_* A) \to \pi_* A.$$

The argument proceeds as follows: Form the bi-cosimplicial object $B$ with

$$B^{p,q} = Q\bar{S}^{p+1}A_q^*.$$

Filtering $B$ by degree in $p$, we obtain a spectral sequence

$$\pi^p \pi^q B \Rightarrow \pi^{p+q} diag[Q\bar{S}_{\cdot,\cdot}A^*].$$

But, since

$$diag[Q\bar{S}_{\cdot,\cdot}A^*] = Q(diag(\bar{S}_{\cdot,\cdot}A))^*,$$

we have that

$$\pi^{p+q} diag[Q\bar{S}_{\cdot,\cdot}A^*] \cong H_Q^{p+q} A.$$

Thus the spectral sequence abuts to the promised object and we need only identify the $E_2$-term. However

$$\begin{aligned} E_1^{p,q} &\cong \pi^q Q\bar{S}^{p+1}A^* \\ &\cong \pi^q Hom_{\mathbf{F}_2}(\bar{S}^p A, \mathbf{F}_2) \\ &\cong Hom_{\mathbf{F}_2}(\pi_q \bar{S}^p A, \mathbf{F}_2) \\ &\cong Hom_{\mathbf{F}_2}(\bar{\mathfrak{S}}^p(\pi_* A)_q, \mathbf{F}_2) \\ &\cong Hom_{\mathcal{AD}}(\bar{\mathfrak{S}}^{p+1}(\pi_* A), \Lambda_q). \end{aligned}$$

Hence

$$\begin{aligned} E_2^{p,q} &\cong \pi^p Hom_{\mathcal{AD}}(\bar{\mathcal{S}}.(\pi_* A), \Lambda_q) \\ &\cong Ext^p_{\mathcal{AD}}(\pi_* A, \Lambda_q). \end{aligned}$$

This identifies the $E_2$-term and completes the proof.

In order to make this spectral sequence more accessible, we now develop methods for approaching the the $E_2$-term. The first such method is to produce a composite functor spectral sequence. For this we need some notation. First we note that if $A \in s\mathcal{A}$ then $\pi_* A$ is an algebra over the higher divided power operations of section 2. Because, for $x, y \in \pi_* A$,

$$\begin{aligned} \delta_i(xy) &= x^2 \delta_i y \qquad \text{if } x \in \pi_0 A; \\ &= y^2 \delta_i x \qquad \text{if } y \in \pi_0 A; \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

we have that the graded vector space of indecomposables $Q\pi_* A$ is actually a module over the higher divided power operations. We could define the category of such modules by some suitable cotriple, but it seems more natural to do the following.

**Definition 12.3:** Define the category $\mathcal{UD}$ of unstable modules over the higher divided squares as follows: $M \in \mathcal{UD}$ is a graded vector space equipped with homomorphisms

$$\delta_i : M_p \to M_{p+i}, \qquad 2 \le i \le p$$

so that the relations of Theorem 2.5 hold. That is, if $i < 2j$, then

$$\delta_i\delta_j(x) = \sum_{i+1/2 \le s \le i+j/3} \binom{j-i+s-1}{j-s} \delta_{i+j-s}\delta_s(x).$$

A morphism in $\mathcal{UD}$ is a vector space map that commutes with the operations $\delta_i$.

**Examples 12.4:1.)** If $A \in s\mathcal{A}$, then $Q\pi_*A \in \mathcal{UD}$.
2.) If $\Lambda, \Gamma \in \mathcal{AD}$ then $\Lambda \otimes \Gamma \in \mathcal{UD}$ via the formula

$$\begin{aligned} \delta_i(x \otimes y) &= x^2 \otimes \delta_i(y) \qquad \text{if } x \in \Lambda_0 \\ &= \delta_i(x) \otimes y^2 \qquad \text{if } y \in \Gamma_0 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then, if $\Lambda \in \mathcal{AD}$ the multiplication map $\Lambda \otimes \Lambda \to \Lambda$ is a morphism in $\mathcal{AD}$.

Now, if $W \in n\mathbf{F}_2$ and $\mathfrak{S}(W) \in \mathcal{AD}$ is the resulting free algebra over the operations $\delta_i$, that $Q\mathfrak{S}(W) \in \mathcal{UD}$ is projective; indeed

$$Hom_{\mathcal{UD}}(Q\mathfrak{S}(W), M) \cong Hom_{n\mathbf{F}_2}(W, M)$$

so the functor $\mathcal{P}(\ ) = Q\mathfrak{S}(\ )$ is left adjoint to forgetful functor. Hence $\mathcal{UD}$ has enough projectives.

Next, if $\Gamma \in \mathcal{AD}$, let $L_*^{\mathfrak{S}}Q\Gamma$ be the left derived functors of the indecomposables functor with respect to the cotriple induced by $\mathfrak{S}$; that is,

$$L_s^{\mathfrak{S}}Q\Gamma \cong \pi_s Q\bar{\mathfrak{S}}.\Gamma$$

where $\mathfrak{S}.\Gamma \to \Gamma$ is the canonical simplicial resolution of $\Gamma$.

Finally, let $\Sigma^n\mathbf{F}_2 \in \mathcal{UD}$ be the trivial module of dimension one over $\mathbf{F}_2$ concentrated in degree $n$. Here is the composite functor spectral sequence. It is based on the observation that

$$Hom_{\mathcal{AD}}(\ , \Lambda_n) \cong Hom_{\mathcal{UD}}(\ , \Sigma^n\mathbf{F}_2) \circ Q(\ ).$$

**Proposition 12.5:** For $\Gamma \in \mathcal{AD}$, there is a spectral sequence

$$Ext^p_{\mathcal{UD}}(L^{\mathfrak{S}}_q Q\Gamma, \Sigma^n \mathbf{F}_2) \Rightarrow Ext^{p+q}_{\mathcal{AD}}(\Gamma, \Lambda_n).$$

**Proof:** The argument, which I saw first as Proposition 2.13 of [16], is standard. The forgetful functor $\mathcal{UD} \to n\mathbf{F}_2$ has left adjoint $\mathcal{P}$, as described above. Let $\bar{\mathcal{P}} : \mathcal{UD} \to \mathcal{UD}$ be the resulting cotriple. Then if $N \in \mathcal{UD}$, we may form the augmented simplicial object $\bar{\mathcal{P}}.(N) \to N$ and

$$Ext^*_{\mathcal{UD}}(N, M) = \pi^* Hom_{\mathcal{UD}}(\bar{\mathcal{P}}.(N), M).$$

Now let $\Gamma \in \mathcal{AD}$ and $\bar{\mathfrak{S}}.\Gamma \to \Gamma$ be the acyclic simplicial resolution of $\Gamma$, and form the bi-cosimplicial vector space $C$ with

$$C^{p,q} = Hom_{\mathcal{UD}}(\bar{\mathcal{P}}_p(Q\bar{\mathfrak{S}}_q\Gamma), \Sigma^n \mathbf{F}_2).$$

Filtering $C$ by degree in $p$, we obtain a spectral sequence with

$$E_1^{p,q} \cong Hom_{\mathcal{UD}}(\bar{\mathcal{P}}_p(L^{\mathfrak{S}}_q\Gamma), \Sigma^n \mathbf{F}_2).$$

This follows from the fact that $\bar{\mathcal{P}}$ is an exact functor and the definitions. From this we conclude that the $E_2$ term of the spectral sequence is as described in the statement of the theorem. To determine what the spectral sequence abuts to, filter $C$ by degree in $q$. Then we get a spectral sequence with

$$E_1^{p,q} \cong Ext^p_{\mathcal{UD}}(Q\bar{\mathfrak{S}}_q\Gamma, \Sigma^n \mathbf{F}_2).$$

Since $Q\bar{\mathfrak{S}}_q\Gamma$ is projective in $\mathcal{UD}$ $E_1^{p,q} = 0$ if $p > 0$ and

$$\begin{aligned} E_1^{0,q} &\cong Hom_{\mathcal{UD}}(Q\bar{\mathfrak{S}}_q\Gamma, \Sigma^n \mathbf{F}_2) \\ &\cong Hom_{\mathcal{AD}}(\bar{\mathfrak{S}}_q\Gamma, \Lambda_n) \end{aligned}$$

Thus $E_2^{p,q} = 0$ if $p > 0$ and

$$E_2^{0,q} \cong Ext^q_{\mathcal{AD}}(\Gamma, \Lambda_n).$$

This completes the argument.

Notice that there is an edge homomorphism

$$Ext^p_{\mathcal{UD}}(Q\Gamma, \Sigma^n \mathbf{F}_2) \to Ext^p_{\mathcal{AD}}(\Gamma, \Lambda_n).$$

We would like to know when this is an isomorphism.

Now, the spectral sequence of 12.5 would not be a step forward except for the fact that there is class of algebras for which $L^{\mathfrak{S}}_q Q$ vanishes for $q > 0$. These will include $\pi_*\Sigma A$, where $\Sigma A$, $A \in s\mathcal{A}$ is the suspension of $A$. The first point to make is that these functors depend only the algebra structure of the argument. Define a triple $\mathcal{T} : n\mathbf{F}_2 \to n\mathbf{F}_2$ by

$$\mathcal{T}(W) = S(W_0) \otimes [\otimes_{p>0} \Lambda(W_p)]$$

where $S(\ )$ and $\Lambda(\ )$ are the symmetric and exterior algebra respectively. Notice that, by Proposition 2.7, $\mathfrak{S}W$ isomorphic, as an algebra, to $\mathcal{T}(\mathcal{P}W)$. Since $\mathcal{T}$ is a triple, we have a category $\mathbf{T}$ of $\mathcal{T}$-algebras and $\mathcal{T} : n\mathbf{F}_2 \to \mathbf{T}$ is left adjoint to the forgetful functor. We may regard $\mathbf{T}$ as the category of graded algebras $\Gamma$ so that if $X \in \Gamma_p$, $p > 0$, then $x^2 = 0$. There is a forgetful functor $\mathcal{AD} \to \mathbf{T}$. Finally, if $\Gamma \in \mathbf{T}$, we can form the vector space $Q\Gamma$ of indecomposables and we can form the derived functors

$$L^{\mathcal{T}}_* Q\Gamma.$$

**Proposition 12.6:** Let $\Gamma \in \mathcal{AD}$. Then there is an isomorphism of vector spaces

$$L^{\mathfrak{S}}_q Q\Gamma \cong L^{\mathcal{T}}_q Q\Gamma.$$

In particular, if there is an isomorphism of algebras $\Gamma \cong \mathcal{T}W$ for some $L^{\mathfrak{S}}_q Q\Gamma = 0$ for $q > 0$ and

$$L^{\mathfrak{S}}_0 Q\Gamma \cong Q\Gamma.$$

**Proof:** The second clause follows from the first. The first isomorphism is a direct consequence of Proposition 2.11 of [16]. The proof goes, in outline, as follows: Let $\bar{\mathfrak{S}}$ and $\bar{\mathcal{T}}$ denote the obvious cotriples. Form the bisimplicial object

$$Q(\bar{\mathcal{T}}^{p+1}\bar{\mathfrak{S}}^{q+1}\Gamma).$$

Filtering by degree in $q$ we obtain a spectral sequence with $E^2_{p,q} = 0$ if $p > 0$ and $E^2_{0,q} \cong L^{\mathfrak{S}}_q Q(\Gamma)$. Here we use the fact that $\mathfrak{S}W \cong \mathcal{T}W'$ for some $W'$. Then, filtering by degree in $p$, we obtain a spectral sequence with $E^2_{p,q} = 0$ if $q > 0$ and $E^2_{p,0} \cong L^{\mathcal{T}}_q Q(\Gamma)$. Here we use that $\pi_* \bar{\mathfrak{S}}.\Lambda \cong \Gamma$ via a canonical contraction and, hence, that $\mathcal{T}\mathfrak{S}.\Gamma$ has a contraction.

When we have an algebra $\Gamma \in \mathcal{AD}$ so that there is an isomorphism of algebras $\Gamma \cong \mathcal{T}W$ for some $W$, then 12.6 guarantees that the spectral sequence of 12.5 collapses and yields an isomorphism

$$Ext^p_{\mathcal{UD}}(Q\Gamma, \Sigma^n \mathbf{F}_2) \cong Ext^p_{\mathcal{AD}}(\Gamma, \Lambda_n).$$

This is obviously a simplification, returning the right hand-side of this equation to the realm of abelian homological algebra. In addition, under the hypothesis that $\pi_* A \cong \mathcal{T}W$ for some $W$, the spectral sequence of 12.2 becomes

$$Ext^p_{\mathcal{UD}}(Q\pi_* A, \Sigma^q \mathbf{F}_2) \Rightarrow H^{p+q}_Q A. \tag{12.7}$$

This will be the case for $\Sigma A$, $A \in s\mathcal{A}$. Indeed, if $\Gamma \in \mathcal{AD}$ is the algebra underlying a graded Hopf algebra that is connected in the sense that $\Gamma_0 \cong \mathbf{F}_2$ generated by the unit, then Borel's structure theorem for Hopf algebras implies that $\Gamma \cong \mathcal{T}W$ for some graded vector space $W$. We will explore computing the $E_2$-term of the spectral sequence (12.7) in the next section.

**Remark 12.8:** Let $\Gamma \in \mathcal{A}$ be a commutative $\mathbf{F}_2$-algebra. Regard $\Gamma$ as a constant simplicial object in $s\mathcal{A}$. Then

$$\pi_* \Sigma\Gamma \cong Tor^{\Gamma}_*(\mathbf{F}_2, \mathbf{F}_2)$$

and

$$H^n_Q \Sigma\Gamma \cong H^{n-1}_Q \Gamma.$$

Thus 12.7 yields a spectral sequence

$$Ext^p_{\mathcal{UD}}(QTor^{\Gamma}_*(\mathbf{F}_2, \mathbf{F}_2), \Sigma^q \mathbf{F}_2) \Rightarrow H^{p+q-1}_Q \Gamma.$$

This is similar to the spectral sequence written down by Miller in [18].

As an interesting contrast, André and Quillen worked out the $H^*_Q\Gamma$ in characteristic 0 — that is, for some field $k$ of characteristic 0 — and obtained that

$$H^*_Q\Gamma \cong [H^Q_*\Gamma]^*$$

(from the definitions) and

$$H^Q_n\Gamma \cong [QTor^\Gamma_*(k,k)]_{n+1};$$

that is, the indecomposables in $Tor$ in degree $n+1$. Thus $H^*_Q\Gamma$ is dual to indecomposables in $Tor$. There are no higher divided power operations in characteristic 0, so there is no need for a spectral sequence as in (12.7).

## 13. A Koszul resolution for computing $Ext_{\mathcal{UD}}$

In the previous section we saw that we could often approach $H^*_Q A$ through the computation of $Ext$ in the category $\mathcal{UD}$. In this section, we apply the work of Priddy [19] to produce a canonical chain complex for computing these $Ext$ groups.

Let $\mathcal{P} : n\mathbf{F}_2 \to \mathcal{UD}$ be the left adjoint to the forgetful functor and $\bar{\mathcal{P}} : \mathcal{UD} \to \mathcal{UD}$ the resulting cotriple. Then, for $M \in \mathcal{UD}$, we compute $Ext^*_{\mathcal{UD}}(M, \Sigma^q\mathbf{F}_2)$ by the equation

$$Ext^p_{\mathcal{UD}}(M, \Sigma^q\mathbf{F}_2) = \pi^p Hom_{\mathcal{UD}}(\bar{\mathcal{P}}.(M), \Sigma^q\mathbf{F}_2).$$

Now $\Sigma^q\mathbf{F}_2 \in \mathcal{UD}$ is simply the trivial module of dimension one over $\mathbf{F}_2$ concentrated in degree $q$. By allowing $q$ to vary, we obtain a graded object $Ext^p_{\mathcal{UD}}(M, \mathbf{F}_2)$ with

$$Ext^p_{\mathcal{U}}(M, \mathbf{F}_2)^q = Ext^p_{\mathcal{UD}}(M, \Sigma^q\mathbf{F}_2).$$

It is this object that the Koszul resolution will compute. Note that we can similarly define $Hom_{\mathcal{UD}}(M, \mathbf{F}_2)$.

Dually, we have a notion of $Tor$ in $\mathcal{UD}$. For $M \in \mathcal{UD}$, we specify two canonical map $\bar{\mathcal{P}}(M) \to M$. The first is the usual projection $\epsilon : \bar{\mathcal{P}}(M) \to M$ adjoint to the identity $M \to M$ in $n\mathbf{F}_2$. The second is obtained as follows: if

$V \in n\mathbf{F}_2$ we may regard $V \in \mathcal{UD}$ as a trivial module and obtain a projection $\epsilon : \bar{\mathcal{P}}(V) \to V$. Applying this to $M$ regarded as an object of $n\mathbf{F}_2$ we obtain a projection $\epsilon' : \bar{\mathcal{P}}(M) \to M$ in $n\mathbf{F}_2$. Then we define $\mathbf{F}_2 \otimes_{\mathcal{UD}} M$ by the coequalizer diagram

$$(13.1) \qquad \bar{\mathcal{P}}(M) \overset{\epsilon,\epsilon'}{\rightrightarrows} M \to \mathbf{F}_2 \otimes_{\mathcal{UD}} M.$$

Then we have

$$(13.2) \qquad Tor_p^{\mathcal{UD}}(M, \mathbf{F}_2) = \pi_p \mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}.(M)$$

and

$$[Tor_p^{\mathcal{UD}}(M, \mathbf{F}_2)]^* \cong Ext^p_{\mathcal{UD}}(M, \mathbf{F}_2).$$

If $V$ is a bigraded vector space, we can define a filtration on $\mathcal{P}(V)$ by

$$(13.3) \qquad F_1^q \mathcal{P}(V) = \{\delta_{i_1} \dots \delta_m(x) \ : \ x \in V \text{ and } m \le q\}.$$

Notice that $F_1^q \mathcal{P}(V) \subseteq F_1^{q+1} \mathcal{P}(V)$ and

$$\delta_i F_1^q \mathcal{P}(V) \subseteq F_1^{q+1} \mathcal{P}(V).$$

Therefore, the associated graded (actually bigraded) object $E_0 \mathcal{P}(V)$ has an action by the operations

$$\delta_i : E_0^q \mathcal{P}(V) \to E_0^{q+1} \mathcal{P}(V).$$

increasing filtration degree by one and subject to all the axioms of (13.3). We call the category of such modules $E_0 \mathcal{UD}$.

Furthermore, if $V$ is itself filtered by a filtration

$$\dots \subseteq F_2^q V \subseteq F_2^{q+1} V \subseteq \dots \subseteq V,$$

we can define a filtration on $\mathcal{P}(V)$ by

$$F^q \mathcal{P}(V) = \sum_{a+b=q} F_1^a \mathcal{P}(F_2^b V) \subseteq \mathcal{P}(V)$$

where $F_1$ is the filtration above. Thus, if $M \in \mathcal{UD}$, we can recursively define a filtration on $\bar{\mathcal{P}}_p(M) = \bar{\mathcal{P}}^{p+1}(M)$ by

$$F^s\bar{\mathcal{P}}_0(M) = F_1^s\bar{\mathcal{P}}(M)$$

as in (12.3) and

$$F^s\bar{\mathcal{P}}_p(M) = \sum_{a+b=s} F_1^a\bar{\mathcal{P}}(F^b\bar{\mathcal{P}}_{p-1}(M)).$$

It is simple to check that the face and degeneracy operators

$$d_i : \bar{\mathcal{P}}_p(M) \to \bar{\mathcal{P}}_{p-1}(M)$$

and

$$s_i : \bar{\mathcal{P}}_p(M) \to \bar{\mathcal{P}}_{p+1}(M)$$

preserve these filtrations and the associated graded yields a simplicial object $E_0\bar{\mathcal{P}}.(M)$ in $E_0\mathcal{UD}$. Furthermore, the canonical contraction of $\bar{\mathcal{P}}.(M)$ also preserves this filtration, so we have

$$\pi_* E_0\bar{\mathcal{P}}.(M) \cong M$$

concentrated in $\pi_0 E_0\bar{\mathcal{P}}.(M)$ and filtration degree 0. In fact, if we regard $M$ as an object in $E_0\mathcal{UD}$ concentrated in filtration degree 0, then

$$E_0\bar{\mathcal{P}}.(M) \to M$$

is the canonical projective simplicial resolution of $M$ in $E_0\mathcal{UD}$. In $E_0\mathcal{UD}$, $M$ is a trivial module because the operations $\delta_i$ shift filtration degree.

Therefore, if we define $\mathsf{F}_2 \otimes_{E_0\mathcal{UD}} (\ \ )$ by analogy with (13.1), then

$$\pi_p\mathsf{F}_2 \otimes_{E_0\mathcal{UD}} E_0\bar{\mathcal{P}}.(M) = Tor_p^{E_0\mathcal{UD}}(\mathsf{F}_2, M).$$

Denote by

$$Tor_p^{E_0\mathcal{UD}}(\mathsf{F}_2, M)_q$$

the elements of filtration degree $q$. Then the filtration above yields a spectral sequence

$$E_1^{p,q} = Tor_p^{E_0\mathcal{UD}}(\mathsf{F}_2, M)_q \Rightarrow Tor_p^{\mathcal{UD}}(\mathsf{F}_2, M)$$

with differentials

$$d_r : E_r^{p,q} \to E_r^{p-1,q-r}.$$

Dualizing, we get a spectral sequence

$$Ext^p_{E_0\mathcal{UD}}(M, \mathsf{F}_2)_q \Rightarrow Ext^p_{\mathcal{UD}}(M, \mathsf{F}_2).$$

Priddy's results, suitably adjusted, include the following.

**Proposition 13.4:** If $p \neq q$, then

$$Tor_p^{E_0\mathcal{UD}}(M, \mathsf{F}_2)_q = 0.$$

We Priddy's argument applies immediately to prove this result. In addition, the argument for this case can be adapted immediately from the argument given in [13]. The next result follows immediately from the spectral sequence above.

**Corollary 13.5:** Define a cochain complex for $M \in \mathcal{UD}$ by

$$C^p(M) = Ext^p_{E_0\mathcal{UD}}(M, \mathsf{F}_2)_p = E^1_{p,p}$$

and

$$d^* = d^1 : C^p(M) \to C^{p+1}(M).$$

Then

$$H^*(C(M), d^*) \cong Ext^*_{\mathcal{UD}}(M, \mathsf{F}_2).$$

To give specific details about the chain complex in 13.5, we need to develop some notation. Recall that a composition

$$\delta_I = \delta_{i_1} \dots \delta_{i_m} \tag{13.6}$$

is *admissible* if $i_k \geq 2i_{k+1}$ for all $k$. The excess of $I$ was defined in section 3. Define the *length* of $\delta_I$ to be $m$.

For $M \in \mathcal{UD}$, $\bar{\mathcal{P}}(M)$ is spanned by elements which we will write

$$[\delta_I | x]$$

where, if $x \in M$ has degree t, then $\delta_I$ is admissible and $e(I) \leq t$ . If $I = \phi$, we write $[1|x]$. Then $E_0^q\bar{\mathcal{P}}(M)$ is spanned by elements of the form

$$[\delta_I | x]$$

with $x \in M$, the operation admissible, $e(I) \leq deg(x)$, and length exactly $q$. If $x$ has degree $t$, then this element has degree

$$t + \sum i_k. \tag{13.7}$$

Then, recursively, we see that $E_0^q\bar{\mathcal{P}}_p(M)$ is spanned by elements of the form

$$[\delta_{I_0} | \delta_{I_1} | \ldots | \delta_{I_p} | x]$$

with $x \in M$, the operations admissible, excess determined by the case $p = 0$ and the sum of the lengths of the $\delta_{I_k}$ exactly $q$. From this it follows that $(\mathbf{F}_2 \otimes_{E_0\mathcal{UD}} E_0\bar{\mathcal{P}}_p(M))_q$ – the elements of filtration degree $q$ – is spanned by the residue classes of elements of the form

$$[1 | \delta_{I_1} | \ldots | \delta_{I_p} | x] \tag{13.8}$$

subject to the same conditions; in particular, if $x$ has degree $t$, then

$$e(I_k) \leq t + \sum_{r>k} (\sum I_r) \tag{13.9}$$

where we write $\sum I$ for $\sum i_r$ with $I = (i_1, \ldots, i_m)$.

It remains only to give a basis for $C^*(M)$ and a formula for $d^*$, so that one can compute with the cochain complex of 13.5. First of all,

$$C^0(M) = M^*$$

is the graded $\mathbf{F}_2$-dual of $M$. Next consider the case where $M = \Sigma^t\mathbf{F}_2$ is of dimension 1 over $\mathbf{F}_2$ concentrated in bidegree $t$. Let $\iota \in \Sigma^t\mathbf{F}_2$ be the non-zero element. Then one easily sees that $Ext^1_{E_0\mathcal{UD}}(\Sigma^t \mathbb{F}_2, \mathbb{F}_2)$ has a basis given by the residue classes of

$$\gamma^i(\iota^*) = [1|\delta_i|\iota]^*, \qquad 2 \leq i \leq s$$

These have filtration degree 1. Then these elements define linear operations, for all $M \in \mathcal{UD}$ and $p \geq 0$

$$\begin{aligned}\gamma^i : C^p(M)_t =& Ext^p_{E_0\mathcal{UD}}(M, \mathbb{F}_2)_{(p,t)} \\ & \to Ext^{p+1}_{E_0\mathcal{UD}}(M, \mathbb{F}_2)_{(p+1,t+i)} = C^{p+1}(M)_{t+i}\end{aligned}$$

for $2 \leq i \leq t$.

**Proposition 13.10:** Let $\{x_\alpha\} \subseteq M^*$ be a homogeneous basis for the $\mathbb{F}_2$-dual of $M \in \mathcal{U}$. Then a basis for $C^p(M)$ is given by all elements of the form

$$\gamma^{i_1} \cdots \gamma^{i_p}(x_\alpha)$$

with $x_\alpha$ in the basis and

1.) $2 \leq i_k < 2i_{k+1}$ for all $k$;

3.) if $x_\alpha$ has degree $t$, then $i_p \leq t$.

Furthermore, if $x_\alpha$ has bidegree $t$ then this element has degree

$$t + i_i + \cdots + i_p.$$

This can be proved exactly as in section 3 of Priddy's paper, or one can use the exact same argument as outlined in [13].

This result suggest that there are relations among the operations $\gamma^i$; following Priddy's line of argument we see that these relations are:

**Lemma 13.11:** If $i \geq 2j$, then

$$\gamma^i \gamma^j = \sum_{s=i-j+1}^{i+j-2} \binom{2s-i-1}{s-j} \gamma^{i+j-s} \gamma^s.$$

This is, of course, the same formula as for the relations among the operations $P^i$ in $H^*_Q$

Finally, we can determine the differential $d^* : C^p(M) \to C^{p+1}(M)$. First we consider the differential

$$d^* : C^p(\Sigma^t \mathbf{F}_2) \to C^{p+1}(\Sigma^t \mathbf{F}_2).$$

One can easily check that if $p = 1$, then

$$d^*\gamma^i(\iota^*) = 0 \tag{13.12}$$

This formula follow from the Adem relations of Theorem 2.5.3 and a standard dualization argument. Then, for $p > 1$, $d^*$ "acts as a derivation"; that is,

$$d^*\gamma^I(\iota^*) = \sum \gamma^{i_1} \cdots (d^*\gamma^{i_k}) \cdots \gamma^{i_p}(\iota^*)$$

or

$$d^*\gamma^I(\iota^*) = 0. \tag{13.13}$$

for all $I$.

One can apply the relations above for further computations. Now, for general $M \in \mathcal{UD}$, the operations $\delta_i : M \to M$ determine dual operations

$$\delta_i : (M^*)_t \to (M^*)_{t-i}$$

We write these operations on the right. Thus, for example, if $x \in M^*$ is of degree $t$, then $x\delta_i$ is of bidegree $t-i$. Then $d^* : C^p(M) \to C^{p+1}(M)$ is now determined by the formula

$$d^*(\gamma^I(x)) = \sum_{i \geq 2} \gamma^I \gamma^i (x\delta_i) \tag{13.14.1}$$

and the relations above. In particular, $d^* : M^* \to C^1(M)$ is given by the formula

$$d^*(x) = \sum_{i \geq 2} \gamma^i (x\delta_i). \tag{13.14.2}$$

**Remark 13.15:** 1.) This is a "lambda algebra" for computing $Ext_{\mathcal{UD}}$, very similar to the lambda algebra often used to make calculations of $Ext$ in the category of unstable modules over the Steenrod algebra.

2.) If $x \in C^p(M)$ is a cocycle in this cochain complex, then $\gamma^i(x) \in C^p(M)$ is also a cocycle. Thus we have an induced operation

$$\gamma^i : Ext^p_{\mathcal{UD}}(M, \Sigma^q \mathbf{F}_2) \to Ext^{p+1}_{\mathcal{UD}}(M, \Sigma^{q+i}\mathbf{F}_2).$$

**Computation 13.16:** Let $K(n)_+ \in s\mathcal{A}$ be the universal object representing cohomology. We wish to compute $H^*_Q K(n)_+$. Consider Miller's spectral sequence

$$Ext^p_{\mathcal{AD}}(\pi_* K(n)_+, \Lambda_q) \Rightarrow H^{p+q}_Q K(n)_+.$$

Now $\pi_* K(n)_+ \cong \Lambda(x_n)$ — the exterior algebra on an element of degree $n$. Thus, if $n > 1$ 12.5 and 12.8 imply that

$$Ext^p_{\mathcal{AD}}(\pi_* K(n)_+, \Lambda_q) \cong Ext^p_{\mathcal{UD}}(\Sigma^n \mathbf{F}_2, \Sigma^q \mathbf{F}_2).$$

Since $\Sigma^n \mathbf{F}_2$ is a trivial module in $\mathcal{UD}$, the differential of 13.14 is zero and we have an isomorphism

$$Ext^p_{\mathcal{UD}}(\Sigma^n \mathbf{F}_2, \mathbf{F}_2) \cong C^p(\Sigma^n \mathbf{F}_2^*).$$

Thus a basis for $Ext^p_{\mathcal{UD}}(\Sigma^n \mathbf{F}_2, \Sigma^q \mathbf{F}_2)$ is all elements of the form

$$\gamma^{i_1} \cdots \gamma^{i_p}(\iota^*)$$

where $\iota^*$ is the non-zero element in $\Sigma^n \mathbf{F}_2^*$, $2 \leq i_k < 2i_{k+1}$ for all $k$, and $i_p \leq n$. Thus, if we can show that Miller's spectral sequence collapses in this case, we have proved Theorem 11.1.

We now write down and study an explicit chain equivalence

(13.17) $$\Phi : Hom_{\mathcal{UD}}(\bar{P}.(M), \mathbf{F}_2) \cong [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{P}.(M)]^* \longrightarrow C^*(M).$$

If $M \in \mathcal{UD}$ and $\delta_I = \delta_{i_1} \cdots \delta_{i_s}$ is an admissible composition of higher divided squares, then we can define an injective homomorphism

$$\delta^I : M^* \to \bar{\mathcal{P}}(M)^*$$

by

$$\langle \delta^I(y), [\delta_J|x] \rangle = \begin{cases} 0, & \text{if } I \neq J; \\ \langle y, x \rangle, & \text{if } I = J. \end{cases}$$

Here $\langle\ ,\ \rangle$ is the pairing between a vector space and its dual. In this formula $I$ might be empty, in which case we write $\delta^I = 1$. Notice that there is a factoring

$$\begin{array}{ccc} M^* & \to & [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}(M)]^* \\ \downarrow = & & \downarrow \\ M^* & \xrightarrow{1} & \bar{\mathcal{P}}(M)^* \end{array}$$

so that, by restricting the range, 1 defines an isomorphism

$$1 : M^* \to [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}(M)]^*.$$

Then, arguing as in 13.8, we see that

$$[\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(M)]^*$$

is spanned by elements of the form

$$1 \circ \delta^{I_1} \circ \cdots \circ \delta^{I_p}(y)$$

where $y \in M^*$. Let us agree, for now, to write this element as

$$\delta^{I_1} \circ \cdots \circ \delta^{I_p}(y),$$

dropping the "1" at the beginning.

We now define $\Phi$ as in 13.17 by

(13.18.1) $$\Phi(\delta^{i_1} \circ \cdots \circ \delta^{i_p}(y)) = \gamma^{i_1} \cdots \gamma^{i_p}(y)$$

and

(13.18.2) $$\Phi(\delta^{I_1} \circ \cdots \circ \delta^{I_p}(y)) = 0 \qquad \text{otherwise.}$$

To see that $\Phi$ is a cochain map is a matter of routine, if lengthy calculation. We will gleefully leave this to the reader after supplying a few pointers. (The less intrepid, can look up the dual calculation in [19], section 4). We have the following formulas for

$$d_i^* : [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(M)]^* \to [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(M)]^*.$$

$$d_0^*(\delta^{I_1} \circ \cdots \circ \delta^{I_p}(y)) = 1 \circ \delta^{I_1} \circ \cdots \circ \delta^{I_p}(y) \tag{13.19.1}$$

$$d_{p+1}^*(\delta^{I_1} \circ \cdots \circ \delta^{I_p}(y)) = \sum_I \delta^{I_1} \circ \cdots \circ \delta^{I_p} \delta^I(y\delta_I) \tag{13.19.2}$$

and, for $1 \leq t \leq p$
(13.19.3)

$$d_t^*(\delta^{I_1} \circ \cdots \circ \delta^{I_p}(y)) = \sum \alpha(J,K,I_t)\delta^{I_1} \circ \cdots \circ \delta^{I_{t-1}} \circ \delta^J \circ \delta^K \circ \cdots \circ \delta^{I_p}(y).$$

In the last equation, $\alpha(J,K,I_t) \in \mathbb{F}_2$ are numbers determined by the equations

$$\delta_I \delta_J = \sum \alpha(J,K,L)\delta_L$$

and the sum in (13.19.3) is over all admissible sequences $I$ and $J$, including the empty sequence. The sum in (13.19.2) is over all admissible sequences, also including the empty sequence. Of crucial importance to any calculation proving that the map $\Phi$ is a chain map is the case where $I_t = \{i,j\}$. Since $I_t$ is admissible $i \geq 2j$. Then it is a calculation similar to the one done in the proof of 9.6 to show that we have the following specialization of (13.19.3) in the case where $I_t = \{i,j\}$:

$$\begin{aligned}
d_t^*&(\delta^{I_1} \circ \cdots \circ \delta^{I_p}(y)) \\
&= \delta^{I_1} \circ \cdots \circ \delta^{I_{t-1}} \circ \delta^{I_t} \circ 1 \circ \cdots \circ \delta^{I_p}(y) \\
&\quad + \delta^{I_1} \circ \cdots \circ \delta^{I_{t-1}} \circ 1 \circ \delta^{I_t} \circ \cdots \circ \delta^{I_p}(y) \\
&\quad + \delta^{I_1} \circ \cdots \circ \delta^{I_{t-1}} \circ \delta^i \circ \delta^j \circ \cdots \circ \delta^{I_p}(y) \\
&\quad + \sum_{s=i-j+1}^{i+j-2} \binom{2s-i-1}{s-j} \delta^{i_1} \circ \cdots \circ \delta^{I_{t-1}} \circ \delta^{i+j-s} \circ \delta^s \circ \cdots \circ \delta^{I_p}(y).
\end{aligned}$$

It is exactly this formula that is needed to make $\Phi$ a chain map.

The fact that $\Phi$ is a cochain equivalence follows from the facts that

1.) $[\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}.(\cdot)]^*$ and $C^*(\cdot)$ are exact functors to the category of cochain complexes,

2.) $H^s[\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}.(M)]^* = H^s C^*(M) = 0$ if $M$ is projective in $\mathcal{UD}$ and $s > 0$, and

3.) $\Phi : H^0[\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}.(M)]^* \to H^0 C^*(M)$ is an isomorphism for all $M \in \mathcal{UD}$.

We can now use $\Phi$ to give another description of the operation

$$\gamma^i : Ext^p_{\mathcal{UD}}(M, \Sigma^q \mathbf{F}_2) \to Ext^{p+1}_{\mathcal{UD}}(M, \Sigma^{q+i}\mathbf{F}_2) \tag{13.20}$$

of 13.15.2. For a fixed $i$, $\delta^i$ defines a map

$$\delta^i : \bar{\mathcal{P}}^p(M)^* \to \bar{\mathcal{P}}^{p+1}(M)^*$$

and thus we get a commutative diagram

$$\begin{array}{ccc} \bar{\mathcal{P}}^p(M)^* & \xrightarrow{\delta^i} & \bar{\mathcal{P}}^{p+1}(M)^* \\ \downarrow 1 & & \downarrow 1 \\ [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(M)]^* & \xrightarrow{\hat{\delta}^i} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(M)]^*. \end{array}$$

In fact, this diagram and the fact that the vertical maps are isomorphisms defines $\hat{\delta}^i$.

Now, let $x \in C^p(M)$ be a cocycle and

$$y \in [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(M)]^*$$

a cocycle so that $\Phi(y) = x$. Then a calculation with the formulas of 13.19 shows that

$$\hat{\delta}^i(y) \in [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(M)]^*$$

is a cocycle and the definition of $\Phi$ demonstrates that

$$\Phi(\hat{\delta}^i(y)) = \gamma^i(x).$$

Hence

$$\hat{\delta}^i : [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(M)]^* \to [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(M)]^* \tag{13.21}$$

induces the operation $\gamma^i$ of 13.20.

It will be useful to have a formula for for $\hat{\delta}^i$, and we obtain one right from the definition (see after 13.17). If

$$z = [1|\delta_{I_1}|\cdots|\delta_{I_p}|x] \in \mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(M),$$

then

$$\langle \hat{\delta}^i(y), z\rangle = \begin{cases} 0, & \text{if } I_1 \neq \{i\}; \\ \langle y, [1|\delta_{I_2}|\cdots|\delta_{I_p}|x]\rangle, & \text{if } I_1 = \{i\}. \end{cases} \tag{13.22}$$

## 14. Operations in the reverse Adams spectral sequence

In section 12 we constructed, for $A \in s\mathcal{A}$, a reverse Adams spectral sequence

$$Ext^p_{\mathcal{AD}}(\pi_*A, \Lambda_q) \Rightarrow H^{p+q}_Q A$$

and a spectral sequence

$$Ext^p_{\mathcal{UD}}(L^{\mathfrak{S}}_r Q\pi_*A, \Sigma^q\mathbf{F}_2) \Rightarrow Ext^{p+r}_{\mathcal{AD}}(\pi_*A, \Lambda_q).$$

The latter spectral sequence degenerated under favorable conditions. In section 13, we produced a chain complex for computing

$$Ext^p_{\mathcal{UD}}(M, \Sigma^q\mathbf{F}_2).$$

We complete the circle of ideas by putting enough structure into the reverse Adams spectral sequence to insure that it will collapse in interesting cases. The method will be this: let

$$P^i : H^n_Q A \to H^{n+i+1}_Q A$$

be the operations in $H^*_Q$. We will describe how these operations are detected in the spectral sequence and use that information to make a computation. We will use the work of W. Singer [23]. Most of the section wil be spent proving the following.

**Theorem 14.1:** Let $A \in s\mathcal{A}$. There are operations

$$P^i : Ext^p_{\mathcal{AD}}(\pi_* A, \Lambda_q) \to Ext^{p+1}_{\mathcal{AD}}(\pi_* A, \Lambda_{q+i})$$

so that

1.) if $x \in Ext^p_{\mathcal{AD}}(\pi_* A, \Lambda_q)$ survives to $E_\infty$ in the reverse Adams spectral sequence, then $P^i(x)$ survives to $E_\infty$; and

2.) if $\alpha \in H^{p+q}_Q A$ is detected by $x \in Ext^p_{\mathcal{AD}}(\pi_* A, \Lambda_q)$, then $P^i(\alpha) \in H^{p+q+i+1}_Q A$ is detected by $P^i(x) \in Ext^{p+1}_{\mathcal{AD}}(\pi_* A, \Lambda_{q+i})$.

To make computations possible we need to know how the operations of 14.1 commute with the edge homomorphism

$$e : Ext^p_{\mathcal{UD}}(Q\pi_* A, \Sigma^q \mathbf{F}_2) \to Ext^p_{\mathcal{AD}}(\pi_* A, \Lambda_q).$$

**Theorem 14.2:** Let

$$\gamma^i : Ext^p_{\mathcal{UD}}(Q\pi_* A, \Sigma^q \mathbf{F}_2) \to Ext^{p+1}_{\mathcal{UD}}(Q\pi_* A, \Sigma^{q+i} \mathbf{F}_2)$$

be the operations of the previous section. Then

$$e(\gamma^i(x)) = P^i(e(x)).$$

We can use these two results to prove Theorem 11.1, which we now restate.

**Theorem 14.3:** Let $n \geq 1$ and let $\iota_n \in H^n_Q K(n)_+$ be the universal cohomology class of degree $n$. Then a basis for $H^*_Q K(n)_+$ is given by all allowable compositions

$$P^I(\iota_n) = P^{i_1} \cdots P^{i_k}(\iota_n)$$

with $s \geq 0$, $i_t \geq 2$ for all $t$, and $i_k \leq n$.

**Proof:** Consider Miller's reverse Adams spectral sequence

$$Ext^p_{\mathcal{AD}}(\pi_* K(n)_+, \Lambda_q) \Rightarrow H^{p+q}_Q K(n)_+.$$

Since $n \geq 1$, $\pi_* K(n)_+ \cong \Lambda(x_n)$, an exterior algebra on a generator of degree $n$, and

$$e : Ext^p_{\mathcal{UD}}(\Sigma^n \mathbf{F}_2, \Sigma^q \mathbf{F}_2) \to Ext^p_{\mathcal{AD}}(\pi_* K(n)_+, \Lambda_q)$$

is an isomorphism. By 13.6, a basis for the target of $e$ is given by all symbols of the form

$$\gamma^{i_1} \cdots \gamma^{i_k}(\iota^*)$$

with the composition allowable, $s \geq 0$, $i_t \geq 2$ for all $t$, and $i_k \leq n$. Here, of course,

$$\iota^* \in Ext^0_{\mathcal{UD}}(\Sigma^n \mathbf{F}_2, \Sigma^n \mathbf{F}_2)$$

is the non-zero element. Since $\iota^*$ detects $\iota_n \in H^n_Q K(n)_+$, 14.1 and 14.2 together imply that $\gamma^{i_1} \cdots \gamma^{i_k}(\iota^*)$ survives to $E_\infty$ and detects $P^{i_1} \cdots P^{i_k}(\iota_n)$. So the reverse Adams spectral sequence collapses, and the result follows.

We now must prove 14.1 and 14.2. The rest of the section will be taken up with the proof of 14.1 and most of that proof will be taken up with making the proper definition of the operations at $E_2$. Then 14.2 will be proved in the next section.

The spectral sequence discussed in 14.2 is that of 12.2; it is obtained by analyzing the bisimplicial vector space $Q\bar{S}_{\cdot,\cdot}A$, where

$$(14.4.1) \qquad Q\bar{S}_{p,q}A = QS^{p+1}A_q \cong I\bar{S}^p A_q$$

More explicitly, the spectral sequence of 12.2 is given by

$$(14.4.2) \qquad \pi^p \pi^q (Q\bar{S}_{\cdot,\cdot}A)^* \Rightarrow \pi^{p+q} diag(Q\bar{S}_{\cdot,\cdot}A)^*.$$

We then identified the $E_2$ term and the abuttment. We will work with this description.

Implicit in the description of the spectral sequence of 14.4.2 is the relationship between the total chain complex of a bisimplicial vector space and its diagonal. If we let $C(Q\bar{S}_{\cdot,\cdot}A)$ be the total complex of $Q\bar{S}_{\cdot,\cdot}A$ and $C(diagQ\bar{S}_{\cdot,\cdot}A)$ the chain complex of the simplicial vector space $diagQ\bar{S}_{\cdot,\cdot}A$, then there is a chain equivalence

$$\Delta : C(Q\bar{S}_{\cdot,\cdot}A) \to C(diagQ\bar{S}_{\cdot,\cdot}A).$$

In fact, this is true of any bisimplicial vector space and $\Delta$ can be written down explicitly in terms of $(p,q)$-shuffles — which are defined using the horizontal and vertical degeneracy operators in a bisimplicial vector space.

The first step, then, is to define a coproduct on $C(Q\bar{S}.,.A)$ and to explore its properties. Since, for a fixed $q$, $\bar{S}.,_q = \bar{S}.A_q$ is an almost-free simplicial algebra, we obtain a coproduct from section 5

$$\psi_{\bar{S}.A_q} : C(Q\bar{S}.,_qA) \to C(Q\bar{S}.,_qA \otimes Q\bar{S}.,_qA) \tag{14.5.1}$$

of degree $-1$. Since this is natural in $A_q$, we obtain a coproduct

$$\psi_{\bar{S}.,.A} : C(Q\bar{S}.,.A) \to C(Q\bar{S}.,.A \otimes Q\bar{S}.,.A) \tag{14.5.2}$$

of degree $-1$ in $p$ and degree $0$ in $q$.

In addition, let $B = diag\bar{S}.,.A$. Then $B$ is an almost-free simplicial algebra and, hence, supports a coproduct

$$\psi_B : C(QB) \to C(QB \otimes QB)$$

of degree $-1$. Since

$$QB = Qdiag\bar{S}.,.A = diagQ\bar{S}.,.A$$

this is relevant; indeed, this coproduct is used to define the product and operations on $H^*_QA$. The following result says that the coproduct (14.5.2) can be used to study this coproduct on the diagonal.

**Lemma 14.6:** There is a chain homotopy commutative diagram, where the vertical maps are an Eilenberg-MacLane chain equivalences

$$\begin{array}{ccc} C(QSA) & \xrightarrow{\psi_{\bar{S}.,.A}} & C(Q\bar{S}.,.A \otimes Q\bar{S}.,.A) \\ \downarrow \Delta & & \downarrow \Delta \\ C(QB)) & \xrightarrow{\psi_B} & C(QB \otimes QB). \end{array}$$

Here $B = diaq\bar{S}.,.A$ and

$$QB \otimes QB = diag[Q\bar{S}.,.A \otimes QSA].$$

**Proof:** In fact, if we take $\Delta$ to be the map defined using $(p,q)$-shuffles, the diagram will actually commute. To see this, recall that $\psi_{\bar{S}.A_q}$, as in (14.5.1) is defined by taking the product of

$$(d_0 \otimes d_0)\psi, d_0\psi \in Hom_{\mathcal{A}}(\bar{S}_p A_q, \bar{S}_{p-1}A_q \otimes \bar{S}_{p-1}A_q)$$

where $\psi : \bar{S}.A_q \to \bar{S}.A_q \otimes \bar{S}.A_q$ is the Hopf algebra diagonal. $\psi_B$ is defined similarly. Since the shuffles are defined using the degeneracies, a routine diagram chase, the simplicial identities, and the definitions imply the result.

To apply this result and to prove Theorem 14.1, we need to define the operations in the spectral sequence 14.4.2. To do this, we use the method of Singer, who construct chain level higher Eilenberg-Zilber maps for the chain complex associated to a bisimplicial vector space.

First a convention: let $V \in ss\mathbf{F}_2$ be a bisimplicial vector space. Then $V = \{V_{p,q}\}$ is set of vector spaces, one for each pair of non-negative integers (p,q), connected by various face and degeneracy operators. Let

$$d_i^h : V_{p,q} \to V_{p-1,q}$$

be the *horizontal* face operators and

$$d_i^v : V_{p,q} \to V_{p,q-1}$$

the *vertical* face operators. This establishes a horizontal and a vertical direction in $V$. So, for example, the sum of the horizontal face operators yields a horizontal differential $\partial^h : V_{p,q} \to V_{p-1,q}$.

Now let $\{D_i\}$ be a set of natural higher Eilenberg-Zilber maps, as in 3.6. Then, for bisimplicial vector spaces $V$ and $W$, if we fix $p$, we have that $V_{p,\cdot}$ and $W_{p,\cdot}$ are simplicial vector spaces. Therefore, $D_i$ determines a map

$$D_i^v : C(V_{p,\cdot} \otimes W_{p,\cdot}) \to C(V_{p,\cdot}) \otimes C(W_{p,\cdot})$$

so that $D_0^v$ is a chain equivalence and for $i > 0$,

$$\partial^v D_i^v + D_i^v \partial^v = D_{i-1}^v + TD_{i-1}^v T.$$

Here $T$ is the switch map. Since this is a natural construction

$$\partial^h D_i^v = D_i^v \partial^h$$

for all $i$. Also, for fixed $j$ and $k$, $V_{\cdot,j}$ and $W_{\cdot,k}$ are simplicial vector spaces, so $D_i$ determines a map

$$D_i^h : C(V_{\cdot,j} \otimes W_{\cdot,k}) \to C(V_{\cdot,j}) \otimes C(W_{\cdot,k})$$

so that $D_0^h$ is a chain equivalence and for $i > 0$,

$$\partial^h D_i^h + D_i^h \partial^h = D_{i-1}^h + T D_{i-1}^h T.$$

Again, this is natural in $j$ and $k$. Thus, following Singer, if we define

$$K_i : C(V \otimes W) \to C(V) \otimes C(W)$$

by

$$K_i = \sum_{s+t=i} T^t D_s^h T^t D_t^v \tag{14.7}$$

we easily compute that $K_0$ is a chain equivalence and for $i > 0$

$$\partial K_i + K_i \partial = K_{i-1} + T K_{i-1} T$$

where $\partial$ denotes the total differential.

Now let $A \in s\mathcal{A}$ and $\bar{S}_{\cdot,\cdot}A$ the associated bisimplicial algebra. Define operations

$$P^i : H^n C(Q\bar{S}_{\cdot,\cdot}A)^* \to H^{n+i+1} C(Q\bar{S}_{\cdot,\cdot}A)^* \tag{14.8.1}$$

as follows. Let $\psi_{\bar{S}_{\cdot,\cdot}A}$ be the chain map of 14.6 and define

$$\Theta^i : C(Q\bar{S}_{\cdot,\cdot}A)^* \to C(Q\bar{S}_{\cdot,\cdot}A)^*$$

by

$$\Theta^i(\alpha) = \psi^*_{\bar{S}_{\cdot,\cdot}A} K^*_{n-i}(\alpha \otimes \alpha) + \psi^*_{\bar{S}_{\cdot,\cdot}A} K^*_{n-i+1}(\alpha \otimes \partial\alpha) \tag{14.8.2}$$

for $\alpha$ of total degree $n$. Then $\Theta^i$ is a quadratic chain map and induces the homomorphic operations of 14.8.2. Compare 5.10.

That these operations are the correct object of study is a consequence of the following.

**Proposition 14.9:** Let

$$\Delta^* : H^*_Q A \cong H^* Q diag \bar{S}_{.,.} A^* \to H^* C(Q\bar{S}_{.,.}A)^*$$

be the Eilenberg-Maclane isomorphism. Then

$$\Delta^* P^i = P^i \Delta^*.$$

**Proof:** Consider the following diagram, where $V$ and $W$ are bisimplicial vector spaces:

$$\begin{array}{ccc} C(V \otimes W) & \xrightarrow{K_i} & C(V) \otimes C(W) \\ \downarrow \Delta & & \downarrow \Delta\otimes\Delta \\ C(diagV \otimes diagW) & \xrightarrow{D_i} & C(diagV) \otimes C(diagW). \end{array}$$

In light of 14.6, the definitions of 14.8 and the definitions of 5.10, it is sufficient to construct maps

$$E_i : C(V \otimes W) \to C(diagV) \otimes C(diagW)$$

so that $E_0$ and for $i \geq 0$

$$\partial E_{i+1} + E_{i+1}\partial = E_i + TE_iT + (\Delta \otimes \Delta)K_i + D_i\Delta.$$

Then we could set $V = W = Q\bar{S}_{.,.}A$. To see the existence of the $E_i$ we use a standard acyclic models argument. First note that since $\Delta$ is a chain map, we have that $D_0\Delta$ and $(\Delta \otimes \Delta)K_0$ are chain maps and for $i \geq 0$,

$$\partial D_{i+1}\Delta + D_{i+1}\Delta\partial = D_i\Delta + TD_i\Delta T$$

and

$$\partial(\Delta \otimes \Delta)K_{i+1} + (\Delta \otimes \Delta)K_{i+1}\partial = (\Delta \otimes \Delta)K_i + T(\Delta \otimes \Delta)K_iT.$$

Now one proceeds exactly as in the proof of Lemma 9.5 of [24]. Incidentally, if $\mathbf{F}_2\Delta[p] \in s\mathbf{F}_2$ is the standard $p$-simplex, so that

$$Hom_{s\mathbf{F}_2}(\mathbf{F}_2\Delta[p], U) \cong U_p$$

then the bisimplicial vector spaces

$$\mathbf{F}_2\Delta[p]\hat{\otimes}\mathbf{F}_2\Delta[q]$$

with

$$(\mathbf{F}_2\Delta[p]\hat{\otimes}\mathbf{F}_2\Delta[q])_{s,t} = \mathbf{F}_2\Delta[p]_s \otimes \mathbf{F}_2\Delta[q]_t$$

are the acyclic models in the category $ss\mathbf{F}_2$ of bisimplicial vector spaces; indeed,

$$Hom_{ss\mathbf{F}_2}(\mathbf{F}_2\Delta[p]\hat{\otimes}\mathbf{F}_2\Delta[q], V) \cong V_{p,q}.$$

We'd next like to see how the operations $P^i$ of 14.8 behave with respect to the filtration of $C(Q\bar{S}_{\cdot,\cdot}A)^*$ used to produce the spectral sequence of 14.4.2. This was filtration by degree in $p$. To do this, Singer points out the existence of "special" higher Eilenber-Zilber maps; that is, he notices that for simplicial vector spaces $U_1$ and $U_2$ we can make a choice of maps

$$D_i : C(U_1 \otimes U_2)_n \to [C(U_1) \otimes C(U_2)]_{n+i}$$

so that

$$D_i = 0$$

if $i > n$. Now let $V$ be a bisimplicial vector space and filter $C(V)^*$ by degree in $p$. Denote this filtration by $F^*V$. Then if we use the special Eilenber-Zilber maps to define the homomorphisms $K_i$ of 14.7, then we can conclude that

$$K_i^* F^{2p}C(V)^* \otimes C(W)^* \subseteq F^pC(V \otimes W)^*$$

so that the maps $\Theta^i$ of (14.8.2) induce maps

(14.10.1) $$\Theta^i : F^pC(Q\bar{S}_{\cdot,\cdot}A)^* \to F^{p+1}C(Q\bar{S}_{\cdot,\cdot}A)^*$$

and, hence, operations

$$P^i : \pi^p \pi^q (Q\bar{S}_{\cdot,\cdot}A)^* \to \pi^{p+1}\pi^{q+i}(Q\bar{S}_{\cdot,\cdot}A)^*. \tag{14.10.2}$$

**Proof of 14.1:** The operations are defined in 14.10. Theorem 14.1.1 follows from that fact that $\Theta^i$ is a chain map. Theorem 14.1.2 follows from 14.9.

To finish this section and prepare for the next, we examine an alternative definition of the operations at $E_2$ — that is, of the operations in 14.10.2. If we fix $p$, then the naturality of the construction of $\psi_{\bar{S}_{\cdot,\cdot}A}$ implies that we obtain a map of simplicial vector spaces

$$\psi_{\bar{S}_{\cdot,\cdot}A} : Q\bar{S}_{p+1,\cdot}A \to Q\bar{S}_{p,\cdot}A \otimes Q\bar{S}_{p,\cdot}A.$$

This maps is cocommutative, so we obtain operations

$$\mathrm{Sq}^i : \pi^q(Q\bar{S}_{p,\cdot}A)^* \to \pi^{q+i}(Q\bar{S}_{p+1,\cdot}A)^*.$$

These operations are natural. Singer ([23], Theorem 5) the implies that

$$P^i = \pi^*\mathrm{Sq}^i : \pi^p\pi^q(Q\bar{S}_{\cdot,\cdot}A)^* \to \pi^{p+1}\pi^{q+i}(Q\bar{S}_{\cdot,\cdot}A)^*. \tag{14.11}$$

If we preferred, we could work directly with homotopy, rather than dualizing. Using the methods of section 3, we see that $\psi_{\bar{S}_{\cdot,\cdot}A}$ induces a map

$$\pi_*\psi_{\bar{S}_{\cdot,\cdot}A} : \pi_*Q\bar{S}_{p+1,\cdot}A \to \pi_*S^2Q\bar{S}_{p,\cdot}A$$

and we could write

$$\pi_*\psi_{\bar{S}_{\cdot,\cdot}A}(x) = \sum \pi_*(1+T)(y_j \otimes z_j) + \sum \sigma_i(x\mathrm{Sq}^i) \tag{14.12}$$

for some $y_j, z_j \in \pi_*Q\bar{S}_{p,\cdot}A$. The dual of the $(\cdot)\mathrm{Sq}^i$ could then be used in the formula 14.11 to define $P^i$ at $E_2$.

## 15. The proof of Theorem 14.2

We devote this section to proving the following result, which is a restatement of 14.2.

**Theorem 15.1:** For all $A \in s\mathcal{A}$ and all $i$, there is a commutative diagram

$$\begin{array}{ccc} Ext^p_{\mathcal{UD}}(Q\pi_*A, \Sigma^q \mathbf{F}_2) & \xrightarrow{e} & Ext^p_{\mathcal{AD}}(\pi_*A, \Lambda_q) \\ \downarrow \gamma^i & & \downarrow P^i \\ Ext^{p+1}_{\mathcal{UD}}(Q\pi_*A, \Sigma^{q+i}\mathbf{F}_2) & \xrightarrow{e} & Ext^{p+1}_{\mathcal{AD}}(\pi_*A, \Lambda_{q+i}) \end{array}$$

Given that the relations among the $\gamma^i$ (13.11) and the relations among the $P^i$ (9.1) are identical, and that $e$ is an isomorphim in the case of the universal example $K(n)_+$ when $n > 0$, it would be surprising if some result such as 15.1 were not true. However, it is quite tedious to prove. The difficulty is that we do not have a very good hold on $e$ — it is defined in terms of a filtration on a bisimplicial vector space — and that the definition of the operations $P^i$ is rather complicated. We begin our attack on 15.1 by rectifying the former of the two problems.

Let $\Gamma \in \mathcal{AD}$. Then the natural map $I\Gamma \to Q\Gamma$ from the augmentation ideal to the indecomposables induces a natural map

$$\bar{\mathfrak{S}}\Gamma = \mathfrak{S}(I\Gamma) \to \mathfrak{S}(Q\Gamma)$$

and, hence, a natural map defined by the composition

$$Q\bar{\mathfrak{S}}\Gamma \to Q\mathfrak{S}(Q\Gamma) \cong \bar{\mathcal{P}}(Q\Gamma).$$

Thus we get a natural transformation of functors

$$\rho : Q\bar{\mathfrak{S}}(\cdot) \to \bar{\mathcal{P}}(Q\cdot).$$

We'd like to extend this to a morphism of simplicial objects

$$(15.2) \qquad \rho_\cdot : Q\bar{\mathfrak{S}}_\cdot\Gamma \to \bar{\mathcal{P}}_\cdot(Q\Gamma).$$

To do this let

$$\epsilon : \bar{\mathfrak{S}} \to 1 \qquad \text{and} \qquad \eta : \bar{\mathfrak{S}} \to \bar{\mathfrak{S}}^2$$

and

$$\epsilon : \bar{\mathcal{P}} \to 1 \qquad \text{and} \qquad \eta : \bar{\mathcal{P}} \to \bar{\mathcal{P}}^2$$

be the structure maps of the cotriples $\bar{\mathfrak{S}}$ and $\bar{\mathcal{P}}$ respectively. The simplicial objects of (15.2) are determined by these structure maps. Compare the proof of 1.4.

It is a simple matter to show that there is a commutative diagram

$$\text{(15.3.1)} \qquad \begin{array}{ccc} Q\bar{\mathfrak{S}}\Gamma & \xrightarrow{\rho} & \bar{\mathcal{P}}(Q\Gamma) \\ \downarrow Q\epsilon & & \downarrow \epsilon \\ Q\Gamma & \xrightarrow{=} & Q\Gamma. \end{array}$$

Now define a map $\rho_1$ by the composition

$$Q\bar{\mathfrak{S}}^2\Gamma = Q\bar{\mathfrak{S}}(\bar{\mathfrak{S}}\Gamma) \xrightarrow{\rho\bar{\mathfrak{S}}\Gamma} \bar{\mathcal{P}}(Q\bar{\mathfrak{S}}\Gamma) \xrightarrow{\bar{\mathcal{P}}\rho\Gamma} \bar{\mathcal{P}}^2(Q\Gamma).$$

The definitions now imply the existence of a commutative diagram

$$\text{(15.3.2)} \qquad \begin{array}{ccc} Q\bar{\mathfrak{S}}\Gamma & \xrightarrow{\rho} & \bar{\mathcal{P}}(Q\Gamma) \\ \downarrow Q\eta & & \downarrow \eta \\ Q\bar{\mathfrak{S}}^2\Gamma & \xrightarrow{\rho_1} & \bar{\mathcal{P}}^2(Q\Gamma). \end{array}$$

Suppose, recursively, that we have defined a map

$$\rho_n : Q\bar{\mathfrak{S}}_n\Gamma \to \bar{\mathcal{P}}_n(Q\Gamma).$$

Then, define $\rho_{n+1}$ to be the composition

$$\text{(15.3.3)} \quad Q\bar{\mathfrak{S}}_{n+1}\Gamma = Q\bar{\mathfrak{S}}(\bar{\mathfrak{S}}_n\Gamma) \xrightarrow{\rho} \bar{\mathcal{P}}(Q\bar{\mathfrak{S}}_n\Gamma) \xrightarrow{\rho_n} \bar{\mathcal{P}}(\bar{\mathcal{P}}_n(Q\Gamma)) = \bar{\mathcal{P}}_{n+1}(Q\Gamma).$$

Set $\rho_0 = \rho$. Then the diagrams of 15.3.1 and 15.3.2 imply that we have defined a map of simplicial objects

$$\rho_\cdot : Q\bar{\mathfrak{S}}_\cdot\Gamma \to \bar{\mathcal{P}}_\cdot(Q\Gamma).$$

Now for any $\Gamma \in \mathcal{AD}$, there is a natural isomorphism

$$Hom_{\mathcal{UD}}(Q\bar{\mathfrak{S}}_\cdot\Gamma, \Sigma^q\mathbf{F}_2) \cong Hom_{\mathcal{AD}}(\bar{\mathfrak{S}}_\cdot\Gamma, \Lambda_q).$$

Therefore the following is at least plausible.

**Proposition 15.4:** Let $\Gamma \in \mathcal{AD}$. Then the edge homomorphism

$$e : Ext^p_{\mathcal{UD}}(Q\Gamma, \Sigma^q \mathbf{F}_2) \to Ext^p_{\mathcal{AD}}(\Gamma, \Lambda_q)$$

is induced by the map

$$\rho^*_. : Hom_{\mathcal{UD}}(\bar{\mathcal{P}}.(Q\Gamma), \Sigma^q \mathbf{F}_2) \to Hom_{\mathcal{UD}}(Q\bar{\mathfrak{S}}.(\Gamma), \Sigma^q \mathbf{F}_2).$$

**Proof:** Consider the bisimplicial module $\bar{\mathcal{P}}.Q\bar{\mathfrak{S}}.(\Gamma)$ used in the proof of 12.5. We refer to that proof freely here. Then

$$[\bar{\mathcal{P}}.Q\bar{\mathfrak{S}}.(\Gamma)]_{p,q} = \bar{\mathcal{P}}_p Q\bar{\mathfrak{S}}_q(\Gamma).$$

We will call $p$ the horizontal direction and $q$ the vertical direction. Then there is a horizontal augmentation

$$\epsilon^h : \bar{\mathcal{P}}.Q\bar{\mathfrak{S}}.(\Gamma) \to Q\bar{\mathfrak{S}}.\Gamma$$

that induces a chain equivalence between the total complex of this bisimplicial vector module and the chain complex associated to the simplicial module $Q\bar{\mathfrak{S}}.\Gamma$. This can be proved as in 12.5; we will construct a specific chain homotopy below. There is also a vertical augmentation

$$\epsilon^v : \bar{\mathcal{P}}.Q\bar{\mathfrak{S}}.(\Gamma) \to \bar{\mathcal{P}}.(Q\Gamma)$$

and the edge homomorphism $e$ is defined by the diagram

$$Hom_{\mathcal{UD}}(\bar{\mathcal{P}}.(Q\Gamma), \mathbf{F}_2) \xrightarrow{(\epsilon^v)^*} Hom_{\mathcal{UD}}(\bar{\mathcal{P}}.Q\bar{\mathfrak{S}}.(\Gamma), \mathbf{F}_2) \xleftarrow{(\epsilon^h)^*} Hom_{\mathcal{UD}}(Q\bar{\mathfrak{S}}.\Gamma, \mathbf{F}_2)$$

using the fact that the second map induces a cochain equivalence.

For a simplicial (or bisimplicial) vector space $V$, let $C(V)$ be the chain complex (or total complex) associated to $V$. To prove the result, we define and investigate a chain map

$$\rho : C(Q\bar{\mathfrak{S}}.\Gamma) \to C(\bar{\mathcal{P}}.Q\bar{\mathfrak{S}}.(\Gamma))$$

that is chain inverse to $\epsilon^h$.

Since, for any $\Lambda \in \mathcal{AD}$, we have an isomorphism $Q\bar{\mathfrak{S}}\Lambda \cong \bar{\mathcal{P}}(I\Lambda)$, the projection

$$\epsilon : \bar{\mathcal{P}}Q\bar{\mathfrak{S}}\Lambda \to Q\bar{\mathfrak{S}}\Lambda$$

has a section

$$s_0 : Q\bar{\mathfrak{S}}\Lambda \to \bar{\mathcal{P}}Q\bar{\mathfrak{S}}\Lambda$$

so that $\epsilon s_0 = id$. Then, we can define

$$s_{n+1} : \bar{\mathcal{P}}_nQ\bar{\mathfrak{S}}\Lambda \to \bar{\mathcal{P}}_{n+1}Q\bar{\mathfrak{S}}\Lambda$$

by $s_{n+1} = \bar{\mathcal{P}}^{n+1}s_0$. As the name would indicate, $s_{n+1}$ acts as an extra degeneracy in the simplicial module $\bar{\mathcal{P}}.Q\bar{\mathfrak{S}}\Lambda$; thus, it is routine to check that

$$\partial s_{n+1} + s_n\partial = id : \bar{\mathcal{P}}_nQ\bar{\mathfrak{S}}\Lambda \to \bar{\mathcal{P}}_nQ\bar{\mathfrak{S}}\Lambda.$$

Therefore we have defined a contraction of $\bar{\mathcal{P}}.Q\bar{\mathfrak{S}}\Lambda$. These maps $s_{n+1}$ are natural in $\Lambda$. Thus, for every $q$, we obtain maps

$$s^h_{n+1} : \bar{\mathcal{P}}_n\bar{\mathfrak{S}}_q\Lambda \to \bar{\mathcal{P}}_{n+1}\bar{\mathfrak{S}}_q\Lambda$$

that commute with

$$d^v_i : \bar{\mathcal{P}}_nQ\bar{\mathfrak{S}}_q\Lambda \to \bar{\mathcal{P}}_nQ\bar{\mathfrak{S}}_{q-1}\Lambda$$

when $1 \le i \le q$. These maps do not neccesarily commute with $d^v_0$. It is the existence of these maps $s^h_{n+1}$ that show that $\epsilon^h$ defines a chain equivalence, as asserted above.

Recursively define, for $0 \le k \le q$, maps

$$\rho^k_q : Q\bar{\mathfrak{S}}_q\Gamma \to \bar{\mathcal{P}}_kQ\bar{\mathfrak{S}}_{q-k}\Lambda$$

by $\rho^0_q = s^h_o$ and $\rho^k_q = s^h_k d^v_0 \rho^{k-1}_q$. We next inductively show that

$$\partial^h \rho^{k+1}_q = \partial^v \rho^k_q + \rho^k_{q-1}.$$

This is true for $k = 0$ by direct calculation and for $k > 0$ by

$$\begin{aligned}
\partial^h \rho_q^{k+1} &= \partial^h s_{k+1}^h d_0^v \rho_q^k \\
&= s_k^h d_0^v \partial^h \rho_q^k + d_0^v \rho_q^k \\
&= s_k^h d_0^v \partial^v \rho_q^{k-1} + s_k^h d_0^v \rho_{q-1}^{k-1} + d_0^v s_k^h d_0^v \rho_q^{k-1} \\
&= \partial^v \rho_q^k + \rho_{q-1}^k
\end{aligned}$$

making liberal use of the simplicial identities and the fact that $s_k^h$ commutes with $d_i^v$ for $i > 0$. Now define

$$\rho : Q\bar{\mathfrak{S}}_q\Gamma \to C(\bar{\mathcal{P}}.Q\bar{\mathfrak{S}}\cdot\Lambda) = \oplus_k \bar{\mathcal{P}}_k Q\bar{\mathfrak{S}}_{q-k}\Lambda$$

by

$$\rho(x) = (\rho_q^0(x), \ldots, \rho_q^q(x)).$$

The tedious calculation above shows that $\rho$ is a chain map

$$\rho : C(Q\bar{\mathfrak{S}}.\Gamma) \to C(\bar{\mathcal{P}}.Q\bar{\mathfrak{S}}\cdot\Lambda).$$

Since

$$\epsilon^h \rho = id : C(Q\bar{\mathfrak{S}}.\Gamma) \to C(Q\bar{\mathfrak{S}}.\Gamma)$$

we have that $\rho$ is a chain equivalence and that

$$\epsilon^v \rho : C(Q\bar{\mathfrak{S}}.\Gamma) \to C(\bar{\mathcal{P}}.(Q\Gamma))$$

can – after applying $Hom_{\mathcal{UD}}(\cdot, \mathbf{F}_2)$ – be used to compute the edge homomorphism. However

$$\epsilon^v \rho = \epsilon^v \rho_q^q = \rho_q$$

where the last map is as in 15.3.3. So the result follows.

We now embark on the proof of 15.1. This will occupy the rest of the section.

Let $A \in s\mathcal{A}$ be a simplicial algebra and $\bar{S}_{\cdot,\cdot}A$ the associated bisimplicial algebra. In 14.11 we noticed that there were operations

$$\text{(15.5.1)} \qquad \mathrm{Sq}^i : \pi^q(Q\bar{S}_{p,\cdot}A)^* \to \pi^{q+i}(Q\bar{S}_{p+1,\cdot}A)^*$$

that could be used to compute the operations $P^i$ on $Ext_{\mathcal{AD}}(\pi_*A, \Lambda_*)$. Now, for any $A \in s\mathcal{A}$, there is a natural isomorphism

$$\pi_* Q\bar{S}A \cong \mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}(\pi_*A).$$

Thus we get naturally defined operations

$$\mathrm{Sq}^i : [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_p(\pi_*A)]^* \to [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_{p+1}(\pi_*A)]^*. \tag{15.5.2}$$

Since, for any $M \in \mathcal{UD}$, we have

$$Hom_{\mathcal{UD}}(M, \Sigma^q\mathbf{F}_2) \cong Hom_{n\mathbf{F}_2}(\mathbf{F}_2 \otimes_{\mathcal{UD}} M, \Sigma^q\mathbf{F}_2)$$

these are exactly the operations that we want to study.

The operations $\gamma^i$ on $Ext_{\mathcal{UD}}(Q\pi_*A, \mathbf{F}_2)$ were defined in 13.15.2; in addition, we showed that the could be computed using the naturally defined homomorphisms of 13.21:

$$\hat{\delta}^i : [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(Q\pi_*A)]^* \to [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(Q\pi_*A)]^*. \tag{15.6.1}$$

In fact, there is a commutative diagram

$$\begin{array}{ccc} [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(Q\pi_*A)]^* & \xrightarrow{\hat{\delta}^i} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(Q\pi_*A)]^* \\ \downarrow \Phi & & \downarrow \Phi \\ C^p(Q\pi_*A) & \xrightarrow{\gamma^i} & C^{p+1}(Q\pi_*A). \end{array} \tag{15.6.2}$$

Here $\Phi$ is the canonical cochain equivalence and $\gamma^i$ induces the operations $\gamma^i$ on $Ext_{\mathcal{UD}}$. Thus, to prove 15.1 we must contemplate the diagram

$$\begin{array}{ccc} C^p(Q\pi_*A) & \xrightarrow{\gamma^i} & C^{p+1}(Q\pi_*A) \\ \uparrow \Phi & & \uparrow \Phi \\ [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(Q\pi_*A)]^* & \xrightarrow{\hat{\delta}^i} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(Q\pi_*A)]^* \\ \downarrow \rho^*_\cdot & & \downarrow \rho^*_\cdot \\ [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_p(\pi_*A)]^* & \xrightarrow{\mathrm{Sq}^i} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_{p+1}(\pi_*A)]^*. \end{array} \tag{15.7}$$

As far as I know, this diagram does not actually commute, which means that we have to resort to subterfuge to show that it commutes up to homotopy.

The crucial observation is this: we noted that there were isomorphisms

$$\begin{aligned} \mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathfrak{S}}_{p+1}(\pi_* A) &\cong \pi_* Q\bar{S}_{p+1} A \\ &\cong \pi_* I\bar{S}_p A \\ &\cong \pi_* I\bar{S}^{p+1} A. \end{aligned}$$

There is a canonical split inclusion for any simplicial vector space $V$

$$S_2 V \xrightarrow{\subseteq} IS(V)$$

where $S_2V$ is the vector space of coinvariants of $V \otimes V$ under the action of $\Sigma_2$ that switches the factors. Therefore, for $A \in s\mathcal{A}$, there is a split inclusion

$$i : S_2 I\bar{S}^p A \xrightarrow{\subseteq} I\bar{S}^{p+1} A. \tag{15.8.1}$$

Hence, there is a split inclusion

$$j : (S_2)^{p+1} IA \xrightarrow{\subseteq} I\bar{S}^{p+1} A. \tag{15.8.2}$$

Applying homotopy we obtain a split inclusion

$$j_* : (\mathfrak{S}_2)^{p+1}(I\pi_* A) \cong \pi_*(S_2)^{p+1} IA \to \pi_* I\bar{S}^{p+1} A \cong \mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathfrak{S}}_{p+1}(\pi_* A)$$

or, by dualizing, a split surjection

$$j^* : [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathfrak{S}}_{p+1}(\pi_* A)]^* \to [(\mathfrak{S}_2)^{p+1}(I\pi_* A)]^*.$$

Here $\mathfrak{S}_2$ is the functor so that for any simplicial vector space $V$, $\pi_* S_2(V) \cong \mathfrak{S}_2(\pi_* V)$.

**Lemma 15.9:** There is a map

$$\lambda : C^{p+1}(Q\pi_* A) \to [\mathfrak{S}_2(I\pi_* A)]^*$$

so that the following diagram commutes

$$\begin{array}{ccc} [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}^{p+1}(Q\pi_* A)]^* & \xrightarrow{\rho_*^*} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_{p+1}(\pi_* A)^* \\ \downarrow \Phi & & \downarrow j^* \\ C^{p+1}(Q\pi_* A) & \xrightarrow{\lambda} & [\mathfrak{S}_2^{p+1}(I\pi_* A)]^*. \end{array}$$

**Proof:** We need only show that if $\Phi(y) = 0$, then $j^*\rho_\cdot^*(y) = 0$. This follows from the definition of $\Phi$ (13.18) and the description of $\mathfrak{S}_2$ given in 3.5.

Because of this lemma, we can extend the diagram 15.7 to a diagram

$$\begin{array}{ccccc}
C^p(Q\pi_*A) & \xleftarrow{\Phi} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(Q\pi_*A)]^* & \xrightarrow{\rho_\cdot^*} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_p(\pi_*A)]^* \\
\downarrow \gamma^i & & \downarrow \hat{\delta}^i & & \downarrow \mathrm{Sq}^i \\
C^{p+1}(Q\pi_*A) & \xleftarrow{\Phi} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_{p+1}(Q\pi_*A)]^* & \xrightarrow{\rho_\cdot^*} & [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_{p+1}(\pi_*A)]^* \\
\downarrow = & & \downarrow \Phi & & \downarrow j^* \\
C^{p+1}(Q\pi_*A) & \xleftarrow{=} & C^{p+1}(Q\pi_*A) & \xrightarrow{\lambda} & [\mathfrak{S}_2^{p+1}(I\pi_*A)]^*.
\end{array}$$

Theorem 15.1 now follows immediately from this diagram and the following lemma.

**Lemma 15.10:** $j^*\mathrm{Sq}^i\rho_\cdot^* = \lambda\Phi\hat{\delta}^i$.

**Proof:** We actually prove slightly more. Since $\lambda\Phi = j^*\rho_\cdot^*$, it is sufficient to show that $j^*\mathrm{Sq}^i\rho_\cdot^* = j^*\rho_\cdot^*\hat{\delta}^i$. Now, since

$$j : S_2^{p+1}IA \to I\bar{S}^{p+1}A$$

factors as

$$S_2^{p+1}IA \to S_2I\bar{S}^pA \to I\bar{S}^{p+1}A$$

we have that $j^*$ factors as

$$[\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_{p+1}(\pi_*A)]^* \xrightarrow{\bar{j}^*} [\mathfrak{S}_2(\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_p(\pi_*A)]^* \longrightarrow [\mathfrak{S}_2^{p+1}(I\pi_*A)]^*.$$

We will show that

$$\bar{j}^*\mathrm{Sq}^i\rho_\cdot^* = \bar{j}^*\rho_\cdot^*\hat{\delta}^i.$$

But $\mathfrak{S}_2(W)$ was computed in 3.5; it is spanned by all elements of the form

$$uv \qquad \text{and} \qquad \delta_j(v)$$

where $u, v \in W$. Let $\langle\ ,\ \rangle$ be the pairing betwen a vector space and its dual. If we can show that for all $z \in [\mathbf{F}_2 \otimes_{\mathcal{UD}} \bar{\mathcal{P}}_p(Q\pi_*A)]^*$ and all $u, v \in \mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_p(\pi_*A)$, that

(15.11.1) $$\langle\bar{j}^*\mathrm{Sq}^i\rho_\cdot^*(z), uv\rangle = 0 = \langle\bar{j}^*\rho_\cdot^*\hat{\delta}^i(z), uv\rangle$$

and

$$(15.11.2) \qquad \langle \bar{j}^*\mathrm{Sq}^i\rho_\cdot^*(z), \delta_j(v)\rangle = \langle \bar{j}^*\rho_\cdot^*\hat{\delta}^i(z), \delta_j(v)\rangle = \begin{cases} 0, & \text{if } i \neq j; \\ \langle \rho_\cdot^* z, v\rangle, & \text{if } i = j \end{cases}$$

then we will have accomplished out goal. These formulas will be verified in Lemmas 15.12 and 15.14 below.

**Lemma 15.12:** For all $z$ and $u, v$

$$\langle \bar{j}^*\rho_\cdot^*\hat{\delta}^i(z), uv\rangle = 0$$
$$\langle \bar{j}^*\rho_\cdot^*\hat{\delta}^i(z), \delta_j(v)\rangle = \begin{cases} 0, & \text{if } i \neq j; \\ \langle \rho_\cdot^* z, v\rangle, & \text{if } i = j. \end{cases}$$

**Proof:** Let $\bar{j}_*$ be the map whose dual is $\bar{j}^*$:

$$\bar{j}_* : \mathfrak{S}_2(\mathbf{F}_2 \otimes_{\mathcal{UD}} Q(\bar{\mathfrak{S}}_p(\pi_* A)) \to \mathbf{F}_2 \otimes_{\mathcal{UD}} Q(\bar{\mathfrak{S}}_{p+1}(\pi_* A)).$$

Then $\rho.\bar{j}_*(uv) = 0$ and

$$\langle \hat{\delta}^i(z), \rho.\bar{j}_*\delta_j(v)\rangle = \begin{cases} 0, & i \neq j; \\ \langle z, \rho.v\rangle, & i = j \end{cases}$$

by 13.22. The result follows.

The proof of the other half of the equalities of 15.11 requires that we have a formula for the composition

$$[\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_p(\pi_* A)]^* \xrightarrow{\mathrm{Sq}^i} [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_{p+1}(\pi_* A)]^* \xrightarrow{\bar{j}^*} [\mathfrak{S}_2(\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\bar{\mathfrak{S}}_p(\pi_* A))]^*.$$

**Lemma 15.13:** For all $w$ and $u, v$ we have

$$\langle \bar{j}^*\mathrm{Sq}^i(w), uv\rangle = 0$$
$$\langle \bar{j}^*\mathrm{Sq}^i(w), \delta_j(v)\rangle = \begin{cases} 0, & \text{if } i \neq j; \\ \langle w, v\rangle, & \text{if } i = j. \end{cases}$$

The proof is below.

**Lemma 15.14:** For all $z$ and $u, v$, we have

$$\langle \bar{j}^* \mathrm{Sq}^i \rho_\cdot^*(z), uv\rangle = 0$$
$$\langle \bar{j}^* \mathrm{Sq}^i \rho_\cdot^*(z), \delta_j(v)\rangle = \begin{cases} 0, & \text{if } i \neq j; \\ \langle \rho_\cdot^*(z), v\rangle, & \text{if } i = j. \end{cases}$$

**Proof:** This second formula follows immediately from Lemma 15.13 by setting $w = \rho_\cdot^*(z)$.

**Proof of 15.13:** By 13.12, the operations can be defined by considering

$$\pi_*\psi_{\bar{S}_{\cdot,\cdot}A} : \pi_* Q\bar{S}_{p+1,\cdot}A \to \pi_* S^2 Q\bar{S}_{p,\cdot}A$$

and using the formula

$$\pi_*\psi_{\bar{S}_{\cdot,\cdot}A}(x) = \sum_j \pi_*(1+T)(y_j \otimes z_j) + \sum_i \sigma_i(x\mathrm{Sq}^i). \tag{15.15}$$

Thus to study $\bar{j}^*\mathrm{Sq}^i$, we must consider the composition

$$S_2 Q\bar{S}_p A_q \xrightarrow{\bar{j}} Q\bar{S}_{p+1}A_q \xrightarrow{\psi_{\bar{S}_{\cdot,\cdot}A}} S^2 Q\bar{S}_p A_q$$

for fixed $p$ and $q$. We claim that

$$\psi_{\bar{S}_{\cdot,\cdot}A}\bar{j} = tr : S_2 Q\bar{S}_p A_q \to S^2 Q\bar{S}_p A_q$$

where $tr : S_2 V \to S^2 V$ is the map induced by $1 + T : V \otimes V \to V \otimes V$. If this is the case, then

$$\pi_*(\psi_{\bar{S}_{\cdot,\cdot}A}\bar{j})(x) = \pi_* tr(x).$$

Then Lemma 3.9 implies that

$$\begin{aligned} \pi_*(\psi_{\bar{S}_{\cdot,\cdot}A}\bar{j})(uv) &= \pi_*(1+T)(u \otimes v) \\ \pi_*(\psi_{\bar{S}_{\cdot,\cdot}A}\bar{j})(\delta_j(v)) &= \sigma_j(v). \end{aligned} \tag{15.16}$$

Then the result will follow by comparing the formulas 15.15 and 15.16.

To prove $\psi_{\bar{S}.,.A}\bar{j} = tr$, we will construct a commutative diagram

$$\begin{array}{lccccc} & S_2Q\bar{S}_pA_q & \xrightarrow{k} & \bar{S}_{p+1}A_q & \xrightarrow{\partial\psi+\psi\partial} & S^2\bar{S}_pA_q \\ & \uparrow = & & \uparrow & & \uparrow \\ (15.17) & S_2Q\bar{S}_pA_q & \xrightarrow{\bar{k}} & I\bar{S}_{p+1}A_q & \xrightarrow{H} & S^2I\bar{S}_pA_q \\ & \downarrow = & & \downarrow & & \downarrow \\ & S_2Q\bar{S}_pA_q & \xrightarrow{\bar{j}} & Q\bar{S}_{p+1}A_q & \xrightarrow{\psi_{S.,.A}} & S^2Q\bar{S}_pA_q \end{array}$$

where

15.17.1) the squares of the right column are constructed using Lemma 7.12;

15.17.2) $\psi : \bar{S}.A_q \to \bar{S}.A_q \otimes \bar{S}.A_q$ is the Hopf algebra diagonal that commutes withh all face and degeneracy maps except $d_0$;

15.17.3) $\partial\psi + \psi\partial = (d_0 \otimes d_0)\psi + \psi d_0$; and

15.17.4) if $x \in S_2Q\bar{S}_pA_q$, then

$$k(x) \in P\bar{S}_{p+1}A_q$$

where $P$ denotes the primitives and

$$d_0k(x) = x \in S_2Q\bar{S}_pA_q \cong S_2I\bar{S}_{p-1}A_q \subseteq \bar{S}_pA_q.$$

Only the existence of $k$ and the properties of 15.17.4 are not an immediate consequence of Lemma 7.12. Assuming, for the moment that we have constructed $k$, we can proceed as follows: let $x \in S_2Q\bar{S}_pA_q$. Then we may write

$$x = \sum_i y_i^2 + \sum_j w_jz_j$$

where $y_i, w_j, z_j \in Q\bar{S}_pA_q$. Then, using 15.17.4, we can compute that

$$\begin{aligned}\psi d_0k(x) &= \psi(\sum y_i^2 + \sum w_jz_j) \\ &= x \otimes 1 + 1 \otimes x + \sum w_j \otimes z_j + z_j \otimes w_j\end{aligned}$$

and

$$\begin{aligned}(d_0 \otimes d_0)\psi k(x) &= (d_0 \otimes d_0)(k(x) \otimes 1 + 1 \otimes k(x)) \\ &= x \otimes 1 + 1 \otimes x.\end{aligned}$$

Hence

$$\begin{aligned}(\partial\psi + \psi\partial)(k(x)) &= \sum w_j \otimes z_j + z_j \otimes w_j \\ &= tr(x).\end{aligned}$$

Therefore, the result will follow from the diagram 15.17.

To construct $k$, we record the following observations.

$$\begin{aligned}Q\bar{S}_{p+1}A_q &\cong I\bar{S}_pA_q \\ &= \oplus_{n\geq 1}S_n(I\bar{S}_{p-1}A_q) \\ &\cong \oplus_{n\geq 1}S_n(Q\bar{S}_pA_q)\end{aligned}$$

and $\bar{\jmath}$ is defined by inclusion into the factor $n = 2$. Furthermore

$$I\bar{S}_{p+1}A_q \cong \oplus_{n\geq 1}S_n(I\bar{S}_pA_q).$$

We define

$$\bar{k} : S_2Q\bar{S}_pA_q \to I\bar{S}_{p+1}A_q$$

by the composition

$$S_2Q\bar{S}_pA_q \xrightarrow{\bar{\jmath}} I\bar{S}_pA_q = S_1(I\bar{S}_pA_q) \xrightarrow{\subseteq} I\bar{S}_{p+1}A_q$$

and we define $k$ by the diagram 15.17. That diagram certainly commutes and 15.17.4 follows from the facts that

$$V = S_1 \subseteq PS(V)$$

for any vector space $V$, and that the composite

$$IA = S_1IA \subseteq S(A) \xrightarrow{\epsilon} A$$

is the inclusion of the augmentation ideal.

## Appendix A: André's product on $H^*_Q \Sigma A$

In [2], André defined a product, for $A \in s\mathcal{A}$,

$$\langle\ ,\ \rangle : H^m_Q \Sigma A \otimes H^n_Q \Sigma A \to H^{m+n}_Q \Sigma A$$

where $\Sigma : s\mathcal{A} \to s\mathcal{A}$ is the suspension functor. This product is bilinear, commutative, and satisfies the Jacobi identity. Furthermore, using the isomorphism $H^*_Q A \cong H^{*+1}_Q \Sigma A$, we see that this product defines a product

$$H^m_Q A \otimes H^n_Q A \to H^{m+n+1}_Q A.$$

We will show that this product is the same as ours.

We begin with a recapitulation of André's results. In section 4, we noted that $H^*_Q \Sigma A$ could be computed using derivations. In like manner, it can be shown that $H^Q_* \Sigma A$ can be computed using Kaehler differentials. Specifically, let $A \in s\mathcal{A}$. Then, factor the augmentation $\epsilon : A \to \mathbf{F}_2$ as

$$A \xrightarrow{i} X \xrightarrow{\epsilon} \mathbf{F}_2$$

where $i$ is almost-free and $\epsilon$ is an acyclic fibration. This may be done functorially in $A$. Then let $J(X)$ be the kernel of the $A$-multiplication

$$X \otimes_A X \to X$$

and define the Kaehler differentials by

$$\Omega(X) = J(X)/J(X)^2.$$

We should really write $\Omega_A(X)$ for $\Omega(X)$ to emphasize that $\Omega(X)$ is functorial in $A$ as well as in $X$, but we prefer the lighter notation.

André proves the following result.

**Lemma A.1.**1.) $J(X)$ is both a left and right $X$-module.

2.) These two module structures agree on $\Omega(X)$.

3.) If we define $d : X \to \Omega(X)$ by letting $d(x)$ be the residue class of the element $x \otimes 1 + 1 \otimes x$, then $d$ is an $A$-derivation:

$$d(xy) = xd(y) + yd(x)$$

and

$$d(ax) = ad(x) \text{ for all } a \in A.$$

4.) The homomorphism

$$Hom_X(\Omega(X), W) \to Der_A(X, W)$$

given by

$$f \longmapsto f \circ d$$

is a natural isomorphism for all $\pi_0 X \cong \mathbf{F}_2$ modules $W$.

If $W$ is a $\pi_0 X \cong \mathbf{F}_2$ modules, then the composition

$$X_n \xrightarrow{d_0^n} X_0 \to \pi_0 X \cong \mathbf{F}_2$$

makes $W$ an $X_n$ module. Of course, this composition is just the augmentation. The isomorphism of A.1.4 is an isomorphism of simplicial vector spaces. Notice that A.1.4 says that differentials represent derivations, as they should.

Now we make a computation. If $W$ is a $\pi_0 X \cong \mathbf{F}_2$ module, since $W$ is an $X$-module via the augmentation $\epsilon : X \to \mathbf{F}_2$, we have that

$$Hom_X(\Omega(X), W) \cong Hom_{\mathbf{F}_2}(\mathbf{F}_2 \otimes_X \Omega(X), W).$$

Furthermore,

$$\begin{aligned} Der_A(X, W) &\cong Der_{\mathbf{F}_2}(\mathbf{F}_2 \otimes_A X, W) \\ &\cong Hom_{\mathbf{F}_2}(Q(\mathbf{F}_2 \otimes_A X), W). \end{aligned}$$

Thus, A.1.4 implies the following result.

**Lemma A.2:** There is a natural isomorphism of simplicial vector spaces

$$\mathbf{F}_2 \otimes_X \Omega(X) \cong Q(\mathbf{F}_2 \otimes_A X)$$

so that

$$\pi_* \mathbf{F}_2 \otimes_X \Omega(X) \cong H_*^Q \Sigma A.$$

**Proof:** By definition, $\Sigma A = \mathbf{F}_2 \otimes_A X$.

But André notes that the situation is actually somewhat simpler than one might have hoped. We will use the following lemma repeatedly; it is due to Quillen.

**Lemma A.3:** [20,II.6.6.b] Let $A \in s\mathcal{A}$ and let $N$ be an almost-free simplicial $A$-module. Let $M$ be a simplicial $A$-module. Then there is a spectral sequence

$$Tor_p^{\pi_* A}(\pi_* M, \pi_* N)_q \Rightarrow \pi_{p+q} M \otimes_A N.$$

**Lemma A.4:** If $A \in s\mathcal{A}$ is almost-free, the quotient map

$$\Omega(X) \to \mathbf{F}_2 \otimes_X \Omega(X)$$

induces a natural isomorphism in homotopy, and there are natural isomorphisms

$$H_*^Q \Sigma A \cong \pi_* \Omega(X)$$

and

$$H_Q^* \Sigma A \cong \pi^* Hom_{\mathbf{F}_2}(\Omega(X), \mathbf{F}_2).$$

**Proof:** If $A$ is almost-free, then $X$ is almost-free and each $X_n$ is a free $\mathbf{F}_2$-algebra. Thus André's calculations [2,Corrollaire 2] implies that we have an isomorphism of $X_n$-modules

$$\Omega(X_n) \cong X_n \otimes Q(\mathbf{F}_2 \otimes_{A_n} X_n).$$

This isomorphism is natural with respect to maps of *free* algebras so that $\Omega(X)$ is an almost-free simplicial $X$-module. So Lemma A.3 applies and and we have

$$\pi_* \Omega(X) \cong \pi_*(\mathbf{F}_2 \otimes_X \Omega(X)).$$

The isomorphisms about homology and cohomology follow from A.2.

Because of the hypotheses of this last result, we work freely in the homotopy category associated to $s\mathcal{A}$, so that we may replace, at will, any object in $s\mathcal{A}$ by an almost-free object.

We can use Lemma A.4 to give a description of the Hurewicz map

$$h : I\pi_*\Sigma A \to H_*^Q \Sigma A.$$

Consider the exact sequence

$$0 \to J(X) \to X \otimes_A X \to X \to 0.$$

Since $\pi_* X \cong \mathbf{F}_2$, we have that $\pi_* J(X) \cong I\pi_* X \otimes_A X$. If $A$ is almost-free, then the quotient map

$$\epsilon \otimes 1 : X \otimes_A X \to \mathbf{F}_2 \otimes_A X = \Sigma A$$

is a weak equivalence, by Lemma A.3. Thus, $\pi_* J(X) \cong I\pi_*\Sigma A$. The quotient map $J(X) \to \Omega(X)$ now induces the Hurewicz map.

To define his product, André first defines a coproduct in homology, then dualizes. We have noted that $\pi_*\Sigma A$ has the structure of a Hopf algebra. The diagonal is obtained as follows. We just saw that the quotient map $X \otimes_A X \to \mathbf{F}_2 \otimes_A X$ is a weak equivalence in $s\mathcal{A}$. The same will be true of the composite

$$1 \otimes \epsilon \otimes 1 : X \otimes_A X \otimes_A X \to X \otimes_A \mathbf{F}_2 \otimes_A X \cong X \otimes_A \mathbf{F}_2 \otimes \mathbf{F}_2 \otimes_A X \cong \Sigma A \otimes \Sigma A.$$

Then in [12] it was proved that the diagonal in $\pi_*\Sigma A$ obtained by applying homotopy to the map

$$1 \otimes \eta \otimes 1 : X \otimes_A X \to X \otimes_A X \otimes_A X$$

produces a Hopf algebra structure on $\pi_*\Sigma A$. Here $\eta : \mathbf{F}_2 \to X$ is the unit map. The induced diagonal

$$\psi : \pi_*\Sigma A \to \pi_*\Sigma A \otimes \pi_*\Sigma A$$

is not necessarily cocommutative; therefore we obtain a possibly non-trivial commutator coproduct

$$\varphi = \psi + T\psi : \pi_*\Sigma A \to \pi_*\Sigma A \otimes \pi_*\Sigma A.$$

where $T$ is the switch map. Since, for $x \in I\pi_*\Sigma A$, we have

$$\psi(x) = x \otimes 1 + 1 \otimes x + \sum y_i \otimes z_i$$

we see that

$$\varphi(x) = \sum y_i \otimes z_i + z_i \otimes y_i \in I\pi_*\Sigma A \otimes I\pi_*\Sigma A$$

so that $\varphi$ induces a map

$$\varphi : I\pi_*\Sigma A \to I\pi_*\Sigma A \otimes I\pi_*\Sigma A.$$

Now recall that $\pi_* J(X) \cong I\pi_*\Sigma X$, where $J(X)$ is the kernel of the multiplication map. We could use this fact to construct $\varphi$ directly.

Define the reduced diagonal

$$\bar{\psi} : \pi_*\Sigma A \to \pi_*\Sigma A \otimes \pi_*\Sigma A$$

to the homotopy of

$$\xi = [1 \otimes \eta \otimes 1 + \eta \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \eta] : X \otimes_A X \to X \otimes_A X \otimes_A X.$$

One checks that if $\psi(x) = x \otimes 1 + 1 \otimes x + \sum y_i \otimes z_i$, then

$$\bar{\psi} = \sum y_i \otimes z_i.$$

Hence the name. Notice that

$$\bar{\psi} + T\bar{\psi} = \psi + T\psi = \varphi.$$

Now we claim that there is a commutative diagram

$$\begin{array}{ccccc} J(X) & \xrightarrow{\xi} & J(x) \otimes_X J(X) & & \\ \downarrow & & \downarrow & & \\ X \otimes_A X & \xrightarrow{\xi} & X \otimes_A X \otimes_X X \otimes_A X & \cong & X \otimes_A X \otimes_A X \end{array}$$

where the vertical maps are induced by the inclusions. To see this, let $\sum x_i \otimes y_i \in J(X)$. Then $\sum x_i y_i = 0$ and

$$(A.5)\qquad \begin{aligned} \xi(\sum x_i \otimes y_i) &= \sum x_i \otimes 1 \otimes y_i + 1 \otimes x_i \otimes y_i + x_i \otimes y_i \otimes 1 \\ &= \sum x_i \otimes 1 \otimes 1 \otimes y_i + 1 \otimes x_i \otimes 1 \otimes y_i \\ &\qquad + x_i \otimes 1 \otimes y_i \otimes 1 + 1 \otimes x_i \otimes y_i \otimes 1 \\ &= \sum (x_i \otimes 1 + 1 \otimes x_i) \otimes (y_i \otimes 1 + 1 \otimes y_i). \end{aligned}$$

This implies the existence of the diagram, and we may take $\varphi$ to be map obtained by applying homotopy to the composite

$$J(X) \xrightarrow{\xi} J(X) \otimes_X J(X) \xrightarrow{1+T} J(X) \otimes_X J(X).$$

Next we claim that there is a morphism

$$\mu : \Omega(X) \to \Omega(X) \otimes_X \Omega(X)$$

so that the following diagram commutes:

$$(A.6)\qquad \begin{array}{ccc} J(X) & \xrightarrow{(1+T)\xi} & J(X) \otimes_X J(X) \\ \downarrow & & \downarrow \\ \Omega(A) & \xrightarrow{\mu} & \Omega(X) \otimes_X \Omega(X). \end{array}$$

To construct $\mu$, consider

$$d \otimes d : X \otimes_A X \to \Omega(X) \otimes_X \Omega(X)$$

where $d$ is the universal derivation of A.1.3. We will see that on $J(X)^2$

$$(1 + T)(d \otimes d) = 0.$$

If this is the case, the we obtain an induced map $\mu$ and a quick glance at the formula (A.5) implies the commutativity of diagram A.6.

To prove the claim, we compute. Let

$$\sum x_i \otimes y_i, \ \sum w_j \otimes z_j \in J(X).$$

Then

$$\begin{aligned}(d\otimes d)(\sum x_iw_j\otimes y_iz_j) &= \sum y_id(x_i)\otimes w_jd(z_j)+\sum z_jd(w_j)\otimes x_id(y_i)\\ &= \sum x_id(y_i)\otimes d(w_j)z_j+\sum w_jd(z_j)\otimes d(x_i)y_i\\ &= \sum z_jd(y_i)\otimes x_id(w_j)+\sum y_id(z_j)\otimes w_jd(x_i)\\ &= T(d\otimes d)(\sum x_iw_j\otimes y_iz_j).\end{aligned}$$

This is exactly what was needed.

Notice that, by construction, this coproduct $\mu$ is commutative; that is,

$$\mu = T\mu : \Omega(X)\to\Omega(X)\otimes_X\Omega(X).$$

Notice also that Lemmas A.3 and A.4 imply that the quotient map

$$\begin{aligned}\rho : \Omega(X)\otimes_X\Omega(X) &\to \mathbf{F}_2\otimes_X\Omega(X)\otimes_X\Omega(X)\\ &\cong Q(\mathbf{F}_2\otimes_A X)\otimes Q(\mathbf{F}_2\otimes_A X)\end{aligned}$$

induces an isomorphism in homotopy.

We now come to the product defined by André.

**Definition A.7:** Define a product

$$\langle\ ,\ \rangle : H^*_Q\Sigma A\otimes H^*_Q\Sigma A\to H^*_Q\Sigma A$$

as follows: if $x,y\in H^*_Q\Sigma A$ are represented by $\alpha,\beta\in Q(\mathbf{F}_2\otimes_A X)^*$ respectively, then $\langle x,y\rangle$ is represented by

$$\mu^*\rho^*D_0^*(\alpha\otimes\beta)$$

where $D_0$ is an Eilenberg-Zilber chain equivalence.

We can also define operations

$$\mathrm{Sq}^i : H^n_Q\Sigma A\to H^{n+i}_Q\Sigma A.$$

As always, this is done by defining a quadratic chain map

$$\Theta^i : Q(\mathbf{F}_2\otimes_A X)^*\to Q(\mathbf{F}_2\otimes_A X)^*$$

by

$$\Theta^i(\alpha) = \mu^*\rho^*D^*_{n-i}(\alpha\otimes\alpha) + \mu^*\rho^*D^*_{n-i+1}(\alpha\otimes\partial\alpha)$$

where $\alpha \in Q(\mathbf{F}_2 \otimes_A X)^*$ is of degree $n$ and $D_k$ are higher Eilenberg-Zilber maps. Then $\Theta^i$ induces $\mathrm{Sq}^i$ in cohomology.

Of course $\langle\ ,\ \rangle$ is commutative, bilinear, and adds degree:

$$\langle\ ,\ \rangle : H^m_Q\Sigma A \otimes H^n_Q\Sigma A \to H^{n+m}_Q\Sigma A.$$

The operation $\mathrm{Sq}^i$ is a homomorphism and

$$\mathrm{Sq}^i(x) = \begin{cases} \langle x, x\rangle, & \text{if } i = deg(x); \\ 0, & \text{if } i > deg(x). \end{cases}$$

This product could be called a Samelson product, just as we called $[\ ,\ ]$ on $H^*_QA$ a Whitehead product. We would like to show that there is some relation between the Whitehead product and the Samelson product. The first thing we must do is show that the Samelson product is non-trivial.

**Lemma A.8:** Suppose $\pi_*A$ is fo finite type and let $\langle,\rangle$ stand for both the product of $H^*_Q\Sigma A$ and the commutator product on $\pi^*\Sigma A^*$. If

$$h^* : H^*_Q\Sigma A \to \pi^*\Sigma A^*$$

is the Hurewicz homomorphism, then

$$h^*\langle x, y\rangle = \langle h^*x, h^*y\rangle.$$

**Proof:** This follows from A.6, using the description of the Hurewicz homomorphism give after A.4.

If we take $A = (K(m) \times K(n))_+$ to be the universal example and

$$j_m, j_n \in H^*_Q\Sigma A$$

to be the suspension of the universal classes $\iota_m$ and $\iota_n$ respectively, then we say in 11.8, that $\langle h^*j_m, h^*j_n\rangle \neq 0$. Hence $\langle j_m, j_n\rangle \neq 0$.

Now for any $A \in s\mathcal{A}$ let

$$\partial : H_Q^n A \to H_Q^{n+1} \Sigma A$$

be the natural isomorphism. Let [ , ] be the Whitehead product on $H_Q^* A$.

**Proposition A.9:** For all $x, y \in H_Q^* A$

$$\partial[x, y] = \langle \partial x, \partial y \rangle.$$

**Proof:** Because both products are natural, we need only show this result for the universal example $A = (K(m) \times K(n))_+$ and $x = \iota_m$ and $y = \iota_n$. In 11.7 we proved that there was a unique non-zero class in $H_Q^{m+n+1} A$ that passed to zero in $H_Q^*$ under the maps induced by the two inclusions

$$f_1 : K(m)_+ \to (K(m) \times K(n))_+ = A$$

$$f_2 : K(n)_+ \to (K(m) \times K(n))_+ = A.$$

Since the classes $[\iota_m, \iota_n]$ and $\partial^{-1}\langle \partial\iota_m, \partial\iota_n \rangle$ both statisfy these conditions, the result follows.

A similar result holds for the operations. However, we must state a lemma first.

**Lemma A.10:** The operations $\mathrm{Sq}^i$ commute with the suspension isomorphisms

$$\partial : H_Q^n \Sigma A \xrightarrow{\cong} H_Q^{n+1} \Sigma^2 A;$$

that is,

$$\partial \mathrm{Sq}^i(x) = \mathrm{Sq}^i(\partial x).$$

The proof of this lemma is rather involved – although very similar to the proof of 5.12. We postpone the argument until after we state and prove the result that we really want.

**Proposition A.11:** Let $P^i$ be the operations of $H^*_Q A$ and let $\mathrm{Sq}^i$ be the operations derived from André's product on $H^*_Q \Sigma A$. Then for all $x \in H^*_Q A$

$$\partial P^i(x) = \mathrm{Sq}^{i+1}(\partial x).$$

**Proof:** We apply 11.14. Define for $A \in s\mathcal{A}$ and $x \in H^*_Q A$

$$\bar{P}^i(x) = \partial^{-1}\mathrm{Sq}^{i+1}(\partial x).$$

Then if $x \in H^n_Q A$

$$\begin{aligned} \bar{P}^n(X) &= \partial^{-1}\mathrm{Sq}^{n+1}(\partial x) \\ &= \partial^{-1}\langle \partial x, \partial x\rangle \\ &= [x, x] \end{aligned}$$

by A.9. Also

$$\begin{aligned} \partial\bar{P}^i(X) &= \mathrm{Sq}^{i+1}(\partial x) \\ &= \partial^{-1}\mathrm{Sq}^{i+1}(\partial^2 x) \\ &= \bar{P}^i(\partial x) \end{aligned}$$

by the previous lemma. Thus 11.14 implies that $\bar{P}^i(x) = P^i(x)$ and the result follows.

To prove Lemma A.10, we have to make specific some of our general constructions. We will use the Eilenberg-MacLane $W$-construction, as spelled out in [17].

**Lemma A.12:** There is a functor

$$W : s\mathcal{A} \to s\mathcal{A}$$

equipped with a natural transformation $i : 1 \to W$ so that:

1.) if $A \in s\mathcal{A}$ is almost-free, then

$$A \xrightarrow{i_A} WA \xrightarrow{\epsilon} \mathbf{F}_2$$

is an almost-free map followed by an acyclic fibration;

2.) if $f : A \to B$ is an almost-free morphism in $s\mathcal{A}$, then

$$Wf : WA \to WB$$

is an almost-free morphism;

3.) if $\bar{W}(\ )$ is the functor $\bar{W}A = \mathbf{F}_2 \otimes_A WA$ then there is a weak equivalence

$$\Sigma A \simeq \bar{W}A :$$

4.) and if $f : A \to B$ is an almost-free morphism, then

$$\bar{W}f : \bar{W}A \to \bar{W}B$$

is an almost-free morphism.

**Proof:** This follows by inspection of the definition of $WA$ given in [17]. The fact that $\Sigma A \simeq \bar{W}A$ follows from part 1.

The upshot of this lemma is that we may use $WA$ for $X$ in the definition of $\Omega(X)$. So, in particular,

$$H^*_Q\Sigma A \cong \pi_*\Omega(WA)$$

for $A \in s\mathcal{A}$ almost-free.

Now consider the diagram, for $A \in s\mathcal{A}$ almost-free:

$$\begin{array}{ccccc}
A & \xrightarrow{i_A} & WA & \longrightarrow & \bar{W}A \\
\downarrow i_A & & \downarrow i_{WA} & & \downarrow Wi_A \\
WA & \xrightarrow{i_{WA}} & W^2A & \longrightarrow & \bar{W}WA \\
 & & & & \downarrow \\
 & & & & \mathbf{F}_2 \otimes_{\bar{W}A} \bar{W}WA.
\end{array}$$

By Lemma A.12, all the labeled maps are almost-free. Since

$$\bar{W}WA \simeq \Sigma WA$$

and $\pi_*WA \cong \mathbf{F}_2$, we have that

$$\epsilon : \bar{W}WA \to \mathbf{F}_2$$

is an acyclic fibration. Thus we have a weak equivalence

$$\mathbf{F}_2 \otimes_{\bar{W}A} \bar{W}WA \simeq \Sigma^2 A.$$

Since $\bar{W}i_A$ is almost-free, there is a short exact sequence

$$0 \to Q\bar{W}A \xrightarrow{Q\bar{W}i_A} Q\bar{W}WA \to Q(\mathbf{F}_2 \otimes_{\bar{W}A} \bar{W}WA) \to 0$$

and the dual of this sequence may be used to compute

$$\partial : H^n_Q \Sigma A \to H^{n+1}_Q \Sigma^2 A.$$

In fact, upon dualizing, we obtain a diagram
(A.13)

$$\begin{array}{ccccccccc}
0 & \to & Q(\mathbf{F}_2 \otimes_{\bar{W}A} \bar{W}WA)^* & \to & Q\bar{W}WA^* & \to & Q\bar{W}A^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Omega(\bar{W}WA)^* & \xrightarrow{g} & \Omega(W^2A) & \xrightarrow{f} & \Omega(WA)^* & &
\end{array}$$

The vertical maps are all weak equivalences and the bottom row, although not exact, has the property that $g$ is an injection, $f$ is a surjection, and $fg = 0$. Let $K$ be the kernel of $f$. Then there is an injection

$$\Omega(\bar{W}WA)^* \to K$$

and the five lemma and A.13 imply that

$$\pi^* K \cong H^*_Q \Sigma^2 A.$$

Hence the short exact sequence

$$0 \to K \to \Omega(W^2A)^* \xrightarrow{f} \Omega(WA)^* \to 0$$

may also be used to compute $\partial : H^n_Q \Sigma A \to H^*_Q \Sigma^2 A$.

**Proof of Lemma A.10:** Let $x \in H^n_Q \Sigma A$ be represented by $\alpha \in Q\bar{W}A^*$. Then, using Definition A.7, $\mathrm{Sq}^i(x)$ is represented $\Theta^i(\alpha)$. Choose $\beta \in Q\bar{W}WA^*$ that maps to $\alpha$ and $\gamma \in Q(\mathbf{F}_2 \otimes_{\bar{W}A} \bar{W}WA)^*$ that maps to $\partial\beta$. Then $\partial x \in H^{n+1}_Q \Sigma^2 A$ is represented by $\gamma$ and, hence, $\mathrm{Sq}^i(\partial x)$ is represented by $\Theta^i(\gamma)$. Now a simple calculation using fact that $\partial\Theta^i = \Theta^i\partial$ shows that $\partial\mathrm{Sq}^i(x)$ is also represented by $\Theta^i(\gamma)$. The result follows.

## Appendix B: An EHP sequence

With an eye to future applications, we use this section to write down and examine a cofibration sequence of simplicial algebras that is analogous in many ways to the EHP sequence in classical homotopy theory. We compute, among other things, the associated long exact sequence in cohomology and show that it is in fact, short exact.

Let $K(n)_+ \in s\mathcal{A}$ be the universal example for cohomology.

**Theorem B.1:** In the homotopy category associated to $s\mathcal{A}$, there is a cofibration sequence for $n \geq 1$:

$$\Sigma K(2n-1)_+ \xrightarrow{H} \Sigma K(n-1)_+ \xrightarrow{E} K(n)_+$$

where

$$E \in [\Sigma K(n-1)_+, K(n)_+]_{s\mathcal{A}} \cong H^n_Q \Sigma K(n-1)_+ \cong H^{n-1}_Q K(n-1)_+ \cong \mathbf{F}_2$$

is the unique non-trivial map.

We prove this at the end of the section, preferring to forge ahead with applications. Because of Theorem B.1, the homotopy cofiber of $E$ is $\Sigma^2 K(2n-1)_+$ and we have a homotopy cofiber sequence

$$\Sigma K(n-1)_+ \xrightarrow{E} K(n)_+ \xrightarrow{P} \Sigma^2 K(2n-1)_+.$$

Let $\mathcal{W}$ be the category of Definition G, Chapter 1.

**Corollary B.2:** If $n \geq 2$, there is a short exact seqeunce in $\mathcal{W}$:

$$0 \to H^*_Q \Sigma^2 K(2n-1)_+ \xrightarrow{P^*} H^*_Q K(n)_+ \xrightarrow{E^*} H^*_Q \Sigma K(n-1)_+ \to 0.$$

Indeed, if $\iota_m \in H^m_Q K(m)_+$ is the universal cohomology class of degree $m$ and $P^I = P^{i_1} \cdots P^{i_s}$ is some sequence of cohomology operations, then

$$E^* P^I(\iota_n) = P^I(\iota_{n-1})$$

and

$$P^* P^I(\iota_{2n-1}) = P^I P^n(\iota_n)$$
$$= P^I[\iota_n, \iota_n].$$

**Corollary B.2:** There is a short exact sequence in $\mathcal{W}$:

$$0 \to H^*_Q K(1)_+ \xrightarrow{E^*} H^*_Q \Sigma K(0)_+ \xrightarrow{H^*} H^*_Q \Sigma K(1)_+ \to 0.$$

Both B.2 and B.3 are obvious from B.1 and the calculations of the later sections of this paper. In fact B.2 may be strengthened. There is a weak equivalence

$$\Sigma K(0)_+ \xrightarrow{\simeq} K(1)_+ \otimes \Sigma K(1)_+$$

given by taking the composite

$$\Sigma K(0)_+ \xrightarrow{\psi} \Sigma K(0)_+ \otimes \Sigma K(0)_+ \xrightarrow{E \otimes \Sigma \beta} K(1)_+ \otimes \Sigma K(1)_+$$

where $\beta \in [K(0)_+, K(1)_+]_{s\mathcal{A}} \cong H^1_Q K(0)_+$ classifies $\beta(\iota_0)$ and $\psi$ is the comultiplication.

We wish to expand this calculation somewhat to say something about the relationship between the homology operations $\delta_i$ of sections 2 and 3 and the cohomology operations $P^i$. Let $S(n)$ be the sphere object that represents homotopy. We consider the homotopy cofiber sequence, for $n > 0$,

$$(B.4) \qquad S(n) \xrightarrow{e} K(n)_+ \xrightarrow{f} F(n)$$

where

$$e \in [S(n), K(n)_+]_{s\mathcal{A}} \cong \pi_n K(n)_+ \cong H^n_Q S(n) \cong \mathbf{F}_2$$

is the unique non-zero class. First notice that we can use 4.7 to compute $\pi_* F(n)$. Indeed, the extension of the cofibration sequence B.4 yields a map

$$g : F(n) \to \Sigma S(n) \simeq S(n+1)$$

and

$$\pi_* S(n+1) \cong \Gamma(\delta_I(j_{n+1}))$$

where the $\Gamma$ denotes the divided power algebra, $\delta_I = \delta_{i_1} \cdots \delta_{i_s}$ is admissible, $s \geq 0$, and $e(I) < n+1$. (See sections 2 and 3.) Then

$$\pi_* g : \pi_* F(n) \to \pi_* S(n)$$

is an injection and defines an isomorphism

$$\pi_* F(n) \cong \Gamma(\delta_I(j_{n+1}))$$

where $\delta_I = \delta_{i_1} \cdots \delta_{i_s}$ is admissible, $s \geq 1$, and $e(I) < n+1$. In particular, in the notation of section 12, $\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\pi_* F(n)$ is spanned by the residue classes of the elements $\delta_j(j_{n+1})$, $2 \leq j \leq n$. Let

$$y_j \in [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\pi_* F(n)]_{n+j+1}$$

with $2 \leq j \leq n$ be this residue class.

On the other hand, since $H_Q^* S(n) \cong \mathbf{F}_2$ concentrated in degree $n$, we have

$$e^* P^i(\iota_n) = 0$$

and, hence, for $2 \leq i \leq n$ there is a class

$$\alpha_i \in H_Q^{n+i+1} F(n)$$

so that $f^* \alpha_i = P^i(\iota_n)$. We wish to show that $\alpha_i$ and $y_i$ are intimately connected.

We have seen that the Hurewicz homomorphism defines a map

$$\begin{aligned} h^* : H_Q^* F(n) &\to Hom_{\mathcal{UD}}(Q\pi_* F(n), \mathbf{F}_2) \\ &\cong [\mathbf{F}_2 \otimes_{\mathcal{UD}} Q\pi_* F(n)]^* \end{aligned}$$

and the following seems entirely reasonable.

**Proposition B.5:** $h^* \alpha_i$ is dual to $y_i$; that is,

$$\langle h^* \alpha_i, y_i \rangle = 1.$$

**Proof:** We first consider the case where $i = n$. In this instance, we examine the diagram of cofibration sequences

$$\begin{array}{ccccccc} S(n) & \xrightarrow{e} & K(n)_+ & \xrightarrow{f} & F(n) & \to & S(n+1) \\ \downarrow \Sigma e & & \downarrow & & \downarrow & & \downarrow \Sigma^2 e \\ \Sigma K(n-1)_+ & \xrightarrow{E} & K(n)_+ & \xrightarrow{P} & \Sigma^2 K(2n-1)_+ & \xrightarrow{\Sigma H} & \Sigma^2 K(n-1)_+. \end{array}$$

We have extended the sequences one step to the right. Then we notice that

$$\pi_* \Sigma^2 e : \pi_* S(n+1) \to \pi_* \Sigma^2 K(n-1)_+$$

identifies the target as the ring $\pi_* S(n+1)$ modulo the ideal generated by the elements $\delta_I(j_{n+1})$ where $e(I) < n$. The result then follows by a diagram chase using Corollary B.2.

For $i < n$, we examine the diagram of cofibration sequences

$$\begin{array}{ccccc} S(n) = \Sigma^{n-i} S(i) & \to & \Sigma^{n-i} K(i)_+ & \to & \Sigma^{n-i} F(i) \\ \downarrow \simeq & & \downarrow E^{n-i} & & \downarrow \\ S(n) & \to & K(n)_+ & \to & F(n). \end{array}$$

A simple diagram chase now proves the result.

This leaves the proof of Theorem B.1. We begin with the following preliminary result. Notice that it is a consequence of 4.9 that

$$\pi_* \Sigma K(m)_+ \cong \Gamma(y)$$

where $\Gamma$ denotes the divided power algebra and $y \in \pi_{m+1} \Sigma K(m)_+$.

**Lemma B.6:** There exists a map $H : \Sigma K(2n-1)_+ \to \Sigma K(n-1)_+$ so that

$$0 \neq \pi_* H : \pi_{2n} \Sigma K(2n-1)_+ \to \pi_{2n} \Sigma K(n-1)_+.$$

**Proof:** We use the Hilton-Milnor Theorem, and it is convenient to compute in cohomotopy. We know, using 4.9, that

$$\pi^* \Sigma (K(n-1) \times K(n-1))^*_+ \cong T(y_1, y_2)$$

where $y_1, y_2 \in \pi^n\Sigma(K(n-1) \times K(n-1))_+^*$ are induced from the two projections and $T$ denotes the tensor algebra. We have constructed a map

$$f : \Sigma(K(n-1) \times K(n-1))_+ \to \Sigma K(2n-1)_+$$

so that under

$$\pi^* f^* : \pi^*\Sigma K(2n-1)_+ \cong \mathsf{F}_2[z] \to \pi^*\Sigma(K(n-1) \times K(n-1))_+$$

we have

$$\pi^* f^*(z) = [y_1, y_2] = y_1 y_2 + y_2 y_1.$$

The Hilton-Milnor Theorem implies that $f$ has a retraction g:

$$g : \Sigma K(2n-1)_+ \to \Sigma(K(n-1) \times K(n-1))_+$$

so that $gf = id$. Therefore

$$\pi^* g^*[y_1, y_2] = z.$$

Now let

$$p_1, p_2 : (K(n-1) \times K(n-1))_+ \to K(n-1)_+$$

be the two projections and let

$$p_1 + p_2 : (K(n-1) \times K(n-1))_+ \to K(n-1)_+$$

be the sum of the two projections. Then we can construct $H$ as follows: if $\pi^* g^* y_1^2 \neq 0$, set $H = \Sigma p_1 \circ g$; similarly, if $\pi^* g^* y_2^2 \neq 0$, set $H = \Sigma p_2 \circ g$. But if $\pi^* g^* y_1^2 = \pi^* g^* y_2^2 = 0$, set $H = \Sigma(p_1 + p_2) \circ g$.

**Proof of Theorem B.1:** Let $H : \Sigma K(2n-1)_+ \to \Sigma K(n-1)_+$ be as in the previous lemma. Since any morphism in $s\mathcal{A}$ induces a morphism of divided power algebras in homotopy, $\pi_* H$ must be an injection. Let $X$ be the homotopy cofiber of $H$; that is, there is a homotopy cofiber sequence

$$\Sigma K(2n-1)_+ \to \Sigma K(n-1)_+ \to X.$$

By 4.7, $\pi_* X \cong \Lambda(x)$ where $\Lambda$ denotes the exterior algebra and $x \in \pi_n X$. Also

$$H^n_Q X \cong H^n_Q \Sigma K(2n-1)_+$$

so that there is a non-trivial map $X \to K(n)_+$. This map must be a weak equivalence. Finally, the composite

$$\Sigma K(n-1)_+ \to X \to K(n)_+$$

is non-trivial in $H^*_Q$ and, hence, must be $E$.

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Quillen and André have rigorized and explored a notion of cohomology of commutative algebras or, more generally, simplicial commutative algebras. They were able to do a number of systematic calculations, especially when concerned with a local ring with residue field of characteristic 0, but the case when the characteristic was non-zero remained a problem. However, for certain applications — for example, to homotopy theory — the non-zero characteristic case is vital. In this paper we explore André-Quillen cohomology of supplemented algebras over the field $\mathbf{F}_2$ of two elements, and completely determine the structure of this cohomology, including a product and "Steenrod" operations. A necessary part of the program is a complete examination of the homotopy theory of simplicial algebras. For this we draw on the work of many authors.