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G. R. ROBINSON
R. STASZEWSKI

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MORE ON ALPERIN'S CONJECTURE

by G. R. ROBINSON AND R. STASZEWSKI

INTRODUCTION: We assume that the reader is familiar with the results, notation, and methods of [5]. Results from that paper (and minor variants thereof) will sometimes be quoted without explicit reference.

Our main aim in this paper is to try to understand the relevance of Clifford - theoretic techniques to Alperin's conjecture. Thus we are concerned with the effect of the presence of normal subgroups when trying to prove Alperin's conjecture. Once more, we try to maintain the dual viewpoint of applying our results to groups for which Alperin's conjecture is known to be valid, and trying to prove the conjecture in general (or to at least obtain some control of the structure of a minimal counterexample). Thus, for example, all the results of the first section are valid for p-solvable groups, since Alperin's conjecture is known to be valid for p-solvable groups (for the prime p), (see Okuyama [7]).

The main result of the first section is that Alperin's conjecture is equivalent to an apparently stronger conjecture which seems more compatible with the presence of normal subgroups.

In the second section, we prove that a minimal counterexample to Alperin's conjecture (for the prime p) has no normal subgroup of index p.

In the third section, we propose an "equivariant" form of Alperin's conjecture, that is a form of Alperin's conjecture which predicts compatibility with the action of a group of automorphisms. As far as we can tell at present, this conjecture is genuinely stronger than Alperin's conjecture.

In his Arcata article [1], Alperin suggests that in trying to prove his conjecture, other, more general, conjectures might naturally arise and need to be proved along the way. We believe that the results and methods of this article are the beginnings of a fulfilment of that prediction.

NOTATION: Throughout, p denotes a fixed prime, and k is the algebraic closure of GF(p). When dealing with the complexes $\mathcal{P}, \mathcal{N}, \mathcal{E}, \mathcal{U}$ of [5], it will sometimes be necessary to indicate the group from which the subgroups involved are taken, so we may speak of $\mathcal{P}(G)$, etc.

When Q is a p-subgroup of the finite group G, and B is a sum of blocks of kG, we let $f_0^{(B)}(N_G(Q)/_Q)$ denote the number of (isomorphism types of) projective simple $kN_G(Q)/_Q$-modules in Brauer correspondents of B.

SECTION ONE: THE CONJECTURE & NORMAL SUBGROUPS.

Let G be a finite group for which every proper section of G satisfies Alperin's conjecture for the prime p (for every p-block). Let $H(\neq G)$ be a normal subgroup of G, and let B be a block of kG which does not lie over blocks of defect 0 of kH, say B lies over the block b of kH.

The following lemma is well-known:

LEMMA 1.1: Let X be a finite group, B be a block of kX with defect group D. Let Q be a subgroup of D such that $kN_X(Q)/_Q$ has a projective simple module, S say, which lies in a Brauer correspondent of B. Then there is a conjugate, Q_1, of Q such that $Q_1 \subset D$ and $C_D(Q_1) \subset Q_1$.

PROOF: Let b be the block of $kN_X(Q)$ containing S. Then S lies over a projective simple $kQC_X(Q)/_Q$-module, so b lies over a block of defect Z(Q) of $k\,C_X(Q)$, say b*. Then $(Q,b^*) \supset (1,B)$. Let (D_1,b') be a maximal B-subpair with $(Q,b^*) \subset (D_1,b')$. We claim that $C_{D_1}(Q) \subset Q$. Otherwise, there is a B-subpair $(QC_{D_1}(Q),b'')$ with $(Q,b^*) \subsetneq (QC_{D_1}(Q),b'') \subset (D_1,b')$, contrary to the fact that b* is a block of defect Z(Q) of $C_X(Q)$. Thus $C_{D_1}(Q) \subset Q$, and the result follows as $D_1 = D^x$ for some $x \in X$.

COROLLARY 1.2: $\sum_{C\in\mathcal{P}(G)/_G} (-1)^{|C|}\, \ell(B_C) = \sum_{C\in\mathcal{P}(H)/_G} (-1)^{|C|}\, \ell(B_C)$.

PROOF: Let D be a defect group for B. Then $D\cap H \neq 1_G$, as B does not lie over blocks of defect 0 of kH. Thus $Z(D) \cap H \neq 1_G$, as $D\cap H \lhd D$.

Hence whenever Q is a p-subgroup of G with $Q\cap H = 1_G$, we have $f_0^{(B)}(N_G(Q)/_Q) = 0$ (for when Q_1 is a conjugate of Q with $Q_1 \subseteq D$, we have $Z(D) \cap H \subset C_D(Q_1)$, but $Z(D) \cap H \not\subseteq Q_1$).

Now $\sum_{C\in\mathcal{P}(G)/_G} (-1)^{|C|}\, \ell(B_C) = \sum_{C\in\mathcal{N}(G)/_G} (-1)^{|C|}\, \ell(B_C)$, and

similarly $\sum_{C\in\mathcal{P}(H)/_G} (-1)^{|C|}\, \ell(B_C) = \sum_{C\in\mathcal{N}(H)/_G} (-1)^{|C|}\, \ell(B_C)$. By an argument similar to that of Proposition 3.3 of [5] we may pair off orbits of chains of $\mathcal{N}(G)\backslash\mathcal{N}(H)$ whose first non-trivial term meets H non-trivially (that is to say given such a chain $C = Q_0 < Q_1 < \ldots < Q_n$ with

$Q_1 \cap H \neq 1_G$ and $Q_n \not\leq H$, form the chain C' as follows: let Q_i be the first term of C for which $Q_i \not\leq H$. If $Q_{i-1} = Q_i \cap H$, delete Q_{i-1} from C, whilst if $Q_{i-1} \neq Q_i \cap H$, insert $Q_i \cap H$ into C (between Q_{i-1} and Q_i). Then we pair the orbit of C with that of C').

We thus obtain:

$$\sum_{C \in \mathcal{N}(G)/G} (-1)^{|C|} \ell(B_C) = \sum_{C \in \mathcal{N}(H)/G} (-1)^{|C|} \ell(B_C) +$$

$$\sum_{(Q)} \sum_{\{\text{chains } C \text{ whose first non-trivial term is } Q\}/N_G(Q)} (-1)^{|C|} \ell(B_C)$$

where (Q) runs over a set of representatives for the conjugacy classes of p-subgroups Q, of G with $Q \cap H = 1_G \neq Q$.

Since Alperin's conjecture holds within proper sections of G, we obtain $\sum_{C \in \mathcal{P}(G)/G} (-1)^{|C|} \ell(B_C) =$

$$\sum_{C \in \mathcal{P}(H)/G} (-1)^{|C|} \ell(B_C) - \sum_{(Q) \text{ as above}} f_o^{(B)}(N_G(Q)/Q).$$

Since $f_o^{(B)}(N_G(Q)/Q) = 0$ whenever $Q \cap H = 1_G$, the result follows.

COROLLARY 1.3: We have

$$\sum_{C \in \mathcal{P}(H)/G} (-1)^{|C|} \ell(B_C) = \ell(B) - \sum_{(Q)} \sum_{(b')} \ell(b')$$

where Q runs over p-subgroups of H (up to G-conjugacy) and b' runs over blocks of $N_G(Q)/Q$, with $b'^G = B$, lying over blocks of defect 0 of $kN_H(Q)/Q$.

<u>PROOF</u>: Let Q be a non-trivial p-subgroup of H. Then Alperin's conjecture holds for blocks of $kN_G(Q)/Q$. Let $X = N_G(Q)/Q$, $Y = N_H(Q)/Q$.

Let b' be a block of kX with $b'^G = B$. Suppose that b' does <u>not</u> lie over blocks of defect 0 of kY. Then, as in Corollary 1.2, we obtain

$$\sum_{C \in \mathcal{N}(X)/X} (-1)^{|C|} \ell(b'_C) = \sum_{C \in \mathcal{N}(Y)/X} (-1)^{|C|} \ell(b'_C)$$

On the other hand, b' is of Lefschetz type, since b' has positive defect and Alperin's conjecture holds for all blocks of all proper sections of G. Thus $\sum_{C \in \mathcal{N}(Y)/X} (-1)^{|C|} \ell(b'_C) = 0$.

If b' <u>does</u> lie over blocks of defect 0 of kY, then it is clear that $\sum_{C \in \mathcal{N}(Y)/X} (-1)^{|C|} \ell(b'_C) = \ell(b')$, since $b'_C = 0$ unless C is the chain $\{1_Y\}$.

It follows, then, that the contribution to $\sum_{C \in \mathcal{N}(H)/_G} (-1)^{|C|} \ell(B_C)$ from chains whose first non-trivial term is (G-conjugate to) Q is $-\sum_{b'} \ell(b')$, where b' runs over blocks of $kN_G(Q)/_Q$ with $b'^G = B$, lying over blocks of defect 0 of $kN_H(Q)/_Q$. The result now follows.

COROLLARY 1.4 : B is of Lefschetz type if and only if $\ell(B) = \sum_{(Q)} \sum_{(b')} \ell(b')$, where Q runs over p-subgroups of H (up to G-conjugacy), and b' runs over blocks of $kN_G(Q)/_Q$ with $b'^G = B$ lying over blocks of defect 0 of $kN_H(Q)/_Q$.

We can now state:

PROPOSITION 1.5 (ANOTHER FORMULATION OF ALPERIN'S CONJECTURE): The following are equivalent:

i) Whenever X is a finite group, and B is a block of kX, B is of Alperin type.

ii) Whenever X is a finite group, $Y \lhd X$ and B is a block of kX we have $\ell(B) = \sum_{(Q)} \sum_{(b')} \ell(b')$, where Q runs over p-subgroups of Y up to X-conjugacy, and b' runs over blocks of $kN_X(Q)/_Q$, with $b'^X = B$, lying over blocks of defect 0 of $kN_Y(Q)/_Q$.

PROOF : It is clear that ii) implies i) (taking Y = X). The results of this section show that i) implies ii) (upon noting that ii) holds vacuously if B lies over blocks of defect 0 of kY).

Now suppose that b has defect group P. Then there is a unique block, B*, of $kN_G(P)$ with $B^{*G} = B$ and B* lies over b* (the Brauer correspondent of b in $kN_H(P)$) (Harris-Knörr [4]).

COROLLARY 1.6 : Suppose that one of the following occurs:

i) b is nilpotent

ii) P is Abelian.

iii) $P \cap P^h = 1_H$ for all $h \in H \backslash N_H(P)$.

Then B is of Lefschetz type if and only if $\ell(B) = \ell(B^*)$.

PROOF : By assumption, Alperin's conjecture holds within H. Thus $f_o^{(b)}(N_H(P)/_P) = \ell(b)$ and $f_o^{(b)}(N_H(Q)/_Q) = 0$ whenever Q is a p-subgroup of H not H-conjugate to P in any of the cases listed (see [1]). The same applies to any G-conjugate of b (using the appropriate G-conjugate of P). It readily follows that the formula appearing in the statement of Corollary 1.4 reduces to : $\ell(B) = \ell(B^*)$ in the cases listed.

REMARKS : In fact, by a theorem of Külshammer and Puig, [6], it is the case that $\ell(B) = \ell(B^*)$ if b is nilpotent. It also follows from their result and the results of this section and [5] that the formula of Proposition 1.5 part ii) holds when B lies over nilpotent blocks of kY.

We also consider that Corollary 1.4 can be considered a reduction towards a proof of Alperin's conjecture. If we were considering G as a minimal counterexample to Alperin's conjecture, then to prove that B was of Lefschetz type it would be sufficient to verify the formula of Corollary 1.4 for <u>some</u> $H \lhd G$ such that B did not lie over blocks of defect zero of kH. (In fact if B lies over blocks of defect 0 of kH for some $H \lhd G$, then conventional Clifford theoretic reductions may be applied (unless H is a central p'-subgroup)).

To illustrate how this type of reduction may be applied, we outline an alternative proof of Alperin's conjecture for p-solvable groups. Let G be a p-solvable group such that Alperin's conjecture is valid for every p-block of every proper section of G. Let B be a block of positive defect of kG. We wish to prove that B is of Lefschetz type.

i) We may suppose that $O_p(G) = 1_G$.

ii) We may suppose that B lies over a G-stable block of $O_{p'}(G)$.

iii) We choose $H \lhd G$ such that $H/O_{p'}(G)$ is elementary Abelian, $p \,\big|\, |H|$.

We let $P \in Syl_p(H)$. Then $G = O_{p'}(G)\, N_G(P)$.

iv) We may suppose that $H \neq G$ (otherwise B is nilpotent, and hence of Lefschetz type).

v) By Corollary 1.6, it suffices to prove that $\ell(B) = \ell(B^*)$ where B^* is the unique block of $kN_G(P)$ with $B^{*G} = B$.

vi) By a theorem of Dade ([3]), we do have $\ell(B) = \ell(B^*)$ (or we could again apply the theorem of Külshammer-Puig mentioned above).

Once more, we point out that the formulae appearing in Proposition 1.5 parts i) and ii) are now proven to be valid for all p-solvable groups.

SECTION TWO : THE CASE $G \neq O^p(G)$.

Let G, H, B, b be as before, except that we now allow the possibility that b (but not B) has defect 0. We assume in addition that $[G : H] = p$. We will prove that B is of Lefschetz type. If $O_p(G) \neq 1$ this is clearly true, so we assume that $O_p(G) = 1_G$. Also, if b <u>does</u> have defect 0, then B has defect 1 and B is of Alperin type (so also of Lefschetz type), so we assume from now on that b has positive defect.

NOTATION : When $C = Q_o < Q_1 \ldots < Q_n$ is a chain in $\mathcal{P}(G)$, we let

$B_c(H) = Br_{Q_n}(1_B)\ kH_c$ (which we view as a G_c-module via conjugation).

The next result follows easily via minor variants of the arguments of section 4 of [5].

LEMMA 2.1 : $\sum_{C \in \mathcal{N}(G)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_c}^{G}(B_c(H))$ is a virtual projective module in the Green ring of kG.

REMARKS: An argument like that used in the proof of Corollary 1.2 can be used to show that $\sum_{C \in \mathcal{N}(G)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_c}^{G}(B_c(H)) =$

$$\sum_{C \in \mathcal{N}(H)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_c}^{G}(B_c(H)) \quad + \quad \sum_{\{C \in \mathcal{N}(G) : Q_1 \cap H = 1_G\}/G} (-1)^{|C|}\ \mathrm{Ind}_{G_c}^{G}(B_c(H))$$

Furthermore, if Q is a p-subgroup of G with $Q \cap H = 1_G$, then $|Q| = p$ and $N_G(Q) = Q \times C_H(Q)$.

LEMMA 2.2 : $\sum_{\{C \in \mathcal{N}(G) : Q_1 \cap H = 1_G\}/G} (-1)^{|C|}\ \mathrm{Ind}_{G_c}^{G}(B_c(H))$ involves no Scott Module.

<u>PROOF</u> : Let Q be a non-trivial p-subgroup of G with $Q \cap H = 1_G$. Let $N = N_G(Q)$, $B' = Br_Q(1_B)kN$.

It suffices to prove that no Scott module occurs in

$\sum_{C \in \mathcal{N}(C_H(Q))/C_H(Q)} (-1)^{|C|+1}\ \mathrm{Ind}_{N_c}^{N}(B'_c(C_H(Q)))$. This last module may be viewed as a virtual projective $kN/_Q$-module, or as a virtual Q-projective N-module (such that Q acts trivially on all modules involved). Thus the only Scott modules it can involve are Scott modules with vertex Q.

Since Alperin's conjecture holds for all blocks of $kC_H(Q)$ (and its sections) there are no $C_H(Q)$-fixed points on $\sum_{C \in \mathcal{N}(C_H(Q))/C_H(Q)} (-1)^{|C|}\ \mathrm{Ind}_{N_c}^{N}(B'(C_H(Q)))$ (for as B does not have defect group Q, $B'(C_H(Q))$ is a sum of blocks of positive defect of $kC_H(Q)$, each of which is of Lefschetz type). Hence the Scott module with vertex Q is not involved in $\sum_{C \in \mathcal{N}(C_H(Q))/C_H(Q)} (-1)^{|C|}\ \mathrm{Ind}_{N_c}^{N}(B'(C_H(Q)))$ (for all modules under consideration are trivial source modules, so the total number of fixed-points on the above virtual module is the number of Scott modules involved). The proof of Lemma 2.2 is complete.

REMARK : We note that we may also conclude from the proof of Lemma 2.2

that $\sum_{\{C\in\mathcal{N}(G):Q_1\cap H = 1_G\}/_G} (-1)^{|C|} \ \mathrm{Ind}_{G_C}^{G}(B_C(H))$ only involves modules whose vertices intersect H trivially. Hence it follows that

$\mathrm{Res}_H^G(\sum_{C\in\mathcal{N}(H)/_G} (-1)^{|C|} \ \mathrm{Ind}_{G_C}^{G}(B_C(H)))$ and $\mathrm{Res}_H^G(\sum_{\{C\in\mathcal{N}(G):Q_1\cap H = 1_G\}/_G} (-1)^{|C|} \ \mathrm{Ind}_{G_C}^{G}(B_C(H)))$

are both virtual projective modules in the Green ring for kH.

Also we remark that as $\sum_{C\in\mathcal{N}(G)/G} (-1)^{|C|} \ \mathrm{Ind}_{G_C}^{G}(B_C(H))$ is a virtually projective kG-module, and as $\sum_{\{C\in\mathcal{N}(G):Q_1\cap H = 1_G\}/_G} (-1)^{|C|} \ \mathrm{Ind}_{G_C}^{G}(B_C(H))$ involves no Scott module, the projective cover of the trivial module is the only Scott module which can be involved in $\sum_{C\in\mathcal{N}(H)/_G} (-1)^{|C|} \ \mathrm{Ind}_{G_C}^{G}(B_C(H))$.

LEMMA 2.3 : Let X be a finite group, $Y \triangleleft X$, and M be an indecomposable trivial source kX-module with vertex Q.

If the projective cover of the trivial module occurs as a summand of $\mathrm{Res}_Y^X(M)$, then $Q\cap Y = 1_X$. Furthermore, if we also have $X = YQ$, then M is the Scott module with vertex Q.

<u>PROOF</u> : We know that $\mathrm{Res}_Y^X(M) \mid \mathrm{Res}_Y^X(\mathrm{Ind}_Q^X(k))$, and that

$$\mathrm{Res}_Y^X(\mathrm{Ind}_Q^X(k)) \cong \bigoplus_{\text{certain } x} \mathrm{Ind}_{Q^x\cap Y}^{Y}(k).$$

Now for $x \in X$, the Scott module with vertex $Q^x\cap Y$ is the unique Scott module occurring as a summand of $\mathrm{Ind}_{Q^x\cap Y}^{Y}(k)$. Hence if the projective of the trivial module occurs as a summand of $\mathrm{Res}_Y^X(M)$, then there must be some $x \in X$ with $Q^x\cap Y = 1_X$, so $Q\cap Y = 1_X$ as $Y \triangleleft X$. If, in addition, we also have $X = YQ$, then $\mathrm{Res}_Y^X(\mathrm{Ind}_Q^X(k)) \cong kY$, so the projective cover of the trivial module occurs just once as a summand of $\mathrm{Res}_Y^X(\mathrm{Ind}_Q^X(k))$. However, let V be the Scott module with vertex Q. Then $\mathrm{Res}_Y^X(V)$ is projective, and Y certainly has a fixed-point on $\mathrm{Res}_Y^X(V)$, so the projective cover of the trivial module does occur as a summand of $\mathrm{Res}_Y^X(V)$. The result follows.

The next result is well-known, and is an easy consequence of Green's indecomposability theorem.

LEMMA 2.4 : Let X be a finite group, $Y \triangleleft\triangleleft X$ with $[X:Y]$ a power of p. Let $P_1(X)$ denote the projective cover of the trivial kX module (similarly for Y). Then:

i) $\mathrm{Ind}_Y^X(P_1(Y)) \cong P_1(X)$.

ii) $\mathrm{Res}_Y^X(P_1(X)) \cong \bigoplus_{[X:Y] \text{ copies}} P_1(Y)$.

PROPOSITION 2.5 : There are no G-fixed-points on $\sum_{C\in\mathcal{N}(G)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H))$.

PROOF : We know that

$$\sum_{C\in\mathcal{N}(G)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H)) = \sum_{C\in\mathcal{N}(H)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H))$$

$$+ \sum_{\{C\in\mathcal{N}(G):Q_1\cap H = 1_G\}/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H)).$$

Furthermore, we know that the second sum on the right hand side involves no Scott module, and that the only Scott module which can be involved in the first sum on the right hand side is the projective cover of the trivial module.

Since $\mathrm{Res}_H^G(\sum_{C\in\mathcal{N}(G)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H))$ is virtually projective, we may count the number of H-fixed points by counting the multiplicity of the projective cover of the trivial kH-module. Since $[G:H] = p$, it follows from Lemmas 2.2 and 2.3 that there are no H-fixed points on $\mathrm{Res}_H^G [\sum_{\{C\in\mathcal{N}(G):Q_1\cap H = 1_G\}/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H))]$(recalling that this last virtual module is virtually projective in the Green ring for kH).

Also, from Lemmas 2.2, 2.3 and 2.4, it follows that the number of H-fixed points on $\mathrm{Res}_H^G(\sum_{C\in\mathcal{N}(H)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H)))$ is p times the number of G-fixed points on $\sum_{C\in\mathcal{N}(H)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H))$.

However, there are no H-fixed points on $\mathrm{Res}_H^G(\sum_{C\in\mathcal{N}(H)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H))$, since

$$\mathrm{Res}_H^G(\sum_{C\in\mathcal{N}(H)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C(H))) = \sum_{C\in\mathcal{N}(H)/H} (-1)^{|C|}\ \mathrm{Ind}_{H_C}^{H}(B_C(H))$$

(and since B(H) is a sum of blocks of positive defect of kH, each of which is of Lefschetz type as Alperin's conjecture is valid for every p-block of every section of H).

PROPOSITION 2.6 : B is of Lefschetz type.

PROOF : We need to prove that there are no G-fixed points on $\sum_{C\in\mathcal{N}(G)/G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C)$. Since this last virtual module is virtually

projective, the number of fixed-points we are interested in is
$\sum_{C\in\mathcal{N}(G)/G} (-1)^{|C|} \dim_k(\mathrm{Tr}_1^{G_C}(B_C))$.

For any $C\in\mathcal{N}(G)$, $\mathrm{Tr}_1^{G_C}(B_C) = \mathrm{Tr}_1^{G_C}(B_C(H))$ (for $G_C\backslash H_C$ consists of p-singular elements), so the number we wish to calculate is
$\sum_{C\in\mathcal{N}(G)/G} (-1)^{|C|} \dim_k(\mathrm{Tr}_1^{G_C}(B_C(H)))$ which is the number of G-fixed points on $\sum_{C\in\mathcal{N}(G)/G} (-1)^{|C|} \mathrm{Ind}_{G_C}^{G}(B_C(H))$
(as this last virtual module is also virtually projective), and we know that this number is 0 by Proposition 2.5. The proof is complete.

We may summarize the results of this section by :

COROLLARY 2.7 : Let X be a minimal counterexample to Alperin's conjecture for the prime p. Then $X = O^P(X)$.

SECTION THREE : AN EQUIVARIANT FORMULATION

We know that Alperin's conjecture (for the prime p) holds for all finite groups if and only if whenever X is a finite group and B is a block of positive defect of kX, there are no X-fixed-points on
$\sum_{C\in\mathcal{P}(X)/X} (-1)^{|C|} \mathrm{Ind}_{X_C}^{X}(B_C)$.

Now suppose that there is $Y \lhd X$, and that B is a minimal X-invariant sum of blocks of kY. When $C = Q_0 < Q_1 \ldots < Q_n$ is a chain in $\mathcal{P}(X)$, we let $B_C = \mathrm{Br}_{Q_n}(1_B)\, kY_C$, which is an X_C-stable sum of blocks of kY_C, so can be viewed as an X_C-module under conjugation. We note that there are no primitive idempotents in $\mathrm{Tr}_1^X(B)$ unless B is a sum of blocks of defect 0 of kY, each of inertial index prime to p, in which case there is just one such idempotent.

Our main result of this section is:

PROPOSITION 3.1 (EQUIVARIANT FORMS OF ALPERIN'S CONJECTURE) :
The following three statements are equivalent

i) Whenever X,Y,B are as above, the number of X-fixed points on
$\sum_{C\in\mathcal{P}(X)/X} (-1)^{|C|} \mathrm{Ind}_{X_C}^{X}(B_C)$ is the number of primitive idempotents in $\mathrm{Tr}_1^X(B)$.

ii) Whenever X,Y,B are as above, the number of X-fixed points on
$\sum_{C\in\mathcal{P}(Y)/X} (-1)^{|C|} \mathrm{Ind}_{X_C}^{X}(B_C)$ is the number of primitive idempotents in $\mathrm{Tr}_1^Y(B)^X$
(ie is 1 if B is a sum of blocks of defect 0 of kY, 0 otherwise).

iii) Whenever X,Y,B are as above,
(X-orbits of (isomorphism types of) simple B-modules) =

$\sum\limits_{(Q)}$ # [$N_X(Q)/_Q$-orbits of projective simple $N_Y(Q)/_Q$-modules in Brauer correspondents of B], where Q runs over p-subgroups of Y, (up to X-conjugacy).

REMARKS : We note that the truth of any of the statements i), ii), or iii) above implies Alperin's original conjecture (upon taking Y = X).

To prove Proposition 3.1, we consider a finite group G such that whenever X is a proper section of G, $Y \lhd X$, and B is a minimal X-stable sum of blocks of kY, the formulae of i), ii), and iii) above are all valid for the triple (X,Y,B). We choose a normal subgroup $H \lhd G$, and we further assume (as we may) that whenever $L \lhd G$ with $|L| < |H|$, and B′ is a minimal G-stable sum of blocks of kL, then the formulae appearing in i), ii) and iii) above are all valid for the triple (G,L,B′). Then to prove Proposition 3.1, it suffices to prove that if B is a minimal G-stable sum of blocks of kH, and the triple (G,H,B) satisfies the formula in one of parts i), ii) or iii) above, then it satisfies the other two formulae.

NOTATION : From now on, the triple (G,H,B) is fixed. We let $B^* = 1_B\, kH_{p'}$ (where $kH_{p'}$ denotes the k-span of the p-regular elements of H). Thus B^* is a G-module via conjugation action. We let $\ell^*(B)$ denote $\dim_k(B^{*G})$. Then minor variants of arguments of Brauer show that $\ell^*(B)$ is the number of G-orbits of (isomorphism types of) simple B-modules. We let $k^*(B)$ denote $\dim_k(B^G)$.

When Q is a p-subgroup of G, we let $m^{(1)}_{B^*}(Q)$ denote $\dim_k(\mathrm{Tr}^G_Q(B^{*Q}) \,/\, \sum\limits_{P<Q} \mathrm{Tr}^G_P(B^{*P}))$. We note that minor variants of arguments from the theory of lower defect groups show that $\ell^*(B) = \sum\limits_{(Q)} m^{(1)}_{B^*}(Q)$, where Q runs over p-subgroups of G (up to conjugacy), (e.g. Olsson [8], or Broué [2]).

REMARKS : Once more, $\sum\limits_{C\in \mathcal{P}(G)/_G} (-1)^{|C|}\ \mathrm{Ind}^G_{G_C}(B_C)$ is virtually projective in the Green ring for kG. Hence the number of G-fixed points on this virtual module is

$$\sum_{C\in \mathcal{P}(G)/_G} (-1)^{|C|}\ \dim_k(\mathrm{Tr}_1^{G_C}(B_C))\ \Big[= \sum_{C\in \mathcal{P}(G)/_G} (-1)^{|C|}\ m^{(1)}_{B^*_C}(1_G)\ \Big]$$

LEMMA 3.2 : $\sum\limits_{C\in \mathcal{P}(G)/_G} (-1)^{|C|}\ \ell^*(B_C) = \sum\limits_{C\in \mathcal{P}(G)/_G} (-1)^{|C|}\ k^*(B_C)$.

PROOF : We need to prove that

$$\sum_{C \in \mathcal{P}(G)/_G} (-1)^{|C|} \ell^*(B_C) = \sum_{C \in \mathcal{P}(G)/_G} (-1)^{|C|} m^{(1)}_{B^*_C}(1_G).$$

It suffices to prove that whenever $\mathfrak{Q}$ is a conjugacy class of non-trivial p-subgroups of G, then

$$\sum_{C \in \mathcal{P}(G)/_G} (-1)^{|C|} \sum_{\{Q \in \mathfrak{Q}:\ Q \lhd G_C\}/_{G_C}} m^{(1)}_{B^*_C}(Q) = 0.$$

Equivalently, we need to prove that

$$\sum_{C \in \mathcal{P}(G)^Q/_{N_G(Q)}} (-1)^{|C|} m^{(1)}_{B^*_C}(Q) = 0 \text{ for } Q \in \mathfrak{Q}.$$

We prove that $\sum_{C \in \mathcal{P}(G)^Q/_{N_G(Q)}} (-1)^{|C|} m^{(1)}_{B^*_C}(Q) = 0$

whenever Q is a non-trivial p-subgroup of G. Let Q be such a p-subgroup. We note that (as is well-known), $m^{(1)}_{B^*_C}(Q) = \dim_k(\mathrm{Tr}_Q^{N_G(Q)}(\mathrm{Br}_Q(B^{*Q})))$

We may pair off $N_G(Q)$-orbits of chains in $\mathcal{P}(G)^Q$ as follows: let $C = Q_0 < Q_1 \ldots < Q_n$ be such a chain. Suppose that $Q \nless Q_i$, but that $Q < Q_{i+1}$ (in case $i \neq n$). If $QQ_i = Q_{i+1}$, delete Q_{i+1} from C, whilst if $QQ_i < Q_{i+1}$, insert QQ_i into C between Q_i and Q_{i+1}. Let C′ be the chain so obtained. Then $(C')' = C$, and $N_C = N_{C'}$ (where N denotes $N_G(Q)$). Furthermore, from the remark above, we see that $m^{(1)}_{B^*_C}(Q) = m^{(1)}_{B^*_{C'}}(Q)$, and it follows that $\sum_{C \in \mathcal{P}(G)^Q/_{N_G(Q)}} (-1)^{|C|} m^{(1)}_{B^*_C}(Q) = 0$, as required to complete the proof.

LEMMA 3.3 : $\sum_{C \in \mathcal{P}(H)/_G} (-1)^{|C|} k^*(B_C) = \sum_{C \in \mathcal{P}(H)/_G} (-1)^{|C|} \ell^*(B_C)$.

PROOF : We know that

$\sum_{C \in \mathcal{P}(G)/_G} (-1)^{|C|} k^*(B_C)$ is the number of G-fixed points on

$$\sum_{C \in \mathcal{P}(G)/_G} (-1)^{|C|} \mathrm{Ind}_{G_C}^G(B_C)$$

$$\left[= \sum_{C \in \mathcal{N}(H)/_G} (-1)^{|C|} \mathrm{Ind}_{G_C}^G(B_C) + \sum_{\{C \in \mathcal{N}(G):Q_1 \cap H = 1_G\}/_G} (-1)^{|C|} \mathrm{Ind}_{G_C}^G(B_C) \right].$$

Similarly, $\sum_{C \in \mathcal{P}(G)/_G} \ell^*(B_C)$ is the number of G-fixed points on

$$\sum_{C\in\mathcal{N}(H)/G}(-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C^*)\ +\ \sum_{\{C\in\mathcal{N}(G):Q_1\cap H=1_G\}/G}(-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C^*)$$

By Lemma 3.2, it suffices to prove that whenever Q is a non-trivial p-subgroup of G with $Q\cap H = 1_G$, the numbers of $N_G(Q)$-fixed points on $\sum_{C\in\mathcal{N}(\overline{X})/\overline{X}}(-1)^{|C|}\ \mathrm{Ind}_{\overline{X}_C}^{\overline{X}}(\overline{B(Q)}_C^*)$ and on $\sum_{C\in\mathcal{N}(\overline{X})/\overline{X}}(-1)^{|C|}\ \mathrm{Ind}_{\overline{X}_C}^{\overline{X}}(\overline{B(Q)}_C)$ are equal (where $X = N_G(Q)$, $B(Q) = Br_Q(1_B).kN_H(Q)$, and $^{-}$ denotes images under the natural algebra epimorphism from kX onto $kX/_Q$). It is necessary to observe here that $Q\cap N_H(Q) = 1$, and that $[Q,N_H(Q)] \leq H\cap Q = 1_H$, so that Q centralizes $N_H(Q)$.

That these two numbers are equal follows from Lemma 3.2 (applied with $\overline{X}$ in place of G, $\overline{N_H(Q)}$ in place of H, and $\overline{B(Q)}$ in place of B). The proof of Lemma 3.3 is complete.

REMARK : So far, we have not used any of the properties of G assumed at the beginning of the section, so that Lemmas 3.2 and 3.3 are valid for arbitrary finite groups. However, the next result (which completes the proof of Proposition 3.1) certainly requires the properties attributed to G.

PROPOSITION 3.4 :

(i) $\sum_{C\in\mathcal{P}(G)/G}(-1)^{|C|}\ k*(B_C)\ =\ \sum_{C\in\mathcal{P}(H)/G}(-1)^{|C|}\ k*(B_C)-$

$\sum_{(Q)}$ # (primitive idempotents in $Tr_Q^G(B^Q)/\sum_{P<Q} Tr_P^G(B^P)$), where Q runs over non-trivial p-subgroups of G with $Q\cap H = 1_G$ (up to conjugacy).

ii) $\sum_{C\in\mathcal{P}(H)/G}(-1)^{|C|}\ k*(B_C)\ =\ \ell*(B)\ -\ \sum_{(Q)}$ # $[N_G(Q)/_Q$-orbits on projective simple $kN_H(Q)/_Q$-modules in Brauer correspondents of B] where Q runs over <u>non-trivial</u> p-subgroups of H, up to G-conjugacy.

iii) If the triple (G,H,B) satisfies any one of the formulae appearing in Proposition 3.1, then it satisfies the other two.

<u>PROOF</u>:

i) $\sum_{C\in\mathcal{P}(G)/G}(-1)^{|C|}\ k*(B_C)$ is the number of G-fixed points on

$$\sum_{C\in\mathcal{P}(G)/G}(-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C)$$

$$\Big[-\sum_{C\in \mathcal{N}(H)/_G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C) + \sum_{\{C\in \mathcal{N}(G):Q_1\cap H = 1_G\}/_G} (-1)^{|C|}\ \mathrm{Ind}_{G_C}^{G}(B_C) \Big]$$

Choose a p-subgroup, Q of G with $Q\cap H = 1_G (\neq Q)$. Then (with notation as in Lemma 3.3), the contribution to $\sum_{C\in \mathcal{P}(G)/_G} (-1)^{|C|}\ k^*(B_C)$ from chains starting with (conjugates of) Q is $\sum_{C\in \mathcal{N}(\bar{X})/_{\bar{X}}} (-1)^{|C|+1}\ k^*(\overline{B(Q)}_C)$.

By the choice of G, we know that $\sum_{C\in \mathcal{N}(\bar{X})/_{\bar{X}}} (-1)^{|C|}\ k^*(\overline{B(Q)}_C)$ is the number of primitive idempotents in $\mathrm{Tr}_1^{\bar{X}}(\overline{B(Q)})$, ie the number of primitive idempotents of $\mathrm{Tr}_Q^{N_G(Q)}(B(Q))$. By a Theorem of Puig [9], this last number is the number of primitive idempotents of $\mathrm{Tr}_Q^G(B^Q) / \sum_{P<Q} \mathrm{Tr}_P^G(B^P)$ (ie. the number of primitive idempotents of B^G with defect group Q, viewing B as a G-algebra).

Since Q was arbitrary (subject to $Q\cap H = 1_G \neq Q$), part i) follows.

ii) We have $\sum_{C\in \mathcal{P}(H)/_G} (-1)^{|C|}\ k^*(B_C) = \sum_{C\in \mathcal{P}(H)/_G} (-1)^{|C|}\ \ell^*(B_C)$.

Let Q be a non-trivial p-subgroup of H, and consider the contribution to $\sum_{C\in \mathcal{P}(H)/_G} (-1)^{|C|}\ \ell^*(B_C)$ from chains starting with conjugates of Q. By minor variants of familiar arguments this is $-\sum_{C\in \mathcal{N}(\bar{Y})/_{\bar{X}}} (-1)^{|C|}\ \ell^*(\overline{B(Q)}_C)$ $\Big[= -\sum_{C\in \mathcal{N}(\bar{Y})/_{\bar{X}}} (-1)^{|C|}\ k^*(\overline{B(Q)}_C) \Big]$ (where $X = N_G(Q)$, $Y = N_H(Q)$, $^-$ denotes images under the natural epimorphism from kX to $kX/_Q$, and where B(Q) denotes $\mathrm{Br}_Q(1_B).kY$).

The choice of G tells us that this contribution is $-\ \#$ ($N_G(Q)/_Q$-orbits on projective simple $\overline{B(Q)}$-modules) so part ii) follows.

iii) We may rewrite the equation of part i) above as:

$$\sum_{C\in \mathcal{N}(H)/_G} (-1)^{|C|}\ k^*(B_C) = \sum_{C\in \mathcal{N}(G)/_G} (-1)^{|C|}\ k^*(B_C) + \sum_{(Q)} \#(\text{primitive idempotents of } B^G \text{ with defect group } Q)$$

where Q runs over non-trivial p-subgroups of G with $Q\cap H = 1_G$ (up to G-conjugacy).

From general considerations on G-algebras, we also have : #[primitive idempotents of $\mathrm{Tr}_1^H(B)^G$]

$= \sum_Q$ #[primitive idempotents of B^G with defect group Q]

where Q runs over p-subgroups of G with $Q\cap H = 1_G$ (up to G-conjugacy, and <u>including</u> the trivial subgroup).

It follows, then, that

$$\sum_{C\in \mathcal{P}(H)/G} (-1)^{|C|} k^*(B_C) = \#[\text{primitive idempotents in } Tr_1^H(B)^G]$$

if and only if $\sum_{C\in \mathcal{P}(G)/G} (-1)^{|C|} k^*(B_C) =$

$$\#[\text{primitive idempotents in } Tr_1^G(B)].$$

It remains to prove that

$$\sum_{C\in \mathcal{P}(H)/G} (-1)^{|C|} k^*(B_C) = \#[\text{primitive idempotents in } Tr_1^H(B)^G]$$

if and only if

#(G-orbits on simple B-modules)$(=\ell^*(B))$

$= \sum_{(Q)}$ #($N_G(Q)/_Q$-orbits on projective simple $kN_H(Q)/_Q$-modules in Brauer correspondents of B).

If B is a sum of blocks of defect 0 of kH, then we have

$$\sum_{C\in \mathcal{N}(H)/G} (-1)^{|C|} k^*(B_C) = k^*(B) = 1 = \#[\text{primitive idempotents of } Tr_1^H(B)^G].$$

Also #(G-orbits of simple B-modules) (=1)

$= \sum_{(Q)}$ #[$N_G(Q)/_Q$-orbits on projective simple $kN_H(Q)/_Q$-modules in Brauer correspondents of B] where Q runs over p-subgroups of H (up to G-conjugacy).

Thus we may suppose that B is a sum of blocks of positive defect of kH. In that case, there are no primitive idempotents in $Tr_1^H(B)^G$, and our problem reduces to showing that $\sum_{C\in \mathcal{N}(H)/G} (-1)^{|C|} k^*(B_C) = 0$ if and only if $\ell^*(B) = \sum_{(Q)}$ #[$N_G(Q)/_Q$-orbits on projective simple $kN_H(Q)/_Q$-modules in Brauer correspondents of B].

where Q runs over p-subgroups of H (up to G-conjugacy). This follows from part ii) above, since the subgroup 1_G makes no contribution to the right hand side of the second equation in this case.

The proof of Proposition 3.4 is complete.

We now consider some circumstances under which the formulae of Proposition 3.1 can be shown to hold:

PROPOSITION 3.5 : Let X be a finite group, $Y \lhd X$, B be a minimal X-stable sum of blocks of defect 0 of kY. Then the triple (X,Y,B) satisfies all the formulae appearing in Proposition 3.1.

PROOF: Only the formula of part i) presents any difficulty. We know that $\sum_{C\in \mathcal{P}(X)/X} (-1)^{|C|} k^*(B_C) = \sum_{C\in \mathcal{P}(Y)/X} (-1)^{|C|} \dim_k(Tr_1^{X_C}(B_C)) + \sum_{\{C\in \mathcal{P}(X):Q_1\cap Y = 1_X\}/X} (-1)^{|C|} \dim_k(Tr_1^{X_C}(B_C))$. Let C be a chain whose first non-trivial term, Q say, interesects Y trivially. Then $Q \leq X_C$, and $[Y_C,Q] \leq Q\cap Y$, so that $Tr_1^{X_C}(B_C) = 0$.

Since $B_C = 0$ whenever C is a non-trivial chain in $\mathcal{P}(Y)$, we need to prove that #(primitive idempotents in $Tr_1^X(B)$) = $\dim_k(Tr_1^X(B))$. This is true, since $Tr_1^X(B)$ is a subalgebra of the semi-simple commutative algebra Z(B).

PROPOSITION 3.6 : Suppose that $O_p(G)\neq 1_G$. Then (G,H,B) satisfies the formulae of Proposition 3.1.

PROOF : It suffices to prove that (G,H,B) satisfies the formula of part i) of Proposition 3.1.

However, since $O_p(G)\neq 1_G$, there are no idempotents in $Tr_1^G(kG)$, so certainly no primitive idempotents in $Tr_1^G(B)$. Also, $\sum_{C\in \mathcal{P}(G)/G} (-1)^{|C|} \mathrm{Ind}_{G_C}^G (B_C) = 0$ so that the formula of part i) is satisfied.

REMARK : In fact, we do not need any of our assumptions on G to prove that (G,H,B) satisfies the formulae of parts i) and iii) of Proposition 3.1, but these assumptions are necessary to prove that (G,H,B) satisfies the formula of part ii) in the case $O_p(G)\cap H = 1_G$.

The proof of the following result is straightforward, so we omit it:

LEMMA 3.7 : Let X be a finite group, M,N be normal subgroups of X with $M \leq N$ and N/M a p-group. Then there is a one-to-one correspondence between X-orbits of projective simple kM-modules of inertial index prime to p and X-orbits of projective simple kN-modules of inertial index prime to p.

PROPOSITION 3.8 : Suppose that $H \neq O^p(H)$. Then the triple (G,H,B) satisfies all the formulae of Proposition 3.1.

PROOF : By Proposition 3.4, it suffices to prove that (G,H,B) satisfies the formula of part i) of Proposition 3.1. Let $L = O^p(H)$(so $L < H$), and let $B' = 1_B kL$, (a minimal G-invariant sum of blocks of kL). By hypothesis, we know that the number of G-fixed points on $\sum_{C \in \mathcal{S}(X)/X} (-1)^{|C|} \mathrm{Ind}_{G_C}^{G}(B'_C)$ is the number of primitive idempotents in $Tr_1^G(B')$, which is the number of G-orbits of projective simple B'-modules of inertial index prime to p. By Lemma 3.7, this is also the number of G-orbits of projective simple B-modules of inertial index prime to p, (which is the number of primitive idempotents of $Tr_1^G(B)$).

Now $\sum_{C \in \mathcal{P}(X)/X} (-1)^{|C|} \mathrm{Ind}_{G_C}^{G}(B_C)$ and $\sum_{C \in \mathcal{P}(X)/X} (-1)^{|C|} \mathrm{Ind}_{G_C}^{G}(B'_C)$ are both virtual projectives. Hence the number of G-fixed points on $\sum_{C \in \mathcal{P}(X)/X} (-1)^{|C|} \mathrm{Ind}_{G_C}^{G}(B_C)$ is $\sum_{C \in \mathcal{P}(X)/X} (-1)^{|C|} \dim_k(Tr_1^{G_C}(B_C))$ $= \sum_{C \in \mathcal{S}(X)/X} (-1)^{|C|} \dim_k(Tr_1^{G_C}(B'_C))$ (as $H_C \backslash L_C$ consists of p-singular elements), and this last number is also the number of G-fixed points on $\sum_{C \in \mathcal{P}(X)/X} (-1)^{|C|} \mathrm{Ind}_{G_C}^{G}(B'_C)$. The result follows.

The methods and results of this section can be readily adapted to yield:

PROPOSITION 3.9: Let X be a finite group, Y be a p-nilpotent normal subgroup of X, B be a minimal X-invariant sum of blocks of kY. Then the triple (X,Y,B) satisfies all the formulae of Proposition 3.1.

LEMMA 3.10: Let X be a finite group, $Y \lhd X$, L be a (not necessarily normal) subgroup of X, M be a right kL-module. Then $\dim_k(Tr_1^{L \cap Y}(M)^L)$ $= \dim_k(Tr_1^Y(\mathrm{Ind}_L^X(M))^X)$.

<u>PROOF</u> : We know that $\mathrm{Ind}_L^X(M)$ has an L-summand, say M', isomorphic to M, and that Tr_L^X induces a vector space isomorphism between M'^L and $\mathrm{Ind}_L^X(M)^X$. We claim that Tr_1^X induces an isomorphism between $Tr_1^{Y \cap L}(M')^L$ and $Tr_1^Y(\mathrm{Ind}_L^X(M))^X$.

Let ϕ be an L-projection from $\mathrm{Ind}_L^X(M)$ onto M'. Then given $v \in \mathrm{Ind}_L^X(M)^X$, $v\phi$ is the <u>unique</u> element of M'^L such that $v = Tr_L^X(v\phi)$.

Suppose then that $v = Tr_1^Y(w)$. Then we have $v\phi = Tr_1^{Y \cap L}(w'\phi)$

where $w' = \sum_{x \in Y/Y \cap L} wx$. This proves that $\mathrm{Tr}_1^Y(\mathrm{Ind}_L^X(M))^X \subset \mathrm{Tr}_L^X(\mathrm{Tr}_1^{Y\cap L}(M')^L)$.

On the other hand, choose $m' \in \mathrm{Tr}_1^{Y\cap L}(M')^L$, say $m' = \mathrm{Tr}_1^{Y\cap L}(w)$ for some $w \in M'$. Then we have $\mathrm{Tr}_L^X(m') = \sum_{x \in L\backslash X/Y} \mathrm{Tr}_{L^x \cap Y}^Y(m x) =$ $\sum_{x \in L\backslash X/Y} \mathrm{Tr}_1^Y(wx)$. Thus $\mathrm{Tr}_L^X(\mathrm{Tr}_1^{Y\cap L}(M')^L) \subset \mathrm{Tr}_1^Y(\mathrm{Ind}_L^X(M))^X$, so the result follows.

COROLLARY 3.11: Let X be a finite group, $Y \lhd X$, B be a minimal X-invariant sum of blocks of kY. Then the number of X-fixed points on $\sum_{C \in \mathcal{P}(Y)/X} (-1)^{|C|} \mathrm{Ind}_{X_C}^X(B_C)$ is $\sum_{C \in \mathcal{P}(Y)/X} (-1)^{|C|} \dim_k(\mathrm{Tr}_1^{Y_C}(B_C)^{X_C})$

<u>PROOF</u> : We have seen that $\sum_{C \in \mathcal{P}(Y)/X} (-1)^{|C|} \mathrm{Ind}_{X_C}^X(B_C)$ is an alternating sum of modules whose vertices intersect Y trivially. It follows that the number of X-fixed points on the above virtual module is $\sum_{C \in \mathcal{P}(Y)/X} (-1)^{|C|} \dim_k(\mathrm{Tr}_1^Y(\mathrm{Ind}_{X_C}^X(B_C))^X) = \sum_{C \in \mathcal{P}(Y)/X} (-1)^{|C|} \dim_k(\mathrm{Tr}_1^{Y_C}(B_C)^{X_C})$ (using Lemma 3.10).

We can now strengthen Proposition 3.9.

PROPOSITION 3.12: Let X be a finite group, $Y \lhd X$, B be a minimal X-invariant sum of blocks of kY. Suppose that B is a sum of nilpotent blocks of kY. Then the triple (X,Y,B) satisfies all the formulae appearing in Proposition 3.1.

<u>PROOF</u> : We may, and do, assume that B is a sum of blocks of positive defect.

i) We know that the number of X-fixed points on

$$\sum_{C \in \mathcal{P}(X)/X} (-1)^{|C|} \mathrm{Ind}_{X_C}^X(B_C) =$$

$$\sum_{C \in \mathcal{P}(Y)/X} (-1)^{|C|} \dim_k(\mathrm{Tr}_1^{X_C}(B_C)) \quad + \sum_{\{C \in \mathcal{P}(X): Q_1 \cap Y = 1_X\}/X} (-1)^{|C|} \dim_k(\mathrm{Tr}_1^{X_C}(B_C))$$

Let C be a chain in $\mathcal{P}(X)$ whose first non-trivial term is Q with $Q \cap Y = 1_X$. Then $[Y_C, Q] \leq Q \cap Y = 1_X$, so that $\mathrm{Tr}_1^{X_C}(B_C) = 0$, as $Q \leq X_C$.

On the other hand, as in Proposition 5.2 of [5], whenever C is a chain in $\mathcal{S}(Y)$, we already have $Tr_1^{Y_C}(B_C)) = 0$, so we see that # (X-fixed points on $\sum_{C\in\mathcal{S}(X)/X} (-1)^{|C|} \operatorname{Ind}_{X_C}^{X}(B_C)) = 0$, which is the number of primitive idempotents in $Tr_1^X(B)$.

ii) The number of X-fixed points on $\sum_{C\in\mathcal{S}(Y)/X} (-1)^{|C|} \operatorname{Ind}_{X_C}^{X}(B_C)$ is $\sum_{C\in\mathcal{S}(Y)/X} (-1)^{|C|} \dim_k(Tr_1^{Y_C}(B_C)^{X_C})$, which is 0 by the remarks above. Also, #(primitive idempotents in $Tr_1^Y(B)^X) = 0$.

iii) Let b be a block of kY which occurs as a summand of B, and let P be a defect group for b. Then as Alperin's conjecture holds for b we see that $f_o^{(b)}(N_Y(Q)/_Q) = 0$ unless Q is Y-conjugate to P, whilst $f_o^{(b)}(N_Y(P)/_P) = 1$. Since similar statements hold for X-conjugates of b, the formula of part iii) of Proposition 3.1 reduces to

#(X-orbits of simple B-modules) = #($N_X(P)/_P$-orbits of projective simple $kN_Y(P)/_P$-modules in Brauer correspondents of B).

The left hand number is 1, as b has a unique simple module.

Also, the number of projective simple $kN_Y(P)/_P$-modules in Brauer correspondents of B is the number of X-conjugates of b which have defect group P. A Frattini-type argument shows that $N_X(P)$ transitively permutes such conjugates of b, and hence also permutes their Brauer correspondents transitively. Thus the right hand number is also 1, and the proof of Proposition 3.12 is complete.

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Robinson, Geoffrey R.
Department of Mathematics
UMIST
PO Box 88
Manchester M60 1QD
England.

* Staszewski, Reiner
Fachbereich Mathematik
Universität Essen
D 4300 Essen
Bundesrepublik Deutschland

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