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TYPE I DEGENERATION OF KUNEV SURFACES

Sampei USUI

Dedicated to Professor Friedrich Hirzebruch
on the occasion of his sixtieth birthday

Introduction.

In this article we determine completely the main components of type I degenerations of Kunev surfaces, i.e., degenerations of Kunev surfaces with finite local monodromy. The main results here were already announced in [Us.4] only with some idea of froofs.

A *Kunev surface* X is defined as a canonical surface, i.e., canonical model of a surfaces of general type, with $\chi(\mathcal{O}_X) = 2$ and $(\omega_X)^2 = 1$, ω_X : the dualizing sheaf, which has an involution σ such that $Y' := X/\sigma$ is a K3 surface with rational double points (R.D.P., for short). It is well-known that X has only R.D.P. hence ω_X is a line bundle. It is also known that the linear system $|\omega_X^{\otimes 2}|$ gives a finite Galois cover $f : X \longrightarrow P^2$,

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factoring through Y', with Galois group $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$ whose branch locus consists of two cubics ΣC_j and a line L satisfying the condition:

(0.1) ΣC_j has only simple singularities. $C_1 \cap C_2 \cap L = \phi$.

The pull-back of L on the minimal model Y of Y' is reduced. Conversely, given two cubics ΣC_j and a line L on P^2 satisfying (0.1), we can reconstruct a Kunev surface X in the following way:

i) Take a double cover Y' of P^2 branched along ΣC_j. Then Y' is a K3 surface only with R.D.P.

ii) Let Y be the minimal model of Y'. Set $\alpha_1 : Y \longrightarrow P^2$. Let E_i $(1 \le i \le 9)$ be the exceptional curves for α_1 whose multiplicity in $\alpha_1{}^*C_j$ is odd. These are called *distinguished (-2)-curves*.

iii) Take a double cover X' of Y branched along $\alpha_1{}^*L + \Sigma E_i$. Then the canonical model X of X' becomes a Kunev surface.

By the structure of Kunev surfaces above, we can construct their coarse moduli space $\mathfrak{M}$ in two ways; by the geometric invariant theory applied for the branch loci $\Sigma C_j + L$, and by the period map for K3 surfaces Y. In order to see it more precisely, set

$\mathcal{G} := \{\Sigma C_j \in \mathrm{Sym}|\mathcal{O}_{P^2}(3)| \mid \Sigma C_j$ has only simple singularities$\}$,

$\mathcal{K}^* := \mathcal{G} \times |\mathcal{O}_{P^2}(1)|$,

$\mathcal{K} := \{\Sigma C_j + L \mid \Sigma C_j + L$ satisfies (0.1)$\}$.

Recall the fact that a plane sextic curve is properly stable with respect to the natural action of $SL_3(\mathbf{C})$ if and only if it has only simple singularities (cf. [H.2], [Sh]). Hence we can see in the first method that

$\mathfrak{N} := \mathcal{G}/SL_3(\mathbf{C})$

is the coarse moduli space of triples $(Y', \alpha_1^* \mathcal{O}_{P^2}(1), \Sigma_1^9 E_i)$, which are called *K3 surfaces of Kunev type*, and that the coarse moduli space of Kunev surfaces is

$\mathfrak{M} = \mathfrak{K}/SL_3(\mathbf{C})$.

On the other hand, by the second method, the projection

$\Phi_2 : \mathfrak{M} \longrightarrow \mathfrak{N}$

can be seen as a period map of the second cohomology for Kunev surfaces. This is proved by suitable versions of the Torelli theorem and surjectivity of the period map for K3 surfaces of Kunev type and the lattice theory of Nikulin in [T.2] and [Mo.2] (there are some ambiguous points in the former; the latter is rigorous) (cf. (2.8)). This together with the Kulikov list of degenerations of K3 surfaces ([Ku], (PP]) implies

$\mathfrak{M}^* := \mathfrak{K}^*/SL_3(\mathbf{C})$

is a partial compactification of $\mathfrak{M}$ obtained by adding those points which correspond to type I degenerations of Kunev surfaces.

Now we define two functions on $\mathfrak{K}^*$ by

$m(\Sigma C_j, L) := \Sigma_{P \in P^2} \min\{I(P, L \cap C_j) \mid j = 1, 2\}$,

$n(\Sigma C_j, L) := \#\{\text{triple points of } C_j \text{ on } L, \ j = 1, 2\}$,

where $I(P, L \cap C_j)$ is the intersection multiplicity of L and C_j at $P \in P^2$. It is easy to see that the value of $m(\ )$ (resp. $n(\ )$) is 0, 1, 2 or 3 (resp. 0, 1 or 2). These functions induce ones on $\mathfrak{M}^*$ and they define two stratifications:

$\mathfrak{M}^* = \mathcal{S}_0 \amalg \mathcal{S}_1 \amalg \mathcal{S}_2$ where $\mathcal{S}_m = \{s \in \mathfrak{M}^* \mid m = \min\{2, m(s)\}\}$,

$\mathfrak{M}^* = \mathcal{T}_0 \amalg \mathcal{T}_1 \amalg \mathcal{T}_2$ where $\mathcal{T}_n = \{s \in \mathfrak{M}^* \mid n = n(s)\}$.

The main result in this paper is stated as

(0.2) In the above notation, the partial compactification $\mathfrak{M}^*$ is divided into five parts by the above stratifications and they correspond to two series of degenerations:

$$\begin{array}{ccccc} & & \mathscr{S}_1 = \left\{\begin{array}{l}\text{numerical K3 surfaces}\\ \text{with one double fiber}\end{array}\right\} & \longrightarrow & \mathscr{S}_2 = \{\text{K3 surfaces}\} \\ & \nearrow & & & \\ \mathscr{S}_0 \cap \mathscr{T}_0 = \mathfrak{M} = \{\text{Kunev surfaces}\} & & & & \\ & \searrow & & & \\ & & \mathscr{T}_1 \cap \mathscr{S}_0 = \left\{\begin{array}{l}\text{elliptic surfaces}\\ \text{with } p_g = q = 1\end{array}\right\} & \longrightarrow & \mathscr{T}_2 = \left\{\begin{array}{l}\text{splitting abelian}\\ \text{surfaces}\end{array}\right\} \end{array}$$

The period maps of the second cohomology for respective surfaces are essentially equal to the restrictions of the projection

$$\Phi_2 : \mathfrak{M}^* \longrightarrow \mathfrak{N}$$

by the Mayer-Vietoris sequence and the Clemens-Schmid sequence and $\Phi_2|_{\mathscr{S}_m}$, $\Phi_2|_{\mathscr{T}_n}$ and $\Phi_2|_{\mathscr{S}_m \cap \mathscr{T}_n}$ have pure relative dimension $2 - m$, $2 - n$ and $2 - (m + n)$ respectively (Theorem (2.6), Corollary (2.10)).

Here we use the terminology a *numerical K3 surface* with one double fiber, which means a minimal elliptic surface with $p_g = 1$, $q = 0$ and $c_1^2 = 0$ and with one double fiber. For $\Sigma C_j + L \in \mathscr{K}^*$, the minimal model $\hat{X}$ of the corresponding surface can be obtained in an analogous way as the reconstruction (i)-(iii) above of Kunev surfaces, i.e., we can construct a diagram:

$$
(0.3)\qquad
\begin{tikzcd}
 & X' \arrow[d] & X^* \arrow[l] \arrow[r] \arrow[d] & \hat{X} \\
Y' \arrow[d] & Y \arrow[l] \arrow[dl, "\alpha_1"'] \arrow[d] & Y^* \arrow[l] & \\
P^2 & P^* \arrow[l] & &
\end{tikzcd}
$$

where Y' is the double cover of P^2 branched along ΣC_j, Y is the canonical resolution of $Y' \longrightarrow P^2$, X' is the double cover of Y branched along $\alpha_1^*L + \Sigma_1^9 E_i$, X^* is the canonical resolution of $X' \longrightarrow Y$, and $\hat{X}$ is the minimal model of X^*. Diagram (0.3) suggests an idea of a proof of (0.2). The essential part is the computation of the branch locus $\alpha_1^*L + \Sigma E_i$ on the minimal K3 surface Y.

Historically the phenomenon of appearrence of positive dimensional fibers of a period map is first observed for Kunev surfaces in [T.1], [Us.1] and [Us.2] (for Todorov surfaces, in [T.2]) then for elliptic surfaces with $p_g = q = 1$ in [Sa.M]. It is new for numerical K3 surfaces with one double fiber. The present result (0.2) explains uniformly these phenomena by degeneration (Corollary (2.13)).

We explain here the background of Kunev surfaces. The minimal model of a Kunev surface is simply connected surface with $p_g = c_1^2 = 1$. Let $\tilde{\mathfrak{M}}$ be the coarse moduli sapce of surfaces with $p_g = c_1^2 = 1$, then $\tilde{\mathfrak{M}}$ is irreducible, rational and with $\dim \tilde{\mathfrak{M}} = 18$ which contains Kunev locus $\mathfrak{M}$ with codimension 6 ([Ca.1], [Ca.2]). On the Hodge theoretic view-point, these surfaces are interesting materials. After Kunev constructed an example of Kunev surface as a counterexample to the infinfitesimal Torelli theorem, the following results are known:

(0.4) The generic infinitesimal Torelli theorem holds for surfaces in $\tilde{\mathfrak{M}}$ ([Ca.1]).

(0.5) The period map Φ_2 of surfaces in $\tilde{\mathfrak{M}}$ has some positive dimensional fibers ([T.1], [Us.1], [Us.2]; [T.1] treats only Kunev surfaces).

(0.6) $\mathfrak{M}$ in $\tilde{\mathfrak{M}}$ is characterized by $\dim \Phi_2^{-1}\Phi_2([X]) = 2$, which is the maximal dimension of the fibers of Φ_2 ([Us.1]).

(0.7) The infinitesimal mixed Torelli theorem holds for pairs (X,C) of surfaces X in $\tilde{\mathfrak{M}}$ and their canonical curves C ([Us.3]).

(0.8) The generic mixed Torelli theorem holds for Kunev surfaces ([L], [SSU]; there is a point about monodromy which is not clear in [L]).

(0.9) There exists a Zariski open subset W of $\mathfrak{M}$ such that $\Phi^{-1}\Phi(W) = W$, where $\Phi : \tilde{\mathfrak{M}} \longrightarrow \Gamma\backslash D$ is the mixed period map ([SSU]).

Hence, in order to solve the mixed Torelli problem for surfaces in $\tilde{\mathfrak{M}}$ via Kunev locus $\mathfrak{M}$, it is necessary to study the following:

(0.10) A compactification of the mixed period map $\Phi : \tilde{\mathfrak{M}} \longrightarrow \Gamma\backslash D$.

(0.11) The monodromy Γ in (0.9), where we used a geometric one.

(For a general reference of the above as well as for the terminology such as *mixed* period map, *mixed* Torelli etc., see [SSU].) Problem (0.10) is one of the motivations of the present work. Our result here is not its answer but a by-product.

Section 1 is preliminaries. We shall recall the canonical resolution of a double cover and related results, the Clemens-Schmid sequence and monodromy criteria, the canonical bundle formula for elliptic surfaces and definitions of Kunev surfaces and numerical K3 surfaces and some of their properties for our later use.

In Section 2, we shall construct an integral family of surfaces $f : \mathcal{X} \longrightarrow U$ over a fixed K3 surface of Kunev type, which is a type I degeneration of Kunev surfaces. We shall state Main Theorem (2.6) and explain this result perspectively in the framework of a type I partial compactification $\mathfrak{M}^*$ of the coarse moduli space $\mathfrak{M}$ of Kunev surfaces (Corollary (2.10)). We shall also explain uniformly the phenomenon of appearence of positive dimensional fibers of the period maps for Kunev surfaces, numerical K3 surfaces with one double fiber and elliptic surfaces with $p_g = q = 1$. The main part of the proof of Theorem (2.6) will be postponed to Sections 4 and 5.

In Section 3, we shall study locally over the singular point P of $\Sigma C_j + L \subset P^2$ for $\Sigma C_j + L \in \mathcal{K}^*$ and give tables of configurations of $\Sigma C_j + L$, the branch loci $B_Y(P)$ on the minimal K3 surfaces Y and the canonical divisors $K_{\hat{X}_1}(P)$ of type I degenerations of Kunev surfaces corresponding to $\Sigma C_j + L$. All of these will be described locally over the critical points P in this section. The result here plays the key role in the proofs of Theorem (2.6.3).

Section 4 contains a proof of Theorem (2.6.3). We shall use the local classification in Section 3 as well as an elliptic fibration on the minimal model $\hat{X}$ induced by the pencil of lines on P^2 through a critical point P of $\Sigma C_j + L$ for $\Sigma C_j + L \in \mathcal{G}_m \cup \mathcal{T}_n$ $(m > 0,\ n > 0)$.

Section 5 contains tables of global configurations of $\Sigma C_j + L \in \mathcal{K}^*$, the branch loci B_Y on the minimal K3 surfaces Y and the canonical divisors of the minimal model $\hat{X}$ of type I degenerations of Kunev surfaces corresponding to $\Sigma C_j + L$. These tables give another proof of Theorem (2.6.3), which is clumsy but elementary and fruitful.

We use the following terminology:

(-1)-curve : an irreducible exceptional curve of the first kind on a smooth surface.

(-2)-curve : an irreducible rational curve with self-intersection -2 on a smooth surface, i.e., a nodal curve.

(n)-(bi)section : a (bi)section of a fibration on a smooth surface with self-intersection n.

ADDED IN PROOF: Because of the reason of publication, we shall publish Section 5 separately elsewhere. The present article consists of Sections 1-4, which is logically self-contained. The author is grateful to the referee for pointing out a careless mistake in (2.5) as well as typographical errors in the old version.

1. Preliminaries.

(1.1) *Canonical resolution.* In this subsection, we shall summarize the process of a canonical resolution and related results in [H.1] in a slightly general form for our later use.

Let Y be a smooth surface, $B = \Sigma\, b_i D_i$ an effective divisor on Y and $\mathcal{F}$ a line bundle on Y such that $\mathcal{O}_Y(B) = \mathcal{F}^{\otimes 2}$. Then we can associate the double cover $X = \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{F}^{-1}) \longrightarrow Y$ branched along B, where $\mathcal{O}_Y \oplus \mathcal{F}^{-1}$ is endowed an $\mathcal{O}_Y$-algebra structure by $s : \mathcal{F}^{\otimes(-2)} \longrightarrow \mathcal{O}_Y$ for $s \in H^0(Y, \mathcal{F}^{\otimes 2})$ with $\{s = 0\} = B$. If B is non-reduced (resp. reduced but singular), X is non-normal (resp. has isolated singularities).

The process to obtain the canonical resolution X^* of X is as follows:

0) Set $Y_0 = Y$, $B_0 = B_{\mathrm{odd,red}} := B - 2\Sigma[b_i/2]D_i$ and $\mathcal{F}_0 = \mathcal{F} \otimes \mathcal{O}_Y(-\Sigma[b_i/2]D_i)$, and take the double cover $X_0 = \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{F}_0^{-1})$ branched along B_0. Let $p_0 : X_0 \longrightarrow X$ be the birational morphism induced by $\mathcal{F}_0 \hookrightarrow \mathcal{F}$.

i) Let $q_1 : Y_1 \longrightarrow Y$ be a blowing-up with center at a singular point P_1 of B_0. Let e_1 be the multiplicity of $P_1 \in B_0$ and $E_1 = q_1^{-1}(P_1)$ the exceptional divisor. Set $B_1 = q_1^*B_0 - 2[e_1/2]E_1$ and $\mathcal{F}_1 = q_1^*\mathcal{F}_0 \otimes \mathcal{O}_{Y_1}(-[e_1/2]E_1)$ and take the double cover $X_1 = \mathrm{Spec}(\mathcal{O}_{Y_1} \oplus \mathcal{F}_1^{-1})$. Let $p_1 : X_1 \longrightarrow X_0$ be the birational morphism induced by $\mathcal{F}_1 \hookrightarrow q_1^*\mathcal{F}_0$.

After a finite number, say n, of repetition of the process i), we get a non-singular model $X^* := X_n$ which is called the *canonical* *resolution* of X. The whole procedure is given by the diagram:

$$
\begin{array}{lccccccccc}
 & X & \xleftarrow{p_0} & X_0 & \xleftarrow{p_1} & X_1 & \xleftarrow{p_2} & \cdots & \xleftarrow{p_n} & X_n = X^* \\
(1.1.1) & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
 & Y & = & Y_0 & \xleftarrow{q_1} & Y_1 & \xleftarrow{q_2} & \cdots & \xleftarrow{q_n} & Y_n =: Y^*
\end{array}
$$

$$B_i = q_i^* B_{i-1} - 2[e_i/2]E_i$$

$$\mathcal{F}_i = q_i^* \mathcal{F}_{i-1} \otimes \mathcal{O}_Y(-[e_i/2]E_i) \qquad (1 \le i \le n)$$

A singularity at a point on a reduced curve is called *simple* if its multiplicity is not greater than three and if it is not an infinitely near triple point. Note that, in the procedure of the canonical resolution (1.1.1), the curve B_0 has at most simple singularities if and only if $[e_i/2] = 1$ for all i.

(1.1.2) Lemma. In the above notation, if B_0 has at most simple singularities, then the canonical resolution of X coincides with the minimal resolution of X and we have

$$K_{X^*} = \varphi^*(K_Y \otimes \mathcal{F}_0), \qquad \text{where } \varphi : X^* \longrightarrow Y.$$

If, moreover, Y is a minimal K3 surface and $p_g(X^*) = 1$, then

$$q(X^*) = -((B_0)^2/8 + 2).$$

Proof. The first assertion follows from $K_{Y_i} \otimes \mathcal{F}_i = q_i^*(K_{Y_{i-1}} \otimes \mathcal{F}_{i-1})$ and this follows from the remark just before this lemma (cf. [H.1]). We shall prove the second assertion. By construction,

$$h^i(\mathcal{O}_{X^*}) = h^i(\mathcal{O}_{Y_n}) + h^i(\mathcal{F}_n^{-1}) = h^i(\mathcal{O}_{Y_n}) + h^{2-i}(K_{Y_n} \otimes \mathcal{F}_n)$$

$$= h^i(\mathcal{O}_Y) + h^{2-i}(K_Y \otimes \mathcal{F}_0) = h^i(\mathcal{O}_Y) + h^{2-i}(\mathcal{F}_0).$$

Since $h^i(\mathcal{O}_{X^*}) = h^i(\mathcal{O}_Y) = 1$ for $i = 0, 2$, we see $h^i(\mathcal{F}_0) = 0$ for $i = 0, 2$. Hence, by the Riemann-Roch theorem on Y, we have

$$q(X^*) = h^1(\mathcal{F}_0) = -\chi(\mathcal{F}_0) = -((\mathcal{F}_0)^2/2 + \chi(\mathcal{O}_Y)) = -((B_0)^2/8 + 2).$$

QED.

(1.2) *Clemens-Schmid sequence and monodromy criteria.* We shall summarize the Clemens-Schmid sequence and related results ([Cl.1], [Sc]) in the form for our later use. There are good expositions on this topic in [P] and [Mo.1].

Let U be the unit disk. Let $f : \mathfrak{X} \longrightarrow U$ be a semi-stable degeneration of surfaces, i.e., f is a proper flat holomorphic map, $\mathfrak{X}$ is a Kähler manifold, $X_t := f^{-1}(t)$ is smooth for $t \neq 0$ and $X_0 := f^{-1}(0) = \Sigma V_i$ is a reduced divisor with simple normal crossings. In this situation, we have the Clemens-Schmid exact sequence

(1.2.1) $$0 \longrightarrow H^0_{lim} \xrightarrow{\beta} H_4 \xrightarrow{\alpha} H^2 \xrightarrow{\iota} H^2_{lim} \xrightarrow{N} H^2_{lim} \xrightarrow{\beta} H_2$$

where

$H^i_{lim} = H^i(X_t, \mathbb{Q})$ endowed with the limiting mixed Hodge structure,

$H^i = H^i(X_0, \mathbb{Q})$ endowed with the functorial mixed Hodge structure of Deligne [D],

$H_i = H_i(X_0, \mathbb{Q})$ endowed with the dual mixed Hodge structure,

$N = \log T$ for the local monodromy T acting on H^i_{lim},

β, α, ι and N are morphisms of mixed Hodge structure of type $(-2,-2)$, $(3,3)$, $(0,0)$ and $(-1,-1)$ respectively.

As a corollary of the Clemens-Schmid sequence, we have:

(1.2.2) Lemma-Definition. In the above notation, $N^3 = 0$ and $p_g(X_t) \geq \Sigma\, p_g(V_i)$ always hold. $N^2 = 0$ if and only if $H^2(\Gamma) = 0$ for the dual graph Γ of $X_0 = \Sigma V_i$. $N = 0$ if and only if $p_g(X_t)$

$= \Sigma\, p_g(V_i)$. The semi-stable degeneration $f : \mathfrak{X} \longrightarrow U$ is called type I (resp. II, III) if $N = 0$ (resp. $N \neq 0$ and $N^2 = 0$, $N^2 \neq 0$ and $N^3 = 0$).

(1.3) *Some results for elliptic fibrations.* We include here the canonical bundle formula [Ko.2, Theorem 12] and the positivity of the direct image of relative dualizing sheaf [Ue, Remark in Appendix] for elliptic fibrations for our later use.

Let X be a non-singular compact complex surface and let $f : X \longrightarrow \Delta$ be a *relatively minimal* elliptic fibration, i.e., a general fiber of f is a non-singular elliptic curve and no fiber of f contains (-1)-curves.

(1.3.1) *Canonical bundle formula.* The canonical bundle K_X of an elliptic surface X has the form

$$K_X = f^*(K_\Delta \otimes (R^1 f_* \mathcal{O}_X)^{-1}) \otimes \mathcal{O}_X(\Sigma(m_i - 1)F_i)$$

where $m_i F_i$, $i = 1, 2, \ldots, n$, are all multiple fibers. The line bundle $R^1 f_* \mathcal{O}_X$ is dual to $f_* \omega_{X/\Delta}$ where $\omega_{X/\Delta} = K_X \otimes (f^* K_\Delta)^{-1}$, and

$$\deg R^1 f_* \mathcal{O}_X = -\chi(\mathcal{O}_X).$$

A simpler proof of the above formula can be found in [Ue, Appendix].

For the degree of the line bundle $R^1 f_* \mathcal{O}_X$, or equivalently of $f_* \omega_{X/\Delta}$, we can see more:

(1.3.2) *Positivity of* $f_* \omega_{X/\Delta}$. We have

$$\deg f_* \omega_{X/\Delta} \geq 0.$$

The equality holds if and only if the elliptic fibration $f : X \longrightarrow \Delta$

has constant J-invariant and has only multiple singular fibers of type ${}_mI_0$ (for the notation, see [Ko.1]).

There is a full proof of (1.3.2) in [BPV, p.110] by reducing the assertion to the case of a semi-stable fibration.

By the definition of the Kodaira dimension, the following assertion can be obtained as an exercise of intersection theory (for a proof, see, e.g., [BPV, p.194]).

(1.3.3) If a non-singular compact complex surface X has Kodaira dimension $\kappa(X) = 1$, then X is an elliptic surface.

(1.4) *Some surfaces and their properties.* We include here the definitions of somewhat unfamiliar surfaces and their properties which will appear later.

(1.4.1) Definition. A *Kunev surface* X is a canonical surface with $\chi(\mathcal{O}_X) = 2$ and $(\omega_X)^2 = 1$, ω_X the dualizing sheaf, which has an involution σ such that X/σ is a K3 surface with at most rational double points (R.D.P. for short).

Let $\hat{X}$ be the minimal model of a Kunev surface X. The following properties are known ([Ca.1]):

(1.4.2) $\hat{X}$ is simply connected. $p_g(\hat{X}) = 1$. $c_1^2(\hat{X}) = 1$.

(1.4.3) The canonical model X can be represented as a weighted complete intersection of type $(6,6)$ in $\mathbf{P}(1,2,2,3,3)$ with at most R.D.P., whose partially normalized equations are

$$f = z_3^2 + f^{(3)}(x_0^2, y_1, y_2)$$

$$g = z_4^2 + g^{(3)}(x_0^2, y_1, y_2)$$

where $\deg x_0 = 1$, $\deg y_i = 2$ $(i = 1, 2)$, $\deg z_i = 3$ $(i = 3, 4)$, and $f^{(3)}$ and $g^{(3)}$ are cubics in $y_0 := x_0^2$, y_1, y_2.

(1.4.4) Definition. A minimal surface X is called a *numerical K3 surface* if $p_g = 1$, $q = 0$ and $c_1^2 = 0$.

The following are known:

(1.4.5) Every simply connected numerical K3 surface X belong to one of two oriented homotopy types according to its Whitney class, i.e., $c_1(X)$ mod 2 ([Mi]).

(1.4.6) A simply connected numerical K3 surface is characterized as either a K3 surface or an elliptic surface with $p_g = 1$ and $q = 0$ which has at most two multiple fibers and, in the case that there are two, their multiplicities are mutually prime ([Ko.3, Proposition 1, Lemma 6]).

(1.4.7) Remark. Kodaira [Ko.3] called a simply connected surface with the same oriented homotopy type as a K3 surface a *homotopy K3 surface*. By definition, a homotopy K3 surface (resp. K3 surface) is equivalent to a simply connected numerical K3 surface with $c_1(X) \equiv 0 \bmod 2$ (resp. $c_1(X) = 0$). While we shall come across numerical K3 surfaces with one double fiber later.

2. Construction of families of surfaces and statements of the main results.

(2.0) In this section, we shall construct families of surfaces which are degenerations of Kunev surfaces over a fixed K3 surface and state the main results. We postpone the proof of Theorem (2.6.3) in Section 4 and Section 5, where we shall give two different proofs after a preparation in Section 3.

(2.1) Let X be a Kunev surface defined in (1.4.1). Then by (1.4.3) the bicanonical bundle $\omega_X^{\otimes 2}$ gives a Galois cover $X \longrightarrow \mathbf{P}^2$ with Galois group $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$. The branch locus consists of two cubics $C_1 = \{f^{(3)}(y_0, y_1, y_2) = 0\}$ and $C_2 = \{g^{(3)}(y_0, y_1, y_2) = 0\}$ and a line $L = \{y_0 = 0\}$. The K3 surface $Y' := X/\sigma$ can be seen as a weighted complete intersection of type (6) in $\mathbf{P}(1,1,1,3)$ defined by an equation $h = u_3^2 + f^{(3)}(y_0, y_1, y_2)\, g^{(3)}(y_0, y_1, y_2)$ with $\deg y_i = 1$ $(0 \le i \le 2)$ and $\deg u_3 = 3$. By construction, the K3 surface Y' with R.D.P. is the double cover of $\mathbf{P}^2$ branched along the two cubics ΣC_j, hence ΣC_j on $\mathbf{P}^2$ has only simple singularities.

(2.2) For sextic curves on $\mathbf{P}^2$, curves with at most simple singularities coincide with properly stable curves with respect to the action of $SL_3(\mathbf{C})$ ([H.2], [Sh]). Set

$\mathcal{G} = \{\Sigma C_j \in \mathrm{Sym}^2|\mathcal{O}_{\mathbf{P}^2}(3)| \mid \Sigma C_j$ has only simple singularities$\}$

$\mathfrak{N} = \mathcal{G}/SL_3(\mathbf{C})$

Then, as a consequence of Theorem (2.6.3) below $\mathfrak{N}$ can be seen as the coarse moduli space of the polarized K3 surfaces with R.D.P. which are quotients of Kunev surfaces X by their involution σ

plus the data of the distinguished (-2)-curves defined in (2.4.2) below (cf. (2.7), (2.8) below). We call the K3 surfaces equipped with these data *K3 surfaces of Kunev type*. We have a projection $p : \mathfrak{M} \longrightarrow \mathfrak{N}$, $[X] \longmapsto [X/\sigma]$.

(2.3) For any fixed $\Sigma C_j \in \mathscr{G}$, we define functions in $t \in \check{P}^2$ by

$$m(t) = \Sigma_{P \in P^2} \min\{I(P, L_t \cap C_j) \mid j = 1, 2\}, \quad \text{and}$$

$$n(t) = \#\{\text{triple points of } C_j \text{ on } L_t, \ j = 1, 2\}.$$

Notice that if C_j has a triple point then C_j consists of three distinct lines with a common point. These functions define two stratifications of $\check{P}^2$:

$$\check{P}^2 = S_0 \amalg S_1 \amalg S_2, \quad \text{where} \quad S_m = \{t \in \check{P}^2 \mid m = \min\{2, m(t)\}\}.$$

$$\check{P}^2 = T_0 \amalg T_1 \amalg T_2, \quad \text{where} \quad T_n = \{t \in \check{P}^2 \mid n = n(t)\}.$$

Notice that $\operatorname{codim} S_m = m$, $\operatorname{codim} T_0 = 0$, and $\operatorname{codim} T_n = n$ if T_n is non-empty $(n = 1, 2)$.

(2.4) For $\Sigma C_j \in \mathscr{G}$, we denote by Y the minimal K3 surface which is obtained as the minimal resolution of the double cover of P^2 branched along ΣC_j. Let $\alpha_1 : Y \longrightarrow P^2$ be the projection and E_i be the exceptional curves for α_1, i.e., (-2)-curves. Then we have the following lemma whose proof is easy and we omit it.

(2.4.1) Lemma The sets $\{E_i \mid$ the multiplicity of E_i in the total transform $\alpha_1^* C_j$ is odd$\}$ $(j = 1, 2)$ coincide and the number of their elements is nine.

(2.4.2) Remark The nine (-2)-curves in the above lemma is an equivalent datum to the one of the *distinguished partial desingularization* of a K3 surface of Kunev (more generally, Todorov) type in [Mo.2]. We call the former the *distinguished (-2)-curves*. They appeared in A.D.E. configuration of exceptional curves over R.D.P. as in Table (3.2.2) in Section 3.

(2.5) Let M be a line on $\check{P}^2$ which is not contained in $(C_1 \cap C_2)^\vee$, and let $U := \check{P}^2 - M$ be the affine plane. We reorder the numbering so that E_i $(1 \le i \le 9)$ are the nine distinguished (-2)-curves on Y, and set $\mathcal{E}_i = U \times E_i$ $(1 \le i \le 9)$. Denote by $\mathcal{L} \subset U \times P^2$ the total space of the universal family of lines on P^2 over U. We can construct families of surfaces $f : \mathfrak{X} \longrightarrow U$ and $\tilde{f} : \tilde{\mathfrak{X}} \longrightarrow U$ in the following way:

0) Set $\alpha = 1 \times \alpha_1 : U \times Y \longrightarrow U \times P^2$.

i) Let $\beta : \mathfrak{Y} \longrightarrow U \times Y$ be the blowing-up along $\alpha^{-1}\mathcal{L} \cap (\Sigma_1^9 \mathcal{E}_i)$. Denote by $\mathcal{W}_i$ $(1 \le i \le 9)$ the exceptional divisors.

ii) Take the double cover $\gamma : \tilde{\mathfrak{X}}' \longrightarrow \mathfrak{Y}$ branched along $(\alpha\beta)^{-1}\mathcal{L} + \beta^{-1}(\Sigma\, \mathcal{E}_i)$.

iii) Let $\delta : \tilde{\mathfrak{X}}' \longrightarrow \tilde{\mathfrak{X}}$ be the contraction of $(\beta\gamma)^{-1}(\Sigma\, \mathcal{E}_i)$.

iv) Let $\varepsilon : \tilde{\mathfrak{X}} \longrightarrow \mathfrak{X}$ be the contraction of $\delta\gamma^{-1}(\Sigma\, \mathcal{W}_i)$.

(In the notation above, we use $\alpha^{-1}\mathcal{L}$ etc. as the proper transforms.) Set $\mathcal{L}_{\tilde{\mathfrak{X}}} = (\delta(\alpha\beta\gamma)^{-1}\mathcal{L}$ with reduced structure$)$ and $\mathcal{W}_{\tilde{\mathfrak{X}},i} = \delta\gamma^{-1}\mathcal{W}_i$.

(2.6) Theorem. In the above notation, $f : \mathfrak{X} \longrightarrow U$ is an integral family of degenerations of Kunev surfaces over the fixed $\Sigma\, C_j \in \mathfrak{N}$. This family has the following properties:

(1) The singularity of the total space $\mathfrak{X}$ consists of disjoint nine compounds Veronese cone over $(S_1 \amalg S_2) \cap U = (C_1 \cap C_2)^{\vee} \cap U$, i.e., analytically isomorphic to the product of a line and the cone over the Veronese embedding of $P^2 \subset P^5$ by $|\mathcal{O}_{P^2}(2)|$. $\varepsilon : \tilde{\mathfrak{X}} \longrightarrow \mathfrak{X}$ is a desingularization and the exceptional divisor $\mathcal{W}_{\mathfrak{X},i}$ is a family of P^2 over a line in $(C_1 \cap C_2)^{\vee} \cap U$ $(1 \le i \le 9)$. $K_{\tilde{\mathfrak{X}}} = \mathcal{L}_{\tilde{\mathfrak{X}}} + \Sigma\, \mathcal{W}_{\mathfrak{X},i}$.

(2) The fiber $\tilde{X}_t := \tilde{f}^{-1}(t) = V_t + \Sigma\, W_{i,t}$, where V_t is the main component, i.e., the component with $p_g = 1$, and $W_{i,t} := \mathcal{W}_{\mathfrak{X},i}|\tilde{X}_t$. Hence the dualizing sheaf of V_t coincides with $\mathcal{O}(\mathcal{L}_{\tilde{\mathfrak{X}}}|V_t)$.

(3) V_t is a (singular) Kunev surface, numerical K3 surface with one double fiber, K3 surface, elliptic surface with $p_g = q = 1$, or splitting abelian surface according to $t \in S_0 \cap T_0$, S_1, S_2, $S_0 \cap T_1$, or T_2.

Proof of (1) and (2). In the notation in (2.5), notice that $\mathcal{L}$ and $U \times (\Sigma\, C_j)$ intersect transversally on $U \times P^2$ hence so do $\alpha^{-1}\mathcal{L}$ and $(\Sigma_1^9\, \mathcal{E}_i)$ on $U \times Y$. This implies that $\alpha^{-1}\mathcal{L} \cap (\Sigma\, \mathcal{E}_i)$ consists of nine disjoint P^1-bundles over the lines $\check{P}_k \cap U$ on U, $P_k \in C_1 \cap C_2$. Therefore the branch locus $(\alpha\beta)^{-1}\mathcal{L} + \beta^{-1}(\Sigma\, \mathcal{E}_i)$ on $\mathcal{Y}$ is a smooth divisor. It follows that $\tilde{\mathfrak{X}}'$ is smooth.

Since $\mathcal{E}_i = U \times E_i$ and E_i is a (-2)-curve on Y, we see $N_{\mathcal{E}_i/U\times Y} \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{P^1}(-2)$. Hence $\beta^{-1}\mathcal{E}_i$ on $\mathcal{Y}$ is a P^1-bundle whose normal bundle restricted to any fiber is isomorphic to $\mathcal{O}_{P^1}(-2)$. This implies that $(\beta\gamma)^{-1}\mathcal{E}_i$ on $\tilde{\mathfrak{X}}'$ is a P^1-bundle over U whose normal bundle restricted to any fiber is isomorphic to $\mathcal{O}_{P^1}(-1)$.

Thus we get a smooth variety $\hat{\mathfrak{X}}$ in Step (iii).

The P^1-bundle $\alpha^{-1}\mathcal{L} \cap \mathcal{E}_i$ over the line $\check{P}_k \cap U$ has $N_{\alpha^{-1}\mathcal{L}\cap\mathcal{E}_i/U\times Y} \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-2)$, because $N_{E_i/U\times Y} \simeq \mathcal{O}_{P^1}^{\oplus 2} \oplus \mathcal{O}_{P^1}(-2)$ and $\alpha^{-1}\mathcal{L} \cap \mathcal{E}_i \cap (H \times Y) = E_i$ transversally, for any line H on U other than $\check{P}_k \cap U$, and $N_{H\times Y/U\times Y} \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{P^1}$. Hence $\mathcal{W}_i$ is a Σ_2-bundle over $\check{P}_k \cap U$, where $\Sigma_2 := \mathrm{Proj}(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-2))$. This implies that $\gamma^{-1}\mathcal{W}_i$ is a Σ_1-bundle over $\check{P}_k \cap U$ intersecting with $(\beta\gamma)^{-1}\mathcal{E}_i$ along the P^1-bundle over $\check{P}_k \cap U$ whose fiber is the (-1)-section on Σ_1. Thus we get a P^2-bundle $\delta\gamma^{-1}\mathcal{W}_i$ over the line $\check{P}_k \cap U$ on $\hat{\mathfrak{X}}$ in Step (iii).

Since $N_{H\times Y/U\times Y} \otimes \mathcal{O}_{E_i} \simeq \mathcal{O}_{P^1}$ as above, $\beta^{-1}(H \times Y)$ intersects with $\mathcal{W}_i$ along a (2)-section on Σ_2 over the point $H \cap \check{P}_k$, hence $(\beta\gamma)^{-1}(H \times Y)$ intersects with $\gamma^{-1}\mathcal{W}_i$ along a (4)-bisection on Σ_1. Therefore $\delta(\beta\gamma)^{-1}(H \times Y)$ intersects with $\delta\gamma^{-1}\mathcal{W}_i$ along a conic on P^2 over the point $H \cap \check{P}_k$. Thus we see that $\delta\gamma^{-1}\mathcal{W}_i$ contracts to a compound Veronese cone over the line $\check{P}_k \cap U$ in Step (iv).

Now the other assertions in (1) and (2) follow easily by the adjunction formula. QED.

(2.7) Set

$$\mathcal{K}^* := \{\Sigma C_j + L \in \mathrm{Sym}^2|\mathcal{O}_{P^2}(3)| \times |\mathcal{O}_{P^2}(1)| \mid \Sigma C_j \in \mathcal{G}\}$$

(for the notation $\mathcal{G}$, see (2.2)).

Now we consider the functions $m(t)$ and $n(t)$ in (2.3) as functions $m(\Sigma C_j, L)$ and $n(\Sigma C_j, L)$ on $\mathcal{K}^*$ and define

$$\mathcal{K} := \{\Sigma C_j + L \in \mathcal{K}^* \mid m(\Sigma C_j, L) = n(\Sigma C_j, L) = 0\}$$

$\mathcal{K}^{\circ} := \{\Sigma\, C_j + L \in \mathcal{K} \mid$ both cubics C_j are smooth and
they intersect transversally.
L and $\Sigma\, C_j$ intersect transversally.$\}$

By the natural action of $SL_3(\mathbf{C})$, we can take their quotients (cf. (2.2)):

$$\mathfrak{M}^{*} := \mathcal{K}^{*}/SL_3(\mathbf{C}), \quad \mathfrak{M} = \mathcal{K}/SL_3(\mathbf{C}), \quad \mathfrak{M}^{\circ} := \mathcal{K}^{\circ}/SL_3(\mathbf{C}),$$

where the midle equality is a consequence of Theorem (2.6.3).

(2.8) In order to explain Theorem (2.6.3) more perspectively, we recall here briefly the construction of the coarse moduli space of Kunev (more generally, Todorov) surfaces by the period map after [Mo.2].

For the economy of pages, we take a reference point $0 = (\Sigma\, C_j,\ L) \in \mathcal{K}^{\circ}$ and construct the following diagram in a similar way as (0)-(iv) in (2.5):

(2.8.1)
$$\begin{tikzcd}
X_0 \arrow[d] & X_0^{*} \arrow[l, "\pi"'] \arrow[d, "\varphi"] \\
Y'_0 \arrow[d] & Y_0 \arrow[l] \arrow[d] \arrow[dl, "\alpha_1"'] \\
\mathbf{P}^2 & \mathbf{P}^{*} \arrow[l]
\end{tikzcd}$$

where Y'_0 is the double cover of $\mathbf{P}^2$ branched along $\Sigma\, C_{j,0}$,

Y_0 is the minimal resolution of Y'_0 on which sit the nine distinguished (-2)-curves $\Sigma\, E_{i,0}$ coming from the nine ordinary double points on Y'_0,

X_0^{*} is the double cover of the minimal K3 surface Y_0 branched along $B_{Y_0} := \alpha_1^{*}L_0 + \Sigma\, E_{i,0}$, and

X_0 is the contraction of the nine (-1)-curves on X_0^{*}

lying over $\Sigma E_{i,0}$ on Y_0.

Set

$$\Lambda := H^2(Y_0, \mathbb{Z})$$

$$\lambda := \text{class}(\alpha_1^* \mathcal{O}_{\mathbf{P}^2}(1)) \in \Lambda$$

$$N := \{\xi \in \Lambda \mid \xi\cdot\lambda = \xi\cdot E_i = 0 \quad (1 \le i \le 9)\}$$

Notice that

$$\{\omega \in \mathbf{P}(N \otimes \mathbf{C}) \mid \omega\cdot\omega = 0, \quad \omega\cdot\bar{\omega} > 0\}$$

has two connected components, interchanging by complex conjugation. Choose the component D containing $H^{2,0}(Y_0)$, a *period domain*. This choice is called the *sign structure*.

Now let Y' be any K3 surface with R.D.P., $\mu : Y \longrightarrow Y'$ the minimal resolution and $\{D_k\}$ the set of exceptional (-2)-curves for μ. Set

$$I^2(Y') := \{\xi \in H^2(Y, \mathbb{Z}) \mid \xi\cdot D_k = 0 \quad \text{for all} \quad k\}.$$

A *marking* of Y' is an embedding of lattice

$$\varphi_0 : I^2(Y') \hookrightarrow \Lambda$$

for which there exists an isometry $\varphi : H^2(Y, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ such that $\varphi|_{I^2(Y')} = \varphi_0$.

Glueing together local deformations by virtue of a suitable versions of the Torelli theorem and surjectivity of the period map for K3 surfaces with R.D.P., we can construct the universal family $g : \mathcal{Y}' \longrightarrow D$ of marked K3 surfaces of Kunev type and a relatively ample line bundle $L_{\mathcal{Y}'}$ on $\mathcal{Y}'$ whose first Chern class on each fiber is mapped to λ by the marking. Here the markings of the fibers are required to have images in the span of λ and N, and to send the holomorphic 2-forms on the minimal model of each fiber into D (cf. [Mo.2, §7]). This yields a $\mathbf{P}^2$-bundle

$$P(g_*L_{\mathcal{Y}'}) \longrightarrow D.$$

Let $\mathcal{V}$ be the Zariski open set of $P(g_*L_{\mathcal{Y}'})$ consisting of those points (ω, L_ω) which satisfies the condition: the pull-back μ^*L_ω of the divisor L_ω on the minimal model $\mu : Y_\omega \longrightarrow Y'_\omega$ of the K3 surface has at most simple singularities and it is disjoint from the distinguished (-2)-curves on Y_ω.

We denote by $\tilde{\Gamma}$ the subgroup of the orthogonal group $O(\Lambda)$ of the K3 lattice Λ consisting of those elements which preserves the polarization λ, the (unordered) set of distinguished (-2)-curves $\{E_1, \ldots, E_9\}$ and the sign structure. By definition there is the natural homomorphism $\tilde{\Gamma} \longrightarrow O(N)/\{\pm 1\}$, where $O(N)$ is the orthogonal group of the lattice N. We denote its image by Γ. Then we can see that the action of Γ on D lifts to the P^2-bundle $P(g_*L_{\mathcal{Y}'}) \longrightarrow D$ which preserves the open set $\mathcal{V}$ and that the quotients $\mathcal{V}/\Gamma \longrightarrow D/\Gamma$ are the coarse moduli spaces of Kunev surfaces and K3 surfaces of Kunev type, which are irreducible (cf. [Mo.2, (7.3), (7.5), (7.8)]).

(2.9) Thus we get the coarse moduli spaces in two ways, via geometric invariant theory and via period map:

$$\begin{array}{lcl}
\mathfrak{M} = \mathcal{X}/SL_3(\mathbf{C}) & \simeq & \mathcal{V}/\Gamma \\
\cap & & \cap \\
\mathfrak{M}^* = \mathcal{X}^*/SL_3(\mathbf{C}) & \simeq & P(g_*L_{\mathcal{Y}'})/\Gamma \\
\downarrow p & & \downarrow \\
\mathfrak{N} = \mathcal{Y}/SL_3(\mathbf{C}) & \simeq & D/\Gamma
\end{array}$$

As a consequence, we see in particular that the partial compactification $\mathfrak{M}^*$ of $\mathfrak{M}$ consists of all the points whose period is an interior point of $D/\Gamma = \mathfrak{N}$, i.e., type I degenerations.

By construction, the functions $m(\Sigma C_j, L)$ and $n(\Sigma C_j, L)$ on $\mathcal{K}^*$ defined in (2.7) and (2.3) induce ones on $P(g_* L_{\mathcal{Y}'})$ and on $\mathfrak{M}^*$, and these functions on $\mathfrak{M}^*$ define two stratifications of $\mathfrak{M}^*$ as in (2.3):

$\mathfrak{M}^* = \mathcal{S}_0 \amalg \mathcal{S}_1 \amalg \mathcal{S}_2$ where $\mathcal{S}_m = \{s \in \mathfrak{M}^* \mid m = \min\{2, m(s)\}\}$

$\mathfrak{M}^* = \mathcal{T}_0 \amalg \mathcal{T}_1 \amalg \mathcal{T}_2$ where $\mathcal{T}_n = \{s \in \mathfrak{M}^* \mid n = n(s)\}$

Theorem(2.6.3) implies:

(2.10) Corollary. The partial compactification $\mathfrak{M}^*$ of the coarse moduli space $\mathfrak{M}$ of Kunev surfaces consists of all the points of type I degenerations and $\mathfrak{M}^*$ is divided into five parts

$$\mathcal{S}_0 \cap \mathcal{T}_0 = \mathfrak{M}, \quad \mathcal{S}_1, \quad \mathcal{S}_2, \quad \mathcal{T}_1 \cap \mathcal{S}_0, \quad \mathcal{T}_2$$

whose points correspond to Kunev surfaces, numerical K3 surfaces with one double fiber, K3 surfaces, elliptic surfaces with $p_g = q = 1$, and splitting abelian surfaces respectively. $\mathfrak{M}^\circ$ is a Zariski open subset of $\mathfrak{M}$ consisting of those points which correspond to smooth Kunev surfaces, i.e., the canonical model is smooth.

(2.11) In the remaining part of this section, we shall explain uniformly by Theorem (2.6.3) the appearance of positive dimensional fibers of the period map for the second cohomology of Kunev surfaces, numerical K3 surfaces with one double fiber and elliptic surfaces with $p_g = q = 1$. These phenomena were observed separately before in [T.1], [Us.1], [Us.2] for the first surfaces and in [Sa.M] for the third. It is new for the second surfaces.

Let $f : \mathfrak{X} \longrightarrow U$ and $\tilde{f} : \tilde{\mathfrak{X}} \longrightarrow U$ be the families of degenerations of Kunev surfaces constructed in (2.5) for a fixed

$\Sigma C_j \in \mathcal{G}$. Starting from these, we can construct semi-stable degenerations as follows (cf. [Us.5]):

(2.11.1) *Case* $t \in S_1$: We may assume that ΣC_j are smooth cubics intersecting transversally because other cases are limit of this. For a general point $t_0 \in S_1$, say $t_0 \in \check{P} \subset (C_1 \cap C_2)^\vee$, let U' be a small polydisk neighborhood with center $(0,0) = t_0 \in U$. Then the restriction over U' of the family $\tilde{f} : \tilde{X} \longrightarrow U$ gives a semi-stable degeneration of Kunev surfaces over U' whose singular fibers lie over the line $\check{P} \cap U' = \{(t,0) \mid |t| < 1\}$. For $(t,0) \in \check{P} \cap U'$, the fiber $\tilde{X}_{t,0} := \tilde{f}^{-1}(t,0) = V_{t,0} + W_{t,0}$ where $V_{t,0}$ is a minimal numerical K3 surface with one double fiber and $W_{t,0} \simeq P^2$. The double locus $V_{t,0} \cap W_{t,0}$ is a smooth bisection with self-intersection -4 on $V_{t,0}$ and a smooth conic on $W_{t,0}$.

(2.11.2) *Case* $t \in T_1 \cap S_0$: For a general point $t_0 \in T_1$, say $t_0 \in \check{Q}$ for the triple point Q of C_j, take a small polydisk neighborhood U' with center $(0,0) = t_0 \in U$. Then the restriction over U' of the family $f : \mathfrak{X} \longrightarrow U$ (equivalently, $\tilde{f} : \tilde{\mathfrak{X}} \longrightarrow U$) gives a degeneration of Kunev surfaces over U' whose singular fibers are non-normal and lie over the line $\check{Q} \cap U'$. Extending the base to the double cover $\pi : U'_{\frac{1}{2}} \longrightarrow U'$ branched along the line $\check{Q} \cap U'$ we can construct a semi-stable family $\hat{f} : \hat{\mathfrak{X}} \longrightarrow U'_{\frac{1}{2}}$ whose singular fibers lie over the line $\pi^{-1}(\check{Q} \cap U') = \{(s,0) \mid |s| < 1\}$. For $(s,0) \in \pi^{-1}(\check{Q} \cap U')$, the fiber $\hat{X}_{s,0} = \hat{f}^{-1}(s,0) = \hat{V}_{s,0} + \hat{W}_{s,0}$ where $\hat{V}_{s,0}$ is a minimal elliptic surface with $p_g = q = 1$ and with a section which is a smooth elliptic curve with self-intersection

-1 and $\hat{W}_{s,0}$ is a rational surface constructed, for example, from P^2 by blowing-up twice at each of the four 2-torsion points on a smooth cubic endowed with a well-known abelian group structure. The double locus $\hat{V}_{s,0} \cap \hat{W}_{s,0}$ is the section mentioned above on $\hat{V}_{s,0}$ and the proper transform of the above cubic on $\hat{W}_{s,0}$.

(2.12) Recall the spectral sequence for a reduced simple normal crossing variety $Z = \Sigma Z_k$:

$$E_1^{p,q} = H^q(Z^{[p]}, \mathbb{Q}) \Longrightarrow E^{p+q} = H^{p+q}(Z, \mathbb{Q})$$

where $Z^{[p]} = \coprod_{k_0 \leq \ldots \leq k_p} X_{k_0} \cap \ldots \cap X_{k_p}$.

It is known that it degenerates at $E_2 = E_\infty$ (cf. [D], [GS]). Applying this to $Z = \tilde{X}_{t,0}$ or $\hat{X}_{s,0}$, the singular fibers of the semi-stable degenerations in (2.11), we can observe easily in both cases that $E_2^{2,0} = E_2^{1,1} = E_2^{1,2} = 0$ hence we have an exact sequence

(2.12.1) $\quad 0 \longrightarrow H^2(Z) \xrightarrow{\nu} H^2(Z_1) \oplus H^2(Z_2) \longrightarrow H^2(Z_1 \cap Z_2) \longrightarrow 0.$

On the other hand, since the local monodromies of the semi-stable families in question are trivial, the Clemens-Schmid sequence (1.2.1) becomes in both cases

(2.12.2) $\quad 0 \longrightarrow H^0_{\mathrm{lim}} \longrightarrow H_4 \longrightarrow H^2 \xrightarrow{\iota} H^2_{\mathrm{lim}} \longrightarrow 0.$

The morphism of Hodge structure (H.S. for short) ν in (2.12.1) relates the variation of Hodge structure (V.H.S. for short) associated to the smooth family $\{V_{t,0}\}_{|t|<1}$ of numerical K3 surfaces with one double fiber (resp. $\{\hat{V}_{s,0}\}_{|s|<1}$ of elliptic surfaces with $p_g = q = 1$) with the V.H.S. associated to the flat family $\{\tilde{X}_{t,0}\}_{|t|<1}$ (resp. $\{\hat{X}_{s,0}\}_{|s|<1}$) and they coincide essentially because $W_{t,0}$ (resp. $\hat{W}_{s,0}$) is a rational surface hence its associated V.H.S. is trivial. While the morphism ι in

(2.12.2) relates the V.H.S. associated to the flat family $\{\tilde{X}_{t,0}\}$ (resp. $\{\hat{X}_{s,0}\}$) with the variation of limiting H.S. associated to the 2-parameter family of semi-stable degeneration of Kunev surfaces $\{\tilde{X}_{t,t'}\}$ (resp. $\{\hat{X}_{s,s'}\}$), taking limit as $t' \longrightarrow 0$ (resp. $s' \longrightarrow 0$), and they coincide essentially because H_4 in (2.12.2) carries a trivial H.S. in both cases.

Thus we get:

(2.13) Corollary. In the above notation, the following assertions hold and they are related by degeneration as above:

(1) The 2-parameter smooth families $\{\tilde{X}_{t,t'}\}_{t'\neq 0}$ and $\{\hat{X}_{s,s'}\}_{s'\neq 0}$ of minimal Kunev surfaces have 2-dimensional moduli and the associated V.H.S. are trivial.

(2) The 1-parameter smooth family $\{V_{t,0}\}$ of minimal numerical K3 surfaces with one double fiber has 1-dimensional moduli and the associated V.H.S. is trivial.

(3) The 1-parameter smooth family $\{\hat{V}_{s,0}\}$ of minimal elliptic surfaces with $p_g = q = 1$ has 1-dimensional moduli and the associated V.H.S. is trivial.

Proof. The assertion on the V.H.S. has already proved before the corollary. As for the assertion on the moduli, the case (1) is obvious by construction (cf. (2.2)). The case (2) follows from an observation that the moduli of the double fiber of $V_{t,0}$ varies (cf. Proposition (4.3) and its proof). The case (3) follows from an observation that the moduli of the section of $\hat{V}_{s,0}$ varies (cf. the proof of Proposition (4.3)). QED.

3. Local study over critical points.

(3.0) Let $\Sigma_1^2 C_j$ be two cubics on P^2 with at most simple singularities, i.e., $\Sigma C_j \in \mathscr{G}$ in the notation of (2.2), and let L be a line. In this section we study locally over the singular points of $\Sigma C_j + L$ on P^2. The tables obtained in this section will play the key role in both proofs of Theorem (2.6.3) in Sections 4 and 5.

(3.1) For the cubics ΣC_j, we constructed the minimal K3 surface Y and the families of surfaces $\tilde{f}' : \mathfrak{X}' \longrightarrow U$, $\tilde{f} : \mathfrak{X} \longrightarrow U$ and $f : \mathfrak{X} \longrightarrow U$ in (2.5). Let V', V and X be the main components of the fibers $\tilde{f}'^{-1}(t)$, $\tilde{f}^{-1}(t)$ and $f^{-1}(t)$ over the point $t \in U$, $L_t = L$, respectively. Then the morphisms γ, δ and ε in Steps (ii), (iii) and (iv) in (2.5) induce the morphisms (abuse of the notation):

(3.1.1) $\qquad Y \xleftarrow{\gamma} V' \xrightarrow{\delta} V \xrightarrow{\varepsilon} X.$

By construction, we see that δ and ε in (3.1.1) are birational morphisms and that γ is the finite double cover branched along $B'_Y := \alpha_1^*L + \Sigma_1^9 E_i - 2\Sigma_i E_i$, where in the last term the index i runs over the set $\{i \mid 1 \le i \le 9,\ E_i \subset \alpha_1^*L\}$. Here we use the notation $\alpha_1 : Y \longrightarrow P^2$, the canonical resolution of the double cover Y' of P^2 branched along ΣC_j, and E_i $(1 \le i \le 9)$, the nine distinguished (-2)-curves, in (2.4).

The minimal model $\hat{X}$ of X is obtained by the successive contraction of (-1)-curves, starting from the canonical resolution X^* of the double cover γ in (3.1.1). This procedure is indicated by the diagram:

$$\text{(3.1.2)} \qquad \begin{tikzcd} & V' \arrow[d] & X^* \arrow[l] \arrow[d] \arrow[dl, "\varphi"'] \arrow[r, "\pi_1"] \arrow[rr, bend right, "\pi"'] & \hat{X}_1 \arrow[r, "\pi_2"] & \hat{X} \\ Y' \arrow[d] & Y \arrow[l] \arrow[d] \arrow[dl, "\alpha_1"'] & Y^* \arrow[l] & & \\ P^2 & P^* \arrow[l] & & & \end{tikzcd}$$

where $\pi_1 : X^* \longrightarrow \hat{X}_1$ is the successive contraction of the (-1)-curves each of which is mapped to a singular points of $\Sigma C_j + L$ on P^2 and $\pi_2 : \hat{X}_1 \longrightarrow \hat{X}$ is the successive contraction of the (-1)-curves each of which is mapped onto the line L on P^2.

We use the notation:

(3.1.3) $\qquad B_Y := (\alpha_1^* L + \Sigma_1^q E_i)_{odd,red} = (B'_Y)_{odd,red}$.

Here, for an effective divisor D, $(D)_{odd,red}$ means the reduced divisor whose support consists of those components with odd multiplicity in D.

(3.2) Notice that, in Diagram (3.1.2), all the processes but $\pi_2 : \hat{X}_1 \longrightarrow \hat{X}$ are local over a singular point of $\Sigma C_j + L$ on P^2. For a singular point $P \in \mathrm{Sing}(\Sigma C_j + L)$, we denote by $\alpha_1^* L(P)$ (resp. $B_Y(P)$, $K_{\hat{X}_1}(P)$) the pull-back of the line $\alpha_1^* L$ on Y (resp. the divisor B_Y on Y in (3.1.3), the canonical divisor $K_{\hat{X}_1}$ of $\hat{X}_1$) restricted over an open neighborhood of the point $P \in P^2$. We can classify the singular points $P \in \mathrm{Sing}(\Sigma C_j + L)$, where ΣC_j has at most simple singularities, and compute the divisors $\alpha_1^* L(P)$, $B_Y(P)$ and $K_{\hat{X}_1}(P) = \pi_1 \varphi^* B_Y(P)/2$ locally over the point P. Note that the last equality follows from the observation that $B_Y(P)$ has at most simple singularities which is a consequence of the computations.

All of these classification and computations are elementary, hence we give here the tables. For the computation of $B_Y(P)$, we use Table (3.2.2) below of the distinguished (-2)-curves.

In order to divide the cases, we define functions $m_P(\Sigma\, C_j,\, L)$ and $n_P(\Sigma\, C_j,\, L)$ in $P \in \mathbf{P}^2$ and $(\Sigma\, C_j,\, L) \in \mathcal{K}^*$ by

$$m_P(\Sigma\, C_j,\, L) = \min\{I(P,\, L \cap C_j) \mid j = 1,\, 2\},$$

$$n_P(\Sigma\, C_j,\, L) = \begin{cases} 1 & \text{if } \operatorname{mult}_P C_j = 3 \text{ for } j = 1 \text{ or } 2 \text{ and if } P \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the summations of these functions over $P \in \mathbf{P}^2$ give

$$m(\Sigma\, C_j,\, L) = \Sigma_{P \in \mathbf{P}^2}\, m_P(\Sigma\, C_j,\, L),$$

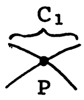
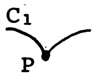
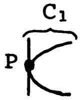
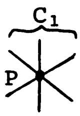
$$n(\Sigma\, C_j,\, L) = \Sigma_{P \in \mathbf{P}^2}\, n_P(\Sigma\, C_j,\, L).$$

(see (2.7)). We also use the following notation:

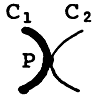
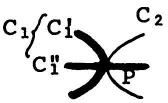
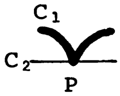
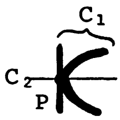
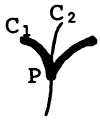
$L_Y := \alpha_1^{-1} L$ the proper transform of L on Y,

$L_{\hat{X}_1} := \pi_1(\alpha_1 \varphi)^{-1} L$ the proper transform of L on $\hat{X}_1$.

(3.2.1) Case $m_P(\Sigma\, C_j,\, L) = n_P(\Sigma\, C_j,\, L) = 0$ and $P \in \operatorname{Sing} C_1 - (C_2 + L)$:

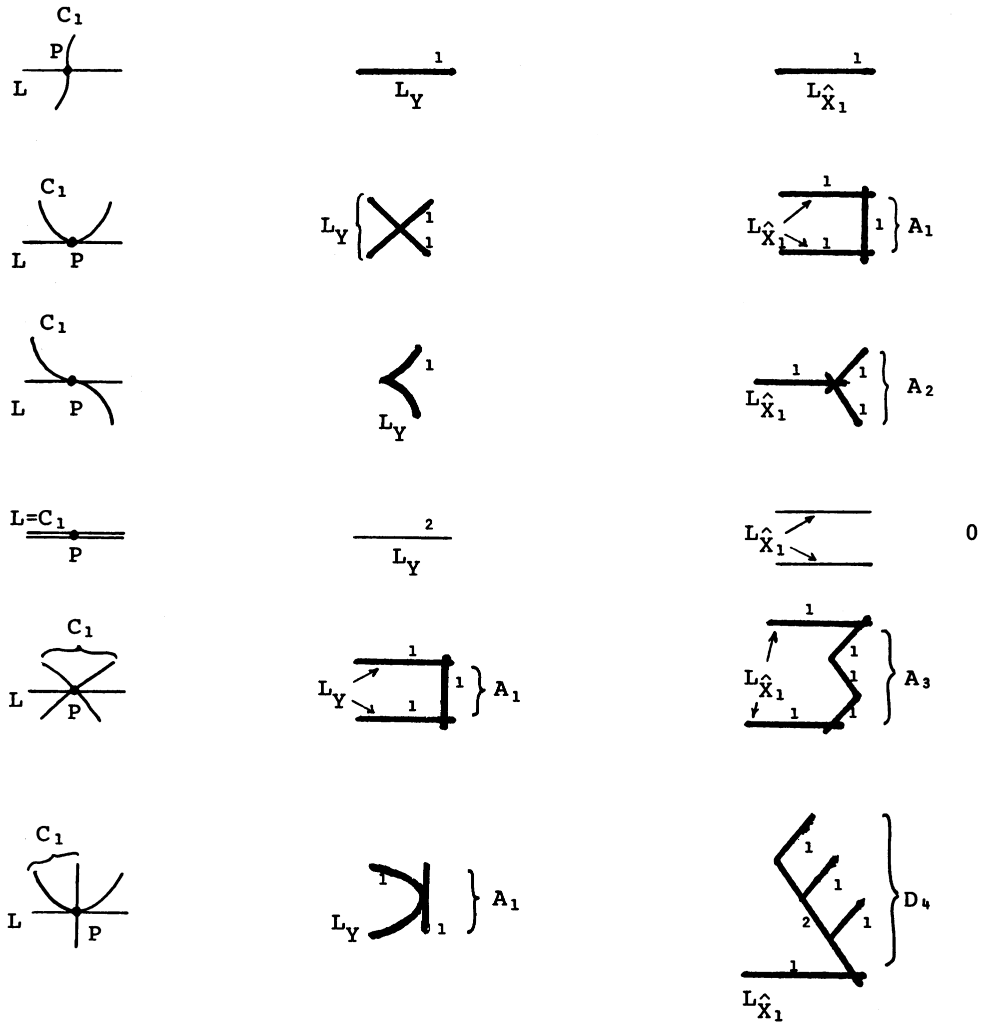
on $\mathbf{P}^2$	exceptional curves on Y		$B_Y(P)$	exceptional curves on $\hat{X}_1$	$K_{\hat{X}_1}(P)$
		A_1	0	$2A_1$	0
		A_2	0	$2A_2$	0
		A_3	0	$2A_3$	0
		D_4	0	$2D_4$	0

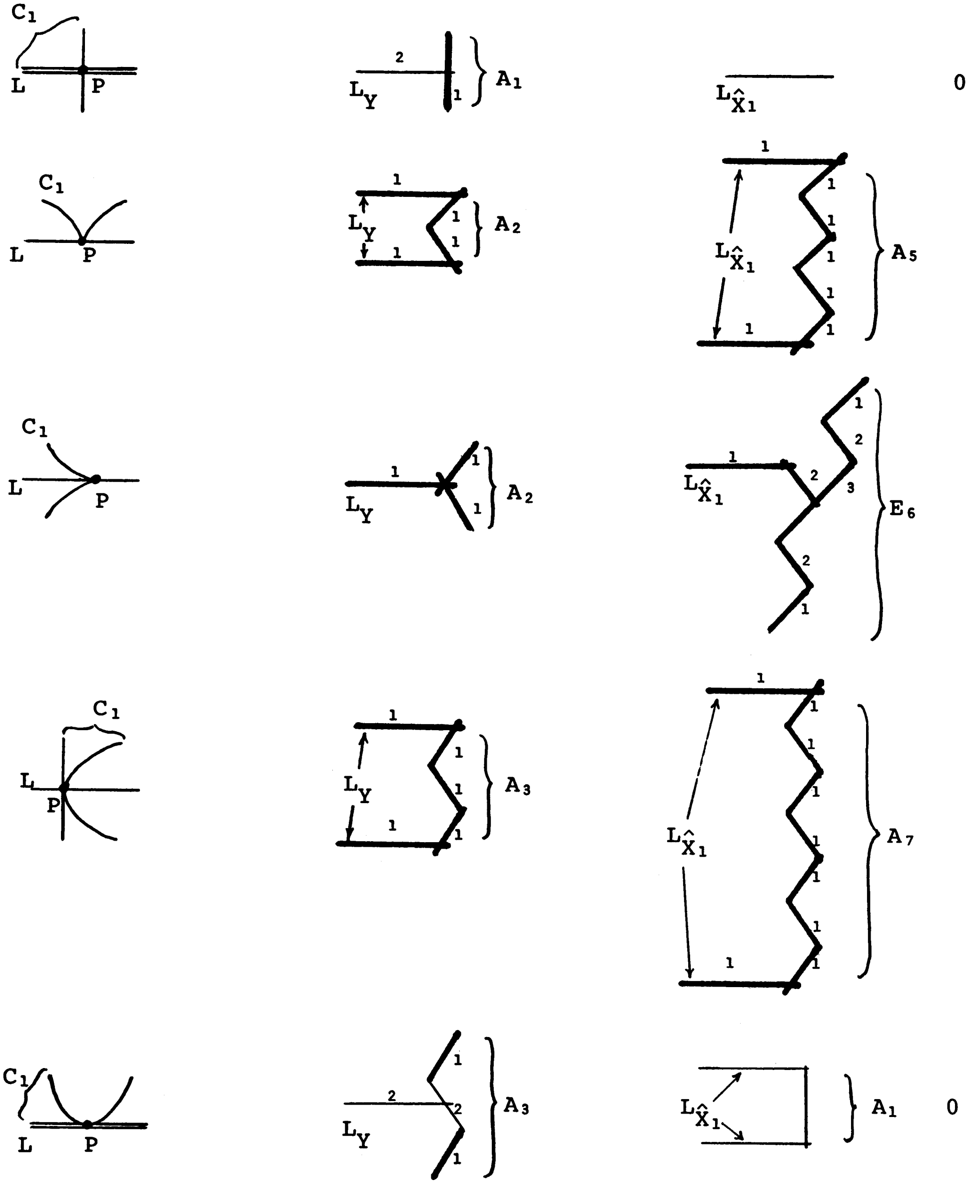
(3.2.2) Case $m_P(\Sigma C_j, L) = n_P(\Sigma C_j, L) = 0$, $P \in C_1 \cap C_2 - L$ (distinguished (-2)-curves):

on $\mathbf{P}^2$	exceptional curves on Y, $B_Y(P)$: bold lines		exceptional curves on $\hat{X}_1$	$K_{\hat{X}_1}(P)$
 $I(P, C_1 \cap C_2) = a$		A_{2a-1}	A_{a-1}	0
 $I(P, C_1' \cap C_2) = a$		D_{2a+2}	D_{a+2}	0
		D_5	A_5	0
		D_6	A_7	0
		E_7	E_6	0

(3.2.3) Case $m_P(\Sigma C_j, L) = n_P(\Sigma C_j, L) = 0$, $P \in L$:

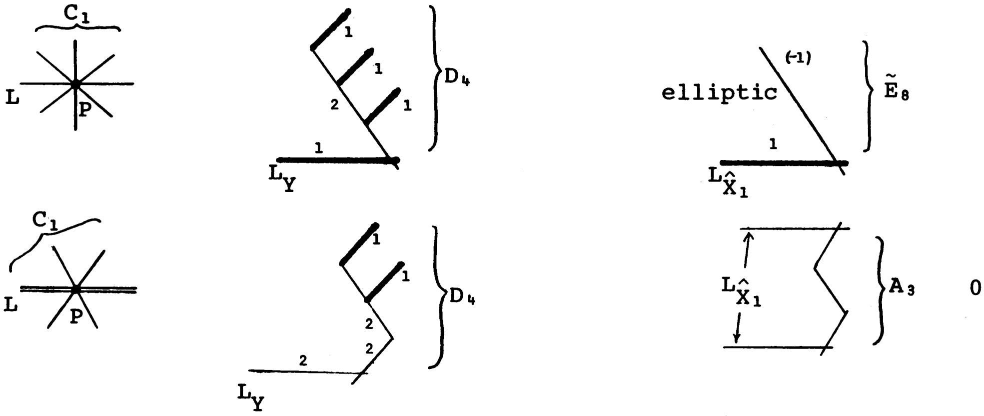
on $\mathbb{P}^2$	$\alpha_1^* L$ with multiplicity, $B_Y(P)$: bold curves	$\pi_1(\alpha_1\varphi)^*L$ on $\hat{X}_1$, $K_{\hat{X}_1}(P)$: bold curves with multiplicity




C1
L
P
Ly
A1
LX̂1
0
A2
A5
E6
A3
A7

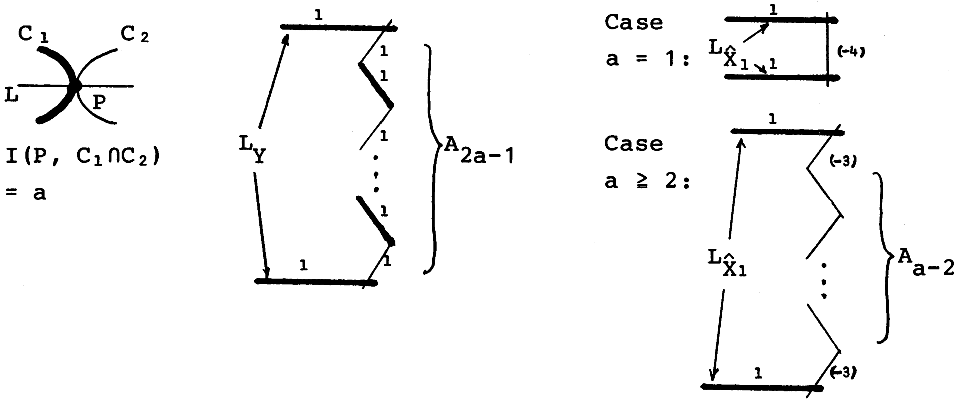
(3.2.4) Case $n_P(\Sigma C_j, L) = 1$:

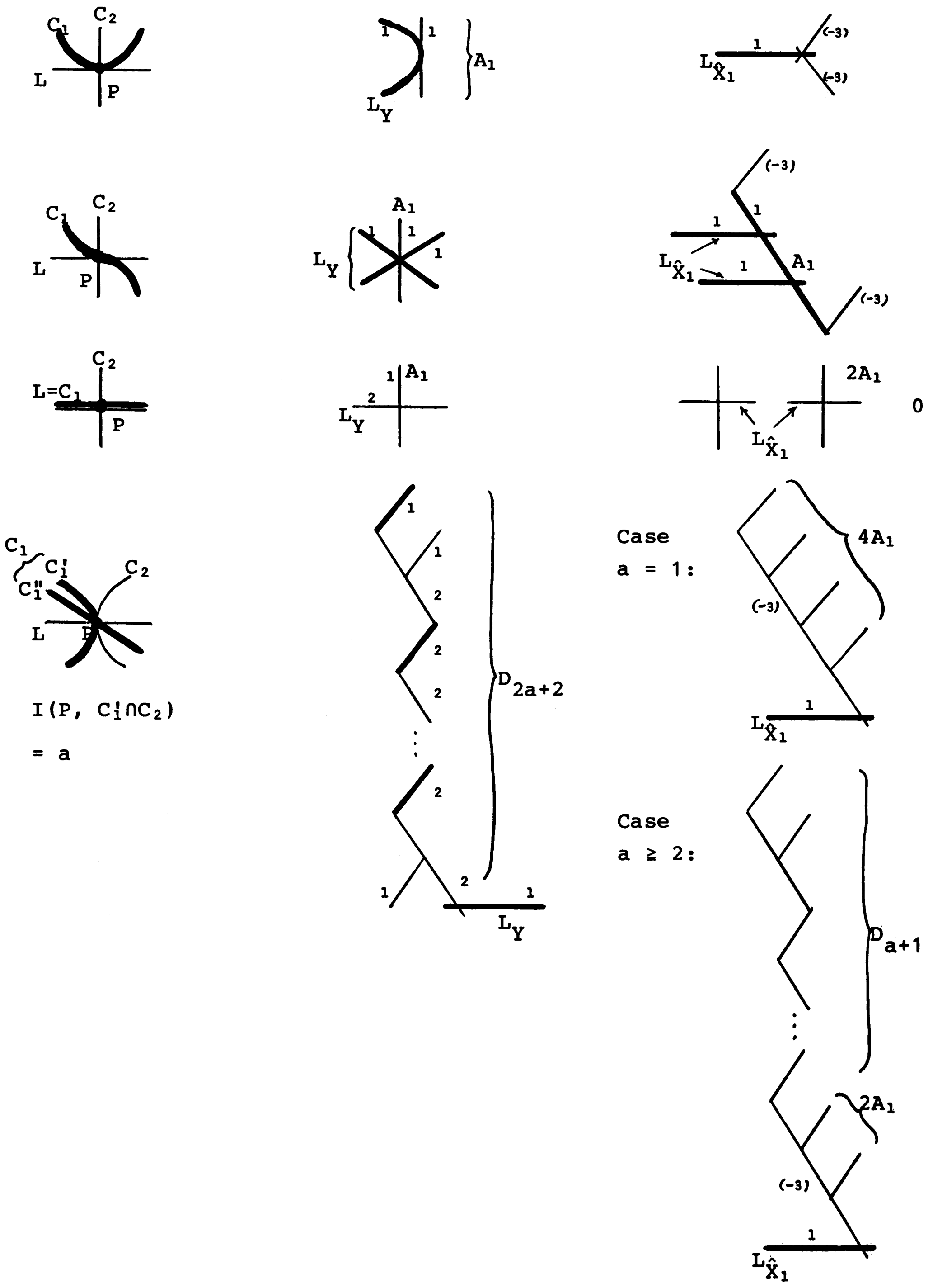
on $\mathbf{P}^2$	$\alpha_1^* L$ with multiplicity, $B_Y(P)$: bold curves	$\pi_1(\alpha_1\varphi)^* L$ on $\hat{X}_1$, $K_{\hat{X}_1}(P)$: bold curves with multiplicity, ( ) : self-intersection

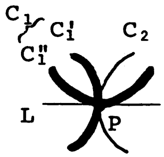


(3.2.5) Case $m_P(\Sigma C_j, L) = 1$:

on $\mathbf{P}^2$	$\alpha_1^* L$ with multiplicity, $B_Y(P)$: bold curves	$\pi_1(\alpha_1\varphi)^* L$ on X , $K_{\hat{X}_1}(P)$: bold curves with multiplicity, ( ) : self-intersection

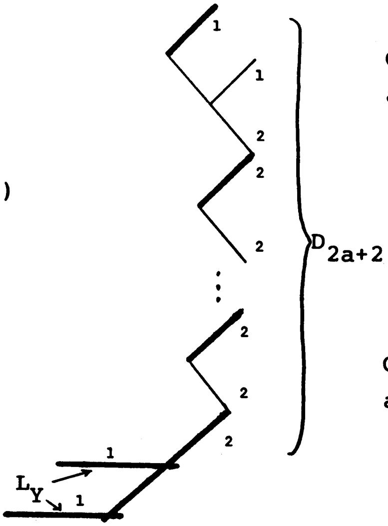



C1
C2
L
P
1
1
A1
LY
1
(-3)
(-3)
LX̂1
C1
C2
L
P
A1
LY
1
1
1
(-3)
1
1
LX̂1
1
A1
(-3)
L=C1
C2
P
1
A1
2
LY
2A1
LX̂1
0
C1
C'1
C2
C''1
L
P
I(P, C'1∩C2)
= a
1
1
2
2
2
D2a+2
2
1
2
1
LY
Case
a = 1:
4A1
(-3)
1
LX̂1
Case
a ≥ 2:
Da+1
2A1
(-3)
1
LX̂1



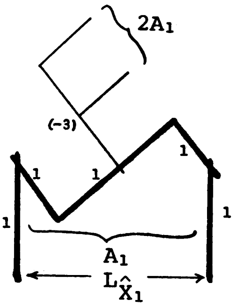
$I(P, C_1' \cap C_2)$

$= a$



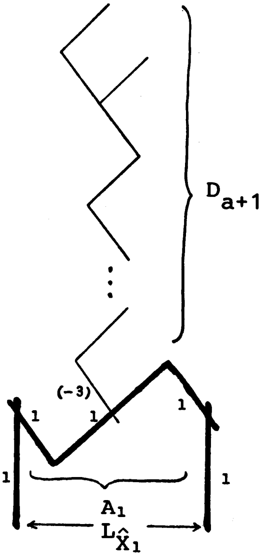
Case

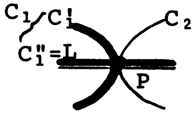
$a = 1$:



Case

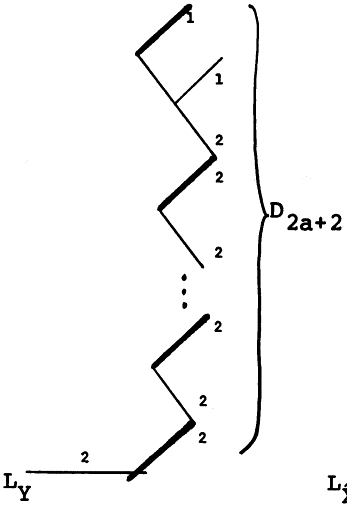
$a \geq 2$:

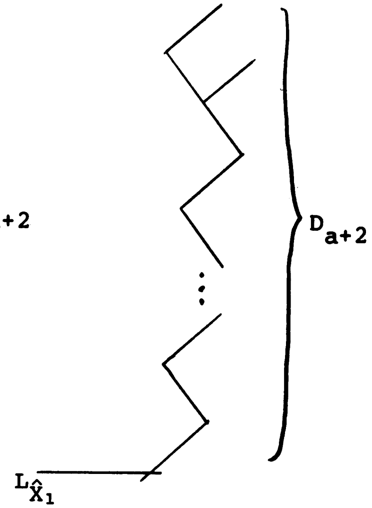




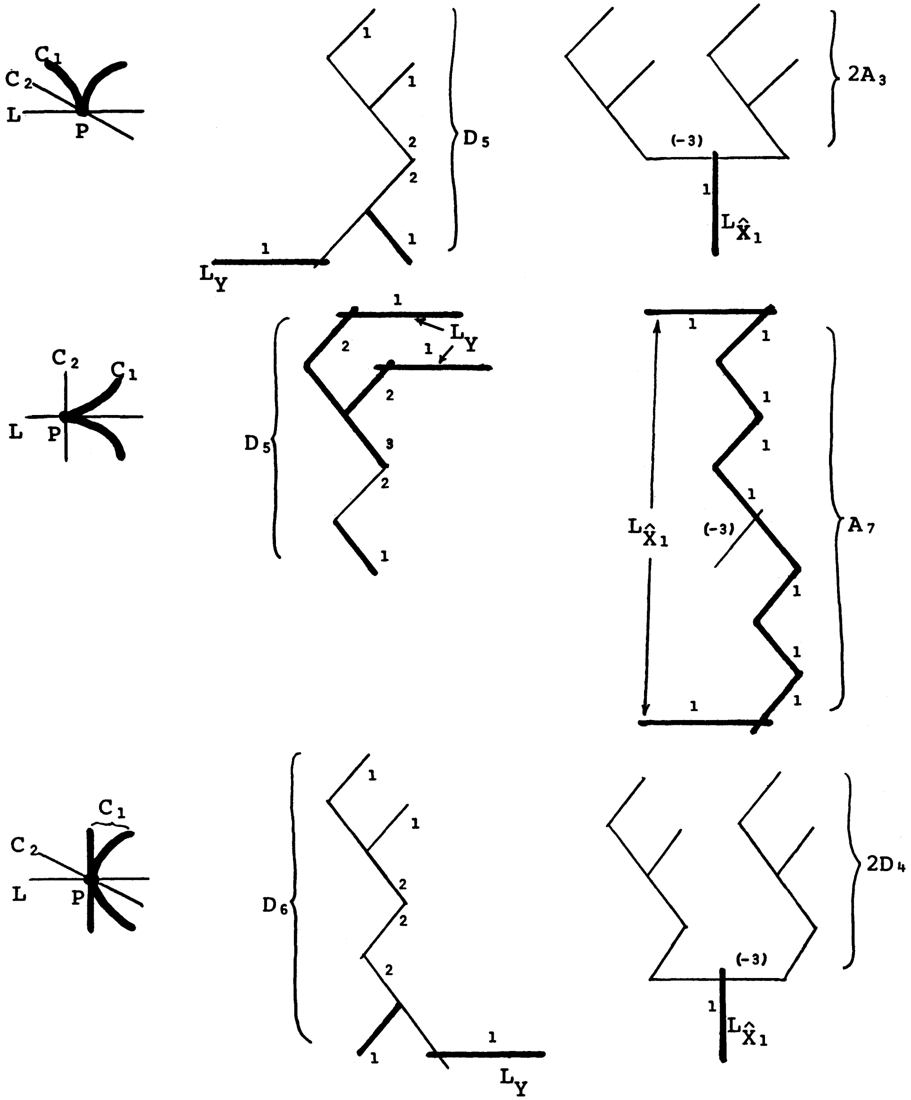
$I(P, C_1' \cap C_2)$

$= a$

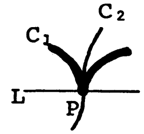


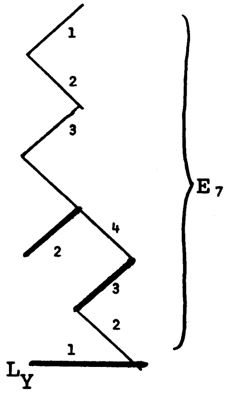


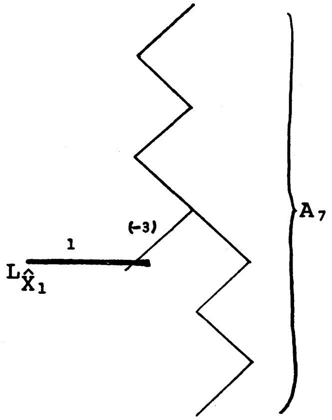
0



C1
C2
L
P
D5
2A3
(-3)
$L_{\hat{X}_1}$
L_Y
A7
D6
2D4







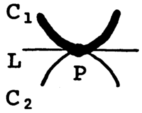
(3.2.6) Case $m_P(\Sigma\ C_j,\ L) \geq 2$:

on $\mathbf{P}^2$

Set $a := I(P,\ C_1 \cap C_2)$

$b := I(P,\ L \cap C_1)$

$c := I(P,\ L \cap C_2)$

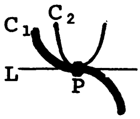


and assume $b \geq c$, then $3 \geq b \geq c \geq 2$.

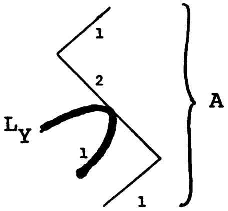
on $\mathbf{P}^2$

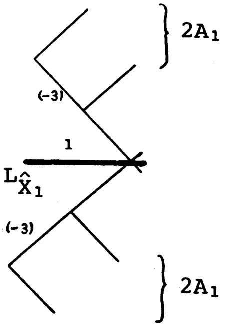
$\alpha_1^* L$ with multiplicity, $B_Y(P)$: bold curves

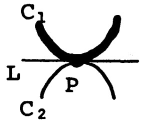
$\pi_1(\alpha_1\varphi)^* L$ on $\hat{X}_1$, $K_{\hat{X}_1}(P)$: bold curves with multiplicity, ( ) : self-intersection



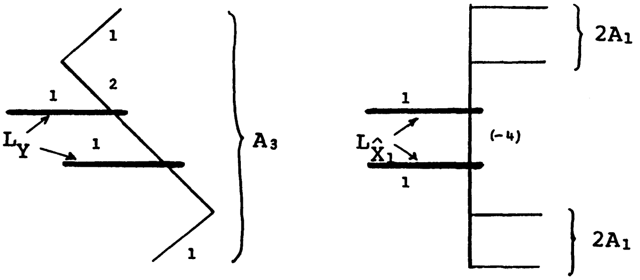
(a;b,c)
= (2;3,2)



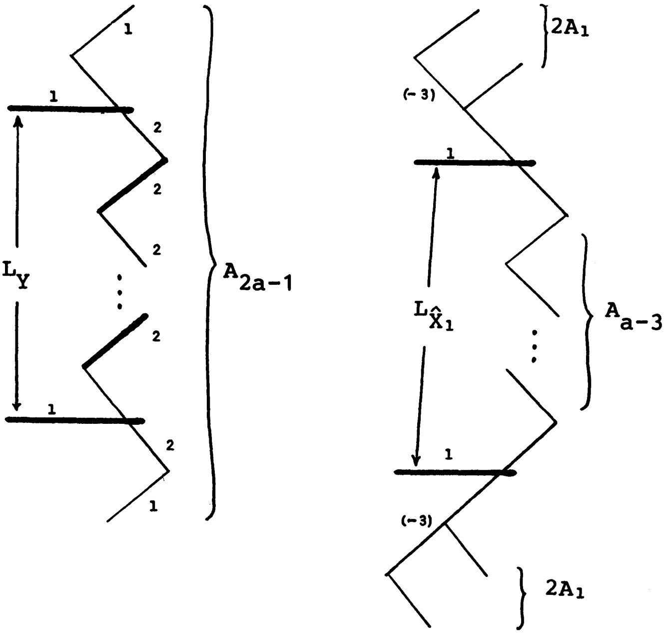


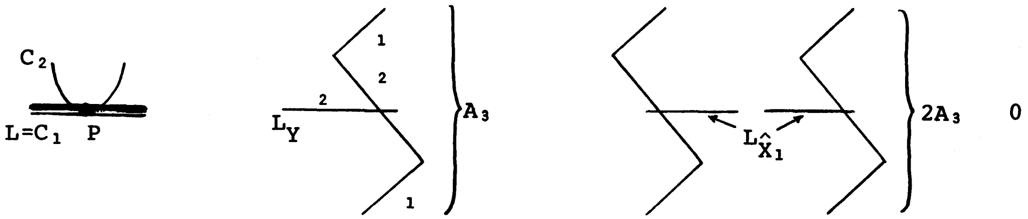


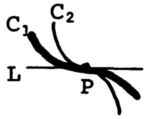
Case (a;b,c)
= (2;2,2):



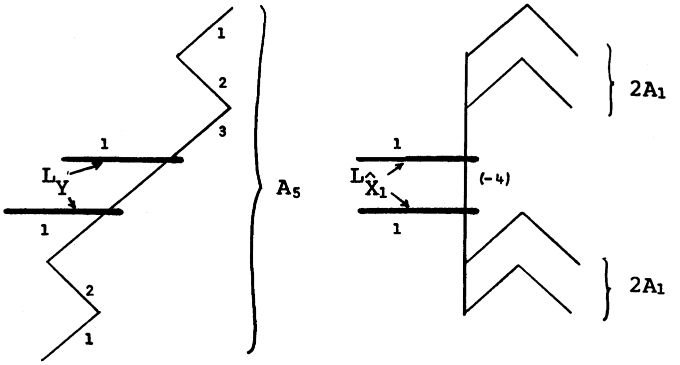
Case (a;b,c)
= (a;2,2)
with a ≥ 3:



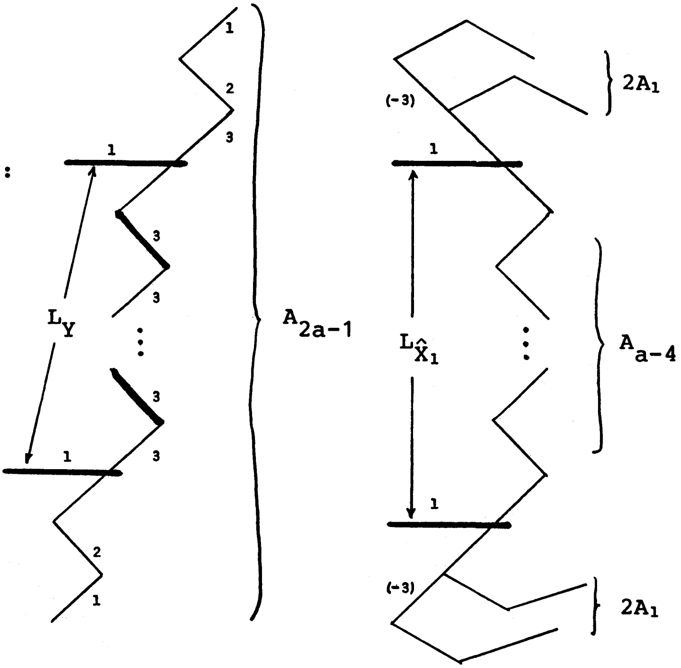


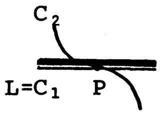


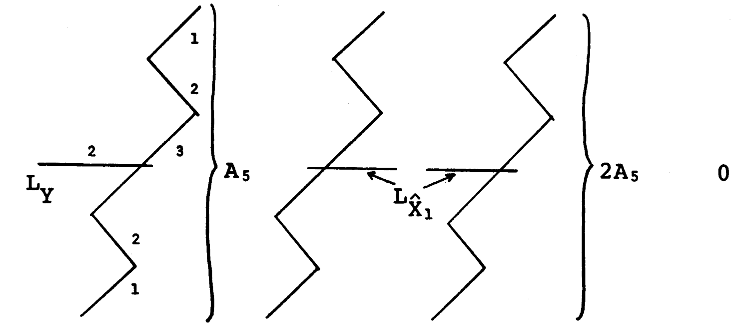
Case
(a;b,c)
= (3;3,3):

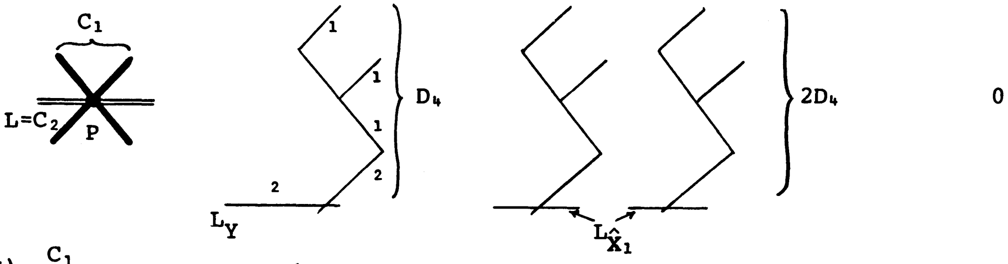


Case
(a;b,c)
= (a;3,3)
with a ≥ 4:

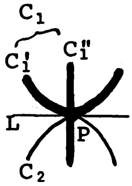




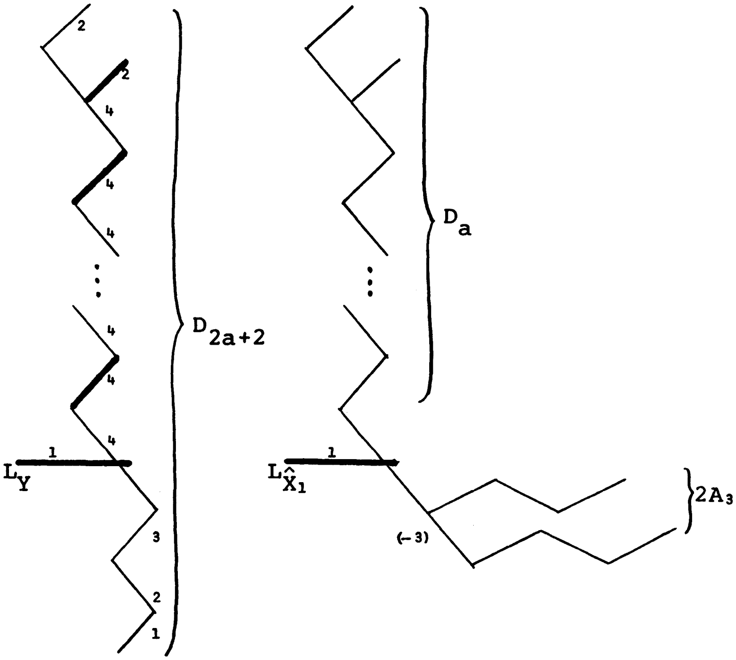




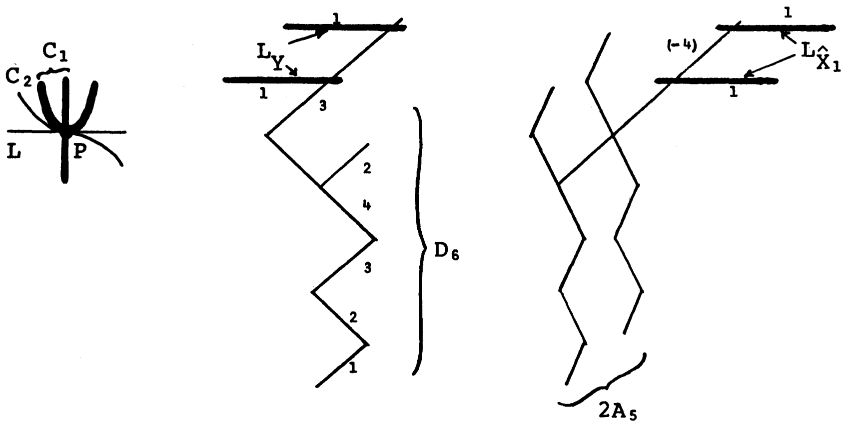
(*)

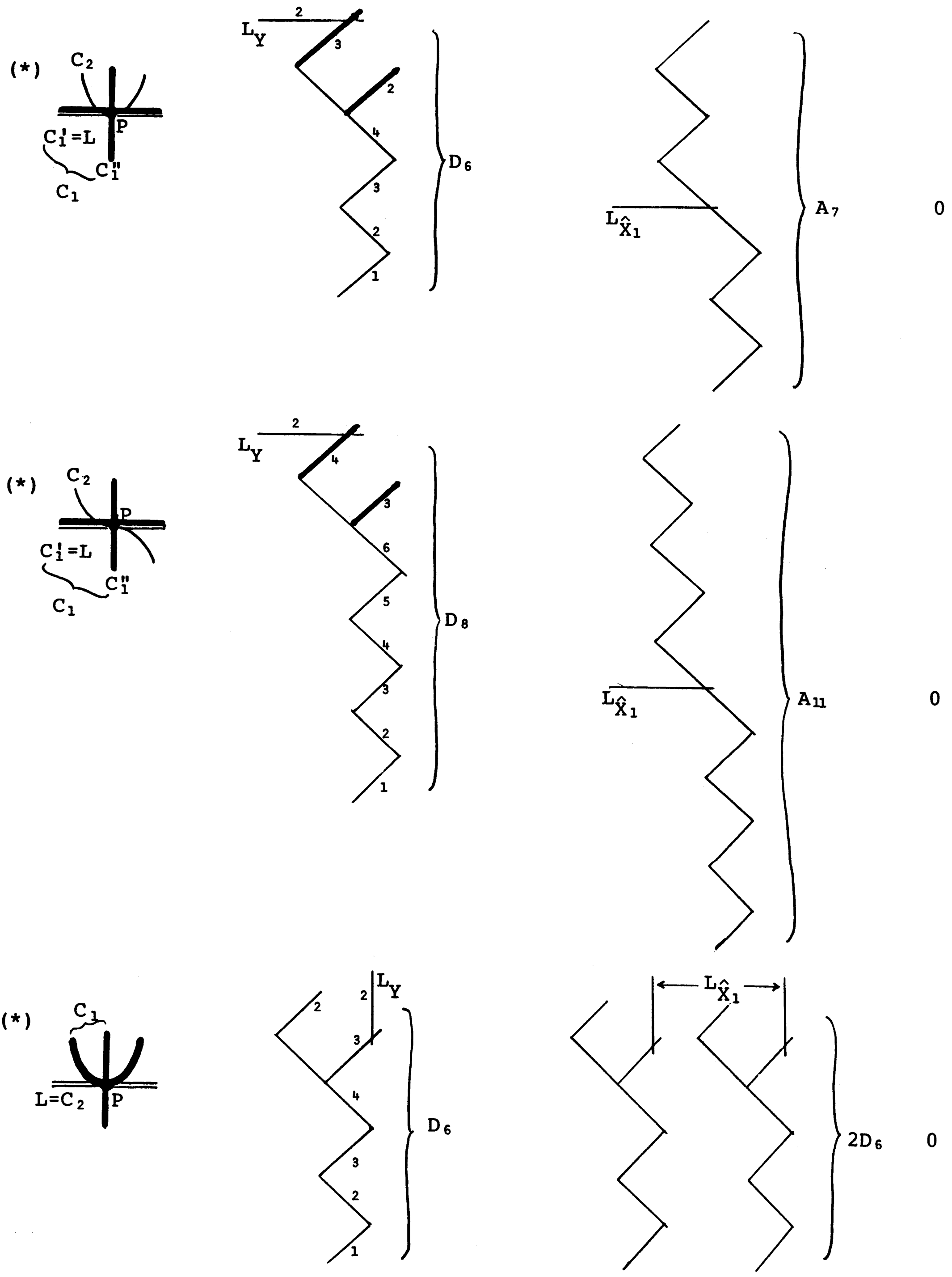


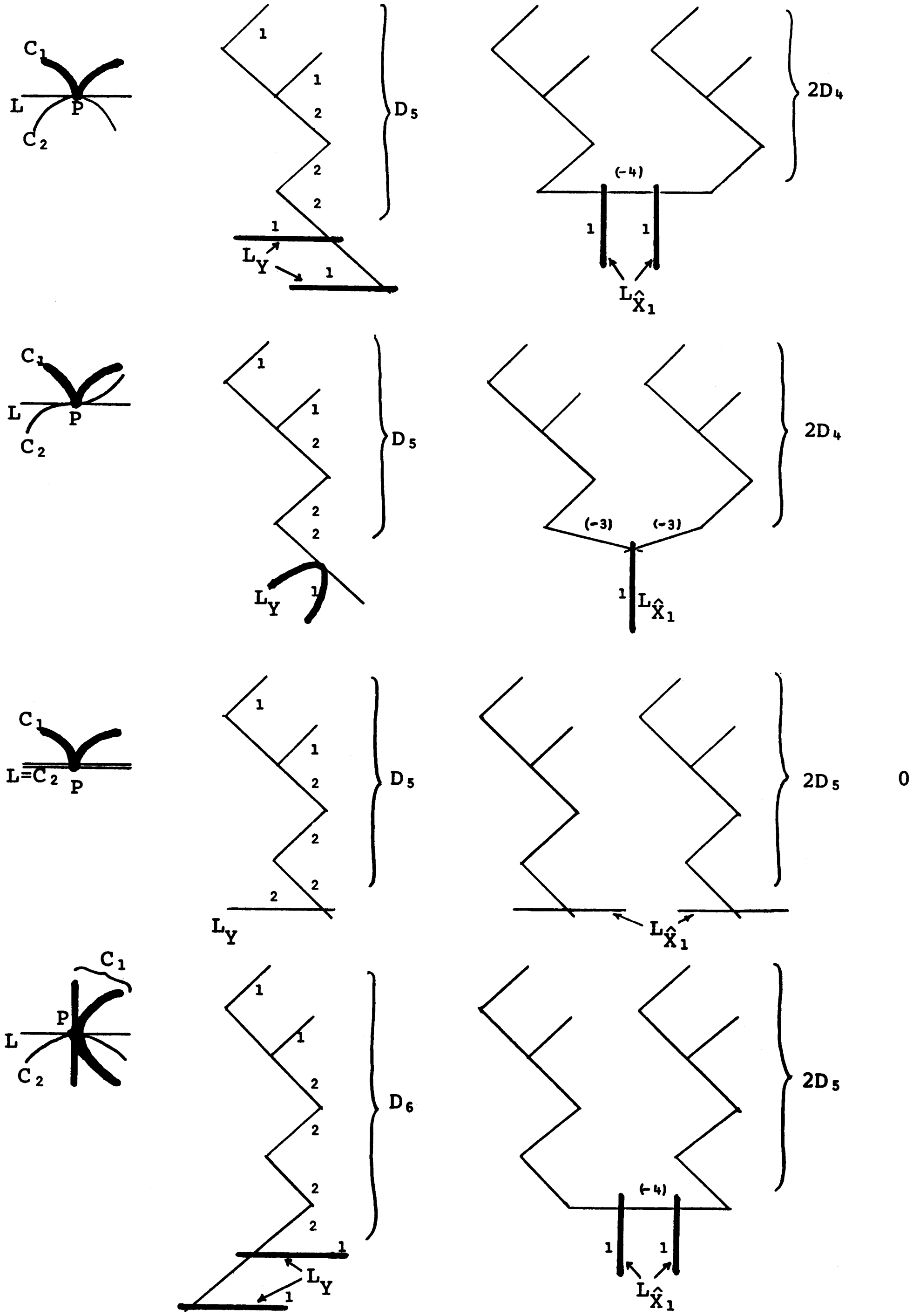
$a = I(P,\ C_1' \cap C_2)$
$\geq I(P,\ L \cap C_2)$
$= 2$



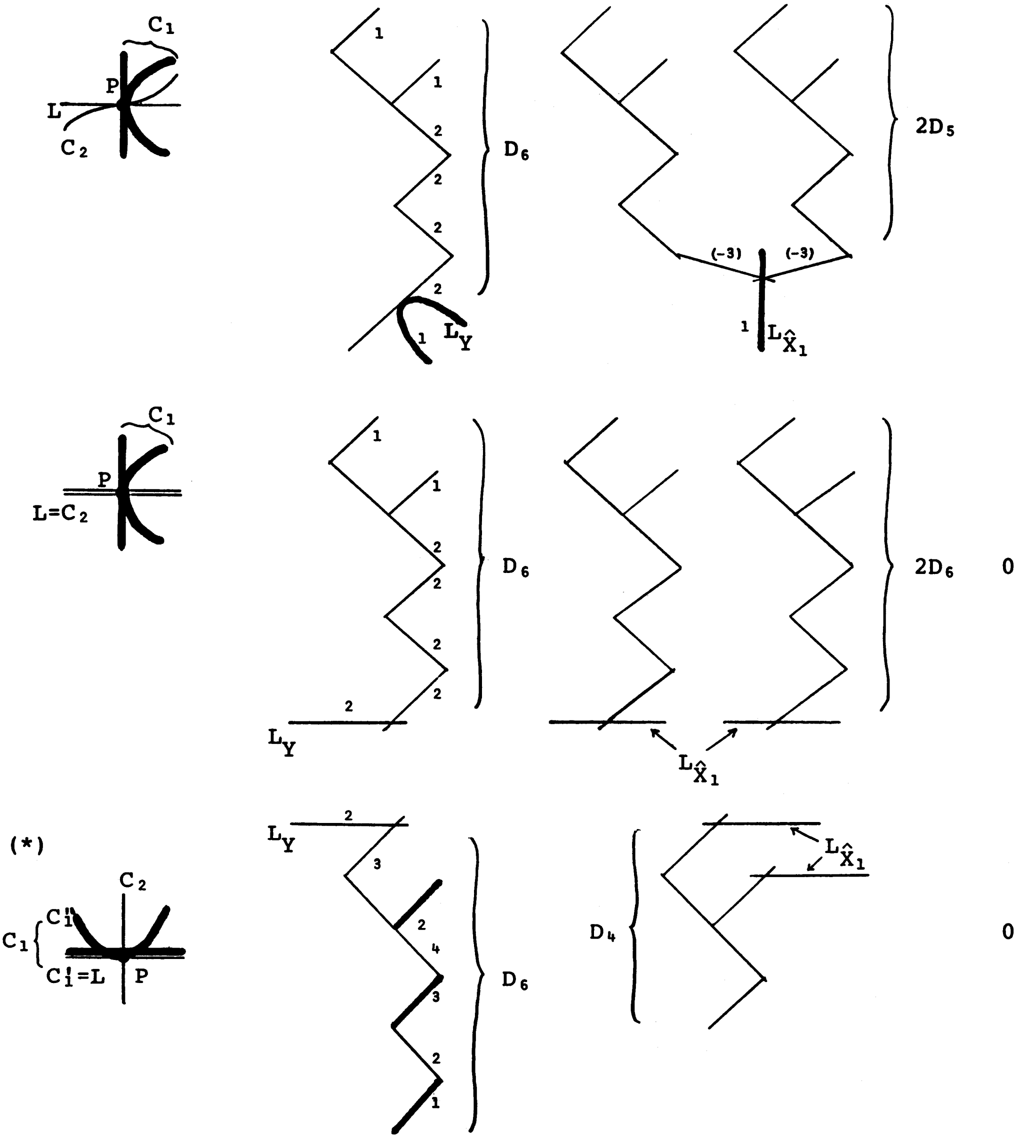
(*)



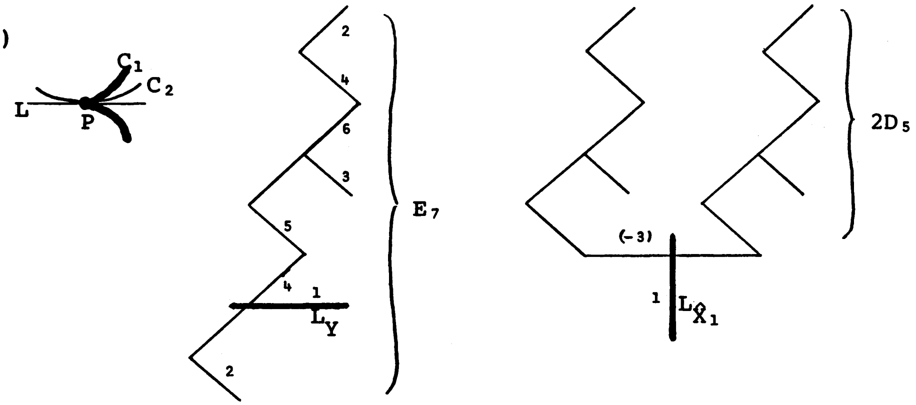

(*)
C2
C1' = L
P
C1''
C1
LY
D6
LX̂1
A7
0
(*)
C2
P
C1' = L
C1''
C1
LY
D8
LX̂1
A11
0
(*)
C1
L=C2
P
LY
D6
LX̂1
2D6
0



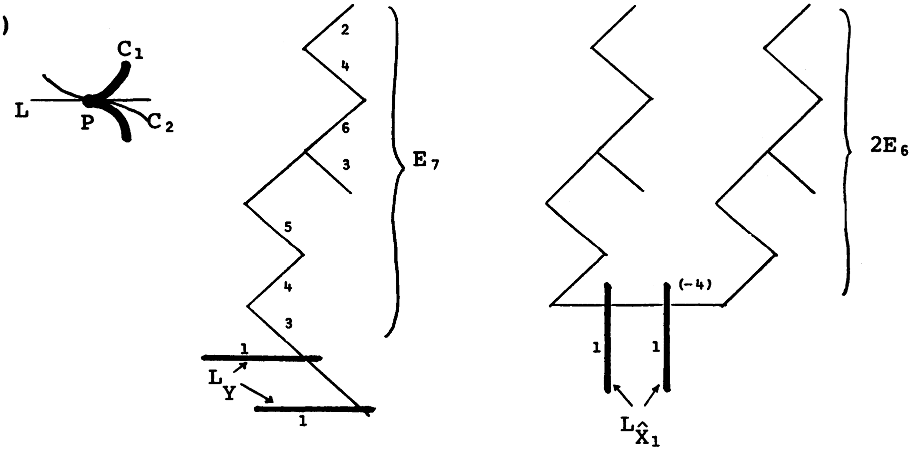
C1
L
P
C2
D5
2D4
(-4)
LY
LX̂1
(-3)
(-3)
L=C2
2D5
0
D6


C1
P
L
C2
D6
2D5
(-3)
(-3)
LY
LX̂1
L=C2
2D6
0
(*)
C1
C1'
C1'=L
D4

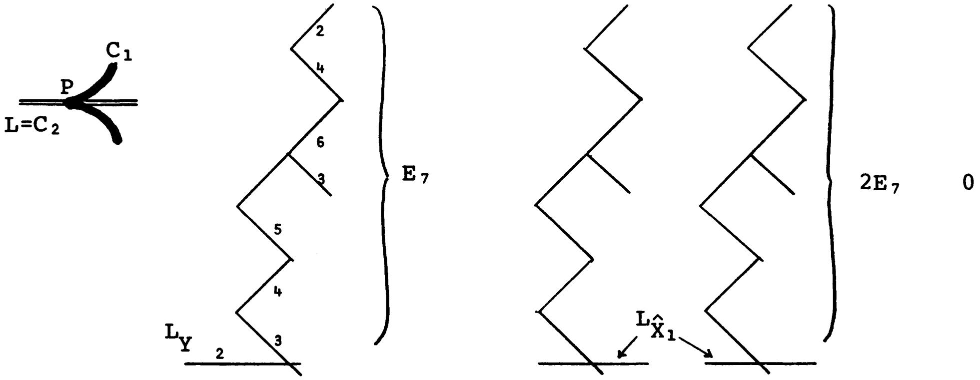
(*)



(*)



(*)



(Curves with (*) are unstable as plane curves of degree 7, cf. [Se].)

(3.3) Observation. We employ the above notation. By Tables in (3.2), we can observe the following:

(1) In all cases, the divisor B_Y on the minimal K3 surface Y has only simple singularities and the canonical divisor $K_{\hat{X}}$ of the minimal model $\hat{X}$ is connected and not multiple.

(2) In the case $m(\Sigma C_j, L) = n(\Sigma C_j, L) = 0$, $\alpha_1^* L$ has only simple singularities on the minimal K3 surface Y and the morphism $\pi : X^* \longrightarrow \hat{X}$ in Diagram (3.1.2) contracts only the nine (-1)-curves coming from the nine distinguished (-2)-curves on Y.

(3.4) Proposition. In the above notation, if $m(\Sigma C_j, L) = n(\Sigma C_j, L) = 0$, the corresponding $\hat{X}$ is the minimal model of a Kunev surface.

Proof. We use the notation in Diagram (3.1.2). Denote by $\mathcal{F}$ the line bundle on Y such that $\mathcal{O}_Y(B_Y) = \mathcal{F}^{\otimes 2}$. Then, since $V' = \mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{F}^{-1})$, we have

$$\chi(\mathcal{O}_{\hat{X}}) = \chi(\mathcal{O}_{V'}) = \chi(\mathcal{O}_Y) + \chi(\mathcal{F}^{-1}) = 2 + \chi(\mathcal{F}).$$

By the Riemann-Roch theorem on Y,

$$\chi(\mathcal{F}) = (\mathcal{F})^2/2 + \chi(\mathcal{O}_Y) = (B_Y)^2/8 + 2 = \{2 + 9(-2)\}/8 + 2 = 0.$$

On the other hand, since $K_{X^*} = \varphi^*\mathcal{F}$ by Observation (3.3.2) and Lemma (1.1.2), we see

$$c_1^2(\hat{X}) = c_1^2(X^*) + 9 = (\varphi^*\mathcal{F})^2 + 9 = 2(\mathcal{F})^2 + 9 = (B_Y)^2/2 + 9$$
$$= \{2 + 9(-2)\}/2 + 9 = 1.$$

Let X be the canonical model of $\hat{X}$. Then, by construction and Observation (3,3,2), the bicanonical map f of $\hat{X}$ is a morphism which factors as

$$f : \hat{X} \xrightarrow{f_1} X \xrightarrow{f_2} Y' \xrightarrow{f_3} P^2$$

where f_1 is a birational morphism and f_2 and f_3 are finite double covers. Hence X is a Kunev surface with an involution σ which is the covering transformation of $f_2 : X \longrightarrow Y'$. QED.

As a corollary, we have the following result, which will be used in Sections 4 and 5:

(3.4.1) Corollary. We use the above notation and the notation in (2.7). For any $(\Sigma C_j, L) \in \mathcal{K}^*$, the corresponding minimal model $\hat{X}$ has $p_g(\hat{X}) = 1$.

Proof. We use the flat family of surfaces $f : \mathfrak{X} \longrightarrow U$ constructed in (2.5). Take a small disk U' in U with center $0 = (\Sigma C_j, L) \in U$ such that $(\Sigma C_j) \cap L$ are six nodes for all $t \in U' - \{0\}$, and denote by $f_{U'} : \mathfrak{X}_{U'} \longrightarrow U'$ the restriction of the family f over U'. Then, by construction and Proposition (3.4), the fibers of $f_{U'}$ over all $t \in U' - \{0\}$ are desingularizations of Kunev surfaces. Let $\hat{f} : \hat{\mathfrak{X}} \longrightarrow U'_r$ be a semi-stable reduction with base extension $U'_r \longrightarrow U'$ of $f_{U'} : \mathfrak{X}_{U'} \longrightarrow U'$ (cf. [Mu]) and let $\hat{f}^{-1}(0) = \Sigma \hat{V}_k$ be the decomposition of the central fiber. Then we see

$$1 \le p_g(\hat{X}) \le \Sigma\, p_g(\hat{V}_k) \le p_g(\hat{f}^{-1}(\hat{t})) = 1 \qquad \text{for } \hat{t} \ne 0.$$

For the first inequality, we use the fact that the minimal model $\hat{X}$ carries a holomorphic 2-form coming from one on a K3 surface Y. The second inequality follows from the fact that there is a component $\hat{V}_k$ dominating X, and the third follows from (1.2.2). QED.

4. Elliptic fibrations in case $m(t) > 0$ or $n(t) > 0$.

(4.0) We continue to use the notation in the previous sections. Throughout this section we assume that $\Sigma C_j \in \mathscr{G}$, i.e., the sum of two cubics ΣC_j on P^2 which has at most simple singularities. In the case that the functions $m(t) > 0$ or $n(t) > 0$ (see (2.3)), the pencil of lines through a critical point on P^2 induces elliptic fibrations both on the minimal K3 surface Y and on the minimal model $\hat{X}$. We shall study these elliptic fibrations in this section. This together with Proposition (3.4) gives a proof of Theorem (2.6.3).

(4.1) We first treat the case $n(t) > 0$ and $m(t) = 0$. Recall that in case $n(t) = 1$ and $m(t) = 0$ one of the cubics on P^2, say C_1, consists of three different lines passing through a common point Q_1 and the line $L := L_t$ also passes Q_1 but L is not a component of C_1 nor passes the triple point of C_2 if exists. In case $n(t) = 2$ each cubic C_j on P^2 consists of three different lines passing through a common point Q_j $(j = 1, 2)$, $Q_1 \neq Q_2$, $L = L_t$ is the line joining these two points Q_1 and Q_2, and the seven lines $\Sigma C_j + L$ are different.

(4.2) Proposition. In the notation in (4.1), if $n(t) = 1$ and $m(t) = 0$, the pencil of lines through Q_1 on P^2 induces an elliptic fibration both on the minimal K3 surface Y and on the minimal model $\hat{X}$ with section. The section on Y is a (-2)-curve and that on $\hat{X}$ is a smooth elliptic curve with self-intersection -1. These elliptic fibrations have constant J-invariants if and only if the other cubic C_2 has also a triple point. In any case, $\hat{X}$ is an elliptic surface with $\kappa(\hat{X}) = p_g(\hat{X}) = q(\hat{X}) = 1$.

Proof. $p_g(\hat{X}) = 1$ is already known in Corollary (3.4.1). Let $\check{Q}_1$ be the pencil of lines through the point Q_1 on P^2.

Following the procedure of Diagram (3.1.2), we shall first prove that $\check{Q}_1$ induces elliptic fibrations on Y and on $\hat{X}$. Let $\Sigma_0^3 D_i'$ be the exceptional curves on P^* over the point Q_1 on P^2 such that $D_0'\cdot D_i' = 1$ $(i = 1, 2, 3)$. Then the branch locus B_{P^*} on P^* becomes $B_{P^*} = D_0' + q^{-1}C_2 + D''$, where $q^{-1}C_2$ is the proper transform of C_2 by $q : P^* \longrightarrow P^2$ and D'' is the effective divisor defined by the above equation. For a line $M \in \check{Q}_1$, the proper transform $q^{-1}M$ intersects with B_{P^*} at four distinct points provided that M is not contained in C_1 nor passes a singular point of ΣC_j other than Q_1 nor touches C_2. Hence these lines $M \in \check{Q}_1$ become smooth irreducible elliptic curves on Y. This shows that the pencil of lines $\check{Q}_1$ on P^2 induces an elliptic fibration on Y. This fibration has a section D which is the component of the ramification divisor on Y lying over D_0'. D is a (-2)-curve.

Since the branch locus on Y is $B_Y = (\alpha_1^*L + \Sigma_1^9 E_i)_{odd,red}$ (see (3.1.3)), the branch locus on Y^* is contained in a finite number of fibers of the elliptic fibration on Y^*. Therefore the elliptic fibration on Y^* induces one on X^*. The canonical divisor K_{X^*} of X^* is contained in $\varphi^*B_Y/2$ (actually they coincide because B_Y has at most simple singularities, which is a consequence of the local classification in Section 3). Hence the exceptional divisor for $\pi : X^* \longrightarrow \hat{X}$ is contained in a finite number of fibers on X^*. Thus we get an elliptic fibration on $\hat{X}$.

Next we shall prove that the elliptic fibration on $\hat{X}$ has a section which is a smooth elliptic curve. For this purpose, note that

$$\alpha_1^{-1}L + 2D + \Sigma_1^3 D_i + F = \alpha_1^* L$$

$$2(\alpha^{-1}C_1)_{red} + 6D + 2\Sigma_1^3 D_i + \Sigma_1^3 E_i + 2G = \alpha_1^* C_1$$

where $\alpha_1^{-1}(\ )$ means the proper transform, D_i is the pull-back of D_i' on Y $(i = 1, 2, 3)$ and F and G are the effective divisors defined by the above equations. From this we get

$$(4.2.1) \quad B_Y = (\alpha_1^* L + \Sigma_1^3 E_i)_{odd,red} = \alpha_1^{-1}L + \Sigma_1^3 D_i + F + \Sigma_1^3 E_i$$
$$= \alpha_1^*(L + C_1) - 2(4D + (\alpha_1^{-1}C_1)_{red} + \Sigma_1^3 D_i + G).$$

This shows that B_Y is linearly equivalent to twice of a divisor whose support is contained in a finite number of fibers on Y. This property is preserved on Y^* and we see that $X^* = \mathrm{Spec}(\mathcal{O}_{Y^*} \oplus \mathcal{F}^{-1})$ for a line bundle $\mathcal{F}$ on Y^* whose restriction to a fiber on Y^* is trivial. This implies that the pull-back of the fibers on Y^*, appart from B_{Y^*}, divide into two disjoint copies on X^* hence $D^* := \varphi^* D$ is a section and so is $\hat{D} := \pi D^*$. $\hat{D}$ is isomorphic to D^* and D^* is a smooth elliptic curve with self-intersection -4 on Y^* because D is a (-2)-curve on Y whose neighborhood is isomorphic to one on Y^* and $D^* \longrightarrow D$ is a double cover branched four different points $D \cap (\alpha_1^{-1}L + \Sigma_1^3 D_i)$.

For the assertion on J-invariant on $\hat{X}$, it is enough to show it on Y because most of the fibers on Y divide into two copies on $\hat{X}$. We recall here an elementary fact that for a line $M \in \check{Q}_1$

the cross-ratio of the branch points on M, i.e., the points $M \cap C_2$ and Q_1, gives the J-invariant of the elliptic curve on Y induced by M up to ordering of the four points (cf., e.g., [Cl.2]). It is easy to see that these cross-ratio upto ordering are constant if and only if C_2 is concurrent three lines. Thus we get our assertion.

We shall now compute $q(\hat{X})$ by using a theorem of Ueno (1.3.2) and the Leray spectral sequence applying to the elliptic fibration $f : \hat{X} \longrightarrow \hat{D}$. In order to check the condition of the above theorem, the only thing we should do is that the elliptic fibration on $\hat{X}$ has singular fibers other than ${}_mI_0$ in the case that the two cubics C_1 and C_2 are pairs of concurrent three lines. But in this case we can perform easily the procedure of Diagram (3.1.2) and we see that there are two singular fibers of type I_0^* on $\hat{X}$ coming from the line joining two triple points Q_1 and Q_2 on P^2.

Finally we shall prove that the section $\hat{D}$ on $\hat{X}$ has self-intersection -1. By Observation (3.3.1), the canonical bundle $K_{\hat{X}}$ is connected and not multiple. Hence $K_{\hat{X}}$ consists of one fiber by the canonical bundle formula (1.3.1), because we have already known that the base curve, i.e., the section $\hat{D}$, is an elliptic curve and $p_g(\hat{X}) = q(\hat{X}) = 1$. Now $(\hat{D})^2 = -1$ follows from the adjunction formula $(K_{\hat{X}} + \hat{D})\cdot\hat{D} = \deg K_{\hat{D}} = 0$. QED.

(4.3) Remark. A smooth elliptic curve with self-intersection -1 on a minimal surface is the exceptional divisor of the minimal resolution of a *simple elliptic singularity of type* $\tilde{E}_8$ in the sense of K. Saito (cf. [Sa.K]).

(4.4) Proposition. In the notation in (4.1), if $n(t) = 2$, the minimal model $\hat{X}$ is isomorphic to a product $\hat{D}_1 \times \hat{D}_2$ of two smooth elliptic curves $\hat{D}_j$ $(j = 1, 2)$, whose two trivial elliptic fibrations coincide with those induced by the pencils of lines through the point Q_j $(j = 1, 2)$ on P^2.

Proof. In the present case, we can go on the same line as the proof of Proposition (4.2). Actually it is simpler than before because the configuration of the two cubics ΣC_j and the line L is unique. We do not repeat it here. Consequently the two pencils of lines $\check{Q}_j$ through the triple point Q_j of C_j induce two elliptic fiber bundles with a section $\hat{D}_j$ coming from the first order infinitely near point of Q_j, which becomes a fiber of the other elliptic fiber bundle $(j = 1, 2)$. Hence the projections induce an isomorphism $\hat{X} \xrightarrow{\sim} \hat{D}_1 \times \hat{D}_2$. QED.

(4.5) Remark. Proposition (4.4) shows that if the two cubics C_1 and C_2 consist of two pairs of concurrent three lines and ΣC_j has at most simple singularities then the minimal K3 surface Y is an elliptic Kummer surface associated to the splitting abelian surface $\hat{X} \simeq \hat{D}_1 \times \hat{D}_2$ obtained in that proposition.

(4.6) Next we deal with the case $m(t) > 0$. In this case the sextic ΣC_j has at most simple singularities and the line $L = L_t$ passes through common points P_i of two cubics C_1 and C_2. $m(t) = 1$ if and only if the number $\#\{P_i\} = 1$ and L is transversal to one of C_j $(j = 1, 2)$ at P_1.

(4.7) Proposition. In the notation in (4.6), if $m(t) = 1$, the pencil of lines through P_1 on $\mathbf{P}^2$ induces elliptic fibrations both on the minimal K3 surface Y and on the minimal model $\hat{X}$ over a rational curve with non-constant J-invariant. The latter has one double fiber. Hence $\hat{X}$ is a numerical K3 surface with one double fiber.

Proof. $p_g(\hat{X}) = 1$ is already known in Corollary (3.4.1). Let $\check{P}_1$ be the pencil of lines through the point P_1 on $\mathbf{P}^2$. The argument in the present case is similar to that in the proof of Proposition (4.2) but there are some points essentially different, hence we shall write down a full proof.

As before, following the procedure of Diagram (3.1.2), we shall first prove that the pencil of lines $\check{P}_1$ induces elliptic fibrations both on Y and on $\hat{X}$. Let $\Sigma_1 \longrightarrow \mathbf{P}^2$ be the blowing-up at the point P_1 and let D_1 be the exceptional curve. Then the pencil of lines $\check{P}_1$ induces the ruling of Σ_1. By construction we have a commutative diagram:

$$\begin{tikzcd} & & Y \arrow[dll, "\alpha_1"'] \arrow[dl, "\alpha'_1"'] \arrow[d, "g"] \\ \mathbf{P}^2 & \Sigma_1 \arrow[l, "q_1"'] & \mathbf{P}^* \arrow[l, "q_2"'] \arrow[ll, bend left, "q"] \end{tikzcd}$$

Because of the procedure of the canonical resolution, the proper transform $D' := q_2^{-1}D_1$ does not appear in the branch locus $B_{\mathbf{P}^*}$ on

P^* if P_1 is a double point of ΣC_j on P^2, while D' remains as a component of B_{P^*} if P_1 is a triple point of ΣC_j. Set

(4.7.1) $B_{P^*} = \Sigma q^{-1}C_j + F'$ in double point case, and

$B_{P^*} = \Sigma q^{-1}C_j + D' + F''$ in triple point case.

Then we see in both cases that, for a fiber M on Σ_1, $q_2^{-1}M$ intersects with B_{P^*} at four distinct points provided that M does not touch $\Sigma q^{-1}C_j$ nor passes a singular point of $\Sigma q^{-1}C_j$. Hence, for these fibers M on Σ_1, $\alpha_1^{\prime *}M$ are smooth irreducible curve on Y. This shows that the pencil of lines $\check{P}_1$ on P^2 induces an elliptic fibration $Y \xrightarrow{\alpha_1'} \Sigma_1 \xrightarrow{pr} D_1$. By the local classification (3.2.5), we can observe that $D := (g^{-1}D')_{red} = \alpha_1'^{-1}D_1$, which is a component in the exceptional divisor for α_1 meeting with $\alpha_1^{-1}L$, does not appear in the branch locus $B_Y = (\alpha_1^*L + \Sigma E_i)_{odd,red}$ on Y. Hence the support of B_Y is contained in a finite number of fibers. Therefore the elliptic fibration on Y induces one on X^* then on $\hat{X}$ by the same argument in the proof of Proposition (4.2).

Next we shall find out the base curve of the elliptic fibration on $\hat{X}$. We observe again the local classification (3.2.5) or its process to get the following:

$$\alpha_1^*L = \alpha_1^{-1}L + g^*D' + E,$$

(4.7.2) $$\alpha_1^*C_2 = 2(\alpha_1^{-1}C_2)_{red} + g^*D' + F,$$

$D \subset \Sigma_1^9 E_i$ and $g^*D' = D$ if P_1 is a double point of ΣC_j,

$D \not\subset \Sigma_1^9 E_i$ and $g^*D' = 2D$ if P_1 is a triple point.

Here E and F are effective divisors defined by the above equations. Notice that their supports are contained in a finite number of fibers on Y. From (4.7.2) we get:

$$\begin{aligned} B_Y &= (\alpha_1^* L + \Sigma_1^9 E_i)_{odd,red} \\ &= \alpha_1^*(L + C_2) - 2((\alpha_1^{-1}C_2)_{red} + g^*D' + G) \\ &\equiv 2(2\alpha_1^* H - (\alpha_1^{-1}C_2)_{red} - g^*D' - G) \\ &\equiv 2(\alpha_1^* H - (\alpha_1^{-1}C_2)_{red} - G'), \end{aligned}$$

where H is a line on P^2 and G and G' are some divisors on Y whose supports are contained in a finite number of fibers. Set $\mathcal{F}'$ $:= \mathcal{O}_Y(\alpha_1^* H - (\alpha_1^{-1}C_2)_{red} - G')$. By (4.7.1), we see that the restriction of the line bundle $\mathcal{F}'$ to a smooth fiber on Y is non-trivial 2-torsion in its Pic. This property is preserved by the line bundle $\mathcal{F}$ with $\mathcal{F}^{\otimes 2} = \mathcal{O}_{Y^*}(B_{Y^*})$ on Y^*. This implies that the pull-back of the fibers on Y^* are still connected on $\hat{X}$. Thus we see that D_1 is the base curve of the elliptic fibration on $\hat{X}$.

For the J-invariant, we use the argument of the cross-ratio as in the proof of Proposition (4.2). Note that a smooth fiber on $\hat{X}$ are isogeneous to the corresponding fiber on Y. Hence it is enough to show that the J-invariant on Y is non-constant. If this is constant, then $\Sigma\, C_j$ should contain concurrent four lines. But this contradicts to our assumption that $\Sigma\, C_j$ has at most simple singularities.

Now we see $q(\hat{X}) = h^1(\mathcal{O}_{D_1}) = 0$ by the same reasoning in the proof of Proposition (4.2).

As for the multiple fibers of the elliptic fibration $f : \hat{X} \longrightarrow D_1$, we use the canonical bundle formula (1.3.1)

$$K_{\hat{X}} = f^*Z + \Sigma(m_i - 1)F_i.$$

Since $\deg Z = \chi(\mathcal{O}_{\hat{X}}) - 2\chi(\mathcal{O}_{D_1}) = 0$, f has only one double fiber.

QED.

(4.8) Proposition. In the notation in (4.6), if $m(t) \geq 2$, the pencils of lines through P_i on P^2 induces elliptic fibrations both on the minimal K3 surface Y and on the minimal model $\hat{X}$ over a rational curve with non-constant J-invariant and without multiple fibers. The canonical divisor $K_{\hat{X}} = 0$, hence $\hat{X}$ is a K3 surface.

Proof. It is enough to show $K_{\hat{X}} = 0$. In fact, we can prove the assertions on elliptic fibrations in the same way as the proof of Proposition (4.7) and the assertion on multiple fibers follow from $K_{\hat{X}} = 0$ by the canonical bundle formula (1.3.1).

In order to see $K_{\hat{X}} = 0$, we divide the cases:

(a) L is a component of ΣC_j.

(b) $m(t) = 2$.

(c) $m(t) = 3$ and not the case (a).

In case (a), $K_{\hat{X}} = 0$ follows from the local classification in Section 3. By the local classification, we can observe that the proper transform $\alpha_1^{-1}L$ on Y is one (-2)-curve (resp. two (-2)-curves) in case (b) (resp. case (c)), and that in both cases $B_Y = (\alpha_1^{*}L + \Sigma E_i)_{odd,red}$ consists of disjoint (-2)-curves. From these observations, we get $K_{\hat{X}} = 0$ in these cases. QED.

(4.9) Remark. A more sophisticated proof of Proposition (4.8) will be given by using Kulikov's list of degenerations of K3 surfaces ([Ku], [PP]), i.e., by virtue of this list it is enough to show that $\hat{X}$ is a K3 surface in generic case with $m(\Sigma C_j, L) = 2$ and in this case the verification is easy. We omit the details.

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