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ADMISSIBLE MODULES ON A SYMMETRIC SPACE

Victor GINSBURG

to I.M.Gel'fand on the occasion of his 75th birthday

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0. Introduction

This paper arose from an attempt to get better understanding of the notion of Character sheaves introduced by G. Lusztig. It is beyond any doubt that Character sheaves play a fundamental role in the Representation theory: they are closely related to irreducible characters of (finite) Chevalley groups, to unipotent representations of complex reductive groups, and, perhaps, to many other matters as well. Unfortunately, there was no simple definition of a character sheaf. Lusztig tried various ones (see [Lu1, Lu2, Lu4 ch. 13]), but all of them seem to be far too complicated for such a basic object and, moreover, it was unclear (while known from tables) why these definitions were equivalent. The definition 1.2 below, given in terms of D-modules, is, I believe, the simplest possible one. Also, it became gradual-

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ly apparent, that replacing the group G by an arbitrary symmetric variety G/K provides a natural setting for the subject. In short, our conclusion can be informally summarized as follows: the class of D-modules arising from character sheaves (in the "G/K-setting") is essentially the same as the class of differential systems (i.e. D-modules) satisfied by K-finite matrix coefficients.

Two ingredients of our approach are especially important to be mentioned here. The first one is the Harish-Chandra functor, taking D-modules on G/K to D-modules on the Flag manifold. The second is a compactification of G/K introduced by De Concini and Procesi [DP] under the name of "complete symmetric variety".

I am grateful to A. Beilinson for explaining to me the definition of a regular compactification.

Quite recently, I have received a preprint by Mirkovic-Vilonen [MiVi] containing a different approach to some of the results of this papers. In particular, one should consult [MiVi] for a new simple characteristic free definition of character sheaves due to Lustig.

1. Basic definitions and main results

1.1. Let G denote a connected complex reductive Lie group. Let Θ be an involutive automorphism of G and let $K := G^{\Theta}$ denote the Θ-fixed point subgroup of G. Let $X = G/K$ be the "complex symmetric variety" associated to the pair $(G; \Theta)$.

Lemma 1.1. (i) K is a reductive subgroup of G;

(ii) G/K is a smooth affine algebraic variety.

For a proof of (i) the reader is referred to [Helg]. (ii) follows from (i) since G/K is an orbit-space of the reductive group K acting freely on G, an affine variety. Such an orbit-space is known to be an affine variety with its regular ring $\mathbb{C}[G/K]$ being equal to $\mathbb{C}[G]^K$.

Here are 2 basic examples of symmetric pairs (G, K).

(i) Given a complex reductive group K set $G = K \times K$ and let θ be the involution on G defined as $\theta : (a, b) \mapsto (b, a)$. Then $G^\theta = K$ (= the diagonal of $K \times K$). Such a symmetric pair $(G = K \times K, K)$ will be referred to as a diagonal pair. Consider the map $r : G \longrightarrow K$, $r(a, b) := a \cdot b^{-1}$. The map r clearly factors through G/K and gives an isomorphism $G/K \cong K$. The left G-action on G/K corresponds to the $K \times K$-action on K by left and right translation.

(ii) Let θ be an involution on a complex reductive group G, $K = G^\theta$, and $G_{\mathbb{R}}$ a real form of G such that $K_{\mathbb{R}} := K \cap G_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$. Then, G/K may be viewed as a "complexification" of the symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$.

1.2. Let $D(X)$ be the ring of global algebraic differential operators on X, let g and k be the Lie algebras of G and K and let $U(g)$, $U(k)$ denote the respective enveloping algebras. The action of G on $X = G/K$ by left translation gives rise to a Lie algebra homomorphism: $g \longrightarrow$ "algebraic vector fields on X". The Lie algebra homomorphism can be naturally extended to an algebra homomorphism: $U(g) \longrightarrow D(X)$. Hence, any $D(X)$-module may be viewed as a $U(g)$-module, via the above homomorphism, and also as a module over a subalgebra of $U(g)$ (e.g. $U(k)$), by restriction.

Let $Z(g)$ denote the center of $U(g)$.

<u>Definition 1.2</u>. A finitely generated $D(X)$-module M is said to be <u>admissible</u> if it is locally-finite both as a $U(k)$-module and as a $Z(g)$-module, that is

$$\dim U(k)\cdot m < \infty \quad \text{and} \quad \dim Z(g)\cdot m < \infty \quad \text{for any} \quad m \in M.$$

We remark that speaking about $D(X)$-modules is the same as speaking about quasi-coherent sheaves of D-modules on X, for X is an affine variety. Yet, there is no obvious way to rephrase the definition of admissible modules in terms of sheaves of D-modules because the algebras U(k) and Z(g) have global nature. A local (actually, even microlocal) characterisation of admissible modules is provided by theorem 1.4.2 below.

Let f be a Z(g)-finite C^∞-function on $G_{\mathbb{R}}$, a real form of G. If f is right-$K_{\mathbb{R}}$-invariant and left $K_{\mathbb{R}}$-finite, then the D(G/K)-module generated by f is admissible. In particular, f satisfies a holonomic system with regular singularities (thm. 1.4.2). The assumption of the right $K_{\mathbb{R}}$-invariance can be easily replaced by the right $K_{\mathbb{R}}$-finiteness.

1.3. Let T be a maximal torus of G, $X(T) := \mathrm{Hom}(T, \mathbb{C}^*)$ the lattice of weights, t the Lie algebra of T, and t^* the dual of t. We view X(T) as a lattice in t^*. Let W be the Weyl group of (G, T), acting on t, t^*, X(T), etc. We form the semidirect product $W_a = W \ltimes X(T)$, called the affine Weyl group. There is a natural W_a-action on t^* by affine transformations.

Maximal ideals of Z(G) can (and will) be parametrized by points of the orbit-space t^*/W via the Harish-Chandra homomorphism $Z(g) \cong \mathbb{C}[t^*]^W$. Let I_λ denote the maximal ideal of Z(g) corresponding to a point $\lambda \in t^*$.

Given a locally-finite Z(g)-module M, one has a root space decomposition

$$V = \bigoplus_{\lambda \in t^*/W} V_\lambda , \quad V_\lambda = \left\{ v \in V \mid I_\lambda^n \cdot v = 0, \ n \gg 0 \right\}$$

The module V is said to have a <u>central character</u> $\bar{\lambda} \in t^*/W_a$ if all the roots λ in the above decomposition belong to the W_a-orbit in t^* corresponding to $\bar{\lambda}$.

<u>Remark 1.3.1</u>. Let $T^* := t^*/X(T)$ be the torus dual to T and let G^* be the complex reductive Lie group containing T^* as a maximal

torus and dual to G in the sense of Langlands [Lan]. There are natural orbit space isomorphisms:

$$t^*/W_a \cong T^*/W \cong \text{the set of semisimple conjugacy classes in } G^*.$$

Thus, there is a bijective correspondence between the set t^*/W_a of all central characters and the set of semisimple conjugacy classes of the dual group.

Admissible modules form an abelian category Admiss(X). Let Admiss(X, $\bar{\lambda}$) denote the full subcategory of Admiss(X), consisting of those modules that have a central character $\bar{\lambda} \in t^*/W_a$. The following result will be proved in section 2.

Theorem 1.3.2. $\text{Admiss}(X) = \bigoplus_{\bar{\lambda} \in t^*/W_a} \text{Admiss}(X, \bar{\lambda})$, i.e.

(i) any admissible module V is isomorphic to a finite direct sum of admissible modules that have central character and

(ii) $\text{Hom}(V_1, V_2) = 0$, provided V_1 and V_2 have different central characters.

Remark. In section 8 we'll prove a much stronger result, saying that $\text{Ext}^*_{D(X)}(V_1, V_2) = 0$ for any admissible modules V_1 and V_2 with different central characters. Here the Ext-group is computed in the ambient category of all D(X)-modules.

1.4. Let g^* be the dual of g and $k^\perp$ the annihilator of k in g^*. The subspace $k^\perp \subset g^*$ is stable under the coadjoint action of K on g^*. Further, let T^*X be the cotangent bundle on X. We observe that T^*_eX, the cotangent space at the base point $e \in G/K$, can be naturally identified with $(g/k)^* \simeq k^\perp$. Hence, there is a vector bundle isomorphism:

$$T^*X = G \times_K k^\perp \qquad (1.4.1)$$

The G-action on X induces a hamiltonian action of G on T^*X. This latter one gives rise to a moment map $\mu : T^*X \to g^*$. Using the isomorphism (1.4.1), the map μ can be described explicitly as follows:

$$\mu : G \times_K k^{\perp} \ni (x, \lambda) \longmapsto x \cdot \lambda \cdot x^{-1} \in g^*$$

Let SSV denote the characteristic variety of a finitely-generated D(X)-module V and let $N_g \subset g^*$ be the nilpotent cone (= the zero-variety of the set of invariant polynomials on g^* without constant term).

Theorem 1.4.2. The following conditions are equivalent

(i) V is an admissible D(X)-module;

(ii) V is a regular holonomic D(X)-module such that

$$SSV \subset \mu^{-1}(N_g \cap k^{\perp}) \qquad (*)$$

The proof of theorem 1.4.2 is rather long. The implication (i) $\Rightarrow$(ii) is proved in section 3 with the exception of the fact that an admissible module has regular singularities. The regularity follows from theorem 8.5.1 on the Harish-Chandra transformation. The proof of the implication (ii) $\Rightarrow$(i) is given in section 3. It heavily depends, however, on results of sections 4, 5 and of Appendix A, concerning characteristic varieties of D-modules on a regular compactification.

Remarks. (a) Write the Cartan decomposition $g = k \oplus p$, where p is the (-I)-eigenspace of the involution on g induced by θ. Let g^* be identified with g via a θ-invariant Killing form isomorphism. Then we have: $k^{\perp} \cong p$ and N_g = variety of nilpotents in g, so that $N_g \cap k^{\perp} = N_p$ (= the subvariety of nilpotent elements of p). The moment map (1.4.1) turns into the map:

$$\mu : T^*X = G \times_K p \longrightarrow g, \quad (x, v) \mapsto x \cdot v \cdot x^{-1}. \qquad (1.4.3)$$

(b) An estimate similar to 1.4.2 (*) was first considered by G. Laumon [Lal] in connection with the work of V. Drinfeld [Dr] on the Langlands' conjecture.

1.5.* We'll see in section 3 that $\mu^{-1}(N_g \cap k^{\perp})$ is a Lagrangian subvariety of T^*X. This observation suggests the following problem.

Let H and K be Lie subgroups of a Lie group G, let g, h, k be the corresponding Lie algebras, and N an Ad G-stable conic subvariety of g^*. Set $X = G/H$ and let $\mu : T^*X \to g^*$ be the moment map. The problem is to classify (for a given G) all the triples (H, K, N), such that $\mu^{-1}(N \cap k^{\perp})$ is a Lagrangian subvariety of T^*X, and to study the category of holonomic modules on X whose characteristic varieties are contained in $\mu^{-1}(N \cap k^{\perp})$. If $N = O$ is a single coadjoint orbit in g^*, then we have the following general criterion

Proposition 1.5.1. The following properties are equivalent

(i) $\mu^{-1}(O \cap k^{\perp})$ is a Lagrangian (resp. isotropic, coisotropic) subvariety of T^*X;

(ii) both $O \cap h^{\perp}$ and $O \cap k^{\perp}$ are Lagrangian (resp. isotropic, coisotropic) subvarieties of the orbit O, viewed as a symplectic manifold. □

Here are some interesting examples of triples (H, K, N) for a reductive group G. The case $H = K = G^{\theta}$ and $N = N_g$ is just that of admissible modules. Further, if H is a maximal unipotent subgroup of G, $K = G^{\theta}$ and $N = N_g$, the category in question is the category of Harish-Chandra modules. Next, set $G = SL_n(\mathbb{C})$, $X = G/H$ = Grassmann manifold, K = diagonal matrices, N = matrices of rank ≤ 1 (= the closure of the minimal orbit in g^*). Then, one can show, that $\mu^{-1}(N \cap k^{\perp})$ = the closure of the union of conormal bundles to all 1-codimensional B-orbits in X, where B runs over the set of n! Borel subgroups of G containing K. The category of holonomic modules on X arising in this case is formed by D-modules generated by the generalized hypergeometric functions in the sense of I.M. Gel'fand et al [G], [GG]. There is a similar example for $G = Sp_{2n}(\mathbb{C})$ and X = Lagrangian grassmannian.

1.6. Two final sections are devoted to a more detailed study of the diagonal case. Let K be a connected reductive group. As was explained in example (i) of n.1.1, an admissible module in the diagonal case can be viewed as an Ad K-monodromic D-module on the group K itself. Using the Harish-Chandra transform of section 8, we prove

Theorem 1.6.1. The following conditions on an irreducible Ad K-equivariant D(K)-module V are equivalent:

(i) V is an admissible module with central character $\bar{\lambda} \in t^*/W_a$ of finite order in T^* (cf. n. 1.3);

(ii) The perverse sheaf DR(V), associated to V, is a character sheaf in the sense of [Lu2].

2. Admissible (g, k)-modules

2.1. Let $U_o(g) \subset U_1(g) \subset \ldots$ be the standard increasing filtration on U(g). By the Poincare-Birkhoff-Witt theorem we have Gr U(g) = $\mathbb{C}[g^*]$. Given a finitely-generated U(g)-module V, let SSV denote the characteristic variety of V, i.e. the support of the graded $\mathbb{C}[g^*]$ = (= Gr U(g))-module associated with a good filtration on V.

A finitely-generated U(g)-module V is called a (g, k)-module if the U(k)-action on V is locally-finite. We remark that the U(k)-action is not required to be completely reducible.

Proposition 2.1.1. The following properties of a (g, k)-module

V are equivalent:

(i) Any irreducible representation of k occurs in V, viewed as a U(k)-module, with finite multiplicity;

(ii) V is a locally-finite Z(g)-module;

(iii) $SSV \subset N_p$ $(= N_g \cap k^{\perp})$.

Remarks. (i) A finitely-generated U(g)-module V, which is a locally-finite Z(g)-module, is necessarily Z(g)-finite, i.e. there is an ideal in Z(g) of finite codimension, annihilating V.

(ii) Let us clarify the meaning of condition (i) of the Proposition. Our module V is the union of an increasing sequence $V_o \subset V_1 \subset V_2 \subset \ldots$ of finite-dimensional U(k)-stable subspaces V_i. The condition means that for any irreducible (finite-dimensional) k-module E the multiplicity sequence $(V_o : E) \leqslant (V_1 : E) \leqslant \ldots$ is bounded. In that case there is a canonical infinite direct sum decomposition $V = \bigoplus_{\alpha} V(\alpha)$ into finite-dimensional k-isotypical components $V(\alpha)$. The components are uniquely defined by the following properties: (a) $V(\alpha)$ is a k-module with all of its simple subquotients being isomorphic to each other; (b) different $V(\alpha)$'s are disjoint, i.e. have no isomorphic subquotients. □

A (g, k)-module V is said to be admissible if it satisfies the equivalent properties (i)-(iii) of Proposition 2.1.1.

Lemma 2.1.2. Let V be an U(g) (resp. D(X))-module, generated by a finite-dimensional subspace V_o. Suppose that V_o is both U(k)-stable and Z(g)-stable subspace of V. Then we have

$$SSV \subset N_p \quad (\text{resp. } SSV \subset \mu^{-1}(N_p)).$$

Proof of the Lemma. Let $Z_+(g)$ denote the augmentation ideal of Z(g) and let $Z_+(g)_i = Z_+(g) \cap U_i(g)$ be the standard induced filtration on $Z_+(g)$. The zero-variety of the subset $\mathrm{Gr}\, Z_+(g) \subset \mathrm{Gr}\, U(g) = \mathbf{C}[g^*]$ is known to be equal to N_g, the nilpotent variety.

Now let V be as in the Lemma. On V define the good filtration

$V_i = U_i(g)\cdot V_o$ (resp. $V_i = D_i(X)\cdot V_o$ where $D_i(X)$ denotes the standard filtration on $D(X)$). It's easy to see that $k\cdot V_i \subset V_i$ and $Z_+(g)\cdot V_i \subset V_i$. Hence, Gr V is annihilated both by k and by Gr $Z_+(g)$. So, supp Gr V is contained in $k^\perp \cap N_g$ (resp. $\mu^{-1}(k^\perp \cap N_g)$), and the statement follows. □

Proof of Proposition 2.1.1. We begin with an observation that a finitely-generated U(g)-module V is both U(k)- and Z(g)-locally-finite iff it is generated by a U(k)-stable and Z(g)-stable finite-dimensional subspace V_o. This, combined with lemma 2.1.2, yields the implication: (ii) $\Longrightarrow$ (iii).

The implication: (i) $\Longrightarrow$ (ii) is clear, since the action of Z(g) on V commutes with that of U(k), hence preserves each k-isotypical component $V(\alpha)$, mentioned in remark (ii) after proposition 2.1.1.

We now turn to the implication (iii) $\Longrightarrow$ (i). For an U(k)-module M, which is the union of a sequence of finite-dimensional k-submodules, consider the following property

(*) Any irreducible representation of k occurs in M with finite multiplicity.

We have to show that the property (*) holds for V, provided SSV $\subset N_p$. This will be done in four steps.

Let $J \subset \mathbb{C}[g]$ be the ideal of all polynomials on g vanishing on N_p, so that $\mathbb{C}[N_p] = \mathbb{C}[g]/J$ is the ring of regular functions on N_p. The ideal J is clearly stable under the adjoint k-action on $\mathbb{C}[g]$ since N_p is an Ad K-stable subvariety of g. So, $\mathbb{C}[N_p]$ is a locally-finite U(k)-module.

Step 1. The property (*) holds for $M = \mathbb{C}[N_p]$.

This claim is essentially due to Kostant-Rallis. In effect, it was shown in [KoRa] that any irreducible K-module occurs in $\mathbb{C}[N_p]$, a completely-reducible K-module, with finite multiplicity. If we now turn from the K-action to that of the Lie algebra k, we see that only finitely many copies of the simple k-modules with highest weight of fixed modulus may occur in $\mathbb{C}[N_p]$. The claim follows.

Step 2. Let M be a finite-generated $\mathbb{C}[N_p]$-module with a locally-finite k-action compatible with that on $\mathbb{C}[N_p]$. Then, the property (*) holds for M. To prove this claim, pick up a finite-dimensional k-stable generating subspace $M_o \subset M$. Then M is k-isomorphic to a quotient of $\mathbb{C}[N_p] \otimes M_o$ so that it suffices to prove (*) for $\mathbb{C}[N_p] \otimes M_o$. Let $\overline{M}_o$ be the semisimplification of M_o, a semisimple k-module. Replacing M_o by $\overline{M}_o$ does not affect multiplicities. Hence, for the multiplicity of a simple k-module E we have

$$(\mathbb{C}[N_p]\otimes\overline{M}_o : E) = \mathrm{Hom}_k(E, \mathbb{C}[N_p] \otimes \overline{M}_o) = \mathrm{Hom}_k(E\otimes M_o^*, \mathbb{C}[N_p])$$

The last term is finite-dimensional by Step 1.

Step 3. Let M be a finitely-generated $\mathbb{C}[g]$-module endowed with a locally-finite k-action compatible with the adjoint one on $\mathbb{C}[g]$. If supp $M \subset N_p$ then M satisfies (*).

To prove this, observe that if M is supported on N_p then it is killed by some power J^n of the ideal J annihilating N_p. Since J is an ad k-stable ideal, the J-adic filtration

$$M \supset J\cdot M \supset J^2\cdot M \supset \ldots \supset J^n . M = 0$$

is a k-stable finite filtration on M. Furthermore, the successive quotients $J^i . M/J^{i+1} . M$ are annihilated by J, hence, may be regarded as $\mathbb{C}[N_p]$-modules. The statement now follows from Step 2.

Step 4. Finally, let V be a (g, k)-module. Choose a finite-dimensional k-stable generating subspace $V_o \subset V$ and let $V_i = U_i(g)\cdot V_o$ be a k-stable good filtration on V. Now, if SSV $\subset N_p$, then the associated graded $\mathbb{C}[g]$-module Gr V is supported on N_p, hence satisfies (*) by step 3. But, applying the functor Gr does not affect multiplicities. So, (*) holds for V iff it holds for Gr V. That completes the proof. □

2.2. Given g-modules E and V we let $E \otimes V$ denote the g-module induced by the diagonal action of g. The following result is fairly well-known (see e.g. [Kos])

<u>Proposition 2.2</u>. Let E be a finite-dimensional g-module.

(i) If V is an admissible (g, k)-module, then so is $E \otimes V$;

(ii) Suppose that E arises from a representation of the group G. If V has a central character $\bar{\lambda} \in t^*/W_a$ (see n.1.3), then $E \otimes V$ has the same central character $\bar{\lambda}$. □

2.3. <u>Proof of theorem 1.3.2</u>. Let the ring D(X) be viewed as a G-module with respect to the adjoint action. The action being locally finite, we have an infinite direct sum decomposition $D(X) = \oplus E_i$ into finite-dimensional G-modules E_i.

Now let V be an admissible D(X)-module and let V_o be a Z(g)-stable finite-dimensional subspace of V, that generates V. Using the root-space decomposition of V_o, one obtains a finite direct sum decomposition $U(g)\cdot V_o = \oplus V_{\bar{\lambda}}$, where $V_{\bar{\lambda}}$ is an U(g)-module with central character $\bar{\lambda} \in t^*/W_a$.

The D(X)-action on V gives rise to a surjective g-module homomorphism $D(X) \otimes U(g)\cdot V_o \longrightarrow V$ where the g-module structure on D(X) is induced by the adjoint action. Hence, there is a surjective homomorphism:

$$(\oplus E_i) \otimes (\oplus V_{\bar{\lambda}}) \longrightarrow V$$

Proposition 2.2 (ii) shows that the module $(\oplus E_i) \otimes V$ has the central character $\bar{\lambda}$. So, V is a finite sum (automatically direct) of the D(X)-modules $D(X)\cdot V_{\bar{\lambda}}$ that have (different) central characters. □

3. <u>Outline of the proof of theorem 1.4.2</u>.

3.1. <u>Proposition 3.1.1</u>. $\mu^{-1}(N_g \cap k^{\perp})$ is a Lagrangian subvariety of $T^*(G/K)$.

<u>Proof</u>. Since N_g is the union of finitely many G-orbits, it suffices to show (prop. 1.5.1) that for any nilpotent orbit $O \subset N_g$ the subvariety $O \cap k^{\perp}$ is Lagrangian. We identify $k^{\perp}$ with p via

a θ-invariant Killing form isomorphism $g^* \cong g$. Given a smooth point $a \in O \cap p$, we claim that the tangent space to $O \cap p$ at a equals $[k, a]$ (where brackets stand for the (co-)adjoint action). To check this, take a tangent vector $[x, a]$, $x \in g$, and write $x = x_k + x_p$, $x_k \in k$, $x_p \in p$. Since $a \in p$ we have $[x_k, a] \in p$, $[x_p, a] \in k$. But $[x, a] \in p$, hence $[x_p, a] = 0$. Thus, $[x, a] = [x_k, a] \in [k, a]$ and the claim follows.

Obviously, $[k, a]$ is an isotropic subspace: $\langle a, [x_1, x_2]\rangle = 0$ for any $x_1, x_2 \in k$. Next, suppose that $\langle a, [x, k]\rangle = 0$ for some $x \in g$. Then, $\langle [x, a], k\rangle = 0$, so that $[x, a] \in p$. Writing $x = x_k + x_p$ as above shows that $[x, a] = [x_k, a] \in [k, a]$. Whence, $[k, a]$ is a coisotropic subspace. Q.E.D.

<u>Sketch of proof of Proposition 1.5.1</u>. Given $a \in O \cap k^\perp \cap h^\perp$, let $S_k = \{x \in g \mid x\cdot a \in k^\perp\}$ and $S_h = \{x \in g \mid x\cdot a \in h^\perp\}$, where $x\cdot a$ denotes the coadjoint action of x. Clearly, the tangent space to $O \cap k^\perp$ at a equals $S_k\cdot a$.

Further, let $X = G/H$ and let λ be a point in the cotangent space at the base point $1 \in G/H$. We can write the tangent vector to T^*X at λ in the form $x\cdot\lambda - \alpha$, where $x\cdot\lambda$ denotes the action of $x \in g$ on λ and $\alpha \in T^*_\lambda X = h^\perp$ is a "vertical" tangent vector. Assume now that $\mu(\lambda) = a$. One shows easily that the tangent space to $\mu^{-1}(a)$ (at λ) is formed by the vectors $s_h\cdot\lambda - \alpha$, where $s_h \in S_h$ and $\alpha = s_h\cdot a$. Whence

$$T_\lambda(\mu^{-1}(O \cap k^\perp)) = \{(s_k + s_h)\cdot\lambda - \alpha \mid s_k \in S_k,\ s_h \in S_h,\ \alpha = s_h\cdot a\}$$

Now, the standard symplectic form on T^*X is given by the formulas:

$$\omega(x\cdot\lambda, y\cdot\lambda) = \langle a, [x, y]\rangle, \quad \omega(x\cdot\lambda, \alpha) = \alpha(x), \quad x, y \in g,$$

Using these formulas one finds that (for $\alpha = s_h\cdot a$, $\alpha' = s_h'\cdot a$):

$$\omega((s_k + s_h)\cdot\lambda - \alpha,\ (s'_k + s'_h)\cdot\lambda - \alpha') = \langle a, [s_k, s'_k] + \langle a, [s_h, s'_h]\rangle$$

Thus, $\mu^{-1}(O \cap k^{\perp})$ is isotropic iff so are $O \cap k^{\perp}$ and $O \cap h^{\perp}$.

The rest of the proof is left to the reader. □

3.2. Proof of theorem 1.4.2. We first show the part of the implication: (i) $\Longrightarrow$ (ii) of the theorem, saying that if V is an admissible D(X)-module then V is a holonomic module and

$$SSV \subset \mu^{-1}(N_p). \tag{3.2.1}$$

The regularity of V will be proved later, in section 8.

The inclusion (3.2.1) is an immediate consequence of lemma 2.1.2. The holonomicity of V then follows from proposition 3.1.1.

The implication: (ii) $\Longrightarrow$ (i) of the theorem follows from Proposition 2.1.1 (ii) and the following

Lemma 3.2.2. Let V be a regular holonomic D(X)-module such that $\mu(SSV) \subset N_p$. Then, any finitely-generated U(g)-submodule $M \subset V$ is an admissible (g, k)-module.

Proof. We shall see (cf. lemma 3.5.6 (ii)) that the variety $\mu^{-1}(p)$ equals the union of conormal bundles to K-orbits in X. By our assumptions, SSV is contained in that variety. Hence, V is a K-monodromic module (see Appendix B) so that the U(k)-action on V is locally-finite. Thus, M is a (g, k)-module.

Next, we have $SSM \subset \mu(SSV)$, provided the group G is of adjoint type. This estimate on SSM follows from the existence of a nice compactification of G/K (Proposition 6.1) combined with a general theorem 4.3.3 (see sections 4 and 5). The estimate yields $SSM \subset N_p$ completing the proof in adjoint case (cf. 2.1.1 (iii)).

In the general case consider a diagram

$$\begin{array}{ccc} G_2 = G_1 \times C & \xleftarrow{\ i\ } & G_1 \\ p_2 \downarrow & & \downarrow p_1 \\ G & & G' \end{array} \tag{3.2.3}$$

Here G' is the derived group of G, the group G_1 (resp. G_2) is a finite central extension of G' (resp. G), so that p_1 (resp. p_2) denotes the natural projection, and C is a torus. To arrange such a diagram, one can take, for instance, G_1 to be the connected and simply-connected covering of G' and C to be the connected center of G. In that case the homomorphism $G_2 = G_1 \times C \longrightarrow G$ is induced by the natural inclusion $C \hookrightarrow G$ and by the homomorphism $G_1 \longrightarrow G$, arising from a Lie algebra splitting homomorphism Lie G' $\longrightarrow$ Lie G.

The construction, just given, shows, that an involution Θ on G can be extended to involutions on G', G_1 and G_2, so that the diagram (3.2.3) becomes a diagram of symmetric pairs. Hence, it gives rise to a similar diagram of symmetric spaces:

$$\begin{array}{ccc} X_2 = X_1 \times (C/C^{\Theta}) & \overset{i}{\hookleftarrow} & X_1 \\ p_2 \big\downarrow & & \big\downarrow p_1 \\ X & & X' \end{array} \qquad (3.2.4)$$

Given a D(X)-module V, let $V_2 = p_2^* V = \mathbb{C}[X_2] \otimes_{\mathbb{C}[X]} V$ be a $D(X_2)$-module (here $\mathbb{C}[X]$ denotes the ring of regular functions on X, etc.), let $V_1 = i^* V_2$ be the restriction of V_2 to X_1, a $D(X_1)$-module, and $V' = (p_1)_* V_1$ a D(X')-module. We identify V with an U(g)-submodule of V_2 via the embedding $p_2^* : v \mapsto 1 \otimes v$ and also identify V' with V_1, for p_1 is an etale covering. It can be easily verified that all the modules V', V_1 and V_2 satisfy the assumptions of lemma 3.2.2 if so does V. In that case V_2 is a C-monodromic module.

Now let M be a finitely-generated U(g)-submodule of V. We'll view M as a submodule of V_2 via the embedding p_2^*. Let c = Lie C. The U(c)-action on M is locally-finite since V_2 is a C-monodromic module. Hence, there is a finite root-space decomposition $M = \oplus M_{\alpha}$, $\alpha \in c^*$. Each M_{α} is, clearly, an U(g)-submodule of M and it suffices to prove the lemma for the modules M_{α}.

Let $i^* : V_2 \longrightarrow V_1$ be the restriction map and $i^* M$ the image of a certain M_α. It's not hard to show that the restriction map $M_\alpha \longrightarrow i^*M_\alpha$ is, in fact, a $U(g_1)$-module isomorphism (where $g_1 = \text{Lie } G_1$). But now, i^*M_α may be viewed as an $U(g_1)$-submodule of the $D(X')$-module V'. Hence, i^*M_α is an admissible (g_1, k_1)-module, for G' is an adjoint group. So, $M_\alpha \cong i^*M$ is an admissible (g, k)-module. □

3.3. Let $P_1 = \{y \in G \mid \Theta(y) = y^{-1}\}$. We define a G-action on P_1 by $h : y \longmapsto h\cdot y\cdot\Theta(h)^{-1}$, $h \in G$. The restriction of the G-action to the subgroup K clearly reduces to the adjoint K-action on P_1.

Let P denote the identity component of P_1. This is a G-stable part of P_1 and we shall prove below the following

Lemma 3.3.0. The "Lang map" $G \ni y \longmapsto y\cdot\Theta(y)^{-1}$ gives rise to a G-equivariant isomorphism $G/K \xrightarrow{\sim} P$.

We record a few well-known properties of P (see e.g. [Helg]).

Let A be a maximal torus contained in P (such a torus is called "split").

Lemma 3.3.1. The set of semisimple elements of P equals Ad K·A, the K-saturation of A under the adjoint action.

Lemma 3.3.2. For an element $y \in P$ there is a Jordan decomposition:

$y = s\cdot u = u\cdot s$, s semisimple, u unipotent and $s, u \in P$. □

Lemma 3.3.3. The number of unipotent K-conjugacy classes in P is finite. □

Lemma 3.3.4. The exponential map $\exp : p \longrightarrow P$ gives an isomorphism of N_p onto the subvariety of unipotent elements of P. □

Proof of Lemma 3.3.0. The map $y \longrightarrow y\cdot\Theta(y)^{-1}$ obviously factors through G/K and the resulting map is clearly injective. Its image belongs to P_1 and, in effect, to P since G is connected. It suffices to prove the surjectivity. Let $y_1 \in P$. Write $y_1 = s_1\cdot u_1$ (lemma 3.3.2). Since a torus is a divisible group, lemma 3.3.1 shows that one can find $s \in P$ such that $s^2 = s_1$. Similarly, there is $u \in P$ such that $u^2 = u_1$ (lemma 3.3.4). Then, for $y = u\cdot s$ we have

$$y\cdot\Theta(y)^{-1} = u\cdot s\cdot\Theta(s)^{-1}\cdot\Theta(u)^{-1} = u\cdot s^2\cdot u = u\cdot s_1\cdot u = s_1\cdot u^2 = s_1\cdot u_1 = y_1,$$

where we have used that u - being a function of u_1 - commutes with s_1.

3.4. Let $Z_G^o(S)$ denote the identity component of the centralizer in G of a subset $S \subset G$. A subgroup of G of the form $Z_G^o(s)$, where s is a semisimple element of P, is said to be a relevant Levi subgroup of G. It is a Θ-stable reductive subgroup of G. The number of K-conjugacy classes of relevant Levi subgroup of G is finite.

Given a relevant Levi subgroup L, set $K_L = L^\Theta = L \cap K$, $P_L :=$ the identity component of $(P \cap L)$, $Z_L = \mathrm{Center}(L) \cap P_L$. It is clear, that the variety P_L is stable under the adjoint action of K_L and under multiplication by elements of Z_L^o, the identity component of the group Z_L.

We introduce a stratification $P = \bigsqcup P_r$ by smooth Ad K-stable locally-closed subvarieties P_r. The index r runs through the set of K-conjugacy classes of triples (L, Z, O) where L is a relevant Levi subgroup of G, Z is a connected component of the group Z_L and O is a unipotent K_L-conjugacy class in P_L. The stratum $P_{L,Z,O}$ is defined as follows (cf. [Lu1]). Let $Z^{reg} = \{s \in Z \mid Z_G^o(s) = L\}$. Clearly Z^{reg} is a Zariski-open part of Z. We put:

$$P_{L,Z,O} = \text{Ad K-saturation of } (Z^{reg}\cdot O) = \{k\cdot s\cdot u\cdot k^{-1},\ k \in K,\ s \in Z^{reg},\ u \in O\}$$

Given an element $y \in P$, write $y = s\cdot u$ (lemma 3.3.2) and let $L = Z_G^o(s)$, a relevant Levi subgroup. It is clear that u is a unipotent element of P_L. Furthermore, if A is a split torus in P containing s (lemma 3.3.1), then $A \subset P_L$. Hence, $s \in Z_L = \mathrm{Center}(L) \cap P_L$. So, if Z is the connected component of Z_L that contains s and O = Ad $K_L\cdot u$, then we have $y \in P_{L,Z,O}$.

3.5. Let L be a relevant Levi subgroup of G. A unipotent element $u \in P_L$ (or its conjugacy class) is called <u>distinguished</u> if u does not belong to any proper relevant Levi subgroup of L. A stratum $P_{L,Z,O}$ is called distinguished if O is a distinguished conjugacy class in P_L.

Now let T^*P be the cotangent bundle to P, let $\Lambda \subset T^*P$ be the union of conormal bundles to all the strata $P_{L,Z,O}$, and let Λ_o be a part of Λ, equal to the closure of the union of conormal bundles to all the distinguished strata. By definition, Λ and Λ_o are Lagrangian subvarieties of T^*P.

Let $\mu : T^*P \longrightarrow g^*$ be the moment map, arising from the G-action on P, defined in n.3.3. The importance of the lagrangian subvariety Λ_o becomes clear from the following

<u>Proposition 3.5.1</u>. $\mu^{-1}(N_p) = \Lambda_o$.

To prove the proposition we have to analyze the geometry of T^*P first. Identify TG with the trivial bundle $g \times G$ via right translations. Lemma 3.3.0 shows that P is a smooth subvariety of G, so that its tangent bundle is a subbundle of $TG_{|P}$. More precisely, we have

$$TP = \{(v, y) \in g \times P \mid \Theta(v) = -y^{-1}\cdot v\cdot y\} \qquad (3.5.2)$$

Observe, that the tangent space to P at the identity equals p.

<u>Lemma 3.5.3</u>. The restriction of the (Θ-invariant) Killing form to a tangent space T_yP, $y \in P$ is a non-degenerate form.

<u>Proof</u>. One decomposes the Lie algebra g into the orthogonal direct sum $g = g_+ \oplus g_-$ where $g_\pm = \{v \in g \mid \Theta(v) = \pm y^{-1}\cdot v\cdot y\}$. Hence, the restriction of the Killing form to $g_- = T_yP$ is a non-degenerate form.

Let $v \in g$ and let $v(y) \in T_yP$ denote the value at $y \in P$ of the vector field on P, arising from v via the G-action. It's easy to see that

$$v(y) = v - y\cdot\Theta(v)\cdot y^{-1} \qquad (3.5.4)$$

We now identify T^*P with TP via the Killing form (cf. lemma 3.5.3). From (3.5.2) and (3.5.4) one finds the following formula for the moment map $\mu : T^*P \longrightarrow g$ (notation of (3.5.2)):

$$\mu(v, y) = v - y^{-1}\cdot\Theta(v)\cdot y = 2\cdot v \qquad (3.5.5)$$

In what follows we adopt the following notation. Given a subset $S \subset G$ and an element y of either G or g let $Z_S(y) = \{s \in S \mid s\cdot y\cdot s^{-1} = y\}$. Similar notation is used if $S \subset g$.

Lemma 3.5.6. Let $y \in P$. Then: (i) $p \cap T_y^*P = Z_p(y)$;
(ii) the conormal space at $y \in P$ to the adjoint K-orbit through y equals

$$\mu^{-1}(p) \cap T_y^*P = p \cap T_y^*P.$$

Proof. Formula (3.5.2) yields:

$$p \cap T_y^*P = \{v \in T_y^*P \mid \Theta(v) = -v\} = \{v \in g \mid -v = \Theta(v) = = -y^{-1}\cdot v\cdot y\} = \{v \in p \mid v = y\cdot v\cdot y^{-1}\} = Z_p(y)$$

and the statement (i) follows. Part (ii) is clear. $\square$

3.6. Let l be the Lie algebra of a relevant Levi subgroup L, $[l, l]$ the derived Lie algebra, z_L the Lie algebra of the group Z_L and $z_L^{\perp}$ the orthogonal complement to z_L in g with respect to the Killing form. It is clear that

$$l \cap z_L^{\perp} \cap p = [l, l] \cap p \qquad (3.6.1)$$

The proof of the following result is left to the reader.

Lemma 3.6.2. A unipotent element $u \in P_L$ is distinguished iff the space $[l, l] \cap Z_l(u) \cap p$ is contained in the nilpotent variety of l. $\square$

Proof of proposition 3.5.1 is based on the following result (recall that Λ denotes the union of conormal bundles to all the strata).

Lemma 3.6.3. (i) $\mu^{-1}(N_p) \subset \Lambda$;
(ii) The conormal bundle to a stratum $P_{L,Z,O}$ is contained in $\mu^{-1}(N_p)$

iff O is a distinguished conjugacy class.

Proof. Let $y \in P_{L,Z,O}$. We may (and will) assume, by Ad K-equivariance, that $y = s \cdot u \in Z^{reg}.O$. Then, the tangent space at y to the stratum $P_{L,Z,O}$ equals

$$E_y = z_L + \text{tangent space at } y \text{ to the Ad K-orbit through } y \tag{3.6.4}$$

Let $E_y^{\perp} \subset T_y^*P$ denote the conormal subspace at y to the stratum $P_{L,Z,O}$. It follows from (3.6.4) and lemma 3.5.6 that we have

$$E_y^{\perp} = z_L^{\perp} \cap p \cap T_yP = z_L^{\perp} \cap z_p(y)$$

Recall now that $y = s \cdot u$ $(s \in Z^{reg},\ u \in O)$ is a Jordan decomposition. Hence,

$$z_p(y) = z_p(s) \cap z_p(u) = l \cap z_p(u) = z_l(u) \cap p \tag{3.6.5}$$

Thus, we obtain

$$E_y^{\perp} = z_L^{\perp} \cap z_l(u) \cap p \overset{(3.6.1)}{=} [l, l] \cap z_l(u) \cap p \tag{3.6.6}$$

Further, it follows from (3.5.2) and lemma 3.5.6 that we have

$$\mu^{-1}(N_p) = \{(v, y) \in p \times \mathbf{P} \mid v \in z_g(y) \cap N_g \} \tag{3.6.7}$$

This formula, combined with (3.6.5), yields

$$\mu^{-1}(N_p) \cap T_y^*P = z_p(y) \cap N_g = N_g \cap z_l(u) \cap p.$$

Using an obvious equality $N_g \cap l = N_g \cap [l, l]$ one finds

$$\mu^{-1}(N_p) \cap T_y^*P = N_g \cap [l, l] \cap z_l(u) \cap p \tag{3.6.8}$$

The first part of lemma 3.6.3 now becomes clear from looking at (3.6.6) and (3.6.8), and part (ii) follows from lemma 3.6.2. $\square$

We are ready to prove proposition 3.5.1. Lemma 3.6.3 (i) shows that $\mu^{-1}(N_p)$ is a subvariety of the lagrangian variety Λ. It follows from Proposition 3.1.1 that each irreducible component of $\mu^{-1}(N_p)$ is also a lagrangian subvariety. Hence, $\mu^{-1}(N_p)$ is the closure of those conormal bundles that are contained in $\mu^{-1}(N_p)$. It remains to apply lemma 3.6.3 (ii). □

4. Regular compactification

4.1. Let Z be a smooth complex algebraic variety, $\mathcal{O}_Z$ the sheaf of regular functions on Z and T_Z the tangent sheaf on Z. Further, let $Y \subset Z$ be a normal crossing divisor with defining ideal $J_Y \subset \mathcal{O}_Z$ and let $T_{Z,Y}$ denote the sheaf of those vector fields on Z that preserve J_Y, i.e. $T_{Z,Y} = \{v \in T_Z \mid v \cdot J_Y \subset J_Y\}$. More geometrically, the sheaf $T_{Z,Y}$ consists of the vector fields that are tangent to Y at generic points of Y.

Lemma 4.1. $T_{Z,Y}$ is a locally-free sheaf of $\mathcal{O}_Z$-modules.

Proof. Let $t_1, \ldots, t_n$ be local coordinates on Z so that Y is locally defined by the equation $t_1 \cdot \ldots \cdot t_k = 0$. Then, the vector fields $t_1 \cdot \partial/\partial t_1, \ldots, t_k \cdot \partial/\partial t_k, \partial/\partial t_{k+1}, \ldots, \partial/\partial t_n$ form a local $\mathcal{O}_Z$-basis of $T_{Z,Y}$. □

There is a natural Lie algebra structure on T_Z given by the Lie bracket. It is clear that the subsheaf $T_{Z,Y} \subset T_Z$ is a Lie subalgebra of T_Z.

4.2. Let G be an algebraic group acting on a smooth algebraic variety Z. Let Y be a G-stable normal crossing divisor in Z and let g denote the Lie algebra of G. The infinitesimal g-action on Z gives rise to a Lie algebra homomorphism $g \longrightarrow \Gamma(Z, T_{Z,Y})$. For any point $z \in Z$ we get, by restriction, a linear map from g into $T_{Z,Y,z}$, the fibre of $T_{Z,Y}$ at z.

The following definition is due to A. Beilinson.

Definition 4.2.1. The action of G on Z is said to be G-regular with respect to Y if the map $g \longrightarrow T_{Z,Y,z}$ is surjective for any point $z \in Z$.

Let Z be an irreducible variety with G-regular action with respect to Y. Note that the restriction of the sheaf $T_{Z,Y}$ to $Z \setminus Y$ coincides with the tangent sheaf on $Z \setminus Y$. Hence, the following readily follows from definition 4.2.1.

Lemma 4.2.2. $Z \setminus Y$ is the unique open G-orbit in Z.

The G-orbit structure of Y can be described as follows. For each $n = 1, 2, \ldots$ let Y_n denote the subvariety of all points $y \in Y$ that have the following property: y belongs to the intersection of n irreducible components of $Y \cap U$, where U is a small open neighborhood of y.

Lemma 4.2.3.* Each irreducible component of $Y_n \setminus Y_{n+1}$ is a single G-orbit. In particular, the number of G-orbits in Z is finite.

Proof is left to the reader. □

4.3. Let G be an algebraic group and $X = G/K$, a homogeneous G-variety.

Definition 4.3.1. An equivariant embedding $X \hookrightarrow \bar{X}$ into a G-variety $\bar{X}$ is called a regular compactification of X if the following holds:

(i) $\bar{X}$ is a smooth compact algebraic variety;

(ii) the image of X is a Zariski-open part of $\bar{X}$ and $\bar{X} \setminus X$ is a normal crossing divisor in $\bar{X}$;

(iii) the action of G on $\bar{X}$ is G-regular with respect to $\bar{X} \setminus X$.

Example 4.3.2. Let T be a torus and $\bar{T}$ a smooth toroidal compactification of T. Then, $T \hookrightarrow \bar{T}$ is a T-regular compactification. □

Now let V be a regular holonomic D-module on $X = G/K$, a homogeneous G-variety. Let g denote the Lie algebra of G and D_X the sheaf of regular differential operators on X. We have a natural homomorphism $U(g) \to \Gamma(X, D_X)$ so that $\Gamma(X, V)$ may be viewed as a $U(g)$-module. Let $\mu : T^*X \to g^*$ be the moment map.

Here is the main result of this section

Theorem 4.3.3. Suppose that X has a regular compactification.

Then, for any finitely-generated U(g)-submodule $V_o \subset \Gamma(X, V)$ we have

$$SSV_o \subset \mu(SSV)$$

and the equality holds, provided V_o generates V, i.e. $V = D_X \cdot V_o$.

A proof will be given in the next section.

Remark.* There are certain infinite dimensional varieties related to representation theory of the Virasoro algebra in the same way (see [BeSch] and [BeFe] as the symmetric space G/K (or the Flag manifold) is related to the representation theory of the group G. These are the moduli spaces of algebraic curves with some additional data (namely a number of points of the curve and infinite jets of parameters at these points). Such a moduli space is non-compact but can be shown to have a smooth "compactification" (by "stable" curves) which is regular (with respect to the Virasoro action) in the sense of definition 4.2.1.

A similar picture seems to be true for adelic groups, where moduli spaces of vector bundles on a curve play the role of G/K (see [Lal]). Moduli spaces of that kind are also expected to have a regular compactification.

5. D-modules on a regular compactification

This section is devoted mainly to the proof of theorem 4.3.3. We begin, however, with some general results that might be of independent interest.

5.1. We keep to the notation of n.4.1, so that Z is a smooth variety, Y is a normal crossing divisor in Z and $T_{Z,Y}$ is the sheaf of vector fields on Z that preserve Y. Let $T^*_{Z,Y}$ be the dual sheaf. The sheaf $T^*_{Z,Y}$ is a locally-free sheaf of $\mathcal{O}_Z$-modules (lemma 4.1), hence, the sheaf of sections of an algebraic vector bundle on Z. The vector bundle is denoted $T^*(Z, Y)$ and is called the logarithmic cotangent bundle on Z (with respect to Y), for the 1-forms $t_1^{-1} \cdot dt_1, \ldots, t_k^{-1} \cdot dt_k, dt_{k+1}, \ldots, dt_n$ (notation of the proof of lemma 4.1) provide a local basis for its sheaf of sections.

Let D_Z be the sheaf of algebraic differential operators on Z and let $D_{Z,Y}$ be the subsheaf of the operators that preserve the J_Y-adic filtration on $\mathcal{O}_Z$, i.e.

$$D_{Z,Y} = \{P \in D_Z \mid P \cdot J_Y^n \subset J_Y^n, \quad n = 1, 2, \ldots\}$$

One can show that $D_{Z,Y}$ equals the $\mathcal{O}_Z$-subalgebra of D_Z generated by the sheaf $T_{Z,Y}$.

Let $\mathcal{O}_Z = F_o D_Z \subset F_1 D_Z \subset \ldots$ be the standard "order" filtration on D_Z and $F_i D_{Z,Y} = F_i D_Z \cap D_{Z,Y}$ the induced filtration on $D_{Z,Y}$. Let Gr D_Z and Gr $D_{Z,Y}$ denote the associated graded $\mathcal{O}_Z$-algebras.

<u>Proposition 5.1</u>. (i) $\mathrm{Gr}\ D_Z \cong \pi_\cdot \mathcal{O}_{T^*Z}$;
(ii) $\mathrm{Gr}\ D_{Z,Y} \cong \pi_\cdot \mathcal{O}_{T^*(Z,\ Y)}$

(in either case the symbol π stands for the projection of the vector bundle to the base Z.)

Part (i) of the Proposition is, of course, well known. Part (ii) follows from a local coordinate computation. □

5.2. Let G be an algebraic group acting on a smooth variety Z. Let g be the Lie algebra of G and let $x \cdot f$ denote the Lie derivative of $f \in \mathcal{O}_Z$ along the vector field on Z corresponding to $x \in g$. We endow $\underline{g} = \mathcal{O}_Z \otimes g$, a free $\mathcal{O}_Z$-sheaf, with a Lie algebra structure as follows:

$$[f_1 \otimes x_1,\ f_2 \otimes x_2] = f_1 \cdot f_2 \otimes [x_1, x_2] + f_1 \cdot (x_1 \cdot f_2) \otimes x_2 - f_2 \cdot (x_2 \cdot f_1) \otimes x_1$$

To understand the meaning of this formula it's instructive to put $f_2 = 1$.

Now let Y be a G-stable normal crossing divisor in Z and let $g \longrightarrow \Gamma(Z,\ T_{Z,Y})$ be the natural Lie algebra homomorphism, considered in n.4.2. We extend it, by $\mathcal{O}_Z$-linearity, to a sheaf morphism $\underline{g} \longrightarrow T_{Z,Y}$ (it turns out to be a Lie algebra homomorphism).

Suppose, further, that the G-action on Z is G-regular. Then, the morphism $\underline{g} \longrightarrow T_{Z,Y}$ is surjective and we get an exact sequence of Lie algebra sheaves:

$$0 \longrightarrow \underline{k} \longrightarrow \underline{g} \longrightarrow T_{Z,Y} \longrightarrow 0 \qquad (5.2.1)$$

where $\underline{k} = \ker(\underline{g} \longrightarrow T_{Z,Y})$. The sheaves $\underline{g}$ and $T_{Z,Y}$ being locally-free, the same is true for $\underline{k}$. Hence, the geometric fibres of $\underline{k}$ form a smooth family $\{k_z, z \in Z\}$ of Lie subalgebras of g. We remark, that if $z \in Z \setminus Y$, then (see lemma 4.2.2) k_z is the Lie algebra of the isotropy group of z.

Let $k_z^{\perp}$ denote the annihilator of k_z in g^* and let $\underline{k}^{\perp}$ be the vector bundle on Z with fibres $k_z^{\perp}$. Dualizing (5.2.1) yields

Lemma 5.2.2. Given a G-regular action on Z, one has a G-equivariant vector bundle isomorphism

$$T^*(Z, Y) \cong \underline{k}^{\perp} = \{(\lambda, z) \in g^* \times Z \mid \lambda \in k_z^{\perp}\}$$

Remark 5.2.3.* Let $r = \operatorname{rank} k$ and Gr(g) the Grassmannian of r-dimensional subspaces of g. The assignment $z \mapsto k_z$ gives rise to a G-equivariant map $\gamma : Z \longrightarrow Gr(g)$, a kind of the Gauss map.

Now, let G/K be a homogeneous space and $\gamma : G/K \longrightarrow Gr(g)$ the Gauss map. Let $\overline{G/K}$ denote the closure in Gr(g) of the image of G/K. (If G/K is a symmetric space, then $\overline{G/K}$ is exactly the compactification of proposition 4.4, as explained after the proposition.) If $\overline{G/K}$ is itself a regular compactification of G/K, then it is the minimal regular compactification. That is, for any other regular compactification Z of G/K the Gauss map $\gamma : Z \longrightarrow Gr(g)$ gives rise to a natural proper surjective morphism $Z \longrightarrow \overline{G/K}$ inscribing into the diagram:

$$\begin{array}{ccc} & \nearrow & Z \\ G/K & & \downarrow \\ & \searrow & \overline{G/K} \end{array}$$

5.3. To an arbitrary G-action on Z and a G-stable normal crossing divisor $Y \subset Z$ one can associate a moment map $\mu : T^*(Z, Y) \longrightarrow g^*$. For $z \in Z$, the restriction of μ to the fibre $T_z^*(Z, Y) = T^*_{Z,Y,z}$ is defined as the linear map $\mu : T^*_{Z,Y,z} \longrightarrow g^*$, obtained by dualizing the map $g \longrightarrow T_{Z,Y,z}$ of n.4.2.

Next, consider the standard increasing filtration on U(g), the enveloping algebra of g, and the filtration on $D_{Z,Y}$, introduced in n.5.1. The Lie algebra homomorphism $g \longrightarrow \Gamma(Z, T_{Z,Y})$ can be uniquely extended to a filtration preserving algebra homomorphism $U(g) \longrightarrow \Gamma(Z, D_{Z,Y})$ and, hence, to the associated graded algebra homomorphism $\mathrm{Gr}\, U(g) \longrightarrow \Gamma(Z, \mathrm{Gr}\, D_{Z,Y})$. Recall now that $\mathrm{Gr}\, U(g) = \mathbb{C}[g^*]$ (Poincaré-Birkhoff-Witt theorem) and $\mathrm{Gr}\, D_{Z,Y} = \pi_* \mathcal{O}_{T^*(Z,Y)}$ (Lemma 5.1 (ii)). We conclude that a G-action on Z gives rise to an algebra homomorphism $\mathbb{C}[g^*] \longrightarrow$ regular functions on $T^*(Z, Y)$.

Lemma 5.3.1. The homomorphism $\mathbb{C}[g^*] \longrightarrow \mathcal{O}(T^*(Z, Y))$ is induced by the moment map $\mu : T^*(Z, Y) \longrightarrow g^*$. $\square$

This lemma might be used as an alternative definition of μ.

In case of a G-regular action the moment map μ can be described explicitly, using lemma 5.2.2. Namely, it assigns the point $\lambda \in g^*$ to a pair $(\lambda, z) \in \underline{k}^{\perp}$. As a consequence, we get

Corollary 5.3.2. The moment map μ is proper, provided Z is a compact variety with G-regular action. $\square$

5.4. Given a $D_{Z,Y}$-module M let $SSM \subset T^*(Z, Y)$ denote the characteristic variety of M. The variety SSM is, by definition, the support of the graded $\mathrm{Gr}\, D_{Z,Y}$ $(\cong \pi_* \mathcal{O}_{T^*(Z, Y)})$-module, associated with a good filtration on M.

Proposition 5.4. Let a group G act G-regularly on a compact variety Z with a G-stable normal crossing divisor Y. Then, for a coherent $D_{Z,Y}$-module M, the space $\Gamma(Z, M)$ is a finitely-generated U(g)-module and we have

$$SS\,\Gamma(Z, M) = \mu\,(SSM).$$

In the special case $Y = \emptyset$ the Proposition is due to Borho-Brylinski [BB]. The proof of the general case goes along the same line as follows. Let $M_o \subset M_1 \subset \ldots$ be a good filtration on M and $\Gamma_i(Z, M) := \Gamma(Z, M_i)$ the induced filtration on $\Gamma(Z, M)$. Then, we have

$$\mathrm{Gr}\,\Gamma(Z, M) \subset \Gamma(Z, \mathrm{Gr}\, M) = \Gamma(g^*, \mu_* \mathrm{Gr}\, M)$$

The sheaf $\mu_*(\mathrm{Gr}\, M)$ here is a $\mathcal{O}_{g^*}$-coherent sheaf since μ is a proper morphism (corollary 5.3.2). So, $\Gamma(g^*, \mu_* \mathrm{Gr}\, M)$, and hence $\mathrm{Gr}\,\Gamma(Z, M)$, are finitely-generated $U(g)$-modules. Thus, the above defined filtration on $\Gamma(Z, M)$ is good and we have

$$SS\,\Gamma(Z, M) = \mathrm{supp}\,\mathrm{Gr}\,\Gamma(Z, M) \subset \mathrm{supp}(\mu_*\,\mathrm{Gr}\,M) = \mu(SSM)$$

The proof of the opposite inclusion is easy (cf. the proof of lemma 2.1.2). □

5.5. Proof of theorem 4.3.3. Let $j : X \hookrightarrow Z$ be a regular compactification of X, so that $Y = Z \setminus X$ is a normal crossing divisor. Let $D_{Z,Y}\cdot V_o$ be the $D_{Z,Y}$-submodule of j_*V generated by V_o. It is clear that $V_o \subset \Gamma(Z, D_{Z,Y}\cdot V_o)$. Hence, Proposition 5.4 yields:

$$SSV_o \subset SS\,\Gamma(Z, D_{Z,Y}\cdot V_o) = \mu(SS(D_{Z,Y}\cdot V_o)) \qquad (5.5.1)$$

Applying now theorem A1.2 (of Appendix A) to $M = D_{Z,Y}\cdot V_o$, we obtain

$$\mu(SS(D_{Z,Y}\cdot V_o)) \subset \mu(\overline{SSV}) \subset \overline{\mu(SSV)} \qquad (5.5.2)$$

Combining (5.5.1) with (5.5.2), we get $SSV_o \subset \overline{\mu(SSV)}$, the estimate of the theorem.

If V_o generates V, then one can show, as in the proof of lemma 2.1.2, that $SSV \subset \mu^{-1}(SSV_o)$. Whence, the opposite inclusion: $\mu(SSV) \subset SSV_o$. So, the equality holds. □

5.6.* We conclude this section with a few general remarks concerning the algebra $D_{Z,Y}$ for a smooth variety Z with G-regular action.

Set $\underline{U(g)} = U(g) \otimes \mathcal{O}_Z$. We endow the sheaf $\underline{U(g)}$ with an algebra structure as follows ($x_1, x_2 \in g$, $f_1, f_2 \in \mathcal{O}_Z$):

$$x_1 \otimes f_1 \cdot x_2 \otimes f_2 = (x_1\cdot x_2) \otimes (f_1\cdot f_2) + x_1 \otimes (x_2\cdot f_1)\cdot f_2$$

The algebra $\underline{U(g)}$ is a sheaf-theoretic version of enveloping algebra of the Lie algebra sheaf $\underline{g}$ introduced in n.5.2. So, there is a natural morphism $\underline{g} \to \underline{U(g)}$. Furthermore, the Lie algebra map $\underline{g} \to T_{Z,Y}$ extends naturally to an algebra homomorphism $\underline{U(g)} \to D_{Z,Y}$. Now, the G-action on Z being G-regular, the Lie algebra exact sequence (5.2.1) yields the following exact sequence of algebras:

$$0 \longrightarrow \underline{k}\cdot\underline{U(g)} \longrightarrow \underline{U(g)} \longrightarrow D_{Z,Y} \longrightarrow 0 \qquad (5.6.1)$$

Here $\underline{k}\cdot\underline{U(g)}$ denotes the right ideal of $\underline{U(g)}$ generated by $\underline{k}$. We note that $\underline{k} = \ker(\underline{g} \longrightarrow T_{Z,Y})$ is an ideal of $\underline{g}$, so that $\underline{k}\cdot\underline{U(g)}$ is actually a two-sided ideal. Thus, we obtain the following

Proposition 5.6.2. For a variety Z with a G-regular action we have an algebra isomorphism

$$D_{Z,Y} \cong \underline{k}\cdot\underline{U(g)} \setminus \underline{U(g)}. \qquad \square$$

6. Complete Symmetric variety

6.1. Let G be a connected complex semisimple Lie group of adjoint type and let $X = G/K$, $K = G^{\theta}$ be the complex symmetric space associated with an involution θ on G. In this section we will prove the following result, used in the proof of theorem 1.4.2.

Proposition 6.1. The G-space $X = G/K$ has a regular compactification $\overline{X}$.

The compactification $\overline{X}$ was introduced by DeConcini and Procesi [DP]. The most simple way to define $\overline{X}$ is as follows. Let k be the Lie algebra of K $(= G^{\theta})$ and $r = \dim k$. Let Gr(g) be the Grassmann manifold of r-dimensional subspaces of g and let $x_o \in Gr(g)$ be the point corresponding to the subalgebra $k \subset g$. The adjoint G-action on g gives rise to a natural action of G on Gr(g). The isotropy subgroup of the point x_o contains the group K and it is, actually, equal to K, provided G is adjoint. Hence, the orbit $G\cdot x_o$ is isomorphic to $X = G/K$. The compactification $\overline{X}$ is defined to be the closure in Gr(g) of the orbit $G\cdot x_o$.

6.2. Proof of Proposition 6.1 is based on another realization of $\overline{X}$ which is also due to [DP]. Let E be an irreducible finite-dimensional representation of G with a non-trivial subspace E^K of K-fixed vectors. Then, it is well-known, that $\dim E^K = 1$. Let $\mathbb{P}(E)$

denote the projective space of 1-dimensional subspaces of E and let x_o be the point of $\mathbb{P}(E)$ corresponding to the subspace E^K. The group G acts on $\mathbb{P}(E)$ and one can prove that the isotropy group of x_o equals K, provided the group G is adjoint and the representation E is sufficiently generic. In that case the closure in $\mathbb{P}(E)$ of the orbit $G\cdot x_o$ turns out to be isomorphic to $\bar{X}$.

To prove the Proposition we pick up a maximal "split" torus $A \subset G$ and set $L = Z_G(A)$. Let $L\cdot U$ be a parabolic subgroup of G such that $U \cap \Theta(U) = \{1\}$ and let x_1 be the unique point in $\mathbb{P}(E)$ fixed by the group $L\cdot\Theta(U)$. The point x_1 has a standard affine neighborhood $V \subset \mathbb{P}(E)$, defined as the set of all points $y \in \mathbb{P}(E)$ such that $\lim_{a\to\infty} (\exp a)\cdot y = x_1$, when $a \in \operatorname{Lie} A$ approaches ∞ inside the Weyl chamber corresponding to $\Theta(U)$. One shows [DP] that $V \cap \bar{X}$ is an U-stable Zariski-open part of $\bar{X}$, containing x_o.

Let $S = \overline{A\cdot x_o}$ be the closure in $V \cap \bar{X}$ of the A-orbit through x_o. DeConcini and Procesi have proved the following:

(i) S is a smooth toroidal variety ($\cong \mathbb{C}^n$, $n = \dim A$);

(ii) The action of U gives an isomorphism $U \times S \xrightarrow{\sim} V \cap \bar{X}$;

(iii) any point of $\bar{X}$ is contained in $g\cdot(V \cap \bar{X})$, the translation of $V \cap \bar{X}$ by an appropriate $g \in G$.

The properties (ii), (iii) show that $\bar{X}$ is locally-isomorphic to $U \times S$. Hence, $\bar{X}$ is smooth. Moreover, the local G-orbit structure of $\bar{X}$ is essentially the same (up to the factor U) as the A-orbit structure of S, a toroidal variety. It follows, that $V \cap (\bar{X} - X) \cong U \times (S - A\cdot x_o)$ is a normal crossing divisor [DP] and that the G-action on $\bar{X}$ is G-regular. □

6.3.* It was shown in [DP] that $Q := G\cdot x_1$ is the unique closed G-orbit in $\bar{X}$. Let $T_Q X$ denote the normal cone at Q of the open G-orbit $G/K = X \subset \bar{X}$. Then, one can prove the following (where $L = Z_G(A)$ and $M = Z_K(A)$):

Proposition 6.3.1. The natural fibration $T_Q X \longrightarrow Q$ is G-equivariantly isomorphic to the projection $G/M \cdot U \longrightarrow G/L \cdot U$. □

This proposition might be helpful in trying to find a geometric construction of the Jacquet functor [CasColl] in terms of the Verdier specialization.

7. A saturation theorem for Harish-Chandra modules

The saturation theorem stated below relates the characteristic variety of a Harish-Chandra module with the characteristic variety of its annihilator. This result has no connection to the main subject of the paper and can be omitted without trouble. We begin, however, by recalling the Beilinson-Bernstein theory that will be indispensable in the rest of the paper.

7.1. Let G be a complex connected reductive Lie group, B a Borel subgroup of G, U the unipotent radical of B and T a maximal torus of B, so that $B = T \cdot U$. Set $Y = G/U$. The group T normalizes U, so that there is a right T-action on Y that commutes with the natural left G-action and makes the projection $\pi : Y = G/U \longrightarrow G/B$ into a principal T-bundle.

Let $t = \text{Lie } T$. A coherent D_Y-module M is said to be admissible if it is smooth along the fibres of π, i.e. if the action of t on $\pi_\bullet M$, arising from the right T-action on Y, is locally-finite.

Let t^* be the dual of t, $T^* := t^*/X(T)$ (see 1.3.1), and $\bar{\lambda} \in T^*$. We say that an admissible D_Y-module M has monodromy $\bar{\lambda}$ if all the eigenvalues of t, acting on $\pi_\bullet M$, belong to the coset $\bar{\lambda} \bmod X(T)$.

Let $g = \text{Lie } G$. Given $\lambda \in t^*$, let I_λ denote the maximal ideal of $Z(g)$, the center of the enveloping algebra, corresponding to λ via the Harish-Chandra homomorphism.

Theorem 7.1.1 [BeBe]. Let $\lambda \in t^*$ be dominant and regular and let $\bar{\lambda}$ be its image in T^*. Then, the category of finitely-generated $U(g)$-modules annihilated by some power I_λ^n is equivalent to the category of admissible D_Y-modyles with monodromy $\bar{\lambda}$.

Remarks 7.1.2. (a) The equivalence of the theorem assigns to a D_Y-module M the U(g)-module $\Gamma(Y, M)_\lambda$. Here $\Gamma(Y, \cdot)$ denotes the global sections functor and the subscript λ denotes taking the λ-isotypical component of the root-space decomposition with respect to the (locally-finite) t-action on $\Gamma(Y, M)$ arising from the right T-action on Y.

(b) The left G-action and the right T-action on Y give rise to an algebra homomorphism $U(g) \otimes U(t) \longrightarrow D(Y)$. The image of this homomorphism clearly consists T-right-invariant operators and one can prove [BB2] that

$$D(Y)^T \cong U(g) \otimes_{Z(g)} U(t) \qquad (7.1.3)$$

where Z(g) is viewed as a subalgebra of U(t) via the Harish-Chandra homomorphism, and $D(Y)^T$ denotes the ring of right T-invariant differential operators.

(c) It follows from (7.1.3) that the algebra $D(Y)^T$ is a Galois extension of U(g) with the Galois group being isomorphic to W, the Weyl group of (g, t). The extension is clearly unramified over regular points of Specm Z(g). Thus, given regular dominant $\lambda \in t^*$ and an U(g)-module M, annihilated by some power of I_λ, there is a unique way to regard M as a $D(Y)^T$-module, by requiring that all the eigenvalues of U(t), the second factor in (7.1.3), on M are equal to λ.

Using this convention one can describe the equivalence of theorem 7.1.1 in the direction opposite to that of remark (a). It assigns to a U(g)-module M the D_Y-module $D_Y \otimes_{D(y)^T} M$.

7.2. Now let (G, θ) be a symmetric pair and $K = G^\theta$. By a (g, K)-module we mean, as usual, a finitely-generated U(g)-module M with an algebraic K-action which is compatible with the g-module structure in the following sense:

(i) The differential of the K-action coincides with restriction of the g-action to the subalgebra $k = \text{Lie } K$;

(ii) $(\text{Ad } h \cdot x)\cdot m = h\cdot(x\cdot(h^{-1}\cdot m))$, $h \in K$, $x \in g$, $m \in M$.

Any (g, K)-module is obviously a (g, k)-module in the sense of §2.

7.3.* Let Ann M denote the annihilator in U(g) of a (g, K)-module M. We suppose that $\text{Ann } M \supset I_\lambda^n$, $n \gg 0$. If M and U(g)/Ann M are viewed as $D(Y)^T$-modules, as explained in remark 7.1.2 (c), then $U(g)/\text{Ann } M = D(Y)^T/J$, where J is the annihilator of M in $D(Y)^T$. So, the D_Y-modules corresponding to M and U(g)/Ann M, are

$$D_Y \otimes_{D(Y)^T} M \quad \text{and} \quad D_Y \otimes_{D(Y)^T} (U(g)/\text{Ann } M) \cong D_Y/D_Y\cdot J$$

We have the following saturation theorem:

Theorem 7.3.1. $SS(D_Y/D_Y\cdot J) = \overline{G\cdot SS(D_Y \otimes_{D(Y)^T} M)}$,

where the RHS stands for the closure of the G-saturation of the characteristic variety of $D_Y \otimes M$. □

Corollary 7.3.2. $SS(U(g)/\text{Ann } M) = \overline{G\cdot SSM}$.

Corollary 7.3.3. Let M be a simple (g, K)-module. Then, G·SSM is closure of a single nilpotent orbit in g^*.

Corollary 7.3.2 follows from [BB1] (cf. also [Gi3, prop. 8.2]) and theorem 7.3.1. Corollary 7.3.3 follows from 7.3.2 and from the irreducibility of the characteristic variety of a primitive ideal (see e.g. [BB1], [Gi3]).

In the diagonal case (see n.1.1) theorem 7.3.1 has been established in [BB3] and in [Gi3]. The proof of theorem 7.3.1 is based on a relative version of theorem 4.3.3 and is similar to that of [Gi3, prop. 8.2]. We omit the details.

8. Harish-Chandra functor

8.1. Let an algebraic group A act on an algebraic variety Y. Following [Lu2] consider the diagram:

$$A \xleftarrow{\;p\;} A \times Y \xrightarrow{\;q\;} Y \times Y \qquad (8.1.1)$$

where p denotes the first projection and the map q is defined by $q(a, y) = (a \cdot y, y)$.

For a variety Z, let $D^b(Z)$ denote the bounded derived category of constructible complexes on Z (or of complexes of D-modules with holonomic cohomology). We define a <u>Harish-Chandra</u> <u>functor</u> $\widetilde{HC}: D^b(A) \longrightarrow D^b(Y \times Y)$ and a <u>Character</u> <u>functor</u> $\widetilde{CH} : D^b(Y \times Y) \longrightarrow D^b(A)$ as follows (see the diagram (8.1.1)):

$$\widetilde{HC} = q_* \cdot p^! \qquad \text{and} \qquad \widetilde{CH} = p_! \cdot q^* \tag{8.1.2}$$

<u>Proposition 8.1.3</u>. The functor $\widetilde{CH}$ is the left adjoint of $\widetilde{HC}$.

<u>Proof</u>. For $V \in D^b(A)$ and $M \in D^b(Y \times Y)$ we have

$$\mathrm{Hom}(M, \widetilde{HC}(V)) = \mathrm{Hom}(M, q_* \cdot p^! V) = \mathrm{Hom}(q^* M, p^! V) =$$

$$= \mathrm{Hom}(p_! \cdot q^* M, V) = \mathrm{Hom}(\widetilde{CH}(M), V). \quad \square$$

8.2. In the setup of n.8.1 we define a convolution structure on $D^b(A)$, and a similar one on $D^b(Y \times Y)$, as follows. Let $m : A \times A \longrightarrow A$ denote the multiplication map. Given $V_1, V_2 \in D^b(A)$, we set

$$V_1 * V_2 = m_*(V_1 \boxtimes V_2) \in D^b(A)$$

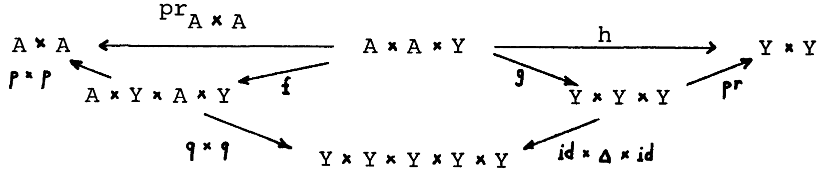
Now, let $\Delta : Y \hookrightarrow Y \times Y$ denote the diagonal embedding and pr: $Y \times Y \times Y \longrightarrow Y \times Y$ the projection along the middle factor. Given $M, M' \in D^b(Y \times Y)$, we set

$$M * M' = pr_*((id_Y \times \Delta \times id_Y)^! (M \boxtimes M')) \in D^b(Y \times Y)$$

The convolution structures on $D^b(A)$ and on $D^b(Y \times Y)$ satisfy the natural associativity law and we have

<u>Proposition 8.2.1</u>. The Harish-Chandra functor $\widetilde{HC} : D^b(A) \longrightarrow D^b(Y \times Y)$ is a "homomorphism", i.e. $\widetilde{HC}(V_1 * V_2) = \widetilde{HC}(V_1) * \widetilde{HC}(V_2)$.

<u>Proof</u>. Consider the following commutative diagram:



where the maps f, g and h are defined as follows: $f(a_1, a_2, y) = (a_1, a_2\cdot y, a_2, y)$; $g(a_1, a_2, y) = (a_1\cdot a_2\cdot y, a_2\cdot y, y)$ and $h(a_1, a_2, y) = (a_1\cdot a_2\cdot y, y)$.

The definitions of the maps in the top row of the diagram yield: $\widetilde{HC}(V_1 * V_2) = h_* \cdot pr^!_{A\times A}(V_1 \boxtimes V_2)$.

On the other hand, the complex $\widetilde{HC}(V_1) * \widetilde{HC}(V_2)$ is obtained by going along the two other sides of the big triangle, that is

$$\widetilde{HC}(V_1) * \widetilde{HC}(V_2) =$$
$$= pr_* \circ (id_Y \times \Delta \times id_Y)^! \circ (q\times q)_* \circ (p\times p)^! (V_1 \boxtimes V_1) \qquad (8.2.2)$$

Now, the definitions of the maps f and g show, that the square with the vertex $A\times A\times Y$ at the top and the vertex $Y\times Y\times Y\times Y$ at the bottom is a Cartesian square. Hence, the base change theorem yields $(id_Y \times \Delta \times id_Y)^! \cdot (q\times q)_* = g_* \cdot f^!$. So, the right-hand side of (8.2.2) can be rewritten as

$$pr_* \cdot g_* \cdot f^! \cdot (p\times p)^! (V_1 \boxtimes V_2) = h_* \cdot pr^!_{A\times A}(V_1 \boxtimes V_2) = \widetilde{HC}(V_1 \boxtimes V_2)$$

Q.E.D.

8.3. We now specify the choice of A and Y. Let A = G be a connected complex reductive Lie group. Let B be a Borel subgroup of G, U the unipotent radical of B and T a maximal torus of B, so that $B = T\cdot U$. We set $Y = G/U$, a homogeneous G-variety.

Let us clarify the meaning of the functors $\widetilde{HC}$ and $\widetilde{CH}$ in our present setting. First, recall that to a constructible complex M one can associate a constructible function $\chi(M, \cdot)$, whose value at a point z equals

$$\chi(M, z) = \sum (-1)^i \cdot \dim \underline{H}^i M\,|_z$$

The definition of the functors $\widetilde{HC}$ and $\widetilde{CH}$ can be carried over to the case of constructible functions. So, given constructible functions f_1 and f_2 on Y and applying the character functor to the function $f_1 \boxtimes f_2$ on $Y\times Y$, we get

$$\widetilde{CH}(f_1 \boxtimes f_2)(u) = \int_Y f_1(u\cdot y)\cdot f_2(y)\,dy \qquad u \in G \tag{8.3.1}$$

where $\int$ denotes the direct image of a constructible function to the point. This expression looks like a "matrix coefficient", provided the functions f_1, f_2 are viewed as "vectors" of a principal series representation.

To give a similar meaning to the functor $\widetilde{HC}$ we observe the following trivial

Lemma 8.3.2. The map $q : G \times Y \longrightarrow Y \times Y$ is a smooth submersion and its fibre over a point $(y_1\cdot U,\ y_2\cdot U)$, $y_1, y_2 \in G$ equals $y_1\cdot U\cdot y_2^{-1} \times y_2\cdot U$. □

Hence, for a constructible function f on G we can write

$$\widetilde{HC}(f)(y_1\cdot U,\ y_2\cdot U) = \int_U f(y_1\cdot u\cdot y_2^{-1})\,du$$

This formula explains the name of the functor $\widetilde{HC}$.

8.4. Recall (see n.7.1), that there is a well-defined T-action on Y "on the right", that commutes with the left G-action. Let $(Y \times Y)/T$ denote the quotient of $Y \times Y$ modulo the diagonal T-action on the right.

It is convenient for many purposes to modify the diagram (8.1.1) slightly, replacing it by a more economical diagram

$$G \xleftarrow{\ p\ } G \times (Y/T) \xrightarrow{\ q\ } (Y \times Y)/T \tag{8.4.1}$$

(the "new" map q here is obtained by dividing both ends of the arrow $G \times Y \longrightarrow Y \times Y$ in (8.1.1) by the right T-action). Using the diagram (8.4.1) we now define the modified functors

$$HC : D^b(G) \longrightarrow D^b((Y \times Y)/T) \quad \text{and} \quad CH : D^b((Y \times Y)/T) \longrightarrow D^b(G)$$

by similar formulas $HC = q_*\cdot p^!$ and $CH = p_! \cdot q^*$.

The functor CH is the left adjoint of HC (an analogue of proposition 8.1.3). Furthermore, there is a natural convolution structure on

$D^b((Y \times Y)/T)$ and an analogue of proposition 8.2.1 holds. To define the convolution structure, one should identify the category $D^b((Y \times Y)/T)$ with the category of complexes on $Y \times Y$ that come from $(Y \times Y)/T$, i.e. with the category of T-equivariant complexes on $Y \times Y$ (cf. Appendix B). The ordinary formula $V_1 * V_2 = m_*(V_1 \boxtimes V_2)$ can be applied in this equivariant setting to define the convolution.

8.5. It is, of course, interesting to know to what extent a complex V on G can be reconstructed from its Harish-Chandra transform HC(V). For that purpose we consider the composition $CH \circ HC : D^b(G) \longrightarrow D^b(G)$.

Theorem 8.5.1. The identity functor $Id_{D^b(G)}$ is a direct summand of the functor $CH \circ HC$.

Corollary 8.5.2. The Harish-Chandra functor is fully faithful, i.e. (i) $V \neq 0 \Longrightarrow HC(V) \neq 0$ and

(ii) the natural homomorphism $Hom(V_1, V_2) \longrightarrow Hom(HC(V_1), HC(V_2))$ is injective. □

Remark. Theorem 8.5.1 provides, in effect, the canonical splitting of the natural functor morphism $CH \circ HC \longrightarrow id_{D^b(G)}$ that arises due to the adjointness of CH and HC (prop. 8.1.3). □

Recall that $Y/T = G/B$ is the Flag manifold associated to the group G. Set

$$Z = \left\{ (u, y) \in G \times (Y/T) \;\middle|\; u \in y \cdot U \cdot y^{-1} \right\} \tag{8.5.3}$$

The second projection $pr_{Y/T} : Z \longrightarrow Y/T$ is a fibration with fibre U. The first projection $pr_G : Z \longrightarrow G$ is the so-called "Springer resolution" of the unipotent variety of G. Let $\underline{Spr} = (pr_G)_* \underline{\mathbb{C}}_Z[\dim Z]$ be the direct image to G of the constant sheaf on Z, shifted by dim Z. The proof of theorem 8.5.1 is based on the following

Lemma 8.5.4. For $V \in D^b(G)$ we have

$$CH \circ HC(V) = V * \underline{Spr}$$

<u>Remark 8.5.5</u>. Let $\underline{\mathbb{C}}_e$ denote the skyscraper sheaf at the identity point of G. Lemma 8.5.4, applied to $V = \underline{\mathbb{C}}_e$, yields <u>Spr</u> $= CH \circ HC(\underline{\mathbb{C}}_e)$.

<u>Proof of lemma 8.5.4</u>. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & G \times (Y/T) & \xleftarrow{\ id_G \times pr_{Y/T}\ } & G \times Z & \xrightarrow{\ id_G \times pr_G\ } & G \times G \\
 {\scriptstyle p}\swarrow & \downarrow{\scriptstyle q} & \textcircled{1} & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle m} \\
 G & (Y \times Y)/T & \xleftarrow{\ q\ } & G \times (Y/T) & \xrightarrow{\ p\ } & G
\end{array}$$

In the diagram, the variety Z is defined by (8.5.3) and the map h is defined by $h(g_1, g, y) = (g_1 \cdot g, y)$, $g_1 \in G$, $(g, y) \in Z$.

The key point is that $(g, y) \in Z$ iff $g \cdot y = y$ (cf. lemma 8.3.2). It follows easily, that the square ① of the diagram is a Cartesian square. Moreover, the map q is a smooth morphism with (dim U)-dimensional fibre. Hence,

$$q^* \cdot q_* = q^! \cdot q_* [-2 \cdot \dim U] = h_* \cdot (id_G \times pr_{Y/T})^! [-2 \cdot \dim U]$$

So, for a complex V on the group G (in the left corner of the diagram) we have (taking into account that $p_! = p_*$, for p is proper):

$$CH \circ HC(V) = p_! \cdot q^* \cdot q_* \cdot p^!(V) = p_* \cdot h_* \cdot (id_G \times pr_{Y/T})^! \cdot p^!(V)[-2 \cdot \dim U] =$$

$$= p_* \cdot h_* (V \boxtimes \underline{\mathbb{C}}_Z[\dim Z]) = m_* \cdot (id_G \times pr_G)_* (V \boxtimes \underline{\mathbb{C}}_Z[\dim Z]) =$$

$$= m_*(V \boxtimes \underline{Spr}) = V * \underline{Spr}.$$

Q.E.D.

<u>Proof of theorem 8.5.1</u>. It is known [BM], that the complex <u>Spr</u> is a semisimple perverse sheaf on G and that the skyscraper sheaf $\underline{\mathbb{C}}_e$ is a direct summand of <u>Spr</u> (occurring with multiplicity one). Hence, the functor $V \longmapsto V * \underline{\mathbb{C}}_e$ is a direct summand of the functor $V \longmapsto V * \underline{Spr} = CH \circ HC(V)$. The trivial observation: $V * \underline{\mathbb{C}}_e = V$ completes the proof. □

8.7. In this n^o we'll establish connection between admissible modules on a symmetric space G/K and the Harish-Chandra functor. So, it will be convenient for us here to adopt the language of D-modules.

The right T-action on Y gives rise to a right action on $(Y \times Y)/T$ of the group $(T \times T)/T \cong T$, commuting with the natural left $G \times G$-action. So, given a subgroup $H \subset G \times G$, one has an $H \times T$-action on $(Y \times Y)/T$. Set $h = \text{Lie } H$, $t = \text{Lie } T$. An $H \times T$-monodromic (cf. Appendix B) D-module M on $(Y \times Y)/T$ will be referred to as an <u>h-admissible</u> module. If, in addition, M is H-equivariant (see n. B2), it will be referred to as an <u>H-admissible</u> module. Let (cf. Appendix B)

$$D^b \text{ Admiss}(*, h) := D^b (*, h \times t), \qquad D^b \text{ Admiss}(*, H) := D^b(*, H \times t)$$

denote the corresponding "derived" categories (here $*$ stands for $(Y \times Y)/T$ and the symbol $H \times t$ in the last expression signifies $H \times T$-monodromicity + H-equivariance). In the equivariant case one may understand $D^b(*, H \times t)$ either as the "right" equivariant derived category, defined by Bernstein-Lunts (see [MiVi]) and Beilinson-Ginsburg (see [Gi3]), or as the "naive" equivariant derived category in the sense of n. B2. The "naive" definition gives "wrong" equivariant Ext-groups and is therefore inadequate in general. It's sufficient however for the limited purposes of this paper since we do not care about Ext's here and are always dealing with individual objects ((9.2.5) is the only exception, for the category as a whole is involved there).

We turn now to D-modules on the group G. We let a subgroup $H \subset G \times G$ act on G by $H \ni (h_1, h_2) : x \mapsto h_1 \cdot x \cdot h_2^{-1}$. A D(G)-module is said to be h-admissible if it is locally-finite as a Z(g)-module and as a U(h)-module. An h-admissible module is said to be H-admissible if it is equipped with an algebraic H-action, compatible with that of h. Any h-admissible (resp. H-admissible) D-module is H-monodromic (resp. H-equivariant). Let D^b Admiss(G, h) and D^b Admiss(G, H) denote the corresponding "derived" categories in the sense of n.B1-B2.

Proposition 8.7.1. The functors HC and CH give rise to the following functors:

$$D^b \text{ Admiss}(G, h) \underset{HC}{\overset{CH}{\rightleftarrows}} D^b \text{ Admiss}((Y \times Y)/T, h)$$

and to similar functors in the equivariant setting.

These functors preserve central characters.

Proof. Define an H-action on $G \times (Y/T)$ as follows: $H \ni (h_1, h_2) : (u, y) \mapsto (h_1 \cdot u \cdot h_2^{-1}, \ h_2 \cdot y)$. Then, the maps p and q in the diagram (8.4.1) become H-equivariant. Hence, it follows from results of Appendix B that the functors HC and CH carry H-monodromic (resp. H-equivariant) objects into H-monodromic (resp. equivariant) ones.

Proofs of the statements about admissibility and central characters are postponed until n. 10.2. □

8.8. Let $K = G^\Theta$ be the subgroup associated to a symmetric pair (G, Θ) and $k = \text{Lie } K$. Let M and N be k-admissible irreducible D_Y-modules. Then, the group $\pi_1(T)$ acts on M and on N by one-dimensional monodromy representations (corresponding to the principal T-bundle $Y \twoheadrightarrow Y/T$). If these monodromy representations on M and N are opposite to each other, then the module $M \times N$ on $Y \times Y$ is the pullback of a D-module on $(Y \times Y)/T$, that will be also denoted by $M \times N$. The complex $CH(M \times N)$ is a well-defined $k \times k$-admissible complex on G, by Proposition 8.7.1, applied to $H = K \times K$. Let $CH^i(M \times N)$ denote its i-th cohomology module.

Proposition 8.8.1. (i) Any irreducible $k \times k$-admissible D(G)-module V is a subquotient of $CH^i(M \times N)$ for some irreducible k-admissible D_Y-modules M, N with opposite monodromies and an integer $i \geq 0$;

(ii) If V is $K \times K$-admissible, then M, N can be chosen to be K-admissible D_Y-modules.

Proof. By theorem 8.5.1 we know that V is a direct summand of

$CH \circ HC(V)$. The standard spectral sequence for composition of functors shows that V is a subquotient of $CH^i \circ HC^j(V)$.

We observe further, that $HC^j(V)$ is a $k \times k$-admissible D-module on $(Y \times Y)/T$ (proposition 8.7.1). Hence, $HC^j(V)$ is a holonomic module with finite Jordan-Halder series. It follows easily, that V is actually a subquotient of $CH^i(L)$ for an irreducible $k \times k$-admissible module L. But the $K \times K$-orbit structure of $Y \times Y$ is the product of K-orbit structures on the factors. So, any irreducible $k \times k$-admissible D-module on $(Y \times Y)/T$ is of the form $L = M \times N$, where M and N are irreducible k-admissible D(Y)-modules. The statement (i) follows. Part (ii) is proved in a similar way, using proposition 8.7.1 (ii). □

Corollary 8.8.2. Any $k \times k$-admissible G-module is a regular holonomic module.

Proof follows from a version of the Beilinson-Bernstein result (cf. [BeBe]), saying that any $k \times k$-admissible D-module on $(Y \times Y)/T$ is regular holonomic. □

8.9. Let $pr : G \to G/K$ be the natural projection. Given an admissible D(G/K)-module V, let $pr^* V = \mathbb{C}[G] \otimes_{\mathbb{C}[G/K]} V$ be its pull-back to G. One can show, that $pr^* V$ is a $k \times k$-admissible D(G)-module. Moreover, one obtains in this way an equivalence of categories of admissible D(G/K)-modules and of $k \times K$-admissible D(G)-modules.

We are now able to complete the proof of theorem 1.4.2 by showing that

Corollary 8.9.1. Any admissible D(G/K)-module is regular.

Proof. It is clear that a D(G/K)-module V is regular iff so is $pr^* V$. It remains to apply corollary 8.8.2. □

8.10. We will use monodromy on $(Y \times Y)/T$ to prove the following

Proposition 8.10.1. Let V_1 and V_2 be $k \times k$-admissible D(G)-modules which have central characters (cf. n.1.3). If the central characters are different, then $Ext^*_{D(G)}(V_1, V_2) = 0$.

Proof. Let M_1, M_2 be admissible D-modules on $(Y \times Y)/T$ with different central characters. Then, they have different monodromy eigenvalues (with respect to the right T-action). It follows easily that $Ext^*(M_1, M_2) = 0$, where Ext is computed in the category of D-modules. A spectral sequence argument shows that a similar vanishing statement holds for complexes that have different central characters on cohomology.

Now, let V_1, V_2 be $k \times k$-admissible $D(G)$-modules with different central characters. Then, $HC(V_1)$ and $HC(V_2)$ have different central characters by proposition 8.7.1. Hence, $Ext^*(HC(V_1), HC(V_2)) = 0$. Corollary 8.5.2 (ii) completes the proof. □

9. Diagonal case (character sheaves)

In this section and the one that follows we study the diagonal case, that is the case of a reductive group G acting on itself by left and right translation (from now on we change the notation, as compared with that of example (i) of n.1.1, so that the group in question will be denoted by G and not by K).

9.1. Harish-Chandra functor in the diagonal case. We keep the notation of n.7.1 so that $B = T \cdot U$, $Y = G/U$, etc. Let D^b Admiss(G) (resp. D^b Admiss($(Y \times Y)/T$)) be the category formed by Ad G-equivariant (resp. G-equivariant with respect to the diagonal G-action on $(Y \times Y)/T$) complexes of D-modules with admissible cohomology in the sense of n.1.2 (resp. n.8.7).

We define functors HC and CH as it has been done in n.8.4, using the diagram (8.4.1). It follows from lemma 8.7.1 (i) (applied to $K = G_\Delta$) that the functors give rise to the following ones:

$$D^b \text{ Admiss}(G) \underset{HC}{\overset{CH}{\rightleftarrows}} D^b \text{ Admiss}((Y \times Y)/T) \qquad (9.1.1)$$

The only non-obvious point in proving (9.1.1) is to show that the functors HC and CH preserve the property of being $Z(g)$-locally-finite module. This can be done in the same way as in the proof of proposition 8.7.2 (see n. 10.3 below).

Proposition 9.1.2. Any irreducible admissible $D(G)$-module is a subquotient of a cohomology group of $CH(M)$, where M is a G-admissible module on $(Y \times Y)/T$.

Proof follows from theorem 8.5.1 and an argument similar to that, used in the proof of proposition 8.8.1. □

Theorem 1.6.1 is a simple consequence of proposition 9.1.2. In fact, Lusztig has defined the character sheaves as simple subquotients of complexes CH(M) for a certain finite collection of modules M. However, the modules M, used by Lusztig, contain as a subquotient every irreducible G-admissible D-module on $(Y \times Y)/T$. So, the class of modules on G, defined by Lusztig, is the same as that arising from proposition 9.1.2.

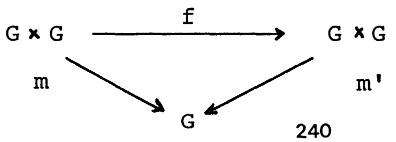
9.2. Relation to Hecke algebra. Recall the convolution structures on $D^b(G)$ and on $D^b((Y \times Y)/T)$ defined in n.8.2. We claim that the categories D^b Admiss$((Y \times Y)/T)$ and D^b Admiss(G) are stable under the convolution. The only problem is to show that the property of "being Z(g)-locally-finite" is preserved by the convolution. For D-modules on $(Y \times Y)/T$, this property is equivalent to "being T-monodromic" (with respect to the right T-action) which is clearly compatible with convolution. To deal with D-modules on G we use the Harish-Chandra functor. Let $V_1, V_2 \in D^b$ Admiss(G). Then, $HC(V_1)$, $HC(V_2) \in D^b$ Admiss$((Y \times Y)/T)$. Hence, $HC(V_1) * HC(V_2) \in D^b$ Admiss$((Y \times Y)/T)$, so that $CH(HC(V_1) * HC(V_2)) \in D^b$ Admiss(G). But we know, that $V_1 * V_2$ is a direct summand of $CH \circ HC(V_1 * V_2) = CH(HC(V_1) * HC(V_2))$. Whence, $V_1 * V_2$ D^b Admiss(G). □

An object M of a category with convolution structure is said to be central if $M * N = N * M$ for any other object N of the category. The category is said to be commutative if all its objects are central.

Proposition 9.2.1. (i) The category D^b Admiss(G) is commutative;

(ii) For any $V \in D^b$ Admiss(G), HC(V) is a central object of D^b Admiss$((Y * Y)/T)$.

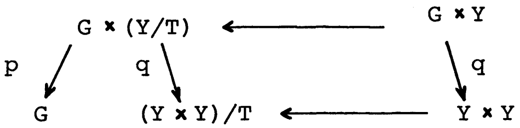
Proof. (i) Consider the commutative diagram:



where the maps m, m' and f are defined as follows: $m(u_1, u_2) = u_1 \cdot u_2$; $m'(u_1, u_2) = u_2 \cdot u_1$; and $f(u_1, u_2) = (u_1, u_1 \cdot u_2 \cdot u_1^{-1})$. It is clear, that for $V_1, V_2 \in D^b$ Admiss(G) we have $V_1 * V_2 = m_*(V_1 \boxtimes V_2)$ and $V_2 * V_1 = m'(V_1 \boxtimes V_2)$. On the other hand, the Ad G-equivariance of V_2 yields $f_*(V_1 \boxtimes V_2) = V_1 \boxtimes V_2$. Hence, we get

$$V_1 * V_2 = m_*(V_1 \boxtimes V_2) = m'_* \cdot f_*(V_1 \boxtimes V_2) = m'_*(V_1 \boxtimes V_2) = V_2 * V_1$$

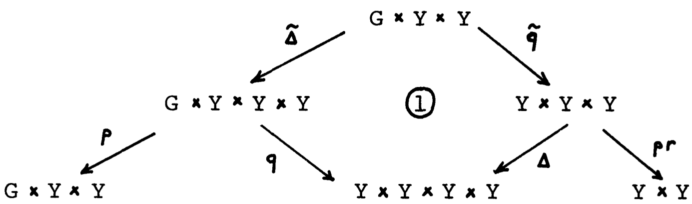
(ii) Consider the following diagram



(9.2.2)

where horisontal arrows are the natural projections. Next, recall, that the convolution of complexes on $(Y \times Y)/T$ is defined in terms of their pull-backs on $Y \times Y$. The diagram (9.2.2) shows, that in a convolution computation, the complex HC(V) should be viewed as a complex on $Y \times Y$ and the Harish-Chandra functor should be viewed as being defined by the diagram (8.1.1) (for A = G) and not by the modified diagram (8.4.1). So, the varieties $(Y \times Y)/T$ and Y/T are replaced by $Y \times Y$ and Y respectively.

Now let $V \in D^b$ Admiss(G) and $M \in D^b$ Admiss$(Y \times Y)$. To compute $HC(V) * M$ we consider the following diagram:



(9.2.3)

The arrows of the diagram are defined as follows: p is the projection along the last factor; $q(u, y_1, y_2, y_3) = (u \cdot y_1, y_1, y_2, y_3)$; $\Delta(y_1, y_2, y_3) = (y_1, y_2, y_2, y_3)$; pr is the projection along the middle factor; $\tilde{\Delta}(u, y_1, y_2) = (u, y_1, y_1, y_2)$ and $\tilde{q}(u, y_1, y_2) = (u \cdot y_1, y_1, y_2)$. Using the diagram, one can write

$$HC(V) * M = pr_* \cdot \Delta^! \cdot q_* \cdot p^!(V \boxtimes M)$$

Now, we observe that the square ① in the centre of (9.2.3) is a Cartesian square. Hence, $\Delta^! \cdot q_* = \tilde{q}_* \cdot \tilde{\Delta}^!$ and the diagram (9.2.3) yields

$$HC(V) * M = f_*(V \boxtimes M)$$

where the map $f : G \times Y \times Y \longrightarrow Y \times Y$ is defined by $f(u, y_1, y_2) = (u \cdot y_1, y_2)$.

An argument involving a similar diagram, yields

$$M * HC(V) = (pr_{Y \times Y})_* \cdot h^*(V \boxtimes M)$$

where $pr_{Y \times Y}$ denotes the projection $G \times Y \times Y \longrightarrow Y \times Y$ and h is an automorphism of $G \times Y \times Y$ defined by $h(u, y_1, y_2) = (u, y_1, u \cdot y_2)$. To compare $HC(V) * M$ with $M * HC(V)$ we consider the following commutative diagram

$$\begin{array}{ccccc} G \times Y \times Y & \xrightarrow{r} & G \times Y \times Y & \xrightarrow{h^{-1}} & G \times Y \times Y \\ & {\scriptstyle f} \searrow & & \swarrow {\scriptstyle pr_{Y \times Y}} & \\ & & Y \times Y & & \end{array}$$

where the automorphism r is defined by $r(u, y_1, y_2) = (u, u \cdot y_1, u \cdot y_2)$. The complex M being G-equivariant, we have $r_*(V \boxtimes M) = V \boxtimes M$. Whence, we obtain

$$HC(V) * M = f_*(V \boxtimes M) = (pr_{Y \times Y})_* \cdot h_*^{-1} \cdot r_*(V \boxtimes M) =$$

$$= (pr_{Y \times Y})_* \cdot h_*^{-1}(V \boxtimes M) = M * HC(V). \qquad \text{Q.E.D.}$$

To proceed further we need the notion of a <u>mixed</u> module on a smooth algebraic variety. The reader may think of a mixed module either as a mixed l-adic perverse sheaf in the sense of [BBD] or as a mixed Hodge module in the sense of [Sa]. We prefer the latter. So, given a complex algebraic variety Z, let $D^b_{mix}(Z)$ denote the bounded derived category of complexes of mixed Hodge modules on Z.

We are mainly interested, of course, in some categories of mixed modules either on the reductive group G or on the Flag manifold G/B. Let D^b Admiss$_o$(G) denote the full subcategory of $D^b_{mix}(G)$ formed by Ad G-equivariant complexes, whose cohomology modules are admissible modules with the <u>trivial</u> <u>central</u> <u>character</u>. Also, let $\overline{Y} = Y/T = G/B$ denote the Flag manifold and let D^b Admiss$(\overline{Y} \times \overline{Y})$ denote the subcategory of $D^b_{mix}(\overline{Y} \times \overline{Y})$ formed by G-equivariant complexes. We define the Harish-Chandra functor

$$HC_o : D^b \text{ Admiss}_o(G) \longrightarrow D^b \text{ Admiss } (\overline{Y} \times \overline{Y}) \qquad (9.2.4)$$

using the diagram (8.1.1) for A = G and Y = G/B. The functor HC_o is compatible with convolution structures and an analogue of proposition 9.2.1 holds for HC_o.

Further, given a category C, let K(C) denote its Grothendieck group. A convolution structure on C gives rise to an algebra structure on K(C). It is known, for example, that the Grothendieck group K(D^b Admiss $(\overline{Y} \times \overline{Y})$) is isomorphic to the Hecke algebra H associated to the Weyl group of G. Hence, the functor (9.2.4) induces an algebra homomorphism

$$HC_o : K(D^b \text{ Admiss}_o(G)) \longrightarrow \text{Center } (H) \qquad (9.2.5)$$

Let $\hat{G}_o$ be the set of irreducible G-admissible modules on G (i.e. characters sheaves) with the trivial central character. Using a case by case argument, Lusztig [Lu2] has attached to each character sheaf $V \in \hat{G}_o$ a two-sided cell c(V) in the Weyl group (cf. [KL]). The assignment $V \longmapsto c(V)$ has played a crucial role in the classification of character sheaves, carried out in [Lu2]. It seems likely that the two-sided cell c(V) is closely connected to the two-sided ideal of the Hecke algebra H, generated by the element $HC_o(V) \in$ Center(H) (see (9.2.5)).

Appendix A. A characteristic variety theorem

A1. We keep to the notation of n.5.1, so that Z is a smooth algebraic variety with a normal crossing divisor Y. Let $X := Z \setminus Y$ be a Zariski-open part of Z and $j : X \hookrightarrow Z$ the inclusion. There is a natural isomorphism

$$T^{*}X \cong T^{*}(Z, Y)\big|_{X} \tag{A1.1}$$

Let V be a regular holonomic D_X-module and let SSV be the characteristic variety of V, viewed as a non-closed subset of $T^{*}(Z, Y)$ via (A1.1). We have the following

Theorem A1.2. Let M be a $D_{Z,Y}$-coherent submodule of $j_{*}V$. Then

$$SSM \subset \overline{SSV}$$

and the equality holds provided M generates V, i.e. $V = D_X \cdot M$.

A2. The rest of the appendix is devoted to the proof of theorem

A1.2. The problem being local, we assume from now on that the divisor Y is defined by an equation $f = 0$. Observe that $f \cdot D_{Z,Y} \cdot f^{-1} = D_{Z,Y}$.

A $D_{Z,Y}$-coherent submodule $M \subset j_* V$ is said to be a <u>lattice</u> in V if $V = D_X \cdot M$. Clearly, theorem A1.2 is equivalent to the following

<u>Proposition A2.1</u>. For any lattice M, the variety SSM has no irreducible components over the divisor $Y = \{f = 0\}$.

Before going into proof we record some elementary properties of lattices.

A2(i) If M is a lattice, then $\bigcup_{m \geq 0} f^{-m} \cdot M = V$

A2(ii) Given two lattices L and M, one can find integers $n, m \geq 0$ such that $f^n \cdot L \subset M \subset f^{-m} \cdot L$

A2(iii) Let $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ be an exact sequence of D_X-modules. Given a lattice L in V, let L'' be its image in V'' and $L' = L \cap V'$. Then, we have an exact sequence of lattices:

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$$

Moreover, the theorem holds for L iff it holds for L' and L''.

A3. <u>Construction a la Malgrange</u>. Let $f_1, \ldots, f_r$ be regular functions on a smooth algebraic veriety Z_o and $Y_o = \{f_1 \cdot \ldots \cdot f_r = 0\}$, a divisor in Z_o. We set $Z = \mathbb{C}^r \times Z_o$ and let $i : Z_o \hookrightarrow Z$ be the graph embedding $i : z \longmapsto (f_1(z), \ldots, f_r(z), z)$. Let $t_1, \ldots, t_r$ denote the functions on $\mathbb{C}^r \times Z_o$ pulled back from standard coordinate functions on $\mathbb{C}^r$. It is clear that $Y = \{t_1 \cdot \ldots \cdot t_r = 0\}$ is a normal crossing divisor in Z and we have

$$i(Y_o) = i(Z_o) \cap Y$$

Set $X_o = Z_o \setminus Y_o$, $X = Z \setminus Y$.

Now, given a D-module V_o on X_o, let $V = i_* V_o$. Following Malgrange [Ma], we give another interpretation of V. To do that, we introduce an auxiliary polynomial ring $\mathbb{C}[s_1, \ldots, s_n]$ and set $D_{X_o}[s_1, \ldots, s_r] = D_{X_o} \otimes \mathbb{C}[s_1, \ldots, s_r]$. The sheaf $V_o \cdot f_1^s \cdot \ldots \cdot f_r^s[s_1, \ldots, s_r]$ has a natural $D_{X_o}[s_1, \ldots, s_r]$-module structure. We abbreviate notation: $V_o \cdot f^s := V_o \cdot f_1^s \cdot \ldots \cdot f_r^s[s_1, \ldots, s_r]$. Viewing D_{Z_o} as a subsheaf of D_Z that acts along the second factor, we have

Proposition A3.1. There is a natural D_{X_o}-module isomorphism

$$V(= i_* V_o) \cong V_o \cdot f^s ,$$

such that

(i) the action of $t_j \cdot \partial/\partial t_j$ on $i_* V_o$ corresponds to the multiplication by $(-s_j)$;

(ii) the action of t_j on $i_* V_o$ corresponds to the automorphism of $V_o \cdot f^s$ induced by the shift $s_j \longmapsto s_j + 1$.

In the special case $r = 1$ the proposition is due to Malgrange [Ma]. General case is handled in the same way. □

A $D_{Z_o}[s_1, \ldots, s_r]$-coherent submodule $L \subset V_o \cdot f^s$ is said to be a lattice if it is stable under the automorphisms of $V_o \cdot f^s$, described in A3.1(ii), and $\bigcup_{m_1,\ldots,m_r \geqslant 0} f_1^{-m_1} \cdot \ldots \cdot f_r^{-m_r} \cdot L = V_o \cdot f^s$. It follows from proposition A3.1 that $D_{Z,Y}$-lattices in $i_* V_o$ correspond to lattices in $V_o \cdot f^s$.

The ring $D_{Z_o}[s_1, \ldots, s_r]$ has a natural increasing filtration, the product of standard increasing filtrations on D_{Z_o} and $\mathbb{C}[s_1, \ldots, s_r]$. Clearly, Spec $D_{Z_o}[s_1, \ldots, s_r] = \mathbb{C}^r \times T^* Z_o$, so that the characteristic variety of a $D_{Z_o}[s_1, \ldots, s_r]$-module is a subvariety of $\mathbb{C}^r \times T^* Z_o$. From proposition A3.1 one easily derives

Corollary A3.2. The following statements are equivalent

(a) Theorem A1.2 holds for $V = i_* V_o$;

(b) If L is a lattice in $V_o \cdot f^s$, then SSL has no irreducible components over the divisor Y_o.

In the special case $r = 1$ the statement A3.2(b) is a key result of my paper [Gi1]. So, theorem A1.2 may be viewed as a generalization of [Gi1, thm 2.3]. The strategy of the proof of theorem A1.2 will, in fact, be similar to that of [Gi1].

A4. We now turn to the proof of proposition A2.1. Let z_o be a point of the normal crossing divisor $Y = \{f = 0\}$. On a neighborhood of z_o in Z we choose a local direct product decomposition $Z = T \times U$, local coordinate functions $t = (t_1, \ldots, t_r)$ on T and $u = (u_1, \ldots, u_m)$ on U, such that in the local coordinates (t, u) on Z we have $f(t, u) = t_1 \cdot \ldots \cdot t_r$ and $z_o = (0, u_o)$. Let Y_T denote the divisor in T defined by the equation $t_1 \cdot \ldots \cdot t_r = 0$. It is clear, that $Y = Y_T \times U$ and $T^*(Z, Y) = T^*(T, Y_T) \times T^*U$.

Further, let V be a regular holonomic D_X-module and let M be a $D_{Z,Y}$-lattice in V. We are going to prove the following

Proposition A4.1. Let S be an irreducible component of SSM contained in $T_O^*(T, Y_T) \times T^*U$. Then, $S \subset \{0\} \times T^*U$. Furthermore, S is an isotropic subvariety (viewed as a subvariety of T^*U).

We first consider a special case of proposition A4.1, arising from the graph construction of n.A3. More specifically, we assume given the following data

(a) $f_1, \ldots, f_r$, regular functions on a smooth variety Z_o, such that $\{f_1 \cdot \ldots \cdot f_r = 0\}$ is a normal crossing divisor; a point $u_o \in Z_o$ such that $f_1(u_o) = \ldots = f_r(u_o) = 0$;

(b) V_o, a smooth regular holonomic module (i.e. a regular local system) on $X_o = Z_o \setminus \{f_1 \cdot \ldots \cdot f_r = 0\}$;

(c) $i : Z_o \hookrightarrow \mathbb{C}^r \times Z_o$, the graph embedding $i(z) = (f_1(z), \ldots, f_r(z), z)$. Set $T = \mathbb{C}^r$, $U = Z_o$, $Z = T \times U = \mathbb{C}^r \times Z_o$, $z_o = i(u_o) = (0, u_o)$.

Lemma A4.2. Proposition A4.1 holds for any lattice M in $i_* V_o$.

Proof. The construction "a la Malgrange" (see proposition A3.1) translates the statement of the lemma into the following one:

Let L be a $D_{Z_o}[s_1, \ldots, s_r]$-lattice in $V_o \cdot f^s$ and let S be an irreducible component of L over the divisor $\{f_1 \cdot \ldots \cdot f_r = 0\}$. Then, $S \subset \{0\} \times T^*Z_o \subset \mathbb{C}^r \times T^*Z_o$ and S is an isotropic subvariety (viewed as a subvariety of T^*Z_o).

This statement can be verified directly, using local coordinates on Z_o adapted to the normal crossing divisor $f_1 \cdot \ldots \cdot f_r = 0$. We omit the details. □

A5. Let Z be a smooth proper variety with a normal crossing divisor Y. Let X_o be a smooth locally-closed subvariety of $Z \setminus Y$ and V_o a regular local system on X_o. Let $i : X_o \hookrightarrow Z$ denote the inclusion. We assume i to be affine, so that $i_* V_o$ is a D_Z-module.

<u>Lemma A5.1</u>. Proposition A4.1 holds for $V = i_* V_o$.

<u>Proof</u>. Let $\overline{X}_o$ be the closure of X_o in Z and let $p_o : Z_o \longrightarrow Z$ be a resolution of singularities of $\overline{X}_o$, so that $Z_o \setminus X_o$ is a normal crossing divisor.

Recall the notation introduced at the beginning of n.A4. We pull the coordinate functions (t, u) back on Z_o and denote the result by the same letters. We have the following commutative diagram:

$$\begin{array}{ccccc} Z_o & \overset{f}{\hookrightarrow} & T \times Z_o & \overset{h}{\hookrightarrow} & T \times U \times Z_o \\ & \searrow^{p_o} & & & \downarrow p \\ & & & & T \times U = Z \end{array}$$

where f denotes the graph embedding $f(z) = (t(z), z)$, h denotes the graph embedding $h(t, z) = (t, u(z), z)$ and p is the natural projection.

We have $V = i_* V_o = p_* \cdot h_* \cdot f_* V_o$. Next, note that proposition A4.1 holds for the module $f_* V_o$ by lemma A4.2. It follows easily, that the same holds for $V_1 := h_* \cdot f_* V_o$.

Let L_1 be a lattice in V_1 and let L be the image of the natural morphism $p_* L_1 \longrightarrow p_* V_1 = V$. We observe that the map p is proper since Z and, hence Z_o, are proper varieties. Furthermore, the divisor $Y_T \times U \times Z_o$ in $T \times U \times Z_o$ equals $p^{-1}(Y)$, so that no logarithmic cotangent bundles along the fibre direction (of the projection p) are involved. It follows, that L is a lattice in V and that the standard estimate on $SS(p_* L_1)$ in terms of SSL_1 (see [Ka]) can be applied. The estimate shows that lemma A5.1 holds for the lattice L (cf. [Gil, corollary 2.9] for a similar argument).

To complete the proof we have to verify proposition A4.1 for any lattice in $V = i_* V_o$. But it follows from the property A2(ii) that proposition A4.1 holds for every lattice, provided it holds for a particular one, e.g.L.

A6. We come to final stages of the proof of proposition A4.1 and of proposition A2.1. The argument copies that of [Gil].

The proof of proposition A4.1 is completed as follows. Property A2(iii) shows that it suffices to establish the result for a family of holonomic D_Z-modules that contain any irreducible regular holonomic module as a subquotient. Such a family is provided by the modules of the form $i_* V_o$ considered in n.A5. Lemma A5.1 completes the proof.

Proof of proposition A2.1. Let M be a lattice and S an irreducible component of SSM contained in $T^*(Z, Y)_{|Y}$, where Y is the normal crossing divisor. Let $z_o \in Y$ be a generic point in the image of S (under the projection to Y). Let $Z = T \times U$ be the direct product decomposition considered in n.A4 and let $\mathcal{E}_U$ be the sheaf of microdifferential operators on T^*U. It follows from proposition A4.1 that $\mathcal{E}_U \otimes_{D_U} M$ is a non-zero holonomic $\mathcal{E}_U$-module on a neighborhood of S. On the other hand, the module $\mathcal{E}_U \otimes M$ has a natural action of the operators t_i and $t_i \cdot \partial/\partial t_i$, $i = 1,\dots,r$. So, proposition 5.11 of [Ka] yields $\mathcal{E}_U \otimes M = 0$. The contradiction shows that $S = 0$. □

Appendix B: Equivariant and monodromic modules

B1. Let G be an algebraic group acting on a smooth algebraic variety Z and let $p,q : G \times Z \longrightarrow Z$ denote the second projection and the "action-map" $q(u, z) = u \cdot z$ respectively.

Definition B1.1. A constructible complex A on Z is called G-monodromic if the following equivalent conditions hold:

(i) The restriction of A on any G-orbit is a locally-constant complex;

(ii) The complex q^*A is locally-constant along the fibres of the projection p.

Now let M be a holonomic D_Z-module and let DR(M) denote its DeRham complex. The module M is called monodromic if D-module counterparts of B1.1 (i)-(ii) hold. A regular module M is monodromic iff so is DR(M).

Let g be the Lie algebra of the group G and let $U(g) \longrightarrow \Gamma(Z, D_Z)$ be the natural algebra homomorphism induced by "infinitesimal" g-action on Z.

Proposition B1.2. If M is a regular G-monodromic D-module, then the natural U(g)-action on $\Gamma(Z, M)$ is locally-finite. The converse is true for regular modules, provided Z is affine.

The key point of the proof is the existence of a natural global good filtration - the so-called Kashiwara-Kawai filtration - on a regular holonomic D-module. Furthermore, the graded sheaf Gr M on T^*Z associated with the Kashiwara-Kawai filtration is known to be reduced. It follows, that if M is G-monodromic, then the filtration on M is g-stable. This yields that the U(g)-action on $\Gamma(Z, M)$ is locally-finite. $\square$

B1.3. Observe, that the property of "being monodromic" depends solely on the infinitesimal g-action without mentioning the group G. So, we let $D^b(Z, g)$ denote the triangulated category formed by bounded complexes of D_Z-modules with G-monodromic cohomology. Monodromic D_Z-modules form a heart of $D^b(Z, g)$.

B2. Definition B2.1. A constructible complex A (or a regular holonomic complex of D_Z-modules) is said to be a G-equivariant complex if a quasi-isomorphism $p^*A \cong q^*A$ is given in such a way that it satisfies a natural cocycle condition.

For the purposes of this paper it suffices to consider a "naive" equivariant derived category $D^b_{naive}(Z, G)$ defined as follows. Its objects are G-equivariant complexes and its morphisms are those morphisms $A \to B$ in $D^b(Z)$ that induce a commutative diagram:

$$\begin{array}{ccc} p^*A & \longrightarrow & p^*B \\ \wr\| & & \wr\| \\ q^*A & \longrightarrow & q^*B \end{array} \qquad \text{(B2.2)}$$

Clearly, $D^b_{naive}(Z, G)$ is an additive category, containing the category of equivariant D_Z-modules as an abelian subcategory. The word "naive" stemms from the fact that $D^b_{naive}(Z, G)$ is <u>not</u> a triangulated category. The point is that given a diagram (B2.2), one can not define a <u>natural</u> quasi-isomorphism $p^* \operatorname{cone}(A \to B) \cong q^* \operatorname{cone}(A \to B)$. For a reasonable definition of derived category of equivariant complexes the reader is referred to [Gi2, nn. 7-8].

Any G-equivariant D-module M is G-monodromic so that there is a natural locally-finite $U(g)$-action on $\Gamma(Z, M)$.

<u>Proposition B2.3</u>. If M is equivariant, then the $U(g)$-action on $\Gamma(Z, M)$ can be naturally integrated to a G-action. The converse is true, provided Z is affine. □

B3. Let $f : Z \to \overline{Z}$ be a principal G-bundle and M a monodromic D_Z-module. Then, there is a natural monodromy action of $\pi_1(G)$ on $f.M$. Suppose that the group G is linear, so that f is an affine morphism.

<u>Proposition B3.1</u>. The following conditions are equivalent

(i) M is G-equivariant;

(ii) The monodromy action on f.M is trivial;

(iii) $M = f^*\overline{M}$ is the pull-back of a D-module on $\overline{Z}$.

Next, suppose that G is a connected group, H is a closed subgroup of G and $Z = G/H$. There is a natural group homomorphism $\pi_1(Z) \to \pi_0(H)$ arising from the fibration $G \to G/H$.

<u>Proposition B3.2</u>. (i) G-monodromic D-modules on G/H are in (1-1)-correspondence with finite-dimensional representations of $\pi_1(G/H)$.

(ii) A monodromic module on G/H is equivariant iff it comes from a representation of $\pi_o(H)$ via the homomorphism $\pi_1(G/H) \longrightarrow \pi_o(H)$.

Let $F : X \longrightarrow Y$ be a G-morphism between G-varieties.

Proposition B3.3. The natural functors F_*, F^*, $F_!$, $F^!$ take monodromic (resp. equivariant) complexes into similar ones. □

B4. Recall the notation of n. B1.

Proposition B4.1. (J. Bernstein). For any regular D-module M on Z, $p_* \cdot q^* M$ is a G-equivariant complex on Z.

Remarks. (a) One can see, that the complex $p_* \cdot q^* M$ comes equipped with a natural structure of an equivariant complex in the sense of [Gi2, n.7] (which is stronger than the definition B2.1).

(b) The functor $p_* \cdot q^*$ has been used by Bernstein to give a D-module definition of the Zuckerman functor. The recent approach to the Zuckerman functor due to Duflo-Vergne [DV] is, actually, nothing but the globalization of the Bernstein construction.

Unfortunately, the functor $p_* \cdot q^*$ does not give rise to a functor between abelian categories, for it takes D-modules into complexes of D-modules. Suppose, however, that G = U is a unipotent group. Then, we have

Proposition B4.2. There is an exact functor J, taking holonomic D_Z-modules into U-equivariant D_Z-modules.

The functor J is a D-module analogue of the Jacquet functor of the I-adic completion, where I is the augmentation ideal in the enveloping algebra of Lie U (see [CasColl]).

Proof of Proposition B4.2 isby induction on dim U. Let $\{0\} = U_o \subset U_1 \subset \cdots \subset U_n = U$ be an increasing sequence of normal subgroups of U such that $U_{i+1}/U_i \cong \mathbb{A}^1$. Suppose, inductively, that we have constructed the functor J_i corresponding to the group U_i. To define J_{i+1}, consider the action-mapping $q : U_{i+1} \times Z \longrightarrow Z$ and the "partial" projection $p_{i+1/i} : U_{i+1} \times Z \longrightarrow (U_{i+1}/U_i) \times Z$. Let M be a D_Z-module. It easily follows from U_i-equivariance of $J_i M$, that there is a unique D-module $\tilde{M}$ on $(U_{i+1}/U_i) \times Z$ such that $p^*_{i+1/i}\tilde{M} = q^*(J_i M)$.

We now identify $U_{i+1}/U_i \cong \mathbb{A}^1$ with the "finite" part of the projective line $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Accordingly, we view $(U_{i+1}/U_i) \times Z$ as the finite part of $\mathbb{P}^1 \times Z$. Let Ψ denote the nearby-cycles functor with respect to $\{\infty\} \times Z \subset \mathbb{P}^1 \times Z$, the divisor at infinity. We set

$$J_{i+1}(M) := \Psi(\tilde{M})$$

Clearly, J_{i+1} is an exact functor, taking D-modules on Z to D-modules on $\{\infty\} \times Z \cong Z$. Verification of the U_{i+1}-equivariance of $J_{i+1}(M)$ is left to the reader. □

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