ANTHONY JOSEPH

The primitive spectrum of an enveloping algebra

Astérisque, tome 173-174 (1989), p. 13-53

<http://www.numdam.org/item?id=AST_1989__173-174__13_0>

Société Mathématique de France
Astérisque 173-174 (1989), p.13-53

THE PRIMITIVE SPECTRUM OF AN ENVELOPING ALGEBRA

Anthony Joseph

Chapter 1.

The base field k will be assumed to be algebraically closed and of characteristic zero throughout.

1.1. Ring theory.

1.1.1. Let A be a ring with identity 1. Define the (left) primitive spectrum $Prim\ A$ of A to be the set of annihilators of simple (left) A modules. Since any maximal two-sided ideal is contained in a maximal left ideal (via Zorn's Lemma) we conclude that $Prim\ A \supset Max\ A$. Define the prime spectrum $Spec\ A$ of A to be the set of all two-sided ideals P of A for which $IJ \subset P$ where $I,\ J$ are two-sided ideals of A, implies either $I \subset P$ or $J \subset P$. Here we can take $I = AaA,\ \ J = AbA$ so the condition is equivalent to $aAb \subset P$ implies $a \in P$ or $b \in P$. Thus the set $Spec_c A$ of all two-sided ideals P for which A/P is an integral domain, is contained in $Spec\ A$ with equality for A commutative. One easily checks that $Spec\ A \supset Prim\ A$ and we set $Prim_c A = Prim\ A \cap Spec_c A$, the elements of which are called completely prime, primitive ideals.

1.1.2. Let $\mathcal{S}$ be a subset of $Spec\ A$. Define the closure $\bar{\mathcal{S}}$ of $\mathcal{S}$ to the set of all prime ideals containing $I(\mathcal{S}) := \bigcap_{P \in \mathcal{S}} P$. One checks that $\mathcal{S} \subset \bar{\mathcal{S}},\ \bar{\mathcal{S}} = \bar{\bar{\mathcal{S}}},\ \overline{\mathcal{S}_1 \cup \mathcal{S}_2} = \bar{\mathcal{S}}_1 \cup \bar{\mathcal{S}}_2$ and hence that closure defines a topology on $Spec\ A$. It is called the Jacobson topology of $Spec\ A$ and coincides with the Zariski topology when A is commutative. One may note that $\mathcal{S} \subset Spec\ A$ is irreducible $\Longleftrightarrow I(\mathcal{S})$ is prime. One calls $I(Spec\ A) =: N(A)$ the nilradical of A.

1.1.3. The study of $Spec\ A$ may be viewed as an extension of algebraic geometry and that of $Prim\ A$ as abstract representation theory. One cannot get too far for an arbitrary ring. Call A (left) noetherian if every increasing sequence of (left) ideals is stationary. This property passes to quotients. From now on we shall assume that A is both left and right noetherian. For such a ring $N(A)$ is nilpotent and a finite intersection of minimal primes ([9], 3.1.10).

1.1.4. Call A a prime (resp. primitive) ring if O (i.e. the zero ideal) is a prime (resp. primitive) ideal. Let A be a prime, noetherian ring. A famous result of Goldie ([10], Chap. 4) asserts that the set S of regular elements of A (i.e. non-zero divisors) is Ore in A and so we can form the *ring of fractions Fract* A of A by adjoining the inverses of elements in S. This is a simple, artinian ring and so by the Wedderburn-Artin theorem a matrix ring $M_n(K)$ over a skew field K. One calls n (resp. K) the Goldie rank $rk\ A$ (resp. Goldie skew field) of A. By the Faith-Utumi lemma ([10], p.72) one may characterize $rk\ A$ as the maximal degree of nilpotence of elements of A, i.e. $rk\ A = Sup\{n | x^n = 0,\ x^{n-1} \neq 0 \text{ for some } x \in A\}$. In particular $P \in Spec\ A$ is completely prime if and only if $x^2 \in P$ implies $x \in P$. This last fact was used by Vogan ([30], 7.12) to show that unitary irreducible representations of complex groups lead to completely prime, primitive ideals through a simple manipulation using the positive definiteness of the hermitian form.

1.1.5. Let A be a k-algebra. Let V be a finite dimensional subspace of A which we can conveniently assume to contain the identity. For each $\ell \in \mathbb{N}^+$ set $V^\ell = k\{v_1 v_2 \cdots v_\ell | v_i \in V\}$. If A is finitely generated we can assume that V contains a generating set and we define the Gelfand-Kirillov dimension $d(A)$ of A through

$$d(A) := \overline{\lim_{\ell \to \infty}} \frac{log\ dim\ V^\ell}{log\ \ell}$$

which one checks is independent of the choice of V. A similar definition can be given for any finitely generated A module M. (For further details, see [23].)

Assume that A is a prime (noetherian) ring. A key point in the proof of Goldie's theorem is that any two-sided ideal $I \neq 0$ contains a regular element. This has the consequence that $d(A/I) \leq d(A) - 1$, which generalizes the reduction in dimension on passage to a subvariety which occurs in the context of algebraic geometry.

1.1.6. Let A be a k algebra. Recall that we are assuming k to be algebraically closed. Let M, N be simple A modules. If M, N are isomorphic A modules, then trivially $Ann\ M = Ann\ N$. Conversely suppose that $P := Ann\ M = Ann\ N$ has finite codimension. Then A/P is a prime ring finite dimensional over k, hence simple artinian. We conclude that M, N viewed as A/P modules are isomorphic, hence isomorphic as A modules.

The above result fails miserably when *codim* P is not finite. Assume A is an integral domain and consider cyclic A modules $M := A/Aa$, $N := A/Ab$. We have an exact sequence $0 \to A \xrightarrow{\varphi} A \to A/Aa \to 0$ where φ denotes right multiplication by a. This gives an exact sequence

$$Hom(A,N) \xrightarrow{\tilde{\varphi}} Hom(A,N) \longrightarrow Ext^1(M,N) \longrightarrow$$

from which we conclude that $Ext^1(M,N) = Coker\ \tilde{\varphi} = A/(aA + Ab)$.

Now let A denote the ring $k[x,y]$ with relation $xy - yx = 1$. This can be viewed as the ring of differential operators on the affine line $\mathbb{A}^1$ with the subring $k[y]$ identified with regular functions and $x = \frac{d}{dy}$. It is called the Weyl algebra A_1 of index 1 over k. One defines the Weyl algebra A_n of index n over k through $A_n = A_1 \otimes A_1 \otimes \cdots \otimes A_1$ (n times). Recalling that *char* $k = 0$ by assumption one easily checks that A_n is a simple integral domain. Moreover $d(A_n) = 2n$ and so A_n admits a skew field of fractions called the Weyl skew field.

Let p be a polynomial of odd degree ≥ 1. One checks that $I := A(x^2 + p(y))$ is a maximal left ideal of A and that $dim\ Ext^1(M,M) = deg\ p$, where $M = A/I$ (using the above formula for Ext^1). This shows that the set $\hat{A}$ of equivalence classes of isomorphic simple A modules is not only infinite; but has infinitely many orbits under the action of *Aut* A (itself an "infinite dimensional Lie group", see [8]). On the other hand since A is simple, *Prim* A is reduced to a single point.

1.2. Enveloping algebras.

1.2.1. Let $\underline{g}$ be a finite dimensional k - Lie algebra and $U(\underline{g})$ its enveloping algebra. Our basic aim is to describe *Prim* $U(\underline{g})$ ultimately as a topological space. As in 1.1.6 for the Weyl algebra this will only give a rough description of the isomorphism classes of simple $U(\underline{g})$ modules. However many of these simple modules are of little interest and can be eliminated if we require that they "lift" to continuous representations of the corresponding Lie group. Alternatively we may attempt to reduce the discrepancy between primitive ideals and simple modules to the case of the Weyl algebra.

1.2.2. For the moment let us limit ourselves to the description of *Prim* $U(\underline{g})$. For $\underline{g}$ solvable ([9], 3.7.2) one has that *Prim* $U(\underline{g})$ = *Prim*$_c$ $U(\underline{g})$. This can be viewed as a generalization of Lie's theorem to the case of infinite dimensional modules! Indeed the proof relies heavily on the applying Lie's theorem to the adjoint action of $\underline{g}$ on a given prime quotient of $U(\underline{g})$.

Let G denote the adjoint group of $\underline{g}$. It acts on $\underline{g}^*$ by transport of structure. A basic question is to show that one has a map $\underline{g}^*/G$ to $Prim_c\, U(\underline{g})$, which in the best situation should be a topological isomorphism and constructed by appropriate induction. Roughly speaking this does work out, except for some significant obstructions provided by the semisimple case. To illustrate this we review the highlights of the nilpotent case which is particularly simple. The results for the solvable case are described in [5], [9], Chaps. 5, 6 and for the general case in [27]. The results of [27] are reviewed in [21], Sect. 4 and [37].

1.2.3. Let $\underline{g}$ be a nilpotent (and finite dimensional) Lie algebra. Choose $f \in \underline{g}^*$ and let G_f denote the stabilizer of $f \in G$. The G orbit G/G_f has the structure of a G-equivariant symplectic manifold and this suggests the construction of a corresponding primitive ideal following the passage from classical to quantum mechanics. To do this we must eliminate half the phase space variables or "polarize" the symplectic manifold G/G_f. This can be interpreted as finding a subgroup P of G containing G_f such that $dim\ G/P = (1/2)\ dim(G/G_f)$ and such that f still defines a character on P. Let $\underline{p}$ denote the Lie algebra of P. Since $([x,y],f) = 0,\ \forall\, x,y \in \underline{p}$ there exists a one dimension $\underline{p}$ module k_f in which $x \in \underline{p}$ acts by multiplication through $f(x)$. Set $I(f) = Ann\left(U(\underline{g}) \otimes_{U(\underline{p})} k_f\right)$. One may show ([9], Chaps. 5, 6) that $I(f) \in Prim\ U(\underline{g})$, that $I(f)$ is independent of the choice of polarization and that the map $f \mapsto I(f)$ factors to a topological isomorphism of $\underline{g}^*/G$ onto $Prim\ U(\underline{g})$. Finally ([9], 4.9.23) each primitive quotient $U(\underline{g})/I(f)$ is isomorphic to a Weyl algebra of index $\frac{1}{2}\ dim\ G/G_f$. We remark that if $\underline{g}$ is a real nilpotent Lie algebra, then the unitary irreducible representations of G are also indexed by $\underline{g}^*/G$.

1.3. The semisimple case - preliminary theory.

1.3.1. Let $Z(\underline{g})$ denote the centre of $U(\underline{g})$. Let M be a simple $U(\underline{g})$ module. By Quillen's lemma ([9], 2.6.4) each $x \in End_{\underline{g}}M$ is algebraic over k and hence $End_{\underline{g}}M$ reduces to scalars. We conclude that $P \mapsto P \cap Z(\underline{g})$ defines a map of $Prim\ U(\underline{g})$ into $Max\ Z(\underline{g})$.

If $\underline{g}$ is nilpotent, each $P \in Prim\ U(\underline{g})$ is a maximal ideal ([9], 4.7.4). This fails in general. For $\underline{g}$ semisimple we shall see that the above map has finite fibres. Even this is false for $\underline{g}$ solvable.

1.3.2. Take $P \in Spec\ U(\underline{g})$ and let $C(P;\underline{g})$ denote the centre of $Fract(U(\underline{g})/P)$. One may show that $P \in Prim\ U(\underline{g})$ if and only if $C(P;\underline{g})$ reduces to scalars ([26], Sect. 4). This shows that $Prim\ U(\underline{g})$ is the same for left and right simple modules. If the radical of $\underline{g}$ is nilpotent it is enough that $U(\underline{g})/P$ to have trivial centre for a prime ideal P to be primitive. For $\underline{g}$ semisimple this holds if $P \cap Z(\underline{g}) \in Max\ Z(\underline{g})$. Finally we remark that every prime ideal is an intersection of primitive ideals ([9], 3.1.15).

1.3.3. From now on we assume $\underline{g}$ semisimple. Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$, $R \subset \underline{h}^*$ the set of non-zero roots, R^+ a choice of positive roots, $B \subset R^+$ the corresponding set of simple roots, W the (Weyl) group generated by the reflections $s_\alpha : \alpha \in R^+$ and ρ the half sum of the positive roots. For each $\alpha \in R$, let X_α be the corresponding element of a Chevalley basis for $\underline{g}$ and $X \mapsto {}^tX$ the corresponding Chevalley antiautomorphism defined by ${}^tX_\alpha = X_{-\alpha} : \alpha \in R$. Set $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. Let $\underline{g} = \underline{n}^+ \oplus h \oplus \underline{n}^-$ be the triangular decomposition of $\underline{g}$ defined by the above choices.

Call $\lambda \in \underline{h}^*$ dominant if $(\lambda,\alpha^\vee) \notin -1,-2,\cdots,\ \forall\ \alpha \in R^+$ and regular if $(\lambda,\alpha) \neq 0,\ \forall\ \alpha \in R$. For each $\lambda \in \underline{h}^*$ let $k_{\lambda-\rho}$ denote the one dimensional $\underline{b} := \underline{h} \oplus \underline{n}^+$ module with highest weight $\lambda - \rho$ and set $M(\lambda) = U(\underline{g}) \otimes_{U(\underline{b})} k_{\lambda-\rho}$ which is generally called a Verma module. Unlike the nilpotent case we do not get all primitive ideals as annihilators of induced modules; in fact a modest change of point of view is necessary. For the moment we note that $(Ann\ M(\lambda)) \cap Z(\underline{g}) \in Max\ Z(\underline{g})$ and the resulting element $\chi_\lambda \in Max\ Z(\underline{g})$ is independent of the choice of λ in its Weyl group orbit $\hat{\lambda}$. On the other hand $M(\lambda)$ identifies with $U(\underline{n}^-)$ as a $U(\underline{n}^-)$ module and so $End_{\underline{n}^-}M(\lambda) \cong U(\underline{n}^-)$ which is an integral domain. Since the diagonal action of $\underline{n}^-$ on $U(\underline{g})/Ann\ M(\lambda)$ is locally nilpotent it follows that the latter is an integral domain and so $Ann\ M(\lambda) \in Spec_c\ U(\underline{g})$. By the remark in 1.3.2 it follows that $Ann\ M(\lambda) \in Prim_c\ U(\underline{g})$ even though $M(\lambda)$ may not be simple (though it is always of finite length). One may show ([9], 8.4.3) that $Ann\ M(\lambda) = \chi_\lambda\ U(\underline{g})$ and so is minimal as a primitive ideal. On the other hand the reductivity of $\underline{g}$ implies that there is a unique maximal two-sided ideal of $U(\underline{g})$ containing $\chi_\lambda\ U(\underline{g})$.

1.3.4. Let $\mathcal{P}$ be the projection of $U(\underline{g})$ onto $U(\underline{h})$ defined by the decomposition $U(\underline{g}) = U(\underline{h}) \oplus (\underline{n}^- U(\underline{g}) + U(\underline{g})\underline{n}^+)$. Since $\underline{h}$ is commutative, $U(\underline{h})$ identifies with the set of all polynomial functions on $\underline{h}^*$. For each $\lambda \in \underline{h}^*$ we define a bilinear form $<,>_\lambda$ on $U(\underline{g})$ through $< a, b >_\lambda = \mathcal{P}({}^t ab)(\lambda - \rho)$. One easily checks that $<,>_\lambda$ is symmetric and $\underline{g}$ invariant with respect to the Chevalley antiautomorphism. Again

$$ker <,>_\lambda \supset U(\underline{g})\underline{n}^+ + \sum_{H \in \underline{h}} U(\underline{g})(H - (\lambda - \rho)(H)) = Ann_{U(\underline{g})}(1 \otimes k_{\lambda - \rho})$$

and so factors to a form on the Verma module $M(\lambda)$. It is called the contravariant form on $M(\lambda)$.

We define the $\mathcal{O}$ category to be the category of $U(\underline{g})$ modules M satisfying

(i) $dim\ U(\underline{b})m < \infty, \qquad \forall\ m \in M$.

(ii) $M = \oplus_{\mu \in \underline{h}^*} M_\mu$ *where each* $M_\mu := \{m \in M | Hm = \mu(H)m\}$ *is finite dimensional.*

(iii) $dim(Z(\underline{g})/Ann_{Z(\underline{g})} M) < \infty$.

This category is closed under subquotients, finite direct sums, but not necessarily extensions. One checks that $M(\mu) \in Ob\mathcal{O}, \ \forall\ \mu \in \underline{h}^*$.

Given $M \in Ob\mathcal{O}$ define $\delta(M) = \{\xi \in M^* | U(\underline{b})\xi < \infty\}$ where the action of $\underline{g}$ on M^* is taken with respect to the Chevalley antiautomorphism. One checks that $\delta(M) \in Ob\mathcal{O}$ and δ is contravariant, exact and $\delta^2 = 1d$. The above construction of a non-zero contravariant form on $M(\lambda)$ defines a non-zero map $M(\lambda) \to \delta M(\lambda)$ whose image we denote by $L(\lambda)$. One checks that $L(\lambda)$ is simple and indeed is the unique simple quotient of $M(\lambda)$. Furthermore $J(\lambda) := Ann\ L(\lambda) \in Prim\ U(\underline{g})$ and satisfies ${}^t J(\lambda) = J(\lambda)$. Clearly each simple module in $Ob\mathcal{O}$ is of this form. Making use of primary decomposition with respect to $Z(\underline{g})$, it follows that each $M \in Ob\mathcal{O}$ has finite length. Furthermore both $[L(\lambda)] : \lambda \in \underline{h}^*$ and $[M(\lambda)] : \lambda \in \underline{h}^*$ are bases for the Grothendieck group of objects in $\mathcal{O}$. We shall prove Duflo's theorem (1.3.7) which asserts that every primitive ideal is some $J(\lambda)$.

1.3.5. Let M, N be $U(\underline{g})$ modules. Define $Hom_{\mathbb{C}}(M, N)$ as a $U(\underline{g}) \otimes U(\underline{g})$ module through $((a \otimes b) \cdot x)m = ({}^t\check{a} x \check{b})m$ where $a \mapsto \check{a}$ denotes the principal antiautomorphism. Define $j : \underline{g} \to \underline{g} \times \underline{g}$ through $j(X) = (X, {}^t\check{X}), \ \forall\ X \in \underline{g}$ and set $\underline{k} = j(\underline{g})$. Define $(M \otimes N)^*$ as a $U(\underline{g}) \otimes U(\underline{g})$ through the principal antiautomorphism. Let $L(M, N)$ (resp. $L(M \otimes N)^*$)

denote the corresponding submodule of locally $\underline{k}$ finite elements. Now take $M, N \in Ob\mathcal{O}$. The canonical isomorphism $Hom_k(M,\ Hom_k(N,k)) \xrightarrow{\sim} Hom_k(N \otimes M, k)$ gives an isomorphism $L(M, \delta N) \xrightarrow{\sim} L(N \otimes M)^*$ by taking $\underline{k}$ locally finite parts. We set $L(M(\lambda) \otimes M(\mu))^* = L(-\lambda, -\mu)$. It identifies with the module obtained by coinduction from the one dimensional representation $k_{-(\lambda-\rho)} \otimes k_{-(\mu-\rho)}$ of $\underline{b} \times \underline{b}$ and taking $\underline{k}$ locally finite parts. It is called a principal series module. One has a non-degenerate bilinear pairing $L(\lambda, \mu) \times L(-\lambda, -\mu) \to k$ which is $\underline{g} \times \underline{g}$ invariant with respect to the principal antiautomorphism (see [9], 9.6.9 for the "diagonal" $\lambda = \mu$ case).

1.3.6. Take $\lambda \in \underline{h}^*$ dominant. Then $M(-\lambda)$ is a simple module so $M(-\lambda) \cong \delta M(-\lambda)$ (1.3.4). By Kostant's theorem ([9], 9.6.6) the action of $U(\underline{g})$ on $M(-\lambda)$ defines an isomorphism of $U(\underline{g})/J(-\lambda)$ onto $L(M(-\lambda), M(-\lambda)) \cong L(\lambda, \lambda)$. (This observation is due to N. Conze [34], 6.9). Given M a submodule of $M(\lambda)$ define $\varphi_M : L(M(\lambda) \otimes M(\lambda))^* \to L(M(\lambda) \otimes M)^*$ by restriction. By the remarks in 1.3.4 one has $Ann\ M(\lambda) = \chi_\lambda U(\underline{g}) = (\chi_{-\lambda} U(\underline{g}))^\vee = J(-\lambda)^\vee$. Thus given a two-sided ideal J of $U(\underline{g})/Ann\ M(\lambda)$, it follows that $\overset{\vee}{J}$ identifies with a $U(\underline{g}) \otimes U(\underline{g})$ submodule of $L(\lambda, \lambda)$.

LEMMA. - *The orthogonal of $\overset{\vee}{J}$ in $L(-\lambda, -\lambda)$ coincides with $ker\ \varphi_{JM(\lambda)}$. In particular the map $J \mapsto JM(\lambda)$ is injective.*

By definition,

$$\begin{aligned} ker\ \varphi_{JM(\lambda)} &= \{x \in L(-\lambda,-\lambda) | x(m \otimes n) = 0,\ \forall\, m \in M(\lambda),\ \forall\, n \in JM(\lambda)\}\ . \\ &= \{x \in L(-\lambda,-\lambda) | (1 \otimes \overset{\vee}{J})x = 0\},\ \text{ by transposition },\\ &= \{x \in L(-\lambda,-\lambda) | < x,\ (1 \otimes J)y >= 0,\ \forall\, y \in L(\lambda,\lambda)\}\ . \\ &= \{x \in L(-\lambda,-\lambda) | < x, \overset{\vee}{J} >= 0\}\ ,\ \text{ as required }. \end{aligned}$$

1.3.7. We easily conclude from 1.3.6 that $J = Ann(M(\lambda)/JM(\lambda))$. Now assume that $J \in Spec\ (U(\underline{g})/Ann\ M(\lambda))$. Since $M(\lambda)/JM(\lambda)$ has finite length it follows that $J = Ann\ L(\mu)$ for some simple subquotient of $M(\lambda)/JM(\lambda)$. This proves Duflo's theorem [3.5], namely

THEOREM. - *The map $\lambda \mapsto J(\lambda)$ of $\underline{h}^*$ into Prim $U(\underline{g})$ is surjective. In particular the map $J \mapsto J \cap Z(\underline{g})$ of Prim $U(\underline{g})$ into Max $Z(\underline{g})$ has fibres being the distinct $J(w\lambda)$: $w \in W$.*

1.3.8. One may improve 1.3.7 to give an equivalence of categories. Let $\mathcal{H}$ denote category of all $U(\underline{g}) \otimes U(\underline{g})$ modules V such that

(i) $dim\, U(\underline{k})v < \infty, \ \forall\, v \in V$.

(ii) $V = \bigoplus_{\sigma \in \underline{k}^\wedge} V_\sigma$ *where* V_σ *denotes the direct sum of all simple finite dimensional* $\underline{k}$ *modules of type* σ *and* $dim\ V_\sigma < \infty$.

(iii) $dim(Z(\underline{g}) \otimes Z(\underline{g}))/Ann_{Z(\underline{g})\otimes Z(\underline{g})} V < \infty$.

(Of course (i) $\Longrightarrow$ (ii) but we keep the above formulation to emphasize the analogy with the $\mathcal{O}$ category).

Now fix $\lambda \in \underline{h}^*$ dominant and let $\mathcal{H}_{\hat{\lambda}}$ (resp. $\mathcal{H}_{\hat{\lambda}}^{\infty}$) denote the subcategory of $\mathcal{H}$ of $U(\underline{g}) \otimes U(\underline{g})$ modules annihilated by $1 \otimes \chi_\lambda^\vee$ (resp. a power of the latter). Given $M \in Ob\mathcal{O}$ we have $L(M(\lambda), M) \in Ob\mathcal{H}_{\hat{\lambda}}$. (Here one checks property (ii) by taking $M = \delta M(\mu)$ and using Frobenius reciprocity). We remark that any exact sequence $0 \to N \to N' \to M(\lambda) \to 0$ in $\mathcal{O}$ must split by primary decomposition and the dominance of λ. Hence $M(\lambda)$ is projective in $\mathcal{O}$ and so it is easy to deduce that $T : M \mapsto L(M(\lambda), M)$ is an exact functor. A standard canonical isomorphism shows that it admits $T' : V \mapsto V \otimes_{U(\underline{g})} M(\lambda)$ as an adjoint functor. Now let $L = M_1/M_2$ be a simple subquotient of $M(\lambda)$. We claim that $V := L(M(\lambda), L)$ is either zero or a simple module. By the exactness of T one has $L(M(\lambda), L) \xleftarrow{\sim} L(M(\lambda), M_1)/L(M(\lambda), M_2)$. Set $J_i = L(M(\lambda), M_i) : i = 1, 2$ which identify with submodules of $L(M(\lambda), M(\lambda))$. Using the projectivity of $M(\lambda)$ one may compute the multiplicity of a $\underline{k}$ type E in $L(M(\lambda), M(\lambda))$ to be exactly $dim\ E^{\underline{h}}$ and this coincides with the multiplicity of E in the submodule $U(\underline{g})/Ann\ M(\lambda)$ by Kostant's theorem. Hence J_1, J_2 identify with ideals of $U(\underline{g})/Ann\ M(\lambda)$ and the assertion results from 1.3.6.

Since $L(M(\lambda), M(\lambda))$ identifies with $A := U(\underline{g})/Ann\ M(\lambda)$ it follows that TT' acts like the identity on A, hence on any module of the form $E \otimes A$ where E is finite dimensional. Take $V \in Ob\mathcal{H}_{\hat{\lambda}}$ to be finitely generated as a $U(\underline{g}) \otimes U(\underline{g})$ module. By (ii), V is finitely generated as a left $U(\underline{g})$ module. This leads to an exact sequence $E_1 \otimes A \to E_2 \otimes A \to V \to 0$ and right exactness of TT' implies that TT' acts like the identity on any finitely generated module in $Ob\mathcal{H}_\lambda$. In particular every simple object in $Ob\mathcal{H}_\lambda$ takes the form $L(M(\lambda), L(\mu)) : \mu \in \underline{h}^*$. (Moreover since $L(M(\lambda), L(\mu))$ is a submodule of the principal series module $L(M(\lambda), \delta M(\mu)) \cong L(-\mu, -\lambda)$, Frobenius reciprocity implies that $\mu - \lambda \in P(R) := \{\nu \in \underline{h}^* | (\nu, \alpha^\vee) \in \mathbb{Z}, \forall \alpha \in R\}$.) In particular the number of isomorphism classes of simple objects in $\mathcal{H}$ in which $Z(\underline{g}) \otimes Z(\underline{g})$ acts by a fixed scalar is bounded by *card* W. We conclude that each $V \in Ob\mathcal{H}$ has finite length using property 1.3.8(ii).

Finally take λ regular and $\mu \in \lambda + P(R)$. One shows that the $\underline{k}$ type with extreme weight $\lambda - \mu$ which occurs in $L(M(\lambda), \delta M(\mu))$ by Frobenius reciprocity, cannot occur in any $L(M(\lambda), \delta M(\mu'))$ with $\mu \neq \mu' \in \mu - \mathbb{N}B$. (This is easy for very dominant λ. It is then deduced for λ dominant, regular by the translation principle - see 2.1.1.). It therefore must occur in $L(M(\lambda), L(\mu))$ which is hence non-zero. Let $\mathcal{O}_\Lambda$ denote the subcategory of $\mathcal{O}$ of all modules whose weights lie in $\Lambda := \lambda + P(R)$.

THEOREM. - *Take $\lambda \in \underline{h}^*$ dominant and regular. The functor $M \mapsto L(M(\lambda), M)$ with adjoint $V \mapsto V \otimes_{U(\underline{g})} M(\lambda)$ defines an equivalence of categories from $\mathcal{O}_\Lambda$ to $\mathcal{H}_{\hat{\lambda}}$.*

Remark. - The above material was drawn from [4,13].

1.3.9. In the above theorem one has $d(T(M)) = 2d(M)$. Geometrically the above result corresponds to the bijective relation between B orbits in the flag manifold G/B and K orbits in $G/B \times G/B$ which K denotes the diagonal copy of $G \times G$ and B a Borel subgroup of G. In the above our restriction to $\mathcal{H}_{\hat{\lambda}}$ may be considered rather unnatural. Indeed let $\mathcal{O}_\Lambda^\infty$ be the category obtained from $\mathcal{O}_\Lambda$ by admitting extensions and ${}_{\hat{\mu}}^{\infty}\mathcal{O}_\Lambda^\infty$: $\hat{\mu} \in \underline{h}^*/W$ the corresponding $Z(\underline{g})$ primary component. Soergel [28] has shown that the categories ${}_{\hat{\mu}}^{\infty}\mathcal{O}_\Lambda^\infty$ and ${}_{\hat{\mu}}^{\infty}\mathcal{H}_{\hat{\lambda}}^\infty$ are equivalent for λ regular, where ${}_{\hat{\mu}}^{\infty}\mathcal{H}_{\hat{\lambda}}^\infty$ denotes the corresponding $Z(\underline{g}) \otimes Z(\underline{g})$ primary component with respect to the maximal ideal defined by $(\hat{\mu}, \hat{\lambda})$. (See 3.1.9 below).

Chapter 2.

2.1. Duflo's involutions.

2.1.1. We saw in 1.3.7 that the maps $\lambda \mapsto J(\lambda) \to \chi_\lambda$ of $\underline{h}^*$ to $Prim\ U(\underline{g})$ to $Max\ Z(\underline{g})$ are both surjective. A basic question now is to describe how $Prim\ U(\underline{g})$ sits between $\underline{h}^*$ and $\underline{h}^*/W$. The result turns out to depend on the extent to which λ is integral. In fact the case λ integral extends to the general case by replacing W by its Weyl subgroup W_λ generated by those $s_\alpha : (\alpha^\vee, \lambda) \in \mathbb{Z}$. For expository purposes we shall therefore restrict to the case λ integral i.e. when $\lambda \in P(R)$. Here we can take $\lambda \in P(R)^+ := \{\mu \in P(R) | \mu$ is dominant$\}$ without loss of generality. Set $P(R)^{++} = P(R)^+ + \rho$ which is just the set of regular elements of $P(R)^+$. The fibres for $\lambda \in P(R)^+$ are a degeneration of those for $\lambda \in P(R)^{++}$ so it is convenient to also assume λ regular. Set $\underline{X}_\lambda := \{J(w\lambda) | w \in W\}$. This is the inverse image of χ_λ in $Prim\ U(\underline{g})$. A result of Borho-Jantzen ([6], Sect. 2) asserts that $\underline{X}_\lambda$ viewed as an ordered set is independent of the choice of $\lambda \in P(R)^{++}$ and suggests that its structure should be expressible in terms of the combinatorics of W. This is just one of many such results called translation principles.

2.1.2. From now on we fix $\lambda \in P(R)^{++}$. We have seen that $[M(w\lambda)]$, $[L(w\lambda)] : w \in W$ form two bases of the Grothendieck group of the subcategory of $\mathcal{O}$ of all modules annihilated by a power of χ_λ. Identify $[M(w\lambda)]$ with w and then $[L(w\lambda)]$ by linearity with an element $a(w) \in \mathbb{Z}W$. The $a(w) : w \in W$ play a fundamental role in the description of $\underline{X}_\lambda$. The truth of the Kazhdan-Lusztig conjecture gives moreover a recipe for their computation [3,7,22]. (The results of [3] are reviewed and extended in Kashiwara's lectures appearing in this collection.)

2.1.3. Recall (1.3.2) that $J \mapsto \bar{J} := J/Ann\ M(\lambda)$ sets up to bijection between $\underline{X}_\lambda$ and $Spec(U(\underline{g})/Ann\ M(\lambda))$. As before we set $A_\lambda = U(\underline{g})/Ann\ M(\lambda)$ (or simply, A). By 1.3.8, A is artinian as a bimodule. We conclude that to each $P \in Spec\ A$ there exists a unique minimal ideal Q of A strictly containing P. As a bimodule Q/P is simple and coincides with the socle of $U(\underline{g})/P$. By the inequality in 1.1.5 we have $d(U(\underline{g})/Q) < d(U(\underline{g})/P)$. **More generally we** shall say that a $U(\underline{g})$ module M is quasi-simple if M has finite length, $Soc\ M$ is simple and

$d(M/Soc\ M) < d(M)$. We have seen that every prime quotient of A is quasi-simple and the converse is easily established. Then by 1.3.8 and 1.3.9 the quasi-simple quotients of $M(\lambda)$ take the form $M(\lambda)/PM(\lambda) : P \in Spec\ A_\lambda$ and index the elements of $Spec\ A_\lambda$. We can write $Soc(M(\lambda)/PM(\lambda)) = L(\sigma_P\lambda)$ for some $\sigma_P \in W$. We shall soon see that σ_P is an involution.

2.1.4. One of the advantages of the equivalence of categories theorem is that bimodules have clearly more structure. In particular we have an automorphism η of $U(\underline{g}) \otimes U(\underline{g})$ defined by $\eta(a \otimes b) = b \otimes a$. For each $U(\underline{g}) \otimes U(\underline{g})$ module V let V^η denote the $U(\underline{g}) \otimes U(\underline{g})$ module obtained by transport of structure. One checks that $V \in Ob\mathcal{H} \Rightarrow V^\eta \in Ob\mathcal{H}$. It is clear that $L(-\mu, -\nu)^\eta = L(-\nu, -\mu)$. Now recall (with $\lambda \in P(R)^{++}$) that $\delta M(w\lambda)$ admits $L(w\lambda)$ as its socle and so $L(M(\lambda), \delta M(w\lambda)) \cong L(-w\lambda, -\lambda)$ admits a simple socle which we denote by $V(-w\lambda, -\lambda)$. We shall prove in the sequel (3.2.11(ii)) that $L(-w\lambda, -\lambda) \cong L(-\lambda, -w^{-1}\lambda)$. This gives $L(-w\lambda, -\lambda)^\eta \cong L(-w^{-1}\lambda, -\lambda)$. The latter has $V(-w^{-1}\lambda, -\lambda)$ as its simple socle and so we conclude that $V(-w\lambda, -\lambda)^\eta \cong V(-w^{-1}\lambda, -\lambda)$. Let $\bar{J}$ be an ideal of $U(\underline{g})/Ann\ M(\lambda)$ viewed as a submodule of $End_k M(\lambda)$. Since $Ann\ M(\lambda)$ is a primitive ideal (1.3.3) one has ${}^t(Ann\ M(\lambda)) = Ann\ M(\lambda)$ by 1.3.4 and 1.3.7. Thus ${}^t\bar{J}$ is defined and one easily checks that ${}^t\bar{J}$ is isomorphic to $\bar{J}^\eta$ as a $U(\underline{g}) \otimes U(\underline{g})$ module. Now suppose $P := \bar{J} \in Spec(U(\underline{g})/Ann\ M(\lambda))$ and define Q as 2.1.3. Then by 1.3.4 we have ${}^tP = P$ and so ${}^tQ = Q$. Consequently $Q/P \cong L(M(\lambda), L(\sigma_P\lambda)) = V(-\sigma_P\lambda, -\lambda)$ is isomorphic to $V(-\sigma_P^{-1}\lambda, -\lambda)$ and hence $\sigma_P = \sigma_P^{-1}$. We call σ_p the Duflo involution associated to P. Let Σ^0 denote the set of all Duflo involutions. We have proved

PROPOSITION. - *The map $P \mapsto \sigma_P$ is a bijection of $Spec(U(\underline{g})/Ann\ M(\lambda))$ onto Σ^0.*

Here we recall that $\lambda \in P(R)^{++}$. By translation principles Σ^0 is independent of the choice of λ.

2.1.5. The Duflo involutions play a key role in the study of $Prim\ U(\underline{g})$ and are also of some importance in the study of the so-called Hecke algebra. This will be noted in the sequel.

2.2. Some applications of Gelfand-Kirillov dimension.

2.2.1. We prove below two lemmas on Gelfand-Kirillov dimension which simplify considerably subsequent analysis.

LEMMA. - *Let M, N be simple $U(\underline{g})$ modules. If $L(M,N) \neq 0$, then $d(M) = d(N)$.*

Let E be a finite dimensional $U(\underline{g})$ module. One easily shows that $d(E \otimes M) = d(M)$. Now if $L(M,N) \neq 0$, we have $0 \neq Hom_{\underline{g}}(E, Hom_{\mathbb{C}}(M,N)) \cong Hom_{\underline{g}}(E \otimes M, N)$, for some E finite dimensional and hence N is a quotient of $E \otimes M$. Consequently $d(N) \leq d(E \otimes M) = d(M)$. Again $Hom_{\underline{g}}(E, \ Hom_{\mathbb{C}}(M,N)) \cong Hom_{\underline{g}}(M, \ E^* \otimes N)$ and so M is a submodule of $E^* \otimes N$. Consequently $d(M) \leq d(E^* \otimes N) = d(N)$, proving the assertion.

2.2.2. **LEMMA.** - *Any Verma module $M(\mu)$ is quasi-simple.*

As a $U(\underline{n}^-)$ module $M(\mu)$ is isomorphic to $U(\underline{n}^-)$. Since $U(\underline{n}^-)$ is an integral domain of finite Gelfand-Kirillov dimension, it follows that $d(U(\underline{n}^-)/U(\underline{n}^-)a) < d(U(\underline{n}^-))$ for any $a \in U(\underline{n}^-)$ different to O. A fortiori $d(M(\mu)/M) < d(M(\mu))$ for any non-zero $U(\underline{g})$ submodule M of $M(\mu)$. (We remark that G.K. dimension calculated with respect to $U(\underline{g})$ and $U(\underline{n}^-)$ coincide for modules with a locally finite $U(\underline{b})$ action - e.g. in the $\mathcal{O}$ category.) It follows that any two non-zero submodules of $M(\mu)$ have non-zero intersection and since $M(\mu)$ has finite length, this proves the assertion.

2.2.3. It is clear from 2.2.2 that the socle L of any Verma module $M(\mu)$ is again a Verma module. Indeed L is clearly a simple highest weight module say $L(\nu)$ and the canonical surjection $M(\nu) \twoheadrightarrow L(\nu)$ is an isomorphism because $d(L(\nu)) = d(M(\mu)) = d(M(\nu))$. More generally any $\underline{g}$ homomorphism between Verma modules is necessarily injective and determined up to scalars.

2.2.4. Let $M(\mu) : \mu \in \underline{h}^*$ be a Verma module and fix a highest weight vector e_μ (of weight $\mu - \rho$) for $M(\mu)$. Let α be a simple root. Set $k := (\alpha^\vee, \mu)$ and assume that k is an integer ≥ 0. One easily checks that $x^k_{-\alpha} e_\mu$ is a highest weight vector and has weight $s_\alpha \mu - \rho$. This gives an embedding of $M(s_\alpha \mu)$ into $M(\mu)$. Now recall our convention that $\lambda \in P(R)^{++}$ and let w_0 be the longest element of W. It follows that for all $w \in W$ we have an embedding $M(w_0 \lambda) \hookrightarrow M(w\lambda)$. Since $M(w_0\lambda)$ is a simple module one has $Soc\ M(w\lambda) = M(w_0\lambda)$.

LEMMA. - *For all $w \in W$ the embedding $U(\underline{g})/Ann\ M(w\lambda) \hookrightarrow L(M(w\lambda), M(w\lambda))$ is an isomorphism.*

By 2.2.1, 2.2.2 and 2.2.3 we conclude that $L(M(w\lambda)/M(w_0\lambda), M(w\lambda)) = 0$ and hence that the map $L(M(w\lambda), M(w\lambda)) \to L(M(w_0\lambda), M(w\lambda))$ is an embedding which by 2.2.1 and 2.2.2 has image in $L(M(w_0\lambda), M(w_0\lambda))$. As we have seen the latter coincide with $U(\underline{g})/Ann\ M(w_0\lambda)$ by Kostant's theorem and the simplicity of $M(w_0\lambda)$.

2.2.5. We note without proof the following generalization of Bernstein's inequality ([23]).

LEMMA. - *For any finitely generated $U(\underline{g})$ module M one has $2d(M) \geq d(U(\underline{g})/Ann\ M)$.*

2.2.6. **PROPOSITION.** - *Let M, N be simple $U(\underline{g})$ modules. Then $L(M, N) \in Ob\mathcal{H}$.*

Property (i) of 1.3.8 holds by construction, whilst property (iii) is obvious. It remains to prove property (ii). We must show that if $\{\varphi_i\}_{i \in I}$ is a basis for $Hom_{\underline{g}}(M, E^* \otimes N) : E^*$ finite dimensional then *card* $I < \infty$. Since $End_{\underline{g}} M$ reduces to scalars by Quillen's lemma, one easily checks that $\Sigma_{i \in F} \varphi_i(M)$ is a direct sum for any finite subset F of I. Since we can assume $d(M) = d(N)$ by 2.2.1 we conclude that $e(M)(card\ F) \leq e(N)\ dim\ E^*$ (where $e(\cdot)$ denotes Bernstein multiplicity) and this bound proves the lemma.

Remark. Let M, N be simple $U(\underline{g})$ modules. Does $L(M, N) \neq 0$ imply $L(N, M) \neq 0$? This holds in the $\mathcal{O}$ category because $\delta N \cong N$ for N simple, whereas $L(\delta M, \delta N) \cong L(N, M)^\eta$.

2.2.7. **COROLLARY.** - *Let M be a simple $U(\underline{g})$ module. Then $L(M, M)$ is a primitive noetherian ring. Furthermore the embedding $U(\underline{g})/Ann\ M \hookrightarrow L(M, M)$ defines an embedding of rings of fractions.*

Set $A = U(\underline{g})/Ann\ M$, $A' = L(M, M)$. Since $A' \in Ob\mathcal{H}$, it has finite length and so by property 1.3.8(ii) it is generated as a $U(\underline{g}) - U(\underline{g})$ bimodule (and hence as a left $U(\underline{g})$ module) by a finite dimensional $\underline{k}$ stable subspace. Then the noetherianity of A implies the noetherianity of A'. Now let $s \in A$ be regular. We show that s is regular in A'. Choose $a \in A'$ such that $sa = 0$. Then Aa is a quotient of A/As and so $d(Aa) \leq d(A/As) < d(A)$

by the regularity of s. Yet $d(AaA) = d(Aa)$ by the $\underline{k}$ finiteness of A'. Set $V = AaA$ and suppose $V \neq 0$. Then $VM = M$ by the simplicity of M and so $Ann\ V = Ann\ M$ as a left $U(\underline{g})$ module. Similarly $Ann\ V = Ann\ M$ as a right $U(\underline{g})$ module. Since $Ann\ M$ is a prime ideal and V has finite length we conclude that V has a simple subquotient V_0 with left and right annihilators equal to $Ann\ M$. The corresponding module L_0 in the $\mathcal{O}$ category satisfies $Ann\ L_0 = Ann\ M$ and so $d(V_0) = 2d(L_0) \geq d(U(\underline{g})/Ann\ L_0) = d(A)$, which contradicts $d(V) < d(A)$ and so proves that $a = 0$. Similarly $as = 0 : a \in A'$ implies $a = 0$.

Finally let S denote the set of regular elements of A. Then flatness of localization gives an embedding $Fract\ A = S^{-1}A \hookrightarrow S^{-1}A' := S^{-1}A \otimes_A A'$. By the above the latter is a noetherian module over the simple artinian ring $S^{-1}A$, hence artinian from which it follows that $S^{-1}A'$ identifies with $Fract\ A'$.

Remark. Set $z_M = \frac{rk\ L(M,M)}{rk(U(\underline{g})/Ann\ M)}$, which is defined and a positive integer by the above result. It is an interesting invariant of M. Let $M = L(w\lambda) : \lambda \in P(R)^{++}$. Then z_M depends only on w and one may show [20] that z_M divides the order of a certain finite group associated to the two-sided cell (see 3.2.16) containing w.

2.2.8. A key fact which leads to an in depth analysis of $rk(U(\underline{g})/J(w\lambda)$ is the following.

LEMMA. - *Let σ be a Duflo involution. Then the embedding $Fract(U(\underline{g})/Ann\ L(\sigma\lambda)) \hookrightarrow Fract\ L(L(\sigma\lambda), L(\sigma\lambda))$ is an isomorphism.*

Set $P = Ann\ L(\sigma\lambda)$, $N = M(\lambda)/PM(\lambda)$. Recall that N is quasi-simple with socle $L(\sigma\lambda)$. Since $M(\lambda)$ is projective in $\mathcal{O}$, the natural map $L(M(\lambda), M(\lambda)) \rightarrow L(M(\lambda), N)$ is surjective. By 2.2.4 we deduce a surjective map $U(\underline{g})/Ann\ M(\lambda) \twoheadrightarrow L(M(\lambda), N)$ coming from the action of $U(\underline{g})$ on $M(\lambda)$. On the other hand the action of $U(\underline{g})$ on its quotient N gives embeddings $U(\underline{g})/Ann\ N \hookrightarrow L(N,N) \hookrightarrow L(M(\lambda), N)$ and so we conclude that we have an isomorphism $U(\underline{g})/Ann\ N \xrightarrow{\sim} L(N,N)$. Furthermore the isomorphism $L(N,N) \xrightarrow{\sim} L(M(\lambda), N)$ defines an isomorphism $L(N, L(\sigma\lambda)) \xrightarrow{\sim} L(M(\lambda), L(\sigma\lambda))$, since the latter is a simple module.

Since $d(N/L(\sigma\lambda)) < d(L(\sigma\lambda))$ it follows by 2.2.1 that the natural map $L(N, L(\sigma\lambda)) \xrightarrow{\varphi} L(L(\sigma\lambda), L(\sigma\lambda)) =: A'$ is an embedding. Let I denote its image which identifies with a left ideal of A' and is a right $U(\underline{g})$ module. Let J denote the right annihilator of I in A'. Then J is a left $U(\underline{g})$ module and $J \neq 0$ implies $JL(\sigma\lambda) = L(\sigma\lambda)$ and so

$IL(\sigma\lambda) = 0$ contradicting that φ is an embedding and $L(N, L(\sigma\lambda)) \neq 0$. Since A' is a prime, noetherian ring (2.2.7) we conclude that I is an essential left ideal of A'. Yet I identifies with the socle of the prime noetherian subring $U(\underline{g})/Ann\ L(\sigma\lambda)$ viewed as a $U(\underline{g})$ bimodule and so the assertion follows.

Remarks. The above result was drawn from [15],I. We saw that $Soc\ L(L(\sigma\lambda), L(\sigma\lambda)) = L(M(\lambda), L(\sigma\lambda))$. This suggests that $\sigma \in \Sigma^0$ may lead to an idempotent. In fact Σ^0 leads to certain important idempotents in the Hecke algebra. For further details see [18-20,24,25].

2.2.9. Let M be a finitely generated $U(\underline{g})$ module. We recall that $2d(M) \geq d(U(\underline{g})/Ann\ M) \geq d(M)$. We call M *holonomic* if $2d(M) = d(U(\underline{g})/Ann\ M)$ and *strongly holonomic* if this equality holds for all subquotients of M.

We remark that for $U(\underline{g})$, Gelfand-Kirillov dimension has the following two properties (see [23] for example).

(i) $d(M) = max\{d(N), d(M/N)\}$, *for any submodule* N *of* M *(partivity).*

(ii) *If* $M = M_1 \supset M_2 \supset \cdots$, *then* $d(M_i/M_{i+1}) < d(M)$, $\forall\ i >> 0$ *(finite partivity).*

Both are proved by passing to $gr\ U(\underline{g})$ and using the Hilbert-Samuel polynomial. Even (i) fails for arbitrary finitely generated rings.

Call M d-homogeneous (resp. d-critical) if $d(M) = d(N)$ (resp. $d(M/N) < d(M)$) for any non-zero submodule N of M. By (i), d-critical $\Rightarrow$ d-homogeneous and these properties pass to submodules. Then by (ii), every module M has a d-critical submodule. Noetherianity implies that we can find a "d-critical chain" $M = M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_{n+1} = 0$ such that M_i/M_{i+1} is d-critical for each $1 \leq i \leq n$.

Gelfand-Kirillov dimension satisfies ideal invariance, namely $d(I \otimes_{U(\underline{g})} M) \leq d(M)$ for every two-sided I of $U(\underline{g})$. Assume that $IJ \subset Ann\ M$, for two-sided ideals I, J of $U(\underline{g})$. Then we have a surjective map $I \otimes_{U(\underline{g})} (M/JM) \to IM$ and hence $d(IM) \leq d(I \otimes_{U(\underline{g})} M/JM) \leq d(M/JM)$. Consequently if M is d-critical either $IM = 0$ or $JM = 0$, that is $Ann\ M \in Spec\ U(\underline{g})$.

Let Hol denote the category of finitely generated modules M for which

(i) *M is strongly holonomic.*

(ii) $Ann_{Z(\underline{g})}M$ *has finite codimension in* $Z(\underline{g})$.

PROPOSITION. - *Each* $M \in Ob\ Hol$ *has finite length.*

We can assume that $Ann_{Z(\underline{g})}M \in Max\ Z(\underline{g})$ and M is d-critical without loss of generality. Consequently there exists $\lambda \in \underline{h}^*$ dominant such that M is a module over $A := U(\underline{g})/Ann\ M(\lambda)$ and $P := Ann_A M \in Spec\ A$. Define $Q \supsetneq P$ as in 2.1.3. Clearly $Q^2 = Q$. Set $N = QM$, which is a non-zero submodule of M. The assertion will result from noetherian induction if we can show that N is simple. Let N_0 be a proper non-zero submodule of N. Then $d(N/N_0) < d(N)$ because N is d-critical. Since M is strongly holonomic we obtain $d(U(\underline{g})/Ann(N/N_0)) = 2d(N/N_0) = 2d(N) = 2d(M) = d(U(\underline{g})/Ann\ M)$ and consequently $Ann\ (N/N_0) \supsetneq P$ which implies $Ann(N/N_0) \supset Q$. Hence $QN \subset N_0$ which contradicts $QN = Q^2M = QM = N$.

2.2.10. It is easy to show that modules in $\mathcal{O}_\Lambda^\infty$ (notation 1.3.9.) are holonomic, hence strongly holonomic. Using the fidelity of the Casselman functor Vogan ([39], Cor. 4.7) showed that the category of $(\underline{g}, \underline{k})$ admissible modules with central character are strongly holonomic. (Here $\underline{k}$ is the set of fixed points of an involution of $\underline{g}$.) More generally Gabber (unpublished) proved the following. Let $\mathcal{V}(M)$ denote the associated variety of M (defined by taking a good filtration on M and setting $\mathcal{V}(M) = \{$set of zeros of Ann gr M$\}$).

THEOREM. - *Suppose* $\mathcal{V}(M) \cap \underline{n}^- = \{0\}$ *and (ii) above holds. Then M is holonomic. (In particular* $M \in Ob\ Hol$*).*

We sketch very briefly the proof which is not particularly difficult. We can assume $Ann_{Z(\underline{g})}M \in Max\ Z(\underline{g})$ without loss of generality. Then $\mathcal{V}(M) \subset \mathcal{V}(Ann_{U(\underline{g})}M) \subset \mathcal{N}$ (variety of nilpotent elements). Now one knows (Spaltenstein [38], 2.8) for any nilpotent orbit $\mathcal{C}$ that $dim(\mathcal{C} \cap \underline{n}^+) = \frac{1}{2}\ dim\ \mathcal{C}$. Since $\mathcal{N}/G$ is finite this implies that for any G stable subvariety $\mathcal{W} \subset \mathcal{N}$ one has $dim(\mathcal{W} \cap \underline{n}^+) = \frac{1}{2}\ dim\ \mathcal{W}$. Taking $\mathcal{W} = \mathcal{V}(Ann_{U(\underline{g})}M)$ this implies the required assertion if $\mathcal{V}(M) \subset \underline{n}^+$, noting that $d(M) = dim\ \mathcal{V}(M)$ and $d(U(\underline{g})/Ann\ M) = dim\ \mathcal{V}(Ann_{U(\underline{g})}M)$. The general case obtains by deforming $\mathcal{V}(M)$ into $\underline{n}^+$. In fact choose $H \in \underline{h}$ such that $\alpha(H) \in \mathbb{N}^+$ for every $\alpha \in R^+$ and define $\varphi : k^* \mapsto End_k \underline{g}$ by $\varphi(t)X = t^{-m}X$ when $[H, X] = mX$. Set $\mathcal{S} = \{(t, \varphi(t)y) : t \neq 0,\ y \in \mathcal{V}(M)\}$ and

$\mathcal{V}_0 = \bar{\mathcal{S}} \cap \{\{0\} \times \underline{g}^*\}$. One checks that $dim\ \mathcal{V}_0 = dim\ \mathcal{V}(M)$. Moreover identifying $\mathcal{V}_0$ with its image under the canonical projection $\pi : \mathbb{A}^1 \times \underline{g}^* \to \underline{g}^*$ one checks that the hypothesis implies that $\mathcal{V}_0 \subset \underline{b}$. On the other hand $\mathcal{V}_0 \subset \mathcal{W} \subset \mathcal{N}$ and so $\mathcal{V}_0 \subset (\underline{n}^+ \cap \mathcal{W})$ as required.

Remark. Suppose M is $(\underline{g}, \underline{k})$ admissible and $dim(Z(\underline{g})/Ann_{Z(\underline{g})}M) < \infty$. We can always choose $\underline{h}$, R^+ such that $\underline{k} + \underline{b}^- = \underline{g}$ (where $\underline{b}^- = \underline{h} + \underline{n}^-$). Since $dim\ U(\underline{k})m < \infty$, $\forall\ m \in M$, it follows that $\mathcal{V}(M) \subset \underline{k}^\perp$. Yet $\underline{k}^\perp \cap (\underline{b}^-)^\perp = 0$ and $(\underline{b}^-)^\perp = \underline{n}^-$, so M satisfies the hypothesis of the theorem. Of course, this result also applies if $M \in \mathcal{O}_\Lambda^\infty$ and more generally for so-called Whittaker modules.

2.3. The Goldie rank additivity principle.

2.3.1. As before we fix $\lambda \in P(R)^{++}$ and $\mu \in P(R)$. Set $V = L(M(\lambda), L(\mu))$ which by 1.3.8 is a simple $U(\underline{g})$ bimodule. Consider V as a left $A' := L(L(\mu), L(\mu))$ and right $U(\underline{g})$ module. Since $L(\mu)$ is simple it follows that left multiplication defines an embedding $A' \hookrightarrow End_{U(\underline{g})}V$.

PROPOSITION. - *One has* $A' \xrightarrow{\sim} End_{U(\underline{g})}V$.

Let us first show that $End_{U(\underline{g})}V$ is $\underline{k}$ locally finite. This means the following. Given $\theta \in End_{U(\underline{g})}V$ and $X \in \underline{g}$, we define $\theta_X \in End_{U(\underline{g})}V$ through $\theta_X(v) = X\theta(v) - \theta(Xv)$. We must show that this action is locally finite. Set $[X, v] = Xv - vX$. Then

$$(*) \qquad \theta_X(v) = [X, \theta(v)] - \theta([X, v]) \ .$$

Now since V is simple and locally $\underline{k}$ finite, there exists a finite dimensional subspace V_0 which is $\underline{k}$ stable and generates V as a bimodule, hence as a right $U(\underline{g})$ module. Obviously $\theta(V_0) = 0 \Rightarrow \theta(V) = 0 \Rightarrow \theta = 0$. Now by (*) we have

$$\theta_X(V_0) \subset [X, \theta(V_0)] + \theta(V_0)$$

from which we conclude that $\theta_X(V_0) \subset U(\underline{k})\theta(V_0)$ which is a finite dimensional subspace of V. Combined with the previous observation this proves the required assertion.

To prove the proposition it remains to compare $\underline{k}$ types. Indeed we must show that

$$(**) \qquad Hom_{\underline{g}\times\underline{g}}(E \otimes k,\ End_k V) \cong Hom_{\underline{g}}(E,\ End_k L(\mu))$$

for every finite dimensional simple $\underline{g}$ module. Here the $\underline{g}$ (or $\underline{g} \times \underline{g}$) action on $End_k(\cdot)$ means the diagonal action. In fact the left hand side of (**) is isomorphic to

$$\begin{aligned} Hom_{\underline{g}\times\underline{g}}(V \otimes (E \otimes k), V) &\cong Hom_{\underline{g}\times\underline{g}}(L(M(\lambda), L(\mu) \otimes E), V), \\ &\cong Hom_{\underline{g}\times\underline{g}}(T(L(\mu) \otimes E), T(L(\mu))), \\ &\cong Hom_{\underline{g}}(L(\mu) \otimes E, L(\mu)) \end{aligned}$$

by the equivalence of categories theorem. The latter is isomorphic to the right hand side of (**).

2.3.2. Retain the above notation and let E be a finite dimensional $\underline{g}$ module. Appropriate identifications give

COROLLARY. - *One has* $A' \otimes End\ E \xrightarrow{\sim} End_{U(\underline{g})} L(M(\lambda), E \otimes L(\mu))$.

2.3.3. Let M be a $U(\underline{g})$ module of finite length. We call (see 2.1.3) $M = M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_{n+1} = 0$ a quasi-composition series for M if $d(M_i/M_{i+1}) = d(M)$ and M_i/M_{i+1} is quasi-simple for each i. One easily checks that M admits a quasi-composition series under the additional hypothesis that M is d-homogeneous. Now take $M = E \otimes L(\mu)$. As in 2.2.1 it follows that M is d-homogeneous and we set $L_i = Soc(M_i/M_{i+1})$ which is a simple highest weight module. It is clear that the L_i are exactly the simple subquotients of M satisfying $d(L_i) = d(M)$.

THEOREM. - *One has*

$$rk\ L(L(\mu), L(\mu))\ dim\ E = \sum_i rk\ L(L_i, L_i)\ .$$

We sketch briefly how 2.3.3 results from 2.3.2. The quasi-composition series for $E \otimes L(\mu)$ gives rise by the equivalence of categories theorem to a quasi-composition series for $V' := L(M(\lambda), E \otimes L(\mu))$ with factors $V_i := L(M(\lambda), M_i/M_{i+1})$. Let I denote the annihilator of $L(M(\lambda), E \otimes L(\mu))$ considered as a right $U(\underline{g})$ module and set $A = U(\underline{g})/I$. Clearly I coincides with the annihilator of $L(M(\lambda), L(\mu))$ as a right $U(\underline{g})$ module. The latter is a simple $U(\underline{g})$ bimodule and using η and the equivalence of categories theorem we may compute I and show in particular that it is primitive. Let S denote the set of regular elements of A. By Goldie's theorem we may form the simple artinian ring AS^{-1}. Then the right

AS^{-1} module $V'S^{-1}$ inherits a composition series with the V_iS^{-1} as factors (terms of lower Gelfand-Kirillov dimension disappear on passage to fractions). As in 2.2.7 one shows that $V'S^{-1}$ is a left $Fract\ A' \otimes End\ E$ module and by 2.3.2 that left multiplication gives an isomorphism

$$Fract\ A' \otimes End\ E \xrightarrow{\sim} End_{AS^{-1}}(V'S^{-1}) \ .$$

Consequently the length of $V'S^{-1}$ as a right AS^{-1} module coincides with $rk(A' \otimes End\ E) = rk\ A'\ dim\ E$. On the other hand set $A'_i = L(L_i, L_i), \forall\ i$. Then one shows that each V_iS^{-1} is a left $Fract\ A'_i$ module and by 2.3.1 that left multiplication gives an isomorphism

$$Fract\ A'_i \xrightarrow{\sim} End_{AS^{-1}}(V_iS^{-1}) \ .$$

Consequently the length of V_iS^{-1} as a right AS^{-1} coincides with $rk\ A'_i$. Finally comparison of lengths gives the assertion of the theorem.

2.3.4. For each $w \in W$ we define the Goldie rank functions

$$p_w(\mu) = rk\left(U(\underline{g})/Ann\ L(w\mu)\right), \quad q_w(\mu) = rk\ L\left(L(w\mu), L(w\mu)\right)\ \forall\ \ \mu \in P(R)^+ \ .$$

COROLLARY. -

(i) q_w *extends to a polynomial on* $\underline{h}^*$.

(ii) p_w *extends to a polynomial on* $\underline{h}^*$.

(iii) $z_w := q_w/p_w \in \mathbb{N}^+$.

Fix $d \in \mathbb{N}$ and let $\mathcal{O}(d)$ denote the subcategory of all $M \in Ob\mathcal{O}$ satisfying $d(M) \le d$. Let $\mathcal{C}(d)$ denote the set of all formal characters of modules in $\mathcal{O}(d)$. This admits the $ch\ L(\nu) : d(L(\nu)) \le d$ as a basis. Define an additive function g_d on $\mathcal{C}(d)$ by

$$g_d(ch\ L(\nu)) = \begin{cases} 0 & : d(L(\nu)) < d, \\ rk\ L(L(\nu), L(\nu)) & : d(L(\nu)) = d \ . \end{cases}$$

It follows from 2.3.3 that

$$(1) \qquad g_d((ch\ E)\chi) = dim\ E \quad g_d(\chi)$$

for all $\chi \in \mathcal{C}(d)$ and all finite dimensional $U(\underline{g})$ modules E.

Now each $w \in W$ we define $\chi_w(\mu) := ch\ L(w\mu) : \mu \in P(R)^{++}$. As noted in 2.1.2 there exist $a(w,y) \in \mathbb{Z}$ such that

$$\chi_w(\mu) = \sum_{y \in W} a(w,y) ch\ M(y\mu) \ .$$

(By the translation principle, the $a(w,y)$ do not depend on μ). Now the right hand side is defined for all $\mu \in P(R)$ and we use this to define $\chi_w(\mu), \forall\ \ \mu \in P(R)$. (Here one should remark that if $\chi_w(\mu) \neq 0 : \mu \in P(R)^+$ then $\chi_w(\mu) = ch\ L(w\mu)$; it is may also vanish for $\mu \in P(R)^+$). From the formula

$$ch\ M(y\mu) = \frac{e^{y\mu-\rho}}{\Pi_{\alpha \in R^+}(1-e^{-\alpha})}$$

one checks for all $\nu \in P(R)^+$ that

$$\chi_w(\mu)\left(\sum_{z \in W} e^{z\nu}\right) = \sum_{z \in W} \chi_w(\mu + z\nu) \ . \tag{2}$$

Through the Weyl character formula for $ch\ E$ we obtain from (1) by induction on $0(\nu) := \sum\ k_i$ (where $\nu = \sum\ k_i \bar{\omega}_i \in P(R)^+$) that

$$g_d\left(\left(\sum_{z \in W} e^{z\nu}\right)\chi\right) = |W| g_d(\chi) \ . \tag{3}$$

Now take $w \in W$ and $d = d(L(w\mu)) : \mu \in P(R)^{++}$. (One may show that $d(L(w\mu))$ is independent of the choice of μ by translation principles). Setting $\psi(\mu) = g_d\,(\chi_w(\mu)) : \mu \in P(R)$ we obtain from (2) and (3) that

$$\sum_{z \in W} \psi(\mu + z\nu) = |W|\psi(\mu),\ \forall\ \ \nu \in P(R)^+,\ \ \mu \in P(R) \ .$$

By Pittie's theorem ψ extends to a polynomial on $\underline{h}^*$ (even a W harmonic polynomial). This proves (i). (ii) obtains from (i), 2.1.4 and 2.2.8.

For (iii) one first notes taking $M = L(w\mu)$ in 2.2.7 that $z_w(\mu) := q_w(\mu)/p_w(\mu) \in \mathbb{N}^+$. Simple growth rate estimates which we omit shows that $z_w(\mu)$ is uniformly bounded on

$P(R)^+$ and hence $z_w(\mu)$ is constant on some Zariski dense subset, hence constant by (i) and (ii).

2.3.5. One may compute the q_w rather explicitly, although the details are rather involved. First recall the definition of $a(w) \in \mathbb{Z}W$ given in 2.1.2. We note without proof the following

THEOREM. - *Fix $w \in W$. Let $m(w)$ denote the least integer ≥ 0 such that $a(w^{-1})\rho^{m(w)} \neq 0$. Then*

(i) $d(L(w\lambda)) = |R^+| - m(w)$,

(ii) $q_w = c\, a(w^{-1})\rho^{m(w)}$,

for some non-zero rational scalar c.

__Remarks.__ The results above were drawn from [15] with improvements due to Jantzen [12]. The precise value of c is as yet unknown (see [19], [29]). Note the remarkable equality $deg\ q_w + d(L(w\lambda)) = |R|$ which expresses the not quite true general idea that Goldie rank increases with the size of $J \in \underline{X}_{\hat{\lambda}}$. Note that some information on $Prim\ U(\underline{g})$ is contained in the values of the $a(w)$. This represents a general phenomenon and was an important guideline in the study of $Prim\ U(\underline{g})$. The distinct p_w can be partitioned into subsets which form bases of irreducible representations of W which were shown by Barbasch and Vogan [1,2] to be the special representations in the sense of Lusztig using case by case analysis. This last result may now be proved in a more elegant fashion through a characterization of Duflo involutions provided by ([15],III, 4.9 and [24], Prop. 1.4). Originally this used in particular the truth of the Jantzen conjecture but this can now be replaced by the main result of [11]. This leads to a new characterization of $m(w)$ related to the leading power occurring in character values of Hecke algebras ([20], 2.6) or ([25], Sect. 3) which eventually gives the required result. In more detail for each simple W module E define a_E, b_E as in [36], p.76. One has $a_E \leq b_E$ and one calls E *special* if equality holds. Now fix a left cell $\mathcal{C}$ of W (see 3.2.16 below). Then the associated $\mathbb{Q}$ module $[\mathcal{C}] \otimes_{\mathbb{Z}} \mathbb{Q}$ has a $\mathbb{Q}W$ module structure and admits (with multiplicity one) the Goldie rank module E_0 (which is simple) spanned by the $p_{w^{-1}} : w \in \mathcal{C}$ as a submodule. It is a consequence of 2.3.5 that if E is another simple submodule of $[\mathcal{C}]$ then $b_E > b_{E_0} = m(w), \forall w \in \mathcal{C}$, whereas by ([20], 2.6) or ([25], Sect. 3) one has $a_E = a_{E_0}$. Thus, only the Goldie rank module can be special. To show that it is always special one needs to show that $a_E = m(w)$ for the above choices of E and w. Both integers behave well under induction so one is reduced to the case when the corresponding two-sided cell $\mathcal{DC}$ does not contain an element of the form yw_0 with y contained in a proper Weyl subgroup of W. However, $\mathcal{DC}$ then contains a longest element in a proper Weyl subgroup and in this case the equality is rather easy (via [15], III, 4.2, 4.9). Unfortunately this last fact still needs some case by case analysis for its proof. It is equivalent to saying that the dual of special non-induced orbit is even (all this in sense of [31], Sect. 5). Actually, such an orbit is distinguished ([31], Prop. 5.1).

Chapter 3.

3.1. The ring of projective modules.

3.1.1. As before we fix $\lambda \in P(R)^{++}$. We already remarked (1.3.8) that $M(\lambda)$ is projective in $\mathcal{O}$ (and hence in $\mathcal{O}_\Lambda$). From the natural isomorphism $Hom_{\underline{g}}(E \otimes_k M,\ N) \xrightarrow{\sim} Hom_{\underline{g}}(M, Hom_k(E \otimes N))$ and because $N \in Ob\mathcal{O}$ and E finite dimensional implies $E \otimes N \in Ob\mathcal{O}$ one easily deduces that $E \otimes M(\lambda)$ and its direct summands are projective in $\mathcal{O}_\Lambda$. On the other hand $V \in Ob\mathcal{H}_{\hat{\lambda}}$ has finite length and is hence generated as a bimodule by a finite dimensional subspace E which we can assume to be $\underline{k}$ stable. Then $V = EU(\underline{g})$ and so we have a surjective map $E \otimes_k U(\underline{g}) \twoheadrightarrow V$. Setting $A_\lambda := U(\underline{g})/Ann\ M(\lambda)$, this leads to surjective map $E \otimes_k A_\lambda \twoheadrightarrow V$. Yet $E \otimes_k A_\lambda \cong E \otimes_k L(M(\lambda), M(\lambda)) \cong T(E \otimes M(\lambda))$. By the equivalence of categories theorem we conclude that every $M \in Ob\mathcal{O}_\Lambda$ is the image of some $E \otimes M(\lambda)$ and so $\mathcal{O}_\Lambda$ has enough projectives (and enough injectives using δ). (On the other hand $\mathcal{O}_\Lambda^\infty$ does not have any projectives).

3.1.2. Given $\mu \in \Lambda$, choose $P \in Ob\mathcal{O}_\Lambda$ projective such that $P \overset{\varphi}{\twoheadrightarrow} L(\mu)$. Let $P(\mu)$ denote a minimal submodule of P such that $\varphi' := \varphi|_{P(\mu)} \neq 0$. Clearly $rad\ P(\mu) = L(\mu)$. Since P is projective we have a map $\theta : P \to P(\mu)$ such that $\varphi = \varphi'\theta$. Then $Im\theta \subset P(\mu)$ and $\varphi|_{Im\theta} \neq 0$, from which we conclude that θ is surjective. One easily checks that $\theta|_{P(\mu)} = Id_{P(\mu)}$ and we conclude that $P = P(\mu) \oplus ker\ \theta$. Thus $P(\mu)$ is an indecomposable projective module mapping onto $L(\mu)$. One easily shows that every such module is isomorphic to $P(\mu)$ which is called the *projective cover* of $L(\mu)$. Finally one shows that every indecomposable projective in $\mathcal{O}_\Lambda$ is isomorphic to some $P(\mu)$.

3.1.3. Let us now take $\mu \in P(R)^{++}$ and restrict to the subcategory ${}^{\infty}_{\hat{\mu}}\mathcal{O}_\Lambda$ of objects in $\mathcal{O}_\Lambda$ annihilated by some power of $\chi_{\hat{\mu}}$. The indecomposable projective modules in this subcategory take the form $P(w\mu) : w \in W$.

3.1.4. A module $M \in Ob\mathcal{O}$ is said to have a *Verma flag* if there exists a decreasing sequence $M = M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_{n+1} = 0$ of submodule of M such that each M_i/M_{i+1} is isomorphic to some $M(\mu_i) : \mu_i \in \underline{h}^*$. In this case we write

$$[M : M(\mu)] = card\{i \in \{1,2,\cdots,n\} | \mu = \mu_i\} \ .$$

Since the $[M(\mu)] : \mu \in \underline{h}^*$ form a basis for the Grothendieck group of $\mathcal{O}$, it follows that $[M : M(\mu)]$ is independent of the choice of Verma flag.

Given E a finite dimensional $U(\underline{g})$ we let $\Omega(E)$ denote its set of weights (counted with multiplicities).

LEMMA. - *Take $\mu \in \underline{h}^*$. Then $E \otimes M(\mu)$ has a Verma flag and $[E \otimes_k M(\mu) : M(\nu)] = dim_k E_{\nu-\mu}$.*

This follows from the isomorphism

$$E \otimes_k \left(U(\underline{g}) \bigotimes_{U(\underline{b})} k_{\mu-\rho} \right) \cong U(\underline{g}) \bigotimes_{U(\underline{b})} \left(E|_{\underline{b}} \otimes k_{\mu-\rho} \right)$$

and the exactness of the functor $U(\underline{g}) \otimes_{U(\underline{b})} -$.

3.1.5. Recall that we have an ordering on $\underline{h}^*$ defined by $\mu \geq \nu$ if $\mu - \nu \in \mathbb{N}B$. It is quite trivial to verify that $Ext^1(M(\mu), M(\nu)) = 0$ unless $\mu < \nu$. This has the following consequence. Set $\Omega(M) = \{\mu \in \underline{h}^* | M_\mu \neq 0\}$ and choose $\nu \in \Omega(M) + \rho$ maximal. If M has a Verma flag, then $\ell := [M : M(\nu)] \neq 0$ and the above vanishing of Ext^1 implies that we can find a Verma flag with $M(\nu)^\ell$ as a submodule.

3.1.6. Since $M(\mu)$ is a free (rank 1) $U(\underline{n}^-)$ module, it follows that a module M with a Verma flag is $U(\underline{n}^-)$ free. Thus given a highest weight vector $e \in M$ of weight $\nu - \rho$ we conclude that $U(\underline{g})e \cong M(\nu)$.

LEMMA. - *If $M = N_1 \oplus N_2 \in Ob\mathcal{O}$ has a Verma flag, then so has N_1, N_2. In particular any projective object in $\mathcal{O}$ has a Verma flag.*

Choose $\nu \in \Omega(M) + \rho$ maximal. If say $\nu \in \Omega(N_1) + \rho$, choose a vector e of weight $\nu - \rho$ in N_1. Then $U(\underline{g})e \cong M(\nu)$ and $M/M(\nu) \cong N_1/M(\nu) \oplus N_2$. The result follows by induction on length taking account of 3.1.5.

Remark. If $\varphi : M \to N$ is a map between modules with a Verma flag, then $coker\ \varphi$ has a Verma flag; but $ker\ \varphi$ does not in general.

3.1.7. **LEMMA.** - *For all* $\mu \in \underline{h}^*$, $M,\ P \in Ob\mathcal{O}$, *one has*

(i) $Hom_{\underline{g}}(P(\mu), M) = [M : L(\mu)]$.

(ii) $[P : M(\mu)] = dim\ Hom_{\underline{g}}(P,\ M(\mu))$ *if* P *is projective.*

(i) By additivity on exact sequences, this reduces to the case $M \cong L(\mu)$, for which it is trivial.

(ii) It is enough to prove the assertion for $P = E \otimes M(\lambda)$ (with $\mu \in \Lambda$ - here we need not assume that λ is integral). By 3.1.4 it is enough to show that $dim\ Hom_{\underline{g}}(E \otimes M(\lambda), M(\mu)) = dim\ E_{\mu-\lambda}$. Now $E^* \otimes M(\mu)$ has a Verma flag with terms $M(\xi) : \xi \in \mu + \Omega(E^*)$, and since $M(\lambda)$ is projective we conclude that

$$\begin{aligned} dim\ Hom_{\underline{g}}(E \otimes M(\lambda),\ M(\mu)) &= dim\ Hom_{\underline{g}}(M(\lambda), E^* \otimes M(\mu)) \\ &= dim\ E^*_{\lambda-\mu}\ , \\ &= dim\ E_{\mu-\lambda}\ , \end{aligned}$$

as required.

3.1.8. **COROLLARY.** - (BGG *reciprocity [32]). For all* $w,\ y \in W$ *one has*

$$[P(w\lambda) : M(y\lambda)] = [M(y\lambda) : L(w\lambda)]\ .$$

3.1.9. We define a multiplicative structure on $U(\underline{g})$ bimodules through

$$V_* V' = V \otimes_{U(\underline{g})} V'\ .$$

Given $V,\ V' \in Ob\mathcal{H}$, we claim that $V_*V' \in Ob\mathcal{H}$. Property 1.3.8(iii) is obvious. Recalling that we can write $V = EU(\underline{g})$ for some finite dimensional $\underline{k}$ stable subspace E of V we conclude that V_*V' is an image of $E \otimes_{\underline{k}} V'$ which taking account of the action of $\underline{k}$ makes assertions 1.3.8(i), (ii) obvious. Again it is obvious that ${}^{\infty}_{\hat{\lambda}}\mathcal{H}^{\infty}_{\hat{\lambda}}$ is a multiplicative subset.

To go further we make use of Soergel's improved version of 1.3.8. First set $M^i(\lambda) = U(\underline{g}) \otimes_{U(\underline{b})} U(\underline{b}) / \left(Ann_{U(\underline{b})} k_{\lambda-\rho}\right)^i$, $\forall\, i \in \mathbb{N}^+$. It is clear that we have maps

$$0 \longleftarrow M^1(\lambda) \longleftarrow M^2(\lambda) \longrightarrow \cdots \qquad ,$$

each with kernel a direct sum of $M(\lambda)$. In particular $M^i(\lambda)$ is a non-split extension $M(\lambda)$ lying in ${}^{\infty}_{\hat{\lambda}}\mathcal{O}^{\infty}_{\Lambda}$. The functors T, T' of 1.3.6 are replaced by

$$T_{\infty}M = \varinjlim L(M^i(\lambda), M) : M \in Ob\mathcal{O}^{\infty}_{\Lambda}$$

$$T'_{\infty}V = \varprojlim V \otimes_{U(\underline{g})} M^i(\lambda) : V \in Ob\mathcal{H}^{\infty}_{\hat{\lambda}}$$

They define an equivalence of categories between $\mathcal{O}^{\infty}_{\Lambda}$ and $\mathcal{H}^{\infty}_{\hat{\lambda}}$ which restricts to an equivalence of categories of between ${}^{\infty}_{\hat{\lambda}}\mathcal{O}^{\infty}_{\Lambda}$ and ${}^{\infty}_{\hat{\lambda}}\mathcal{H}^{\infty}_{\hat{\lambda}}$. Set $J_i = Ann\ M^i(\lambda)$ and observe that $T'_{\infty}\left(U(\underline{g})/J_i\right) = M^i(\lambda)$. Also $J_i = (ker\chi_{\lambda})^i U(\underline{g})$ by 1.3.3.

Given $M \in Ob\mathcal{O}^{\infty}_{\Lambda}$ let $M_{\hat{\lambda}}$ denote its $Z(\underline{g})$ primary component lying in $Ob\left({}^{\infty}_{\hat{\lambda}}\mathcal{O}^{\infty}_{\Lambda}\right)$. By 3.1.1 there exists E finite dimensional such that $P(w\lambda)$ is a direct summand of $E \otimes M(\lambda)$. Set $V_w = T_{\infty}(P(w\lambda))$. Then V_w is a direct summand of $E \otimes (U(\underline{g})/J_1)$. By the right action of $Z(\underline{g})$ we may find a direct summand V^i_w of $E \otimes U(\underline{g})/(ker\chi_{\lambda})^i U(\underline{g})$ and maps

$$0 \longleftarrow V^1_w \longleftarrow V^2_w \longleftarrow$$

such that each successive quotient is the corresponding direct sum of the V_w. Define a functor θ_w on ${}^{\infty}_{\hat{\lambda}}\mathcal{H}^{\infty}_{\hat{\lambda}}$ (and hence on ${}^{\infty}_{\hat{\lambda}}\mathcal{O}^{\infty}_{\lambda}$) via $V \mapsto \lim_{\longleftarrow} V^i_{w*}V$. It is a direct summand of the functor $\theta_E : M \mapsto (E \otimes M)_{\hat{\lambda}}$ defined on ${}^{\infty}_{\hat{\lambda}}\mathcal{O}^{\infty}_{\Lambda}$ and hence exact.

3.1.10. Let $b(w) : w \in W$ denote the element of $\mathbb{Z}W$ defined by setting $y = [M(y\lambda)]$ and then $b(w) = [P(w\lambda)]$ (c.f. 2.1.2). By 3.1.8 the $b(w)$ can be computed from the $a(w)$ and vise-versa. The advantage of $b(w)$ is that they satisfy remarkable positivity constraints. First of all the $b(w)$ lie in $\mathbb{N}W$. Secondly

PROPOSITION. - *For all $x, y \in W$ one has*

$$b(x)b(y) = \sum_{z \in W} c^z_{x,y} b(z)$$

with $c^z_{x,y} \in \mathbb{N}$.

The result will follow from the computation of $\theta_x\theta_y$. Since direct sum commutes with tensor product it is enough to compute $\theta_{E_1}\theta_{E_2}$ (notation 3.1.9). Then it is enough to show for the multiplication in the Grothendieck group of ${}^{\infty}_{\hat{\lambda}}\mathcal{O}^{\infty}_{\Lambda}$ defined above that

$$\left[(E_2 \otimes M(\lambda))_{\hat{\lambda}}\right] \; [(E_1 \otimes M(\lambda))]_{\hat{\lambda}} = \left[\left(E_1 \otimes (E_2 \otimes M(\lambda))_{\hat{\lambda}}\right)_{\hat{\lambda}}\right] \quad .$$

Now by 3.1.4 the left hand side equals

$$\sum_{x,y\in W} xy \; dim(E_2)_{x\lambda-\lambda} dim(E_1)_{y\lambda-\lambda}$$
$$= \sum_{z\in W} z \sum_{x\in W} dim(E_2)_{x\lambda-\lambda} dim(E_1)_{z\lambda-x\lambda} \quad ,$$

which by 3.1.4 again, equals the right hand side.

Remark. These positivity constraints were described by an essentially equivalent procedure by Bernstein-Gelfand [4]. One may show that they are the $t = 1$ specialization of deeper positivity constraints described by multiplication of certain basis elements in the Hecke algebra obtained by Springer using the decomposition theorem for perverse sheafs. (See Curtis' lectures in this series.) The latter play a fundamental role in the combinatorics of the Hecke algebra which in turn has important implications for enveloping algebras [20]. So far we do not know how to get these deeper constraints from enveloping algebras alone. This should involve the Jantzen filtration and Hodge decomposition (see Shoji's lectures in this series) in some way.

3.1.11. Take $\alpha \in B$. Then $[P(s_\alpha\lambda) : M(y\lambda)] \neq 0$ implies via 3.1.8 that $y\lambda - s_\alpha\lambda \in \mathbb{N}B$. Hence $s_\alpha y\lambda - \lambda \in \mathbb{N}B + \mathbb{Z}\alpha \cap -\mathbb{N}B = -\mathbb{N}\alpha$ and so $y = s_\alpha$ or 1. Now $[M(\lambda)/M(s_\alpha\lambda) : L(s_\alpha\lambda)] = 0$ because $L(s_\alpha\lambda)$ is $k[X_\alpha]$ free whereas $M(\lambda)/M(s_\alpha\lambda)$ is $k[X_\alpha]$ locally finite. We conclude that $b(s_\alpha) = 1 + s_\alpha$.

One easily checks that $\theta_w M(\lambda) = P(w\lambda)$. Set $\theta_\alpha = \theta_{s_\alpha} \;\forall\; \alpha \in B$. The calculation in 3.1.10 shows that $[\theta_w M] = [M]b(w)$. Furthermore from the canonical isomorphism $Hom_{\underline{g}}(E \otimes M, N) \cong Hom_{\underline{g}}(M, \; E^* \otimes N)$ it follows that θ_y admits an adjoint θ_y^* which is necessarily a direct sum of the $\theta_w : w \in W$. From $dim \; E^*_{w\lambda-\lambda} = dim \; E_{\lambda-w\lambda} = dim \; E_{w^{-1}\lambda-\lambda}$ we easily conclude that $[\theta_y^* M] = [M]b_y^*$ where $*$ is the involution of $\mathbb{Z}W$ defined by linearity and $w \mapsto w^{-1}$. In particular $[\theta^*_\alpha M] = [M](1+s_\alpha)$. Since by 3.1.8 the coefficient of w in $b(w)$ is one we conclude that $\theta^*_\alpha = \theta_\alpha$. Finally we compute $\theta_\alpha M(w\lambda)$. We already have $[\theta_\alpha M(w\lambda)] = w + ws_\alpha$. By the remark in 3.1.5 we can assume that $\theta_\alpha M(w\lambda)$ is an extension of $M(w\lambda)$ by $M(ws_\alpha\lambda)$ where $ws_\alpha\lambda < w\lambda$, i.e. that $ws_\alpha > w$ (Bruhat order). It is a non-trivial extension because $Hom_{\underline{g}}(\theta_\alpha M(w\lambda), L(w\lambda)) = Hom_{\underline{g}}(M(w\lambda), \theta_\alpha L(w\lambda))$ and because $\theta_\alpha L(w\lambda) = \theta_\alpha(M(w\lambda)/M(ws_\alpha\lambda)) = 0$. One may show that $dim\ Ext^1(M(ws_\alpha\lambda), M(w\lambda)) = 1$.

3.2. The Enright Functor.

3.2.1. Fix $\lambda \in \underline{h}^*$ dominant and regular and set $\Lambda = \lambda + P(R)$. As before let $\mathcal{O}_\Lambda$ denote the subcategory of $\mathcal{O}$ of all modules whose weights lie in Λ. It is easy to see that $\mathcal{O}_\Lambda$ is a direct sum of certain primary components of $\mathcal{O}$. Again $E \otimes L \in Ob\mathcal{O}_\Lambda$ given $L \in Ob\mathcal{O}_\Lambda$. Using Frobenius reciprocity one may also characterize $\mathcal{O}_\Lambda$ through

(*) *For each* $L \in Ob\mathcal{O}$ *simple,* $L \in Ob\mathcal{O}_\Lambda \Longleftrightarrow L(M(\lambda), L) \neq 0$.

As before it is convenient to assume $\lambda \in P(R)^{++}$, and we set $A = U(\underline{g})/Ann\ M(\lambda)$.

3.2.2. Recall the equivalence of categories theorem and define for each $\alpha \in B$ a functor C_α on $\mathcal{O}_\Lambda$ through

$$C_\alpha M = L(M(s_\alpha\lambda), M) \otimes_{U(\underline{g})} M(\lambda) .$$

It is clearly covariant and left exact. From the embedding $M(s_\alpha\lambda) \hookrightarrow M(\lambda)$ we have a map $M \to C_\alpha M$. Call M *α-free* if this map is injective and *α-cofree* if the map $\delta M \to C_\alpha \delta M$ is injective. We remark that M is α-free if and only if M is a free $k[X_{-\alpha}]$ module. Set $S = \{X^\ell_{-\alpha} : \ell \in \mathbb{N}\}$. Because $ad_{\underline{g}} X_{-\alpha}$ is nilpotent, S is *Ore* in $U(\underline{g})$ and we may form $S^{-1}M$. Let $\tilde{C}_\alpha M$ denote the submodule $\{m \in S^{-1}M \mid dim\ k[X_\alpha]m < \infty\}$ of $S^{-1}M$. Then $\tilde{C}_\alpha$ is just Deodhar's formulation of Enright's functor. We show (3.2.9) that $\tilde{C}_\alpha$ and C_α coincide on the category of α-free modules (i.e. when $M \to S^{-1}M$ is injective) but in general $\tilde{C}_\alpha = C^2_\alpha$.

It is convenient to define $D^-_\alpha M = Im(M \to C_\alpha M)$ and $D^+_\alpha M = \delta D^-_\alpha(\delta M)$. Then M is α-cofree $\Longleftrightarrow$ the map $D^+_\alpha M \to M$ is surjective.

3.2.3. One nice property of the Enright functor is its behaviour with respect to annihilators. Clearly $Ann(C_\alpha M) \subset Ann\ M$ with equality if M is α-free.

The behaviour of C_α with respect to the Grothendieck group of $\mathcal{O}_\Lambda$ is complicated by the failure of C_α to be exact. However one may relate C_α to the exact functor θ_α of coherent continuation across the α-wall (3.1.11). This has the property that we have an exact sequence

$$(*) \qquad 0 \longrightarrow M(w\lambda) \longrightarrow \theta_\alpha M(w\lambda) \longrightarrow M(ws_\alpha\lambda) \longrightarrow 0$$

whenever $ws_\alpha > w$ (Bruhat order). In the conventions of 2.1.2 we have $[\theta_\alpha M] = [M](1+s_\alpha)$. Taking $w = 1$ in (*) we may apply the equivalence of categories theorem to obtain

$$(**) \qquad 0 \to L(M(\lambda), C_\alpha M) \to L(\theta_\alpha M(\lambda), M) \to L\left(M(\lambda), D_\alpha^+ M\right) \to 0$$

for all $M \in Ob\mathcal{O}_\Lambda$. Actually, using Soergel's improved version of this equivalence of categories theorem, we get an exact functor $\theta'_\alpha := \eta\theta_\alpha\eta$ on $\mathcal{O}_\Lambda^\infty$ and an exact sequence $0 \to C_\alpha M \to \theta'_\alpha M \to D_\alpha^+ M \to 0$. This gives the

LEMMA. - *For all* $M \in Ob\mathcal{O}_\Lambda$,

(i) $[C_\alpha M] + [D_\alpha^+ M] = [\theta'_\alpha M] = (1+s_\alpha)[M]$

(ii) $[C_\alpha M] = s_\alpha[M] \iff M$ *is α-cofree.*

3.2.4. **LEMMA.** - *For each $\alpha \in B$, one has $C_\alpha \delta M(\lambda) = \delta M(s_\alpha\lambda)$.*

One has $L(M(\lambda), C_\alpha\delta M(\lambda)) \cong L(M(s_\alpha\lambda), \delta M(\lambda)) = L(-\lambda, -s_\alpha\lambda)$, whereas $L(M(\lambda), \delta M(s_\alpha\lambda)) = L(-s_\alpha\lambda, -\lambda)$. Since $M(\lambda)$ is projective in $\mathcal{O}_\Lambda$ we have a surjection $L(-\lambda,-\lambda)\xrightarrow{\varphi}L(-s_\alpha\lambda,-\lambda)$ and because $\delta M(\lambda)$ is injective in $\mathcal{O}_\Lambda$ we have a surjection $L(-\lambda,-\lambda)\xrightarrow{\varphi'}L(-\lambda,-s_\alpha\lambda)$. It remains to show that the $ker\ \varphi = ker\ \varphi'$. Set $J = Ann_A(M(\lambda)/M(s_\alpha\lambda))$. Via the analysis in 1.3.6 it is equivalent to show that ${}^tJ = J$. Set $P = Ann_A L(w_\lambda s_\alpha\lambda) \in Spec\ A$. Since $M(\lambda)/M(s_\alpha\lambda)$ vanishes on passage to the α-wall so does every simple subquotient L. We conclude that $Ann\ L \supset P$ by the Borho-Jantzen translation principle (the τ-invariant). By injectivity in 1.3.6 we conclude that $J = P^\ell$ for all $\ell >> 0$. (In fact P is idempotent and $J = P$). Since ${}^tP = P$ the assertion follows.

3.2.5. We need the following properties of C_α and D_α^+. Call N *α-finite* if it is $X_{-\alpha}$ locally finite.

PROPOSITION. -

(i) *$C_\alpha M$ is α-free,* $\forall\ M \in Ob\mathcal{O}_\Lambda$.

(ii) $C_\alpha M = 0$, $\forall\ M \in Ob\mathcal{O}_\Lambda$ *α-finite.*

(iii) *$M/D_\alpha^+ M$ is the largest α-finite quotient of M.*

(iv) $s_\alpha[C_\alpha M] = [D_\alpha^+ M]$.

(v) $L(M(s_\alpha\lambda), M)M(s_\alpha\lambda) = D_\alpha^+ M$, $\forall\ M \in Ob\mathcal{O}_\Lambda$.

(i) By definition N is α-free if the map $N \to C_\alpha N$ is injective. The latter is equivalent to $L(M(\lambda)/M(s_\alpha\lambda), N) = 0$ which holds if N has no simple submodules L which are α-finite. Now

$$\begin{aligned} Hom_{\underline{g}}(L, C_\alpha M) &\cong Hom_{\underline{g}\times\underline{g}}(T(L), T(C_\alpha M)) \quad , \text{ by equivalence of categories },\\ &\hookrightarrow Hom_{\underline{g}\times\underline{g}}(T(L), L(\theta_\alpha M(\lambda), M)) \quad , \text{ by 3.2.3(**)} ,\\ &\cong Hom_{\underline{g}\times\underline{g}}(L(\theta_\alpha M(\lambda), L), T(M)) \quad , \text{ by the} \end{aligned}$$

self-adjointness of θ_α. Again by the self-adjointness of θ_α and because $\theta_\alpha L(\lambda) = 0$ we conclude $L(\lambda)$ is not a quotient of $\theta_\alpha M(\lambda)$ and then from 3.2.3(*) that $\theta_\alpha M(\lambda)$ has no α-finite quotients. Hence $Hom_{\underline{g}}(L, C_\alpha M) = 0$ if L is α-finite. This proves (i).

For (ii) it is enough to observe that $L(M(s_\alpha\lambda), M) = 0$ for M α-finite. For (iii) observe that $ker(M \to C_\alpha M)$ is α-finite and hence so is $(\delta M)/D_\alpha^+(\delta M)$. Moreover $ker(M \to C_\alpha M)$ is by (i) the largest α-finite submodule of M and this gives (iii).

Take M α-finite. By (ii), $C_\alpha M = 0$ and since δM is also α finite, we obtain that $D_\alpha^+ M = 0$. Then from 3.2.3(i) we obtain $(1 + s_\alpha)[M] = 0$. Applying this to $M/D_\alpha^+ M$ and using 3.2.3(i) again gives (iv).

Set $M' = L(M(s_\alpha\lambda), M)M(s_\alpha\lambda)$, which is a submodule of M. Since $L(M(s_\alpha\lambda), N) = 0$, for N α-finite, it follows from (iii) that M' is a submodule of $D_\alpha^+ M$. On the other hand from the definition of M' it follows that the embedding $M' \hookrightarrow M$ gives an isomorphism $L(M(s_\alpha\lambda), M') \xrightarrow{\sim} L(M(s_\alpha\lambda), M)$ and hence an isomorphism $C_\alpha M' \xrightarrow{\sim} C_\alpha M$. By (ii), (iii) this gives an isomorphism $C_\alpha M' \xrightarrow{\sim} C_\alpha D_\alpha^+ M$. By (iii), $D_\alpha^+ M$ is α-cofree and because $M' = L(M(s_\alpha\lambda), M')M(s_\alpha\lambda)$ we conclude that $M' = D_\alpha^+ M'$ and so M' is α-cofree. Then by

3.2.3(ii) the above isomorphism gives $[M'] = [D_\alpha^+ M]$ which combined with embedding $M' \hookrightarrow D_\alpha^+ M$ gives (v).

3.2.6. Take $M, N \in Ob\mathcal{O}_\Lambda$ and $\alpha \in B$. The composition $(a,b) \mapsto ab$ defines a map $L(M,N) \times L(M(s_\alpha\lambda), M) \to L(M(s_\alpha\lambda), N)$ of $U(\underline{g})$ bimodules and hence a map $L(M,N) \times C_\alpha M \to C_\alpha N$ of $U(\underline{g})$ modules. This gives a $U(\underline{g})$ bimodule map $T_\alpha^{M,N} : a \mapsto (b \mapsto ab)$ of $L(M,N)$ into $L(C_\alpha M, C_\alpha N)$.

LEMMA. - *If M is α-cofree, then $T_\alpha^{M,N}$ is injective.*

Take $a \in L(M,N)$. If $a \in ker\ T_\alpha^{M,N}$, then $a(C_\alpha M) = 0$ and so $aL(M(s_\alpha\lambda), M) = 0$. Yet $L(M(s_\alpha\lambda), M)M(s_\alpha\lambda) = D_\alpha^+ M$ by 3.2.5(v) and so $a(D_\alpha^+ M) = 0$ which proves the assertion.

3.2.7. By the equivalence of categories theorem $T_\alpha^{M,N}$ gives a map, for which we shall use the same symbol, of $U(\underline{g})$ modules. When $N = \delta M(\lambda)$ we shall simply denote it by T_α^M. By 3.2.4 we have $L(C_\alpha M, C_\alpha \delta M(\lambda)) \cong L(C_\alpha M, \delta M(s_\alpha\lambda)) \cong L(M(s_\alpha\lambda), \delta C_\alpha M)^\eta \cong L(M(\lambda), C_\alpha \delta C_\alpha M)^\eta$, whereas $L(M, \delta M(\lambda)) \cong L(M(\lambda), \delta M)^\eta$. Thus T_α^M is a $U(\underline{g})$ module map of δM into $C_\alpha \delta C_\alpha M$. This gives a $U(\underline{g})$ module map δT_α^M of $\delta C_\alpha \delta C_\alpha M$ into M.

LEMMA. - *If M is α-cofree, the map δT_α^M is bijective.*

By 3.2.6, δT_α^M is surjective. Since $C_\alpha M$ is α-free by 3.2.5(i) we obtain from 3.2.3(ii) and 3.2.5(iv) that $[\delta C_\alpha \delta C_\alpha M] = [C_\alpha \delta C_\alpha M] = s_\alpha[\delta C_\alpha M] = s_\alpha[C_\alpha M] = [D_\alpha^+ M] = [M]$, by the hypothesis on M. This proves the lemma.

3.2.8. Retain the notation of 3.2.6 and 3.2.7.

THEOREM. - *If M is α-cofree, then $T_\alpha^{M,N}$ is bijective.*

By 3.2.6 the assertion holds when $N = \delta M(\lambda)$. Let E be a finite dimensional $U(\underline{g})$ module. Then $E \otimes -$ is an exact functor on $\mathcal{O}_\Lambda$ commuting with C_α. It easily follows that the assertion holds for $N = E \otimes \delta M(\lambda)$. Now take $N \in Ob\mathcal{O}_\Lambda$ arbitrary. By 3.1.1 we can find E, F finite dimensional and an exact sequence

$$0 \to N \to E \otimes \delta M(\lambda) \to F \otimes \delta M(\lambda) \ .$$

Since the functors $L(M, -)$ and $L(C_\alpha M, -)$ are covariant and left exact, we obtain a diagram of maps with exact rows

$$\begin{array}{ccccccc} 0 & \to & L(M,N) & \to & L(M, E\otimes \delta M(\lambda)) & \to & L(M, F\otimes \delta M(\lambda)) \\ & & \downarrow T_\alpha^{M,N} & & \downarrow T_\alpha^{M,E\otimes\delta M(\lambda)} & & \downarrow T_\alpha^{M,F\otimes\delta M(\lambda)} \\ 0 & \to & L(M, C_\alpha N) & \to & L(M, C_\alpha(E\otimes \delta M(\lambda)) & \to & L(M, C_\alpha(F\otimes \delta M(\lambda)) \end{array}$$

which from the definition of the T_α maps is easily seen to be commutative. Diagram chasing proves the theorem.

3.2.9. Let us show that C_α, $\tilde{C}_\alpha$ coincide on α-free modules. Take M, N to be α-free. Then $\delta D_\alpha^+ M$ and $\tilde{C}_\alpha N$ are α-free and the quotients $\tilde{C}_\alpha M/D_\alpha^+ M$ and $\tilde{C}_\alpha N/D_\alpha^+ N$ are α-finite. It easily follows that the map $L(\tilde{C}_\alpha M,\ \tilde{C}_\alpha N) \to L(D_\alpha^+ M,\ \tilde{C}_\alpha N)$ defined by restriction is injective and has image $L(D_\alpha^+ M,\ D_\alpha^+ N)$. Now take $M = M(\lambda)$. Then $D_\alpha^+ M(\lambda) = M(s_\alpha \lambda)$ (by say 3.2.5(iii)). By reduction to the $\underline{s\ell}(2)$ case one may show that

$$(*) \qquad \tilde{C}_\alpha M(\mu) \cong \begin{cases} M(s_\alpha \mu) & : (\alpha^\vee, \mu) \in \mathbb{N}^- \\ M(\mu) & : \text{otherwise.} \end{cases}$$

In particular $\tilde{C}_\alpha M(\lambda) \cong M(\lambda)$. Hence we obtain an isomorphism $L(M(\lambda), \tilde{C}_\alpha N) \xrightarrow{\sim} L(M(s_\alpha\lambda), D_\alpha^+ M) \xrightarrow{\sim} L(M(s_\alpha\lambda), M)$ since $M/D_\alpha^+ M$ is α-finite and $M(s_\alpha\lambda)$ is α-cofree. This proves our assertion.

3.2.10. Any Verma module is α-free, whilst $M(w\lambda)$ is α-cofree if and only if $s_\alpha w \leq w$ (Bruhat order). Thus if $M(w\lambda)$ is α-cofree we obtain from (*) above that $C_\alpha M(w\lambda) = M(s_\alpha w\lambda)$. By 3.2.8 we conclude that we have an isomorphism $L(M(w\lambda), M) \xrightarrow{\sim} L(M(s_\alpha w\lambda), C_\alpha M)$, for any $M \in Ob\mathcal{O}_\Lambda$. Now define the functor C_w : $w \in W$ on $\mathcal{O}_\Lambda$ through $C_w M = L(M(w^{-1}\lambda), M) \otimes_{U(\underline{g})} M(\lambda)$. The previous observation gives

COROLLARY. - *Let $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ be a reduced decomposition. Then $C_w = C_{\alpha_1} C_{\alpha_2} \cdots C_{\alpha_k}$.*

Remarks. Take $\underline{g} = \underline{s\ell}(2)$. Then $C_\alpha \delta M(\lambda) = \delta M(s_\alpha\lambda) = M(s_\alpha\lambda)$. Yet $C_\alpha M(s_\alpha\lambda) = M(\lambda)$ and so $C_\alpha^2 \neq C_\alpha$. Yet $\tilde{C}_\alpha^2 = \tilde{C}_\alpha$ trivially. Since $C_\alpha M$ is α-free we conclude from 3.2.9 that $\tilde{C}_\alpha C_\alpha = C_\alpha^2$ and $\tilde{C}_\alpha C_\alpha^2 = C_\alpha^3$. This proves that $C_\alpha^3 = C_\alpha^2$. Curiously the $\tilde{C}_\alpha$ do not satisfy the braid relations except on the category $\underline{n}^-$ of free modules.

3.2.11. **COROLLARY.** - *For all $x, y \in W$ one has*

(i) $C_x \delta M(\lambda) = \delta M(x\lambda)$.

(ii) $L(-x\lambda, -\lambda) \cong L(-\lambda, -x^{-1}\lambda)$.

(iii) $T'(L(-x\lambda, -y\lambda)) = C_{y^{-1}} C_x \delta M(\lambda)$.

For (i) it is enough to show that $C_\alpha \delta M(w\lambda) = \delta M(s_\alpha w\lambda)$ when $s_\alpha w > w$ (Bruhat order). Now $\delta M(w\lambda)$ is α-cofree and so by 3.2.3(ii) we obtain $[C_\alpha \delta M(w\lambda)] = s_\alpha[\delta M(w\lambda)] = s_\alpha[M(w\lambda)] = s_\alpha w = [\delta M(w\lambda)]$. On the other hand we saw in the proof of 3.1.5(iii) that $K := Ker(M \to C_\alpha M)$ is the largest α-finite submodule of M. Taking $M = \delta M(w\lambda)$ we get $K = \delta(M(w\lambda)/M(s_\alpha w\lambda))$ and so we deduce an embedding $\delta M(s_\alpha w\lambda) \hookrightarrow C_\alpha \delta M(w\lambda)$. Combined with our previous observation this proves (i).

(ii) By definition $L(M(\lambda), C_x \delta M(\lambda)) = L(M(x^{-1}\lambda), \delta M(\lambda)) = L(-\lambda, -x^{-1}\lambda)$, whereas by (i) we have $L(M(\lambda), C_x \delta M(\lambda)) = L(M(\lambda), \delta M(x\lambda)) = L(-x\lambda, -\lambda)$. This proves (ii).

(iii) Observe that $L(-x\lambda, -y\lambda) = L(M(y\lambda), \delta M(x\lambda)) = L(M(\lambda), C_{y^{-1}} \delta M(x\lambda)) = L(M(\lambda), C_{y^{-1}} C_x \delta M(\lambda))$, by (i). Hence (iii).

Remarks. Since the number of isomorphism classes of principal series modules with a given infinitesimal character is strictly greater that $|W|$ we see from (iii) why the $C_\alpha : \alpha \in B$ cannot also satisfy $C_\alpha^2 = C_\alpha$. Combined with 3.2.8 we can also obtain the intertwining maps of Kunze-Stein for principal series modules.

3.2.12. Set $\overline{M(\mu)} = ker(M(\mu) \to L(\mu))$ and recall that $J(\mu) = Ann\ L(\mu)$.

LEMMA. - *One has* $J(w\lambda)M(\lambda) = C_{w^{-1}}(\overline{M(w\lambda)})$, $\forall\ w \in W$.

By 2.2.4 we have an isomorphism $U(\underline{g})/Ann\ M(w\lambda) \xrightarrow{\sim} L(M(w\lambda), M(w\lambda))$. Recalling that $Ann\ M(w\lambda) = Ann\ M(\lambda)$ we conclude this restricts to an isomorphism $J(w\lambda)/Ann\ M(\lambda) \xrightarrow{\sim} L(M(w\lambda), \overline{M(w\lambda)})$. Thus $J(w\lambda)M(\lambda) = L(M(w\lambda), \overline{M(w\lambda)}) \otimes_{U(\underline{g})} M(\lambda) = C_{w^{-1}}(\overline{M(w\lambda)})$.

Remarks. An interesting open problem is to compute $[J(w\lambda)M(\lambda)]$. Had we been able to successively apply the conclusion of 3.2.3(ii) we would have obtained $[J(w\lambda)M(\lambda)] = w^{-1}[\overline{M(w\lambda)}] = w^{-1}(w - a(w)) = 1 - w^{-1}a(w)$ and so $[M(\lambda)/J(w\lambda)M(\lambda)] = w^{-1}a(w)$. This formula fails in general; but can be shown to hold when $w = w_B w_{B'}$ where $w_{B'} : B' \subset B$ is the longest element in the subgroup $W_{B'}$ of W generated by the $s_\alpha : \alpha \in B'$.

3.2.13. For each $w \in W$, let $\ell(w)$ denote its reduced length. We define a new multiplication on W by

$$s_{\alpha *} w = \begin{cases} w & : \ell(s_\alpha w) < \ell(w) \ , \\ s_\alpha w : \ell(s_\alpha w) > \ell(w) \ . \end{cases}$$

and associativity. This is just the multiplication rule for Enright functors on $\underline{n}^-$ free modules. In particular from 3.2.9 and 3.2.10 one has $C_x M(y w_B \lambda) = M((x_* y) w_B \lambda)$, $\forall\, x, y \in W$. One checks that $x_*^{-1}(y w_B) = w_B$ if and only if $x \geq y$. We conclude that

$$(*) \qquad C_{x^{-1}} M(y\lambda) = M(\lambda) \Longleftrightarrow x \geq y \ \text{(Bruhat order)}.$$

COROLLARY. -

(i) *If $x \geq y$, then there exists an embedding of $M(x\lambda)$ into $M(y\lambda)$.*

(ii) *If $[M(x\lambda) : L(y\lambda)] \neq 0$, then $x \leq y$.*

(i) It is enough to consider the case when $\ell(x) = \ell(y) + 1$. Then we can find $w_1, w_2 \in W$, $\alpha \in B$ such that $x = w_1 s_\alpha w_2$, $y = w_1 w_2$ are reduced expressions (i.e. lengths add). Set $M = C_{w_2} M(w_B \lambda) = M(w_2 w_B \lambda)$. Since M is α-free, the map $M \to C_\alpha M$ is an embedding and left exactness of C_{w_1} gives an embedding $M(y\lambda) = C_{w_1} C_\alpha M = M(x\lambda)$. This gives (i) under appropriate substitutions.

(ii) If $y = 1$, then $x = 1$ trivially. Otherwise choose $\alpha \in B$ such that $s_\alpha y < y$. We claim that $[C_\alpha M(x\lambda) : L(s_\alpha y \lambda)] \neq 0$. This implies eventually that $[C_{y^{-1}} M(x\lambda) : L(\lambda)] \neq 0$, hence $C_{y^{-1}} M(x\lambda) = M(\lambda)$ and so $x \leq y$ by (*), as required. To prove the claim we recall the embedding $D_\alpha^+ M(x\lambda) \hookrightarrow C_\alpha M(x\lambda)$ and so by 3.2.3(i) it is enough to show that if $[L(y\lambda)]$ occurs in the decomposition of $[M(x\lambda)]$, then $[L(s_\alpha y \lambda)]$ occurs in $(1 + s_\alpha)[M(x\lambda)]$. Yet $(1 + s_\alpha) a(y)$ is always a non-negative linear combination of the $a(z) : z \in W$ and so it is enough to observe that $a(s_\alpha y)$ occurs (with coefficient 1) in $(1 + s_\alpha) a(y)$. This is true because $C_\alpha L(y\lambda) \neq 0$ (recall $L(y\lambda)$ is α free and use 3.2.9) and so $C_\alpha M(y\lambda) = M(s_\alpha y \lambda)$ must have a non-zero image in $C_\alpha L(y\lambda)$.

__Remark.__ This result was first proved by Bernstein, Gelfand and Gelfand [33]. The present proof is in the spirit of [35].

3.2.14. **LEMMA.** - *Take $w \in W$. Then $M(\lambda)/J(w\lambda)M(\lambda)$ is the unique smallest submodule of $C_{w^{-1}}L(w\lambda)$ admitting $L(\lambda)$ as a quotient.*

From the exact sequence

$$0 \longrightarrow \overline{M(w\lambda)} \longrightarrow M(w\lambda) \longrightarrow L(w\lambda) \longrightarrow 0$$

the left exactness of C_α, 3.2.9(*), 3.2.10 and 3.2.12 we obtain applying $C_{w^{-1}}$ that

$$0 \longrightarrow J(w\lambda)M(\lambda) \longrightarrow M(\lambda) \longrightarrow C_{w^{-1}}L(w\lambda)$$

is exact. Finally it follows as in the proof of 3.2.13(ii) that $L(\lambda)$ occurs with multiplicity one in $C_{w^{-1}}L(w\lambda)$. This proves the assertion.

3.2.15. Recall that $\{a(w) : w \in W\}$ is a basis for $\mathbb{Z}W$. For each $x \in \mathbb{Z}W$ we can write $x = \Sigma x_w a(w)$ and we set $Supp\ x := \{w | x_w \neq 0\}$. Given $X \subset \mathbb{Z}W$, define $Supp\ X = \bigcup_{x \in X} Supp\ x$. Now for each $w \in W$, $\alpha \in B$, 3.2.3(i) implies that $(1 + s_\alpha)a(w)$ is always a non-negative integer combination of the $a(y) : y \in W$. Thus if we set $\mathcal{D}(w) = Supp(Wa(w))$, then $[\mathcal{D}(w)] := \mathbb{Z}\{a(y) | y \in \mathcal{D}(w)\}$ is a left ideal of $\mathbb{Z}W$. One calls $\mathcal{D}(w)$ (or sometimes $[\mathcal{D}(w)]$ the left cone containing W. Its importance lies in the following.

THEOREM. - *Fix $x, y \in W$. The following two assertions are equivalent.*

(i) $J(x\lambda) \supset J(y\lambda)$.

(ii) $x \in \mathcal{D}(y)$.

(ii) $\Longrightarrow$ (i) because $Ann\ C_\alpha L(y\lambda) \supset Ann\ L(y\lambda)$ and because $s_\alpha[L(y\lambda)] = -[L(y\lambda)]$ if $s_\alpha y > y$ whereas $s_\alpha[L(y\lambda)] = [C_\alpha L(y\lambda)]$ if $s_\alpha y < y$ (3.2.3(ii) and 3.2.9.).

By 3.2.14 and 3.2.3(i) we obtain $Supp\ [M(\lambda)/J(y\lambda)M(\lambda)] \subset [\mathcal{D}(y)]$ whilst the hypothesis $J(x\lambda) \supset J(y\lambda)$ implies $a(x) \in Supp[M(\lambda)/J(x\lambda)M(\lambda)] \subset Supp\ [M(\lambda)/J(y\lambda)M(\lambda)]$. This gives (i) $\Longrightarrow$ (ii).

<u>**Remark.**</u> The above description of *Prim* $U(\underline{g})$ as an ordered set is part of the wider problem of describing *Spec* $U(\underline{g})$ as a topological space.

3.2.16. By 2.2.10 we have $d(U(\underline{g})/J(w\lambda)) = 2d(L(w\lambda)) = 2(|R^+| - m(w))$. By 1.1.5 it follows that if $J(x\lambda) \supset J(y\lambda)$ then $m(x) \leq m(y)$ and strict inclusion holds if and only if a strict inequality holds. We define the left cell $\mathcal{C}(x)$ of W containing x through $\mathcal{C}(x) = \{y \in \mathcal{D}(x) | m(y) = m(x)\}$. It follows from 3.2.15 and the above that

COROLLARY. - *The following two assertions are equivalent.*

(i) $J(x\lambda) = J(y\lambda)$.

(ii) $x \in \mathcal{C}(y)$.

Remark. The above material was drawn from [17,18]. This definition of left cells was first given in [14] where (ii) $\Longrightarrow$ (i) in 3.1.15 was also proved. Vogan ([29], Sect. 3) proved the reverse inclusion. By 3.1.15 the set $[\mathcal{C}(w)] := \mathbb{Z}\{a(y) | y \in \mathcal{C}(w)\}$ is a W module quotient of $[\mathcal{D}(w)]$. Thus we obtain the remarkable fact that the partition of the Weyl group into left cells needed to describe the fibres of the map $Prim\ U(\underline{g}) \to Max\ Z(\underline{g})$ leads to associations of simple W modules into families (i.e. all those lying in the same left cell). For totally different reasons Lusztig gave associations of simple W modules by an inductive rule based on a polynomial associated to unipotent representations of finite Chevalley groups. The above description of left cells and that they lead to Weyl group representations partly motivated the formulation of the Kazhdan-Lusztig conjectures [22] for the $a(w)$. Its truth [3, 7] showed eventually that these two sets of associations of Weyl group representations coincide.

Warning. Association does not mean partition. Indeed call the double cell $\mathcal{DC}(w)$ containing w the smallest union of left cells containing $\mathcal{C}(w)$ which is stable under the map $y \mapsto y^{-1}$. Then every left cell in $\mathcal{DC}(w)$ contains the Goldie rank or special representation associated to $\mathcal{DC}(w)$ (and with multiplicity one). On the other hand the cruder association of simple W modules defined by the double cells is a partition of $\hat{W}$.

3.2.17. The truth of the Kazhdan-Lusztig conjectures (from which the $a(w)$ can be computed) can be shown to be equivalent to the remarkable statement (Vogan, unpublished) that for each $\alpha \in B$ and each α-free simple module $L \in Ob\mathcal{O}$ the cokernel of the map $L \to C_\alpha L$ is semisimple. Assume that this holds and take $L = L(y\lambda)$ with $s_\alpha y < y$ to ensure that $L(y\lambda)$ is α-free. Consider the exact sequence

$$0 \longrightarrow C_\alpha(\overline{M(y\lambda)}) \xrightarrow{\varphi} M(s_\alpha y\lambda) \longrightarrow C_\alpha L(y\lambda) \longrightarrow$$

which results from 3.2.9. Our hypothesis implies that $Coker\ \varphi$ is the extension of $Coker(\overline{M(y\lambda)} \to M(y\lambda))$ by $L(s_\alpha y\lambda)$. Hence

THEOREM. - *Take $\alpha \in B$, $y \in W$ such that $s_\alpha y < y$. Then*

$$\overline{M(s_\alpha y\lambda)} = M(y\lambda) + C_\alpha(\overline{M(y\lambda)}) \ .$$

3.2.18. Retain the above notation and hypotheses and set $k = -(\alpha^\vee, y\lambda)$, $X = X_\alpha$, $Y = X_{-\alpha}$. Let e (resp. f) denote the cannonical generator of $M(s_\alpha y\lambda)$ (resp. $M(y\lambda)$) and $\overline{e}$ (resp. $\overline{f}$) its image in $L(s_\alpha y\lambda)$ (resp. $L(y\lambda)$).

COROLLARY. -

$$Ann_{U(\underline{n}^-)}\overline{e} = U(\underline{n}^-)Y^k + \left(k[Y^{-1}](Ann_{U(\underline{n}^-)}\overline{f})Y^k\right) \cap U(\underline{n}^-) \ .$$

As noted in 2.2.4 we can write $f = Y^k e$. Take $a \in Ann_{U(\underline{n})}\overline{e}$. By 3.2.17 we have $ae = bY^k e + ce$ for some $b, c \in U(\underline{n}^-)$ such that $ce \in C_\alpha(\overline{M(y\lambda)})$. Since $Ann_{U(\underline{n}^-)}e = 0$ it follows that $a = by^k + c$. Consider $m \in C_\alpha(\overline{M(y\lambda)})$ and identify C_α with $\tilde{C}_\alpha$ via 3.2.9. Then there exists $\ell \in \mathbb{N}$ such that $Y^\ell\ m \in \overline{M(y\lambda)}$ and so $Y^\ell m = df = dY^k e$ for some $d \in Ann_{U(\underline{n}^-)}\overline{f}$. Taking $m = ce$ it follows that $c \in Y^{-\ell}(Ann_{U(\underline{n}^-)}\overline{f})Y^k \cap U(\underline{n}^-)$. Conversely suppose $c \in k[Y^{-1}](Ann_{U(\underline{n}^-)}\overline{f})Y^k \cap U(\underline{n}^-)$. Then $ce \in k[Y, Y^{-1}] \otimes_{k[Y]} \overline{M(y\lambda)}$. On the other hand $Xe = 0$, whereas $(ad_{\underline{g}}X)^t c = 0, \forall\ t >> 0$ and so ce is locally X finite. We conclude that $ce \in C_\alpha(\overline{M(y\lambda)})$. This proves the assertion .

Remarks. This result was obtained in [16], 8.3. Let B be the Borel subgroup corresponding to $\underline{b}$. Let $\mathcal{V}(y)$ denote the Zariski closure of the smallest B stable subset containing $\underline{n}^+ \cap y\underline{n}^-$. Let $\mathcal{V}(L(y\lambda))$ be the associated variety of $L(y\lambda)$. One may show that the above result implies that $\mathcal{V}(L(y\lambda)) \supset \overline{\mathcal{V}(y)}$ (but unfortunately this can be a strict inclusion). This has an independent geometric proof based on [3]. In principle the above result gives much finer information.

References.

[1]. D. Barbasch and D. Vogan, *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. 259 (1982), 152 - 199.

[2]. D. Barbasch and D. Vogan, *Primitive ideals and orbital integrals in complex exceptional groups*, J. Algebra 80 (1983), 350 - 382.

[3]. A.A. Beilinson and I.N. Bernstein, *Localization of g modules*, C.R. Acad. Sci. Paris Sér A, 292 (1981), 15 - 18.

[4]. I.N. Bernstein and S.I. Gelfand, *Tensor products of finite and infinite dimensional representations of semi-simple Lie algebras*, Compos. Math. 41 (1980), 245 - 285.

[5]. W. Borho, P. Gabriel and R. Rentschler, *Primideale in Einhüllenden auflösbarer Lie-Algebren*, Lecture Notes in Mathematics 357 (Springer, Berlin, 1973).

[6]. W. Borho and J.C. Jantzen, *Über primitive Ideale in der Einhüllenden einer halbeinfacher Lie Algebra*, Invent. Math. 39 (1977), 1 - 53.

[7]. J.L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. 64 (1981), 387 - 410.

[8]. J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France, 96 (1968), 209 - 242.

[9]. J. Dixmier, *Enveloping algebras*, (North-Holland, New York, 1977).

[10]. I.N. Herstein, *Topics in Ring theory*, (The University of Chicago Press, Chicago, 1969).

[11]. R.S. Irving, *The socle filtration of a Verma module*, Ann. Ec. Norm. Sup. (to appear).

[12]. J.C. Jantzen, *Einhüllenden Algebren halbeinfacher Lie-Algebren*, Ergebnisse der Mathematik (Springer, Berlin, 1983).

[13]. A. Joseph, *Dixmier's problem for Verma and principal series modules*, J. Lond. Math. Soc. 20 (1979), 193-204.

[14]. A. Joseph, *W-module structure in the primitive spectrum of the enveloping algebra of a semisimple Lie algebra*, in Lecture Notes in Mathematics 728, pp. 116 - 135 (Springer, Berlin, 1979).

[15]. A. Joseph, *Goldie rank in the enveloping algebra of a semisimple Lie algebra* I, II, III, J. Algebra, 66 (1980), 269-283, 284-306; 73 (1981), 295 - 326.

[16]. A. Joseph, *On the variety of a highest weight module*, J. Algebra 88 (1984), 238 - 278.

[17]. A. Joseph, *The Enright functor in the Bernstein-Gelfand-Gelfand $\mathcal{O}$ category*, Invent. Math. 67 (1982), 423 - 445.

[18]. A. Joseph, *Completion functors in the $\mathcal{O}$ category*, in Lecture Notes in mathematics, 1020, pp. (Springer, Berlin, 1983).

[19]. A. Joseph, *On the cyclicity of vectors associated with Duflo involutions*, in Lecture Notes in mathematics, 1243, pp. 144 - 188 (Springer, Berlin, 1987).

[20]. A. Joseph, *A sum rule for scale factors in the Goldie rank polynomials*, J. Algebra (to appear).

[21]. A. Joseph, *Primitive ideals in enveloping algebras*, ICM proceedings pp.403 - 414 (Polish Scientific Publishers - North Holland, Warsaw, 1984).

[22]. D.A. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. 53 (1979), 165 - 184.

[23]. G.B. Krause and T.H. Lenagen, *Growth of algebras and Gelfand-Kirillov dimension*, Research Notes in mathematics, 116 (Pitman, London, 1985).

[24]. G. Lusztig, *Cells in affine Weyl groups* II, J. of Algebra, 109 (1987), 536 - 548.

[25]. G. Lusztig, *Leading coefficients of character values of Hecke algebras*, Proceedings of Symposia in Pure Mathematics 47 American Math. Soc. (1987), 235 - 262.

[26]. C. Moeglin, *Idéaux primitifs dans les algebres enveloppantes*, J. Math. pures et appl. 59 (1980), 265 - 336.

[27]. C. Moeglin and R. Rentschler, *Sur la classification des idéaux primitifs des algebres enveloppantes*, Bull. Soc. Math. France, 112 (1984).

[28]. W. Soergel, *Equivalences de certains catégories de g modules*, C.R.Acad. Sci. Série I, 303 (1986), 725 - 728.

[29]. D. Vogan, *Ordering of the primitive spectrum of a semisimple Lie algebra*, Math. Ann. 248 (1980), 195 - 203.

[30]. D. Vogan, *The orbit method and primitive ideals for semisimple Lie algebras*, Canadian Math. Soc. Conference Proceedings, Vol. 5 (1986), 281 - 316.

[31.] D. Barbasch and D. Vogan, *Unipotent representations of complex semisimple groups*, Ann. Math. 121 (1985), 41 - 110.

[32.] I.N. Bernstein, I.M. Gelfand and S. Gelfand, *The structure of representations generated by vectors of largest weight*, Functional analysis and its applications, 5 (1971), 1 - 19.

[33] I.N. Bernstein, I.M. Gelfand and S. Gelfand, *Differential operators on the base affine space and a study of g modules*, in : I.M. Gelfand, ed., Publ. of 1971 Summer School in Math. Janos Bolyai Math. Soc., Budapest, 21 - 64.

[34] N. Conze, *Algèbres d'opérateurs différentiels et quotients des algèbres enveloppantes*, Bull. Soc. Math. France, 102 (1974), 379 - 415.

[35] M. Duflo, *Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple*, Ann. Math. 105 (1977), 107 - 130.

[36] G. Lusztig, *Characters of a reductive group over a finite field*, Ann. of Math. Studies 107, Princeton Univ. Press, 1984.

[37] R. Rentschler, *Primitive ideals in enveloping algebras (General Case)*, in Noetherian rings and their applications (Oberwolfach 1983), Math. Surveys Monographs, 24, Amer. Math. Soc., Providence, R.I. (1987), 37 - 57.

[38] N. Spaltenstein, *Classes Unipotentes et Sous-groupes de Borel*, Lecture Notes in Mathematics, 946 Springer-Verlag, Berlin 1982.

[39.] D. Vogan, *Gelfand-Kirillov dimension for Harish-Chandra modules*, Inv. Math. 48 (1978), 75 - 98.

Department of Theoretical Mathematics
The Weizmann Institute of Science
Rehovot 76100, Israel.

and

Laboratoire de mathématiques fondamentales
(Equipe de recherche associée au CNRS)
Université de Pierre et Marie Curie, France.