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L-Functoriality for dual pairs

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§1 Introduction

The theory of reductive dual pairs is important in the study of automorphic forms. This paper is an attempt to understand this theory in the language of L-groups.

Let (G,G') be a reductive dual pair of subgroups of the symplectic group Sp(2n,$\mathbb{R}$). Let Mp(2n,$\mathbb{R}$) be the metaplectic cover of Sp(2n,$\mathbb{R}$), and let $(\tilde{G},\tilde{G}')$ be the inverse image of (G,G') in Mp(2n,$\mathbb{R}$). Let ω be the oscillator representation of Mp(2n,$\mathbb{R}$). Let $\Pi(G)$ denote the set of equivalence classes of irreducible admissible representations of G. Taking quotients of the restriction of ω to $\tilde{G}\times\tilde{G}'$ establishes a bijection between a subset of $\Pi(\tilde{G})$ and a subset of $\Pi(\tilde{G}')$ [11]. We refer to this as the representation correspondence, and write $\pi\rightarrow\pi'$. It is of great interest to compute this correspondence explicitly; this is known only in a few cases (cf. for example [18], [1], [21]). We say an irreducible representation of $\tilde{G}$ or $\tilde{G}'$ occurs in the representation correspondence if it is contained in this subset, i.e. if it is a quotient of the metaplectic representation restricted to $\tilde{G}$ or $\tilde{G}'$.

Assume for the moment that ω restricted to $\tilde{G}\times\tilde{G}'$ factors to $G\times G'$; then the representation correspondence becomes a bijection between representations of G and of G'. Let ${}^LG$ and ${}^LG'$ denote the L-groups of G and G' respectively. It is natural to conjecture that there is a homomorphism $\gamma:{}^LG\rightarrow{}^LG'$ which "realizes" this correspondence via the principal of functoriality [19]. That is let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$, and let $\Phi(G)$ denote the set of equivalence classes of admissible homomorphisms of $W_{\mathbb{R}}$ into ${}^LG$ ([20], [8]). Given $\varphi\in\Phi(G)$, let $\Pi(\varphi)\subset\Pi(G)$ denote the L-packet associated to φ. We use similar notation for G'. In [19] Langlands conjectured that if $\pi\in\Pi(\varphi)$ occurs in the representation correspondence, then the corresponding representation π' of G' is contained in $\Pi(\gamma\circ\varphi)$. This would compute the representation correspondence (up to L-packets).

This conjecture is false in many known examples (unavailable at the time it was made). Its validity would imply that if π_1 and π_2 occur in the representation correspondence and are contained in the same L-packet, then π'_1 and π'_2 are contained in the same L-packet. Even this weaker statement is false: it may for example happen that π_1 and π_2 are tempered (even discrete series representations), whereas π'_1 is tempered and π'_2 is non-tempered.

A possible explanation for this phenomenon is that L-packets are not the correct notion here, but the larger packets whose existence was conjectured by Arthur [7]. These we call Arthur-packets. Thus let Ψ be an admissible homomorphism $\Psi: W_{\mathbb{R}}\times SL(2,\mathbb{C})\to {}^LG$. Arthur conjectured that attached to Ψ is a finite set of irreducible representations $\Pi(\Psi)$ with properties similar to those for tempered L-packets. We sketch such a definition in §2. Then there is a natural analogue of the preceding conjecture with $W_{\mathbb{R}}$ replaced by $W_{\mathbb{R}}\times SL(2,\mathbb{C})$, and L-packets replaced by Arthur-packets.

The main conjecture of this paper is a slight modification of this picture. We continue to assume that ω factors to $G\times G'$. For each such irreducible pair we define $\gamma: {}^LG\to {}^LG'$ (after possibly exchanging G and G') and a fixed homomorphism $T: SL(2,\mathbb{C})\to {}^LG'$. Given $\Psi: W_{\mathbb{R}}\times SL(2,\mathbb{C})\to {}^LG$, we let $\Psi'(w,g)=(\gamma\circ\Psi)(w,g)T(g)$ $(w\in W_{\mathbb{R}}, g\in SL(2,\mathbb{C}))$. Conjecture A (4.3) says that if $\pi\in\Pi(\Psi)$ occurs in the representation correspondence, then $\pi'\in\Pi(\Psi')$. Thus the conjecture would give a description of the representation pairing, up to Arthur-packets. It does not predict which π are in the domain of the representation correspondence.

The fixed map T plays the role of the "tail" of [19] coming from the Weil-group parameter of ${}^LG$. If G and G' are roughly the same size then T is the identity map, and Conjecture A is closer to Langlands' original conjecture; see the discussion following 4.3.

This conjecture is compatible with all evidence known (to this author). Unfortunately the only cases in which we can prove the conjecture are those for which the representation correspondence is known a priori; hence this does not give any new explicit results of this form. This paper is intended as evidence for the conjectures, with the hope that a general direct proof can be found.

A new explicit result which is included here (in §5) is a generalization of [1] to general groups. That is, we compute the pairing for certain discrete series in the stable range in terms of derived functor modules.

We prove Conjecture A in the following cases. First of all we discuss the discrete series in the stable range just mentioned. If either member of the dual pair is compact the representation correspondence is known ([18], [10]). We use [2] which expresses this result in terms of derived functor modules. We also consider the case of $(O(p,q),Sp(2m,\mathbb{R}))$ with π the trivial representation of $O(p,q)$, which has been discussed in ([21], [17]). Finally we consider the case of $(GL(m,\mathbb{R}),GL(n,\mathbb{R}))$ which is particularly simple and known completely ([21]); Conjecture A is true without qualification in this case.

The first and third cases mentioned are in some sense opposite extremes. Thus if π is a discrete series representation then $\pi\in\Pi(\Psi)$ with $\Psi|_{SL(2,\mathbb{C})}=1$. On the other hand the trivial representation π of $O(p,q)$ is a unipotent representation, i.e. π is contained in a unipotent Arthur-packet: $\Psi|_{\mathbb{C}^*}=1$ ($\mathbb{C}^*\subset W_{\mathbb{R}}$). The corresponding representation π' of $Sp(2m,\mathbb{R})$ is also a unipotent representation.

The conjecture only discusses representations π which occur in Arthur-packets. This is a proper subset of $\Pi(G)$, containing the tempered representations, which should be the set of representations of interest in the theory of automorphic forms for linear groups. In particular these representations are conjectured to be unitary. Together with Conjecture A this would imply that if π is unitary and contained in an Arthur-packet then π' is unitary. This statement is consistent with known (and expected) results. The conjecture does not make any prediction about which representations of G occur in the representation correspondence; it is not enough that π is contained in an Arthur-packet.

A definition of $\Pi(\Psi)$ has not appeared in the literature; in fact its final form has yet to be determined. For the purposes of this paper a definition does exist, and is due to D. Vogan and D. Barbasch. We summarize this in §2. One of the notions which the definition of $\Pi(\Psi)$ requires is that of an E-group for a reductive group [6]. This is a generalization of an L-group and plays the role of the L-group in parametrizing representations of certain algebraic covering groups of G.

In general $\tilde{G}$ and $\tilde{G}'$ are not algebraic groups, and in this case we make no attempt to include them in this scheme unless ω factors to G and G'. One case in which ω does not factor to (G,G') and yet $\tilde{G}$ and $\tilde{G}'$ are algebraic is $(G,G')=(U(p,q),U(r,s))$. If $r+s$ is odd, the group $\tilde{G}$ is an algebraic group, and the representations of $\tilde{G}$ occuring in the representation correspondence are genuine (they do not factor to $U(p,q)$). A similar statement holds by symmetry for $U(r,s)$. These representations are parametrized by maps of $W_{\mathbb{R}}$ or $W_{\mathbb{R}}\times SL(2,\mathbb{C})$ into an E-group ${}^EG$ for G. Thus we state the conjecture with E-groups or L-groups (depending on parity) for G and G' in this case. In §3 we discuss E-groups for unitary groups. We also define an L-group and related notions for $O(p,q)$, which is not standard because $O(p,q)$ is not the real points of a connected algebraic group.

Suppose (G,G') and a representation $\pi'\in\Pi(\Psi')$ occuring in the representation correspondence are given. It may be the case that another representation $\sigma'\in\Pi(\Psi')$ does not occur in the representation correspondence for this pair. One reason is that the L-group side does not distinguish between inner forms: it may be that σ occurs for some dual pair (G^2,G') where G^2 is an inner form of G. Thus it is natural to collect inner forms of a given group together: consider an Arthur-packet to be a set of representations of any of these inner forms, and ask for a bijection between corresponding Arthur-packets of this type. This is the content of Conjecture B (4.5). Furthermore it suggests a role for stable distributions and lifting from endoscopic groups. We discuss these matters briefly in §7.

This paper was motivated in part by the following example. Consider the dual pair $(O(2),Sp(4,\mathbb{R}))$. Let π_+ (resp. π_-, also known as θ_{10}) correspond to the trivial (resp. sign) representation of $O(2)$. Since the trivial and sign representation are in an Arthur-packet for $O(2)$, π_+ and π_- are contained in an Arthur-packet (this is known in this case). Thus the counterexample to the generalized Ramanujan conjecture constructed in [15] by exchanging π_+ and π_- at the infinite place amounts to exchanging two representations in an Arthur-packet. Note that $\pi_\pm$ are not contained in an L-packet: π_- is tempered, whereas π_+ is non-tempered.

Here is some notation we use. We let G be a reductive algebraic group defined over $\mathbb{R}$, and also use G to denote its real points. We write $G(\mathbb{C})$ for the complex points. If G is not $O(p,q)$ it will be (algebraically) connected, and we let ${}^LG$ be an L-group for G. If G is $O(p,q)$ ${}^LG$ is defined in §2. We fix a Cartan involution θ with corresponding maximal compact subgroup K for G, and similarly for G'. Let (G,G') be a reductive dual pair (or dual pair for short) of subgroups of $Sp(2n,\mathbb{R})$. Fix an oscillator representation ω of $Mp(2n,\mathbb{R})$ (there are two, the other is the contragredient ω^*). If (G,G') are not unitary groups, we assume ω factors to (G,G'). Suppose $(G,G')=(U(p,q),U(r,s))$. If ω factors to $U(p,q)$ we let $G=U(p,q)$. If ω does not factor we change notation and let $G=U(p,q)^{\sim}$, which is a subgroup of $Mp(2n,\mathbb{R})$. We use similar notation for G'. Thus in general G is a subgroup of $Sp(2n,\mathbb{R})$ $(G\not\approx U(p,q))$, and is a subgroup of $Sp(2n,\mathbb{R})$ or $Mp(2n,\mathbb{R})$ $(G\approx U(p,q))$.

The organization of this paper is as follows. Section 2 discusses Arthur-packets. In §3 we discuss L-groups and E-groups explicitly for $O(p,q)$ and $U(p,q)$. We define the maps γ and T in each case in §4, and state the main conjectures. In §5 we discuss the discrete series in the stable range for all groups (generalizing [1]). Some readers may be interested in this information independent of the other results of this paper; the presentation of this material was done with this in mind. In §6 we prove Conjectures A and B for some special cases. We conclude with some final remarks and conjectures involving endoscopic groups in §7.

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§2 Arthur-Packets

We sketch a definition of the Arthur-packet $\Pi(\Psi)$ associated to an admissible homomorphism. This definition is still experimental in some of its details; in fact the considerations of this paper are intended partly as a guide to this definition. We will spell this out carefully in the cases we need.

The definition of $\Pi(\Psi)$ has two steps: construction of a unipotent Arthur-packet for G or for the Levi component L of a parabolic subgroup of $G(\mathbb{C})$; and induction from L to G. The induction step is a combination of real parabolic induction, and holomorphic induction from a θ-stable parabolic subalgebra of $\mathfrak{g}$, and is well understood. The first step may be made explicit, for example, when the Arthur-packet of unipotent representations is a single one-dimensional representation. This is the case that is needed for discussion of the pairing of discrete series in the stable range (cf. Theorem 6.1). In section 6 we spell out a special case of some particular unipotent representations of $Sp(2n,\mathbb{R})$.

We use a few notions which are not standard. The most important of these is that of an E-group for G [6]. Recall [20] an L-group for G is (roughly speaking) a group ${}^LG$ which fits in a split exact sequence:

2.1 $\quad {}^LG^0 \to {}^LG \to \Gamma$.

Here Γ is the Galois group of $\mathbb{C}$ over $\mathbb{R}$ which acts on ${}^LG^0$. (The qualification refers to the fact that the splitting must be "admissible", and ${}^LG$ is in fact such a group together with an equivalence class of such splittings). An E-group for G is a group which fits in the exact sequence 2.1, but in which the sequence is not necessarily split. An element $z\in Z({}^LG^0)^\Gamma$ determines such a group up to isomorphism; and we refer to it as the E-group determined by z. In particular ${}^LG$ is the E-group determined by z=1.

Now given $z\in Z({}^LG^0)$, $z^2=1$, we obtain a certain (algebraic) two-fold covering covering group $\tilde{G}=\tilde{G}_z$ of G. If $z=\exp(2\pi i\gamma)$ we write $\tilde{G}_\gamma=\tilde{G}_z$. A representation of $\tilde{G}$ is said to be genuine if it does not factor to G. Then conjugacy classes of admissible homomorphisms $\varphi:W_{\mathbb{R}} \to {}^EG$ parametrize

L-packets of genuine representations of $\tilde{G}$ [6]. If ${}^EG={}^LG$, $\tilde{G}\approx G\times\mathbb{Z}/2\mathbb{Z}$, and genuine representations of $\tilde{G}$ are canonically in bijection with representations of G.

The definitions of [6] are given in slightly different terms; here we use an equivalent version closer to the original definitions of [20] (cf. [6], chapter 9).

Let $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\to {}^LG$ be a (quasi-) admissible homomorphism [7]. Thus first of all Ψ is a continuous group homomorphism. Secondly Ψ restricted to $W_{\mathbb{R}}$ is a tempered admissible homomorphism of $W_{\mathbb{R}}$ in the usual sense [8]: Ψ preserves projection on Γ and the image of $\mathbb{C}^*$ is bounded and consists of semisimple elements. The prefix (quasi-) refers to the fact that we only assume Ψ restricted to $W_{\mathbb{R}}$ is admissible for the quasisplit form of G, i.e. we impose no condition involving parabolic subgroups.

To Ψ we associate an infinitesimal character χ_Ψ of G. The image of $\mathbb{C}^*$ is contained in a Cartan subgroup ${}^LT^0$ of ${}^LG^0$. As in [8] write $\Psi(z)=z^\mu\bar{z}^\nu$ for $\mu,\nu\in X_*({}^LT^0)\otimes\mathbb{C}\approx{}^L\mathfrak{t}\approx\mathfrak{t}^*$, for $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Let $\lambda=d\Psi|_{SL(2,\mathbb{C})}(\mathrm{diag}(\frac{1}{2},-\frac{1}{2}))$; after conjugation we may assume $\lambda\in{}^L\mathfrak{t}\approx\mathfrak{t}^*$. Let χ_Ψ be the infinitesimal character of G corresponding to $\lambda+\mu$ via the Harish-Chandra homomorphism.

Now assume as a first case that the image of $\mathbb{C}^*$ is contained in the center of ${}^LG^0$. Then to Ψ we associate an Arthur-packet of representations of G. These representations are unipotent when restricted to the derived group of G (it is convenient to reserve the term "unipotent" for semi-simple groups). For the moment let Ψ denote Ψ restricted to $SL(2,\mathbb{C})$. By the Jacobson-Morozov theorem Ψ corresponds to a unipotent orbit ${}^L\mathcal{O}_\Psi$ of ${}^LG^0$ (by orbit we will always mean coadjoint orbit in the dual of a Lie algebra or conjugacy class in a Lie group; there will be no danger of confusion). For later use we note that if λ as above is integral then it is singular unless ${}^L\mathcal{O}_\Psi$ is the principal unipotent orbit of ${}^LG^0$, in which case it is the infinitesimal character of the trivial representation. Now ${}^L\mathcal{O}_\Psi$ corresponds to a special unipotent orbit $\mathcal{O}_\Psi$ of G by [23] or ([9], Appendix). For example $\mathcal{O}_\Psi$ is the 0-orbit if ${}^L\mathcal{O}_\Psi$ is the principal unipotent orbit of ${}^LG^0$. Recall that the wave-front set of an irreducible representation π is a finite union of coadjoint G orbits [13]. The following definition is the analogue for real groups of ([9], Definition 1.17).

2.1 Definition:

Suppose $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\to{}^LG$, and the image of $\mathbb{C}^*$ is contained in the center of ${}^LG^0$. Then the (weak) Arthur packet $\Pi(\Psi)$ is the finite set of irreducible representations π of G such that:

(i) <u>The infinitesimal character of π is χ_Ψ</u>,
(ii) <u>The wave-front set of π is equal to the closure of</u> $\mathcal{O}_\Psi$.

We note that (at least in the case when ${}^L\mathcal{O}_\Psi$ an even orbit) condition (ii) is equivalent to (cf. [9], corollary 5.19):

(ii)' The Gelfand-Kirillov dimension of π is minimal among representations satisfying (i).

For example if ${}^L\mathcal{O}_\Psi$ is the principal unipotent orbit, then π is a one-dimensional representation, determined (if G is connected) by its infinitesimal character (cf. Lemma 6.3).

An Arthur-packet is a refinement of a (weak) Arthur-packet. That is, note that the definition of $\Pi(\Psi)$ makes no reference to the element $\Psi(j)$ ($j \in W_{\mathbb{R}}$); incorporating this information decomposes $\Pi(\Psi)$ into a (not necessarily disjoint) union of (true) Arthur-packets. For most of our purposes the above definition is all we need and we drop the prefix. We will have two occasions to refine this slightly (cf. Definitions 6.6 and 6.9).

The packets thus defined are the special unipotent Arthur-packets, referring to the fact that the orbit $\mathcal{O}_\Psi$ is a special unipotent orbit.

Let $\text{Ann}(\pi)$ denote the annihlator of π in the universal enveloping algebra of $\mathfrak{g}$. Suppose ${}^L\mathcal{O}_\Psi$ is an even orbit. Then a useful fact about these Arthur-packets is:

<u>2.2 Lemma</u> ([<u>9</u>], <u>Lemma 5.10</u>):
<u>In the setting of Definition 2.1</u>, <u>Ann</u>(π) <u>is the same for all</u> $\pi \in \Pi(\Psi)$.

The restriction to even orbits in the discussion following Definition 2.1 and in Lemma 2.2 is not essential; it may be removed by considering the integral root system of χ_Ψ (cf. the end of the introduction to [9]).

More general (non-special) unipotent representations of G are not necessarily related to maps $\Psi: W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^LG$ as above, for a discussion of these matters see [26].

A similar definition holds when ${}^LG$ is replaced by ${}^EG$; we obtain representations of a covering group $\tilde{G}$ of G.

More generally if the image of $\mathbb{C}^*$ is not necessarily contained in the center of ${}^LG^0$, let ${}^LC^0$ denote the identity component of the centralizer of the image of $\mathbb{C}^*$ in ${}^LG^0$. Let $y = \Psi(j)$, which normalizes ${}^LC^0$. Let ${}^EC = \langle {}^LC^0, y \rangle \subset {}^LG$, the group generated by ${}^LC^0$ and y. Then ${}^EC$ is the E-group of a connected reductive group C defined over $\mathbb{R}$.

Now $\Psi: W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^{E}C \hookrightarrow {}^{L}G$, and the image of $\mathbb{C}^*$ is contained in the center of ${}^{L}C^0$. By the above construction applied to C we obtain the Arthur-packet $\Pi^C(\Psi)$ of representations of a covering group $\tilde{C}$ determined by ${}^{E}C$.

By [8] conjugacy classes of Levi subgroups of $G(\mathbb{C})$ are in bijection with conjugacy classses of Levi subgroups of ${}^{L}G^0$. Suppose L is a θ-stable Levi subgroup of G such that the conjugacy class of $L(\mathbb{C})$ corresponds to ${}^{L}C^0$. Furthermore assume L is an inner form of C. Let $\Pi^L(\Psi)$ denote the Arthur-packet constructed in the preceding paragraph (taking L=C). This is a finite set of representations of $\tilde{L}$, which are special unipotent when restricted to the derived group. Given $\pi_L \in \Pi^L(\Psi)$, choose a parabolic subgroup $Q(\mathbb{C})=L(\mathbb{C})U(\mathbb{C}) \subset G(\mathbb{C})$. We assume $Q(\mathbb{C})$ is weakly-non-negative in the sense of ([25], Definition 17.1(h)), this is a condition on the imaginary roots of $\mathfrak{u}$.

Let $\mathcal{R}(\pi_L)$ be the derived functor module of π_L . The normalization is as in [6], and is as follows. Let $\mathbb{C}_{\rho(\mathfrak{u})}$ denote the one-dimensional representation of $\tilde{L}_{\rho(\mathfrak{u})}$ with weight $\rho(\mathfrak{u})$ [5]. Here $\tilde{L}_{\rho(\mathfrak{u})}$ is the "metaplectic" cover of L defined by the element $\rho(\mathfrak{u})$. Then $\tilde{L} \approx \tilde{L}_{\rho(\mathfrak{u})}$, and $\pi_L \otimes \mathbb{C}_{\rho(\mathfrak{u})}$ is naturally a representation of L. Let $S=\frac{1}{2}\dim(\mathfrak{k}/\mathfrak{k}\cap\mathfrak{l})$, and let $\mathcal{R}(\pi_L)=\Gamma^S \circ \mathrm{pro}(\pi_L \otimes \mathbb{C}_{\rho(\mathfrak{u})})$ (notation as in [24],6.3.1). This has the same infinitesimal character as π_L. See also ([6], Chapter 8).

The role of E-groups in the definition of $\Pi(\Psi)$ is to make this construction functorial. This could be avoided: one could use ${}^{L}C$ in place of ${}^{E}C$, L in place of $\tilde{L}$, and $2\rho(\mathfrak{u})=\Lambda^{top}(\mathfrak{g}/\mathfrak{q})^*$ in place of $\rho(\mathfrak{u})$. This is particularly simple when $\rho(\mathfrak{u})$ factors to L; this happens for example if G is GL(n) (cf. [25], §6).

The construction of $\mathcal{R}(\pi_L)$ may as usual be broken up into two steps. There exist $L \subset L_\theta \subset G$ with the following properties. There is a real parabolic subgroup P of L_θ containing L as its reductive part, and $\mathfrak{l}_\theta$ is the Levi factor of a θ-stable parabolic subalgebra of $\mathfrak{g}$. Furthermore $\mathcal{R}(\pi_L) \approx \mathcal{R}_\theta \circ \mathrm{Ind}(\pi_L)$, where Ind is ordinary parabolic induction form P to L_θ, and $\mathcal{R}_\theta$ is cohomological parablic induction from the θ-stable parabolic subalgebra $\mathfrak{q}$ (up to one-dimensional twists) ([25],Definitions 17.1 and 5.17).

2.3 Definition:

The Arthur-packet $\Pi^G(\Psi)$ associated to Ψ is the set of irreducible constituents of the modules $\mathcal{R}(\pi_L)$, as L, π_L run over all possible choices given above.

We write $\Pi^G(\Psi)$ if it is necessary to specify the group G.

A similar procedure is used to define $\Pi(\Psi)$ when $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\to {}^{E}G$, where ${}^{E}G$ is the E-group of G determined by some element $z=\exp(2\pi i\gamma)\in Z({}^{L}G^{0})$. Here ${}^{E}L$ is the E-group of L determined by $\rho(\mathfrak{u})+\gamma$. Given π a unipotent representation of $L_{\rho(\mathfrak{u})+\gamma}$, $\pi\otimes\mathbb{C}_{\rho(\mathfrak{u})}$ is naturally a representation of $\tilde{L}_{\gamma}\subset\tilde{G}_{\gamma}$. Then $\mathcal{R}(\pi)=\Gamma^{S}\circ\mathrm{pro}(\pi\otimes\mathbb{C}_{\rho(\mathfrak{u})})$ is a representation of $\tilde{G}_{\gamma}$.The Arthur-packet $\Pi(\Psi)$ is thus a set of representations of $\tilde{G}_{\gamma}$ constructed as above.

We freely identify representations and characters. We may on occasion identify $\Pi(\Psi)$ with a $\mathbb{Z}$-module of virtual characters spanned by the irreducible characters it contains.

Given G, suppose $\{G_i\}$ is a set of groups which are inner forms of G with $G=G_0$. We identify the L-groups (and E-groups) for each G_i with those for G. We also identify the Lie algebras of G_i with $\mathfrak{g}$, universal enveloping algebras, infinitesimal characters, etc. Given $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\to{}^{L}G$, we obtain $\Pi^{G_i}(\Psi)$ as above. In section 7 we will write $\Pi^{\{G_i\}}(\Psi)=\cup_i\Pi^{G_i}(\Psi)$.

§3 L-Groups and E-Groups

We define some L-groups and E-groups which are not standard. First we define the notion of L-group for the group $O(p,q)$. Then we discuss E-groups for the groups $U(p,q)$. The reader may wish to skip this section and refer back to it when necessary.

The notation we use for L-groups is as follows. Let $W_{\mathbb{R}}=\mathbb{C}^*\cup j\mathbb{C}^*$ be the Weil group of $\mathbb{R}$. If G is a connected reductive algebraic group defined over $\mathbb{R}$, we let ${}^LG={}^LG^0\rtimes\Gamma$ be an L-group for G. Here $\Gamma=\{1,\sigma\}$ is the Galois group of $\mathbb{C}$ over $\mathbb{R}$. We write $\times$ in the case when a semi-direct product is actually direct. Let $\Phi(G)$ denote the set of ${}^LG^0$-conjugacy classes of admissible homomorphisms of $W_{\mathbb{R}}$ into ${}^LG$. Given a conjugacy class $\{\varphi\}$ of admissible homomorphisms $W_{\mathbb{R}}\to{}^LG$, we let $\Pi(\{\varphi\})\subset\Pi(G)$ denote the corresponding L-packet. If there is no danger of confusion we write $\Pi(\varphi)$ for $\Pi(\{\varphi\})$. If it is necessary to specify G we will write $\Pi^G(\varphi)$.

Because $G=O(p,q)$ is not the real points of a connected algebraic group, the L-group of G is not defined. We proceed to make such a definition, and show it has the properties required of an L-group.

<u>Case I</u>: $p+q\in 2\mathbb{Z}$.

Let $2n=p+q$. For the L-group of $SO(p,q)$ we take $SO(2n,\mathbb{C})\rtimes\Gamma$, where we define $SO(2n,\mathbb{C})$ with respect to the diagonal form. Let ϵ be the element $\mathrm{diag}(1,1,\dots 1,-1)\in O(2n,\mathbb{C})\setminus SO(2n,\mathbb{C})$. The action of Γ on $SO(2n,\mathbb{C})$ is given by:

3.1 $$\sigma(g)=\begin{cases} g & p-q\equiv 0 \pmod 4 \\ \epsilon g\epsilon^{-1} & p-q\equiv 2 \pmod 4). \end{cases}$$

Note that $O(2n,\mathbb{C})$ is generated by $SO(2n,\mathbb{C})$ and the element ϵ.

3.2 <u>Definition:</u>

<u>Let $G=O(p,q)$, $p+q=2n$. Let ${}^LG^0=O(2n,\mathbb{C})$, and let ${}^LG=O(2n,\mathbb{C})\rtimes\Gamma$, where $O(2n,\mathbb{C})$ is taken with respect to the diagonal form, the action of Γ on $SO(2n,\mathbb{C})$ is as in 3.1, and $\sigma(\epsilon)=\epsilon$.</u>

Now in fact ${}^LO(p,q)\approx{}^LO(p+1,q-1)$, even though ${}^LSO(p,q)\not\approx{}^LSO(p+1,q-1)$. This is reflected in the fact that both ${}^LSO(p,q)$ and ${}^LSO(p+1,q-1)$ embed in ${}^LO(p,q)$. The embedding of ${}^LSO(p,q)$ is the obvious one; we call it ι.

Case II: $p+q\in 2\mathbb{Z}+1$

Let $p+q=2n+1$. For the L-group of $SO(p,q)$ we take $Sp(2n,\mathbb{C})\times\Gamma$.

3.3 Definition:

Let $G=O(p,q)$, $p+q=2n+1$. Let ${}^LG^0=Sp(2n,\mathbb{C})\times\mathbb{Z}/2\mathbb{Z}$, and let ${}^LG=(\ Sp(2n,\mathbb{C})\times\mathbb{Z}/2\mathbb{Z}\ )\times\Gamma$.

Let $\pi:{}^LO(p,q)\to{}^LSO(p,q)$ denote the obvious map.

We now define admissible homomorphisms of the Weil group into these L-groups, and the L-packets corresponding to them.

3.4 Definition:

I. $G=O(p,q)$, $p+q\in 2\mathbb{Z}$

a. An admissible homomorphism $\varphi:W_{\mathbb{R}}\to{}^LG$ is a homomorphism of the form $\varphi=\iota\circ\varphi_0$, for φ_0 an admissible homomorphism $W_{\mathbb{R}}\to{}^LSO(p,q)$.

b. $\Phi(G)=\{$ admissible homormorphisms $W_{\mathbb{R}}\to{}^LG\ \}/$ conjugation by ${}^LG^0$.

II. $G=O(p,q)$, $p+q\in 2\mathbb{Z}+1$

An admissible homomorphism $\varphi:W_{\mathbb{R}}\to{}^LG$ is a homomorphism such that $\varphi_0=\pi\circ\varphi$ is an admissible homomorphism $W_{\mathbb{R}}\to{}^LSO(p,q)$. We define $\Phi(G)$ as in (I,b).

3.5 Definition:

Let $G=O(p,q)$, and let Ind denote induction from $SO(p,q)$ to G. Given an admissible homomorphism $\varphi:W_{\mathbb{R}}\to{}^LG$, we let $\Pi(\varphi)$ be the set of irreducible constituents of $\mathrm{Ind}(\pi)$, as π runs over $\Pi^{SO(p,q)}(\varphi_0)$ (φ_0 as in Definition 3.4). Let $\Pi(\{\varphi\})=\Pi(\varphi)$.

The crucial part of the definitions is that in fact $\Pi(\varphi)$ only depends on the conjugacy class of φ, so that $\Pi(\{\varphi\})$ is well-defined. If n is even there may exist φ and φ' such that φ is conjugate to φ' via ${}^LG^0$, whereas φ_0 is not conjugate to φ'_0 via ${}^LSO(p,q)^0$. In this case, however, the two corresponding distinct L-packets for $SO(p,q)$ induce to the same L-packet for G. We obtain:

3.7 Theorem:

1. *$\{\varphi\}\to\Pi(\{\varphi\})$ is a well-defined map from $\Phi(G)$ to finite subsets of $\Pi(G)$.*
2. *$\Pi(\{\varphi\})\cap\Pi(\{\varphi'\})$ is empty if $\{\varphi\}\neq\{\varphi'\}$.*
3. *$\Pi(G)=\cup\ \Pi(\{\varphi\})$, where this is a (disjoint) union over $\{\varphi\}\in\Phi(G)$.*

We leave the details of the proof to the reader.

We discuss E-groups for $G=U(p,q)$. Let $n=p+q$, and let ${}^LG^0=GL(n,\mathbb{C})$.

3.8 Definition:

Let

$$\epsilon=\begin{pmatrix} & & & & 1\\ & & & -1 & \\ & & 1 & & \\ & \cdot & & & \\ \pm1 & & & & \end{pmatrix}$$

Let ${}^LG={}^LG^0\rtimes\Gamma$, where $\sigma(g)=\epsilon\,{}^tg^{-1}\epsilon^{-1}$.

3.9 Definition:

${}^EG=\langle{}^LG^0,\sigma\rangle$, the group generated by ${}^LG^0$ and an element σ, subject to the relations

i. $\sigma^2=-Id$

ii. $\sigma(g)=\epsilon\,{}^tg^{-1}\epsilon^{-1}$.

By [6] homomorphisms of $W_{\mathbb{R}}$ into ${}^EG$ parametrize L-packets of representations of a certain algebraic two-fold cover $\tilde{G}$ of G. We spell this out in this example, in which things are very simple; the following two results could be taken as definitions (without reference to [6]). Let $\Pi:\tilde{G}\to G$ be the covering map. Given a representation π of G, let $\Pi^{-1}(\pi)$ denote the pullback of π to $\tilde{G}$.

3.10 Lemma:

i. The group $\tilde{G}$ is isomorphic to G itself. The covering map is $\Pi(g)=\det(g)^2g$.

ii. Let $\det^{\frac{1}{2}}$ denote the unique genuine character satisfying $\det(\Pi(g))=(\det^{\frac{1}{2}}(g))^2$.

Then then the representations of G may be canonically identified with the genuine representations of $\tilde{G}$ via $\pi \to \Pi^{-1}(\pi)\otimes \det^{\frac{1}{2}}$.

Let T: ${}^{E}G \to {}^{L}G$ be the group homomorphsim defined by $T(g)=(\det(g)^2 g)\times 1$ ($g\in {}^{L}G^{0}\approx GL(n,\mathbb{C})$), and $T(\sigma)=1\times\sigma$. Given $\varphi:W_{\mathbb{R}} \to {}^{E}G$, let $\varphi'=T\circ\varphi$.

3.11 Theorem:

Let $\varphi:W_{\mathbb{R}} \to {}^{E}G$ be an admissible homomorphism. Let $\Pi(\varphi')$ denote the L-packet of representations of G defined by φ'. Then the L-packet $\Pi^{\tilde{G}}(\varphi)$ of genuine representations of $\tilde{G}$ defined by φ is $\{\pi\otimes\det^{\frac{1}{2}} \mid \pi\in\Pi^{G}(\varphi')\}$.

The proofs of Lemma 3.10 and Theorem 3.11 follow immediately from the definitions of [6]. Everything reduces to the case of U(1), where the result is easy; we leave it to the reader to check this case directly.

§4 Conjectures A and B

In this section we give precise statements of the conjectures. We start by explicitly defining maps ${}^LG \to {}^LG'$. We consider the irreducible reductive dual pairs one at a time.

In this section we reserve the notation ${}^EG$ for a proper E-group, i.e. for ${}^EG \not\simeq {}^LG$. This occurs only for $G=U(p,q)$ and was discussed in this case in the previous section.

Let I_k denote the $k\times k$ identity matrix.

I. <u>$(O(p,q), Sp(2m,\mathbb{R})) \subset Sp(2m(p+q),\mathbb{R})$</u>

Let $n=p+q$. The oscillator representation factors to $Sp(2m,\mathbb{R})$ if and only if $n \in 2\mathbb{Z}$. We assume this holds.

A. L-groups:

a. For the L-group of $Sp(2m,\mathbb{R})$ we take the direct product $SO(2m+1,\mathbb{C})\times\Gamma$, where we take $SO(2m+1,\mathbb{C})$ with respect to the diagonal form.

b. For the L-group of $O(p,q)$ we take $O(2n,\mathbb{C})\rtimes\Gamma$ as described in §3.

B. Mapping of L-groups:

a. <u>$n \le m$</u>

Let $G=O(p,q)$, $G'=Sp(2m,\mathbb{R})$. We define $\gamma: {}^LG \to {}^LG'$, i.e. $O(2n,\mathbb{C})\rtimes\Gamma \to SO(2m+1,\mathbb{C})\times\Gamma$:

4.1 i. $\gamma(g\times 1)=\mathrm{diag}(g,\det(g)I_{2(m-n)+1})\times 1 \quad g\in O(2n,\mathbb{C})$

ii. $\gamma(1\times\sigma)=\begin{cases} I_{2m+1}\times\sigma & p-q\equiv 0 \pmod 4 \\ \mathrm{diag}(\epsilon,-I_{2(n-m)+1})\times\sigma & p-q\equiv 2 \pmod 4 \end{cases}$

(recall $\epsilon=\mathrm{diag}(1,1,\dots,1,-1)$)

b. n>m

Let $G=Sp(2m,\mathbb{R})$, $G'=O(p,q)$. We define $\gamma: {}^LG\to {}^LG'$, i.e. $SO(2m+1,\mathbb{C})\times\Gamma\to O(2n,\mathbb{C})\rtimes\Gamma$:

4.2 i. $\gamma(g\times 1)=\mathrm{diag}(g,I_{2(n-m)-1})\times 1 \quad g\in SO(2m+1,\mathbb{C})$

ii. $\gamma(1\times\sigma)=\begin{cases} I_{2n}\times\sigma & p-q\equiv 0 \pmod 4\\ \epsilon\times\sigma & p-q\equiv 2 \pmod 4\end{cases}$

II. $(U(p,q)\times U(r,s))\subset Sp(2(p+q)(r+s),\mathbb{R})$

Let $m=p+q$, $n=r+s$. The oscillator representation factors to $U(p,q)$ if and only if $n\in 2\mathbb{Z}$. Suppose $n\in 2\mathbb{Z}+1$. Then $U(p,q)^{\sim}$ was defined in section 3. The genuine representations of G are obtained by maps of $W_{\mathbb{R}}$, or $W_{\mathbb{R}}\times SL(2,\mathbb{C})$, into the E-group for G defined in §3. These representations may be thought of as representations whose central character is $\det^{(k+\frac{1}{2})}$, $k\in\mathbb{Z}$.

Thus if $n\in 2\mathbb{Z}+1$ we use the E-group for $U(p,q)$, and if $n\in 2\mathbb{Z}$ we use the L-group described in section 3. By symmetry similar statements hold for $U(r,s)$.

After relabelling if necessary we assume $m\leq n$.

A. L-Groups and E-Groups

The L-group and E-groups of $U(p,q)$ were defined in §3.

B. Mapping of L-groups

Let $G=U(p,q)$, $G'=U(r,s)$, $m\leq n$. We define a map $\gamma: {}^LG$ (or ${}^EG$)$\to {}^LG'$ (or ${}^EG'$).

a. $n\in 2\mathbb{Z}$, $m\in 2\mathbb{Z}$. Let $s=(n-m)/2$. Define $\gamma: {}^LG\to {}^LG'$ via:

i. $\gamma(g\times 1)=\mathrm{diag}(I_s,g,I_s)\times 1 \quad (g\in GL(m,\mathbb{C}))$,

ii. $\gamma(1\times\sigma)=1\times\sigma$.

b. $n\in 2\mathbb{Z}+1$, $m\in 2\mathbb{Z}+1$. Let $s=(n-m)/2$. Define $\gamma: {}^EG\to {}^EG'$ via:

i. $\gamma(g)=\mathrm{diag}(I_s,g,I_s) \quad (g\in GL(m,\mathbb{C}))$,

ii. $\gamma(\sigma)=\epsilon\sigma \quad$ where $\epsilon=\mathrm{diag}(I_s,I_m,-I_s)$.

c. $m\in 2\mathbb{Z}$, $n\in 2\mathbb{Z}+1$. Define $\gamma: {}^EG\to {}^LG'$ via:

i. $\gamma(g)=(\mathrm{diag}(g,I_{n-m})\times 1) \quad (g\in GL(m,\mathbb{C}))$,

ii. $\gamma(\sigma)=(\epsilon\times\sigma) \quad$ where $\epsilon=\begin{pmatrix}0 & I_m\\ I_{n-m} & 0\end{pmatrix}$.

d. $m\in 2\mathbb{Z}+1$, $n\in 2\mathbb{Z}$. Define $\gamma: {}^LG\to {}^EG'$ via:

i. $\gamma(g\times 1)=\mathrm{diag}(g,I_{n-m}) \quad (g\in GL(m,\mathbb{C}))$,

ii. $\gamma(1\times\sigma)=\epsilon\sigma \quad$ where $\epsilon=\begin{pmatrix}0 & I_m\\ I_{n-m} & 0\end{pmatrix}$.

III. $(Sp(p,q),O^*(2n))\subset Sp(4(p+q)n,\mathbb{R})$

This case is very similar to case I.

We recall that $O^*(2n)$ is the group preserving a skew-Hermitian form on $\mathbb{H}^n$ ($\mathbb{H}$ is the quaternions). If $g\in O^*(2n)$, then $\det(g)=1$, where the determinant is taken as a linear transformation of a real vector space of dimension 4n. That is $O^*(2n)=SO^*(2n)$. The group preserving a Hermitian form of signature (p,q) on $\mathbb{H}^{p+q}$ is $Sp(p,q)$.

The oscillator representation factors to yield a bijection between representations of the linear groups.

Let $m=p+q$.

A. L-Groups

a. The group $Sp(p,q)$ is inner to $Sp(2m,\mathbb{R})$: for an L-group we take the direct product $SO(2m+1,\mathbb{C})\times\Gamma$ (cf. I)

b. The group $O^*(2n)$ is inner to $SO(2n,0)$. (This may be seen as follows: $O^*(2n)$ is inner either to $SO(2n,0)$ or $SO(2n-1,1)$; $SO(2n-1,1)$ is ruled out because since $O^*(2n)$ contains a compact Cartan subgroup, so must any inner form of it.

We use the L-group of $SO(2n,0)$, as in section 3: $SO(2n,\mathbb{C})\rtimes\Gamma$, where the action is given by:

i. $\sigma(g)=\begin{cases} g & n\equiv 0 \pmod 2 \\ \epsilon g\epsilon^{-1} & n\equiv 1 \pmod 2 \end{cases}$ $(\epsilon=\mathrm{diag}(1,1,\ldots 1,-1))$.

B. Mapping of L-groups:

This is essentially the same as I.B.

a. $n\le m$

Let $G=O^*(2n)$, $G'=Sp(p,q)$. We define $\gamma: {}^LG\to {}^LG'$, i.e. $SO(2n,\mathbb{C})\rtimes\Gamma\to SO(2m+1,\mathbb{C})\times\Gamma$.

4.1 i. $\gamma(g\times 1)=\mathrm{diag}(g,I_{2(m-n)+1})\times 1 \quad g\in SO(2n,\mathbb{C})$

ii. $\gamma(1\times\sigma)=\begin{cases} I_{2m+1}\times\sigma & n\equiv 0 \pmod 2 \\ \mathrm{diag}(\epsilon,-I_{2(n-m)+1})\times\sigma & n\equiv 1 \pmod 2 \end{cases}$

b. $n>m$

Let $G=Sp(p,q)$, $G'=O^*(2n)$. We define $\gamma: {}^LG\to {}^LG'$, i.e. $SO(2m+1,\mathbb{C})\times\Gamma\to SO(2n,\mathbb{C})\rtimes\Gamma$.

4.2 i. $\gamma(g\times 1)=\mathrm{diag}(g,I_{2(n-m)-1}) \quad g\in SO(2m+1,\mathbb{C})$

ii. $\gamma(1\times\sigma)=\begin{cases} I\times\sigma & n\equiv 0 \pmod 2 \\ \epsilon\times\sigma & n\equiv 1 \pmod 2. \end{cases}$

The preceding groups are referred to as those of type I. Note that (except in the case of $O(p,q)$, both p and q odd, each group has discrete series representations. Also note that in each case one member of the pair may be compact. The following case is of type II.

IV. $(GL(m,\mathbb{R}),GL(n,\mathbb{R}))\subset Sp(2nm,\mathbb{R})$

This is the easiest case of all. We assume $m \leq n$, let $G=GL(m,\mathbb{R})$ and $G'=GL(n,\mathbb{R})$. The oscillator representation factors to G and G'.

A. L-groups:

For the L-group of $GL(k,\mathbb{R})$ we take the direct product $GL(k,\mathbb{C})\times\Gamma$ ($k=m, n$).

B. Mapping of L-groups:

We define $\Upsilon:{}^LG\to{}^LG'$, i.e. $GL(m,\mathbb{C})\times\Gamma\to GL(n,\mathbb{C})\times\Gamma$.

i. $\Upsilon(g\times 1)=\mathrm{diag}(g,I_{n-m})$ $\quad g\in GL(m,\mathbb{C})$

ii. $\Upsilon(1\times\sigma)= I\times\sigma$

In each case Υ is clearly a group homomorpism when restricted to ${}^LG^0$. One checks directly in each case that the relations $\sigma^2=1$ (or $\sigma^2=-1$ if $\sigma\in{}^EG$ for $G=U(p,q)$) and $\sigma g\sigma^{-1}=\sigma(g)$ ($g\in{}^LG^0$) are preserved by Υ. Thus Υ is a group homomorphism.

We are now in a position to state the main conjectures. Let (G,G') be an irreducible reductive dual pair, with G the smaller group. We assume (G,G') is one of the cases treated above in I-IV. Recall (cf. §1)we consider G (resp. G') either as a subgroup of $Sp(2n,\mathbb{R})$ or $Mp(2n,\mathbb{R})$. Let Υ: ${}^LG$ (or ${}^EG$) $\to$ ${}^LG'$ (or ${}^EG'$) be the homomorphism defined above. Let ${}^LH'^0$ be the identity component of the centralizer of $\Upsilon({}^LG^0)$ in ${}^LG'^0$. Let $T:SL(2,\mathbb{C})\to{}^LH'^0$ be the homomorpism corresponding to the principal unipotent orbit in ${}^LH'^0$ (this is defined up to conjugacy by ${}^LH'^0$).

4.3 Conjecture A:

Suppose π is an irreducible representation of G occuring in the representation correspondence for this dual pair. Let π' be the corresponding irreducible representation of G'. Suppose $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\to{}^LG$ (or ${}^EG$) is an admissible homomorphsim, such that π is contained in the corresponding Arthur-packet $\Pi(\Psi)$. Let $\Psi':W_{\mathbb{R}}\times SL(2,\mathbb{C})\to{}^LG'$ (or ${}^EG'$) be defined by:

4.4 i. $\Psi'(w\times 1)=\Upsilon\circ\Psi(w)$ $\quad w\in W_{\mathbb{R}}$

ii. $\Psi'(1\times g)=\Upsilon\circ\Psi(g)T(g)$ $\quad g\in SL(2,\mathbb{C})$

iii. $\Psi'(w\times g)=\Psi'(w\times 1)\Psi'(1\times g)$ $\quad w\in W_{\mathbb{R}}, g\in SL(2,\mathbb{C})$.

Then Ψ' is a (quasi-) admissible homomorphism (cf. §2). Let $\Pi(\Psi')$ denote the corresponding Arthur-packet. Then $\pi'\in\Pi(\Psi')$.

There is a small point to check to see that Ψ' is in fact a group homomorphism: we need to check $\Psi'(g)$ commutes with $\Psi'(w)$ for all

$g\epsilon SL(2,\mathbb{C})$, $w\epsilon W_{\mathbb{R}}$. This is immediate from the definitions for $w\epsilon\mathbb{C}^*$; and may be checked case by case for w=j. It follows immediately that Ψ is a (quasi-) admissible homomorphsim.

As mentioned in the introduction T is similar to the "tail" used in [19], and that if G and G' are roughly the same size (i.e. if ${}^LH'^0$ is abelian) then T is equal to the identity map. Then if π is tempered Conjecture A would imply π' is tempered, and the nature of the representation correspondence for these representations should be quite simple. In particular in this situation Conjecture A is similar to [19]. This is not the case (even when T=1) if π is contained in a non-tempered Arthur-packet. The nature of the representation correspondence can be quite complicated and appparently non-functorial; perhaps functoriality can only be expected to hold on the subset of representations of $\Pi(G)$ and $\Pi(G')$ which occur in Arthur-packets.

Given (G,G'), let $\{(G_i,G')\}$ (i=0,1,...,k) be a set of representatives for the equivalence classes of dual pairs such that G_i is an inner form of G, with G_0=G. For example if $(G,G')=(O(2m,0),Sp(2n,\mathbb{R}))$ let $(G_i,G')=(O(2m-2i,2i),Sp(2n,\mathbb{R}))$ (i=1,2,...,m). As another example if $(G,G')=(U(m),U(r,s))$, then $\{(G_i,G')\}=\{U(m-i,i),U(r,s))\mid i=0,1,...,m\}$.

Note that all (G_i,G') have the same L-groups (or E-groups), maps γ, etc. Given $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\to{}^LG$ (or ${}^EG$) we let $\Pi^{\{G_i\}}(\Psi)$ be a set of irreducible representations of some G_i: $\Pi^{\{G_i\}}(\Psi)=\cup_i\,\Pi^{G_i}(\Psi)$.

<u>4.5 Conjecture B</u>:

<u>Suppose we are in the situation of the preceding conjecture, with (G,G') in the stable range (stable range is defined in §5). Thus we are given $\pi\epsilon\Pi^G(\Psi)$, π occuring in the representation correspondence for G.</u>

<u>1</u>. <u>Suppose σ is an irreducible representation of G_i, for some i, and σ is contained in $\Pi^{G_i}(\Psi)$. Then σ occurs in the representation correspondence for the dual pair (G_i,G'). Let σ' denote the corresponding representation. Then $\sigma'\epsilon\Pi^{G'}(\Psi')$.</u>

<u>2</u>. <u>$\sigma\to\sigma'$ is a bijection between $\Pi^{\{G_i\}}(\Psi)$ and $\Pi^{G'}(\Psi')$.</u>

This conjecture is not true without the assumption of stable range: the trivial representation of O(4) occurs for the dual pair $(O(4),Sp(4,\mathbb{R}))$, but the sgn representation does not. It is not clear what the correct range of validity of this conjecture should be.

We extend Conjecture B to one about endoscopic groups in §7.

§5 Discrete Series in the Stable Range

In this section we explicitly describe the representation correspondence for certain discrete series representations in the stable range. This section may be read independently of the rest of the paper.

A reductive dual pair (G,G') is said to be in the stable range if G is small with respect to G' in a certain sense [12]. We will spell out this condition below for each pair. The discrete series of G then occurs in the representation correspondence. Let π be a discrete series representation of G. The corresponding representation π' of G' may or may not be tempered. Assume further that π' has regular infinitesimal character; this may be expressed as a conditon on the infinitesimal character of π. Then it is possible to identify π' as a representaion with $(\mathfrak{g},K)$-cohomology, i.e. as the derived functor module of a one-dimensional representation of a θ-stable parabolic. We state a precise result for each pair. This was done for the pair $(O(p,q),Sp(2m,\mathbb{R}))$ in [1].

Consider an irreducible type I dual pair (G,G') in the stable range, with G small. If G or G' is O(p,q) we assume that both p and q are even. Then G (resp. G') contains a compact Cartan subgroup $T\subset K$ (resp. $T'\subset K'$) which we fix. Let $\mathfrak{t}$ be the complexified Lie algebra of T. Let W be the the Weyl group $W(\mathfrak{g},\mathfrak{t})$ of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and let W_K be the Weyl group W(K,T) of T in K. We use similar notation for G'. The orbit correspondence establishes a bijection between elliptic coadjoint G orbits, and elliptic coadjoint G' orbits. Equivalently, this is a bijection between $\mathfrak{t}_0{}^*/W_K$ and $\mathfrak{t}_0{}'^*/W_{K'}$. It is convenient to multply by i (the square-root of -1) and obtain $i\mathfrak{t}_0{}^*/W_K \to i\mathfrak{t}_0{}'^*/W_{K'}$; we write this as $\lambda\to\lambda'$ ($\lambda\in i\mathfrak{t}_0{}^*$, $\lambda'\in i\mathfrak{t}_0{}'^*$).

We define some ρ-shifts. Supposing $\mathfrak{h}$ is an ad($\mathfrak{t}$)-stable subspace of $\mathfrak{g}$, such that if α is a weight of $\mathfrak{t}$ acting on $\mathfrak{h}$ then $-\alpha$ is not such a weight, we let $\rho(\mathfrak{h})\in\mathfrak{t}^*$ denote one half the sum of the roots of $\mathfrak{t}$ in $\mathfrak{h}$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Fix a θ-stable Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$, and let $\rho=\rho(\mathfrak{b})$, $\rho_n=\rho(\mathfrak{b}\cap\mathfrak{p})$, and $\rho_c=\rho(\mathfrak{b}\cap\mathfrak{k})$. Suppose $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ is a θ=stable parabolic subalegbra of $\mathfrak{g}$. We assume $\mathfrak{b}$ contains $\mathfrak{u}$, and let $\rho(\mathfrak{l})=\rho(\mathfrak{b}\cap\mathfrak{l})$.

Given $\lambda\in i\mathfrak{t}_0^*$, we let $\mathfrak{q}(\lambda)=\mathfrak{l}(\lambda)\oplus\mathfrak{u}(\lambda)$ be the usual θ-stable parabolic subalgebra associated to λ ([24], Defintion 5.2.1). After conjugating by K we may and do assume λ is dominant for the roots of $\mathfrak{t}$ in $\mathfrak{b}\cap\mathfrak{k}$.

Recall G is either a subgroup of $Sp(2n,\mathbb{R})$ or a subgroup of $Mp(2n,\mathbb{R})$ which is a two-fold cover of such a subgroup (cf. §1). In the latter case, T is a two-fold cover of a torus in $Sp(2n,\mathbb{R})$.

<u>5.1 Definition</u>:

<u>λ is integral if the following two conditions hold</u>:

<u>i.</u> <u>$\lambda+\rho(\mathfrak{u})$ exponentiates to a character of T,</u>

<u>ii.</u> <u>this character is genuine in the case that $G\subset Mp(2n,\mathbb{R})$</u>.

Given $\lambda\in\mathfrak{t}^*$ integral, we define $A(\lambda)=A_{\mathfrak{q}}(\lambda)$ as in [1]. Thus let $\mathfrak{q}=\mathfrak{q}(\lambda)$ and let L be the stabilizer of $\mathfrak{l}$ in G. As in §2 let $L_{\rho(\mathfrak{u})}$ be the "metaplectic" cover of L defined by $\rho(\mathfrak{u})$ [5]. If L is connected, let $\mathbb{C}_\lambda$ denote the unique genuine one-dimensional $L_{\rho(\mathfrak{u})}$ module with weight λ. In the notation of section 2, $A(\lambda)=\mathfrak{R}(\mathbb{C}_\lambda)$. Thus $A(\lambda)$ has the same infinitesimal character as $\mathbb{C}_\lambda$, i.e. $\lambda+\rho(\mathfrak{l})$. In the case of the orthogonal group L may be disconnected. In this case $L_{\rho(\mathfrak{u})}$ is isomorphic to $L\times\mathbb{Z}/2\mathbb{Z}$; take $\mathbb{C}_\lambda$ to be the genuine one-dimensional representation of $L\times\mathbb{Z}/2\mathbb{Z}$ with weight λ, trivial on any factor of the form $O(r,s)$.

<u>5.2 Definition</u>:

<u>λ is good if $(\lambda+\rho(\mathfrak{l}),\alpha)>0$ for all roots of $\mathfrak{t}$ in $\mathfrak{u}$</u>.

If λ is good, $A(\lambda)$ is irreducible, has regular infinitesimal character, and has lowest K-type $\lambda+\rho_n-\rho_c$ (we identify irreducible K-modules with their highest weights). In particular if λ is regular then $A(\lambda)$ is the discrete series representation with Harish-Chandra parameter λ. If λ is good then the following conditions are equivalent: $A(\lambda)$ is tempered, $A(\lambda)$ is a discrete series representation, L is compact.

We use the same notation for G', with ' appended: ρ',ρ'_n, etc.

<u>5.3 Theorem</u>:

<u>Given a stable reductive dual pair (G,G') as above, with G small. Suppose $\lambda\in\mathfrak{t}^*$ is the Harish-Chandra parameter of a discrete series representation $\pi=A(\lambda)$ of G. Let $\lambda'\in\mathfrak{t}'^*$ correspond to λ via the orbit correspondence. Assume λ' is good, and let $\pi'=A(\lambda')$. Then π corresponds to π' in the representation correspondence</u>.

<u>proof</u>:

The proof is exacly as in [1]. It amounts to a few calculations. We sketch the argument below, and then for each irreducible dual pair we tabulate the key steps in the calculation. This is mainly intended as a reference for the explicit representation correspondence in this case, in terms of λ and λ', lowest K-types, infinitesimal character, etc.

Consider the following diagram:

5.4

$$\begin{array}{ccccc} K & & & & H' \\ & G & — & G' & \\ & | & & | & \\ & H & — & K' & \\ K_H & & & & \end{array} \qquad \begin{array}{cccc} \delta & & & \\ & \pi & — & \pi' \\ & | & & || \\ & \sigma & — & \sigma' \\ \tau_H & & & \end{array}$$

Here K and K' are as usual, and K_H is a maximal compact subgroup of H. The groups appearing on the same line are members of a dual pair. Let $\pi=A(\lambda)$ be a discrete series representation of G, and let π' be the corresponding G' module. Assume the infinitesimal character of π' is regular. Let λ' correspond to λ via the orbit correspondence. We are claiming $\pi'\approx A(\lambda')$. By [27] it is enough to show π' has the correct infinitesimal character and contains the lowest K'=type of $A(\lambda')$. Explicitly, it is enough to show:

5.5 i. π' has infinitesimal character $\lambda'+\rho(\mathfrak{l}')$

ii. π' contains the K'-type $\lambda'+\rho'_n-\rho'_c$.

The infinitesimal character of π' is known, and satisfies 5.3(i). Assume λ' is good, and let $\sigma'=\lambda'+\rho'_n-\rho'_c$ be the lowest K'-type of $A(\lambda')$. Let σ denote the corresponding representation of H corresponding to σ' in the dual pair (H,K'). This is known explicitly, since K' is compact. It is enought to show σ' is contained in π' restricted to K', or equivalently that π is contained in σ restricted to G. Let τ_H be the minimal K_H-type of σ. Now the restriction of σ to G is known explicitly by ([14], cf. also [22]): $\sigma\approx \mathrm{Ind}_K{}^G(\tau_H|_K)$. Thus by Frobenious reciprocity it is enough to show $\mathrm{Hom}_K(\tau_H|_K,\pi|_K)\neq 0$. Let δ denote the minimal K-type of π. Then 5.5(ii) follows from:

5.5 (ii') δ is contained in $\tau_H|_K$.

In fact it is enough to show the much weaker statement that the highest weight vector for K_H in τ_H is a highest weight vector for K, with weight δ. We note that in these cases δ and σ' are found in the space H(K,K') of joint harmonics, and δ corresponds to σ' in the sense of [11].

This completes the sketch of the proof; the necessary calculations for each dual pair are completed below.

We identify irreducible modules for a compact group with their highest weights. In the case of the orthogonal group, it may in addition be necessary to specify ±1. Thus if π is an irreducible representation of $O(2n)$ such that π restricted to $SO(2n)$ is irreducible, it is necessary to specify the scalar ±1 by which $\mathrm{diag}(1,1,\ldots,1,-1)$ acts. In our situation this is always +1, and we omit it from the notation.

I. ($\underline{Sp(2m,\mathbb{R}),O(2p,2q)}$):

A maximal compact subgroup K of $Sp(2m,\mathbb{R})$ is isomorphic to $U(m)$. Choose the usual Cartan subgroup T of K, and choose the usual coordinates $\lambda=(a_1,a_2,\ldots,a_m)$ for $i\mathfrak{t}_0{}^*$. We choose positive K-roots so λ is dominant if $a_1\geq a_2\geq\ldots\geq a_m$. We make similar choices for $O(2p,2q)$ and write $(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q)$ for coordinates for $i\mathfrak{t}_0{}^*$ for $O(2p,2q)$. We choose positive K-roots so λ is dominant if $a_1\geq a_2\geq\ldots\geq a_p\geq 0$; $b_1\geq b_2\geq\ldots\geq b_q\geq 0$.

Let $n=p+q$. This pair is in the stable range if and only if $m\leq\min(p,q)$ or $2n\leq m$. We consider the two cases seperately.

A ($\underline{Sp(2m,\mathbb{R}),O(2p,2q)}$), $\underline{m\leq\min(p,q)}$

Let $G=Sp(2m,\mathbb{R})$, $G'=O(2p,2q)$. Then $K\approx U(m)$, $K'\approx O(2p)\times O(2q)$, $H\approx Sp(2m,\mathbb{R})\times Sp(2m,\mathbb{R})$, $K_H\approx U(m)\times U(m)$.

1. Orbit Correspondence:

$\lambda=(a_1,a_2,\ldots,a_k,b_1,b_2,\ldots,b_\ell)\rightarrow\lambda'=(a_1,a_2,\ldots,a_k,0,0,\ldots 0;-b_\ell,-b_{\ell-1},\ldots,-b_1,0,0,\ldots,0)$

$a_1\geq a_2\geq\ldots\geq a_k\geq 0>b_1\geq b_2\geq\ldots\geq b_\ell$.

Then λ is regular if and only if all inequalites are strict, in which case $L=T$ and $L'\approx U(1)^m\times O(2p-2k,2q-2\ell)$. Furthermore the following conditions are equivalent: λ is integral, λ' is integral, all $a_i,b_j\in\mathbb{Z}$. We assume these conditions hold.

2. Infinitesimal Character:

The correspondence on infinitesimal character is given by: $\chi_\gamma\rightarrow\chi_{\gamma'}$, where $\gamma=(c_1,c_2,\ldots,c_m)$, $\gamma'=(c_1,c_2,\ldots,c_m,n-m-1,n-m-2,\ldots,1,0)$.

It follows that 5.5(i) holds.

3. Minimal K-types:

The minimal K-type of the discrete series representation $A(\lambda)$ is δ:

$\delta=(a_1-x_1,a_2-x_2,\ldots,a_k-x_k,b_1+y_1,b_2+y_2,\ldots,b_\ell+y_\ell)+(1,2,\ldots,k,-\ell,-\ell+1,\ldots,-1)$,

where $x_i=|\{\,j\mid a_i+b_j<0\,\}|$, $y_i=|\{\,j\mid b_i+a_j>0\,\}|$.

Now λ' is good if and only if a_k, $b_1 > n-m-1$. Assume this holds. Then $A(\lambda')$ has minimal K'-type $\sigma'=\lambda'+\rho'_n-\rho'_c$, $\sigma'=\sigma'_1\check{\otimes}\sigma'_2$ ($\check{\otimes}$ denotes outer tensor product):

$\sigma'_1=(a_1-x_1,a_2-x_2,\ldots,a_k-x_k,0,0,\ldots,0)+(q-p+1,q-p+2,\ldots,q-p+k,0,0,\ldots,0)$,

$\sigma'_2=(-b_\ell-y_\ell,\ldots,-b_2-y_2,-b_1-y_1,0,0,\ldots,0)+(p-q+1,p-q+2,\ldots,p-q+\ell,0,0,\ldots,0)$.

It is convenient to rewrite this as follows. The lowest K-type of $A(\lambda)$ is δ:

$\delta=(\alpha_1,\alpha_2,\ldots,\alpha_k,\beta_1,\beta_2,\ldots,\beta_\ell)\quad \alpha_1>\alpha_2>\ldots>\alpha_k>0>\beta_1>\beta_2>\ldots>\beta_\ell$.

The lowest K'-type of $A(\lambda')=\sigma'=\sigma'_1\check{\otimes}\sigma'_2$:

$\sigma'_1=(\alpha_1,\alpha_2,\ldots,\alpha_k,0,0,\ldots0)+(q-p,q-p,\ldots,q-p,0,0,\ldots,0)$

$\sigma'_2=(-\beta_\ell,-\beta_{\ell-1},\ldots,-\beta_1,0,0,\ldots,0)+(p-q,p-q,\ldots,p-q,0,0,\ldots,0)$.

4. The H-module σ and the K_H module τ_H:

The H-module σ corresponding to σ' in the dual pair $(H,K')\approx(Sp(2m,\mathbb{R})\times Sp(2m,\mathbb{R}),O(2p)\times O(2q))$ is a discrete series representation with lowest K_H-type $\tau_H=\tau_{H_1}\check{\otimes}\tau_{H_2}$:

$\tau_{H_1}=(a_1-x_1+q+1,a_2-x_2+q+2,\ldots,a_k-x_k+q+k,p,p,\ldots,p)$

$\tau_{H_2}=(-q,-q,\ldots,-q,b_1+y_1-p-\ell,b_2+y_2-p-(\ell-1),\ldots,b_\ell+y_\ell-p-1)$

5. τ_H restricted to K:

Now $K\approx U(n)$ embeds in $K_H\approx U(n)\times U(n)$ on the diagonal. The highest weight vector of τ_H has weight $\tau_{H_1}+\tau_{H_2}=\delta$ for K, as is easily seen, verifying 5.5(ii)'.

B. <u>$(Sp(2m,\mathbb{R}),O(2p,2q))$ $2n\le m$</u> :

Let $G=O(2p,2q)$, $G'=Sp(2m,\mathbb{R})$. Thus $K\approx O(2p)\times O(2q)$, $K'\approx U(m)$, $H\approx U(2p,2q)$, $K_H\approx U(2p)\times U(2q)$.

1. Orbit Correspondence:

$\lambda=(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q)\to\lambda'=(a_1,a_2,\ldots,a_p,0,0,\ldots0,-b_q,-b_{q-1},\ldots,-b_2,-b_1)$,

$a_1\ge a_2\ge\ldots\ge a_p\ge 0,\quad b_1\ge b_2\ge\ldots\ge b_q\ge 0$.

Then λ is regular if and only if $a_i\ne b_j$ for all i,j, and all inequalites are strict except that possibly either $a_p=0$ or $b_q=0$, but not both. Then $L=T$ and assuming $a_p,b_q>0$, $L'\approx U(1)^n\times Sp(2(m-n),\mathbb{R})$. The following conditions are equivalent: λ is integral, λ' is integral, all $a_i,b_j\in\mathbb{Z}$. We assume these conditions hold.

2. Infinitesimal Character:

The correspondence on infinitesimal character is given by: $\chi_\gamma \to \chi_{\gamma'}$:
$\gamma=(c_1,c_2,\dots,c_n)$, $\gamma'=(c_1,c_2,\dots,c_n,m-n,m-n-1,\dots,1)$.
Then 5.5(i) holds.

3. Minimal K-types:
The minimal K-type of the discrete series representation $A(\lambda)$ is $\delta=\delta_1\check{\otimes}\delta_2$:
$\delta_1=(a_1+x_1,a_2+x_2,\dots,a_p+x_p)+(-p+1,-p+2,\dots,0)$
$\delta_2=(b_1+y_1,b_2+y_2,\dots,b_q+y_q)+(-q+1,-q+2,\dots,0)$
where $x_i=|\{\, j \mid a_i-b_j>0 \,\}|$, $y_i=|\{\, j \mid b_i-a_j>0 \,\}|$.

Assume λ' is good; equivalently a_p, $b_q>m-n$. Then $A(\lambda')$ has minimal K'-type $\sigma'=\lambda'+\rho'_n-\rho'_c$:

$$\sigma'=(a_1+x_1,a_2+x_2,\dots,a_p+x_p,0,0,\dots,0,-b_q-y_q,-b_{q-1}-y_{q-1},\dots,-b_1-y_1)+$$
$$(-q+1,-q+2,\dots,-q+p,p-q,p-q,\dots,p-q,p-q,p-(q-1),\dots,p-1)\,.$$

We may rewrite this as follows:
The lowest K-type of $A(\lambda)$ is $\delta=\delta_1\check{\otimes}\delta_2$:
$\delta_1=(\alpha_1,\alpha_2,\dots,\alpha_p)$, $\delta_2=(\beta_1,\beta_2,\dots,\beta_q)$, $\alpha_1>\alpha_2>\dots>\alpha_p>0$, $\beta_1>\beta_2>\dots>\beta_q>0$.

The lowest K'-type of $A(\lambda')$ is:
$\sigma'=(\alpha_1,\alpha_2,\dots,\alpha_p,0,0,\dots0,-\beta_q,-\beta_{q-1},\dots,-\beta_1)+(p-q,p-q,\dots,p-q)$.

4. The H-module σ and the K_H module τ_H:
The H-module σ corresponding to σ' in the dual pair $(H,K')\approx(U(2p,2q),U(m))$ is a discrete series representation with lowest K_H-type $\tau_H=\tau_{H_1}\check{\otimes}\tau_{H_2}$:
$\tau_{H_1}=(a_1-x_1,a_2-x_2,\dots,a_p-x_p,0,0,\dots,0)+(\tfrac12 m+q-p+1,\tfrac12 m+q-p+2,$
$\dots,\tfrac12 m+q,\tfrac12 m,\tfrac12 m,\dots,\tfrac12 m)$
$\tau_{H_2}=(b_1-y_1,b_2-y_2,\dots,b_q-y_q,0,0,\dots,0)+(\tfrac12 m+p-q+1,\tfrac12 m+p-+2,$
$\dots,\tfrac12 m+p,\tfrac12 m,\tfrac12 m,\dots,\tfrac12 m)$

5. τ_H restricted to K:
Now $K\approx O(2p)\times O(2q)$ embeds in $K_H\approx U(2p)\times U(2q)$ in the natural way. It is elementary to check that the highest weight vector of τ_H has weight δ for K, verifying 5.5(ii)'.

II. (U(p,q),U(r,s)):

Let $G=U(p,q)$ $(r+s\in 2\mathbb{Z})$ or $U(p,q)^\sim$ $(r+s\in 2\mathbb{Z}+1)$; let $G'=U(r,s)$ $(p+q\in 2\mathbb{Z})$ or $U(r,s)^\sim$ $(p+q\in 2\mathbb{Z}+1)$ where the coverings are as in §4,(II). We may and do assume $p+q \leq r+s$. This pair is in the stable range if and only if $p+q \leq \min(r,s)$, which we assume. Up to coverings, $K \approx U(p) \times U(q)$, $K' \approx U(r) \times U(s)$, $H \approx U(p,q) \times U(p,q)$, $K_H \approx U(p) \times U(q) \times U(p) \times U(q)$.

Choose the usual compact Cartan subgroup T of G. Choose coordinates $(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots b_q)$ for $i\mathfrak{t}_0{}^*$, and choose positive K-roots as usual. We write $(\ \ldots \ )_{p,q}$ to specify the group if necessary.

1. Orbit Correspondence:
   $\lambda = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_\ell; c_1, c_2, \ldots, c_m, d_1, d_2, \ldots d_n)_{p,q} \rightarrow$
   $\lambda' = (a_1, a_2, \ldots, a_k, 0, 0, \ldots 0, d_1, d_2, \ldots, d_n; c_1, c_2, \ldots, c_m, 0, 0, \ldots, 0, b_1, b_2, \ldots, b_\ell)_{r,s}$
   $a_1 \geq a_2 \geq \ldots \geq a_k \geq 0 > b_1 \geq b_2 \geq \ldots \geq b_\ell \ ; \ c_1 \geq c_2 \geq \ldots \geq c_m \geq 0 > d_1 \geq d_2 \geq \ldots \geq d_n$
   Then λ is regular if and only if all inequalities are strict, except that a single entry may be zero. Then $L=T$ and $L' \approx U(1)^{p+q} \times U(r-k-n, s-\ell-m)$ (assuming no entry is zero). Furthermore λ is integral if and only if λ' is integral, if and only if all entries are equivalent to $\frac{1}{2}(r+s+p+q+1) \mathrm{mod}(\mathbb{Z})$. We assume these conditions all hold.

2. Infinitesimal Character:
   The correspondence on infinitesimal character is given by: $\chi_\gamma \rightarrow \chi_{\gamma'}$, where
   $\gamma = (e_1, e_2, \ldots e_{p+q})_{p,q}$
   $\gamma' = (e_1, e_2, \ldots e_{p+q}, \frac{1}{2}(t-1), \frac{1}{2}(t-3), \ldots, \frac{1}{2}(-t+1))_{r,s}$ for $t=(r+s)-(p+q)$.

3. Minimal K-types:

   The minimal K-type of the discrete series representation $A(\lambda)$ is $\delta = \delta_1 \check{\otimes} \delta_2$:
   $\delta_1 = (a_1 - x_1, a_2 - x_2, \ldots, a_k - x_k, b_1 + z_1, b_2 + z_2, \ldots, b_\ell + z_\ell)_p +$
   $\frac{1}{2}(q-p+1, q-p+3, \ldots, q-p+(2k-1), p-q-(2\ell-1), \ldots, p-q-1)_p$
   $\delta_2 = (c_1 - y_1, c_2 - y_2, \ldots, c_m - y_m, d_1 + w_1, d_2 + w_2, \ldots, d_n + w_n)_q +$
   $\frac{1}{2}(p-q+1, p-q+3, \ldots, p-q+(2m-1), q-p-(2n-1), q-p-(2n-3), \ldots, q-p-1)_q$.
   Here $x_i = |\{ j \mid a_i - c_j < 0 \}|$, $y_i = |\{ j \mid c_i - a_j < 0 \}|$, $z_i = |\{ j \mid b_i - d_j > 0 \}|$, $w_i = |\{ j \mid d_i - b_j > 0 \}|$.

   λ' is good if and only if $a_k, c_m > \frac{1}{2}(t-1)$, $\frac{1}{2}(-t+1) > b_1, d_1$. Assume λ' is good. Then $A(\lambda')$ has minimal K'-type $\sigma' = \lambda' + \rho'_n - \rho'_c = \sigma'_1 \check{\otimes} \sigma'_2$:

$\sigma'_1=(a_1-x_1,a_2-x_2,\ldots,a_k-x_k,0,0,\ldots,0,d_1+w_1,d_2+w_2,\ldots,d_n+w_n)_r+$
$\frac{1}{2}(s-r+1,s-r+3,\ldots,s-r+(2k-1),p-q,p-q,\ldots,p-q,$
$(-s+r)-(2n-1),(-s+r)-(2n-3),\ldots,-s+r-1)_r$

$\sigma'_2=(c_1-y_1,c_2-y_2,\ldots,c_m-y_m,0,0,\ldots0,b_1+z_1,b_2+z_2,\ldots,b_\ell+z_\ell)_s+$
$\frac{1}{2}(r-s+1,r-s+3,\ldots,r-s+(2m-1),q-p,q-p,\ldots,q-p,$
$(-r+s)-(2\ell-1),(-r+s)-(2\ell-3),\ldots,(-r+s)-1)_s$.

It is convenient to rewrite this as follows.

The lowest K-type of $A(\lambda)$ is $\delta=\delta_1\check{\otimes}\delta_2$:

$\delta_1=(\alpha_1,\alpha_2,\ldots,\alpha_k,\beta_1,\beta_2,\ldots,\beta_\ell)_p \quad \alpha_1>\alpha_2>\ldots>\alpha_k>0>\beta_1>\beta_2>\ldots>\beta_\ell$

$\delta_2=(\gamma_1,\gamma_2,\ldots,\gamma_m,\delta_1,\delta_2,\ldots,\delta_n)_q \quad \gamma_1>\gamma_2>\ldots>\gamma_m>0>\delta_1>\delta_2>\ldots>\delta_n$.

The lowest K'-type of $A(\lambda')$ is $\sigma'=\sigma'_1\check{\otimes}\sigma'_2$:

$\sigma'_1=(\alpha_1,\alpha_2,\ldots,\alpha_k,0,0,\ldots0,\delta_1,\delta_2,\ldots,\delta_n)_r$
$+\frac{1}{2}(\ \{\ (p-q)+(s-r)\ \}^k,\ \{\ (p-q)\ \}^{r-k-n},\ \{\ (q-p)+(r-s)\ \}^n\ )_r$

$\sigma'_2=(\gamma_1,\gamma_2,\ldots,\gamma_m,0,0,\ldots,0,\beta_1,\beta_2,\ldots,\beta_\ell)_s$
$+\frac{1}{2}(\ \{\ (q-p)+(r-s)\ \}^m,\ \{\ (q-p)\ \}^{s-m-\ell},\ \{\ (p-q)+(s-r\ \}^\ell\ )_s$

Here $\{\ a\ \}^n$ denotes $(\overbrace{a,a,\ldots,a}^{n})$.

4. The H-module σ and the K_H module τ_H:

The H-module corresponding to σ' in the dual pair (H,K') is a discrete series representation with lowest K_H-type τ_H. Write $\tau_H=\tau_{H_1}\check{\otimes}\tau_{H_2}\check{\otimes}\tau_{H_3}\check{\otimes}\tau_{H_4}$ corrresponding to the decomposition $K_H=U(p)\times U(q)\times U(p)\times U(q)$ (up to covering). Then:

$\tau_{H_1}=(a_1-x_1,a_2-x_2,\ldots,a_k-x_k,0,0,\ldots,0)_p$
$+\frac{1}{2}(s+q-p+1,s+q-p+3,\ldots,s+q-p+(2k-1),r,r,\ldots r)_p$,

$\tau_{H_2}=(0,0,\ldots0,d_1+w_1,d_2+w_2,\ldots,d_k+w_k)_q$
$+\frac{1}{2}(-r,-r,\ldots,-r,s+q-p-(2n-1),s+q-p-(2n-3),\ldots,s+q-p-1)_q$,

$\tau_{H_3}=(c_1-y_1,c_2-y_2,\ldots,c_k-y_k,0,0,\ldots)_p$
$+\frac{1}{2}(p-q+r+1,p-q+r+3,\ldots,p-q+r+(2m-1),s,s,\ldots,s)_p$,

$\tau_{H_4}=(0,0,\ldots0,b_1+z_1,b_2+z_2,\ldots,b_k+z_k)_q$
$+\frac{1}{2}(-s,-s,\ldots,-s,p-q-r-(2\ell-1),p-q-r-(2\ell-3),\ldots,p-q-r-1)_q$.

5. τ_H restricted to K:

Now $K=U(p)\times U(q)$ embeds in K_H as the diagonal, so τ_H restricted to K is isomorphic to $(\tau_{H_1}\otimes\tau_{H_3})_p\check{\otimes}(\tau_{H_2}\otimes\tau_{H_4})_q$. The highest weight vector in τ_H

has weight $(\tau_{H_1}+\tau_{H_3})_p \otimes (\tau_{H_2}+\tau_{H_4})_q$ for K. From (D), this equals δ, the minimal K-type of $A(\lambda)$, verifying 5.5(ii)'.

III. ($SO^*(2n)$, $Sp(p,q)$)

Fix a maximal compact subgroup $K \approx U(n)$ of $SO^*(2n)$. Choose the usual Cartan subgroup T of K, and choose the usual coordinates $(a_1, a_2, \ldots, a_n)$ for $i\mathfrak{t}_0^*$. Make similar choice for $Sp(p,q)$, with $K \approx Sp(p) \times Sp(q)$ Write $(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q)$ for coordinates for $i\mathfrak{t}'_0{}^*$. Let $m=p+q$. This pair is in the stable range if and only if $m \leq [n/2]$ or $n \leq \min(p,q)$. We consider the two cases seperately.

A ($SO^*(2n)$, $Sp(p,q)$) ($m \leq [n/2]$):

Let $G=Sp(p,q)$, $G'=SO^*(2n)$. Then $K \approx Sp(p) \times Sp(q)$, $K' \approx U(n)$, $H \approx U(2p,2q)$, $K_H^{\sim} \approx U(2p) \times U(2q)$.

1. Orbit Correspondence:

$\lambda=(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q) \to \lambda'=(a_1,a_2,\ldots,a_p,0,0,\ldots 0,-b_q,-b_{q-1},\ldots,-b_1)$,
$a_1 \geq a_2 \geq \ldots \geq a_p \geq 0$; $b_1 \geq b_2 \geq \ldots \geq b_q \geq 0$.

Then λ is regular if and only if all inequalites are strict; then $L=T$ and $L' \approx U(1)^m \times SO^*(2(n-m))$. Furthermore λ is integral if and only if λ' is integral, if and only if all $a_i, b_j \in \mathbb{Z}$. We assume these conditions hold.

2. Infinitesimal Character:

The correspondence on infinitesimal character is given by: $\chi_\gamma \to \chi_{\gamma'}$, where: $\gamma=(c_1,c_2,\ldots,c_m)$, $\gamma'=(c_1,c_2,\ldots,c_m,n-m-1,n-m-2,\ldots,1,0)$.

3. Minimal K-types:

The minimal K-type of the discrete series representation $A(\lambda)$ is $\delta=\delta_1 \check{\otimes} \delta_2$:
$\delta_1=(a_1+x_1,a_2+x_2,\ldots,a_p+x_p)+(-p,-p+1,\ldots,-1)$,
$\delta_2=(b_1+y_1,b_2+y_2,\ldots,b_q+y_q)+(-q,-q+1,\ldots,-1)$.
Here $x_i = |\{ j \mid a_i-b_j>0 \}|$, $y_i = |\{ j \mid b_i-a_j>0 \}|$.

Now λ' is good if and only if $a_p, b_q>n-m-1$. Assume this holds. Then $A(\lambda')$ has minimal K'-type $\sigma'=\lambda'+\rho'_n-\rho'_c$:

$\sigma'=(a_1+x_1,a_2+x_2,\ldots,a_p+x_p,0,0,\ldots,0,-b_q-y_q,-b_{q-1}-y_{q-1},\ldots,-b_1-y_1)+$
$(-q,-q+1,\ldots,-q+(p-1),p-q,p-q,\ldots,p-q,p-(q-1),p-(q-2),\ldots,p)$.

It is convenient to rewrite this as follows. The lowest K-type of $A(\lambda)$ is $\delta=\delta_1 \check{\otimes} \delta_2$:

$\delta_1=(\alpha_1,\alpha_2,\ldots,\alpha_p)$, $\delta_2=(\beta_1,\beta_2,\ldots,\beta_q)$ $\alpha_1>\alpha_2>\ldots>\alpha_p>0$, $\beta_1>\beta_2>\ldots>\beta_q>0$.

The lowest K'-type of $A(\lambda')$ is σ': ·

$\sigma'=(\alpha_1,\alpha_2,\dots,\alpha_p,0,0,\dots 0,-\beta_q,-\beta_{q-1},\dots,-\beta_1)+(p-q,p-q,\dots,p-q)$.

4. The H-module σ and the K_H module τ_H:
The H-module corresponding to σ' in the dual pair $(H,K')\approx(U(2p,2q),U(n))$ is a discrete series representation with lowest K_H-type $\tau_H=\tau_{H_1}\check{\otimes}\tau_{H_2}$,
$\tau_{H_1}=(a_1+x_1,a_2+x_2,\dots,a_p+x_p,0,0,\dots,0)+(\frac12 n-p,\frac12 n-(p-1),\dots,\frac12 n-1,\frac12 n,\frac12 n,\dots\frac12 n)$
$\tau_{H_2}=(b_1+y_1,b_2+y_2,\dots,b_q+y_q,0,0,\dots,0)+(\frac12 n-q,\frac12 n-(q-1),\dots,\frac12 n-1,\frac12 n,\frac12 n,\dots\frac12 n)$.

5. τ_H restricted to K
Now $K\approx Sp(p)\times Sp(q)$ embeds in $K_H\approx U(2p)\times U(2q)$, and it is an elementary exercise to see the highest weight vector for τ_H has weight δ, the minimal K-type of $A(\lambda)$, verifying 5.5(ii)'.

B (<u>$SO^*(2n),Sp(p,q)$</u>) (<u>$n\le\min(p,q)$</u>):
Let $G=SO^*(2n)$, $G'\approx Sp(p,q)$. Then $K\approx U(n)$, $K'\approx Sp(p)\times Sp(q)$, $H\approx SO^*(2n)\times SO^*(2n)$, $K_H\approx U(n)\times U(n)$.

1. Orbit Correspondence:
$\lambda=(a_1,a_2,\dots,a_k,b_1,b_2,\dots,b_\ell)\to\lambda'=(a_1,a_2,\dots,a_k,0,0,\dots 0;-b_\ell,-b_{\ell-1},\dots,-b_1,0,0,\dots,0)$
$a_1\ge a_2\ge\dots\ge a_k\ge 0>b_1\ge b_2\ge\dots\ge b_q$.
Then λ is regular if and only if all inequalites are strict except possibly $a_k=0$. Then $L=T$ and $L'\approx U(1)^n\times Sp(p-k,q-\ell)$ (assuming $a_k>0$).
Furthermore the following conditions are equivalent: λ is integral, λ' is integral, all $a_i,b_j\in\mathbb{Z}$. We assume these conditions hold.

2. Infinitesimal Character:
The correspondence on infinitesimal character is given by: $\chi_\gamma\to\chi_{\gamma'}$, where $\gamma=(c_1,c_2,\dots,c_n)$, $\gamma'=(c_1,c_2,\dots,c_n,m-n,m-n-1,\dots,1)$.

3. Minimal K-types:
The minimal K-type of the discrete series representation $A(\lambda)$ is δ:
$\delta=(a_1-x_1,a_2-x_2,\dots,a_k-x_k,b_1+y_1,b_2+y_2,\dots,b_\ell+y_\ell)$
$+(0,1,\dots,k-1,-(\ell-1),-(\ell-2),\dots,0)$,
where $x_i=|\{\, j\mid a_i+b_j<0\,\}|$, $y_i=|\{\, j\mid b_i+a_j>0\,\}|$.

Assume λ' is good, which holds if and only if a_k, $b_\ell>m-n-1$. Then $A(\lambda')$ has minimal K'-type
$\sigma'=\lambda'+\rho'_n-\rho'_c$, $\sigma'=\sigma'_1\otimes\sigma'_2$:
$\sigma'_1=(a_1-x_1,a_2-x_2,\dots,a_k-x_k,0,0,\dots,0)+\ (q-p,q-p+1,\dots,q-p+(k-1),0,0,\dots,0)$,

$\sigma'_2=(-b_\ell-y_\ell,-b_{q-1}-y_{q-1},\dots,-b_1-y_1,0,0,\dots,0)+(p-q,p-q+1,\dots,p-q+(\ell-1),0,0,\dots,0)$.

It is convenient to rewrite this as follows. The lowest K-type of $A(\lambda)$ is :
$\delta=(\alpha_1,\alpha_2,\dots,\alpha_k,\beta_1,\beta_2,\dots,\beta_\ell)$ $,\alpha_1>\alpha_2>\dots>\alpha_k>0>\beta_1>\beta_2>\dots>\beta_\ell$.

The lowest K'-type of $A(\lambda')$ is $\sigma'=\sigma'_1\check{\otimes}\sigma'_2$:
$\sigma'_1=(\alpha_1,\alpha_2,\dots,\alpha_k,0,0,\dots0)+(q-p,q-p,\dots,q-p,0,0,\dots,0)$
$\sigma'_2=(-\beta_\ell,-\beta_{\ell-1},\dots,-\beta_1,0,0,\dots,0)+(p-q,p-q,\dots,p-q,0,0,\dots,0)$.

4. The H-module σ and the K_H module τ_H
The H-module corresponding to σ' in the dual pair $(H,K')\approx(SO^*(2n)\times SO^*(2n),Sp(p)\times Sp(q))$ is a discrete series representation with lowest K_H-type $\tau_H=\tau_{H_1}\check{\otimes}\tau_{H_2}$,
$\tau_{H_1}=(a_1-x_1+q,a_2-x_2+q+1,\dots,a_k-x_k+q+(k-1),p,p,\dots,p)$
$\tau_{H_2}=(-q,-q,\dots,-q,b_1+y_1-p-(\ell-1),b_2+y_2-p-(\ell-2),\dots,b_\ell+y_\ell-p)$.

5. τ_H restricted to K
Now $K\approx U(n)$ embeds in $K_H\approx U(n)\times U(n)$ on the diagonal. The highest vector of τ_H has weight $\tau_{H_1}+\tau_{H_2}=\delta$ for K, as is easily seen, and 5.5(ii)' is verified one last time.

This completes the proof of Theorem 5.3.

§6 Proof of Conjectures A and B in Special Cases

In this section we prove some of the conjectures in the preceding section in special cases. We start off with a discussion of the discrete series in the stable range. We then discuss the case when one member of the dual pair is compact. The third case considered is that of the trivial representation of various orthogonal groups, corresponding to some unipotent representations of $Sp(2m,\mathbb{R})$. We conclude with a discussion of the case of $(GL(m,\mathbb{R}),GL(n,\mathbb{R}))$.

Let (G,G') be an irreducible reductive dual pair, in the stable range with G small, such that G and G' have discrete series representations. Let $\gamma:{}^LG$ (or ${}^EG)\rightarrow{}^LG'$ (or ${}^EG'$) be as in §5. Given $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\rightarrow{}^LG$ (or ${}^EG$), let $\Psi':W_{\mathbb{R}}\times SL(2,\mathbb{C})\rightarrow{}^LG'$ (or ${}^EG'$) be defined as in §4.

<u>6.1 Theorem</u>:

<u>Suppose π is a discrete series representation of G. Let π' be the representation of G' which corresponds to π via the representation correspondence. Assume π' has regular infinitesimal character. Then Conjecture A holds in this case. That is suppose $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\rightarrow{}^LG$ (or ${}^EG$), and $\pi\in\Pi(\Psi)$. Then $\pi'\in\Pi(\Psi')$.</u>

<u>proof</u>:

For the proof we drop the distinction between ${}^EG$ and ${}^LG$. Choose $\delta\in{}^L\mathfrak{g}$ such that ${}^EG$ is the E-group determined by δ (cf. §2), and similarly for G'. Thus $\delta=0$ and ${}^EG={}^LG$ unless $G=U(p,q)^{\sim}$, and a similar statement holds for G'.

Let $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\rightarrow{}^EG$ be an admissible homomorphism such that $\pi\in\Pi(\Psi)$. Since π is a discrete series representation, a possibility for Ψ is the ordinary tempered parameter $\Psi:W_{\mathbb{R}}\rightarrow{}^EG$. We first consider this case. Thus $\Psi|_{SL(2,\mathbb{C})}$ is trivial, and the centralizer of the image of $\mathbb{C}^*$ is a Cartan subgroup

${}^LT^0$ of ${}^LG$. Let $\Psi':W_{\mathbb{R}}\times SL(2,\mathbb{C})\to {}^EG'$ be the map constructed in Conjecture A. We need to show $\pi'\in\Pi(\Psi')$.

Write $\pi\approx A(\lambda)$ as in Theorem 5.3, $\pi'\approx A(\lambda')$. Recall L' is the centralizer of λ' in G', and $\mathfrak{q}'=\mathfrak{l}'+\mathfrak{u}'$ is the θ-stable parabolic subalgebra of $\mathfrak{g}'$ defined by λ'. Then $\mathbb{C}_{\lambda'}$ is a one-dimesional representation of $L'_{\rho(\mathfrak{u}')}$, and $\pi'\approx\mathcal{R}(\mathbb{C}_{\lambda'})$ (cf. the discussion following 5.1). The conclusion will follow from the definition of the Arthur-packet $\Pi(\Psi')$.

Thus we describe $\Pi(\Psi')$ more carefully in this case. Recall ${}^LH'^0$ is the identity component of the centralizer of $\gamma({}^LG^0)$ in ${}^LG'^0$. Let ${}^LT'^0=\gamma({}^LT^0)$. It follows from the above facts and the additional assumption that π' has regular infinitesimal character that the identity component of the centralizer of $\Psi'(\mathbb{C}^*)$ in ${}^LG'^0$ is ${}^LC'^0=\langle {}^LT'^0,{}^LH'^0\rangle$. Let ${}^EC'=\langle {}^LC'^0,\Psi'(j)\rangle$. The following lemma follows immediately from these facts and a simple case-by-case check.

6.2 Lemma:

(i) ${}^LC'^0$ is isomorphic to ${}^LL'^0$

(ii) ${}^EC'$ is isomorphic to the E-group for L' determined by the element $\delta'+\rho(\mathfrak{u}')$.

Now $\Psi':W_{\mathbb{R}}\times SL(2,\mathbb{C})\to {}^EC'$, $\Psi'|_{\mathbb{C}^*}$ is contained in the center of ${}^LC'^0$, and $\Psi'|_{SL(2,\mathbb{C})}$ corresponds to the principal unipotent orbit in ${}^LC'^0$.

6.3 Lemma:

Suppose we are given G, ${}^EG$, and a (quasi-) admissible homomorphism $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\to {}^EG$, such that $\Psi(\mathbb{C}^*)$ is contained in the center of ${}^LG^0$. Furthermore assume $\Psi|_{SL(2,\mathbb{C})}$ corresponds to the principal unipotent orbit in ${}^LG^0$. Then the Arthur-packet of representations $\Pi(\Psi)$ attached to Ψ is a set of one-dimensional representations. If G is connected $\Pi(\Psi)$ consists of a single representation π, which is determined by its infinitesimal character.

proof:

Suppose $\pi\in\Pi(\Psi)$. By definitinon 2.1 the infinitesimal character χ_Ψ when restricted to the semisimple part of G is that of the trivial representation. Furthermore $\mathcal{O}$ in this case is the 0-orbit. The only representations whose wave-front set is contained in the 0-orbit are finite-dimensional representations. Thus π is finite-dimensional; hence π is one-dimensional. The remaining statement follows immediately.

Returning to our discussion, we apply the lemma to C'. Suppose $\pi_{C'}\in\Pi^{C'}(\Psi')$. If C' has a factor isomorphic to O(r,s) we take $\pi_{C'}$ to be trivial on that factor. First suppose ${}^EG'={}^LG'$. By Lemma 6.2 ${}^EC'$ is the E-group determined by the element $\rho(\mathfrak{u}')$. Choose $\mathfrak{q}'=\mathfrak{l}'\oplus\mathfrak{u}'$ as above. By Definition 2.3 $\mathcal{R}(\pi_{C'})$ is contained in $\Pi(\Psi')$. We may choose an isomorphism $L'\approx C'$ to take $\mathbb{C}_{\lambda'}$ to $\pi_{C'}$; this follows immediately from a consideration of infinitesimal characters. Thus $\pi'\approx\mathcal{R}(\mathbb{C}_{\lambda'})\approx\mathcal{R}(\pi_{C'})\in\Pi(\Psi')$, proving the Theorem in this case.

Next consider ${}^EG'\not\approx{}^LG'$, so $G'=U(r,s)\tilde{}$. We need to chase the coverings a bit. In the notation of §5, G' is isomorphic to the covering of U(r,s) determined by δ'. Let L'_0 be the image of L' in U(r,s) under the projection map. Furthermore $L'\approx(L'_0)\tilde{}_{\delta'}$, (the covering of L'_0 determined by δ'). Now $\mathbb{C}_{\lambda'}$ is a genuine representation of $(L')_{\rho(u')}\tilde{}$. It follows that $\mathbb{C}_{\lambda'}\otimes\mathbb{C}_{\rho(u')}$ is naturally a genuine representation of $L'\approx(L'_0)_{\delta'}\subset G'_{\delta'}$. Furthermore $C'\approx L'_0$. By Lemma 6.2 ${}^EC'$ is the E-group for C' determined by $\delta'+\rho(\mathfrak{u}')$. Thus $\pi_{C'}\otimes\mathbb{C}_{\rho(u')}$ may also be considered a genuine representation of $(L'_0)_{\delta'}$, and $\mathcal{R}(\mathbb{C}_{\lambda'})\approx\mathcal{R}(\pi_{C'})$ as before.

Finally we need to consider the case when $\Psi|_{SL(2,\mathbb{C})}$ is non-trivial. Because of the assumption of regular infinitesimal character for G', this differs from the previous case only in compact factors for L and L', which causes no serious difficulty.

Thus suppose Ψ is given and $\pi\in\Pi(\Psi)$. Let L,$\mathfrak{q}$, and π_L be as in the definition of $\Pi(\Psi)$, such that $\pi\approx\mathcal{R}(\pi_L)$. Let $\Psi':W_{\mathbb{R}}\times SL(2,\mathbb{C})\rightarrow{}^EG'$ be the map constructed in §4 corresponding to Ψ. By the preceding discussion we choose $\Psi^{\dagger}$ as above such that $\pi'\in\Pi(\Psi^{\dagger})$. Thus $\pi'\approx\mathcal{R}(\mathbb{C}_{\lambda^{\dagger}})$, for $L^{\dagger}$, $\mathfrak{q}^{\dagger}=\mathfrak{l}^{\dagger}\oplus\mathfrak{u}^{\dagger}$ etc. as above. We need to show $\pi'\in\Pi(\Psi')$.

Recall π has regular integral infinitesimal character, and this is the same as the infinitesimal character of π_L. Thus we have that π_L (restricted to the derived group of L) is a unipotent representation with regular integral infinitesimal character. Hence π_L must be one-dimensional and $\Psi|_{SL(2,\mathbb{C})}$ corresponds to the principal unipotent orbit of ${}^LL^0$ (cf. §2). Since π is a discrete series representation, L must be compact.

Now consider Ψ'. Let ${}^LC'^0$, $\mathfrak{q}'=\mathfrak{l}'+\mathfrak{u}'$, etc. be as in the definition of $\Pi(\Psi')$. From the above discussion we have that $\Psi'|_{SL(2,\mathbb{C})}$ corresponds to the principal unipotent orbit in ${}^LC'^0$. We have $\mathfrak{q}'\supset\mathfrak{q}^{\dagger}$, $L'\supset L^{\dagger}$, and $\mathcal{R}(\mathbb{C}_{\lambda'})\subset\Pi(\Psi')$, for some one-dimensional representation $\mathbb{C}_{\lambda'}$. A case by case check shows that we may take $L'/L^{\dagger}$ compact (by carrying over the fact that L is compact, and using the assumption that π' has regular infinitesimal character). By induction by stages ([24], Theorem 6.3.6) we have $\pi'\approx\mathcal{R}(\mathbb{C}_{\lambda'})\approx\mathcal{R}(\mathbb{C}_{\lambda^{\dagger}})\in\Pi(\Psi')$, completing the proof of the Theorem.

We discuss Conjecture B in this case. Assume G' is connected. Consider $\{G_i\}$ $(0\leq i\leq k)$ as in the statement of that conjecture. Fix a discrete series representation $\pi_0=A(\lambda_0)$ as in Theorem 6.1. Write the corresponding representation π' in the form $A(\lambda'_0)$ as in Theorem 5.3. We identify $\mathfrak{g}$, $\mathfrak{t}$ etc. for $\{G_i\}$. Let χ be the infinitesimal character of π; we think of this as an infinitesimal character of G_i for any i. We note that the condition in Theorem 6.1 on the infinitesimal character of π' may be written as a condition on χ.

Let $\Pi^{G_0}(\Psi)=\{A(w\lambda_0)| w\in W_K\backslash W(\mathfrak{g},\mathfrak{t})$ be the ordinary L-packet of discrete series representations of G_0 containing π_0. Then for $\pi\in\Pi^{G_0}(\Psi)$, π occurs in the representation correspondence. By Theorem 5.3 and the explicit orbit correspondence we see the corresponding representation π' is of the form $A(w'\lambda'_0)$ for some $w'\in W'=W(\mathfrak{g}',\mathfrak{t}')$. Let $\mathfrak{q}'_0=\mathfrak{l}'_0\oplus\mathfrak{u}'_0$ be the parabolic subalgebra of $\mathfrak{g}'$ defined by λ'_0, and let $S=W_{K'}\backslash W'/W(\mathfrak{l}'_0,\mathfrak{t}')$ (with the obvious notation, cf. [4]). It follows from the definition of $\Pi^{G'}(\Psi')$ that $\Pi^{G'}(\Psi')=\{A(w'\lambda'_0) \mid w'\in S\}$. Thus $\pi\to\pi'$ establishes an injection $\Pi^{G_0}(\Psi)\hookrightarrow \Pi^{G'}(\Psi')$.

Furthermore, given $w'\in S$, from the orbit correspondence we see that there exists G_i such that $w'\lambda'_0$ occurs in the orbit correspondence for the pair (G_i,G'), and G_i is determined uniquely. It follows that $\Pi^{\{G_i\}}(\Psi)$ is in bijection with $\Pi^{G'}(\Psi')$.

If G' is O(2p,2q) the same result holds, except that $\Pi^{G'}(\Psi')$ contains more representations, obtained from one-dimensional representations of L' which are non-trivial on an orthogonal group factor O(r,s). Note that in this case $G=Sp(2n,\mathbb{R})$ is the only inner form of G occuring.

This proves:

<u>6.4 Theorem</u>:

<u>Suppose we are in the setting of Theorem 6.1.</u>

(<u>1</u>) <u>Suppose $G'\not\simeq O(2p,2q)$. Then Conjecture B holds</u>: $\underline{\Pi^{\{G_i\}}(\Psi)\to\Pi^{G'}(\Psi')}$ <u>is a bijection.</u>

(<u>2</u>) <u>Suppose $G'=O(2p,2q)$. Then</u> $\underline{\Pi^{G}(\Psi)\hookrightarrow\Pi^{G'}(\Psi')}$ <u>is an injection</u>.

Part (2) could be strengthened by by using the "true" Arthur-packets which are a refinement of these (cf. §2).

We next discuss the case of dual pairs where one member of the pair is compact. This is similar to the preceding case with one important difference. The most interesting cases for the following theorem are for degenerate representations, with (very) singular infinitesimal character, a case which was excluded in Theorem 6.1.

We begin by summarizing the explicit representation correspondence; we will be quite brief, since this is discussed in detail in [2].

The situation is similar to the case of discrete series in the stable range. Suppose G is compact. Then the irreducible representations of G may be thought of as discrete series representations. Suppose π occurs in the representation correspondence, corresponding to π'. Write $\pi\approx A(\lambda)$ as above, with λ regular. Thus $L(\lambda)\approx T$, for T a compact Cartan subgroup of G. Let $\lambda'\epsilon\mathfrak{t}'^*$ correspond to λ via the orbit correspondence. Then, roughly speaking, $\pi'\approx A(\lambda')$.

The qualification is necessary for the following reason. Suppose the small group G is compact. For generic λ as above,with corresponding λ', $\mathfrak{l}(\lambda')$ remains unchanged; call it $\mathfrak{l}_0$ ($\mathfrak{l}_0$ depends only on (G,G')). However for certain values of λ, $\mathfrak{l}(\lambda')$ may increase in size; this happens if λ' becomes singular on a single extra simple root wall, even though λ remains regular. Let $\pi=A(\lambda)$ as above, and let π' correspond to π (if G is O(n) we may only have $\pi'A(\lambda')$, see below). We write $\mathcal{R}_{\mathfrak{l}}$ or $\mathcal{R}_{\mathfrak{l}_0}$ to distinguish between the corresponding functors.

Now there is a one-dimensional representation $\mathbb{C}_{\lambda'}$ of L_0 (or a covering group) such that $\pi'\subset\mathcal{R}_{\mathfrak{l}_0}(\mathbb{C}_{\lambda'})$. Here L_0 satisfies $L_0=L_0(\lambda'_0)$ for some λ'_0 satisfying $\mathfrak{l}(\lambda'_0)\approx\mathfrak{l}_0$. Furthermore $\mathcal{R}_{\mathfrak{l}_0}(\mathbb{C}_{\lambda'})\approx\mathcal{R}_{\mathfrak{l}}(\pi'_L)$ where π'_L is a unipotent representation of $L=L(\lambda')$ (or a covering group).

This situation may occur if (G,G')=(U(m),U(r,s)), m<r+s. In this case $\pi'\approx\mathcal{R}_{\mathfrak{l}_0}(\mathbb{C}_{\lambda'})\approx\mathcal{R}_{\mathfrak{l}}(\pi'_L)$. It may also arise if $(G,G')=(O(m),Sp(2n,\mathbb{R}))$, m<2n. Suppose $A(\lambda)$ is reducible, i.e. $A(\lambda)\approx\pi\oplus\pi_*$, where $\pi_*\approx\pi_*\otimes sgn$. Then either $\pi'\approx\mathcal{R}_{\mathfrak{l}_0}(\mathbb{C}_{\lambda'})\approx\mathcal{R}_{\mathfrak{l}}(\pi'_L)$, or $\mathcal{R}_{\mathfrak{l}}(\mathbb{C}_{\lambda'})\approx\mathcal{R}_{\mathfrak{l}}(\pi'_{L'})\approx\pi'\oplus\pi'_*$, where π'_* corresponds to π_*. The latter case may only occur if m<n. These are the only situations in which this occurs.

The representations π' occuring here are all highest weight modules, so it is not hard to identify them with modules of the form $A(\lambda')$ [3]. However we are making no restriction on the infinitesimal character of π'; it is allowed to be quite singular, and as mentioned above these are the interesting cases.

6.5 Theorem:

Suppose either G or G' is compact. Suppose π corresponds to π' in the representation correspondence. Then π is tempered. Choose $\Psi:W_{\mathbb{R}}\to {}^LG$ (or ${}^EG$) such that $\pi\epsilon\Pi(\Psi)$, this is possible because π is tempered. Then Conjecture A holds in this case: $\pi'\epsilon\Pi(\Psi')$, for Ψ' constructed in §4.

Note there is no restriction on the infinitesimal character of π' as there is in Theorem 6.1. As in Theorem, 6.1, it may be that π' is contained in some other Arthur-packet $\Pi(\Psi^\dagger)$ with $\Psi^\dagger|_{SL(2,\mathbb{C})}$ non-trivial. For example the trivial representation of G when G is compact may be obtained this way. As in Theorem 6.1 the conclusion of the Theorem should still hold; however since we do not restrict to regular infinitesimal character there are many more possibilites here, and we do not pursue this matter. We discuss the case of G=O(n) and π one-dimensional below.

proof:

This follows from the main result of [2] just as Theorem 6.1 follows from §5. Thus suppose π corresponds to π' in the representation correspondence. If G is not compact π is a highest-weight module; in fact it follows from the explicit representation correspondence ([18], [10]) that π is a discrete series representation. This obviously also holds if G is compact. In particular π has regular infinitesimal character.

Suppose $\pi\in\Pi(\Psi)$. Then $\Pi(\Psi')$ is described as in the proof of Theorem 6.1. Except in the exceptional cases mentioned preceding the statement of the Theorem, $\pi'\approx\mathfrak{R}^s(\mathbb{C}_{\lambda'})\approx\mathfrak{R}^s(\pi_{C'})\in\Pi(\Psi')$ as in the proof of Theorem 6.1, proving the Theorem in these cases.

So suppose G=O(m), G'=Sp(2n,$\mathbb{R}$), m=2k<2n. Suppose $A(\lambda)$ is reducible. Write $A(\lambda)\approx\pi_+\oplus\pi_-$, where $\pi_\pm$ has highest weight of the form $(a_1,a_2,\dots,a_{k-1},0)_\pm$ for $a_1\ge a_2\ge\dots\ge a_{k-1}\ge 0$ in the usual notation. Then ${}^EC'$ is isomorphic to the L-group of $C'=U(1)^{k-1}\times Sp(2(n-k+1),\mathbb{R})$, in particular ${}^LC'^0\approx SO(2n-m+3,\mathbb{C})$. Consider the map $\Psi':W_{\mathbb{R}}\times SL(2,\mathbb{C})\to {}^EC'$. Then $\Psi'|_{SL(2,\mathbb{C})}$ corresponds to ${}^LC'^0\bullet\mathcal{O}$, where $\mathcal{O}$ is the prinicpal unipotent orbit of $SO(2,\mathbb{C})\times SO(2n-m+1,\mathbb{C})\subset SO(2n-m+3,\mathbb{C})$. It follows that the irreducible representation of C' with highest weight $(a_1+k-1,a_2+k-2,\dots,a_{k-1}+1)\check{\otimes}(1,1,\dots,1)$ is contained $\Pi^{C'}(\Psi')$. Now π_+ occurs in the representation correspondence, and it follows from [2] that the corresponding representation π'_+ satisfies $\pi'\subset\mathfrak{R}^s(\pi_{C'})$. Furthermore it follows that if π_- occurs also, then $\pi'_+\oplus\pi'_-\approx\mathfrak{R}^s(\pi_{C'})$ (where π'_- corresponds to π'_-). This proves the Theorem in this case.

The case of U(r,s) is similar, and in fact simpler since G is connected. We omit the details. This completes the proof of the Theorem.

Our next case is that of the Type II dual pair (G,G')=(GL(m,$\mathbb{R}$,GL(n,$\mathbb{R}$)), m≤n. The result in this case is complete and simple to state.

We use the usual notation for induced representations of GL(n,$\mathbb{R}$). Thus we let P be a parabolic subgroup of GL(n,$\mathbb{R}$) and write $Ind_P(\sigma)$ for ordinary parabolic induction from P to G, where σ is a representation of the Levi component, extended to P by letting the nilradical act trivially. The Levi component of P will be products of groups of the form GL(,$\mathbb{R}$) embedded on the diagonal.

The representation correspondence is known completely and is very simple [21]. All representations of G occur. Suppose n=m. Then the representation π' of G' corresponding to a given representation π of G is isomorphic to π^* (π^* denotes the contragredient).

Suppose m<n, and let P' denote the upper triangular parabolic subgroup of G' with Levi factor GL(m,$\mathbb{R}$)×GL(n-m,$\mathbb{R}$). Then π' is a constituent of Ind($\pi^*\otimes$trivial). Here π^* is taken as before on the GL(m,$\mathbb{R}$) factor and the trivial representation is taken on GL(n-m,$\mathbb{R}$). If π is unitary then this induced module is irreducible ([25],Theorem 17.6).

Arthur-packets for G=GL(n,$\mathbb{R}$) are described as follows.

6.6 Definition:

Suppose $\Psi: W_{\mathbb{R}}\times SL(2,\mathbb{C}) \to {}^LG$, where Ψ restricted to $\mathbb{C}^*\subset W_{\mathbb{R}}$ is trivial, and Ψ restricted to SL(2,$\mathbb{C}$) corresponds to the principal unipotent orbit. Then $\Pi(\Psi)$ is a singleton: $\Pi(\Psi)$={trivial representation} if $\Psi(j)$=Id, and $\Pi(\Psi)$={sgn representation} if $\Psi(j)$=-Id.

This is a refinement of Definition 2.1; as defined there $\Pi(\Psi)$ consists of both the trivial and the sgn representation.

It follows from this definition that if $\Psi: W_{\mathbb{R}}\times SL(2,\mathbb{C}) \to {}^LG$, with Ψ restricted to $\mathbb{C}^*$ trivial, then there is a parabolic subgroup P such that $\Pi(\Psi)=\{Ind_P(\sigma)\}$, where σ is the trivial or sgn representation of each GL(,$\mathbb{R}$) of P. Finally in general $\Pi(\Psi)=\{Ind_P(\sigma)\}$ is a singleton, where σ is either tempered or the trivial, sgn, or a "Speh" representation on each GL(,$\mathbb{R}$) factor of P. In all cases $Ind_P(\sigma)$ is irreducible. The only fact which we need is the following, which is immediate: if $\pi\in\Pi(\Psi)$, then $\pi^*\in\Pi(\Gamma\circ\Psi)$, where Γ is the outer automorphism of ${}^LG$ given by $g\to {}^tg^{-1}$ ($g\in {}^LG^0$). Note that this property also holds for $\Pi(\Psi)$ defined by 2.1

6.7 Theorem:

Conjecture A is true in the case of (GL(m,$\mathbb{R}$,GL(n,$\mathbb{R}$)).

proof:

Suppose $\pi\in\Pi(\Psi)$, and π' corresponds to π. Assume m=n. Then $\pi'\approx\pi^*$, and identifying ${}^LG$ and ${}^LG'$ we have $\Psi'=\Gamma\circ\Psi$. The result follows immediately in this case from the discussion preceding the Theorem. For m≤n let P' denote a parabolic subgroup of G' with Levi component H'=GL(m,$\mathbb{R}$)×GL(n-m,$\mathbb{R}$)⊂GL(n,$\mathbb{R}$). Now ${}^LH'$ embeds in ${}^LG'$ in the obvious way, and Ψ' factors through ${}^LH'$. It follows immediately from the previous case and the construction of Ψ' that $(\pi^*\check{\otimes}\text{trivial})\in\Pi^{H'}(\Psi')$, and $\Pi(\Psi')=\{\text{Ind}_{P'}(\pi^*\check{\otimes}\text{trivial})\}$. Since $\pi'\approx\text{Ind}_{P'}(\pi^*\check{\otimes}\text{trivial})$ the proof of the Theorem is complete.

We turn next to a consideration of the case of $(O(p,q),Sp(2n,\mathbb{R}))$, $p+q=2k\le 2n$, and the Arthur-packet for $O(p,q)$ consisting of the trivial and the sgn representation of $O(p,q)$. Let $G_j=O(2k-2j,2j)$ $(0\le j\le k)$, and let $\pi_{j,+}$ denote the trivial representation of G_j. Then $\pi_{j,+}$ occurs in the representation correspondence; let $\pi'_{j,+}$ be the corresponding representations. Then $\pi'_{j,+}$ has been computed in [21]. Thus the lowest K'-type of π'_+ is one-dimensional, and is $(k-2j,k-2j,\ldots,k-2j)$. This representation has infinitesimal character $(n-k,n-k-1,\ldots,1,0,-1,-2,\ldots,-(k-1))$. Furthermore, it is not hard to see that the wave front set of π' is contained in the $Sp(2n,\mathbb{C})$ orbit $\mathcal{O}$ through the element γ:

6.8 $$\gamma=\begin{pmatrix} 0_n & X \\ 0_n & 0_n \end{pmatrix} \qquad \text{where } X=\text{diag}(\overbrace{1,1,\ldots,1}^{k},0,0,\ldots,0)\ .$$

Let ${}^L\mathcal{O}'={}^LG'^0\bullet\Omega$, where Ω is the principal unipotent orbit of $SO(2k,\mathbb{C})\times SO(2n-2k+1,\mathbb{C})\subset SO(2n+1,\mathbb{C})\approx{}^LG'^0$. Then $\mathcal{O}'$ is dual to ${}^L\mathcal{O}'$ in the sense of [23]. It follows that $\pi'_{j,+}$ is a unipotent representation corresponding to this orbit.

Now suppose $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\to{}^LG$ with $\Psi|_{W_{\mathbb{R}}}$ trivial, and $\Psi|_{SL(2,\mathbb{C})}$ corresponding to the principal unipotent orbit in ${}^LG^0\approx O(2k,\mathbb{C})$.

<u>6.9 Definition</u>:

<u>The Arthur-packet $\Pi(\Psi)$ for $O(p,q)$ is $\{$trivial,sgn$\}$</u>.

Again this is a refinement of Definition 2.1: as defined in §2, $\Pi(\Psi)=\{\text{trivial},\text{sgn},\pi^1,\pi^2\}$, the four one-dimensional representations of $O(p,q)$.

Now for Ψ' constructed as in §4, $\Psi'|_{SL(2,\mathbb{C})}$ corresponds to ${}^L\mathcal{O}'$. Thus by definition, $\pi'_{j,+}\in\Pi^{G'}(\Psi')$ $(0\le j\le k)$. We have proved:

<u>6.10 Theorem</u>:

<u>Let $(G,G')=(O(p,q),Sp(2n,\mathbb{R}))$, $p+q=2k\le 2n$.</u>

(<u>i</u>) <u>Conjecture A holds for Ψ as above</u>, <u>π equal to the trivial representation of G</u>.

(<u>ii</u>) <u>Conjecture B,1 holds for Ψ as above</u>, <u>σ the trivial representation of G_i</u>, <u>for G_i an inner form of G</u>.

In the next section we will discuss super-stable distributions, of which $\Sigma_i \pi_{i,+}$ is an example. Conjecture 7.5 says in this case that $\Theta'=\Sigma\pi'_{i,+}$ is a stable distribution of $Sp(2n,\mathbb{R})$. For example if $2k=2n$ then Θ' is equal (as a character) to a representation induced from a one-dimensional representation of a maximal parabolic subgroup with Levi component $GL(n,\mathbb{R})$. Since a one-dimensional representation (in fact any representation) of $GL(n,\mathbb{R})$ is stable, and parabolic induction preserves stability, Θ' is stable. These characters are studied from a different point of view in [17].

It is of interest to study $\pi'_{i,-}$ (corresponding the the sgn representation of G_j) and Σ_i ($\pi'_{i,-}$). Recall $\pi'_{i,-}$ is a quotient of a certain reducible module module $\Pi'_{i,-}$, which is itself a quotient of ω restricted to G' [11]. It may be that it is $\Pi'_{i,-}$ rather than $\pi'_{i,-}$ which is the proper object of study.

§7 Concluding Remarks

We conclude with a few further remarks, and a conjecture about endoscopic groups.

The results of this paper may be extended in a number of ways. However, these methods are not intended as a method of proof in general, so pushing these results as far as possible is probably not worth the effort. For example the results on discrete series can certainly be strengthened quite a bit, both to representations with singular infinitesimal character, and to groups outside of the stable range.

A comparison with the extensive results of Moeglin for the dual pair $(O(2p,2q),Sp(2n,\mathbb{R}))$ [21] is very instructive. In broad terms Moeglin's result is the following, which is very similar to the p-adic case treated in [16].

Let $G=Sp(2n,\mathbb{R})$, and $G'=O(2p,2q)$, with $n\le p+q$ and $p\ge q$. Let $G_k=Sp(2k,\mathbb{R})$ $(k=1,2,\ldots,n)$ and let $G'_z=O(2(p-q)+2z,2z)$ $(z=0,1,2,\ldots)$. Consider any irreducible representation π of G_k. Then π occurs in the representation correspondence for the pairs (G_k,G_z) for all $z\ge N(\pi)$, for some $N(\pi)$ depending on π. We refer to this integer $N(\pi)$ as "the first occurence of π". Suppose the first occurence of π were known for all discrete series representations π of G_k, for all $k\le n$. Then the representation correspondence is obtained by real parabolic induction as follows.

Suppose a representation π of G occurs, corresponding to π' of G'. We write (P,σ) for the Langlands parameters of a representation of G. Thus P is a parabolic subgroup of G, and σ is a relative discrete series representation of the Levi component M of P. We use similar notation for G'. Now M is isomorphic to

7.1 $GL(1,\mathbb{R})^r\times GL(2,\mathbb{R})^s\times Sp(2t,\mathbb{R})$

for some r,s and t. Write $\sigma=\sigma_1\check{\otimes}\sigma_2\check{\otimes}\tau$ corresponding to this decomposition. Let $N=N(\tau)$, and let τ' denote the representation of

$G'_N \approx O(2k,2\ell)$ corresponding to τ in the first occurence of τ. Then there is a parabolic subgroup P' of G', such that M' is isomorphic to

7.2 $GL(1,\mathbb{R})^{r'} \times GL(1,\mathbb{R})^{r} \times GL(2,\mathbb{R})^{s} \times O(2k,2\ell)$,

Let $\sigma' = \sigma_0 \check{\otimes} \sigma_1 \check{\otimes} \sigma_2 \check{\otimes} \tau'$, where σ_0 is a certain character of $GL(1,\mathbb{R})^{r'}$. Then π' is a constituent of the module obtained by inducing σ' from P' to G'.

Suppose τ' is a discrete series representation. Then this essentially gives the Langlands parameters of π'. The most severe restriction of ([21], Condition (†)) insures that τ' is a discrete series representation. This forces $k+\ell=t$ or $t+1$.

Thus it appears that the difficulty in computing the Langlands parameters is largely concentrated on the case of first occurence. It seems likely that this is where the derived functors play their essential role. Thus suppose the first occurence of π is naturally described in terms of derived functor modules. Then perhaps the above construction may be pushed to give a complete description of the representation correspondence. This will not be expressed naturally in terms of Langlands parameters, except in the case $k+\ell=t$ or $t+1$ of the preceding paragraph. However it should be compatible with Arthur-parameters.

For example, in § IV.7 and IV.8 Moeglin studies certain discrete series π of $Sp(2n,\mathbb{R})$ for which condition (†) fails. The corresponding representations of $O(2p,2q)$ may be realized in terms of derived functor modules. Given such π there is a natural derived functor module $A(\lambda')$ which has the correct infinitesimal character and minimal K'-type; by ([21], IV.7, (5)') $A(\lambda')$ corresponds to π. This holds even for the representations π' of $O(2p,2q)$ with almost trivial lowest K'-type.

A true understanding of the conjectures should probably involve understanding the oscillator representation itself in terms of L-groups, something which is beyond us at the moment.

We conclude with a conjecture about stable characters and lifting from endoscopic groups. We will gloss over details about embeddings of L-groups, E-groups and covering groups. We hope to discuss this issue in more detail in a later paper.

Suppose we are in the setting of conjecture B, and as in that conjecture there is a bijection $\Pi^{\{G_i\}}(\Psi) \rightarrow \Pi^{G'}(\Psi')$, where $\Pi^{\{G_i\}}(\Psi)$ is considered as a set of representations of $\{G_i\}$. By a virtual character of a group we now allow a finite complex linear combination of irreducible characters.

7.3 Definition:

1. A virtual character of $\{G_i\}$ is a formal sum $\Theta=\Theta_1+\Theta_2+\ldots+\Theta_k$ where Θ_i is a virtual character of G_i.

2. The virtual character Θ corresponds to the virtual character $\Theta'=\Sigma_i a_i\Theta'_i$ of G' if Θ_i corresponds to Θ'_i for all i.

3. The virtual character Θ is super-stable only if Θ_i is stable for all i.

The complete definition of super-stable is stronger than this, which we leave to a later paper and use only this part of the definition. An example of a super-stable distribution is the following. Let $\{G_i\}$ be a complete set of representatives for the isomorphism classes of inner forms of G. Let $\Psi:W_{\mathbb{R}}\rightarrow {}^LG$ be the parameter corresponding to an L-packet $\Pi^G(\Psi)$ of discrete series representations. Let $\Theta=\Sigma_j\Theta_j$ be the stable sum of discrete series characters in $\Pi^G(\Psi)$. The same definition holds for G_i, let Θ_i be the corresponding stable virtual character of G_i. Then $\Sigma_i\,\Theta_i$ is a super-stable virtual character. Another example is $\Sigma_i\,\epsilon_i\pi_i$ where π_i is the trivial representation of G_i, and $\epsilon_i=\pm 1$.

We now consider an Arthur-packet to be a complex vector space of virtual characters (spanned by the irreducible characters in it). In general an Arthur-packet is conjectured to contain a distinguished super-stable virtual character. For example the Arthur-packets of Theorem 6.1 are either discrete series packets, or those discussed in [4]. Thus we have:

7.3 Theorem:

Suppose we are in the setting of Theorem 6.1. Then there is a distinguished super-stable virtual character Θ_0 (resp. Θ'_0) in $\Pi(\Psi)$ (resp. $\Pi(\Psi')$).

In any event, assume there is such a virtual character $\Theta_0\in\Pi(\Psi)$. Write $\Pi(\Psi)=<\pi_{i,j}>$, where (for all i and j) $\pi_{i,j}$ is an irreducible representation of G_i, and < > denotes complex span. For $\delta_{i,j}\in\mathbb{C}$ write $\Theta_0=\Sigma_{i,j}\,\delta_{i,j}\,\pi_{i,j}$; this defines $\{\delta_{i,j}\}$.

Supppose we are in the setting of Conjecture B. Write the bijection $\Pi(\Psi)\rightarrow\Pi(\Psi')$ as $\pi_{i,j}\rightarrow\pi'_{i,j}$. Define $\delta_{i,j}$ and $\delta'_{i,j}$ as above (applied to G or G'). Define a new bijection $\Upsilon:\Pi(\Psi)\rightarrow\Pi(\Psi')$ via $\pi_{i,j}\rightarrow\delta'_{i,j}/\delta_{i,j}\,\pi'_{i,j}$. Thus $\Upsilon(\Theta_0)=\Theta'_0$.

Now suppose H is an endoscopic group for G, and $\Psi:W_{\mathbb{R}}\times SL(2,\mathbb{C})\rightarrow{}^LH\rightarrow{}^LG$. Thus ${}^LH^0$ is the centralizer of a semi-simple element h of ${}^LG^0$. Then $\mathrm{Lift}_H{}^G$ is defined, taking super-stable virtual characters in $\Pi(\Psi)$ to virtual characters in $\Pi(\Psi)$, where these Arthur-packets are considered as spaces of virtual characters of the inner forms of G and H. Let $h^*=\gamma(h)$, and let ${}^LH^{*0}$ be the centralizer of h^* in ${}^LH'^0$. Note that ${}^LH^{*0}$ contains the fixed group ${}^LH'^0$. Suppose ${}^LH'^0$ is extended to ${}^LH'$, with corresponding endoscopic group H'. Then we have $\Psi':W_{\mathbb{R}}\times SL(2,\mathbb{C})\rightarrow {}^LH'\rightarrow {}^LG'$, and we obtain $\Pi^{H'}(\Psi')$. We have the following diagram:

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                            Ψ                                     T
7.4          W_R×SL(2,C) -----> LG -------> LG'  ⊃  LH'0 <--- SL(2,C)
                          `- -   U           U            - - - -
                               `> LH -------> LH' <- -
```

7.5 Conjecture C:

Suppose we are in the setting of Conjecture B. With $\gamma:\Pi(\Psi)\rightarrow\Pi(\Psi')$ as above:

1. $\gamma(\Theta_0)=\Theta'_0$.

2. Suppose $\Theta=Lift_H{}^G(\Theta_H)\in\Pi(\Psi)$, for $\Theta_H\in\Pi^H(\Psi)$ a super-stable virtual character. Then there is a super-stable virtual character $\Theta_{H'}\in\Pi^{H'}(\Psi')$ such that $\gamma(\Theta)=Lift_{H'}{}^{G'}(\Theta_{H'})\in\Pi(\Psi')$.

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