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Astérisque, tome 168 (1988), p. 61-140

<http://www.numdam.org/item?id=AST_1988__168__61_0>

Société Mathématique de France
Astérisque 168 (1988), p.61-140.

GEOMETRY OF ORBITS AND SPRINGER CORRESPONDENCE

Toshiaki Shoji

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INTRODUCTION

As is well-known, irreducible characters of the symmetric group of degree n are in 1-1 correspondence with unipotent conjugacy classes of GL_n; both of them are parametrized by partitions of n. Of course, this is nothing more than a level of parametrization. More conceptual description may be found in the work of Green [12] concerning with Green polynomials arising from the representation theory of $GL_n(\mathbf{F}_q)$, where $\mathbf{F}_q$ is a finite field of q-elements.

In 1976, Springer [39] succeeded in generalizing the above result to the case of connected reductive groups, i.e., he obtained a general relationship between unipotent conjugacy classes of a reductive group and irreducible characters of its Weyl group. Before stating his result more precisely, we prepare some notations. Let G be a connected reductive algebraic group defined over an algebraically closed field k of characteristic $p > 0$ and W be a Weyl group of G. Let $\mathcal{B}$ be the variety of Borel subgroups of G, and for each $g \in G$ we denote by $\mathcal{B}_g$ the subvariety of $\mathcal{B}$ consisting of Borel subgroups which contain g. Let u be a unipotent element in G. Springer constructed in [loc.cit.] the so-called <u>Springer representations</u> of W on ℓ-adic cohomologies $H^i(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell)$. Here ℓ is a prime number not equal to p. Let $A_G(u)$ be the quotient of the centralizer $Z_G(u)$ of u by its connected centralizer $Z_G^0(u)$. Then $A_G(u)$ acts naturally on $H^i(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell)$ commuting with the action of W. We shall consider the top cohomology group $H^{top}(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell)$. It is decomposed as $W \times A_G(u)$-modules,

$$H^{top}(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell) \simeq \bigoplus_\rho V_{u,\rho} \otimes \rho ,$$

where ρ runs over all elements in $A_G(u)^\wedge$, the set of irreducible representations of $A_G(u)$ (up to conjugacy), and $V_{u,\rho}$ is the W-module corresponding to ρ. Let $\mathcal{N}_G$ be the set of pairs (u,ρ), where u runs over a set of representatives of unipotent classes in G and $\rho \in A_G(u)^\wedge$. Springer has proved that $V_{u,\rho}$ is irreducible if it is non-zero, and that the map $(u,\rho) \to V_{u,\rho}$ induces a bijection (the Springer correspondence),

$$\{(u,\rho) \in \mathcal{N}_G \mid V_{u,\rho} \neq 0\} \xrightarrow{\sim} W^\wedge ,$$

where $W^\wedge$ is the set of isomorphism classes of irreducible representations of W.

Springer's original construction relies on a Lie algebra argument in positive characteristic. In particular, it involves the Artin-Schreier covering of k. Thus it works neither for the case where G is defined over $\mathbb{C}$ nor for the case p is too small. However, in [23], Lusztig gave a more elegant construction of Springer representations by making use of the theory of perverse sheaves ([3]), which works simultaneously for G and its Lie algebra $\underline{g}$, and still makes sense in the characteristic 0 case by replacing ℓ-adic cohomologies by classical cohomologies with $\mathbb{C}$-coefficients. We shall see briefly his construction. Let $\tilde{G} = \{(g,B) \in G \times \mathcal{B} \mid g \in B\}$ and let $\pi\colon \tilde{G} \to G$ be the first projection. Let $K = \pi_*\bar{\mathbb{Q}}_\ell$ be the complex in the derived category $\mathcal{D}^b_c(G,\bar{\mathbb{Q}}_\ell)$ of certain ℓ-adic sheaves on G, obtained as the direct image of the constant sheaf $\bar{\mathbb{Q}}_\ell$ on $\tilde{G}$. We denote by $\mathcal{M}G$ the category of perverse sheaves on G, which is a full subcategory of $\mathcal{D}^b_c(G,\bar{\mathbb{Q}}_\ell)$ and forms an abelian category (see §3). Lusztig showed that K turns out to be a semisimple object in $\mathcal{M}G$ up to shift, and that the endomorphism algebra $\mathrm{End}\, K$ in $\mathcal{M}G$ is isomorphic to the group algebra $\bar{\mathbb{Q}}_\ell[W]$ of W. Since $\pi^{-1}(g) \simeq \mathcal{B}_g$, the induced action of W on the stalk $\mathcal{H}^i_g(K)$ at $g \in G$ of i-th cohomology sheaf of K induces an action of W on $H^i(\mathcal{B}_g,\bar{\mathbb{Q}}_\ell)$. This is Lusztig's construction.

In the above setting, the Springer correspondence may be formulated as follows. K is decomposed into simple objects through W-action as

$$K \simeq \bigoplus_{E\in W^\wedge} E \otimes K_E \quad ,$$

where K_E is a simple object in $\mathcal{M}G$ (up to shift) corresponding to E. Let G_{uni} be the set of unipotent elements in G. Borho-MacPherson [6] showed that the restriction $K_E|G_{uni}$ of K_E on G_{uni} is isomorphic to a simple object in $\mathcal{M}G$ of the form $IC(\bar{C},\mathcal{E})$ (extended by 0 on $G - \bar{C}$) up to shift, where $IC(\bar{C},\mathcal{E})$ is the intersection cohomology complex on $\bar{C}$ associated with $(C,\mathcal{E})$ (see §3). Here C is a unipotent class in G and $\bar{C}$ is its Zariski closure in G, and $\mathcal{E}$ is an irreducible G-equivariant local system on C. Since the set of such pairs $(C,\mathcal{E})$ is naturally in 1-1 correspondence with $\mathcal{N}_G$, we have a map $W^\wedge \to \mathcal{N}_G$, which is shown to be injective by [6]. This is nothing but the Springer correspondence.

It should be noticed that the above map $W^{\wedge} \to \mathcal{N}_G$ is not necessarily surjective in general, and there exist some pairs $(C,\mathcal{E})$ which do not correspond to any $E \in W^{\wedge}$. In order to understand these missing pairs $(C,\mathcal{E})$, Lusztig [25] introduced the notion of cuspidal complexes and that of induction of such complexes from Levi subgroups, (analogue of Harish-Chandra's philosophy on cuspidal representations and parabolic inductions of reductive groups over $\mathbb{R}$ or $\mathbb{F}_q$). Thus by decomposing various induced complexes he obtained a certain class of G-equivariant simple perverse sheaves $A(G)_{uni}$, whose elements are called (unipotent) admissible complexes on G. He showed that, for each $A \in A(G)_{uni}$ the restriction $A|G_{uni}$ is isomorphic to $IC(\bar{C},\mathcal{E})$ up to shift for some $(C,\mathcal{E}) \in \mathcal{N}_G$, and that the map $A \to (C,\mathcal{E})$ gives rise to a bijection $A(G)_{uni} \simeq \mathcal{N}_G$, (the generalized Springer correspondence). Note that $A(G)_{uni}$ is parametrized by a disjoint union of irreducible characters of various Coxeter groups, and among them the Springer correspondence plays a role of principal series representations.

The theory of generalized Springer correspondence may be extended to the Lie algebras $\underline{g}$ without difficulty replacing unipotent classes by nilpotent orbits, whenever p is large enough. One of the advantages in the Lie algebra setting is the existence of Fourier transforms of $\bar{\mathbb{Q}}_\ell$-sheaves introduced by Deligne. Let $\mathcal{F}$ be such a Fourier transform on $\underline{g}$, and let $\mathcal{M}\underline{g}$ be the category of perverse sheaves on $\underline{g}$. It is known by Laumon [19] that $\mathcal{F}$ induces an equivalence of categories $\mathcal{F}: \mathcal{M}\underline{g} \to \mathcal{M}\underline{g}$. The significance of the restriction $G \to G_{uni}$ appearing in the generalized Springer correspondence may be understood naturally from a view point of Fourier transforms. Springer's original work [39] is already related to the Fourier transform, even though it is implicit there. Inspired by an idea of Kashiwara [16] concerning with a $\mathcal{D}$-module approach to the Springer correspondence, Brylinski [7] formulated Borho-MacPherson's theorem in terms of Fourier transforms, i.e., he showed that for each K_E (simple object in $\mathcal{M}\underline{g}$ similar to $K_E \in \mathcal{M}G$ as before), $\mathcal{F}K_E$ is isomorphic to $IC(\bar{\mathcal{O}},\mathcal{E})$, up to shift, for some $(\mathcal{O},\mathcal{E})$ in $\mathcal{N}_{\underline{g}}$, (the analogous set for $\underline{g}$ to $\mathcal{N}_G$ with respect to nilpotent orbits in $\underline{g}$). This result was strengthened by Lusztig [28] to the following form, including the case of generalized Springer correspondence for $\underline{g}$. He defined as an analogy of the case of G, a certain class of G-equivariant simple perverse sheaves $A(\underline{g})$, called admissible complexes on

$\underline{g}$, and its subset $A(\underline{g})_{nil}$, nilpotent admissible complexes. We shall say a G-equivariant simple perverse sheaf on $\underline{g}$ is orbital if it is of the form $IC(\bar{\mathcal{O}},\mathcal{E})$, up to shift, extended by 0 on $\underline{g} - \bar{\mathcal{O}}$, where $\mathcal{O}$ is an arbitrary G-orbit in $\underline{g}$ and $\mathcal{E}$ is an irreducible G-equivariant local system on $\mathcal{O}$. He showed that $A(\underline{g})$ and the set of orbital objects in $\mathcal{M}\underline{g}$ are in 1-1 correspondence through $\mathcal{F}$. In particular, $\mathcal{F}$ induces a bijection between $A(\underline{g})_{nil}$ and $\mathcal{N}_{\underline{g}}$.

After Springer's pioneer work, various kind of approaches have been done for the construction of Springer representations in the case of $k = \mathbf{C}$, e.g., the approach given by Slodowy [35] as the monodromy representations associated with the transversal slices of nilpotent orbits in $\underline{g}$, the one given by Kazhdan-Lusztig [22] from a topological point of view. As mentioned before, Hotta-Kashiwara [16] constructed Springer representations by making use of the theory of $\mathcal{D}$-modules, which is closely related to the representation theory of Lie groups. Joseph [18] also constructed them (the ones corresponding to $(u,1) \in \mathcal{N}_G$) by applying his Goldie rank represenations of W in connection with the theory of primitive ideals of the enveloping algebra of $\underline{g}$. In fact, it became to notice, through their works, that Springer representations are closely related with the theory of primitive ideals, the geometry of nilpotent orbits and the characteristic classes associated with certain bundles on $\mathcal{B}$. See Borho [5] for details, and also Ginsburg [11].

The aim of these notes is to give an exposition of the theory of the generalized Springer correspondence and their relations with Fourier transforms as mentioned in the earlier part. Hence, in these notes we concentrate ourselves to the positive characteristic case. Although much part of the discussion on the Springer correspondence is contained in that of the generalized Springer correspondence, we treated the Springer correspondence separately because of its importance. Also we added a chapter concerning generalized Green functions associated with a reductive group over a finite field, which leads to the determination of local intersection cohomologies $\mathcal{H}^i_x(IC(\bar{C},\mathcal{E}))$. Although results of these notes are closely related to the theory of character sheaves developed by Lusztig [26], we do not discuss it here except a brief remark. The reader may consult the report of Mars and Springer on character sheaves in this volume.

I GEOMETRY OF ORBITS

§1. Geometry of $\mathcal{B}_g$

1.1. Let G be a connected reductive algebraic group defined over an algebraically closed field k of characteristic $p \geqq 0$, and let $\mathcal{B}$ be the variety of Borel subgroups of G. We fix $B = UT$ in $\mathcal{B}$, where U is the unipotent radical of B and T is a maximal torus of G contained in B. Let $W = N_G(T)/T$ be the Weyl group of G with respect to T. We denote by G_{uni} the set of unipotent elements in G. For each $g \in G$, put

$$\mathcal{B}_g = \{B_1 \in \mathcal{B} \mid g \in B_1\} \quad .$$

It is a closed subvariety of $\mathcal{B}$. The variety $\mathcal{B}_g$ has been studied extensively, in connection with the study of conjugacy classes of G (or G-orbits of the Lie algebra $\underline{g}$ of G) by many authors. In particular, $\mathcal{B}_g$ is of pure dimension (Spaltenstein [36]), and $\mathcal{B}_g$ is connected if g is unipotent (Steinberg [42]).

We now consider the following varieties,

$$Z = \{(g, B_1, B_2) \in G \times \mathcal{B} \times \mathcal{B} \mid g \in B_1 \cap B_2\} \quad ,$$
$$Z' = \{(g, B_1, B_2) \in G_{uni} \times \mathcal{B} \times \mathcal{B} \mid g \in B_1 \cap B_2\} \; .$$

Let ν_G be the number of positive roots in G, hence $\nu_G = \dim \mathcal{B}$. We have the following result due to Springer, Steinberg and Spaltenstein (see, eg., Spaltenstein [36]).

1.2. **Theorem.** (i) <u>Let</u> C <u>be a conjugacy class of</u> G <u>containing</u> g. <u>Then</u> $\dim \mathcal{B}_g = \nu_G - \frac{1}{2}\dim C$.

(ii) <u>Let</u> C <u>be a unipotent class</u>. <u>Then</u> $\dim (C \cap U) = \frac{1}{2}\dim C$.

(iii) <u>All the irreducible components of</u> Z (<u>resp</u>. Z') <u>are parametrized by</u> W. <u>The variety</u> Z (<u>resp</u>. Z') <u>is of pure dimension with</u> $\dim Z = \dim G$ (<u>resp</u>. $\dim Z' = 2\nu_G$), <u>respectively</u>.

<u>Proof</u>. Although all of these results are well-known now, we give the proof below for the sake of completeness. First we show (iii). Let $p: Z' \to \mathcal{B} \times \mathcal{B}$ be the projection on the second and third factors. G acts diagonally on $\mathcal{B} \times \mathcal{B}$ by conjugation action so that those G-orbits are parametrized by W (Bruhat decomposition). Let $\mathcal{O}_w$ be

the G-orbit corresponding to $w \in W$, i.e., the G-orbit of (B, wBw^{-1}) in $\mathcal{B} \times \mathcal{B}$. Put $Z'_w = p^{-1}(\mathcal{O}_w)$. We have a partition $Z' = \bigcup_{w \in W} Z'_w$. Now, Z'_w is a vector bundle over $G/(B \cap wBw^{-1}) \simeq \mathcal{O}_w$ with fibre isomorphic to $U \cap wUw^{-1}$. Hence Z'_w is irreducible and

$$\begin{aligned} \dim Z'_w &= \dim \mathcal{O}_w + \dim (U \cap wUw^{-1}) \\ &= \dim G - \dim T \\ &= 2\nu_G \ . \end{aligned}$$

Since all the pieces Z'_w have the same dimension $2\nu_G$, we see that $\dim Z' = 2\nu_G$ and that $\{\bar{Z}'_w \mid w \in W\}$ gives all the irreducible components of Z', where $\bar{Z}'_w$ is the Zariski closure of Z'_w in Z'. This proves (iii) for Z'. The case of Z is proved in a similar way. In particular, if we define Z_w as the inverse image of $\mathcal{O}_w$ under the projection $Z \to \mathcal{B} \times \mathcal{B}$, $Z_w \to \mathcal{O}_w$ has a locally trivial fibration with fibre isomorphic to $B \cap wBw^{-1}$. Hence $\{\bar{Z}_w \mid w \in W\}$ gives all the irreducible components of Z.

Nextly, we shall show (i). Let $g = su = us$ be the Jordan decomposition of g, where s is semisimple and u is unipotent, and let $H = Z_G^0(s)$. Then $\mathcal{B}_g$ is isomorphic to the disjoint union of finitely many copies of $\mathcal{B}_u^H$, where the last variety is the similar one as $\mathcal{B}_g$ with respect to the reductive group H and a unipotent element u in H. Thus the proof of (i) is reduced to the case where g is unipotent. We assume that g is unipotent and let C be the class containing g. The following inequality, which is crucial for our arguments in subsequent sections, was proved by Springer and Steinberg ([42]), independently.

(1.2.1) $$\dim \mathcal{B}_g \leq \nu_G - \frac{1}{2} \dim C \ .$$

We show (1.2.1). Let $\psi_1 : Z' \to G_{uni}$ be the first projection. Then each fibre of ψ_1 on C is isomorphic to $\mathcal{B}_g \times \mathcal{B}_g$, and so,

$$\dim \psi_1^{-1}(C) = \dim C + 2 \dim \mathcal{B}_g \ .$$

Since $\dim \psi_1^{-1}(C) \leq \dim Z' = 2\nu_G$ by (iii), we get (1.2.1).

To complete the proof of (i), it is enough to show the following property, which was verified essentially using the classification of unipotent classes of G by Bala-Carter [8, 5.9.6, 5.10.1] in the case where $\operatorname{char}(k) = 0$ or large enough, and by Spaltenstein (e.g., [36]) in small characteristic cases.

(1.2.2) For each unipotent class C in G, there exists $w \in W$ such that $C \cap (U \cap wUw^{-1})$ is open dense in $U \cap wUw^{-1}$.

Finally, we show (ii). Consider the variety $V = \{(x,B_1) \in C \times \mathcal{B} \mid x \in B_1\}$. The projection on the first factor yields $\dim V = \dim C + \dim \mathcal{B}_g$ as each fibre is isomorphic to $\mathcal{B}_g$. On the other hand, the second projection yields $\dim V = \dim \mathcal{B} + \dim (C \cap U)$ since C is a unipotent class. Thus, it follows from (i) that $\dim (C \cap U) = \dim C + \dim \mathcal{B}_g - \nu_G = \frac{1}{2} \dim C$. This proves (ii).

1.3. Theorem 1.2 was generalized by Lusztig to the case of parabolic subgroups. Let P be a parabolic subgroup of G with Levi decomposition $P = LU_P$, where L is a Levi subgroup of P and U_P is the unipotent radical of P. We assume given a conjugacy class C_1 in L and put $\Sigma = Z^0(L)C_1$ where $Z^0(L)$ is the connected center of L. Set

$$Z = \{(g, xP, x'P) \in G \times G/P \times G/P \mid g \in x(\Sigma U_P)x^{-1} \cap x'(\Sigma U_P)x'^{-1}\}$$

$$Z' = \{(g, xP, x'P) \in G \times G/P \times G/P \mid g \in x(C_1 U_P)x^{-1} \cap x'(C_1 U_P)x'^{-1}\}.$$

We also assume given a conjugacy class C in G. The following result was first proved by Springer [40] with some restriction on p, and by Lusztig in the general case extending the argument used in the proof of Theorem 1.2. In the special case where $L = T$, it is reduced to the preceding theorem. Note, in the following theorem, the equality fails in general (for example, C = regular unipotent class, $C_1 = \{1\}$).

1.4. **Theorem** (Lusztig [25]). (i) <u>Given</u> $g \in C$, <u>we have</u>

$$\dim \{xP \in G/P \mid x^{-1}gx \in C_1U_P\} \leq (\nu_G - \tfrac{1}{2} \dim C) - (\nu_L - \tfrac{1}{2} \dim C_1),$$

<u>where</u> ν_L <u>is the number of positive roots in</u> L.

(ii) <u>Given</u> $\bar{g} \in C_1$, <u>we have</u>

$$\dim (C \cap \bar{g}U_P) \leq \tfrac{1}{2}(\dim C - \dim C_1) .$$

(iii) $\dim Z \leq 2\nu_G - 2\nu_L + \dim \Sigma$.

(iv) $\dim Z' \leq 2\nu_G - 2\nu_L + \dim C_1$.

1.5. Returning to the case of Borel subgroups, we shall deduce several properties of $\mathcal{B}_g$ as corollaries of Theorem 1.2. Let $I(\mathcal{B}_g)$ be the set of irreducible components of $\mathcal{B}_g$. All the irreducible components of $\mathcal{B}_g$ have the same dimension (cf. 1.1). Let $A_G(g)$ be the quotient of the centralizer $Z_G(g)$ by its connected centralizer $Z_G^0(g)$. The conjugation action of $Z_G(g)$ on $\mathcal{B}_g$ induces an action of $A_G(g)$ on $I(\mathcal{B}_g)$. We denote by $(I(\mathcal{B}_g) \times I(\mathcal{B}_g))/A_G(g)$ the set of $A_G(g)$-orbits in $I(\mathcal{B}_g) \times I(\mathcal{B}_g)$ under the diagonal action of $A_G(g)$. Then we have

1.6. **Corollary** (Steinberg [43]).

$$\coprod_u (I(\mathcal{B}_u) \times I(\mathcal{B}_u))/A_G(u) \simeq W$$

<u>where</u> u <u>runs over all the representatives of unipotent classes in</u> G.

<u>Proof</u>. Let $\psi_1 : Z' \to G_{uni}$ be as in the proof of theorem 1.2. Then for each unipotent class C containing u, $\psi_1^{-1}(C)$ is isomorphic to $G \times^{Z_G(u)} (\mathcal{B}_u \times \mathcal{B}_u)$. Thus, all the irreducible components of $\psi_1^{-1}(C)$ come from the components of $G \times (\mathcal{B}_u \times \mathcal{B}_u)$, and two components of $G \times (\mathcal{B}_u \times \mathcal{B}_u)$ give the same component of $\psi_1^{-1}(C)$ if and only if they are in the same $Z_G(u)$-orbit. Hence the set of irreducible components of $\psi_1^{-1}(C)$ are in 1-1 correspondence with the set of $A_G(u)$-orbits in $I(\mathcal{B}_u) \times I(\mathcal{B}_u)$. Now it follows from Theorem 1.2 that all the irreducible components of $\psi_1^{-1}(C)$ have the same dimension $2\nu_G = \dim Z'$. So, the left hand side of Corollary 1.6 corresponds bijectively to the set of irreducible components of Z', which is parametrized by W by Theorem 1.2 (iii).

1.7. **Corollary.**

(i) <u>For any</u> $i > 0$, $\dim \{x \in G \mid \dim \mathcal{B}_x \geq \frac{i}{2}\} < \dim G - i$,

(ii) <u>For any</u> $i \geq 0$, $\dim \{x \in G_{uni} \mid \dim \mathcal{B}_x \geq \frac{i}{2}\} \leq 2\nu_G - i$.

<u>Proof</u>. First consider (i). Assume that (i) fails for some $i > 0$. Put $V = \{x \in G \mid \dim \mathcal{B}_x \geq i/2\}$ for this i. V is closed in G. Consider the variety Z defined in 1.1 and let $\psi : Z \to G$ be the first projection. Then by our assumption, the dimension of the inverse image $\psi^{-1}(V)$ of V is at least $(\dim G - i) + i = \dim G$. Since $\dim Z = \dim G$ by Theorem 1.2 (iii), $\psi^{-1}(V)$ contains an

irreducible component of Z, which is necessarily of the form $\bar{Z}_w$ for some $w \in W$ by (the proof of) Theorem 1.2. Hence $\psi^{-1}(V)$ contains a set

$$\{(g,\ xB,\ xwB) \in G \times G/B \times G/B \mid x^{-1}gx \in T_{reg}\}$$

where T_{reg} is the set of regular semisimple elements in T, i.e., the set of $t \in T$ such that $Z_G^0(t) = T$. Then we have $T_{reg} \subset V$. But this implies a contradiction since $\dim \mathcal{B}_x = 0$ for a regular elemnt x.

Next we show (ii). Assume that (ii) fails for some $i \geqq 0$, and put $V' = \{x \in G_{uni} \mid \dim \mathcal{B}_x \geqq i/2\}$ for this i. Let $\psi_1 : Z' \to G_{uni}$ be as before. Then by our assumption, $\dim \psi_1^{-1}(V) > 2\nu_G = \dim Z'$. This is a contradiction.

§2. Some partition of G

2.1. In this section, we shall construct a partition of G into irreducible, locally closed, smooth pieces stable by conjugation by G. This partition will play a fundamental role in Chapter III.
For each $g \in G$, let g_s be its semisimple part. An element $g \in G$ is said to be <u>isolated</u> if $Z_G^0(g_s)$ is not contained in any Levi subgroup of a proper parabolic subgroup of G. Its class is called an isolated class. Note, if G is semisimple, there exist only finitely many isolated classes. Let L be a Levi subgroup of a parabolic subgroup of G and Σ be the inverse image of an isolated class in $L/Z^0(L)$ under the natural map $L \to L/Z^0(L)$. Put

$$\Sigma_{reg} = \{x \in \Sigma \mid Z_G^0(x_s) \subset L\} \quad ,$$

and

$$Y_{(L,\Sigma)} = \bigcup_{g \in G} g(\Sigma_{reg})g^{-1} \quad .$$

Then we have

2.2. **Proposition.** $\quad G = \coprod_{(L,\Sigma)} Y_{(L,\Sigma)}$ (<u>disjoint union</u>), <u>where</u> (L,Σ) <u>runs over all such pairs up to</u> G-<u>conjugacy</u>. <u>Each piece</u> $Y_{(L,\Sigma)}$ <u>is irreducible</u>, <u>locally closed</u>, <u>smooth and stable by conjugation by</u> G. <u>Moreover</u>,

$$\dim Y_{(L,\Sigma)} = \dim G - \dim L + \dim \Sigma \quad .$$

In fact, if we define $H_G(g)$, for each $g \in G$, as a Levi subgroup of some parabolic subgroup of G which is the smallest closed subgroup containing $Z_G^0(g_s)$, then we see that $H_G(g) \subset L$ and $\Sigma_{reg} = \{g \in \Sigma \mid H_G(g) = L\}$. Hence Σ_{reg} is open dense in Σ and so, $Y_{(L,\Sigma)}$ is irreducible. By making use of the Steinberg map $\sigma\colon G \to T/W$, it is verified that $Y_{(L,\Sigma)}$ is locally closed. The map $g \to H_G(g)$ defines a locally trivial fibration $Y_{(L,\Sigma)} \to$ the variety of all conjugates of L, whose fibres are isomorphic to $\bigcup_{n \in N_G(L)/L} n(\Sigma_{reg})n^{-1}$. Since this last variety is a finitely many copies of Σ_{reg}, we see that $Y_{(L,\Sigma)}$ is smooth. The dimension of $Y_{(L,\Sigma)}$ is also determined from this. Conversely, for any $g \in G$, if we put $L = H_G(g)$ and $\Sigma = Z^0(L)C_1$, where C_1 is the class of L containing g, then Σ is isolated modulo $Z^0(L)$ and $g \in Y_{(L,\Sigma)}$. Hence these $Y_{(L,\Sigma)}$ form a partition of G.

2.3. Let $Y = Y_{(L,\Sigma)}$ be as in 2.1. Let us define

$$\tilde{Y} = \{(g,\ xL) \in G \times G/L \mid x^{-1}gx \in \Sigma_{reg}\} ,$$

and

$$X = \{(g,\ xP) \in G \times G/P \mid x^{-1}gx \in \bar{\Sigma}U_P\} ,$$

where $P = LU_P$ is a parabolic subgroup cntaining L and $\bar{\Sigma}$ is the closure of Σ in L. Let $\pi\colon X \to G$ be the projection onto the first factor. π is G-equivariant with respect to a G-action on X given by $g_1\colon (g,xP) \to (g_1gg_1^{-1}, g_1xP)$, and the conjugation action of G on G. Then we have

2.4. **Lemma.** (i) X <u>is irreducible</u>, <u>and</u> $\dim X = \dim Y$.
(ii) π <u>is proper and</u> $\pi(X) = \bar{Y}$ (<u>the closure of</u> Y <u>in</u> G).
(iii) <u>The map</u> $(g,xL) \to (g,xP)$ <u>gives an isomorphism</u> $\tilde{Y} \xrightarrow{\sim} \pi^{-1}(Y) \subset X$

<u>Proof</u>. We show only (i) and (ii). For the proof of (iii), see [25, 4.3]. The second projection $X \to G/P$ is G-equivariant and each fibre is isomorphic to $\bar{\Sigma}U_P$. Hence X is irreducible, and

$$\dim X = \dim G/P + \dim \Sigma + \dim U_P = \dim Y.$$

This proves (i).

For (ii), let $X' = \{(g,xP) \mid x^{-1}gx \in P\}$. Then it is easily verified that X is a closed subvariety of X'. Since the first projection $X' \to G$ is proper, the map $\pi: X \to G$ is also proper and so $\pi(X)$ is closed in G. Since $Y \subset \pi(X)$, we have $\bar{Y} \subset \pi(X)$. Now $\dim \pi(X) \leq \dim X = \dim Y$ by (i). Hence $\bar{Y} = \pi(X)$ as both sets are irreducible of the same dimension. This proves (ii).

2.5. **Corollary.** Assume $\Sigma \cap L_{uni} \neq \phi$. Let $\bar{Y}_{uni} = \bar{Y} \cap G_{uni}$ and $X_{uni} = \pi^{-1}(\bar{Y}_{uni})$. Then $\bar{Y}_{uni}$ and X_{uni} are (non-empty and) irreducible.

Proof. Since $\Sigma \cap L_{uni} \neq \phi$, $\Sigma \cap L_{uni} = C_1$: a unipotent class in L. Then $\bar{\Sigma} \cap L_{uni} = \bar{C}_1$ (the closure of C_1 in L) and

$$\begin{aligned} X_{uni} &= \{(g, xP) \in G_{uni} \times G/P \mid x^{-1}gx \in \bar{\Sigma}U_P\} \\ &= \{(g, xP) \in G \times G/P \mid x^{-1}gx \in \bar{C}_1U_P\}, \end{aligned}$$

which is irreducible by the similar proof as in Lemma 2.4 (i). Thus $\bar{Y}_{uni} = \pi(X_{uni})$ is also irreducible.

II SPRINGER CORRESPONDENCE

From now on, throughout the paper, we assume $k = \bar{\mathbb{F}}_q$, the algebraic closure of a finite field $\mathbb{F}_q$ of characteristic p.

§3. Generalities on perverse sheaves

3.1. In this section, we review briefly the fundamental results on perverse sheaves which will be used later. For the general reference on the theory of perverse sheaves, see Beilinson, Bernstein and Deligne [3].

Let X be an algebraic variety over k. We fix a prime number ℓ ($\ell \neq p$) and let $\bar{\mathbb{Q}}_\ell$ be the algebraic closure of ℓ-adic number field. A $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on X is called a ($\bar{\mathbb{Q}}_\ell$-) local system if $\mathcal{F}$ is locally constant and each stalk $\mathcal{F}_x$ at $x \in X$ is a finite dimensional $\bar{\mathbb{Q}}_\ell$-vector space. A sheaf $\mathcal{F}$ on X is said to be constructible if there exists a filtration $X_0 \subset X_1 \subset \dots \subset X_n = X$ by closed subsets such that the restriction $\mathcal{F}|(X_i - X_{i-1})$ of $\mathcal{F}$ on $X_i - X_{i-1}$ is a local system for each i.

Let $\mathcal{D}(X) = \mathcal{D}(X, \bar{\mathbb{Q}}_\ell)$ be the derived category of complexes of $\bar{\mathbb{Q}}_\ell$-

sheaves on X. For a complex K in $\mathcal{D}(X)$, $\mathcal{H}^iK$ will denote the i-th cohomology sheaf of K. We denote by $\mathcal{D}_c^b(X) = \mathcal{D}_c^b(X,\bar{\mathbb{Q}}_\ell)$ the full subcategory of $\mathcal{D}(X)$ consisting of bounded complexes K such that $\mathcal{H}^iK$ are constructible for any i. For $K \in \mathcal{D}_c^b(X)$, $K[i]$ means the i-th shift, so that $\mathcal{H}^i(K[j]) = \mathcal{H}^{i+j}K$. In the following, we regard a constructible sheaf $\mathcal{E}$ on X as a complex in $\mathcal{D}_c^b(X)$ concentrated to the degree 0.

3.2. Let $D : \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(X)$ be the Verdier dual operator. $K \in \mathcal{D}_c^b(X)$ is called a <u>perverse sheaf</u> if

(3.2.1)
$$\begin{aligned} &\text{(i)} \quad \dim \operatorname{supp} \mathcal{H}^iK \leqq -i\ , \\ &\text{(ii)} \quad \dim \operatorname{supp} \mathcal{H}^iDK \leqq -i \qquad \text{for any } i. \end{aligned}$$

Let $\mathcal{M}X$ be the full subcategory of $\mathcal{D}_c^b(X)$ consisting of perverse sheaves. It is known that $\mathcal{M}X$ is an abelian category [3, 2.14, 1.3.6]. It is artinian and noetherian, hence all objects have finite length [3, 4.3.1].

3.3. Let $Y \subset X$ be a locally closed smooth irreducible subvariety of dimension d of X and $\mathcal{L}$ be a local system on Y. Let $\bar{Y}$ be the algebraic closure of Y in X. Following Goresky-MacPherson [13] and Deligne, one can associate to $\mathcal{L}$, an intersection cohomology complex $IC(\bar{Y},\mathcal{L})$, which is an object in $\mathcal{D}_c^b(\bar{Y})$. $K = IC(\bar{Y},\mathcal{L})[d]$ is characterized by the following properties.

(3.3.1)
$$\begin{aligned} &\text{(i)} \quad \mathcal{H}^iK = 0 \quad \text{if } i < -d\ , \\ &\text{(ii)} \quad \mathcal{H}^{-d}K|Y = \mathcal{L}\ , \\ &\text{(iii)} \quad \dim \operatorname{supp} \mathcal{H}^iK < -i \quad \text{if } i > -d\ , \\ &\text{(iV)} \quad \dim \operatorname{supp} \mathcal{H}^iDK < -i \quad \text{if } i > -d\ . \end{aligned}$$

Thus K is a perverse sheaf. If $\mathcal{L}$ is an irreducible local system on Y, K is a simple object in $\mathcal{M}\bar{Y}$ and the direct image ι_*K is a simple object in $\mathcal{M}X$, where $\iota : \bar{Y} \to X$ is an inclusion. Moreover by [3,4.3.1], all the simple objects in $\mathcal{M}X$ are obtained in this way from some pair $(Y,\mathcal{L})$ as above. Note that ι_*K is a complex obtained from K by extending by 0 outside of $\bar{Y}$. So, we regard, by abbreviation, an object in $\mathcal{M}\bar{Y}$ as an object in $\mathcal{M}X$ by applying ι_*.

3.4. Let $K = IC(\bar{Y},\mathcal{L})[\dim Y]$ be a simple object in $\mathcal{M}X$. K is called a perverse sheaf with finite monodromy if there exists a finite étale covering $\pi: \tilde{Y} \to Y$ such that $\pi^*\mathcal{L}$ is a constant sheaf. For example if Y is smooth, the constant sheaf $\bar{\mathbb{Q}}_\ell$ is a simple perverse sheaf with finite monodromy thanks to 3.3. A complex in $\mathcal{D}^b_c(X)$ is said to be semisimple if it is a direct sum of $B_i[n_i]$, where B_i is a simple perverse sheaf and n_i is a shift. We have the following deep result due to Beilinson, Bernstein, Deligne and Gabber.

Decomposition theorem ([3, 6.2.5])
Let $f: X \to Y$ be a proper morphism and let K be a simple perverse sheaf with finite monodromy. Then the direct image $f_*K \in \mathcal{D}^b_c(Y)$ is semisimple. In particular, if f_*K is a perverse sheaf, it is a semisimple object in $\mathcal{M}Y$.

3.5. If $f: X \to Y$ is a smooth morphism with connected fibre of dimension d, then for each $K \in \mathcal{M}Y$, $f^*K[d]$ is a perverse sheaf on X. Let H be a connected group acting on a variety X. $K \in \mathcal{M}X$ is said to be H-equivariant if there exists an isomorphism between two perverse sheaves $m^*K[\dim H]$ and $p^*K[\dim H]$ on $H \times X$, where $m: H \times X \to X$ is the action of H on X and $p: H \times X \to X$ is the second projection. The notion of H-equivariance of perverse sheaves is compatible with the direct image and the inverse image functors in an appropriate sense. Moreover,

(3.5.1) ([26, (1.9.1)]) If $K \in \mathcal{M}X$ is H-equivariant, then any subquotient $K' \in \mathcal{M}X$ of K is also H-equivariant.

The similar definition of H-equivariance can be applied to the case of local systems on X, i.e., a local system $\mathcal{L}$ on X is said to be H-equivariant if $m^*\mathcal{L} \simeq p^*\mathcal{L}$. Assume H acts transitively on X and let H_x be the isotropy subgroup of H at x in X. Let $\mathcal{L}$ be an H-equivariant local system on X. Then the component group $A(x) = H_x/H_x^0$ acts naturally on the stalk $\mathcal{L}_x$ of $\mathcal{L}$, which gives rise to a $\bar{\mathbb{Q}}_\ell$-representaton of $A(x)$. Conversely, if ρ is an irreducible representation of $A(x)$, we obtain an irreducible H-equivariant local system $\mathcal{L}_\rho$ by decomposing $A(x)$-equivariant local system $\pi_*\bar{\mathbb{Q}}_\ell$, i.e., $\mathcal{L}_\rho = \mathrm{Hom}_{A(x)}(\rho, \pi_*\bar{\mathbb{Q}}_\ell)$. Here $\pi: H/H_x^0 \to H/H_x \simeq X$ is a finite étale covering with group $A(x)$. $A(x)$ acts on the stalk $(\mathcal{L}_\rho)_x$ at x as the dual representation $\rho^\vee$ of ρ. The above procedure defines a 1-1 correspondence between H-equivariant local systems on X and finite dimensional representations of $A(x)$ on $\bar{\mathbb{Q}}_\ell$.

§4. Construction of Springer representations

4.1. Returning to the situation in Chapter I with char(k) = p > 0, we consider a reductive group G and the variety $\mathcal{B}_g$ for $g \in G$. Following Lusztig [23], we shall construct Weyl group representations on ℓ-adic cohomology groups $H^i(\mathcal{B}_g, \bar{\mathbb{Q}}_\ell)$. Our starting point is the Grothendieck-Springer map $\pi: \tilde{G} \to G$, where

$$\tilde{G} = \{(g, xB) \in G \times G/B \mid x^{-1}gx \in B\}$$

and π is the projection onto the first factor. The map π enjoys several nice properties as follows:

- the second projection $\tilde{G} \to G/B$ has a locally trivial fibration each of whose fibre is isomorphic to B, hence $\tilde{G}$ is smooth.
- the map π is proper since $\tilde{G}$ is closed in $G \times G/B$.
- $\pi^{-1}(G_{uni})$ is a vector bundle over G/B with fibre isomorphic to U, via the second projection. Hence $\pi^{-1}(G_{uni})$ is smooth.

In fact the restriction of π to $\pi^{-1}(G_{uni})$ is the resolution of singularity of G_{uni} and is the so-called "Springer resolution" of G_{uni}.

Let G_{reg} be the set of regular semisimple elements of G. (Note $g \in G$ is said to be regular semisimple if $Z_G^0(g)$ is a maximal torus). G_{reg} is open dense in G and in fact coincides with $Y_{(L,\Sigma)}$ with $(L,\Sigma) = (T,T)$ in 2.1. Hence $\pi: X \to G$ in 2.3 coincides with our map $\pi: \tilde{G} \to G$. Put

$$\tilde{G}_{reg} = \{(g, xT) \in G \times G/T \mid x^{-1}gx \in T_{reg}\} ,$$

where $T_{reg} = T \cap G_{reg}$. It follows from Lemma 2.4 (iii) (and is verified easily), that

$$\tilde{G}_{reg} \xrightarrow{\sim} \pi^{-1}(G_{reg}) \tag{4.1.1}$$

via $(g, xT) \to (g, xB)$.

Let $\pi_0: \pi^{-1}(G_{reg}) \to G_{reg}$ be the restriction of π to $\pi^{-1}(G_{reg})$. W acts on $\tilde{G}_{reg}$ from the right by $w: (g, xT) \to (g, xwT)$ and so acts on $\pi^{-1}(G_{reg})$. Then π_0 is W-equivariant with respect to the trivial W-action on G_{reg}, and in fact, $\pi_0: \pi^{-1}(G_{reg}) \to G_{reg}$ turns out to be an unramified covering with Galois group W. Thus the direct image sheaf $\mathcal{L} = (\pi_0)_*\bar{\mathbb{Q}}_\ell$ $(= R^0(\pi_0)_*\bar{\mathbb{Q}}_\ell)$ of the constant sheaf $\bar{\mathbb{Q}}_\ell$ on $\pi^{-1}(G_{reg})$ is a local system on G_{reg} with W-action,

(W acts from the left as sheaf automorphisms on $\mathcal{L}$).

We now consider the intersection cohomology complex $IC(G,\mathcal{L})$ on $\tilde{G}_{reg} = G$ associated with $\mathcal{L}$. Since IC is a fully faithful functor from the category of local systems on G_{reg} to $\mathcal{M}G$, up to shift, it induces an action of W on $IC(G,\mathcal{L})[\dim G] \in \mathcal{M}G$. On the other hand, one can consider the direct image $\pi_*\bar{\mathbb{Q}}_\ell \in \mathcal{D}^b_c(G)$ of the constant sheaf $\bar{\mathbb{Q}}_\ell$ on $\tilde{G}$ under the map π. The following result is crucial for our construction of Weyl group representations.

4.2. Proposition (Lusztig [23]). Let $n = \dim G$. Then

$$IC(G,\mathcal{L})[n] \simeq \pi_*\bar{\mathbb{Q}}_\ell[n] \quad \text{in } \mathcal{M}G.$$

Proof. Put $K = \pi_*\bar{\mathbb{Q}}_\ell[n]$. It is enough to verify the statement corresponding to (3.3.1) for K. Note that $\mathcal{H}^i(\pi_*\bar{\mathbb{Q}}_\ell) \simeq R^i\pi_*\bar{\mathbb{Q}}_\ell$. Thus (i) follows from the fact that the stalk $(R^i\pi_*\bar{\mathbb{Q}}_\ell)_x$ at $x \in G$ is isomorphic to $H^i(\pi^{-1}(x),\bar{\mathbb{Q}}_\ell)$. (ii) is immediate from the proper base change theorem for $\pi\colon \tilde{G} \to G$ and for $G_{reg} \hookrightarrow G$. We have to show (iii) and (iV). Since $\tilde{G}$ is smooth of dimension n, we have $D(\bar{\mathbb{Q}}_\ell[n]) \simeq \bar{\mathbb{Q}}_\ell[n]$. This implies that $DK \simeq K$ since the Verdier dual operator commutes with the direct image for a proper map. Hence it is enough to show (iii). Since $\pi^{-1}(x) \simeq \mathcal{B}_x$ for each $x \in G$, the condition $(R^i\pi_*\bar{\mathbb{Q}}_\ell)_x \neq 0$ implies that $H^i(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell) \neq 0$ and so $i \leqq 2\dim \mathcal{B}_x$. Thus $\operatorname{supp} \mathcal{H}^i(\pi_*\bar{\mathbb{Q}}_\ell)_x$ is contained in a set $\{x \in G \mid i \leqq 2\dim \mathcal{B}_x\}$. Now it follows frim Corollary 1.7 (i) that $\dim \operatorname{supp} \mathcal{H}^i(\pi_*\bar{\mathbb{Q}}_\ell) < n - i$ if $i > 0$. This proves (iii).

4.3. From the argument in 4.1, we know already that W acts on the complex $IC(G,\mathcal{L})$. Hence W acts on $\pi_*\bar{\mathbb{Q}}_\ell$ by Proposition 4.2 and so on each stalk $\mathcal{H}^i_x(\pi_*\bar{\mathbb{Q}}_\ell)$ at $x \in G$ of its cohomology sheaf. Thus we have an action of W on $H^i(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell) \simeq \mathcal{H}^i_x(\pi_*\bar{\mathbb{Q}}_\ell)$. It is called the "Springer representations" of W. As mentioned in Introduction, this kind of representations were first constructed by Springer [39] by a different method under some restriction on p, rather for the varieties $\mathcal{B}_A$ associated to a nilpotent element A in the Lie algebra $\underline{g}$ of G. Note the above construction works for every $p \geqq 0$ and also makes sense for the Lie algebra case (see §5).

4.4 **Remark.** The arguments used to prove Proposition 4.2 is summarized as follows. Let $f\colon X \to Y$ be a proper map of irreducible varieties of the same dimension, say n. We assume that X is smooth. Following Goresky-MacPherson [13], we say that f is <u>small</u> if $\dim \{y \in Y \mid 2 \dim f^{-1}(y) \geq i\} < n-i$ for $i > 0$, and <u>semismall</u> if the last inequality is replaced by $\leq n-i$. Let $\mathcal{L}$ be a local system on X. Since X is smooth, we have $D(\mathcal{L}[n]) \simeq \mathcal{L}^{\vee}[n]$, where $\mathcal{L}^{\vee}$ is the dual local system of $\mathcal{L}$ on X. Now the similar argument as in the proof of Proposition 4.2 implies that $f_*\mathcal{L}[n]$ is a perverse sheaf on Y if f is semismall, and $f_*\mathcal{L}[n]$ satisfies the property (i), (iii) and (iV) in (3.3.1) if f is small.

§5. The action of W on $H^*(\mathcal{B},\bar{\mathbb{Q}}_\ell)$

5.1. We have an action of W on $H^i(\mathcal{B},\bar{\mathbb{Q}}_\ell)$ as a special case $g = 1$ of Springer representations on $H^i(\mathcal{B}_g,\bar{\mathbb{Q}}_\ell)$, which we call the Springer action of W. On the other hand, we have another action of W on $H^i(\mathcal{B},\bar{\mathbb{Q}}_\ell)$ defined as follows. Consider the natural map $G/T \to G/B$. Then G/T is a vector bundle over G/B with fibre isomorphic to U, and so $H^i(G/T,\bar{\mathbb{Q}}_\ell)$ is isomorphic to $H^i(G/B,\bar{\mathbb{Q}}_\ell)$. Hence the right action of W on G/T given by $w\colon gT \to gwT$ gives rise to a left action of W on $H^i(\mathcal{B},\bar{\mathbb{Q}}_\ell)$ via the above isomorphism. We call this the classical action of W. In fact, much is known about the classical action of W. In particular, we have the following fact ([39]). Let V be a finite dimensional $\bar{\mathbb{Q}}_\ell$-space on which W acts as the reflection representation, and let $S(V)$ be the symmetric algebra on V. W acts naturally on $S(V)$. Let I be the ideal of $S(V)$ generated by non-constant homogeneous W-invariant elements of $S(V)$. Then $S(V)/I \simeq \bar{\mathbb{Q}}_\ell[W]$ as W-modules. It turns out that $H^i(\mathcal{B},\bar{\mathbb{Q}}_\ell) = 0$ if i is odd and that $H^*(\mathcal{B},\bar{\mathbb{Q}}_\ell)$ is isomorphic to $S(V)/I$ as graded modules.

5.2. The purpose of this section is to prove the coincidence of these two Weyl group representations on $H^*(\mathcal{B},\bar{\mathbb{Q}}_\ell)$. In order to do this, we shall pass to the Lie algebra situation for a while and consider an analogous construction of Springer representations. Let $\underline{g} = \mathrm{Lie}\, G$ and $\underline{g}_{\mathrm{reg}}$ be the set of regular semisimple elements of $\underline{g}$, i.e., the set of $X \in \underline{g}$ such that $Z_G^0(X)$ is a maximal torus. We assume p large enough so that $\underline{g}_{\mathrm{reg}}$ is open dense in $\underline{g}$. Let

$$\tilde{\underline{g}} = \{(X,gB) \in \underline{g} \times G/B \mid \mathrm{Ad}(g^{-1})X \in \mathrm{Lie}\, B\}$$

and $\pi'\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$ be the first projection. For each $X \in \mathfrak{g}$, $\pi'^{-1}(X)$ is isomorphic to $\mathcal{B}_X = \{B_1 \in \mathcal{B} \mid X \in \operatorname{Lie} B_1\}$. The argument employed in §4 can be applied without change to this case. In particular, W acts on the direct image $\pi'_*\bar{\mathbb{Q}}_\ell$ on $\mathfrak{g}$, and we have an induced W-action on a stalk $\mathcal{H}^i_X(\pi'_*\bar{\mathbb{Q}}_\ell) \simeq H^i(\mathcal{B}_X,\bar{\mathbb{Q}}_\ell)$, which is an analogue of Springer representations of W to the Lie algebra case.

5.3. Assume, for a while, that $G = G_1 \times G_2 \times \dots \times G_k$, where G_i is either GL_n or a simply connected almost simple group not of type A, and also assume that p is good for each G_i. Then, by Bardsley-Richardson [2], there exists a logarithm map $\log\colon G \to \mathfrak{g}$ satisfying the following properties; log is G-equivariant with respect to the adjoint action of G, $\log(1) = 0$, the differential $d(\log)_1\colon \mathfrak{g} \to \mathfrak{g}$ is the identity map, the restriction of log on G_{uni} gives rise to an isomorphism $G_{uni} \xrightarrow{\sim} \mathfrak{g}_{nil}$, where $\mathfrak{g}_{nil}$ is the set of nilpotent elements in $\mathfrak{g}$. Moreover, it is easily checked from the construction of log, that $\dim Z_G(x) = \dim Z_G(\log x)$ and that $\log x \in \mathfrak{g}$ is semisimple if $x \in G$ is semisimple.

Let $\pi\colon \tilde{G} \to G$ and $\pi'\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$ be as before, and consider the complex $\pi_*\bar{\mathbb{Q}}_\ell \in \mathcal{D}^b_c(G)$ and $\pi'_*\bar{\mathbb{Q}}_\ell \in \mathcal{D}^b_c(\mathfrak{g})$. Applying the proper base change theorem to $\tilde{G} \simeq \tilde{\mathfrak{g}} \times_{\mathfrak{g}} G$ (fibre product), we see that $\log^*(\pi'_*\bar{\mathbb{Q}}_\ell) \simeq \pi_*\bar{\mathbb{Q}}_\ell$. Moreover, it follows from the properties of log, that $\log^{-1}(\mathfrak{g}_{reg}) = G_{reg}$. This implies that the two actions of W on $\pi_*\bar{\mathbb{Q}}_\ell$, one is obtained from G_{reg}, and the other is obtained from $\mathfrak{g}_{reg}$ via $\log^*$, coincide each other. In particular, we see that the Springer representations of W on $H^i(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell)$ coincides with those on $H^i(\mathcal{B}_X,\bar{\mathbb{Q}}_\ell)$, where $X = \log x$. This conclusion is still true for a reductive group G in general in an appropriate sense, since the variety $\mathcal{B}_x$ depends only on x mod Z(G) and the Springer representation is independent of an isogeny of groups.

Returning to the previous setting, we consider a reductive group G in general. We shall prove the following fact, which was noticed by Borho-MacPherson in the case $k = \mathbb{C}$, and was proved by Spaltenstein for k of arbitrary characteristic.

5.4. **Proposition** (Borho-MacPherson [6], Spaltenstein [38]).
The Springer action of W on $H^i(\mathcal{B},\bar{\mathbb{Q}}_\ell)$ coincides with the classical action.

Proof. We shall prove this only for the case p is large enough. See the remark below for the general case. In view of the discussion in 5.3, we may consider the Lie algebra case. For a subvariety X of $\underline{g}$, we have a canonical isomorphism $H^i(X,\pi'_*\bar{\mathbb{Q}}_\ell) \simeq H^i(\pi'^{-1}(X),\bar{\mathbb{Q}}_\ell)$. The action of W on $\pi'_*\bar{\mathbb{Q}}_\ell$ induces the action of W on $H^i(X,\pi'_*\bar{\mathbb{Q}}_\ell)$, and so on $H^i(\pi'^{-1}(X))$ (we omit the constant sheaf $\bar{\mathbb{Q}}_\ell$ to simplify the notation). If $X \subset Y$ are subvarieties of $\underline{g}$, the natural map $H^i(\pi'^{-1}(Y)) \to H^i(\pi'^{-1}(X))$ is W-equivariant with respect to these W-actions.

Now assume, for simplicity, that there exists a strongly regular semisimple element ξ in $\underline{t}$ = Lie T (i.e., $Z_G(\xi) = T$). This is certainly the case if p is large enough. Let $\mathcal{O}$ be the G-orbit of ξ in $\underline{g}$ and consider the second projection $p\colon \pi'^{-1}(\mathcal{O}) \to \mathcal{B}$. Then

(5.4.1) The induced map $p^*\colon H^i(\mathcal{B}) \to H^i(\pi'^{-1}(\mathcal{O}))$ is W-equivariant with respect to the Springer action of W.

In fact, this is shown as follows. Note that p is factorized as $p = \bar{p}\circ\iota$, where $\iota\colon \pi'^{-1}(\mathcal{O}) \hookrightarrow \tilde{\underline{g}}$, and $\bar{p}\colon \tilde{\underline{g}} \to \mathcal{B}$ is the second projection. Since ι^* is W-equivariant, we have only to show that $\bar{p}^*$ is W-equivariant. First we note that, if we write $\bar{\iota}\colon \mathcal{B} \xrightarrow{\sim} \pi'^{-1}(0) \hookrightarrow \tilde{\underline{g}}$, then $\bar{p}\circ\bar{\iota} = \mathrm{id}_{\mathcal{B}}$. On the other hand, since $\tilde{\underline{g}}$ is a vector bundle over $\mathcal{B}$ with fibre isomorphic to Lie B, $\bar{p}^*\colon H^i(\mathcal{B}) \to H^i(\tilde{\underline{g}})$ is an isomorphism. Hence $\bar{p}^* = (\bar{\iota}^*)^{-1}$ is W-equivariant as asserted.

Let $\tilde{\underline{g}}_{reg}$ be the variety similar to $\tilde{G}_{reg}$, and let $\tilde{\mathcal{O}}$ be the subvariety of $\tilde{\underline{g}}_{reg}$ corresponding to $\pi'^{-1}(\mathcal{O})$ under the isomorphism $\pi'^{-1}(\underline{g}_{reg}) \simeq \tilde{\underline{g}}_{reg}$ (analogue of (4.1.1)), note that $\mathcal{O} \subset \underline{g}_{reg}$). Hence

$$\tilde{\mathcal{O}} = \{(\mathrm{Ad}(g)\xi,\ gwT) \in \underline{g} \times G/T \mid g \in G,\ w \in W\} \quad .$$

W acts on $\tilde{\mathcal{O}}$ by $w\colon (\mathrm{Ad}(g)\xi,\ gvT) \to (\mathrm{Ad}(g)\xi,\ gvwT)$, which is the restriction of the action of W on $\tilde{\underline{g}}_{reg}$ considered before. Thus the induced action of W on $H^i(\pi'^{-1}(\mathcal{O})) \simeq H^i(\tilde{\mathcal{O}})$ is nothing but the Springer action of W on it. Let $\tilde{p}\colon \tilde{\mathcal{O}} \to G/T$ be the second projection. Then $\tilde{p}$ is W-equivariant with respect to the natural action of W on G/T considered before, and we have the following commutative diagram,

$$\begin{array}{ccc} \tilde{\mathcal{O}} & \longrightarrow & \pi'^{-1}(\mathcal{O}) \\ {\scriptstyle\tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ G/T & \longrightarrow & G/B \simeq \mathcal{B} \end{array}$$

where $G/T \to G/B$ is the natural map. Since the classical action of W comes from the W-action on G/T, we see that $p^*: H^i(\mathcal{B}) \to H^i(\pi'^{-1}(\mathcal{O}))$ is W-equivariant with respect to the classical action of W on $H^i(\mathcal{B})$ also. Hence, to complete the proof, it is enough to show that p^* is injective, or equivalently $\tilde{p}^*: H^i(G/T) \to H^i(\tilde{\mathcal{O}})$ is injective. The latter is shown as follows. For each $w \in W$, let

$$\tilde{\mathcal{O}}_w = \{(\mathrm{Ad}(g)\xi,\ gwT) \in \underline{g} \times G/T \mid g \in G\} \quad .$$

Then $\tilde{\mathcal{O}} = \coprod_{w\in W} \tilde{\mathcal{O}}_w$ (disjoint union), and $\tilde{\mathcal{O}}_w$ are irreducible components of $\tilde{\mathcal{O}}$. Moreover, the second projection $\tilde{p}_w: \tilde{\mathcal{O}}_w \to G/T$ gives rise to an isomorphism between two varieties. Hence, $H^i(\tilde{\mathcal{O}}) \simeq \bigoplus_{w\in W} H^i(\tilde{\mathcal{O}}_w)$ and $\tilde{p}_w^* = \iota_w^* \circ \tilde{p}^*$, where $\iota_w: \tilde{\mathcal{O}}_w \hookrightarrow \tilde{\mathcal{O}}$. This implies that $\tilde{p}^*$ is injective, and we get the proposition.

5.5. **Corollary.** $H^*(\mathcal{B},\bar{\mathbb{Q}}_\ell) \simeq \bar{\mathbb{Q}}_\ell[W]$ as W-modules.

5.6. **Remark.** The argument given in 5.4 works as well to G itself except one step. In the case of groups, the projection $p: \tilde{G} \to \mathcal{B}$ is no longer a vector bundle. So we can not know whether $H^i(\mathcal{B})$ is isomorphic to $H^i(\tilde{G})$ or not, and the proof of the counterpart of (5.4.1) for the group case breaks here. However, Spaltenstein [38] proved (5.4.1) directly for the group case, i.e., he showed that $p^*: H^i(\mathcal{B}) \to H^i(\pi^{-1}(C))$ is W-equivariant for arbitrary characteristic, where C is the conjugacy class containing a strongly regular element t in T. Thus the proposition holds without restriction on p.

§6. Borho-MacPherson's theorem

6.1. In this section, we want to prove the Borho-MacPherson's theorem, which describes the restriction of $\pi_*\bar{\mathbb{Q}}_\ell$ to G_{uni} and also gives the Springer correspondence mentioned in Introduction. Before stating the theorem, we prepare some notations. Let $\mathcal{N}_G$ be the set of all pairs $(C,\mathcal{E})$, where C is a unipotent class in G and $\mathcal{E}$ is a G-equivariant irreducible local system on C. In view of 3.5, $\mathcal{N}_G$ can be regarded as the set of pairs (u,ρ), up to G-conjugate, where u is a unipotent element in G and $\rho \in A_G(u)^\wedge$.

For a given pair $(C,\mathcal{E}) \in \mathcal{N}_G$, the intersection cohomology complex $IC(\bar{C},\mathcal{E})[\dim C]$, extended by 0 on $G_{uni} - \bar{C}$, ($\bar{C}$ is the closure of C in G), is a G-equivariant simple perverse sheaf on G_{uni}, and all the G-equivariant simple perverse sheaves on G_{uni} are obtained in this way from some $(C,\mathcal{E}) \in \mathcal{N}_G$. We can now state

6.2. **Theorem** (Borho-MacPherson [6]).

(i) $\pi_*\bar{\mathbb{Q}}_\ell[2\nu_G]|G_{uni}$ <u>is a semisimple object in</u> $\mathcal{M}G_{uni}$ <u>and is decomposed as</u>

$$\pi_*\bar{\mathbb{Q}}_\ell[2\nu_G]|G_{uni} \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes IC(\bar{C},\mathcal{E})[\dim C] \tag{6.2.1}$$

<u>where</u> $(C,\mathcal{E})$ <u>runs over all the pairs in</u> $\mathcal{N}_G$, <u>and</u> $V_{C,\mathcal{E}}$ <u>is a</u> $\bar{\mathbb{Q}}_\ell$<u>-vector space whose dimension is equal to the multiplicity of the simple object correponding to</u> $(C,\mathcal{E})$.

(ii) <u>The action of</u> W <u>on</u> $\pi_*\bar{\mathbb{Q}}_\ell|G_{uni}$ <u>leads to the action of</u> W <u>on each</u> $V_{C,\mathcal{E}}$ <u>in the right hand side of</u> (6.2.1). <u>Then any</u> W<u>-module</u> $V_{C,\mathcal{E}}$ <u>is irreducible</u>, <u>and the map</u> $(C,\mathcal{E}) \to V_{C,\mathcal{E}}$ <u>gives rise to a bijection</u>,

$$\{(C,\mathcal{E}) \in \mathcal{N}_G \mid V_{C,\mathcal{E}} \neq 0\} \xrightarrow{\sim} W^\wedge \quad \text{(\underline{the Springer correspondence})}.$$

<u>Proof</u>. Let π_1 be the restriction of $\pi: \tilde{G} \to G$ to $\pi^{-1}(G_{uni})$. Then by the proper base change theorem, $\pi_*\bar{\mathbb{Q}}_\ell|G_{uni} \simeq (\pi_1)_*\bar{\mathbb{Q}}_\ell$. Now, $\pi^{-1}(G_{uni})$ is smooth, π_1 is proper and $\dim G_{uni} = 2\nu_G$. Hence it follows from Corollary 1.7 (ii) and Remark 4.4, that $(\pi_1)_*\bar{\mathbb{Q}}_\ell[2\nu_G]$ is a perverse sheaf on G_{uni}, which is G-equivariant since π_1 is so. Put $K_1 = (\pi_1)_*\bar{\mathbb{Q}}_\ell[2\nu_G]$. Then the decomposition theorem in 3.4 asserts that K_1 is a semisimple object in $\mathcal{M}G_{uni}$. Since K_1 is G-equivariant, each direct summand is also G-equivariant (cf. (3.5.1)). Thus, we have a decomposition

$$K_1 = (\pi_1)_*\bar{\mathbb{Q}}_\ell[2\nu_G] \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes IC(\bar{C},\mathcal{E})[\dim C] \quad ,$$

which proves (i).

Since the action of W is given as automorphisms on K_1, it necessarily induces the action of W on the multiplicity spaces, $V_{C,\mathcal{E}}$. On the other hand, the action of W on K_1 induces a map

$$\alpha: \bar{\mathbb{Q}}_\ell[W] \longrightarrow \operatorname{End} K_1 \quad .$$

In order to prove (ii), it is enough to show that α is an isomorphism. First we show that α is injective. Note that each stalk $\mathcal{H}^i_1(K_1)$ of K_1 at $1 \in G$ is isomorphic to $H^{i+2\nu}(\mathcal{B},\bar{\mathbb{Q}}_\ell)$, where $\nu = \nu_G$. Thus we have a natural homomorphism $\operatorname{End} K_1 \to \operatorname{End} H^*(\mathcal{B})$. Combining this map with α, we have

$$\beta: \bar{\mathbb{Q}}_\ell[W] \longrightarrow \operatorname{End} H^*(\mathcal{B}) \quad .$$

But β is nothing but the action of W on $H^*(\mathcal{B})$. Since $H^*(\mathcal{B})$ is a regular W-module by Corollary 5.5, we see that β is injective. Thus α is injective.

Next we show that α is surjective. We already know that α is injective. So, it is enough to show that $\dim \bar{\mathbb{Q}}_\ell[W] \geqq \dim \operatorname{End} K_1$, which , by (6.2.1), is equivalent to

$$\sum_{(C,\mathcal{E})} (\dim V_{C,\mathcal{E}})^2 \leqq |W| \quad . \tag{6.2.2}$$

We shall prove (6.2.2). Take $x \in G_{\text{uni}}$, and let C_x be the class in G containing x. Put $d_x = \dim \mathcal{B}_x$. Since $H^{2d_x}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell) \simeq \mathcal{H}_x^{2d_x}(K_1[-2\nu])$, we have, by (6.2.1),

$$H^{2d_x}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell) \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes \mathcal{H}_x^{2d_x - 2\nu + \dim C} IC(\bar{C},\mathcal{E}) \quad . \tag{6.2.3}$$

Since K_1 is G-equivariant, $Z_G(x)$ acts on each stalk $\mathcal{H}^i_x(K_1)$ at x, which induces an action of $A_G(x)$ on it. This action coincides with the corresponding action of $A_G(x)$ on $H^{i+2\nu}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell)$ induced from the natural action of $Z_G(x)$ on $\mathcal{B}_x$. Let $(C_x,\mathcal{E}_\rho) \in \mathcal{N}_G$ be the pair corresponding to (x,ρ) with $\rho \in A_G(x)^\wedge$. We consider the direct summand in (6.2.1) correponding to $(C_x,\mathcal{E}_\rho)$. Since $2d_x - 2\nu + \dim C_x = 0$ by Theorem 1.2 (i), this direct summand is equal to $\mathcal{H}^0_x(IC(\bar{C}_x,\mathcal{E}_\rho)) = (\mathcal{E}_\rho)_x$, the stalk of $\mathcal{E}_\rho$ at x, which is an irreducible $A_G(x)$-module isomorphic to $\rho^\vee$, the dual of ρ. This implies that the multiplicity of $\rho^\vee$ in $A_G(x)$-module $H^{2d_x}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell)$ is at least $\dim V_{x,\rho}$, where $V_{x,\rho} = V_{C_x,\mathcal{E}_\rho}$. So, if we put $m_{(x,\rho)}$ the multiplicity of $\rho^\vee$ in $H^{2d_x}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell)$, we have

$$\dim V_{x,\rho} \leqq m_{(x,\rho)} \quad .$$

On the other hand,

$$\begin{aligned}\sum_{\rho\varepsilon A_G(x)^\wedge} m^2_{(x,\rho)} &= \dim \mathrm{End}_{A_G(x)}(H^{2d_x}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell)) \\ &= \dim (\mathrm{End}_{\bar{\mathbb{Q}}_\ell} H^{2d_x}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell))^{A_G(x)} \quad .\end{aligned}$$

Now the base of $H^{2d_x}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell)$ is parametrized by the set of irreducible components $I(\mathcal{B}_x)$ of $\mathcal{B}_x$, with the action of $A_G(x)$. Hence the last quantity coincides with the number of $A_G(x)$-orbits on $I(\mathcal{B}_x) \times I(\mathcal{B}_x)$ under the diagonal action. Thus,

$$(6.2.4) \qquad \begin{aligned}\sum_{(x,\rho)} (\dim V_{x,\rho})^2 &\leqq \sum_{(x,\rho)} m^2_{(x,\rho)} \\ &= \sum_{x\varepsilon G_{uni}/\sim} |(I(\mathcal{B}_x) \times I(\mathcal{B}_x))/A_G(x)| \\ &= |W| \quad .\end{aligned}$$

The last equality follows from Corollary 1.6. This proves (6.2.2) and so the theorem.

The theorem implies that the inequality in (6.2.4) is indeed an equality. Hence, $\dim V_{x,\rho} = m_{(x,\rho)}$, and we have the following corollary, which is the original form of the Springer correspondence given in Springer [39].

6.3. **Corollary** (Springer [39]). $H^{2d_x}(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell) \simeq \bigoplus_\rho V_{x,\rho} \otimes \rho$ <u>as</u> $W \times A_G(x)$-<u>modules</u>.

6.4. **Remarks.** (i) It follows from the theorem that the map

$$\beta\cdot\alpha^{-1}: \mathrm{End}\, K_1 \longrightarrow \mathrm{End}\, H^*(\mathcal{B},\bar{\mathbb{Q}}_\ell)$$

obtained by restrcting to the stalk of K_1 at $1 \varepsilon G$ is injective. This implies the uniqueness of Springer representations in the following sense. If we have a W-action on $(\pi_1)_*\bar{\mathbb{Q}}_\ell$ which induces the classical W-action on $H^*(\mathcal{B})$, then this W-action must coincide with the Springer action. Using this property, various Weyl group representations on $H^*(\mathcal{B}_x,\bar{\mathbb{Q}}_\ell)$ constructed by several authors, the original one defined by Springer [39], [40], the one defined by Slodowy [35], and the one by Kazhdan and Lusztig [22], etc., can be identified. (See, Hotta [14],[15]).

(ii) The equality in Theorem 1.2 (i) itself is not so essential compared with the inequality (1.2.1) for the proof of Theorem 6.2. In fact, if we assume only (1.2.1), the left hand side of Corollary 1.6 is replaced by the disjoint union of unipotent classes for which the equality in Theorem 1.2 (i) holds. Thus, even in this case, Borho-MacPherson's theorem holds in a modified form, i.e., it gives a bijective correspondence between $W^\wedge$ and the set of pairs (u,ρ) in $\mathcal{N}_G$ such that u satisfies the equality in Theorem 1.2 (i) and that $V_{x,\rho} \neq 0$.

6.5. The Springer correspondence discussed above may be seen from somewhat different point of view as follows. Since $\pi_0 : \pi^{-1}(G_{reg}) \to G_{reg}$ is an unramified covering with Galois group W, $\mathcal{L} = (\pi_0)_*\bar{\mathbb{Q}}_\ell$ is semisimple and is decomposed as

$$\mathcal{L} \simeq \bigoplus_{E \in W^\wedge} E \otimes \mathcal{L}_E \ ,$$

where $\mathcal{L}_E = \mathrm{Hom}_W(E,\mathcal{L})$ is an irreducible local system corresponding to $E \in W^\wedge$ and W acts only on the multiplicity space E. Thus, from the functoriality of IC, we have

$$IC(G,\mathcal{L})[n] \simeq \bigoplus_{E \in W^\wedge} E \otimes IC(G,\mathcal{L}_E)[n] \quad \text{in } \mathcal{M}G,$$

where $n = \dim G$. The W-action on $IC(G,\mathcal{L})[n]$ is realized by the W-action on the multiplicity spaces E, and Borho-MacPherson's theorem asserts that the restriction of $IC(G,\mathcal{L}_E)$ to G_{uni} is still a perverse sheaf up to shift and

$$IC(G,\mathcal{L}_E)[2\nu_G]|G_{uni} \simeq IC(\bar{C},\mathcal{E})[\dim C]$$

for a unique pair $(C,\mathcal{E}) \in \mathcal{N}_G$. The class of G-equivariant simple perverse sheaves such as $IC(G,\mathcal{L}_E)[n]$ are examples of admissible complexes on G, which is the main objective in the next chapter. In fact, it is expected that there will be a close relationship between admissible complexes and irreducible characters of finite reductive groups $G(\mathbb{F}_q)$ (see Chapter IV).

6.6. Corollary 1.6 has the following cohomological interpretation concerned with Springer representations of W. Consider Z as in 1.1 and let $\psi : Z \to G$ be the first projection. Then $Z \simeq \tilde{G} \times_G \tilde{G}$ (fibre product) and $\psi_!\bar{\mathbb{Q}}_\ell \simeq \pi_*\bar{\mathbb{Q}}_\ell \boxtimes \pi_*\bar{\mathbb{Q}}_\ell$ in $\mathcal{D}^b_c(G)$ for the constant sheaf $\bar{\mathbb{Q}}_\ell$ on Z. Since $\pi_*\bar{\mathbb{Q}}_\ell$ is a W-equivariant complex, $\psi_!\bar{\mathbb{Q}}_\ell$ is $W \times W$-equivariant. Thus $H_c^i(Z,\bar{\mathbb{Q}}_\ell) \simeq \mathbb{H}_c^i(G,\psi_!\bar{\mathbb{Q}}_\ell)$ turns out to be a $W \times W$-module. This procedure also works for $\psi^{-1}(X) \to X$ for any locally closed subset X in G, and one can get a $W \times W$-module $H_c^i(\psi^{-1}(X),\bar{\mathbb{Q}}_\ell)$, whose action commutes with a natural map $H_c^i(\psi^{-1}(X)) \to H_c^i(\psi^{-1}(Y))$ for a closed immersion $Y \hookrightarrow X$. Now let us consider the set $Z' = \psi^{-1}(G_{uni})$. By Theorem 1.2 (iii), Z' is of pure dimension with $\dim Z' = 2\nu_G$, and the number of irreducible components is equal to $|W|$. Thus two-sided W-module $H_c^{4\nu}(Z',\bar{\mathbb{Q}}_\ell)$ has dimension equal to $|W|$. In fact, this two-sided W-representation coincides with the two-sided regular representation of W. This is shown as follows. Let Z'_C be the inverse image of a unipotent class C in G_{uni} under ψ. It follows from the proof of Theorem 1.2 that $\dim Z'_C = 2\nu_G$ for each class C. Thus, using the cohomology long exact sequence, we have

$$H_c^{4\nu}(Z', \bar{\mathbb{Q}}_\ell) \simeq \bigoplus_{C \subset G_{uni}} H_c^{4\nu}(Z'_C, \bar{\mathbb{Q}}_\ell)$$

as $W \times W$-modules. On the other hand, from the proof of Corollary 1.6, Z'_C is of pure dimension and its irreducible components are described by $A_G(u)$-orbits of $I(\mathcal{B}_u) \times I(\mathcal{B}_u)$ for $u \in C$. Thus, we have a natural isomorphism

$$H_c^{4\nu}(Z'_C, \bar{\mathbb{Q}}_\ell) \simeq (H^{2d_u}(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell) \otimes H^{2d_u}(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell))^{A_G(u)} ,$$

which turns out to be $W \times W$-equivariant. Hence if we write

$$H^{2d_u}(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell) \simeq \bigoplus_\rho V_{u,\rho} \otimes \rho ,$$

as in Corollary 6.3, we have

$$H_c^{4\nu}(Z'_C, \bar{\mathbb{Q}}_\ell) \simeq \bigoplus_{(u,\rho)} V_{u,\rho} \otimes V_{u,\rho}$$

as $W \times W$-modules. (Note $W \times A_G(u)$-module $H^{2d_u}(\mathcal{B}_u,\bar{\mathbb{Q}}_\ell)$ is self dual). This shows that $H_c^{4\nu}(Z',\bar{\mathbb{Q}}_\ell)$ is a two-sided regular representation of W. Conversely, if we have a two-sided W-action on each piece $H_c^i(Z'_C,\bar{\mathbb{Q}}_\ell)$ compatible with cohomology long exact sequences and if $H_c^{4\nu}(Z',\bar{\mathbb{Q}}_\ell)$ is a two-sided regular W-module, the above argument proves the "completeness theorem", Corollary 6.3. In fact, this was first pointed out by Springer and was carried out by Kazhdan and Lusztig [22], where they constructed a Weyl group representation from a different point of view, and proved the completeness result.

6.7. The Springer correspondence was determined explicitly by several authors. Assume G is almost simple. Note that the Springer correspondence is independent of the isogeny of groups. Under the condition that p is large enough, Hotta and Springer [17] determined the Springer correspondence in the case where G is of type A. The case where G is of type B, C or D, and the case of type F_4 were treated by the author [31],[32]. The cases E_6, E_7 and E_8 were treated by Alvis and Lusztig [1] and Spaltenstein [37]. The case G_2 is already included in Springer's original paper [39]. In these cases, p is always assumed to be large enough. The case where p is small was treated by Lusztig and Spaltenstein [29] and Spaltenstein [38] in connection with the generalized Springer correspondence (see the next chapter).

Here we give some examples, assuming p is good. If $G = GL_n(k)$, the Springer correspondence is nothing but the one given in the beginning of Introduction as follows. For a partition $\lambda = (\lambda_i)$ of n, we denote by χ_λ the corresponding irreducible character of W, where we normalize it so that χ_λ is a unit character for $\lambda = (n)$, the single partition. Let u_λ be a unipotent element of $GL_n(k)$ whose Jordan's normal form is of type λ. Then $A_G(u) = \{1\}$ for any u, and $u_\lambda \leftrightarrow \chi_\lambda$ gives the Springer correspondence. The case of other classical groups is more complicated (see. §12). Let $\delta = |\mathcal{N}_G| - |W^\wedge|$. Then from the Springer correspondence, $\delta \geqq 0$. We assume that G is of adjoint type. Then $\delta = 0$ if G is of type A, E_6 or E_7. For other classical groups, δ becomes large as the rank of G becomes large. If G is of type G_2, F_4 or E_8, then $\delta = 1$, i.e., there exists a unique pair $(u,\rho) \in \mathcal{N}_G$ which does not correspond to any irreducible characters of W. These elements are described as follows. For G of type G_2 (resp. F_4 or E_8) there exists a unique unipotent element u (up to conjugacy) such that $A_G(u) \simeq S_3$ (resp. S_4 or S_5), where S_i is the symmetric group of degree i. Then (u,ε) is the desired pair, where ε is the sign character of S_i $(i = 3, 4, 5)$. The significance of these missinig pairs will be discussed in the next chapter.

III GENERALIZED SPRINGER CORRESPONDENCE

§7. Cuspidal pairs

7.1. Let $\mathcal{N}_G$ be as in 6.1. As we have seen in 6.7, the natural injection $W^\wedge \to \mathcal{N}_G$ via the Springer correspondence is not necessarily surjective. In order to extend this to the whole of $\mathcal{N}_G$, Lusztig [25] introduced a certain class of G-equivariant perverse sheaves, i.e., admissible complexes on G. Before the construction of admissible complexes, we need the notion of cuspidal pairs.

7.2. **Definition.** A pair $(C,\mathcal{E}) \in \mathcal{N}_G$ is called a cuspidal pair if for any parabolic subgroup $P \neq G$, and for any unipotent class $C' \subset P/U_P$,

(7.2.1) $$H_c^{\dim C - \dim C'}(C \cap \pi_P^{-1}(v),\mathcal{E}) = 0 \quad ,$$

where $\pi_P\colon P \to P/U_P$ is a natural projection and v is an element in C'. $\mathcal{E}$ is called a cuspidal local system if $(C,\mathcal{E})$ is cupidal.

Note, by Theorem 1.4 (ii), $\dim (C \cap \pi_P^{-1}(v)) \leq (\dim C - \dim C')/2$. Hence $H_c^i(C \cap \pi_P^{-1}(v),\mathcal{E}) = 0$ for $i > \dim C - \dim C'$. It is seen if $\mathcal{E}$ is a cuspidal local system on C, then the dual local system $\mathcal{E}^\vee$ of $\mathcal{E}$ is also cuspidal on C.

7.3. **Remark.** Let $(C,\mathcal{E}) \in \mathcal{N}_G$. We choose $u \in C$ and assume that $\mathcal{E}$ corresponds to $\rho \in A_G(u)^\wedge$. Let L be a Levi subgroup of a parabolic subgroup P of G, and let C' be a unipotent class in L. Take $v \in C'$ and set

$$Y_{u,v} = \{gZ_L^0(v)U_P \mid g \in G,\ g^{-1}ug \in vU_P\} \ .$$

$Z_G(u)$ acts on $Y_{u,v}$ by the left multiplication, and so $A_G(u)$ acts on the set $I(Y_{u,v})$ of irreducible components of $Y_{u,v}$ of dimension $s = (\dim Z_G(u) - \dim Z_L(v))/2$. (In fact, $\dim Y_{u,v} \leq s$, see below). Then (7.2.1), for C', L and P in our setting, is equivalent to

(7.3.1) ρ does not appear in the permutation representaion of $A_G(u)$ on the set $I(Y_{u,v})$.

In fact, this is shown as follows. Let

$$\hat{Y}_{u,v} = \{g \in G \mid g^{-1}ug \in vU_P\} \ .$$

Then $Z_G(u) \times Z_L(v)U_P$ acts on $\tilde{Y}_{u,v}$ by $(x,y): g \to xgy^{-1}$, $(x \in Z_G(u)$, $y \in Z_L(v)U_P$, $g \in \tilde{Y}_{u,v})$ and $Y_{u,v}$ is isomorphic to the quotient of $\tilde{Y}_{u,v}$ by $Z_L^0(v)U_P$. Let $\eta: \tilde{C} = G/Z_G^0(u) \to G/Z_G(u) \simeq C$ be an unramified covering of C with Galois group $A_G(u)$. Then $\eta^{-1}(C \cap vU_P)$ may be identified with $\tilde{Y}_{u,v}/Z_G^0(u)$ and we have the following commutative diagram.

$$\begin{tikzcd}
Y_{u,v} & \tilde{Y}_{u,v} \arrow[l, "h"'] \arrow[r, "f"] \arrow[d, "\omega"'] & \tilde{Y}_{u,v}/Z_G^0(u) \arrow[d, "\wr"] \\
& C \cap vU_P & \eta^{-1}(C \cap vU_P) \arrow[l, "\eta"']
\end{tikzcd}$$

where h and f are canonical maps to the quotients, and $\omega: \tilde{Y}_{u,v} \to C \cap vU_P$ is defined by $g \to g^{-1}ug$. Put $\delta = (\dim C - \dim C')/2$. Since we know $\dim (C \cap vU_P) \leqq \delta$, it follows from the diagram that $\dim Y_{u,v} \leqq s$. Let $I(\tilde{Y}_{u,v})$ be the set of irreducible components of $\tilde{Y}_{u,v}$ of dimension $\delta + \dim Z_G(u)$. Then f induces an $A_G(u)$-equivariant bijection between $I(\tilde{Y}_{u,v})$ and the set of irreducible components, say $I_{u,v}$, of $\eta^{-1}(C \cap vU_P)$ of dimenesion δ. Also h induces an $A_G(u)$-equivariant bijection between $I(\tilde{Y}_{u,v})$ and $I(Y_{u,v})$. Hence, we see that $I(Y_{u,v})$ is in 1-1 correspondence with $I_{u,v}$ with $A_G(u)$-action.

On the other hand, since $\mathcal{E} \simeq \mathrm{Hom}_{A_G(u)}(\rho, \eta_*\bar{\mathbb{Q}}_\ell)$, we have

$$\begin{aligned}
H_c^\delta(C \cap vU_P, \mathcal{E}) &\simeq (H_c^\delta(C \cap vU_P, \eta_*\bar{\mathbb{Q}}_\ell) \otimes \rho^\vee)^{A_G(u)} \\
&\simeq (H_c^\delta(\eta^{-1}(C \cap vU_P), \bar{\mathbb{Q}}_\ell) \otimes \rho^\vee)^{A_G(u)} ,
\end{aligned}$$

where $\rho^\vee$ is the dual representation of ρ. Thus (7.2.1) for C', L, P is equivalent to the statement that ρ does not appear in the permutation representation of $A_G(u)$ on $I_{u,v}$, which is also equivalent to (7.3.1) from the above argument.

§8. Admissible complexes

8.1. We are now ready to construct admissible complexes. Let $(C_1, \mathcal{E}_1)$ be a cuspidal pair in a Levi subgroup L of a parabolic subgroup P of G. Let $\Sigma = Z^0(L)C_1$ and let $1 \boxtimes \mathcal{E}_1$ be the irreducible local system on Σ, which we denote also by $\mathcal{E}_1$ by abuse of notation. We associate to (L,Σ) an open subset Σ_{reg}, and a smooth irreducible variety $Y = Y_{(L,\Sigma)} = \bigcup_{g\in G} g\Sigma_{reg}g^{-1}$ as in 2.1. We consider the diagram,

$$\Sigma \xleftarrow{\ \alpha\ } \hat{Y} \xrightarrow{\ \beta\ } \tilde{Y} \xrightarrow{\ \pi_0\ } Y \quad ,$$

where

$$\tilde{Y} = \{(g,xL) \in G \times G/L \mid x^{-1}gx \in \Sigma_{reg}\} \, ,$$
$$\hat{Y} = \{(g,x) \in G \times G \mid x^{-1}gx \in \Sigma_{reg}\}$$

and $\alpha\colon (g,x) \mapsto x^{-1}gx$, $\beta\colon (g,x) \mapsto (g,xL)$, $\pi_0\colon (g,xL) \mapsto g$.

Let $\alpha^*\mathcal{E}_1$ be the pull back of $\mathcal{E}_1$ on Σ to $\hat{Y}$. Since Σ_{reg} is open dense in Σ and $\hat{Y} \simeq \Sigma_{reg} \times G$, we see that $\alpha^*\mathcal{E}_1$ is irreducible. Moreover, $\alpha^*\mathcal{E}_1$ is L-equivariant with respect to the action of L on $\hat{Y}$ by $\ell\colon (g,x) \to (g,x\ell^{-1})$. Since $\tilde{Y}$ is the quotient of Y by the free action of L, there exists a unique irreducible local system $\tilde{\mathcal{E}}_1$ on $\tilde{Y}$ such that $\beta^*\tilde{\mathcal{E}}_1 = \alpha^*\mathcal{E}_1$, (in fact, $\tilde{\mathcal{E}}_1 = R^0\beta_*\alpha^*\mathcal{E}_1$). We consider the direct image $(\pi_0)_*\tilde{\mathcal{E}}_1$ on Y. Since the map π_0 is an unramified covering with Galois group $\mathcal{W}_\Sigma$, the stabilizer of Σ in $N_G(L)/L$, $(\pi_0)_*\tilde{\mathcal{E}}_1$ is a semisimple local system on Y, which is G-equivariant. We now consider the complex $K = IC(\bar{Y},(\pi_0)_*\tilde{\mathcal{E}}_1)[\dim Y]$ on G, extended by 0 on $G - \bar{Y}$. By the functoriality of IC, K is a G-equivariant semisimple perverse sheaf on G. Hence by (3.5.1), each direct summand is G-equivariant.

8.2. **Definition.** A G-equivariant simple perverse sheaf on G is called a (unipotent) admissible complex if it is a direct summand of $K = IC(\bar{Y},(\pi_0)_*\tilde{\mathcal{E}}_1)[\dim Y]$ for some cuspidal pair $(C_1,\mathcal{E}_1)$ in various Levi subgroups L.

Note, if $(C_1,\mathcal{E}_1)$ is a cuspidal pair for G, then $IC(\bar{\Sigma},\mathcal{E}_1)[\dim \Sigma]$ itself is an admissible complex, where $\Sigma = Z^0(G)C_1$ and $\mathcal{E}_1 = 1 \boxtimes \mathcal{E}_1$ as above.

8.3. **Remark.** Cuspidal pairs and admissible complexes are defined in a more general setting in [25], and the ones defined above just cover a part of them. In fact, let $\Sigma \subset G$ be the inverse image of a single conjugacy class in $G/Z^0(G)$, and let $\mathcal{E}$ be a G-equivariant local system on Σ. Then the pair $(\Sigma,\mathcal{E})$ is said to be cuspidal if $\mathcal{E}$ is $Z^0(G)$-equivariant with respect to the action of $Z^0(G)$ on Σ by $z\colon x \to z^n x$ for some integer $n \geq 1$, and if the condition

(7.2.1) holds replacing C by Σ and $\dim C - \dim C'$ by $\dim \Sigma/Z^0(G) - \dim C'$. It is shown that if $(\Sigma,\mathcal{E})$ is a cuspidal pair, then Σ modulo $Z^0(G)$ is an isolated class in $G/Z^0(G)$. In particular, there are only finitely many cuspidal pairs on G if G is semisimple. Admissible complexes in a general sense are the ones defined in a similar way as in 8.1 from these cuspidal pairs $(\Sigma,\mathcal{E})$. Thus, in some sense, cuspidal pairs and admissible complexes defined in 8.2 correspond to the unipotent classes among all the conjugacy classes. In this chapter, we are only concerned with admissible complexes in the sense of 8.2 unless otherwise stated.

8.4. We shall give an alternative construction of the complex $K = IC(\bar{Y},(\pi_0)_*\tilde{\mathcal{E}}_1)[\dim Y]$. Let P be a parabolic subgroup of G having L as a Levi subgroup. Consider the diagram

$$\bar{\Sigma} \xleftarrow{\hat{\alpha}} \hat{X} \xrightarrow{\hat{\beta}} X \xrightarrow{\pi} \bar{Y}$$

where

$$X = \{(g,xP) \in G \times G/P \mid x^{-1}gx \in \bar{\Sigma}U_P\}$$

$$\hat{X} = \{(g,x) \in G \times G \mid x^{-1}gx \in \bar{\Sigma}U_P\}$$

and $\quad \hat{\alpha}(g,x) = x^{-1}gx \bmod U_P,\ \hat{\beta}(g,x) = (g,xP),\ \pi(g,xP) = g$.

Here $\bar{\Sigma}$ is the closure of Σ in L. Note by Lemma 2.4 (ii), π is well-defined.

Now $\bar{\Sigma} = Z^0(L)\bar{C}_1$ has a stratification into finitely many smooth strata, consisting of stratum $\Sigma_\alpha = Z^0(L)C_\alpha$, where C_α is a unipotent class in L contained in $\bar{C}_1$. Hence $\hat{X}$ is stratified with each smooth stratum $\hat{X}_\alpha = \hat{\alpha}^{-1}(\Sigma_\alpha) \simeq G \times \Sigma_\alpha U_P$. The strata $\hat{X}_\alpha$ are P-stable and we have a stratification of $X = \hat{X}/P$ with smooth strata $X_\alpha = \hat{\beta}(\hat{X}_\alpha)$. We denote by $\hat{X}_0$ (resp. X_0) the open strutum in $\hat{X}$ (resp. X) corresponding to $C_\alpha = C_1$. Hence

$$(8.4.1) \qquad X_0 = \{(g,xP) \in G \times G/P \mid x^{-1}gx \in \Sigma U_P\} ,$$

and similarly for $\hat{X}_0$.

Let $\mathcal{E}_1$ be the local system on Σ as before and consider the inverse image $(\hat{\alpha}_0)^*\mathcal{E}_1$ on $\hat{X}_0$, where $\hat{\alpha}_0\colon \hat{X}_0 \to \Sigma$ is the restriction of $\hat{\alpha}$ to $\hat{X}_0$. $(\hat{\alpha}_0)^*\mathcal{E}_1$ is $G \times P$-equivariant and there exists a unique G-equivariant local system $\bar{\mathcal{E}}_1$ on X_0 such that $(\hat{\beta}_0)^*\bar{\mathcal{E}}_1 = (\hat{\alpha}_0)^*\mathcal{E}_1$, where $\hat{\beta}_0\colon \hat{X}_0 \to X_0$ is the restriction of $\hat{\beta}$. $\bar{\mathcal{E}}_1$ gives rise to an extension of $\tilde{\mathcal{E}}_1$ on $\tilde{Y}$ under the isomorphism $\tilde{Y} \simeq \pi^{-1}(Y) \subset X_0$

(cf. Lemma 2.4). Let us denote $\tilde{K} = IC(X,\tilde{\mathcal{E}}_1)[\dim X]$ and consider the direct image $\pi_!\tilde{K} \in \mathcal{D}^b_c(\bar{Y})$. The following result is a generalization of Proposition 4.2.

8.5. **Proposition.** $IC(\bar{Y},(\pi_0)_*\tilde{\mathcal{E}}_1)[\dim Y] \simeq \pi_!\tilde{K}$ <u>in</u> $\mathcal{M}G$, <u>where both sides are regarded as complexes on</u> G <u>extended by</u> 0 <u>on</u> $G - \bar{Y}$.

This is proved essentially in the same spirit as in Proposition 4.2, i.e., we have only to check that $\pi_!\tilde{K}$ satisfies (3.3.1) with $\mathcal{L} = (\pi_0)_*\tilde{\mathcal{E}}_1$. We give below a brief sketch of the proof. First note that $\tilde{K}[-\dim X]|\pi^{-1}(Y) \simeq \tilde{\mathcal{E}}_1|\pi^{-1}(Y) \simeq \tilde{\mathcal{E}}_1$. It follows that $\pi_!\tilde{K}[-\dim Y]|Y \simeq (\pi_0)_*\tilde{\mathcal{E}}_1$ since $\dim X = \dim Y$ by Lemma 2.4. Now $D(\pi_!\tilde{K}) = \pi_!D\tilde{K}$ since π is proper, and $D\tilde{K}$ is isomorphic to $IC(X,\tilde{\mathcal{E}}_1^\vee)[\dim X]$, where $\tilde{\mathcal{E}}_1^\vee$ is the local system on X attached to the dual local system $\mathcal{E}_1^\vee$ on Σ. Hence it is enough to verify (iii) of (3.3.1), i,e., the following statement.

(8.5.1) For any $i > -\dim Y$, we have $\dim \operatorname{supp} \mathcal{H}^i(\pi_!\tilde{K}) < -i$.

For $g \in \bar{Y}$, the stalk $\mathcal{H}^i_g(\pi_!\tilde{K})$ is isomorphic to the hypercohomology $\mathbb{H}^i(\pi^{-1}(g),\tilde{K})$. Then thanks to the the stratification $\pi^{-1}(g) = \bigcup_\alpha \pi^{-1}(g)_\alpha$, where $\pi^{-1}(g)_\alpha = \pi^{-1}(g) \cap X_\alpha$, we may only consider each cohomology $\mathbb{H}^i_c(\pi^{-1}(g)_\alpha,\tilde{K})$, separately. Therefore using the hypercohomology spectral sequence

$$H^i_c(\pi^{-1}(g)_\alpha,\ \mathcal{H}^j\tilde{K}) \Longrightarrow \mathbb{H}^{i+j}_c(\pi^{-1}(g)_\alpha,\ \tilde{K})$$

and also using the support condition (3.2.1) of perverse sheaf for $\tilde{K} \in \mathcal{M}X$, (8.5.1) is reduced to showing the following statemens, (here $\pi^{-1}(g)_0 = \pi^{-1}(g) \cap X_0$).

(8.5.2) For any $i > 0$,

$$\dim \{g \in \bar{Y} \mid \dim \pi^{-1}(g)_\alpha > \tfrac{i}{2} - \tfrac{1}{2}(\dim \Sigma - \dim \Sigma_\alpha)\} < \dim \bar{Y} - i\ , \qquad \text{if } C_\alpha \neq C_1\ ,$$

$$\dim \{g \in \bar{Y} \mid \dim \pi^{-1}(g)_0 \geqq \tfrac{i}{2}\} < \dim \bar{Y} - i, \qquad \text{if } C_\alpha = C_1\ .$$

(8.5.2) is verified in a similar way as in Corollary 1.7 (i) using Theorem 1.4 instead of Theorem 1.2.

8.6. **Remark.** The construction of $\pi_!\tilde{\mathcal{K}}$ given in 8.4 can be seen in a more general setting through the induction ind_P^G of complexes in the sense of Lusztig [26, I, 4]. Consider the diagram

$$L \xleftarrow{\bar{\pi}} V_1 \xrightarrow{\pi'} V_2 \xrightarrow{\pi''} G$$

where

$$V_1 = \{(g, h) \in G \times G \mid h^{-1}gh \in P\} ,$$

$$V_2 = \{(g, hP) \in G \times G/P \mid h^{-1}gh \in P\} ,$$

and $\bar{\pi}(g, h) = \pi_P(h^{-1}gh)$, $\pi'(g, h) = (g, hP)$, $\pi''(g, hP) = g$.

Let A be any L-equivariant perverse sheaf on L. Since $\bar{\pi}$ is P-equivariant with respect to the action of P on V_1, $p: (g,h) \to (g,hp^{-1})$ and on L, $p: \ell \to \pi(p)\ell\pi(p)^{-1}$, $\bar{\pi}^*A[d]$ is a P-equivariant perverse sheaf on V_1, where $d = \dim G + \dim U_P$ is the dimension of the fibre of $\bar{\pi}$. Then there exists a unique perverse sheaf $\tilde{A}$ on V_2 such that $\bar{\pi}^*A[d] = (\pi')^*\tilde{A}[\dim U_P]$ (cf. [26, I, (1.9.3)]). We define the induction $\mathrm{ind}_P^G A$ of A by $\mathrm{ind}_P^G A = (\pi'')_!\tilde{A} \in \mathcal{D}_c^b(G)$. ind_P^G is a functor from the category of L-equivariant perverse sheaves $\mathcal{M}_L(L)$ on L to $\mathcal{D}_c^b(G)$.

If we take a complex $A = IC(\bar{\Sigma},\mathcal{E}_1)[\dim \Sigma] \in \mathcal{M}_L(L)$ associated with a cuspidal pair $(C_1,\mathcal{E}_1)$ on L, then $\tilde{A}$ on V_2 is essentially the same as $\tilde{\mathcal{K}}$ in 8.4. Thus, $\pi_!\tilde{\mathcal{K}}$ is nothing but the induction $\mathrm{ind}_P^G A$ from a cuspidal complex A in L.

§9. Generalized Springer correspondence

9.1. In this section, we shall state a theorem on the generalized Springer correspondence, which is a natural generalization of Borho-MacPherson's theorem in §6.

Before going into details on the generalized Springer correspondence, we need to know about the decomposition of the induced complex $K = IC(\bar{Y},(\pi_0)_*\tilde{\mathcal{E}}_1)[\dim Y]$ into simple objects. For this, we have to consider the endomorphism algebra $\mathrm{End}\, K$ of K in $\mathcal{M}G$. Let $\mathcal{A} = \mathcal{A}_{\mathcal{E}_1} = \mathrm{End}\, K$. Put $\mathcal{L} = (\pi_0)_*\tilde{\mathcal{E}}_1$. Since K is a semisimple object in $\mathcal{M}G$, $\mathcal{A}$ is a semisimple algebra. Moreover, $\mathcal{A}$ is isomorphic to $\mathrm{End}\, \mathcal{L}$. Let

$$N(\mathcal{E}_1) = \{n \in N_G(L) \mid n\Sigma n^{-1} = \Sigma,\ \mathrm{ad}(n)^*\mathcal{E}_1 = \mathcal{E}_1\} ,$$

where $\mathrm{ad}(n)(\ell) = n\ell n^{-1}$ for $\ell \in L$. $N(\mathcal{E}_1)$ contains L and so we obtain a finite group $\mathcal{W} = \mathcal{W}_{\mathcal{E}_1} = N(\mathcal{E}_1)/L$. For each $w \in \mathcal{W}$,, let $\gamma_w\colon \tilde{Y} \to \tilde{Y}$ be an isomorphism defined by $\gamma_w(g,xL) = (g,xn^{-1}L)$, where n is a representative of w in $N(\mathcal{E}_1)$. Let $\mathcal{A}_w$ be the set of homomorphisms of local systems $f\colon \tilde{\mathcal{E}}_1 \to \gamma_w^*\tilde{\mathcal{E}}_1$ over $\tilde{Y}$. As $\tilde{\mathcal{E}}_1$ is irreducible and $\tilde{\mathcal{E}}_1 \simeq \gamma_w^*\tilde{\mathcal{E}}_1$, so $\mathcal{A}_w$ is a one-dimensional $\bar{\mathbb{Q}}_\ell$-vector space. Since $(\pi_0)_*\gamma_w^*\tilde{\mathcal{E}}_1 = (\pi_0)_*\tilde{\mathcal{E}}_1$, the map $f \to \pi_*(f)$ induces an injection $\mathcal{A}_w \hookrightarrow \mathrm{End}_G\,\mathcal{L}$, the algebra of G-equivariant endomorphisms of $\mathcal{L}$. Hence we have an imbedding

$$\bigoplus_{w\in\mathcal{W}} \mathcal{A}_w \to \mathrm{End}_G\,\mathcal{L} \hookrightarrow \mathrm{End}\,\mathcal{L}\ .$$

On the other hand, it is verified that $\dim \mathrm{End}\,\mathcal{L} \leqq |\mathcal{W}|$. It follows that $\mathcal{A} \simeq \bigoplus_{w\in\mathcal{W}} \mathcal{A}_w$ and any endomorphism of K is automatically G-equivariant. The multiplication of $\mathcal{A}$ implies that $\mathcal{A}_w.\mathcal{A}_{w'} = \mathcal{A}_{ww'}$. Thus we have

9.2. **Proposition.** $\mathcal{A} = \mathrm{End}\,K$ <u>is isomorphic to a twisted group algebra</u> $\bar{\mathbb{Q}}_\ell[\mathcal{W}]_t$ <u>and we have</u> $\mathrm{End}\,K = \mathrm{End}_G\,K$.

Now $\mathcal{L}$ is decomposed as

$$\mathcal{L} \simeq \bigoplus_{E\in\mathcal{A}^\wedge} E \otimes \mathcal{L}_E\ ,$$

where $\mathcal{A}^\wedge$ is the set of irreducible representations of $\mathcal{A}$ up to isomorphism and $\mathcal{L}_E = \mathrm{Hom}_{\mathcal{A}}(E,\mathcal{L})$. Similarly for K, we have

$$K \simeq \bigoplus_{E\in\mathcal{A}^\wedge} E \otimes K_E\ ,$$

where $K_E = \mathrm{Hom}_{\mathcal{A}}(E,K) = \mathrm{IC}(\bar{Y},\mathcal{L}_E)[\dim Y]$.

9.3. We denote by $\mathcal{S}_G$ the set of triples $(L,C_1,\mathcal{E}_1)$ up to G-conjugacy, where $(C_1,\mathcal{E}_1)$ is a unipotent cuspidal pair for L. Let $A(G)_{\mathrm{uni}}$ be the set of (isomorphism classes of) unipotent admissible complexes in G, i.e., the set of all K_E as in 9.2 for $E \in \mathcal{A}^\wedge_{\mathcal{E}_1}$ associated with various $(L,C_1,\mathcal{E}_1) \in \mathcal{S}_G$. We say that an admissible complex belongs to $(L,C_1,\mathcal{E}_1)$ if it is a direct summand of K associated with $(L,C_1,\mathcal{E}_1)$. We can now state

9.4. **Theorem** (generalized Springer correspondence [25, Th.6.5]).

(i) Let $(L,C_1,\mathcal{E}_1) \in \mathcal{S}_G$ and let $K \simeq \pi_!\tilde{\mathcal{K}}$ be an induced complex on G associated with it. Then

$$K[-r]|G_{uni} \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes IC(\bar{C},\mathcal{E})[\dim C] ,$$

where $r = \dim Z^0(L) = \dim Y - \dim (Y \cap G_{uni})$, and $V_{C,\mathcal{E}}$ is an irreducible $\mathcal{A}_{\mathcal{E}_1}$-module, and any irreducible $\mathcal{A}_{\mathcal{E}_1}$-module is realized as $V_{C,\mathcal{E}}$ for a unique pair $(C,\mathcal{E})$. In other words, for each $E \in \mathcal{A}^{\wedge}_{\mathcal{E}_1}$, there exists $(C,\mathcal{E}) \in \mathcal{N}_G$ such that $K_E[-r]|G_{uni} \simeq IC(\bar{C},\mathcal{E})[\dim C]$.

(ii) Each $A \in \mathcal{A}(G)_{uni}$ belongs to a unique $(L,C_1,\mathcal{E}_1) \in \mathcal{S}_G$. Hence $\mathcal{A}(G)_{uni} \simeq \coprod \mathcal{A}^{\wedge}_{\mathcal{E}_1}$, where the summation is taken over all $(L,C_1,\mathcal{E}_1) \in \mathcal{S}_G$. The restriction, $A \to A|G_{uni}$, of admissible complexes to the unipotent variety induces, via the correspondence in (i), a bijection $\mathcal{A}(G)_{uni} \xrightarrow{\sim} \mathcal{N}_G$.

(iii) Let $\pi\colon X \to \bar{Y}$ be associated with a triple $(L,C_1,\mathcal{E}_1)$. Let $f\colon X_0 \to \bar{Y}$ be the restriction of π, and consider the sheaf $\bar{\mathcal{E}}_1$ on X_0 as in 8.4. Then $(C,\mathcal{E}) \in \mathcal{N}_G$ corresponds to $(L,C_1,\mathcal{E}_1)$ if and only if $C \subset \bar{Y}$ and $\mathcal{E}_1$ is a direct summand of $R^{2d_C}f_!\bar{\mathcal{E}}_1|C$, where $d_C = (\nu_G - \frac{1}{2}\dim C) - (\nu_L - \frac{1}{2}\dim C_1)$. Moreover,

$$\mathcal{H}^{2d_C}(\pi_!\tilde{\mathcal{K}}[-\dim Y])|C \simeq R^{2d_C}f_!\bar{\mathcal{E}}_1|C \quad .$$

9.5. **Remark.** Concerning the structure of $\mathcal{A}_{\mathcal{E}_1}$, the following fact is known by Lusztig [25, Th.9.2]; $N_G(L)/L$ is a Coxeter group if $(L,C_1,\mathcal{E}_1) \in \mathcal{S}_G$. In this case, $\mathcal{W} = \mathcal{W}_{\mathcal{E}_1} \simeq N_G(L)/L$, and $\mathcal{A}_{\mathcal{E}_1}$ is isomorphic to the group algebra $\bar{\mathbb{Q}}_\ell[\mathcal{W}]$, i.e., there exists a natural basis $\theta_w \in \mathcal{A}_w \subset \mathcal{A}_{\mathcal{E}_1}$ $(w \in \mathcal{W})$ such that $\theta_w\theta_{w'} = \theta_{ww'}$.

9.6. Let Y be associated with $(L,C_1,\mathcal{E}_1) \in \mathcal{S}_G$, and let C be a unipotent class contained in $\bar{Y}$. We choose $u \in C$ and put

$$\mathcal{P}_u^{(C_1)} = \{xP \in G/P \mid x^{-1}ux \in C_1U_P\} .$$

Then $f^{-1}(u) \simeq \mathcal{P}_u^{(C_1)}$ and $\pi^{-1}(u) \simeq \bar{\mathcal{P}}_u^{(C_1)}$, the algebraic closure of $\mathcal{P}_u^{(C_1)}$ in $\mathcal{P} = G/P$. Hence the stalk $\mathcal{H}^i_u(\pi_!\tilde{\mathcal{K}})$ at u of i-th cohomology sheaf of $\pi_!\tilde{\mathcal{K}}$ is isomorphic to $H^i(\bar{\mathcal{P}}_u^{(C_1)},\tilde{\mathcal{K}})$, and the stalk $(R^i f_! \bar{\mathcal{E}}_1)_u$ is isomorphic to $H^i_c(\mathcal{P}_u^{(C_1)}, \bar{\mathcal{E}}_1)$. These cohomologies inherit the structure of $\mathcal{A}_{\mathcal{E}_1} \times A_G(u)$-modules. In the case where $P = B$, $H^i_c(\mathcal{P}_u^{(C_1)}, \bar{\mathcal{E}}_1)$ reduces to $H^i(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell)$. Theorem 9.4 (iii) implies that

$$\mathbb{H}^{2d_C}(\bar{\mathcal{P}}_u^{(C_1)}, \tilde{K}[-\dim Y]) \simeq H_c^{2d_C}(\mathcal{P}_u^{(C_1)}, \bar{\mathcal{E}}_1) \quad .$$

As in the case of Springer correspondence (cf. Cor. 6.3), we have

9.7. Corollary. $$H_c^{2d_C}(\mathcal{P}_u^{(C_1)}, \bar{\mathcal{E}}_1) \simeq \bigoplus_{\rho \in A_G(u)^\wedge} V_{u,\rho} \otimes \rho$$

<u>as</u> $\mathcal{A}_{\mathcal{E}_1} \times A_G(u)$<u>-modules</u>. $V_{u,\rho}$ <u>is an irreducible</u> $\mathcal{A}_{\mathcal{E}_1}$<u>-module</u>, <u>and all the irreducible</u> $\mathcal{A}_{\mathcal{E}_1}$<u>-modules are realized in this way by a unique pair</u> $(C_u, \mathcal{E}_\rho) \in \mathcal{N}_G$.

The proof of Theorem 9.4 and Corollary 9.7 will be given in §11 after some preliminaries in §10.

§10. Sheaves on T/W

10.1. In order to prove Theorem 9.4, we need to know about End K as in the case of Springer correspondence. However, the method used in the proof of Theorem 6.2 can not be applied to our case as we have no good information on the stalks of K. So, we should take an alternative approach, i.e., the one discussed in 6.6. In our case, the corresponding variety is $Z = X \times_{\bar{Y}} X$ (fibre product). We consider two complexes $K = \pi_! \tilde{K}$ and $K' = \pi_! \tilde{K}'$ on $\bar{Y}$, where K (resp. K') is associated with $(C_1, \mathcal{E}_1)$ (resp. $(C_1', \mathcal{E}_1')$) of a Levi subgroup L in P. We form the external tensor product $\tilde{K} \boxtimes \tilde{K}'$, which is a complex on Z. Let $\psi: Z \to \bar{Y}$ be the natural projection associated with π. Then $\mathbb{H}^i_c(Z, \tilde{K} \boxtimes \tilde{K}') \simeq \mathbb{H}^i_c(\bar{Y}, K \otimes K')$ by Künneth formula. Since $K \otimes K'$ has a structure of $\mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}_1'}$-module, we have an action of $\mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}_1'}$ on $\mathbb{H}^i_c(Z, \tilde{K} \boxtimes \tilde{K}')$. Now assume $(C_1', \mathcal{E}_1') = (C_1, \mathcal{E}_1^\vee)$, where $\mathcal{E}_1^\vee$ is the dual local system on C_1. Then $\tilde{K}' \simeq D\tilde{K}$. Let Z' be the inverse image of $\bar{Y} \cap G_{uni}$ under ψ. Then $\mathbb{H}^i_c(Z', \tilde{K} \boxtimes D\tilde{K})$ is, in our setting, the object corresponding to $H^i_c(Z', \bar{\mathbb{Q}}_\ell)$ in 6.6. Thus, in view of the argument in 6.6, the crucial step to the proof of the theorem is to show that $\mathbb{H}_c^{-2r}(Z',\tilde{K} \boxtimes D\tilde{K})$ is a two-sided regular $\mathcal{A}_{\mathcal{E}_1}$-module, i.e., is isomorphic to $\mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}^0_{\mathcal{E}_1}$-module $\mathcal{A}_{\mathcal{E}_1}$. Here $\mathcal{A}^0_{\mathcal{E}_1}$ is the algebra opposed to $\mathcal{A}_{\mathcal{E}_1}$. The idea of Lusztig to prove the above property is to compare the cohomology of Z' with that of $\psi^{-1}(Y)$. Note that the latter set is easily described since $\psi^{-1}(Y) \to Y$ is a principal covering. As Y is open dense in $\bar{Y}$, we would be able to get an information on Z' from that of $\psi^{-1}(Y)$ by the "analytic continuation" such as IC functor. Thus we are led to consider the following situation. Let $\sigma: G \to T/W$ be the Steinberg map which assigns to $g \in G$ the element of T/W corresponding to the conjugacy class of semisimple part of g. Let $\tilde{\sigma} = \sigma \circ \psi: Z \to T/W$ be the composite of two maps. We consider the direct image $\tilde{\sigma}_!(\tilde{K} \boxtimes D\tilde{K})$. Then the stalk of its cohomology sheaf at $a \in T/W$ represents the hypercohomology corresponding to $\tilde{\sigma}^{-1}(a) \cap Z$. Hence by investigating these cohomology sheaves, we should get the continuation from the regular part. This is what we want to do in this section.

10.2. We consider a more general situation. Let $(L,C_1,\mathcal{E}_1)$ and $(L',C_1',\mathcal{E}_1')$ be the triples in $\mathcal{S}_G$ and let P (resp. P') be a parabolic subgroup containing L (resp. L'). We shall denote the various objects associated to $(C_1',\mathcal{E}_1')$ by attaching primes to the notation of corresponding objects for $(C_1,\mathcal{E}_1)$, like $\pi': X' \to \bar{Y}'$. We consider the fibre product $Z = X \times_G X'$ over G and let $\psi = \psi'$: $Z \to \bar{Y} \cap \bar{Y}'$ be the corresponding map. (We assume here $\bar{Y} \cap \bar{Y}' \neq \phi$, otherwise Z is empty.) Let $\tilde{\sigma} = \sigma \circ \psi: Z \to T/W$ be the map as before. Consider the external tensor product $\tilde{K} \boxtimes \tilde{K}'$ on Z and put

$$\tilde{\mathcal{T}} = \mathcal{H}^{-r-r'}(\tilde{\sigma}_!(\tilde{K} \boxtimes \tilde{K}')),$$

$(-r-r')$-th cohomology sheaf of $\tilde{\sigma}_!(\tilde{K} \boxtimes \tilde{K}')$, which is a constructible sheaf on T/W, ($r = \dim Z^0(L)$ and so on for r'). We shall define a similar object $\mathcal{T}$ which is more convenient for our use. Let $Z_0 =$ $= X_0 \times_G X_0'$ be the fibre product of X_0 and X_0' on G, and let ψ_0: $Z_0 \to \bar{Y} \cap \bar{Y}'$ be the canonical map, where X_0 is as in (8.4.1) and similarly for X_0'. Thus,

$$Z = \{(g, xP, x'P') \in G \times G/P \times G/P' \mid g \in x\bar{\Sigma}U_Px^{-1} \cap x'\bar{\Sigma}'U_{P'}x'^{-1}\}$$

$$Z_0 = \{(g, xP, x'P') \in G \times G/P \times G/P' \mid g \in x\Sigma U_Px^{-1} \cap x'\Sigma'U_{P'}x'^{-1}\}$$

Note that the restriction of $(\mathcal{K} \boxtimes \mathcal{K}')[-\dim Y - \dim Y']$ to Z_0 coincides with $\mathcal{E}_1 \otimes \mathcal{E}_1'|Z_0$. Let $\tilde{\sigma}_0: Z_0 \to T/W$ be the restriction of $\tilde{\sigma}$ to Z_0. We put

$$\mathcal{T} = \mathcal{H}^{-r-r'}(\tilde{\sigma}_{0!}(\mathcal{K} \boxtimes \mathcal{K}')) = \mathcal{H}^{2d_0}(\tilde{\sigma}_{0!}(\mathcal{E}_1 \otimes \mathcal{E}_1')) ,$$

where $d_0 = 2\nu_G - \nu_L - \nu_{L'} + \frac{1}{2}(\dim C_1 + \dim C_1')$.

Z_0 is an open subset of Z, and we have a natural map of sheaves $\mathcal{T} \to \tilde{\mathcal{T}}$ on T/W, which will be shown later to be isomorphic.

Let $A_L \subset A = T/W$ be the image of $Z^0(L)$ under σ and similarly for $A_{L'}$. Then it is proved that $\sigma(\bar{Y} \cap \bar{Y}') = A_L \cap A_{L'}$. Hence the supports of $\mathcal{T}$ and $\tilde{\mathcal{T}}$ lie in $A_L \cap A_{L'}$. From now on, we regard $\mathcal{T}$ and $\tilde{\mathcal{T}}$ the sheaves on $A_L \cap A_{L'}$. The sheaf $\mathcal{T}$ is actually easier to deal with than $\tilde{\mathcal{T}}$ since its stalks are represented by usual cohomologies. However, the crucial point is that $\mathcal{T}$ turns out to be a perfect sheaf in the following sense.

10.3. **Definition.** A constructible sheaf $\mathcal{E}$ on an irreducible variety V is said to be perfect if it satisfies the following two conditions,

(i) there exists an open dense smooth subset V_0 of V such that $\mathcal{E}|V_0$ is locally constant and $\mathcal{E} = IC(V, \mathcal{E}|V_0)$,

(ii) the support of any nonzero constructible subsheaf of $\mathcal{E}$ coincides with V.

The following property of perfect sheaves is easily verified.

(10.3.1) If $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \longrightarrow 0$ is an exact sequence of constructible sheaves on V, with $\mathcal{E}_1$ and $\mathcal{E}_3$ perfect, then $\mathcal{E}_2$ is also perfect.

10.4. **Remarks.** (i) Perfect sheaves fit into our requirement for the continuation mentioned above. In fact, if two perfect sheaves are isomorphic on an open dense subset of V, then they are isomorphic on V.

(ii) An important example of perfect sheaves is given as follows. Let $\pi: V' \to V$ be a finite surjective morphism with V' smooth, and let $\mathcal{E}'$ be a local system on V'. Then $\mathcal{E} = \pi_*\mathcal{E}'$ is a perfect sheaf on V. In fact, since π is finite surjective, there exists an open dense smooth subset V_0 of V such that $\mathcal{E}|V_0$ is non-zero and locally constant. Now it follows from the fact that π is a small map (Remark 4.4), we see that $\pi_*\mathcal{E}' \simeq R^0\pi_*\mathcal{E}'$ is isomorphic to $IC(V,\mathcal{E}|V_0)$, which shows (i). (ii) is immediate since $\pi_*\mathcal{E}'$ is the direct image of a local system.

We can now stste

10.5. Theorem. (i) <u>If the pairs</u> (L,C_1), (L',C_1') <u>are not conjugate under an element in G, then</u> $\mathcal{T} = 0$.

(ii) <u>If</u> $L = L'$, $C_1 = C_1'$, <u>then</u> $\mathcal{T}$ <u>is a perfect sheaf on</u> A_L.

(iii) <u>The natural map of sheaves</u> $\mathcal{T} \to \bar{\mathcal{T}}$ <u>is an isomorphism</u>.

<u>Proof</u>. Let $p: Z_0 \to G/P \times G/P'$ be the projection onto the second and third factors. For a G-orbit $\mathcal{O}$ of $G/P \times G/P'$ (G acts by left translation on both factors), let $Z_0^{\mathcal{O}}$ be the inverse image of $\mathcal{O}$ under p, and we denote by $\mathcal{T}_{\mathcal{O}}$ the $2d_0$-th cohomology sheaf of the direct image of $\bar{\mathcal{E}}_1 \boxtimes \bar{\mathcal{E}}_1'|Z_0^{\mathcal{O}}$ under the restriction of $\tilde{\sigma}_0$ to $Z_0^{\mathcal{O}}$. It can be shown that $\mathcal{T}_{\mathcal{O}} = 0$ except for the case that there exists $(xP,x'P') \in \mathcal{O}$ such that $xLx^{-1} = x'L'x'^{-1}$ and $x\Sigma x^{-1} = x'\Sigma'x'^{-1}$. (i) is immediate from this fact. We show (ii). From (10.3.1), it is enough to show that $\mathcal{T}_{\mathcal{O}}$ is a perfect sheaf for each G-orbit $\mathcal{O}$ in $G/P \times G/P'$. Using the above property on $\mathcal{O}$, we may assume that $L = L'$ and that there exists $n \in N_G(L)$ such that $\Sigma = n\Sigma'n^{-1}$. So, in particular, $C_1 = nC_1'n^{-1}$ and $Y = Y'$. We want to show that

$$\mathcal{T}_{\mathcal{O}} \simeq R^{2c}\bar{\sigma}_!(\mathcal{E}_1 \otimes n^*\mathcal{E}_1'), \quad \text{(up to Tate twist)} \tag{10.5.1}$$

where $\bar{\sigma}: \Sigma \to A_L$ is the restriction of σ to Σ and $c = \dim C_1$, $n^*\mathcal{E}_1'$ is the inverse image of $\mathcal{E}_1'$ on Σ' under the map $g \to n^{-1}gn$: $\Sigma \to \Sigma'$.

Note that $p: Z_0^{\mathcal{O}} \to \mathcal{O}$ has a locally trivial fibration with fibre isomorphic to $\Sigma(U_P \cap nU_{P'}n^{-1})$, coming from the stratification of $\mathcal{O} \simeq G/(P \cap nP'n^{-1})$ with respect to the Bruhat decomposition. Thus, we can find an open dense subset V in $Z_0^{\mathcal{O}}$ such that $V \simeq p(V) \times \Sigma(U_P \cap nU_{P'}n^{-1})$ and that $p(V)$ is an affine space. Then we have $\dim (V \cap \tilde{\sigma}^{-1}(a)) = d_0$, $\dim ((Z_0^{\mathcal{O}} - V) \cap \tilde{\sigma}^{-1}(a)) < d_0$ for each $a \in A_L$. Thus,

$$\mathcal{T}_{\mathcal{O}} = R^{2d_0}(\tilde{\sigma}_0')_!(\bar{\mathcal{E}}_1 \boxtimes \bar{\mathcal{E}}_1'|Z_0^{\mathcal{O}}) \simeq R^{2d_0}(\tilde{\sigma}_0'')_!(\bar{\mathcal{E}}_1 \boxtimes \bar{\mathcal{E}}_1'|V) .$$

where $\tilde{\sigma}_0'$ (resp. $\tilde{\sigma}_0''$) is the restriction of $\tilde{\sigma}_0$ to Z_0^0 (resp. V). Let $j: V \to \Sigma$ be the projection onto Σ. Then it is easily checked that $\mathcal{E}_1 \boxtimes \mathcal{E}_1' | V \simeq j^*(\mathcal{E}_1 \otimes n^*\mathcal{E}_1')$. Note $j: V \to \Sigma$ is a trivial vector bundle of rank $\dim G - \dim L = d_0 - c$. It follows that $R^{2d_0-2c} j_! j^*(\mathcal{E}_1 \otimes n^*\mathcal{E}_1')$ is isomorphic to $\mathcal{E}_1 \otimes n^*\mathcal{E}_1'$ on Σ up to a Tate twist. Since $\tilde{\sigma}_0''$ is factorized as $\tilde{\sigma}_0'' = \bar{\sigma}\circ j$, we have

$$R^{2d_0}(\tilde{\sigma}_0'')_!(\mathcal{E}_1 \boxtimes \mathcal{E}_1' | V) \simeq R^{2c}\bar{\sigma}_! R^{2d_0-2c} j_! j^*(\mathcal{E}_1 \otimes n^*\mathcal{E}_1)$$
$$\simeq R^{2c}\bar{\sigma}_!(\mathcal{E}_1 \otimes n^*\mathcal{E}_1') .$$

This proves (10.5.1).

Let $h: \Sigma \simeq Z^0(L) \times C_1 \to Z^0(L)$ be the projection, and set $\mathcal{F} = R^{2c}h_!(\mathcal{E}_1 \otimes n^*\mathcal{E}_1')$. Since $\mathcal{E}_1 \otimes n^*\mathcal{E}_1'$ is the inverse image of $(\mathcal{E}_1 \otimes n^*\mathcal{E}_1')|C_1$ under the projection $\Sigma \to C_1$, $\mathcal{F}$ is a local system on $Z^0(L)$. Let $\sigma': Z^0(L) \to A_L$ be the restriction of σ to $Z^0(L)$. Then $\bar{\sigma}$ is factorized as $\bar{\sigma} = \sigma' \circ h$ and we have $R^{2c}\bar{\sigma}_!(\mathcal{E}_1 \otimes n^*\mathcal{E}_1') \simeq$ $\simeq R^0\sigma_!'\mathcal{F}$. Since σ' is a finite morphism, $\mathcal{T}_0 \simeq R^0\sigma_!'\mathcal{F}$ is a perfect sheaf on A_L by Remark 10.4 (ii). This proves (ii).

Next we show (iii). It is enough to show that $\mathcal{T}_a \to \tilde{\mathcal{T}}_a$ is an isomorphism for each $a \in A_L = \sigma(Z^0(L)) = \sigma(\Sigma)$. Let $Z^a = \tilde{\sigma}^{-1}(a)$ and $Z_0^a = Z^a \cap Z_0$. Then $\mathcal{T}_a \simeq \mathbb{H}_c^{-r-r'}(Z_0^a, \mathcal{K} \boxtimes \mathcal{K}') = H_c^{2d_0}(Z_0^a, \mathcal{E}_1 \boxtimes \mathcal{E}_1')$ and $\tilde{\mathcal{T}}_a \simeq \mathbb{H}_c^{-r-r'}(Z^a, \mathcal{K} \boxtimes \mathcal{K}')$. Thus, we have to show that the natural map induced from $\iota: Z_0^a \hookrightarrow Z^a$,

$$(10.5.3) \qquad \iota^*: \mathbb{H}_c^{-r-r'}(Z_0^a, \mathcal{K} \boxtimes \mathcal{K}') \longrightarrow \mathbb{H}_c^{-r-r'}(Z^a, \mathcal{K} \boxtimes \mathcal{K}')$$

is an isomorphism. Now, given a stratum α of $\bar{\Sigma}$ and a stratum α' of $\bar{\Sigma}'$, we set $Z_{\alpha,\alpha'}^a = Z^a \cap X_\alpha \times_G X_{\alpha'}$, where X_α are as in 8.4. Then the subsets $Z_{\alpha,\alpha'}^a$ form a partition of Z^a into locally closed pieces with Z_0^a $(= Z_{0,0}^a)$ open in Z^a. First we show that ι^* is surjective. For this, it is enough to show that for each $Z_{\alpha,\alpha'}^a \neq Z_0^a$, $\mathbb{H}_c^i(Z_{\alpha,\alpha'}^a, \mathcal{K} \boxtimes \mathcal{K}') = 0$ for $i \geq -(r+r')$. Assume that $\mathbb{H}_c^i(Z_{\alpha,\alpha'}^a, \mathcal{K} \boxtimes \mathcal{K}') \neq 0$. Then, by making use of the hypercohomology spectral sequence, we see that i can be written as $i = i_0+j+j'$, where $i_0 \leq 2 \dim Z_{\alpha,\alpha'}^a$ and $\mathcal{H}^j\mathcal{K}|X_\alpha$, $\mathcal{H}^{j'}\mathcal{K}'|X_{\alpha'}$ are not identically zero. On the other hand, it is shown by Theorem 1.4, that

$$\dim Z_{\alpha,\alpha'}^a \leq d_0 - \tfrac{1}{2}(\dim C_1 - \dim C_\alpha) - \tfrac{1}{2}(\dim C_1' - \dim C_\alpha') .$$

This inequality together with the support condition (3.2.1) for perverse sheaves $\tilde{K}$, $\tilde{K}$ implies that $i < -(r+r')$. This shows the surjectivity. Now, in order to prove that $\mathcal{T} \simeq \tilde{\mathcal{T}}$, it is enough to show that the kernel of $\mathcal{T} \to \tilde{\mathcal{T}}$ is zero. Since we know already that $\mathcal{T}$ is a perfect sheaf, it is enough to show that the support of this kernel is a proper subset of A_L, hence, enough to show that the map ι^* in (10.5.3) is an isomorphism for any $a \in \sigma(Z^0(L)_{reg}) = \sigma(\Sigma_{reg})$. Actually, a more precise argument for $Z^a_{\alpha,\alpha'}$ $(a \in \sigma(\Sigma_{reg}))$ shows that $\mathbb{H}^i_c(Z^a_{\alpha,\alpha'}, \tilde{K} \boxtimes \tilde{K}') = 0$ for $i \geq -(r+r')-1$. This establishes the isomorphism in (10.5.3), and so proves (iii).

10.6. **Corollary.** Assume $P = P'$, $L = L'$ and $\Sigma = \Sigma'$. Then $\mathcal{T} \simeq \bigoplus_{w} \mathcal{T}_{\mathcal{O}(w)}$ as sheaves on A_L, where w runs over $\mathcal{W}_\Sigma$ (cf. 8.1) and $\mathcal{O}(w)$ is the G-orbit on $\mathcal{P} \times \mathcal{P}$ which contains (P, nP), ($n \in N_G(L)$ is a representative of $w \in \mathcal{W}_\Sigma$).

Proof. Since $\mathcal{T}$ and $\bigoplus_{w} \mathcal{T}_{\mathcal{O}(w)}$ are perfect sheaves on A_L, it is enough to show that $\mathcal{T} \simeq \bigoplus_{w} \mathcal{T}_{\mathcal{O}(w)}$ on th set $(A_L)_{reg} = \sigma(\Sigma_{reg})$ which is open dense in $A_L = \sigma(\Sigma)$. Now it is verified that the inverse image $\tilde{\sigma}_0^{-1}(A_L)_{reg}$ of $(A_L)_{reg}$ is isomorphic to the fibre product $\tilde{Y} \times_Y \tilde{Y}$. Let $\tilde{\sigma}_1 : \tilde{Y} \times_Y \tilde{Y} \to (A_L)_{reg}$ be the composite of the projection $\tilde{Y} \times_Y \tilde{Y} \to Y$ and the restriction of σ to Y. Then the restriction of $\mathcal{T}$ to $(A_L)_{reg}$ gives rise to $R^{2d_0}(\tilde{\sigma}_1)_!(\tilde{\mathcal{E}}_1 \boxtimes \tilde{\mathcal{E}}_1')$. Let us consider the intersection $Z_0^{\mathcal{O}} \cap (\tilde{Y} \times_Y \tilde{Y})$ under the above isomorphism, for each G-orbit $\mathcal{O}$ in $\mathcal{P} \times \mathcal{P}$. It follows from the fact that $\tilde{Y} \to Y$ is a principal $\mathcal{W}_\Sigma$-covering, that this intersection is non-empty only if $\mathcal{O} = \mathcal{O}(w)$ for some $w \in \mathcal{W}_\Sigma$ and that, for such $\mathcal{O} = \mathcal{O}(w)$, $Z_0^{\mathcal{O}} \cap (\tilde{Y} \times_Y \tilde{Y})$ is open and closed in $\tilde{Y} \times_Y \tilde{Y}$. This implies that $\mathcal{T} \simeq \bigoplus_{w} \mathcal{T}_{\mathcal{O}(w)}$ on $(A_L)_{reg}$ and so the corollary.

10.7. We shall see more precisely about $\mathcal{T}_{\mathcal{O}(w)}$ in the above proposition. Let $1 \in A_L$ be the element corresponding to the unit element in G. The stalk of $\mathcal{T}_{\mathcal{O}(w)}$ at 1 is equal to $H_c^{2c}(C_1, \mathcal{E}_1 \otimes n^*\mathcal{E}_1')$ by (10.5.1). Since $c = \dim C_1$, this is a one dimensional $\bar{\mathbb{Q}}_\ell$-vector space if $\mathcal{E}_1^\vee \simeq n^*\mathcal{E}_1'$, and is zero otherwise. Hence in the case where $\mathcal{E}_1' = \mathcal{E}_1^\vee$, $\mathcal{T}_{\mathcal{O}(w)} \neq 0$ if and only if $w \in \mathcal{W}_{\mathcal{E}_1}$.

Let us consider $\tilde{\mathcal{T}} = R^{-r-r'}\tilde{\sigma}_!(\tilde{K} \boxtimes \tilde{K}')$. $\tilde{\mathcal{T}}$ has a natural structure of $\mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}_1'}$-modules as discussed in 10.1. The following proposition is now easily verified.

10.8. **Proposition.** Let $\mathcal{T} \simeq \tilde{\mathcal{T}}$ be the sheaf on $A_L \cap A_{L'}$ defined by the set of data, $(\Sigma, L, P, \mathcal{E}_1)$ and $(\Sigma', L', P', \mathcal{E}_1')$. Let $\mathcal{T}_1 = \tilde{\mathcal{T}}_1$ be its stalk at $1 \in A_L$.

(i) If $(\Sigma, L, \mathcal{E}_1)$ and $(\Sigma', L', \mathcal{E}_1'^{\vee})$ are not conjugate by an element of G, then $\tilde{\mathcal{T}}_1 = 0$.

(ii) Assume that $\Sigma = \Sigma'$, $L = L'$, $P = P'$ and $\mathcal{E}_1' = \mathcal{E}_1^{\vee}$. Then $\mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}_1^{\vee}}$-module $\tilde{\mathcal{T}}_1$ is isomorphic to the $\mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}^0_{\mathcal{E}_1}$-module $\mathcal{A}_{\mathcal{E}_1}$ (two sided regular $\mathcal{A}_{\mathcal{E}_1}$-module), where $\mathcal{A}^0_{\mathcal{E}_1}$ is the algebra opposed to $\mathcal{A}_{\mathcal{E}_1}$.

§11. The proof of Theorem 9.4.

11.1. In this section, we shall give a proof of Theorem 9.4. Let us take a triple $(L, C_1, \mathcal{E}_1) \in \mathcal{S}_G$ and let P be a parabolic subgroup containing L. We consider the perverse sheaf $K = \pi_! \tilde{K}$ on G associated with $(L, C_1, \mathcal{E}_1)$, and its restriction $\pi_! \tilde{K} | G_{uni}$ to G_{uni}. First we note that

(11.1.1) $\pi_! \tilde{K}[-r] | G_{uni}$ is a semisimple perverse sheaf on G_{uni}, where $r = \dim Z^0(L)$.

In fact, this is proved as in Proposition 8.5 (see also the proof of Theorem 6.2 (i)), by making use of the decomposition theorem (3.4) if we note that $\pi_1 : X_{uni} \to G_{uni}$ is proper and that $\tilde{K}[-r] | X_{uni} = IC(X_{uni}, \bar{\mathcal{E}}_1 | X^0_{uni})[\dim X_{uni}]$, where $X_{uni} = \pi^{-1}(G_{uni})$, $X^0_{uni} = X_0 \cap X_{uni}$ and π_1 is the restriction of π to X_{uni}.

Now $\pi_! \tilde{K}[-r] | G_{uni}$ is a G-equivariant perverse sheaf on G_{uni} and so is a direct sum of complexes of the form $K(\mathcal{E}) = IC(\bar{C}, \mathcal{E})[\dim C]$ for some $(C, \mathcal{E}) \in \mathcal{N}_G$. We need the following lemma.

11.2. **Lemma.** Let $(C, \mathcal{E})$, $(C', \mathcal{E}') \in \mathcal{N}_G$ be such that $C, C' \subset \bar{Y}_{uni}$ $(= \bar{Y} \cap G_{uni})$. Then we have

$$\dim \mathbb{H}^0_c(\bar{Y}_{uni}, K(\mathcal{E}) \otimes K(\mathcal{E}')) = \begin{cases} 1 & \text{if } C = C' \text{ and } \mathcal{E}^{\vee} \simeq \mathcal{E}' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let C'' be a unipotent class contained in $\bar{C} \cap \bar{C}'$. Then by applying the support condition for perverse sheaves $K(\mathcal{E})$, $K(\mathcal{E}')$ to the hypercohomology spectral sequence, one can show that $\mathbb{H}^0_c(C'', K(\mathcal{E}) \otimes K(\mathcal{E}')) = 0$ if $C'' \neq C$ or $C'' \neq C'$. This implies that $\mathbb{H}^0_c(\bar{Y}_{uni}, K(\mathcal{E}) \otimes K(\mathcal{E}')) = 0$ if $C \neq C'$ and that

$$\mathbb{H}^0_c(\bar{Y}_{uni}, K(\mathcal{E}) \otimes K(\mathcal{E}')) \simeq \mathbb{H}^0_c(C, K(\mathcal{E}) \otimes K(\mathcal{E}')) \quad ,$$

if $C = C'$. The latter space is isomorphic to $H_c^{2\dim C}(C, \mathcal{E} \otimes \mathcal{E}')$, which is one dimensional if $\mathcal{E}^\vee \simeq \mathcal{E}'$ and is zero otherwise. This proves the lemma.

11.3. We now prove (i) of Theorem 9.4. Let $\mathcal{A}_{\mathcal{E}_1} = \mathrm{End}\, K$ be the endomorphism algebra as in 9.1 and set $K_1 = K[-r]|G_{uni}$. We have only to show that the natural map

$$\alpha\colon \mathrm{End}\, K \longrightarrow \mathrm{End}\, K_1$$

is an isomorphism. Let us denote by $n_{\mathcal{E}}$ the multiplicity with which $K(\mathcal{E})$ appears as a direct summand of K_1. Clearly $n_{\mathcal{E}}$ is equal to the multiplicity with which $K(\mathcal{E}^\vee) \simeq D(K(\mathcal{E}))$ appears as a direct summand of DK_1.

Let $Z_{uni} = \psi^{-1}(\bar{Y}_{uni})$ and let $\psi_1\colon Z_{uni} \to \bar{Y}_{uni}$ be the restriction of ψ. Since Z_{uni} is isomorphic to the fibre product $X_{uni} \times_{\bar{Y}_{uni}} X_{uni}$, we have $(\psi_1)_!(\tilde{K} \boxtimes \tilde{K}'|Z_{uni}) \simeq (\pi_!\tilde{K}|Y_{uni}) \otimes (\pi_!\tilde{K}'|Y_{uni})$ for complexes $\tilde{K}$, $\tilde{K}'$ on X. Let $\tilde{\mathcal{T}}$ be the sheaf on T/W as in 10.2 associated with $\tilde{K}$ and $D\tilde{K}$. Then

$$\begin{aligned}\tilde{\mathcal{T}}_1 &\simeq \mathbb{H}_c^{-2r}(Z_{uni}, \tilde{K} \boxtimes D\tilde{K}) \\ &\simeq \mathbb{H}_c^{-2r}(\bar{Y}_{uni}, (K|Y_{uni}) \otimes (K'|Y_{uni})) \ .\end{aligned}$$

So, by Lemma 11.2, we have

$$\dim \tilde{\mathcal{T}}_1 = \dim H^0_c(\bar{Y}_{uni}, K_1 \otimes DK_1) = \sum_{(C,\mathcal{E})} n_{\mathcal{E}}^2 \quad ,$$

where the summation is taken over all $(C, \mathcal{E}) \in \mathcal{N}_G$. Now the left hand side equals to $\dim \mathcal{A}_{\mathcal{E}_1}$ by Proposition 10.8. The right hand side equals to $\dim \mathrm{End}\, K_1$. Thus $\dim \mathrm{End}\, K_1 = \dim \mathrm{End}\, K$. Therefore it is enough to show that α is injective. By composing $\alpha\colon \mathrm{End}\, K \to \mathrm{End}\, K_1$ with the natural map $\mathrm{End}\, K_1 \to \mathrm{End}\, \mathcal{T}_1$, we have an algebra homomorphism

$$\beta\colon \mathrm{End}\ K \longrightarrow \mathrm{End}\ \mathcal{T}_1 .$$

The map β is nothing but the action of $\mathcal{A}_{\mathcal{E}_1}$ on $\mathcal{T}_1$. On the other hand, it follows from Proposition 10.8 that $\mathcal{A}_{\mathcal{E}_1}$-module $\mathcal{T}_1$ is a regular $\mathcal{A}_{\mathcal{E}_1}$-module. Hence β is injective, and so α is injective. This proves that α is an isomorphism, and so (i) follows.

11.4. In order to prove the completeness property ((ii) of Theorem 9.4), we need some detour. First we construct a partition of $\mathcal{N}_G$ parametrized by the set $\mathcal{S}_G$ and then, show that this partition indeed coincides with the one asserted in the theorem.

Assume given a unipotent class C_1 in a Levi subgroup L and a parabolic subgroup P containing L. Let $\mathcal{P}$ be the variety of all parabolic subgroups conjugate to P. For each $P' \in \mathcal{P}$, we denote by $C_{\bar{P}'}$ the unipotent class in $\bar{P}' = P'/U_{P'}$ which corresponds to C_1 under the isomorphism $\bar{P}' \simeq L$. $\pi_{P'}\colon P' \to \bar{P}'$ will denote the natural projection. Let C be a unipotent class in G. We consider the following diagram.

$$\begin{array}{ccc} \tilde{D} = \{(g, P') \in C \times \mathcal{P} \mid \pi_{P'}(g) \in C_{\bar{P}'}\} & \xrightarrow{f_1} & C \\ \Big\downarrow{\scriptstyle f_2} & & \\ D = \{(\bar{g}, P') \mid P' \in \mathcal{P},\ \bar{g} \in C_{\bar{P}'}\} & & \end{array} ,$$

where $f_1(g,P') = g$, $f_2(g,P') = (\pi_{P'}(g),P')$.

G acts naturally on these varieties and f_1, f_2 are G-equivariant. The action of G on C and D are transitive. Let $d_1 = (\nu_G - \frac{1}{2}\dim C) - (\nu_L - \frac{1}{2}\dim C_1)$ and $d_2 = \frac{1}{2}(\dim C - \dim C_1)$. Then all fibres of f_1 have dimension $\leq d_1$ and all fibres of f_2 have dimension $\leq d_2$, and if the equality holds on one hand for some (or all) fibre, then the equality on the other hand also holds. We show the following lemma.

11.5. **Lemma.** Let $\mathcal{E}$ (resp. $\mathcal{E}'$) be a G-equivariant local system on C (resp. D). Then the multiplicity of $\mathcal{E}$ in the G-equivariant local system $R^{2d_1}(f_1)_! f_2^{*}\mathcal{E}'$ on C is the same as the multiplicity of $\mathcal{E}'$ in the G-equivariant local system $R^{2d_2}(f_2)_! f_1^{*}\mathcal{E}'$ on D.

<u>Proof</u>. First we note that $R^{2d_1}(f_1)_!f_2^*\mathcal{E}'$ (resp. $R^{2d_2}(f_2)_!f_1^*\mathcal{E}$) is actually a local system on C (resp. D) since it is constructible and G acts transitively on C (resp. D). Let $u \in C$ and $v \in C_1$ be such that $\pi_P(u)$ corresponds to v under $C_{\bar{P}} \simeq C_1$. Since $\mathcal{E}$ is G-equivariant, we can attach $\rho \in A_G(u)^\wedge$ to $\mathcal{E}$. On the other hand, as the isotropy subgroup of $(\pi_P(u), P)$ in D is equal to $Z_L(v)U_P$, we can attach $\rho' \in A_L(v)^\wedge$ to $\mathcal{E}'$. Now the stalk $(R^{2d_1}(f_1)_!f_2^*\mathcal{E}')_u$ at $u \in C$ has a natural structure of $A_G(u)$-module, and the stalk $(R^{2d_2}(f_2)_!f_1^*\mathcal{E})_\xi$ at $\xi = (\pi_P(u), P)$ in D has a natural structure of $A_L(v)$-module. Then the lemma is equivalent to

$$(11.5.1) \quad \langle \rho, H_c^{2d_1}(f_1^{-1}(u), f_2^*\mathcal{E}')\rangle_{A_G(u)} = \langle \rho', H_c^{2d_2}(f_2^{-1}(\xi), f_1^*\mathcal{E})\rangle_{A_L(v)}$$

where $\langle\ ,\ \rangle_{A_G(u)}$ means the inner product as $A_G(u)$-modules and similarly for $A_L(v)$.

In order to prove (11.5.1), we shall make use of the argument in Remark 7.3. Let $\tilde{Y}_{u,v} = \{g \in G \mid g^{-1}ug \in vU_P\}$ and let $I(\tilde{Y}_{u,v})$ be the set of irreducible components defined in 7.3. $Z_G(u) \times Z_L(v)$ acts on $\tilde{Y}_{u,v}$ by $(z_1, z_2)\colon g \to z_1 g z_2^{-1}$, and induces an action of $A_G(u) \times A_L(v)$ on $I(\tilde{Y}_{u,v})$. We denote by $S_{u,v}$ this permutation representation of $A_G(u) \times A_L(v)$. We consider the right hand side of (11.5.1). It follows that $f_2^{-1}(\xi) \simeq C \cap vU_P$, and the restriction of $f_1^*\mathcal{E}$ on $f_2^{-1}(\xi)$ is equal to the restriction of $\mathcal{E}$ on $C \cap vU_P$. Now $H_c^{2d_2}(C \cap vU_P, \mathcal{E})$ can be described as in Remark 7.3 in terms of the $A_G(u)$-action on $I(\tilde{Y}_{u,v})$. $A_L(v)$-module structure of this cohomology is also described by the $A_L(v)$-action on $I(\tilde{Y}_{u,v})$, and we have

$$\langle \rho', H_c^{2d_2}(C \cap vU_P, \mathcal{E})\rangle_{A_L(v)} = \langle \rho \otimes \rho', S_{u,v}\rangle_{A_G(u) \times A_L(v)} .$$

Similar argument can be applied to the left hand side since

$$f_1^{-1}(u) \simeq \{gP \in G/P \mid g^{-1}ug \in C_1U_P\} \simeq \tilde{Y}_{u,v}/Z_L(v)U_P \quad ,$$

and we see that the left hand side of (11.5.1) is also equal to the multiplicity of $\rho \otimes \rho'$ in $S_{u,v}$. This proves the lemma.

11.6. Let $(C, \mathcal{E})$ be an element in $\mathcal{N}_G$. We shall choose a parabolic subgroup P of G with the following two properties.

(i) there exists a unipotent element v in a Levi subgroup L of P such that $H_c^{\dim C - \dim C_1}(C \cap vU_P, \mathcal{E}) \neq 0$, where C_1 is the class in L containing v.

(ii) P is a minimal parabolic subgroup satisfying (i).

Note that the condition (i) is always satisfied for $P = G$. Let (C_1, L, P) be a triple which satisfies (i), (ii) above. Consider a map $f_2 \colon C \cap C_1U_P \to C_1$ defined by the projection onto L, and let $\mathcal{E}_1$ be an irreducible L-equivariant local system on C_1 which is a direct summand of $R^{\dim C - \dim C_1}(f_2)_!\mathcal{E}$. Then it is easily seen that $(C_1, \mathcal{E}_1)$ is a cuspidal pair on L. Note that, in the setting of Lemma 11.5, $R^{\dim C - \dim C_1}(f_2)_!\mathcal{E}$ is nothing but $R^{2d_2}(f_2)_!f_1^*\mathcal{E}|C_1$ under the inclusion $C_1 \hookrightarrow D$. Now $\mathcal{E}_1$ can be extended uniquely to an irreducible G-equivariant local system $\mathcal{E}_1'$ on D. The inverse image $f_2^*\mathcal{E}_1'$ on $\tilde{D}$ coincides with the restriction to $\tilde{D}$ of $\bar{\mathcal{E}}_1$ on X_0, where $\tilde{D} \hookrightarrow X_0 = \{(g, xP) \in \bar{Y} \times G/P \mid x^{-1}gx \in \Sigma U_P\}$.

Let $f \colon X_0 \to \bar{Y}$ be the first projection as in the theorem. Then by Lemma 11.5, we see that $\mathcal{E}$ is a direct summand of $R^{2d_1}f_!\bar{\mathcal{E}}_1|C$, where $d_1 = (\nu_G - \frac{1}{2}\dim C) - (\nu_L - \frac{1}{2}\dim C_1)$. We now show that the choice of $(L, C_1, \mathcal{E}_1)$ is unique up to G-conjugacy for a given $(C, \mathcal{E})$. In fact, assume we have another triple $(L', C_1', \mathcal{E}_1')$ associated to $(C, \mathcal{E})$. Let $f' \colon X_0' \to \bar{Y}'$ be the map corresponding to a parabolic subgroup P' containing L'. Then $R^{2d_1'}f_!'\bar{\mathcal{E}}_1'|C$ contains $\mathcal{E}$ as a direct summand. Thus, $R^{2d_1}f_!\bar{\mathcal{E}}_1 \otimes R^{2d_1'}f_!'\bar{\mathcal{E}}_1'^{\vee}$ restricted to C contains a constant sheaf $\bar{\mathbb{Q}}_\ell$, and we have

$$H_c^{2\dim C}(C, R^{2d_1}f_!\bar{\mathcal{E}}_1 \otimes R^{2d_1'}f_!'\bar{\mathcal{E}}_1'^{\vee}) \neq 0 \quad . \tag{11.6.1}$$

Using the partition of $Z_{uni}^0 = Z_0 \cap Z_{uni}$, $Z_{uni}^0 = \coprod_{C'} Z_{0,C'}$, where $Z_{0,C'} = \psi_0^{-1}(C')$ for each unipotent class C', and using the Leray spectral sequence for $Z_{0,C'} \to C'$, one can deduce from (11.6.1) that

$$H_c^{2d_0}(Z_{uni}^0, \bar{\mathcal{E}}_1 \boxtimes \bar{\mathcal{E}}_1'^{\vee}) \neq 0 \quad .$$

(Note $d_0 = \dim C + d_1 + d_1'$, and $\dim Z_{0,C'} \leq d_0$). This means that $\mathcal{T}_1 \neq 0$ for $\mathcal{T}$ defined with respect to $(C_1, \mathcal{E}_1)$ and $(C_1', \mathcal{E}_1')$. Hence, by Proposition 10.8, we see that $(L', C_1', \mathcal{E}_1')$ is conjugate to

$(L,C_1,\mathcal{E}_1)$ under G.

The above argument shows that we have a well-defined map

$$\Phi : \mathcal{N}_G \longrightarrow \mathcal{S}_G$$

by $(C,\mathcal{E}) \to (L,C_1,\mathcal{E}_1)$, and so we get a partition of $\mathcal{N}_G$,

$$\mathcal{N}_G = \coprod_{(C_1,\mathcal{E}_1)} \Phi^{-1}((L,C_1,\mathcal{E}_1)) \quad .$$

It is now easy to see that

(11.6.2) $(C,\mathcal{E})$ is in $\Phi^{-1}((L,C_1,\mathcal{E}_1))$ if and only if C is contained in $\bar{Y}$ and $\mathcal{E}$ is a direct summand of $R^{2d_C}f_!\bar{\mathcal{E}}_1|C$, where $d_C = (\nu_G - \frac{1}{2}\dim C) - (\nu_L - \frac{1}{2}\dim C_1)$.

11.7. We shall prove (ii) and (iii) of Theorem 9.4 simultaneously. Let $m_{\mathcal{E}}$ be the multiplicity of $\mathcal{E}$ in $R^{2d_C}f_!\bar{\mathcal{E}}_1|C$ and $n_{\mathcal{E}}$ be that of $K(\mathcal{E})$ in K_1 as before. In order to prove (ii), we have only to show that $m_{\mathcal{E}} = n_{\mathcal{E}}$.

We consider the map

$$(11.6.3) \qquad H_c^{2d_C}(f^{-1}(u),\ \bar{\mathcal{E}}_1) \longrightarrow \mathbb{H}_c^{2d_C}(\pi^{-1}(u),\ \bar{K}[-\dim Y]) \ ,$$

obtained from the natural map of complexes $f_!\bar{\mathcal{E}}_1 = \pi_!(\bar{K}[-\dim Y]|X_0) \to \pi_!(\bar{K}[-\dim Y])$ with respect to the inclusion $X_0 \hookrightarrow X$. This map is surjective. In fact, applying the support condition for $\bar{K}$ to the hypercohomology spectral sequence, we see that $\mathbb{H}_c^{2d_C}(\pi^{-1}(u)\cap X_\alpha,\bar{K}[-\dim Y]) = 0$ for $X_\alpha \neq X_0$. Surjectivity follows from this since $f^{-1}(u) = \pi^{-1}(u)\cap X_0$.

Let $\tilde{m}_{\mathcal{E}}$ be the multiplicity of $\mathcal{E}$ in $R^{2d_C}\pi_!\bar{K}[-\dim Y]$. If we denote by ρ the irreducible $A_G(u)$-module corresponding to $\mathcal{E}$, $m_{\mathcal{E}}$ is the multiplicity of ρ in the $A_G(u)$-module $H_c^{2d_C}(f^{-1}(u),\bar{\mathcal{E}}_1)$ and $\tilde{m}_{\mathcal{E}}$ is the mulptiplicity of ρ in $\mathbb{H}_c^{2d_C}(\pi^{-1}(u),\bar{K}[-\dim Y])$. Thus the surjectivity of (11.6.3) implies that $m_{\mathcal{E}} \geq \tilde{m}_{\mathcal{E}}$.

On the other hand, the direct sum decomposition of K_1 in (i) of Theorem 9.4 implies that $\tilde{m}_{\mathcal{E}} \geq n_{\mathcal{E}}$ as in the case of Springe correspondence. Hence

$$(11.6.4) \qquad \sum_{(C,\mathcal{E})} m_{\mathcal{E}}^2 \ \geq \ \sum_{(C,\mathcal{E})} \tilde{m}_{\mathcal{E}}^2 \ \geq \ \sum_{(C,\mathcal{E})} n_{\mathcal{E}}^2 = \dim \mathcal{A}_{\mathcal{E}_1} \quad .$$

We now prove that

$$\sum_{(C,\mathcal{E})} m_{\mathcal{E}}^2 = \dim A_{\mathcal{E}_1} \quad . \tag{11.6.5}$$

For a fixed unipotent class C, we have, from the definition of $m_{\mathcal{E}}$, that

$$\dim H_c^{2\dim C}(C,\ R^{2d_C}f_!\mathcal{E}_1 \otimes R^{2d_C}f_!\mathcal{E}_1^{\vee}) = \sum_{\mathcal{E}} m_{\mathcal{E}}^2 \quad ,$$

where the summation is taken over all $\mathcal{E}$ on C.

Let $Z_{0,C} = \psi_0^{-1}(C)$ as before. Then $Z_{0,C} \simeq X_{0,C} \times_C X_{0,C}$, where $X_{0,C} = f^{-1}(C)$. Moreover, $d_0 = \dim C + 2d_C$ and all of the fibres of the natural map $Z_{0,C} \to C$ have dimension $\leq 2d_C$. Thus, by using the Leray spectral sequence to $Z_{0,C} \to C$, together with the Künneth formula, we see that

$$H_c^{2d_0}(Z_{0,C},\ \mathcal{E}_1 \boxtimes \mathcal{E}_1^{\vee}) \simeq H_c^{2\dim C}(C,\ R^{2d_C}f_!\mathcal{E}_1 \otimes R^{2d_C}f_!\mathcal{E}_1^{\vee}) \ .$$

Now $Z_{\mathrm{uni}}^0 = \coprod_C Z_{0,C}$ form a partition of Z_{uni}^0 into locally closed pieces of dimension $\leq d_0$ as before. Hence

$$\dim H_c^{2d_0}(Z_{\mathrm{uni}}^0,\ \mathcal{E}_1 \boxtimes \mathcal{E}_1^{\vee}) = \sum_{(C,\mathcal{E})} m_{\mathcal{E}}^2 \quad .$$

Since $H_c^{2d_0}(Z_{\mathrm{uni}}^0,\ \mathcal{E}_1 \boxtimes \mathcal{E}_1^{\vee}) = \mathcal{T}_1$, the left hand side is equal to $\dim \mathcal{T}_1 = \dim A_{\mathcal{E}_1}$ by Proposition 10.8 (ii). This proves (11.6.5). Now it follows from (11.6.4) that $m_{\mathcal{E}} = \tilde{m}_{\mathcal{E}} = n_{\mathcal{E}}$, hence (ii) and (iii) of Theorem 9.4 follows.

Corollary 9.7 is now immediate from this last equality.

§12. **Examples**

12.1. Extending the results stated in Chapter II concerning the Springer correspondence, Lusztig and Spaltenstein [29] determined the generalized Springer correspondence completely in the case of classical groups, including the case of bad characteristic. The generalized Springer correspondence for exceptional groups was determined by Spaltenstein [38] except few ambiguities, i.e., one case where $p = 2$, G is of type E_8 and L is of type E_6, and the other case where G is of type E_6 and L is of type $2A_2$ for all p. In both cases, $N_G(L)/L$ is a dihedral group of order 12 and two irreducible characters of degree 2 of $N_G(L)/L$ could not be distinguished in his result. However, after that, using a property of coefficients of the generalized Green functions, Lusztig [26, V, 24.10] determined the remaining two characters in the latter case subject to the condition that $p \geq 5$.

In this section, we give some examples only in the case where p is good. First we note that cuspidal pairs are determined once we know the generalized Springer correspondence for each Levi subgroup of G. In particular, the number of cuspidal pairs is easily determined. It turns out that cuspidal pairs occur very few in a good characteristic case, i.e., if we fix a character χ of the component group $\Gamma = Z(G)/Z^0(G)$ of the center of G, then there exist at most one cuspidal pair $(C,\mathcal{E}) \in \mathcal{N}_G$ such that Γ acts on ρ according to χ, where $\mathcal{E}$ is the local system on C corresponding to $\rho \in A_G(u)^\wedge$ for $u \in C$. We now assume that G is almost simple, simply connected, and assume p is good. Then the condition on G for the existence of cuspidal pairs is as follows.

Type A_n: χ is order $n+1$

Type B_n: $\chi = 1$, $2n+1 \in \square$
$\chi \neq 1$, $2n+1 \in \triangle$

Type C_n: $\chi = 1$, $n \in \triangle$ and n: even
$\chi \neq 1$, $n \in \triangle$ and n: odd ,

Type D_n: $\chi = 1$, $2n \in \square$ and $n/2$: even
$\chi \neq 1$, $\chi(\varepsilon) = 1$, $2n \in \square$ and $n/2$: odd
$\chi(\varepsilon) \neq 1$, $2n \in \triangle$,

Type E_6: $\chi \neq 1$

Type E_7: $\chi \neq 1$

Type E_8: $\chi = 1$

Type F_4: $\chi = 1$

Type G_2: $\chi = 1$.

Here we use the notation, $\square = \{1,4,9,16, \ldots\}$, $\triangle = \{1,3,6,10, \ldots\}$, and ε denote the non-trivial element in the kernel of the natural map $\mathrm{Spin}_{2n} \to SO_{2n}$.

The case $\chi = 1$ corresponds to the case of adjoint groups. In particular, in the case of exceptional groups, there exist no cuspidal pairs for adjoint groups of type E_6 or E_7, and $(C,\mathcal{E}) = (u,\rho)$ described in 6.6 is the unique cuspidal pair for G of type E_8, F_4 or G_2, respectively. If G is a simply connected group of type E_6 or E_7, cuspidal pairs $(C,\mathcal{E})$ are given as follows; C is the regular unipotent class, $\mathcal{E}$ corresponds to a non-trivial character χ of $\Gamma \simeq A_G(u)$ for $u \in C$.

In the case where $G = SL_n(k)$, $(C,\mathcal{E})$ is cuspidal if and only if C is the regular unipotent class and $\mathcal{E}$ corresponds to a character χ of $\Gamma \simeq A_G(u) \simeq Z/nZ$ of order n for $u \in C$.

12.2. In order to describe the generalized Springer correspondence in the case of remaining classical groups, Lusztig introduced some combinatorial objects, "symbols", which is an analogue of symbols used in classifying the irreducible charcters of $G(\mathbb{F}_q)$ in the case of classical groups ([24]). In the following, we shall consider, as an example, only the case of symplectic groups.

Let $\tilde{\Psi}_{2n}$ be the set of pairs (A,B), where A is a finite subset of $\{0,1,2,\ldots\}$, B is a finite subset of $\{1,2,\ldots\}$, satisfying the following three conditions;

(12.3.1) (i) concecutive integers $\{i,i+1\}$ are not contained in A nor in B,

(ii) $|A|-|B|$ is odd (positive or negative),

(iii) $\sum_{a\in A} a + \sum_{b\in B} b = n + \frac{1}{2}(|A|+|B|)(|A|+|B|-1)$.

Let Ψ_{2n} be the set of equivalence classes in $\tilde{\Psi}_{2n}$ for the equivalence relation generated by shift operation,

$$(A,\ B) \to (\{0\}\cup(A+2),\ \{1\}\cup(B+2)) \quad .$$

The equivalence class of the pair (A,B) is also denoted by (A,B) by abbreviation and called symbols. Two symbols in Ψ_{2n} are said to be similar if they can be represented in the form (A,B), (A',B') such that $A\cup B = A'\cup B'$, $A\cap B = A'\cap B'$, i.e., they consist of the same entries with multiplicities. For each $(A,B) \in \Psi_{2n}$, $d = |A|-|B|$ is called a defect of (A,B). d is an odd integer, and in fact, does not depend on the shift. We have a partition

$$\Psi_{2n} = \coprod_{d:\mathrm{odd}} \Psi_{2n,d} \quad ,$$

where $\Psi_{2n,d}$ is the set of elements of defect d in Ψ_{2n}. On the other hand, we have a canonical bijection

$$\Psi_{2m,1} \xrightarrow{\sim} \Psi_{2m+d(d-1),d} \quad (d:\ \text{odd})$$

defined by

$$(A,B) \to (\{0,2,4,\dots,2d-4\} \cup (A+2d-2),\ B) \quad \text{if } d \geqq 1$$

and by

$$(A,B) \to (A,\ \{1,3,5,\dots,1-2d\} \cup (B+2-2d)) \quad \text{if } d \leqq -1 .$$

Hence, we have a natural bijection

$$(12.2.2) \qquad \Psi_{2n} \simeq \coprod_{d:\text{odd}} \Psi_{2n-d(d-1),1} = \coprod_{j \geqq 0} \Psi_{2n-j(j+1),1} ,$$

by taking $j = d-1$ if $d \geqq 1$ and $j = -d$ if $d \leqq -1$. Here we used the convention that Ψ_{2m} is empty if $m < 0$.

Let $G = Sp_{2n}$. It is shown that there exists a natural bijection between $\mathcal{N}_G$ and Ψ_{2n}. This bijection has the following property; $(C,\mathcal{E})$ and $(C',\mathcal{E}')$ correspond to the same similarity class if and only if $C = C'$. We now consider the generalized Springer correspondence

$$\mathcal{N}_G \xrightarrow{\sim} \coprod_{(C_1,\mathcal{E}_1)} (N_G(L)/L)^\wedge$$

given in Theorem 9.4 (and Remark 9.5). It follows from the list in 12.1 that $(L,C_1,\mathcal{E}_1) \in \mathcal{S}_G$ exist only for $L = L_i$ of type C_{n-i} such that $n-i \in \Delta$, and that exactly one $(C_1,\mathcal{E}_1)$ exists for such an L. For $L = L_i$, $N_G(L_i)/L_i$ is isomorphic to the Weyl group of type C_i, which we denote by W_i. Thus the generalized Springer correspondence is nothing but the bijection

$$(12.2.3) \qquad \mathcal{N}_G \xrightarrow{\sim} \coprod_{j \geqq 0} (W_{n-\frac{1}{2}j(j+1)})^\wedge .$$

In [25, §12], Lusztig defined a bijection between $\Psi_{2m,1}$ and $W_m^\wedge$ for each $m \geqq 1$, and showed that the resulting bijection between Ψ_{2n} and the right hand side of (12.2.3), obtained through the map (12.2.2), coincide with the generalized Springer correspondence under the identification $\mathcal{N}_G \simeq \Psi_{2n}$. Note the bijection $\Psi_{2n,1} \simeq W_n^\wedge$ is nothing but the Springer correspondence.

We shall only give an example below in the case of $G = Sp_6$. For the detailed description of the generalized Springer correspondence, see [loc.cit.]. The correspondence between unipotent classes in G and symbols Ψ_6 are given as follows. In the table below, unipotent classes are expressed by the type of Jordan's normal forms.

unipotent classes	Ψ_6
(6)	$(\{3\}, \phi)$, $(\phi, \{3\})$
(24)	$(\{04\},\{2\})$, $(\{02\},\{4\})$, $(\{0\},\{24\})$, $(\{024\},\phi)$
$(1^2 4)$	$(\{14\},\{1\})$, $(\{1\},\{14\})$
(3^2)	$(\{03\},\{3\})$
(2^3)	$(\{13\},\{2\})$, $(\{2\},\{13\})$
$(1^2 2^2)$	$(\{025\},\{24\})$, $(\{024\},\{25\})$
$(1^4 2)$	$(\{135\},\{13\})$, $(\{13\},\{135\})$
(1^6)	$(\{0246\},\{246\})$

Now Ψ_6 is decomposed as $\Psi_6 = \Psi_{6,1} \coprod \Psi_{6,-1} \coprod \Psi_{6,3}$, and a unique element $(\{024\},\phi)$ in $\Psi_{6,3}$ corresponds to a cuspidal pair $(C, \mathcal{E})$ in G. The generalized Springer correspondence is now given as follows.

$\Psi_{6,1}$	$\hat{W}_3$
$(\{3\},\phi)$	$(3;\ \phi)$
$(\{04\},\{2\})$, $(\{02\},\{4\})$	$(2;\ 1)$, $(\phi;\ 3)$
$(\{14\},\{1\})$	$(12;\ \phi)$
$(\{03\},\{3\})$	$(1;\ 2)$
$(\{13\},\{2\})$	$(1^2;\ 1)$
$(\{025\},\{24\})$, $(\{024\},\{25\})$	$(1;\ 1^2)$, $(\phi;\ 12)$
$(\{135\},\{13\})$	$(1^3;\ \phi)$
$(\{0246\},\{246\})$	$(\phi;\ 1^3)$

$\Psi_{6,-1}$ $\longleftrightarrow$	$\Psi_{4,1}$	$\hat{W}_2$
$(\phi, \{3\})$	$(\{02\}, \{3\})$	$(\phi;\ 2)$
$(\{0\}, \{24\})$	$(\{024\}, \{24\})$	$(\phi;\ 1^2)$
$(\{1\}, \{14\})$	$(\{03\}, \{2\})$	$(1;\ 1)$
—	—	—
$(\{2\}, \{13\})$	$(\{2\}, \phi)$	$(2;\ \phi)$
—	—	—
$(\{13\}, \{135\})$	$(\{13\}, \{1\})$	$(1^2;\ \phi)$
—	—	—

Here, as usual, we expressed irreducible characters of W_m by a pair of partitions $(\alpha; \beta)$ such that $\alpha = (\alpha_i)$, $\beta = (\beta_j)$ and that $\Sigma\, \alpha_i + \Sigma\, \beta_j = m$, where $(m; \phi)$ corresponds to a unit charcter.

12.3. The case of $SO_N(k)$ $(\mathrm{char}(k) \neq 2)$ can be described using similar kind of symbols ([25, §13]). From these considerations, cuspidal pairs for $Sp_{2n}(k)$ and $SO_N(k)$ are determined as follows. First consider $G = Sp_{2n}(k)$. We denote by $\lambda = (1^{i_1}, 2^{i_2}, \ldots)$ a partition of $2n$, i.e., $i_1 \geqq 0$, $i_2 \geqq 0$, ..., and $\lambda = \sum_k i_k k$. Unipotent classes of G are in 1-1 correspondence, through Jordan's normal form, with partitions λ of $2n$ such that i_k is even for odd k. For a unipotent element u in G, $A_G(u)$ (and so $A_G(u)^\wedge$) is identified with the $\mathbb{F}_2$-vector space $\mathbb{F}_2[\Delta_\lambda]$ with basis indexed by the set $\Delta_\lambda = \{k\colon \text{even} \mid i_k > 0\}$, where λ is the partition of $2n$ corresponding to u.

Cuspidal pairs of G are now described as follows. A cuspidal pair $(C, \mathcal{E})$ exists if and only if $n = d(d+1)/2$ for a positive integer d, and is unique if it exists. In that case C corresponds to a partition $\lambda = (2, 4, 6, \ldots)$ and $\rho \in A_G(u)^\wedge$ $(u \in C)$ corresponding to $\mathcal{E}$ is given by $(1, 0, 1, 0, \ldots, 1)$ (resp. $(0, 1, 0, \ldots, 0, 1)$) if d is odd (resp. d is even) under the identification $A_G(u)^\wedge \leftrightarrow \mathbb{F}_2[\Delta_\lambda]$.

Next consider the case $G = SO_N(k)$. In this case, the set of unipotent classes are parametrized by partitions λ of N of the form $\lambda = (1^{i_1}, 2^{i_2}, \ldots)$ such that i_k is even for even k. Let u be a unipotent elemnt in G attached to λ. Then $A_G(u)^\wedge$ may be naturally identified with $\widetilde{\mathbb{F}_2[\Delta_\lambda]}$, where $\Delta_\lambda = \{k\colon \text{odd} \mid i_k > 0\}$ and $\widetilde{\mathbb{F}_2[\Delta_\lambda]}$ is a quotient of the $\mathbb{F}_2$-vector space $\mathbb{F}_2[\Delta_\lambda]$ by the line spanned by the sum of all base vectors. Now G contains a cuspidal pair $(C, \mathcal{E})$ if and only if $N = d^2$ for a positive integer d. In that case, $(C, \mathcal{E})$ is unique, and C is attached to $\lambda = (1, 3, \ldots, 2d-1)$, $\rho \in A_G(u)^\wedge$ corresponding to $\mathcal{E}$ is given by $(1, 0, 1, \ldots) \in \widetilde{\mathbb{F}_2[\Delta_\lambda]}$.

The generalized Springer correspondence in the case of Spin groups $(\mathrm{char}(k) \neq 2)$ or classical groups in $\mathrm{char}(k) = 2$ is also described in terms of symbols of different kinds, which has been treated in [29].

IV GENERALIZED GREEN FUNCTIONS

§13. Green functions and representations of finite groups

13.1. In this chapter, we assume that G is defined over a finite field $\mathbb{F}_q$ with Frobenius map $F: G \to G$. The representation theory of finite reductive groups G^F developed by Lusztig and others is closely related to the theory of admissible complexes on G discussed in Chapter II and III. In this section, we will see it briefly.

After Deligne-Lusztig [10], one can construct a virtual G^F-module $R_T^G(\theta)$ associated with a pair (T,θ) as the alternating sum of certain ℓ-adic cohomologies on which G^F acts. Here T is an F-stable maximal torus of G and $\theta \in \mathrm{Hom}\,(T^F, \bar{\mathbb{Q}}_\ell^*)$. The trace $\mathrm{Tr}(u, R_T^G(\theta))$ for a unipotent element u is independent of the choice of θ, and the function $Q_T^G : u \to \mathrm{Tr}(u, R_T^G(\theta))$ is called a <u>Green function</u> on G associated with T, which is a G^F-invariant function on G_{uni}^F. The significance of Green functions lies in the following character formula which asserts that the determination of characters of $R_T^G(\theta)$ is reduced to the determination of Green functions of various reductive subgroups of G.

(13.1.1) **Character formula** ([10, Th.4.2]). Let $g = su = us$ be the Jordan decomposition of $g \in G^F$, where $s \in G^F$ is semisimple and $u \in G^F$ is unipotent. Then

$$\mathrm{Tr}(g, R_T^G(\theta)) = |Z_G^0(s)^F|^{-1} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} Q_{xTx^{-1}}^{Z_G^0(s)}(u)\theta(x^{-1}sx) \quad .$$

Another important property of Green functions is the following orthogonality relation, which is in fact equivalent to the orthogonality relations of $R_T^G(\theta)$ ([10, Th.6.8]) once we admit (13.1.1).

(13.1.2) **Orthogonality relations** ([10, Th.6.9])

$$\frac{1}{|G^F|} \sum_{u \in G_{uni}^F} Q_T^G(u) Q_{T'}^G(u) = \frac{|N_G(T,T')^F|}{|T^F||T'^F|} \quad ,$$

where T, T' are F-stable maximal tori, and $N_G(T,T') = \{n \in G \mid n^{-1}Tn = T'\}$.

13.2. Let T_0 be an F-stable maximal torus contained in an F-stable Borel subgroup B_0 of G, and let $W = N_G(T_0)/T_0$. Then any F-stable maximal torus is G^F-conjugate to T_w, an F-stable maximal torus obtained from T_0 by twisting by $w \in W$.

Let $\mathcal{B}_g$ be the fixed point subvariety of $\mathcal{B}$ by $g \in G$ as before. If $g \in G^F$, F acts naturally on $\mathcal{B}_g$, and so we have an action of F on $H^i(\mathcal{B}_g, \bar{\mathbb{Q}}_\ell)$ as well as the action of W. Now the following result gives a geometric interpretation of Green functions in terms of Springer representations. (It can be regarded as a generalization of Green's result for $GL_n(\mathbb{F}_q)$ mentioned in Introduction.)

(13.2.1) (Springer [39], Kazhdan [21]). Assume that p and q are large enough. Then for each $u \in G^F_{uni}$,

$$Q^G_{T_w}(u) = \sum_{i \geq 0} (-1)^i \mathrm{Tr}(Fw, H^i(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell)) \quad .$$

In fact, Springer has proved the orthogonality relations for the functions on $\underline{g}^F_{nil}$ defined by (the Lie algebra analogue of) the right hand side of the formula in (13.2.1), by making use of the theory of trigonometric sums on $\underline{g}$, and Kazhdan showed their coincidence with Green functions using those orthogonality relations.

13.3. We shall look over (13.2.1) again from a view point of admissible complexes. First we introduce some notations. Let V be an algebraic variety defined over $\mathbb{F}_q$ and $F: V \to V$ be the Frobenius map. A complex $K \in \mathcal{D}^b_c(V)$ is said to be F-stable if $F^*K \simeq K$. Let K be an F-stable complex and we fix an isomorphism $\phi: F^*K \xrightarrow{\sim} K$. For each $x \in V^F$, ϕ induces a linear automorphism on the stalk $\mathcal{H}^i_x K$ of i-th cohomology sheaf of K, which we denote also by ϕ. Set

(13.3.1) $$\chi_{K,\phi}(x) = \sum_i (-1)^i \mathrm{Tr}(\phi, \mathcal{H}^i_x K) \quad .$$

The function $\chi_{K,\phi} : V^F \to \bar{\mathbb{Q}}_\ell$ is called a <u>characteristic function</u> of K with respect to ϕ.

We consider the complex $K = IC(G, \mathcal{L})[\dim G]$ with $\mathcal{L} = (\pi_0)_* \bar{\mathbb{Q}}_\ell$ on G, (cf. (4.1)). If we take $B = B_0$ and $T = T_0$ in the construction of K given in §4, we have a natural isomorphism $F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ and so an isomorphism $F^*K \xrightarrow{\sim} K$, which we denote by F. The formula in (13.2.1) is now written as

$$(13.3.2) \qquad Q^G_{T_w}(u) = \sum_{i\geq 0}(-1)^i \mathrm{Tr}(Fw, \mathcal{H}^i_u(K[-n]))$$

for $u \in G^F_{uni}$ and $w \in W$, where $n = \dim G$.

We assume, for simplicity, that G is of split type over $\mathbb{F}_q$. Then F acts trivially on W. For each $E \in W^\wedge$, set

$$R^G_E(1) = \frac{1}{|W|}\sum_{w\in W} \mathrm{Tr}(w, E) R^G_{T_w}(1) \quad ,$$

which is an element of the Grothendieck group $\mathcal{R}(G^F)$ of G^F-modules tensored by $\mathbb{Q}$.

On the other hand K is decomposed as $K \simeq \bigoplus_{E\in W^\wedge} E \otimes K_E$, where $K_E = IC(G, \mathcal{L}_E)[n]$ (cf. Remark 6.1, (i)). Hence in this case, (13.3.2) is equivalent to

$$(13.3.3) \qquad \mathrm{Tr}(u, R^G_E(1)) = \chi_{K_E,F}(u) \qquad \text{for } u \in G^F_{uni} \text{ and } E \in W^\wedge.$$

In fact, using his theory of character sheaves [26], Lusztig has shown

13.4. **Theorem** (Lusztig [27]). <u>Assume</u> G <u>is of split type</u>. <u>Then</u> (13.3.3) <u>holds subject to the condition that</u> p <u>is good for</u> G.

13.5. As will be seen in §14 (Th.14.4), the character formula such as (13.1.1) holds also for the characteristic functions $\chi_{K,F}$ without restriction on p. Then the truth of (13.3.3) (even for non-split groups in an appropriate form) is equivalent to the truth of

$$(13.5.1) \quad \mathrm{Tr}(g, R^G_E(1)) = \chi_{K_E,F}(g) \quad \text{for any } g \in G^F \text{ and any } E \in W^\wedge.$$

This formula brings us a hope of describing certain "characters" such as $R^G_E(1)$ of G^F in terms of characteristic functions of certain nice complexes in G. In fact, this point of view is fairly strengthened as follows. In [24], Lusztig defined a certain set of class functions called <u>almost characters</u> of G^F, which form an orthonormal basis of the space of class functions of G^F apart from the orthonormal basis consisiting of irreducible characters. As the transition matrix of these two bases is explicitly described, the determination of (the values of) irreducible characters of G^F is equivalent to that of almost characters of G^F. The $R^G_E(1)$ given above are examples of almost characters. Meanwhile, we consider the set of admissible complexes $\mathcal{A}(G)$ in G (in a general sense, cf. Remark 8.3), and let $\mathcal{A}(G)^F$ be the subset of F-stable admissible complexes. For each $A \in \mathcal{A}(G)^F$, a characteristic function $\chi_{A,\phi}$ turns out to be a class function on G^F since A is G-equivariant. Moreover, as A is a simple perverse sheaf, $\chi_{A,\phi}$ does not depend on the choice of $\phi: F^*A \xrightarrow{\sim} A$ up to a scalar multiple. Note K_E as above are examples of F-stable admissible complexes. We can now state

13.6. **Conjecture** (Lusztig).

$$\{\text{almost characters of } G^F\} = \{\chi_{A,\phi} \mid A \in \mathcal{A}(G)^F\} \quad \text{up to scalar.}$$

In fact, in [26], he verified that the functions in the right hand side form an orthonormal basis of the space of class functions of G^F under a mild restriction on p. Also he showed in [27], that the restriction to G^F_{uni} of these two kinds of functions coincide each other for almost simple groups of split type under a certain condition on p.

§14. Generalized Green functions

14.1. In this section, we shall define generalized Green functions attached to admissible complexes on G, which will play the same role as green functions for the characteristic functions of admissible comlexes. We formulate their fundamental properties established in [26, II] without proof.

Let $K = IC(\bar{Y},(\pi_0)_*\tilde{\mathcal{E}}_1)[\dim Y]$ be a complex constructed in a general setting as in Remark 8.3 associated with $(L,\Sigma,\mathcal{E}_1)$, ($\Sigma = Z^0(L)C$ and a conjugacy class C in L is not necessarily unipotent). Assume that $FL = L$, $F\Sigma = \Sigma$ and that there exists an isomorphism $\phi_0: F^*\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1$ of local systems on Σ. Then one can define in a natural way from the construction in 8.1 (note Y, $\tilde{Y}$ and $\hat{Y}$ have $\mathbf{F}_q$-structures), an isomorphism $\phi: F^*K \xrightarrow{\sim} K$ in $\mathcal{M}G$.

14.2. We assume further that $\Sigma = Z^0(L)C$, where C is a unipotent class. Let $\mathcal{F} = \mathcal{E}_1|C$, an irreducible local system on C, and $\phi_1: F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ be the restriction of $\phi_0: F^*\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1$. A function

$Q^G_{L,C,\mathcal{F},\phi_1}: G^F_{uni} \to \bar{\mathbb{Q}}_\ell$ defined by

$$Q^G_{L,C,\mathcal{F},\phi_1}(u) = \sum_i (-1)^i \mathrm{Tr}(\phi, \mathcal{H}^i_u K) ,$$

is called a <u>generalized Green function</u> on G. One can prove that $Q^G_{L,C,\mathcal{F},\phi_1}$ is independent of the choice of $\mathcal{E}_1$ on Σ and of $\phi_0: F^*\mathcal{E}_1 \tilde{\to} \mathcal{E}_1$ extending $\mathcal{F}$ and ϕ_1, respectively, so the notation makes sense.

14.3. As in the case of Green functions, the important property of the generalized Green function is the character formula, which describes $\chi_{K,\phi}$ in terms of generalized Green functions of smaller groups. First we prepare some notations. Let L, Σ, $\mathcal{E}_1$, ϕ_0 and K be as in 14.1. Let s be a semisimple element in G^F and let u be a unipotent element in G^F such that $su = us$. We denote by Σ_1 the set of semisimple parts of elements in Σ. Take an element $x \in G^F$ such that $x^{-1}sx \in \Sigma_1$ and set $L_x = xLx^{-1} \cap Z^0_G(s)$. Then $s \in L_x$ and L_x is a Levi subgroup of some parabolic subgroup of $Z^0_G(s)$. Let C_x be the set of unipotent elements v in $Z^0_G(s)$ such that $sv \in x\Sigma x^{-1}$. One can show that C_x is a single unipotent class in $Z^0_G(s)$. Let $\mathcal{F}_x$ be the local system on C_x defined as the inverse image of $\mathcal{E}_1$ under the map $C_x \to \Sigma$, $v \mapsto x^{-1}svx$. Since this map is defined over $\mathbb{F}_q$, we have an isomorphism $\phi_x: F^*\mathcal{F}_x \tilde{\to} \mathcal{F}_x$ induced from ϕ_0. We can now state

14.4. **Theorem** (Character formula [26, II, Th.8.5]).
<u>Let the notations be as in 14.3</u>. <u>Then</u>

$$\chi_{K,\phi}(su) = |Z^0_G(s)^F|^{-1} \sum_{\substack{x\in G^F \\ x^{-1}sx\in\Sigma_1}} (|L^F_x|/|L^F|) Q^{Z^0_G(s)}_{L_x,C_x,\mathcal{F}_x,\phi_x}(u) .$$

Note that in the case where $L = T$ is a maximal torus, this formula coincides with (13.1.1) by taking (13.3.2) into account. In fact, the term $\theta(x^{-1}sx)$ in (13.1.1) is involved in the definition of ϕ_x.

The following result is an extension of (13.1.2) to the generalized Green functions.

14.5. **Theorem** (Orthogonality relations [26, II, Cor.9.11]). Assume p is good for each simple factor of exceptional type of G (and no assumption on simple factors of classical type). Then

$$|G^F|^{-1} \sum_{u \in G^F_{\mathrm{uni}}} Q^G_{L,C,\mathcal{F},\phi_1}(u) Q^G_{L',C',\mathcal{F}',\phi_1'}(u) =$$

$$= \begin{cases} |N_G(L)^F/L^F|\,|Z^0(L)^F|^{-1} q^{\dim C - \dim (L/Z^0(L))} \\ \qquad \text{if } L' = L,\ C' = C,\ \mathcal{F}' = \mathcal{F}^\vee,\ \phi_1 = \phi_1^\vee, \\ 0 \qquad \text{if there is no } g \in G^F \text{ such that} \\ \qquad gLg^{-1} = L',\ gCg^{-1} = C',\ \mathrm{ad}(g)^*\mathcal{F}' \simeq \mathcal{F}^\vee, \end{cases}$$

where $\mathcal{F}^\vee$ is the local system on C dual to $\mathcal{F}$, and $\phi_1^\vee: F^*\mathcal{F}^\vee \xrightarrow{\sim} \mathcal{F}^\vee$ is the contragredient of ϕ_1.

14.6. **Remark.** In [26], Theorem 14.5 was proved in a general setting under the assumption that $IC(\bar{\Sigma},\mathcal{E}_1)[\dim \Sigma]$, $IC(\bar{\Sigma}',\mathcal{E}_1')[\dim \Sigma']$ are clean and strongly cuspidal in L and L'; in general, the complex $K_0 = IC(\Sigma,\mathcal{E})[\dim \Sigma]$ on G is said to be strongly cuspidal if $(\pi_P)_! \iota^* K_0 = 0$ for any $P \neq G$, where $\iota: P \hookrightarrow G$ is the inclusion and $\pi_P: P \to P/U_P$ is the canonical map. Also a cuspidal complex $K_0 \in \mathcal{M}G$ is said to be clean if K_0 is zero on $\bar{\Sigma} - \Sigma$, i.e., K_0 is a single complex concentrated on the degree $-(\dim \Sigma)$. It is straighforward that if $K_0 \in \mathcal{M}G$ is strongly cuspidal, then it is cuspidal. However the converse statement is more delicate. In fact, in [loc.cit.], Lusztig defined a certain subset $\hat{G}$ of $A(G)$, the set of character sheaves of G, and proved by a general argument that a cuspidal complex $K_0 \in \mathcal{M}G$ is strongly cuspidal and clean whenever K_0 is a character sheaf. On the other hand, the main result of [loc.cit.] asserts that $\hat{G} = A(G)$ under the condition given in Theorem 14.5. The proof of this last fact is very long and depends on the classification of $\hat{G}$ carried out in [loc.cit.] under the same condition as above. Note that the classification of $\hat{G}$ enables us to conclude that $\hat{G} = A(G)$ since we know already $A(G)$ through the generalized Springer correspondence.

For further details on the theory of character sheaves, see the report of Mars and Springer in this volume.

14.7. Let K be the complex induced from a triple $(L,\Sigma,\mathcal{E}_1)$. We assume that $\Sigma = Z^0(L)C$ for a unipotent class C in L and $\mathcal{E}_1$ is of the form $1 \boxtimes \mathcal{F}$ for a cuspidal pair $(C,\mathcal{F})$ on L. Assume further that the triple $(L,\Sigma,\mathcal{E}_1)$ is F-stable and let $\phi: F^*K \simeq K$ be the isomorphism as before. Let $\mathcal{A} = \bigoplus_{w\in W} \mathcal{A}_w$ be the endomorphism algebra $\operatorname{End} K$ as in 9.1. There exists a canonical vector θ_w in $\mathcal{A}_w$ such that $\theta_w\theta_{w'} = \theta_{ww'}$ (cf. Remark 9.5).

Since K is F-stable, we have a natural automorphism of algebras, $\iota: \mathcal{A} \to \mathcal{A}$ given by $\theta \to \phi\circ F^*(\theta)\circ\phi^{-1}$ $(\theta \in \mathcal{A})$. Let $A \in \mathcal{A}(G)_{uni}$ be a complex isomorphic to a direct summand of K. Then F^*A is also a direct summand of K since K is F-stable. We assume that A is F-stable and choose $\phi_A: F^*A \simeq A$. Let $V_A = \operatorname{Hom}(A,K)$ be the irreducible $\mathcal{A}$-module as before. We define a map $\sigma_A: V_A \to V_A$ by $v \to \phi\circ F^*(v)\circ\phi_A^{-1}$ $(v \in V_A)$, where $F^*(v) \in \operatorname{Hom}(F^*A,F^*K)$. Then it is verified that σ_A is $\mathcal{A}$-semilinear, i.e., $\sigma_A(\theta v) = \iota(\theta)\sigma_A(v)$ for $\theta \in A$, $v \in V_A$. We now consider the decomposition $\bigoplus_A (A \otimes V_A) \simeq K$, which induces, for any $g \in G$ and any integer i,

$$\bigoplus_A (\mathcal{H}^i_g A \otimes V_A) \simeq \mathcal{H}^i_g K \quad . \tag{14.7.1}$$

If we choose $g \in G^F$, ϕ induces $\phi: \mathcal{H}^i_g K \to \mathcal{H}^i_g K$. Then one can verify that ϕ stabilizes $\mathcal{H}^i_g A \otimes V_A$ under the isomorphism (14.7.1) if A is F-stable, and in that case the restriction of ϕ to $\mathcal{H}^i_g A \otimes V_A$ corresponds to $\phi_A \otimes \sigma_A$. On the other hand, if A is not F-stable, ϕ maps $\mathcal{H}^i_g A \otimes V_A$ onto a different direct summand. It follows that

$$\operatorname{Tr}(\phi, \mathcal{H}^i_g K) = \sum_A \operatorname{Tr}(\phi_A, \mathcal{H}^i_g A)\operatorname{Tr}(\sigma_A, V_A) \quad , \tag{14.7.2}$$

where the summation is taken over all $A \in \mathcal{A}(G)^F_{uni}$.

If $\phi: F^*K \simeq K$ is replaced by $\theta_w\circ\phi$ for some $w \in \mathcal{W} = N_G(L)/L$ and $\phi_A: F^*A \simeq A$ remains unchanged, the formula (14.7.2) is changed as follows. ϕ in the left hand side is replaced by $\theta_w\circ\phi$ and the right hand side of σ_A is replaced by $\theta_w\circ\sigma_A$. By making use of orthogonality relations of characters of $\mathcal{W}$ (or rather its ι-twisted version), (14.7.2) is solved for each A, i.e., we have

$$\chi_{A,\phi_A} = \frac{1}{|\mathcal{W}|} \sum_{w\in\mathcal{W}} \operatorname{Tr}((\theta_w\circ\sigma_A)^{-1}, V_A)\chi_{K,\theta_w\circ\phi} \quad . \tag{14.7.3}$$

Note that $\theta_w \circ \phi\colon F^*K \xrightarrow{\sim} K$ is not a standard one, i.e., it is not the induced map from some $\phi_0\colon F^*\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1$. However as in the following way, we can regard this map as a standard one. Fix $w \in \mathcal{W}$ and choose its representative $n \in N_G(L)$. One can find $z \in G$ such that $z^{-1}F(z) = n^{-1}$. Set $L^w = zLz^{-1}$, $\Sigma^w = z\Sigma z^{-1}$ and $\mathcal{E}_1^w = \mathrm{ad}(z^{-1})^*\mathcal{E}_1$. Then $FL^w = L^w$ and $F\Sigma^w = \Sigma^w$. Moreover one can define an isomorphism $\phi_0^w : F^*\mathcal{E}_1^w \xrightarrow{\sim} \mathcal{E}_1^w$ in terms of $\phi_0\colon F^*\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1$ and θ_w. Let K^w be the complex induced from the triple $(L^w, \Sigma^w, \mathcal{E}_1^w)$. We have a standard isomorphism $\phi^w\colon F^*K^w \xrightarrow{\sim} K^w$. Now K^w is naturally isomorphic to K and ϕ^w corresponds to $\theta_w \circ \phi$ under this isomorphism. Thus we have

$$\chi_{K,\theta_w \circ \phi} = \chi_{K^w,\phi^w} \quad ,$$

and the restriction of $\chi_{K,\theta_w\circ\phi}$ on G^F_{uni} can be described by that of χ_{K^w,ϕ^w}, i.e., by the generalized Green functions with respect to $(L^w, C_1^w, \mathcal{E}_1^w)$. Theorem 14.5 now implies the follwing orthogonality relations for $\chi_{A,\phi_A}|G^F_{\mathrm{uni}}$.

14.8. **Corollary** ([26, II, Th.10.9]).

$$|G^F|^{-1} \sum_{u \in G^F_{\mathrm{uni}}} \chi_{A,\phi_A}(u)\chi_{A',\phi_{A'}}(u) =$$

$$= \begin{cases} |\mathcal{W}|^{-1} \displaystyle\sum_{w\in\mathcal{W}} \mathrm{Tr}((\theta_w \circ \sigma_A)^{-1}, V_A)\mathrm{Tr}(\theta_w \circ \sigma_{DA'}, V_{DA'})|Z^0(L^w)^F|^{-1}q^{-c} , \\ \qquad \underline{\text{if}} \ \ L = L',\ C_1 = C_1',\ \mathcal{F} = (\mathcal{F}')^\vee \ \ \underline{\text{and}} \ \ \phi_0 = (\phi_0')^\vee , \\ \\ 0 \qquad \underline{\text{if}} \ \ (L, C_1, \mathcal{E}_1) \ \ \underline{\text{is not G-conjugate to}} \ \ (L', C_1', \mathcal{E}') , \end{cases}$$

<u>where</u> $\phi_0\colon F^*\mathcal{E}_1 \simeq \mathcal{E}_1$ <u>and similarly for</u> $\mathcal{E}_1'$, <u>and</u> $(\mathcal{E}_1')^\vee$ <u>is the dual of</u> $\mathcal{E}_1'$ <u>on</u> C_1', $(\phi_0')^\vee$ <u>is the contragredient of</u> ϕ_0'. $c = \dim G - \dim \mathrm{supp}\, A$. <u>Moreover</u>, $\sigma_{DA'}$ <u>is defined with respect to</u> $\phi^\vee\colon F^*DK \xrightarrow{\sim} DK$, $\phi_{A'}^\vee\colon F^*DA' \xrightarrow{\sim} DA'$, <u>the contragredients of</u> ϕ <u>and</u> $\phi_{A'}$.

§15. Determination of generalized Green functions

15.1. We shall show in this section, following [26, V, 24], that there exists an algorithm of computing generalized Green functions under the assumption of Theorem 14.5. This makes it possible, in particular, to compute all the characteristic functions χ_{A,ϕ_A} for admissible complexes $A \in \mathcal{A}(G)^F$ in view of Theorem 14.4.

15.2. In this section, we use $\mathcal{N}_G$ (resp. $\mathcal{S}_G$) as an index set for parametrizing $\mathcal{A}(G)_{uni}$ through the generalized Springer correspondence, and denote it by I (resp. J). For each $A \in \mathcal{A}(G)_{uni}$ there exist unique $i = (C,\mathcal{E}) \in I = \mathcal{N}_G$ and $j = (L,C_1,\mathcal{E}_1) \in J = \mathcal{S}_G$ such that A is isomorphic to the direct summand A_i of K_j, where K_j is the induced complex associated with the triple j, and A_i is the summand such that $A_i[-r]|G_{uni} \simeq IC(\bar{C},\mathcal{E})[\dim C]$, ($r = \dim Z^0(L)$ as before). Thus we can define a map $\tau: I \to J$ by associating to each $i \in I$ the triple j such that A_i is a direct summand of K_j. We shall define a preorder in I as follows. For given $i = (C,\mathcal{E})$, $i' = (C',\mathcal{E}')$ in I, we say that $i' \leq i$ if $C' \subset \bar{C}$. We say that $i \sim i'$ if $C = C'$ and that $i' < i$ if $C' \subsetneq \bar{C}$.

For each $i = (C,\mathcal{E})$, we denote $i^* = (C,\mathcal{E}^\vee)$, where $\mathcal{E}^\vee$ is the dual local system of $\mathcal{E}$ on C. Similarly, for each $j = (L,C_1,\mathcal{E}_1)$, we denote $j^* = (L,C_1,\mathcal{E}_1^\vee)$, where $\mathcal{E}_1^\vee$ is the dual local system of $\mathcal{E}_1$. The involutions $i \to i^*$, $j \to j^*$ commute with the map $\tau: I \to J$.

15.3. We note that F acts naturally on I and J so that I^F corresponds to $\mathcal{A}(G)^F_{uni}$ and that τ induces a surjection $I^F \to J^F$. For each $A \in \mathcal{A}(G)^F_{uni}$, we shall choose a suitable $\phi_A: F^*A \simeq A$ as follows. First take $j = (L,C_1,\mathcal{E}_1) \in J^F$ and fix $\phi_j: F^*\mathcal{E}_1 \simeq \mathcal{E}_1$ which induces a map of finite order on the stalk at any point in C_1^F. ϕ_j induces a map $\phi_j: F^*K_j \simeq K_j$. For each $i \in I^F$ such that $\tau(i) = j$, let $V_{A_i} = \mathrm{Hom}(A_i,K_j)$ be the irreducible $\mathcal{A}_j$-module, where $\mathcal{A}_j = \mathrm{End}\, K_j \simeq \bar{\mathbb{Q}}_\ell[\mathcal{W}_j]$ with $\mathcal{W}_j = N_G(L)/L$, and let $\sigma_{A_i}: V_{A_i} \to V_{A_i}$ be as in 14.7.

It is possible to choose $\phi_{A_i}: F^*A_i \simeq A_i$ so that $\mathrm{Tr}(\theta_w \circ \sigma_{A_i}, V_{A_i}) \in \mathbb{Z}$, and that $\mathrm{Tr}((\theta_w \circ \sigma_{A_i})^{-1}, V_{A_i}) \in \mathbb{Z}$ for each $w \in \mathcal{W}_j$. We fix such ϕ_{A_i} for each $i \in I^F$. By making use of thus defined ϕ_{A_i}, we shall define a map $\phi: F^*\mathcal{E} \simeq \mathcal{E}$ for each $i = (C,\mathcal{E})$ as follows. Since $A_i[-r]|G_{uni} \simeq IC(\bar{C},\mathcal{E})[\dim C]$, $A_i|C$ is a single complex $\mathcal{E}$ concentrated to the degree $a_0 = -(\dim Z^0(L) + \dim C)$, so $\mathcal{H}^{a_0}A_i|C \simeq \mathcal{E}$. Let $\bar{\phi}_{A_i}: \mathcal{H}^{a_0}(F^*A_i) \to \mathcal{H}^{a_0}A_i$ be the induced map, and we define $\psi: F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ by $\psi = q^{-(a_0+n_0)/2}(\bar{\phi}_{A_i}|C)$, where $n_0 =$ $\dim \operatorname{supp} A_i = \dim Y_{(L,\Sigma)}$. It can be proved that thus defined ψ is of finite order on the stalk at any point in C^F. Let $\phi_j^\vee: F^*\mathcal{E}_1^\vee \xrightarrow{\sim} \mathcal{E}_1^\vee$ be the contragredient of ϕ_j and let $\psi^\vee: F^*\mathcal{E}^\vee \xrightarrow{\sim} \mathcal{E}^\vee$ be the map defined in terms of $(L,C_1,\mathcal{E}_1^\vee)$ and $\phi_j^\vee$. Then $\psi^\vee$ is in fact equal to the contragredient of ψ.

Let us define, for each $i = (C,\mathcal{E}) \in I^F$ and for ψ as above, a function $Y_i: G^F_{\mathrm{uni}} \to \bar{\mathbb{Q}}_\ell$ by

(15.3.1) $$Y_i(g) = \begin{cases} \operatorname{Tr}(\psi, \mathcal{E}_g) & \text{if } g \in C^F, \\ 0 & \text{if } g \notin C^F. \end{cases}$$

Then it is easy to see that

(15.3.2) The functions Y_i $(i \in I^F)$ form a basis for the vector space $\mathcal{V}$ of G^F-invariant functions on G^F_{uni}.

We need another set of functions on G^F_{uni}. For each $i \in I^F$, let

(15.3.3) $$\tilde{Y}_i(g) = \begin{cases} \operatorname{Tr}(\psi^\vee, \mathcal{E}^\vee_g) & \text{if } g \in C^F, \\ 0 & \text{if } g \notin C^F. \end{cases}$$

where $\psi^\vee: F^*\mathcal{E}^\vee \xrightarrow{\sim} \mathcal{E}^\vee$ is the contragredient of $\mathcal{E}$ as before. $\tilde{Y}_i$ $(i \in I^F)$ also gives a basis of $\mathcal{V}$.

15.4. We now proceed to the determination of generalized Green functions. In view of (14.7.2), the determination is equivalent to that of characteristic functions of $A_i \in \mathcal{A}(G)^F_{\mathrm{uni}}$. Let us define for any $i = (C,\mathcal{E}) \in I^F$ a function $X_i \in \mathcal{V}$ by

(15.4.1) $$X_i(g) = \sum_a (-1)^{a+a_0} \operatorname{Tr}(\phi_{A_i}, \mathcal{H}^a_g(A_i))q^{-(a_0+n_0)/2}.$$

Since $\{Y_i\}$ is a basis for $\mathcal{V}$, we can write as

(15.4.2) $$X_i = \sum_{i' \in I^F} P_{i',i} Y_{i'}$$

with $P_{i',i} \in \bar{\mathbb{Q}}_\ell$. Since $A_i[a_0]|G_{\mathrm{uni}} \simeq IC(\bar{C},\mathcal{E})$, it follows that

$$(15.4.3) \qquad P_{i',i} = 0 \quad \text{if} \quad i' \nleq i \quad \text{or} \quad \text{if} \quad i' \sim i,\ i' \neq i \ ,$$
$$P_{i,i} = 1 \ .$$

The second equality follows from the definition of ψ.

Let $\tilde{\phi}_{A_i^*}: F^*A_{i^*} \to A_{i^*}$ be the map defined as $q^{a_0+n_0} h^{-1} \circ \phi_{A_i}^{v} \circ F^*(h)$, where $\phi_{A_i}^{v} : F^*DA_i \to DA_i$ is the contragredient of $\phi_{A_i} : F^*A_i \to A_i$, and $h : A_{i^*} \simeq DA_i$ is an isomorphism, $F^*(h): F^*A_{i^*} \to F^*DA_i$ is defined by h. Then $\tilde{\phi}_{A_i^*}$ is not necessarily equal to $\phi_{A_i^*}$, but $\sigma_{DA_i^*}$ is equal to σ_{A_i}.
We define $\tilde{X}_i \in V$ by

$$(15.4.4) \qquad \tilde{X}_i(g) = \sum_a (-1)^{a+a_0} \operatorname{Tr}(\tilde{\phi}_{A_i^*}, \mathcal{H}_g^a(A_{i^*}))q^{-(a_0+n_0)/2} \ .$$

Then as in the above case, we have

$$(15.4.5) \qquad \tilde{X}_i = \sum_{i' \in I^F} \tilde{P}_{i',i}\tilde{Y}_{i'} \ ,$$

with $\tilde{P}_{i',i} \in \bar{\mathbb{Q}}_\ell$, and similarly as before,

$$(15.4.6) \qquad \tilde{P}_{i',i} = 0 \quad \text{if} \quad i' \nleq i \quad \text{or if} \quad i' \sim i,\ i' \neq i \ ,$$
$$\tilde{P}_{i,i} = 1$$

Let us now introduce a non-degenerate bilinear form on V by

$$(X, X') = \sum_{u \in G^F_{\mathrm{uni}}} X(u)X'(u) \qquad (X, X' \in V) \ .$$

Then Corollary 14.8 is rewritten as

$$(15.4.6) \qquad (X_i, \tilde{X}_{i'}) = \omega_{i,i'} \qquad (i, i' \in I^F) \ ,$$

where

$$\omega_{i,i'} = |\mathcal{W}_j|^{-1} \sum_{w \in \mathcal{W}_j} \operatorname{Tr}((\theta_w\sigma_{A_i})^{-1}, V_{A_i})\operatorname{Tr}(\theta_w\sigma_{A_{i'}}, V_{A_{i'}}) \times$$
$$\times |G^F||Z^0(L^w)^F|^{-1}q^{-\dim G}q^{-(a_0+a_0')/2} \ ,$$

if $\tau(i) = \tau(i') = j = (L, C_1, \mathcal{E}_1)$ with $a_0 = -(\dim Z^0(L) + \dim C)$, $a_0' = -(\dim Z^0(L) + \dim C')$, and is equal to zero if $\tau(i) \neq \tau(i')$.

We note, in particular, that $\omega_{i,i'} = \omega_{i',i}$ is a rational number.

On the other hand, let

$$(15.4.7) \qquad \lambda_{i,i'} = (Y_i, \tilde{Y}_{i'}) .$$

Then $\lambda_{i,i'} = 0$ unless $i \sim i'$. Moreover, since $\{Y_i\}$ and $\{\tilde{Y}_i\}$ are bases of V, the matrix $(\lambda_{i,i'})$ is non-singular.

We can express these relations in a form of matrices of degree $|I^F|$. We give an order in I^F such that it is compatible with the given preorder, and that each equivalence class in I^F form an interval. Let $P = (P_{i,i'})$, $\tilde{P} = (\tilde{P}_{i,i'})$, $\Omega = (\omega_{i,i'})$ and $\Lambda = (\lambda_{i,i'})$. Then by (15.4.2), (15.4.5), (15.4.6) and (15.4.7), we have

$$(15.4.8) \qquad {}^tP\Lambda\tilde{P} = \Omega .$$

We now regard these matrices as block-matrices each of whose block corresponds to an equivalent class for $\sim$ in I^F. Then (15.4.3) and (15.4.6) says that P and $\tilde{P}$ are the block matrices which have zero for each block below the diagonal and have identities in diagonal blocks. Moreover, by the remark after (15.4.7), Λ is a non-singular block matrix, off-diagonal blocks are all zero. On the other hand, we can compute all entries in Ω since we know already the generalized Springer correspondence. It is now easy to see that the formula (15.4.8) regarded as an equation with respect to unknown P, $\tilde{P}$ and Λ, can be solved, and in fact, one can determine P, $\tilde{P}$ and Λ uniquely from the information of Ω. Thus we can verify the following theorem using inductive argument.

15.5. **Theorem** ([26, V, Th.24.4]). P, $\tilde{P}$ and Λ are determined uniquely from (15.4.8), hence so is X_i for each $i \in I^F$. In particular

(i) $P_{i,i'} = \tilde{P}_{i,i'}$, $\lambda_{i,i'} = \lambda_{i',i}$ for all $i, i' \in I^F$ and they are rational numbers.

(ii) $P_{i,i'}$ and $\lambda_{i,i'}$ are zero if $\tau(i) \neq \tau(i')$.

15.6. **Remark.** Let $j_0 = (T,\{1\},\bar{\mathbb{Q}}_\ell) \in J$ and set $I_0 = \tau^{-1}(j_0)$. Then $K_{j_0} = K$ (= $IC(G,\mathcal{L})[\dim G]$) in 13.3 and A_i $(i \in I_0)$ are the direct summand of K_{j_0}. The determination of Green functions of G is equivalent to the determination of characteristic functions for $A_i|G_{uni}$ $(i \in I_0^F)$. Let $P_0 = (P_{i,i'})$ $(i,i' \in I_0^F)$ be the matrix obtained by restricting entries in P to $I_0^F \times I_0^F$, and similarly we define Λ_0 and Ω_0 corresponding to Λ, Ω, respectively. Then Theorem 15.5 implies that

(15.6.1) $$ {}^tP_0\Lambda_0P_0 = \Omega_0 \quad . $$

This is the equation used to determine Green functions. In fact, Green functions (in the sense of the right hand side of (13.2.1)) have been determined by the author [33],[34], and by Beynon-Spaltenstein [4] using a different kind of (but essentially the same as (15.6.1)) equations in the case where p is good for G. In these computations, one had to know the property corresponding to Theorem 15.5 (ii), i.e., if $i = (C,\mathcal{E}) \notin I_0^F$, and $\mathcal{E}$ corresponds to $\rho \in A_G(u)^\wedge$ for $u \in C$, then (*) $\sum_{a\geq 0}(-1)^a\mathrm{Tr}(F,H^a(\mathcal{B}_u,\bar{\mathbb{Q}}_\ell)_\rho) = 0$, or a rather stronger property, (**) $H^a(\mathcal{B}_u,\bar{\mathbb{Q}}_\ell)_\rho = 0$ for any $a \geq 0$. In fact the property (**) was proved by ad hoc method in the case of classical groups, and (*) was verified in the case of exceptional groups using a computer in the course of explicit computation of Green functions. Recently De Concini, Lusztig and Procesi [9] verified (**) for exceptional groups (over $\mathbb{C}$) by a detailed study of $\mathcal{B}_u$, independent of the theory of Green functions. If we extend the scope to generalized Green functions, Theorem 15.5 (ii) is a direct consequence of the orthogonality relations (Theorem 14.5). However as remarked in 14.6, still we need a long way to verify the assumption of Theorem 14.5.

15.7. In Theorem 15.5 (i), we have seen that $P_{i',i}$ are rational numbers. On the other hand, $P_{i',i}$ are algebraic integers since they are alternating sums of eigenvalues of Frobenius map. It follows that $P_{i',i}$ are integers. Let us choose $\mathbb{F}_q$ large enough so that F fixes all elements in I. If we replace $\mathbb{F}_q$ by $\mathbb{F}_{q^n}$ for any integer $n > 0$, we can get a different system of coefficients $P_{i',i}$ with respect to each $\mathbb{F}_{q^n}$-structure. Then it turns out that there exist polynomials $\Pi_{i',i}(\underline{q})$ with integral coefficients such that $P_{i',i}$ with respect to $\mathbb{F}_{q^n}$ is obtained by substituting q^n to $\underline{q}$ in $\Pi_{i',i}(\underline{q})$.

By the way, Lusztig [26] proved the following fact.

15.8. **Proposition.** <u>Assume</u> p <u>is good</u>. <u>For</u> $(C,\mathcal{E}) \in I^F$, <u>let</u> ϕ: $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ <u>be an isomorphism which induces the identity map on the stalk of</u> $\mathcal{E}$ <u>at any point of</u> C^F. <u>Then for any</u> $u \in \bar{C}^F$, <u>the induced map</u> ϕ <u>on</u> $\mathcal{H}_u^a(IC(\bar{C},\mathcal{E}))$ <u>is</u> a-<u>pure</u>, i.e., <u>its eigenvalues are algebraic numbers all of whose complex conjugate have absolute value</u> $q^{a/2}$.

Combining this with the previous result, one gets

15.9. **Corollary** ([26, V, Th.24.8]). <u>For any</u> $i = (C,\mathcal{E}) \in I$, <u>we have</u> $\mathcal{H}^a A_i = 0$ <u>if</u> $a \equiv \dim \operatorname{supp} A_i \pmod 2$ <u>and</u> $\mathcal{H}^a IC(\bar{C},\mathcal{E}) = 0$ <u>if</u> a <u>is odd</u>.

15.10. **Remarks.** (i) For any $(C,\mathcal{E}) \in \mathcal{N}_G$, we know already the characteristic function of $IC(\bar{C},\mathcal{E})$ with respect to each $\mathbf{F}_{q^n}$-structure as polynomials in q^n through the generalized Green functions. Thus Proposition 15.8 together with Corollary 15.9 enables us to determine the local intersection cohomologies $\mathcal{H}^a_u(IC(\bar{C},\mathcal{E}))$ for any $a \in \mathbf{Z}$, $u \in \bar{C}$ completely, at least in the form of algorithm.

(ii) Proposition 15.8 and Corollary 15.9 also implies that $\mathbb{H}^a(\mathcal{P}_u^{(C_1)},\tilde{\mathcal{K}})$ (cf. 9.6) is zero if a is odd and that eigenvalues of F on $\mathbb{H}^{2a}(\mathcal{P}_u^{(C_1)},\tilde{\mathcal{K}})$ have absolute value q^a. This is a generalization of Springer's result ([40]) that $H^a(\mathcal{B}_u,\bar{\mathbb{Q}}_\ell)$ is a-pure. The vanishing of $H^{odd}(\mathcal{B}_u,\bar{\mathbb{Q}}_\ell)$ was known in [4],[34] using the Springer's result. The vanishing of odd cohomology was also proved in [9] directly by analyzing $\mathcal{B}_u$.

V <u>FOURIER TRANSFORMS</u>

§16. Fourier transforms of $\bar{\mathbb{Q}}_\ell$-sheaves

16.1. In this section, we introduce the notion of Fourier transforms of $\bar{\mathbb{Q}}_\ell$-sheaves, and give some of their fundamental properties. For details on this subject the reader may consult Brylinski [7], Katz-Laumon [19].

Let X be an algebraic variety defined over $\mathbb{F}_q$ and E be a vector bundle of rank r over X defined over $\mathbb{F}_q$, $E^\vee$ be the dual bundle of E. We denote by $\mu: (x,y) \to \langle x,y\rangle$ the natural pairing $E \times_X E^\vee \to \mathbb{A}^1$, where $\mathbb{A}^1$ is an affine line over $\mathbb{F}_q$. We fix a non-trivial additive character $\psi: F_q \to \bar{\mathbb{Q}}_\ell^*$. Consider an Artin-Schreier covering $h: \mathbb{A}^1 \to \mathbb{A}^1$, $t \to t^q - t$, and let $h_*\bar{\mathbb{Q}}_\ell$ be the direct image sheaf of $\bar{\mathbb{Q}}_\ell$ on $\mathbb{A}^1$. Then the Galois group $\mathbf{F}_q$ acts on $h_*\bar{\mathbb{Q}}_\ell$, and we denote by $\mathcal{L}_\psi$ the subsheaf of $h_*\bar{\mathbb{Q}}_\ell$ on which $\mathbb{F}_q$ acts as ψ^{-1}. $\mathcal{L}_\psi$ is a local system of rank 1 on $\mathbb{A}^1$. Consider the diagram,

$$\begin{tikzcd} & E \times_X E^\vee \arrow[dl, "p"'] \arrow[dr, "p^\vee"] & \\ E & & E^\vee \end{tikzcd}$$

where p (resp. $p^\vee$) is a natural projection from $E \times_X E^\vee$ to E (resp. $E^\vee$). Following Deligne, we shall define a functor $\mathcal{F} = \mathcal{F}_\psi$: $\mathcal{D}_c^b(E) \to \mathcal{D}_c^b(E^\vee)$ for each $K \in \mathcal{D}_c^b(E)$ by

$$(16.1.1) \qquad \mathcal{F}(K) = p^\vee_!(p^* K \otimes \mu^* \mathcal{L}_\psi)[r] \quad .$$

$\mathcal{F}(K)$ is called the Fourier transform of K. $\mathcal{F}(K)$ is independent of the field extension $\mathbb{F}_q \to \mathbb{F}_{q^n}$ if we replace ψ by $\psi \circ \mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}$.

16.2. Fourier transforms $\mathcal{F}$ can be regarded as a natural generalization of the classical Fourier transforms of functions in the following sense. Let E be a vector space defined over $\mathbb{F}_q$ and $E^\vee$ be its dual space, $(x,y) \to \langle x,y \rangle$, $E \times E^\vee \to \mathbb{A}^1$ be the natural pairing. Assume $K \in \mathcal{D}_c^b(E)$ is F-stable and we fix an isomorphism $F: F^* K \xrightarrow{\sim} K$. Then $\mathcal{F}K \in \mathcal{D}_c^b(E^\vee)$ is also F-stable, and we denote by $F: F^*(\mathcal{F}K) \xrightarrow{\sim} \mathcal{F}K$ the isomorphism induced from F for K. Let χ_K, $\chi_{\mathcal{F}K}$ be the characteristic functions of K, $\mathcal{F}K$ with respect to these F. Then, by using the Lefschetz fixed point formula and the proper base change theorem, we see easily that

$$(16.2.1) \qquad \chi_{\mathcal{F}K}(y) = (-1)^r \sum_{x \in E(\mathbb{F}_q)} \chi_K(x)\psi(\langle x,y \rangle) \qquad (y \in E^\vee) \quad .$$

Hence $\chi_{\mathcal{F}K}$ is the Fourier transform of χ_K in the classical sense.

16.3. The following properties are easily verified.

(16.3.1) Fourier transforms commute with the base change of X in the following sense. Let $f: Y \to X$ be a morphism and let $Y \times_X E \to Y$ be the vector bundle over Y, and $Y \times_X E^\vee \to Y$ be its dual bundle. We write $f_E: Y \times_X E \to E$, $f_{E^\vee}: Y \times_X E^\vee \to E^\vee$. Then for each $K \in \mathcal{D}_c^b(Y \times_X E)$,

$$(f_{E^\vee})_!(\mathcal{F}_Y K) \simeq \mathcal{F}_X((f_E)_! K) \quad ,$$

where $\mathcal{F}_X$ (resp. $\mathcal{F}_Y$) is the Fourier transform with respect to X (resp. Y).

(16.3.2) Let $\rho: E \to F$ be a map of vector bundles over X, and let $\rho^{\vee}: F^{\vee} \to E^{\vee}$ be its transpose. Then for each $K \in \mathcal{D}_c^b(E)$,

$$\mathcal{F}(\rho_! K) \simeq (\rho^{\vee})^* \mathcal{F}K[\text{rank } F - \text{rank } E] .$$

(16.3.3) $\mathcal{F} \circ \mathcal{F} = a^* \bullet (-r)$, where $a: E \to E$ is the antipodal map $x \to -x$, and $(.)$ is the Tate twist.

The following result, due to Laumon, is essential for our later application and makes it possible to consider Fourier transforms on the category of perverse sheaves.

16.4. **Theorem** (Katz-Laumon [19]). Fourier transforms commute with the Verdier dual operator, i.e.,

$$D_{E^{\vee}} \circ \mathcal{F}_{\psi} \simeq \mathcal{F}_{\psi^{-1}} \circ D_E(r) \quad ,$$

where D_E (resp. $D_{E^{\vee}}$) is the Verdier dual operator on E (resp. $E^{\vee}$).

16.5. **Corollary.** If K is a perverse sheaf on E, $\mathcal{F}K$ is a perverse sheaf on $E^{\vee}$. $\mathcal{F}: \mathcal{M}E \to \mathcal{M}E^{\vee}$ induces an equivalence of categories. In particular, if K is a simple perverse sheaf on E, so is $\mathcal{F}K$ on $E^{\vee}$.

§17. The Springer correspondence and Fourier transforms

17.1. Let $\underline{g}$ be the Lie algebra of a reductive group G defined over $\mathbf{F}_q$, and let $\mathcal{B}_A$ be the variety of Borel subgroups of G whose Lie algebra contains a nilpotent element A in $\underline{g}$. The original construction, due to Springer [39], of Springer representations of W on $H^i(\mathcal{B}_A, \bar{\mathbf{Q}}_\ell)$ should be understood from the view point of Fourier transforms of perverse sheaves on $\underline{g}$, although it is not explicitly written there. The reformulation of Springer's construction in terms of Fourier transforms may be found in Brylinski [7]. In this section, we only show that Borho-MacPherson's theorem can be obtained very naturally by applying the Fourier transform to the complex K on $\underline{g}$ induced from Lie T as in §4.

17.2. We assume that p is large enough so that one can identify $\underline{g}$ with the dual space $\underline{g}^{\vee}$ by making use of the Killing form. We follow the notation in 5.2. In particular,

$$\tilde{\underline{g}} = \{(X,gB) \in \underline{g} \times G/B \mid \mathrm{Ad}(g^{-1})X \in \mathrm{Lie}\ B\}$$

and $\pi' : \tilde{\underline{g}} \to \underline{g}$ is the first projection. Let $\underline{b}$ be the Lie algebra of B and let $\underline{b} = \underline{t} \oplus \underline{u}$, where $\underline{t}$ is the Lie algebra of T and $\underline{u}$ is the Lie algebra of U. We assume $\underline{b}$ and $\underline{t}$ are F-stable. Now $\tilde{\underline{g}} \simeq G \times^B \underline{b}$ is a vector bundle over $\mathcal{B}$, and we can regard $\tilde{\underline{g}}$ as a subbundle of the trivial bundle $\underline{g} \times \mathcal{B}$ over $\mathcal{B}$. Let $\tilde{\underline{g}}_{nil} = \pi'^{-1}(\underline{g}_{nil})$ be the pull back of the nilpotent variety $\underline{g}_{nil}$ under π'. Then $\tilde{\underline{g}}_{nil} \simeq G \times^B \underline{u}$ is also a vector bundle over $\mathcal{B}$. We denote by $i: \tilde{\underline{g}} \hookrightarrow \underline{g} \times \mathcal{B}$, $j: \tilde{\underline{g}}_{nil} \hookrightarrow \underline{g} \times \mathcal{B}$ the natural injections. We shall start with the following lemma.

17.3. Lemma. $\mathcal{F}_{\mathcal{B}}(j_*\bar{\mathbb{Q}}_\ell) \simeq i_*\bar{\mathbb{Q}}_\ell[r_0](-\nu_G)$,

<u>where</u> $\mathcal{F}_{\mathcal{B}}$ <u>is the Fourier transform with respect to the trivial bundle</u> $\underline{g} \times \mathcal{B} \to \mathcal{B}$, <u>and</u> $r_0 = \mathrm{rank}\ G$, $\nu_G = \dim \mathcal{B}$.

<u>Proof</u>. First consider the Fourier transform $\mathcal{F}$ on the vector bundle $G \times^B \underline{u}$ over $\mathcal{B}$ and its dual $G \times^B(\underline{g}/\underline{b})$. For a constant sheaf $\bar{\mathbb{Q}}_\ell$ on $G \times^B \underline{u}$, we claim that $\mathcal{F}\bar{\mathbb{Q}}_\ell$ turns out to be a constructible sheaf on $G \times^B(\underline{g}/\underline{b})$ whose support is in $G \times^B\{0\}$, and its restriction to $G \times^B\{0\}$ is a single complex $\bar{\mathbb{Q}}_\ell[-\nu_G](-\nu_G)$. In fact, it is easily checked that the support of $\mathcal{F}\bar{\mathbb{Q}}_\ell$ lies in $G \times^B\{0\}$. It is also verified that $\mu^*\mathcal{L}_\psi$ is a constant sheaf $\bar{\mathbb{Q}}_\ell$ on $(p^\vee)^{-1}(G \times^B\{0\})$, where $p^\vee$ is the projection $(G \times^B \underline{u}) \times_{\mathcal{B}} (G \times^B(\underline{g}/\underline{b})) \to G \times^B(\underline{g}/\underline{b})$. Since $(p^\vee)^{-1}(G \times^B\{0\})$ is a vector bundle over $G \times^B\{0\}$ of rank ν_G, we get the claim.

The lemma now follows from (16.3.2) applied to the map of vector bundles $j: G \times^B \underline{u} \to G \times^B \underline{g} = \underline{g} \times \mathcal{B}$ over $\mathcal{B}$ and its transpose $j^\vee: G \times^B \underline{g} \to G \times^B(\underline{g}/\underline{b})$ since $\dim \underline{g} - \dim(\underline{g}/\underline{b}) - \nu_G = r_0$.

Let $K = \pi'_*\bar{\mathbb{Q}}_\ell[\dim G] \in \mathcal{M}\underline{g}$ be the induced complex as in §4. By applying (16.3.1) to the base change $\mathcal{B} \to \mathrm{Spec}(k)$, we have

17.4. Theorem. $\mathcal{F}(K[-r_0]|\underline{g}_{nil}) \simeq K(-\nu_G)$,

<u>where</u> $K[-r_0]|\underline{g}_{nil}$ <u>is regarded as an object in</u> $\mathcal{D}^b_c(\underline{g})$ <u>by extending</u> 0 <u>on</u> $\underline{g} - \underline{g}_{nil}$, <u>and</u> $\mathcal{F}$ <u>is the Fourier transform</u> $\mathcal{D}^b_c(\underline{g}) \to \mathcal{D}^b_c(\underline{g})$.

Note that K is a semisimple perverse sheaf on $\underline{g}$ and its endomorphism algebra End K is isomorphic to the group algebra

$\bar{\mathbb{Q}}_\ell[W]$ (cf. 6.5). Since $\mathcal{F}$ gives a category equivalence $\mathcal{M}\underline{g} \to \mathcal{M}\underline{g}$, we see that $\mathcal{F}^{-1}K$ is also a semisimple perverse sheaf on $\underline{g}$ and that $\mathrm{End}\,\mathcal{F}^{-1}K \simeq \bar{\mathbb{Q}}_\ell[W]$. Thus we obtain a new proof of (the Lie algebra analogue of) Borho-MacPherson's theorem (Theorem 6.2) without using the decomposition theorem (3.4).

17.5. **Corollary** (Borho-MacPherson's theorem).
$K[-r_0]|\underline{g}_{nil}$ <u>is a semisimple perverse sheaf on</u> $\underline{g}_{nil}$ <u>and</u>

$$\mathrm{End}\,(K[-r_0]|\underline{g}_{nil}) \simeq \bar{\mathbb{Q}}_\ell[W]\ .$$

17.6. Let $K \simeq \bigoplus_{E \in W^\wedge} E \otimes K_E$ be the decomposition of K into simple perverse sheaves as in 6.5. Then $\mathcal{F}^{-1}(K_E)$ is a simple perverse sheaf on $\underline{g}$ which is a direct summand of $K[-r_0]|\underline{g}_{nil}$. On the other hand , we know already by Theorem 6.2 that $K_E[-r_0]|\underline{g}_{nil}$ is also a simple perverse sheaf on $\underline{g}$. However, these two simple objects do not coincide in general. In fact, we have

17.7. **Proposition.** $\mathcal{F}^{-1}(K_E) \simeq K_{\varepsilon\otimes E}[-r_0]|\underline{g}_{nil}$,
<u>where</u> ε <u>is the sign representation of</u> W.

<u>Proof</u>. It is enough to show that the two kinds of W-actions on $K[-r_0]|\underline{g}_{nil}$, one is obtained from K by restricting to $K|\underline{g}_{nil}$, the other is obtained from K by the Fourier transform, coincide up to the multiplication by sign representation. For this, we shall make use of the uniqueness property of Springer representations, $\mathrm{End}\,(K[-r_0]|\underline{g}_{nil}) \simeq \mathrm{End}\,H^*(\mathcal{B},\bar{\mathbb{Q}}_\ell)$, (cf. Remark 6.4 (i)). Hence we have only to compare the W-actions on the stalks at $0 \in \underline{g}$ of K and $\mathcal{F}^{-1}K$. As was verified in Proposition 5.4, the W-action on $\mathcal{H}^i_0(K[-r_0]) \simeq H^{i+2\nu}(\mathcal{B},\bar{\mathbb{Q}}_\ell)$ induced from the W-action on K, is nothing but the classical action of W on $H^{i+2\nu}(\mathcal{B},\bar{\mathbb{Q}}_\ell)$, (here we put $\nu = \nu_G$). On the other hand, since $\mu^*\mathcal{L}_\psi$ is a constant sheaf on $\underline{g} \times \{0\} \subset \underline{g} \times \underline{g}^\vee$, we see easily that

$$\begin{aligned}
\mathcal{H}^i_0(\mathcal{F}^{-1}K) &\simeq \mathbb{H}^i_c(\underline{g}, \pi_*^!\bar{\mathbb{Q}}_\ell)[2n] \qquad (n = \dim G)\\
&\simeq H^{i+2n}_c(\tilde{\underline{g}}, \bar{\mathbb{Q}}_\ell)\\
&\simeq H^{-i}(\tilde{\underline{g}}, \bar{\mathbb{Q}}_\ell) \qquad \text{(by Poincaré duality)}\\
&\simeq H^{-i}(\mathcal{B}, \bar{\mathbb{Q}}_\ell),
\end{aligned}$$

and the W-action on $\mathcal{H}_0^i(\mathcal{F}^{-1}K)$ is nothing but the classical action of W on $H^{-i}(\mathcal{B}, \bar{\mathbb{Q}}_\ell)$. Now, W-module $H^{i+2\nu}(\mathcal{B},\bar{\mathbb{Q}}_\ell)$ is isomorphic to $H^{-i}(\mathcal{B},\bar{\mathbb{Q}}_\ell)$ tensored by sign representation by 5.1. This proves our assertion.

§18. Fourier transforms of admissible complexes on a Lie algebra

18.1. As was seen in the preceding section, the Springer correspondence is explained quite naturally in terms of the Fourier transform of perverse sheaves on the Lie algebra. So far the generalized Springer correspondence may be formulated as well to the Lie algebra case, it is natural to expect that the theory of the previous section should be extended to the case of generalized Springer correspondence. In fact, as mentioned in Introduction, this was carried out by Lusztig [28]. In this section we shall formulate Lusztig's result and give an outline of the proof.

18.2. We shall start with the definition of admissible complexes in $\underline{g}$. Let $\mathcal{N}_{\underline{g}}$ be the set of pairs $(\mathcal{O},\mathcal{E})$, where $\mathcal{O}$ is a nilpotent G-orbit in $\underline{g}$ and $\mathcal{E}$ is an irreducible G-equivariant local system on $\mathcal{O}$. The definition 7.2 of cuspidal pairs makes sense also for the Lie algebra case, and we say that $(\mathcal{O},\mathcal{E}) \in \mathcal{N}_{\underline{g}}$ is a (nilpotent) cuspidal pair if it satisfies the similar condition as in 7.2. The logarithm map $\log\colon G \to \underline{g}$ (cf. 5.3) induces a bijection $\mathcal{N}_G \to \mathcal{N}_{\underline{g}}$, $(C,\mathcal{E}) \to (\mathcal{O},\mathcal{E})$, where $u \in C$, $\log(u) = X \in \mathcal{O}$ and $\mathcal{E} \in A_G(u)^\wedge = A_G(X)^\wedge$. Then under this correspondence, $(C,\mathcal{E})$ is a cuspidal pair if and only if $(\mathcal{O},\mathcal{E})$ is a cuspidal pair, and in fact,

$$\log^* IC(\bar{\mathcal{O}},\mathcal{E}) \simeq IC(\bar{C},\mathcal{E}) \quad .$$

Thus the properties of $IC(\bar{\mathcal{O}},\mathcal{E})$ are all deduced from that of $IC(\bar{C},\mathcal{E})$. In particular,

(18.2.1) There exist at most one cuspidal complex $K_0 = IC(\bar{\mathcal{O}},\mathcal{E})[\dim \mathcal{O}]$ on which the center of G acts by a prescribed character.

(18.2.2) Cuspidal complex K_0 as above is clean, i.e., $K_0|(\bar{\mathcal{O}} - \mathcal{O}) = 0$, or equivalently, $K_0 \simeq \mathcal{E}[\dim \mathcal{O}]$.

Next we give a definition of (not necessarily nilpotent) cuspidgiven in Remark 8.3. Let $\underline{g} = \underline{z} \oplus \underline{g}'$, where $\underline{z}$ is the center of $\underline{g}$ and $\underline{g}'$ is the Lie algebra of $G/Z^0(G)$. We denote by $\mathcal{M}_G(\underline{g})$ the category of G-equivariant perverse sheaves on $\underline{g}$, which is a full subcategory of $\mathcal{M}\underline{g}$. A complex $K \in \mathcal{M}_G(\underline{g})$ is said to be cuspidal if it is of the form $K_1 \boxtimes K_2$, where K_2 is a (nilpotent) cuspidal complex in $\underline{g}'$ and K_1 is a perverse sheaf on $\underline{z}$ which is (up to shift) a local system of the form $h^*\mathcal{L}_\psi$, where $h: \underline{z} \to k$ is a linear form and $\mathcal{L}_\psi$ is as in §17.

18.3. Admissible complexes in $\underline{g}$ are defined in a similar way as in 8.1 and 8.4 with suitable modifications for $Y, X, \ldots$, by using $\Sigma = \underline{z} + \mathcal{O}$. Here we give a description in terms of the induction of complexes (cf. Remark 8.6) as follows. Let P be a parabolic subgroup of G with a Levi subgroup L and the unipotent radical U_P. Let $\underline{p}$, $\underline{\ell}$ and $\underline{u}_P$ be the corresponding Lie algebras, and $\pi_P: \underline{p} \to \underline{\ell}$ be the canonical projection. We consider the following diagram

$$\underline{\ell} \xleftarrow{\bar{\pi}} V_1 \xrightarrow{\pi'} V_2 \xrightarrow{\pi''} \underline{g} \quad ,$$

where

$$V_1 = \{(X, g) \in \underline{g} \times G \mid \mathrm{Ad}(g^{-1})X \in \underline{p}\}$$

$$V_2 = \{(X, gP) \in \underline{g} \times G/P \mid \mathrm{Ad}(g^{-1})X \in \underline{p}\}$$

and $\quad \pi''(X, gP) = X, \quad \pi'(X, g) = (X, gP), \quad \bar{\pi}(X,g) = \pi_P(\mathrm{Ad}(g^{-1})X)$.

As in the same way mentioned in Remark 8.6, we can define an induction functor from $\mathcal{M}_L(\underline{\ell})$ to $\mathcal{D}^b_c(\underline{g})$, which we denote by i_P^G. It is shown, as in the case of groups (cf. 8.5), if K_0 is a cuspidal complex on $\underline{\ell}$, the induced complex $i_P^G(K_0)$ is a G-equivarinat semisimple perverse sheaf on $\underline{g}$. The set of admissible complexes $A(\underline{g})$ on $\underline{g}$ is defined as the set of complexes A which is a direct summand of $i_P^G(K_0)$ for various K_0 and L.

It is proved as in [26, I, 4.2, (4.3.2)] that

(18.3.1) i_P^G is transitive. Hence if $A \in A(\underline{\ell})$, then $i_P^G(A)$ is a semisimple perverse sheaf, and each direct summand is in $A(\underline{g})$ and is not cuspidal.

18.4. A simple object of $\mathcal{M}_G(\underline{g})$ is said to be <u>orbital</u> if it is of the form $IC(\bar{\mathcal{O}},\mathcal{E})[\dim \mathcal{O}]$ for some G-orbit $\mathcal{O}$ in $\underline{g}$ and for a G-equivariant local system $\mathcal{E}$ on $\mathcal{O}$. Note that the Fourier transform $\mathcal{F}$ induces an equivalence of categories $\mathcal{F}: \mathcal{M}_G(\underline{g}) \to \mathcal{M}_G(\underline{g})$. A simple object of $\mathcal{M}_G(\underline{g})$ is said to be antiorbital if it is of the form $\mathcal{F}A$ for an orbital object A in $\mathcal{M}_G(\underline{g})$. We can now state

18.5. **Theorem** (Lusztig [28]). (i) *Let* $A \in \mathcal{M}_G(\underline{g})$ *be a simple object. Then* A *is admissible if and only if* A *is antiorbital.*

(ii) *Assume* G *is semisimple. Then an admissible complex* A *is cuspidal if and only if* $\mathcal{F}A \simeq A$. *In particular,* A *is orbital and antiorbital if* A *is cuspidal.*

The following special case would be worth mentioning.

18.6. **Corollary** (the generalized Springer correspondence). *Let* $A(\underline{g})_{nil}$ *be the set of nilpotent admissible complexes. Then* $\mathcal{F}$ *gives a 1-1 correspondence* $A(\underline{g})_{nil} \to \mathcal{N}_{\underline{g}}$ *by* $A \mapsto \mathcal{F}A \simeq IC(\bar{\mathcal{O}},\mathcal{E})[\dim \mathcal{O}]$ *for* $(\mathcal{O},\mathcal{E}) \in \mathcal{N}_{\underline{g}}$.

One of the main ingredients for the proof is the following lemma.

18.7. **Lemma.** *Let* K, K' *be semisimple objects in* $\mathcal{M}_G(\underline{g})$ *and assume that the support of* K *consists of finitely many* G-*orbits. Assume that* K *and* K' *are* F-*stable. Suppose we can choose* $\phi: F^*K \to K$ *and* $\phi': F^*K' \to K'$ *such that* $\chi_{K,\phi^n} = \chi_{K',\phi'^n}$ *for each integer* $n \geq 1$. *Then* $K \simeq K'$.

In fact, the condition on the characteristic functions implies that the irreducible components of maximal dimension of supp K and supp K' coincide each other. Hence, by induction on dim supp K, the lemma is reduced to showing the following; if $\mathcal{E}$ and $\mathcal{E}'$ are the two G-equivariant local systems on a single G-orbit $\mathcal{O}$ stable by F, with the same characteristic functions, with respect to each F^n, then $\mathcal{E}$ is isomorphic to $\mathcal{E}'$. The last statement follows easily from Lang's theorem for G.

18.8. **Remark.** Lemma 18.7 holds in general without the assumption on G-equivalence for perverse sheaves. In fact, one can show the following. Let V be an irreducible variety and K, K' be semisimple perverse sheaves on V, stable by F. We assume that the characteristic functions for K and K' with respect to F^n coincide for each $n \geq 1$. Then $K \simeq K'$. The proof is reduced ,as in

Lemma 18.7, to the case where $K = \mathcal{E}$, $K' = \mathcal{E}'$ are local systems on V. Then, as Laumon pointed out, our assertion follows from Artin-Čebotarev theorem (Theorem 7 and its corollary) in Serre [30].

18.9. We now assume that the theorem holds for any $L \neq G$. We need two lemmas before the proof of the theorem. Let $K = IC(\bar{\mathcal{O}},\mathcal{E})[\dim \mathcal{O}]$ be an orbital object in $\mathcal{M}_G(\underline{g})$, where $X = S + N \in \mathcal{O}$, S is semisimple, N is nilpotent and $[S,N] = 0$. Let $L = Z_G(S)$ and P be a parabolic subgroup of G with Levi subgroup L. Let $K_0 = IC(\bar{\mathcal{O}}_0,\mathcal{E}_0)[\dim \mathcal{O}_0]$ be an orbital object in $\mathcal{M}_L(\underline{\ell})$, where $\mathcal{O}_0$ is the L-orbit of X in $\underline{\ell}$ and $\mathcal{E}_0$ is the local system on $\mathcal{O}_0$ determined by $\mathcal{E}$ on $\mathcal{O}$, (note $A_G(X) \simeq A_L(X)$).

18.10. **Lemma.** <u>Under the above notation</u>, <u>we have</u>

$$K \simeq \mathcal{F}^{-1}\circ i_P^G(\mathcal{F}K_0) \quad .$$

In fact, by our assumption, $\mathcal{F}K_0$ is admissible. Hence $i_P^G(\mathcal{F}K_0)$ and so the right hand side is a semisimple G-equivariant perverse sheaf. Now thanks to Lemma 18.7, we have only to show that charcteristic functions of complexes on both sides coincide for each F^n, which is done without difficulty by the computation of the classical Fourier transform of functions.

18.11. Let L be a Levi subgroup and let $A \in \mathcal{M}_L(\underline{\ell})$ be a cuspidal object. We still assume that the theorem holds for L. Now A can be written as $A = h^*\mathcal{L}_\psi \boxtimes A_1$, where $h: \underline{z}(\underline{\ell}) \to k$ is a linear map on the center $\underline{z}(\underline{\ell})$ of $\underline{\ell}$ and A_1 is a nilpotent cuspidal object on $\underline{\ell}'$. Let $S \in \underline{z}(\underline{\ell})$ be defined by $h(X) = -\langle S,X\rangle$ for any $X \in \underline{z}(\underline{\ell})$. Then we have the following lemma.

18.12. **Lemma.** <u>Assume that</u> $S \in \underline{z}(\underline{g})$. <u>Then</u>

$$\mathcal{F}(i_P^G A) \simeq i_P^G A[-r]\,|\,(S + \underline{g}_{nil}) \quad ,$$

<u>where the right hand side is a complex on</u> $\underline{g}$ <u>extended by</u> 0 <u>outside of</u> $S + \underline{g}_{nil}$, <u>and</u> $r = \dim Z^0(L)$.

This can be proved also by making use of Lemma 18.7. Note that the special case where $S = 0$, $L = T$ and $A = \bar{\mathbb{Q}}_\ell$ is the case treated in the preceding section. Contrast to the argument given there, the above lemma depends on the extensive use of the decomposition theorem.

18.13. We now give a brief sketch of the proof of Theorem 18.5. We may assume that the theorem is already verified for all Levi subgroups $L \neq G$. We can also assume that G is semisimple. The following two facts are easily proved by making use of Lemma 18.10 and Lemma 18.12 together with the general property of induction functors.

(18.13.1) If $K \in \mathcal{M}_G(\underline{g})$ is orbital and non-cuspidal, then $\mathcal{F}K$ is admissible , non-cuspidal.

(18.13.2) If $A \in \mathcal{M}_G(\underline{g})$ is admissible, non-cuspidal, then A is antiorbital,i.e., $\mathcal{F}A$ is orbital.

In view of (18.13.1), (18.13.2), it remains to show that $\mathcal{F}K \simeq K$ for any cuspidal object $K \in A(\underline{g})_{nil}$. For this, it is enough to show that

(18.13.3) If K is cuspidal in $A(\underline{g})_{nil}$, then $\operatorname{supp} \mathcal{F}K \subset g_{nil}$.

In fact, if (18.13.3) holds, then $\mathcal{F}K$ is a G-equivariant simple perverse sheaf on $\underline{g}_{nil}$ up to shift, so is orbital in $\mathcal{M}_G(\underline{g})$. Then $\mathcal{F}K$ is necessarily cuspidal. In fact, if $\mathcal{F}K$ is non-cuspidal, then $\mathcal{F}\mathcal{F}K \simeq a^*K$ is admissible and non-cuspidal by (18.13.1). Since a^* permutes non-cuspidal admissible complexes, K becomes non-cuspidal, a contradiction. We now see that K in $\mathcal{M}_G(\underline{g})$ is cuspidal if and only if $\mathcal{F}K$ is cuspidal. Since the Fourier transform $\mathcal{F}$ preserves the action of the center of G on K, we have $\mathcal{F}K \simeq K$ by virtue of (18.2.1).

18.14. We shall prove (18.13.3) by making use of some properties of generalized Green functions discussed in §15. Let I be the set of all pairs $(\mathcal{O},\mathcal{E})$, where $\mathcal{O}$ is an arbitrary G-orbit in $\underline{g}$ and $\mathcal{E}$ is an irreducible G-equivariant local system on $\mathcal{O}$. We denote by $K_i = IC(\bar{\mathcal{O}},\mathcal{E})[\dim \mathcal{O}] \in \mathcal{M}_G(\underline{g})$ the object corresponding to $i = (\mathcal{O},\mathcal{E}) \in I$. We fix an $\mathbb{F}_q$-structure on G and on $\underline{g}$ such that the form $\langle\ ,\ \rangle$ is defined over $\mathbb{F}_q$. For each $i = (\mathcal{O},\mathcal{E}) \in I^F$, we choose an isomorphism $\phi_0 \colon F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ and define a function $\underline{Y}_i \colon \underline{g}^F \to \bar{\mathbb{Q}}_\ell$ by

$$\underline{y}_i(X) = \begin{cases} \mathrm{Tr}(\phi_0, \mathcal{E}_X) & \text{if } X \in \mathcal{O}^F, \\ 0 & \text{if } X \in \underline{g}^F - \mathcal{O}^F. \end{cases}$$

Then as in §15, $\underline{y}_i$ $(i \in I^F)$ form a basis for the vector space $\underline{V}$ of all G^F-invariant functions $\underline{g}^F \to \bar{\mathbb{Q}}_\ell$.

For each K_i $(i \in I^F)$, we fix an isomorphism $\phi_i: F^*K_i \simeq K_i$ and define a function $\underline{x}_i: \underline{g}^F \to \bar{\mathbb{Q}}_\ell$ by $\underline{x}_i = \chi_{K_i, \phi_i}$. As in §15, $\underline{x}_i$ $(i \in I^F)$ also form a basis for $\underline{V}$.

Let $(\ , \ )$ be a non-degenerate bilinear form on $\underline{V}$ defined by

$$(f, f') = \sum_{X \in \underline{g}^F} f(X)f'(X) \qquad (f, f' \in \underline{V}) .$$

Let I_0 be the set of all $i \in I$ such that K_i is cuspidal. Then we have

(18.14.1) $\quad \underline{x}_h = c_h \underline{y}_h \quad$ for $h \in I_0^F$, $\quad (c_h \in \bar{\mathbb{Q}}_\ell^*)$.

(18.14.2) $\quad (\underline{y}_h, \underline{y}_i) = 0 \quad$ for $h \in I_0^F$, $i \in I^F - I_0^F$.

In fact, (18.14.1) follows from thr fact that cuspidal complexes are clean (18.2.3). For (18.14.2), note that if $h = (\mathcal{O}, \mathcal{E}) \in I_0$, then $\mathcal{O}$ is a nilpotent orbit. Thus the statement is clear for $i \in I$ which corresponds to a non-nilpotent orbit. If $i \in I^F - I_0^F$ corresponds to a nilpotent orbit, (18.14.2) follows from Theorem 15.5.(ii).

(18.14.3) $\quad (\underline{y}_h, \underline{x}_i) = 0 \quad$ for $h \in I_0^F$, $i \in I^F - I_0^F$.

In fact, by Theorem 15.5 (ii), $\underline{x}_i | \underline{g}^F_{nil}$ is a linear combination of $\underline{y}_j$ for $j \in I^F - I_0^F$. So, (18.14.3) follows from (18.14.2).

For each i, let $\hat{\underline{x}}_i: \underline{g}^F \to \bar{\mathbb{Q}}_\ell$ be the function defined as the characteristic function of $\mathcal{F}K_i$ with respect to the isomorphism $F^*(\mathcal{F}K_i) \simeq \mathcal{F}K_i$ induced from $\phi_i: F^*K_i \simeq K_i$ used in the defintion of $\underline{x}_i$. Thus $\hat{\underline{x}}_i$ is given by the classical Fourier transform of $\underline{x}_i$ as in (16.2.1). We have

18.14.4) $\quad (\underline{y}_h, \hat{\underline{x}}_i) = 0 \quad$ for $h \in I_0^F$, $i \in I^F - I_0^F$.

In fact, by (18.13.2), $\mathcal{F}K_i$ is a non-cuspidal orbital object. Hence, $\hat{\underline{x}}_i | \underline{g}^F_{nil}$ is a linear combination of $\underline{y}_j$ $(j \in I^F - I_0^F)$ by Theorem 15.5 (ii), and (18.14.2) can be applied.

We now define a Fourier transform $f \to \hat{f}$, on $\underline{V}$ by the formula (16.12.1). Then $f \to \hat{f}$ preserves $(\, , \,)$ up to scalar. Hence (18.14.4) implies that $(\hat{\underline{y}}_h, \hat{\underline{x}}_i) = 0$ for $h \in I_0^F$, $i \in I^F - I_0^F$. Since a^* permutes non-cuspidal admissible complexes, the functions $\hat{\underline{x}}_i$ $(i \in I^F - I_0^F)$ coincide with the functions $\underline{x}_i$ $(i \in I^F - I_0^F)$ up to scalars and up to the order. This implies that the space spanned by $\{\hat{\underline{y}}_h \mid h \in I_0^F\}$ is the orthogonal complement of the space spanned by $\{\underline{x}_i \mid i \in I^F - I_0^F\}$, and so coincides with the space spanned by $\{\underline{y}_h \mid h \in I_0^F\}$. It follows from this that $\hat{\underline{y}}_h$ is a linear combination of $\underline{y}_k$, $k \in I_0^F$, and in particular, $\hat{\underline{y}}_h$ vanishes on $\underline{g}^F - \underline{g}^F_{nil}$. Thus by (18.14.1), $\hat{\underline{x}}_h$ vanishes on $\underline{g}^F - \underline{g}^F_{nil}$. This holds for each characteristic function of $\mathcal{F}K_h$ with respect to F^n for any integer $n \geqq 1$. Thus we can conclude that $\operatorname{supp} \mathcal{F}K_h \subset \underline{g}_{nil}$.

18.15. **Remark.** The theorem asserts that the characteristic functions of cuspidal objects in $A(\underline{g})_{nil}$ are stable, up to scalar, by the Fourier transform. In the case of G_2, F_4 or E_8, there exists exactly one such function, and these functions are closesy related to the generalized Gauss sums associated to certain prehomogeneous vector spaces (see Kawanaka [20]).

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