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# HIGHER DIMENSIONAL COMPLEX GEOMETRY

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A Summer Seminar at the University of Utah,
Salt Lake City, 1987

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## Introduction

These notes originated at a seminar that was held during July and August of 1987 at Salt Lake City. The original aim of the seminar was to get an overview of the following three topics:

1. Recent advances in the classification program of three (and higher) dimensional algebraic varieties.
2. Existence of rational curves and other special subvarieties.
3. Existence and nature of special metrics on varieties.

We also hoped to then go further and study the relationships between these three approaches. Time however proved to be insufficient to complete even the limited program.

The first part of the program was considered in detail. In that part, the central theme is the investigation of varieties on which the canonical class is not numerically effective. For smooth threefolds this was done in [M1] and later extended considerably. The original approach of [M1] is geometrically very clear, therefore it is given in detail. Subsequent generalizations were also considered at length.

Considerable attention was paid also to the study of special curves on hypersurfaces and some related examples. There seems to be a lot of experimental evidence to indicate that there is a very close relationship between the Kodaira dimension of a threefold (a property of a threefold from classification theory) and the existence of rational curves. These problems are very interesting but they also seem quite hard. Our contribution in this direction is mostly limited to presenting some examples and conjectures.

In the second direction, one of the questions we were interested in was that of understanding rational curves on quintic hypersurfaces in $\mathbf{P}^4$. Later this was scaled down to understand *lines* on quintic hypersurfaces in $\mathbf{P}^4$, but even this seems a hard problem. We began to understand it more completely only after the seminar had ended (see [J]).

Very little time was left to consider the third direction. We were fortunate to have a series of lectures, but we could not pursue this interesting and important direction in any detail.

The style of the seminars was very informal. We tried to keep them discussion-and-problem oriented. Notes were taken by H. Clemens who typed them up by the next day. These notes constituted the first version of the present text. During the seminar and afterwards, these notes were considerably revised, cut, expanded and edited. During this process we tried to keep the original informality of the talks alive.

The regular participants of the seminar were J. Jimenez, T. Luo, K. Matsuki and the three of us. Several other people joined us for various length of time. A hopefully complete list is:

J. Carlson, L. Ein, M. He, Y. Ma, D. Ortland, S. Pantazis, P. Roberts, D. Toledo, S. Turner, and Stephen Yau. We are very grateful for their contribution to the success of the seminar.

We are especially thankful to those people who gave talks. The following is a list of the lectures of a mathematician other than one of the three of us.

| | |
|---|---|
| J. Carlson: | Maximal variations of Hodge structures; |
| L. Ein: | Submanifolds of generic complete intersections in Grassmanians; |
| L. Ein: | A theorem of Gruson-Lazarsfeld-Peskine and a lemma of Lazarsfeld; |
| K. Matsuki: | Cone Theorem; |
| K. Matsuki: | Non-vanishing Theorem; |
| D. Toledo: | Kähler structures on locally symmetric spaces; |
| D. Toledo: | Proof of Sampson's theorem; |
| D. Toledo: | Abelian subalgebras of Lie algebras. |

At the final editing of these notes some talks were left out. This was the fate of the following talks:

| | |
|---|---|
| H. Clemens: | Abel-Jacobi maps; |
| S. Turner: | Elliptic surfaces in characteristic $p$; |
| S. Yau: | Euler characteristic of Chow varieties. |

These talks were about topics that we had no time to pursue further, and therefore they did not fit neatly into the final version of the notes.

Our aim was to keep the notes advanced enough to be of interest even to the specialists, but understandable enough so that a person with a good general background in algebraic geometry would be able to understand and enjoy them. Especially at the beginning, the lectures are rather informal and concentrate on the geometric picture rather than on a proof that is correct in every technical detail. We hope that this informal introduction to [M1] will be helpful. These matters occupy the first two lectures.

The classification theory of surfaces is reviewed from the point of view of threefold theory in Lecture 3. This leads naturally to the next lecture which is an introduction to the study of cones of curves. Lecture 5 discusses the aims of Mori's program in more detail, concentrating mainly on flips, the presence of which is perhaps the most important difference between algebraic geometry in two and in three dimensions. At the end of this lecture, a table compares the basic results in the birational geometry of surfaces and threefolds. Even though the list was selected with bias, the similarities are striking.

Lecture 6 is a little more technical. It discusses the singularities that arise naturally in the study of *smooth* threefolds. These are the three dimensional analogues of the rational double points of surfaces. Their structure is however more complicated and not completely known.

Lecture 7 discusses extensions of the Cone Theorem to relative situations and equivariant settings. In Lecture 8 we give quick proofs of some vanishing theorems that are needed for the proof of the Cone Theorem.

This leads directly to the next big section, which is the proof of the general Cone Theorem. This is done in Lectures 9-13. Here the proofs are (or at least are intented to be) also technically correct. The proof of the final step (given in Lecture 11) is new, and makes it possible to avoid the rather technical relative case. At least for us, this made the proof much clearer.

The end of the first part of these notes is a discussion of flops and flips. If a rational curve on a quintic threefold in $\mathbf{P}^4$ can be contracted, then it can be flopped. Thus understanding flops yields results about rational curves on quintic threefolds in $\mathbf{P}^4$. The simplest question to which this approach leads is:

> Is it true that, if C is a smooth rational curve on a quintic threefold in $\mathbf{P}^4$ which has normal bundle $\mathcal{O}(2)+\mathcal{O}(-4)$, then some multiple of C moves?

There are no such lines, but this situation can occur for plane conics on some special quintic threefolds in $\mathbf{P}^4$. (A negative answer to the question in that case appears in [C3], written after the completion of these notes.)

Two lectures are devoted to flips. Lecture 14 is a general introduction, and Lecture 15 is an essentially complete proof of the local description of a threefold along a contractible rational curve that has negative intersection with the canonical class. This should give a fairly clear idea of the content of the first seven chapters of [M3], and should enable the reader to go directly to Chapter 8 (after reviewing some additional definitions and statements). Then the introduction of [M3] should give a good idea of how the proof proceeds in the final chapters of [M3]. We hope that this introduction will encourage people to study in more detail the complete proof. Lecture 16 is a short discussion of flops. These are much easier than flips and are very well understood.

Lectures 17-20 are devoted to studying Kähler structures on Riemannian locally symmetric spaces. The results are due to J. Carlson and D. Toledo. Building on results of Eels and Sampson, they give unified proofs of some old and some new results. In short, a compact Riemannian locally symmetric space has a Kählerian complex structure only if it is one of the classically known spaces, in which case the complex structure is the expected one. These lectures show one example of the applications of harmonic maps to complex geometry.

For lack of time we could not go into other questions like one of the ones we originally intended to attack:

> Is there a relationship between the Kähler-Einstein metric of a quintic threefold in $\mathbf{P}^4$ along a rational curve and the deformation theory of that rational curve?

The last part is the study of special curves on general hypersurfaces. In short, these results claim that a general hypersurface of high enough degree does not contain any low genus curves. In Lecture 21, earlier results of Clemens are extended to singular curves. The results are very close to being best possible, but unfortunately they fall short of what we would like to have. Therefore in Lecture 22 we can give only a conjectural discussion concerning quintic hypersurfaces in $\mathbf{P}^4$ and abelian varieties. This would be a very interesting direction to pursue. The above results can be extended to complete intersections in Grassmanian varieties; these generalizations are due to, and were presented by, L. Ein. He also reviewed the proof of the Castelnuovo bound for smooth space curves proved by Gruson-Lazarsfeld-Peskin which was used in the previous lecture.

**Note: In Lectures 1-3 21, and 23-24, we work in arbitrary characteristic, however, in the remaining lectures, characteristic 0 is always assumed.**

Once again we would like to express our thanks to all of the people who contributed to the success of the seminar, and to all those, including F. Serrano-Garcia, P. Roberts, T. Luo, and the referee, who made corrections to, and improvements on, these notes. Partial financial support was provided by NSF under Grant numbers DMS-8702680 and DMS-8707320.

## Notes on Terminology

The following is a list of terminology that is getting to be generally accepted in higher dimensional geometry but may not be well known outside the field.

In pre-Bourbaki algebraic geometry it was customary to use maps that were not defined everywhere. These were called rational maps. We use simply the name **map** for them and they are indicated by a dotted arrow $\dashrightarrow$. A **morphism** is an everywhere defined map of schemes. It is denoted by a solid arrow $\longrightarrow$.

A map $g: X \dashrightarrow Y$ between two varieties is called **birational** if it is an isomorphism between dense open subsets. Two varieties are called **birational** if there is a birational map between them. (Note that we deliberately avoid the old expression "birationally isomorphic" since it is confusing.)

A variety $X$ of dimension $n$ is called **rational** (resp. **ruled**) if it is birational to $\mathbf{P}^n$ (resp. $Y \times \mathbf{P}^1$ for some variety $Y$ of dimension $n-1$).

A variety $X$ of dimension $n$ is called **uniruled** if there is an $n-1$ dimensional variety $Y$ and a map $f: Y \times \mathbf{P}^1 \dashrightarrow X$ which is generically surjective. If $n \leq 2$ then this is equivalent to ruledness, but not in higher dimensions.

A Cartier divisor $D$ on a scheme $V$ is called **nef** if, for every complete curve $C$ contained in $V$, the intersection number $C \cdot D$ is non-negative. This notion is usually used only if $V$ is proper.

A Cartier divisor $D$ on a proper irreducible variety $V$ is called **big** if the map given by the linear system $|mD|$ is birational for $m$ sufficiently large.

A **Q-divisor** is a formal linear combination $D = \sum a_i D_i$, where the $a_i$ are rational numbers and the $D_i$ are irreducible Weil divisors. It is called effective if all the $a_i$ are nonnegative.

A divisor (or, more generally, a **Q**-divisor) $D$ is called **Q-Cartier** if some positive integral multiple $mD$ is Cartier. A **Q**-Cartier **Q**-divisor $D$ is called nef (resp. big, ample,...) if $mD$ is nef (resp. big, ample...).

The **index** of a **Q**-Cartier Weil divisor $D$ is the smallest positive integer $m$ such that $mD$ is Cartier. Then, if $kD$ is Cartier, $k$ is a multiple of $m$. The **index** of a variety $X$ is the index of its canonical divisor $K_X$ (provided that it is defined).

A **divisor with simple normal crossings** on a non-singular variety is a sum of non-singular divisors intersecting transversely with each other.

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## Lecture #1: Finding rational curves when $K_X$ is negative

(1.1) This chapter will serve as a warm-up to the first 16 lectures. In it we explore the general theme:

> How do rational curves on a variety influence the birational geometry of that variety?

We will see that the absence of rational curves has some very pleasant consequences. Later this will be turned around, and we will see that certain complications of birational geometry of a variety X are caused precisely by certain special rational curves on X.

The simplest example is in the theory of surfaces:

If X is a smooth proper surface, then there is a non-trivial birational morphism

$$f: X \longrightarrow Y$$

to a smooth surface Y iff X contains a smooth rational curve with self-intersection -1.

One side of this is easy to generalize as follows:

(1.2) **Proposition:** Let X be smooth of any dimension and $f: Y \longrightarrow X$ a proper birational morphism. For any $x \in X$, either $f^{-1}(x)$ is a point or $f^{-1}(x)$ is covered by rational curves.

**Proof:** Let us consider first the case when X is a surface. We resolve the indeterminacies of $f^{-1}$ by successively blowing-up points of X. At each step we introduce a $\mathbf{P}^1$. Thus every $f^{-1}(x)$ is dominated by a union of some of these $\mathbf{P}^1$'s. By Lüroth's theorem, every $f^{-1}(x)$ is a union of rational curves.

The general case can be proved the same way provided we know how to resolve indeterminacies of maps. However a much weaker version of resolution is sufficient. Since we will use (1.2) later only when X is a surface, we only sketch the proof in the higher-dimensional case:

We may assume that Y is normal. By van der Waerden's theorem, the exceptional set of $f$ is of pure codimension one. Let $E \subseteq Y$ be an irreducible component of the exceptional set. At a generic point $e \in E$, $(Y,E)$ is isomorphic to a succession of blow-ups with smooth centers. Thus there is a rational curve C in E that passes through $e$ such that f(C) is a point. Since a rational curve can specialize only to unions of rational curves, there is a rational curve through every point of E.

(1.3) **Corollary**: Let $g: Z \dashrightarrow X$ be a rational map from a smooth variety. Let

$$Y \subseteq X \times Z$$

be the closure of the graph of $g$, and let $q$ and $p$ be the coordinate projections. Let $S \subseteq Z$ be the set of points where $g$ is not regular. Then $q(p^{-1}S)$ is covered by rational curves.

(1.4) **Corollary**: Let $X$ and $Z$ be algebraic varieties, $Z$ smooth and $X$ proper. If there is a rational map

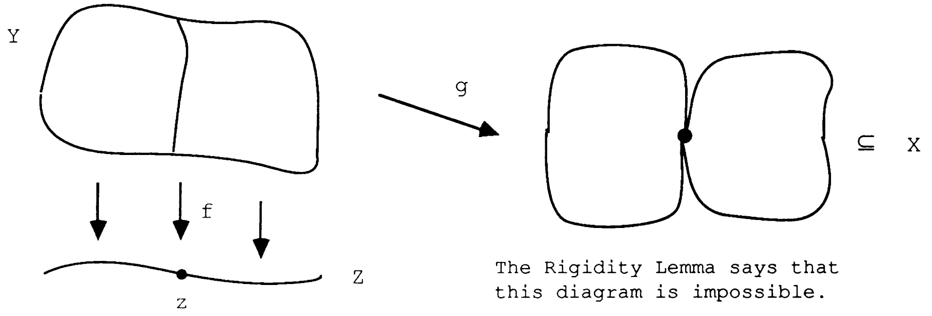
$$g: Z \dashrightarrow X$$

which is not everywhere defined, then $X$ contains a rational curve.

The simplest situation where one could apply this corollary is when $Z$ is a surface which we obtain as a family of curves. In some cases one can assert that a map $g$ as in (1.4) can not be regular:

(1.5) **Rigidity Lemma**: Let $f: Y \longrightarrow Z$ be a proper morphism with connected fibers and assume that $Z$ is connected. If $g: Y \longrightarrow X$ is a morphism and for some $z_0 \in Z$, $g(f^{-1}(z_0))$ is a closed point, then $g(f^{-1}(z))$ is also a closed point for every $z \in Z$.

**Proof**: The set of $z \in Z$ such that $g(f^{-1}(z))$ is a point is clearly closed. Thus it is sufficient to prove that it is also open. Let $U$ be an affine neighborhood of $g(f^{-1}(z_0))$. Then $g^{-1}(U)$ is an open neighborhood of $f^{-1}(z_0)$. Since $f$ is proper, there is a neighborhood $V$ of $z_0$ such that, whenever $z \in V$, then $g^{-1}(U) \supset f^{-1}(z)$. Thus $g(f^{-1}(z))$ is contained in $U$. Since this set is also proper and connected, it is a single point.




The Rigidity Lemma says that this diagram is impossible.

(1.6) **Corollary**: Let X be a proper variety, C a smooth proper curve, $p \in C$ a point, and

$$g_0\colon C \longrightarrow X$$

a non-constant morphism. Assume that there is a non-trivial algebraic family

$$g_t\colon C \longrightarrow X$$

parametrized by a (possibly non-proper) curve $D_0$ such that

$$g_0(p) = g_t(p)$$

for every $t$. Then X contains a rational curve through $g_0(p)$.

**Proof**: We compactify $D_0$ to a proper curve D, and so we have a rational map $g\colon C \times D \dashrightarrow X$. If C itself is rational, then we have our rational curve. Otherwise $g$ must have two-dimensional image, since C can not have a one-parameter family of automorphisms that keep the point p fixed. We claim that $g$ is not a morphism. To see this apply (1.5) to the projection map

$$f\colon C \times D \to C.$$

$f^{-1}(p)$ is mapped to a single point; thus the same holds for every fiber, and the image is one-dimensional, a contradiction. Thus $g$ is not defined somewhere along $\{p\} \times D$. By (1.4), X contains a rational curve. Using (1.3), we see that there is a rational curve through the image of $\{p\} \times D$, that is, through $g_0(p)$.

It is interesting to note that the algebraicity assumption is essential:

(1.7) **Example**: Let E be an elliptic curve and let M be a line bundle of degree $\geq 2$ with generating sections $\sigma$ and $\tau$. In

$$V = M + M,$$

the sections

$$(\sigma,\tau),\ (i\sigma,-i\tau),\ (\tau,-\sigma),\ (i\tau,i\sigma)$$

are everywhere independent over $\mathbf{R}$, thus they generate a "lattice bundle" L over E. Let $X = V/L$ and

$$C = \text{the zero section in } V/L.$$

Then C must move leaving a point fixed by the positivity of the bundle V, yet V/L has no rational curves.

**Conclusion:** The family of deformations of the mapping of C into X (leaving a point of C fixed) has no non-trivial compactifiable subvarieties.

We are ready to formulate and prove the first main result about the existence of rational curves. This first result is of independent interest, even after we consider a later variant which is, in some aspects, considerably sharper.

(1.8) **Theorem**: Let X be a smooth projective variety such that $-K_X$ is ample. Then X contains a rational curve. In fact, through every point of X there is a rational curve D such that

$$D\cdot(-K_X) \leq 1 + \dim X.$$

**Proof**: This will be done in several steps.

(1.9) Step 1: We intend to apply (1.6). Thus we have to find a morphism

$$f: C \longrightarrow X$$

which we will be able to deform. Pick any curve C. If we want to find a rational curve through a given point $x \in X$, then we require C to pass through x and pick $p \in C$ such that its image is x.

(1.10) Step 2: Morphisms f of C into X have a deformation space of dimension

$$\geq (h^0(C, f^*T_X) - h^1(C, f^*T_X)) = f(C)\cdot c_1(X) + (1-g(C))\cdot \dim X$$

by the Riemann-Roch theorem. Since it is dim X conditions to fix the image of the basepoint p under f, morphisms f of C into X sending p to x have a deformation space of dimension

$$\geq (h^0(C, f^*T_X) - h^1(C, f^*T_X)) - \dim X = f(C)\cdot c_1(X) - g(C)\cdot \dim X.$$

Thus whenever the quantity

$$f(C)\cdot c_1(X) - g(C)\cdot \dim X$$

is positive there must be an actual one-parameter family of deformations of the map $f: C \longrightarrow X$ keeping the image of p fixed. By (1.6) therefore, we obtain a rational curve in X through x. We remark that this part of the proof works also for Kähler manifolds, but by (1.7) it fails for arbitrary compact complex manifolds.

(1.11) Step 3: We show how to get $(f(C)\cdot c_1(X) - g(C)\cdot \dim X) > 0$. To do this, we need to get $f(C)\cdot c_1(X)$ big enough. We take cases:

i) $g(C) = 0$. If $f(C)\cdot c_1(X) > 0$, then C moves in X, but we already knew that X has a rational curve through x.

ii) $g(C) = 1$. If $f(C)\cdot c_1(X) > 0$, compose f with the endomorphism of C given by multiplication by the integer n. Then

$$((f \circ n)(C)\cdot c_1(X) - \dim X) = n^2(f(C)\cdot c_1(X)) - \dim X$$

so this time some multiple of C moves (so that one point of some sheet over the image stays fixed).

iii) $g(C) \geq 2$. The problem here is that if, for example, we try to move an m-sheeted unbranched cover of C, we are only guaranteed a deformation space of dimension

$$m[(f(C)\cdot c_1(X) - g(C)\cdot \dim X)] + (m-1)\dim X.$$

This does not necessarily get positive by making $m$ large, even when $f(C)\cdot c_1(X) > 0$.

(1.12) Thus we are in trouble in the case $g(C) > 1$ because C does not admit endomorphisms of high degree. However, there is a situation in which a curve C does in fact admit endomorphisms "of high degree", namely, in finite characteristic. The *Frobenius morphism* is such an endomorphism. We next see how to pass from our original situation to one over a field of characteristic $p > 0$.

(1.13) Step 4: Take a curve C in a smooth manifold X in $\mathbf{P}^n$. First suppose that both C and X are defined by equations with *integral* coefficients:

$$\begin{array}{l} h_1(X_0,\ldots,X_n),\ \ldots,\ h_r(X_0,\ldots,X_n) \text{ define } X \\ c_1(X_0,\ldots,X_n),\ \ldots,\ c_s(X_0,\ldots,X_n) \text{ define } C. \end{array}$$

Let $\mathbf{F}(p)$ be the field with $p$ elements and $\mathbf{F}(p)^\wedge$ its algebraic closure. Then the equations $h_i$ and $c_j$ above define varieties $C_p$ and $X_p$ respectively in the projective space $(\mathbf{F}(p)^\wedge)\mathbf{P}^n$. These varieties are non-singular, and $\dim C_p = 1$, for almost all $p$. The mapping

$$(X_0,\ldots,X_n) \longrightarrow (X_0{}^p,\ldots,X_n{}^p)$$

gives an endomorphism $\mathfrak{f}_p$ of $C_p$, which, although it is injective in a set-theoretic sense, should be thought of as a morphism of degree $p^{\dim C}$. By "generic flatness over Spec $\mathbf{Z}$", $c_1(X_p)$, $g(C_p)$, and $\chi(T_X|C_p)$, are constant for almost all $p$. The dimension of the "base-pointed" deformation space of the morphism

$$\mathfrak{f}_p{}^m : C_p \longrightarrow C_p \longrightarrow X_p$$

has dimension bounded below by

$$p^m(C_p\cdot c_1(X_p)) - g(C_p)\cdot \dim X.$$

So, since $C_p\cdot c_1(X_p)$ is constant (and assumed positive) for almost all $p$, we can pick an $m$ so that the above expression

$$p^m(C_p\cdot c_1(X_p)) - g(C_p)\cdot \dim X$$

is positive for almost all $p$. Then, as in Step 2, we produce a rational curve $R_p$ on $X_p$ for almost all $p$.

(1.14) Suppose now that we are in the general case in which the coefficients of the $h_i$ (defining X in $\mathbf{P^n}$), the $f_j$ (defining C in $\mathbf{P^m}$) and the $g_j$ (defining the graph of the map in $\mathbf{P^n} \times \mathbf{P^m}$) are not integers. In any case, these coefficients generate a finitely generated ring $\mathfrak{R}$ over $\mathbf{Z}$. Let $\mathfrak{p}$ be any maximal ideal in $\mathfrak{R}$. Then $\mathfrak{R}/\mathfrak{p}$ is a finite field (since otherwise we would have a field $\mathbf{Q}[x_1,\ldots,x_r] = \mathbf{Z}[x_1,\ldots,x_r]$ which cannot happen because there are infinitely many prime numbers). So $\mathfrak{R}/\mathfrak{p}$ is isomorphic to $\mathbf{F}(p^k)$, the finite field with $p^k$ elements for some p. In this case, our Frobenius morphism $\mathfrak{f}$ is given by raising the homogeneous coordinates $(X_0,\ldots,X_n)$ of $(\mathbf{F}(p^k)\hat{}\,)\mathbf{P^m}$ to the $p^k$-th power. The rest of the argument proceeds as above, giving us a rational curve $R_{\mathfrak{p}}$, for all closed points $\mathfrak{p}$ in some Zariski open set of Spec $\mathfrak{R}$.

(1.15) Step 5: Now we assume that $c_1(X)$ is ample and that X is embedded by $mc_1(X)$ for some positive integer m. In this step, we wish to replace $R_{\mathfrak{p}}$ with a rational curve $S_{\mathfrak{p}}$ with

$$c_1(X_{\mathfrak{p}}) \cdot S_{\mathfrak{p}} \le \dim X + 1.$$

To do this, notice that, if

$$c_1(X_{\mathfrak{p}}) \cdot R_{\mathfrak{p}} > \dim X + 1,$$

then the morphism from $R_{\mathfrak{p}}$ to $X_{\mathfrak{p}}$ deforms with *two* points fixed in at least a two-parameter family. Since $\mathbf{P^1}$ has only a one-dimensional family of automorphisms leaving two points fixed, the image of $R_{\mathfrak{p}}$ in $X_{\mathfrak{p}}$ must move. As in Step 2, we construct a rational mapping from $D \times R_{\mathfrak{p}}$ into $X_{\mathfrak{p}}$ taking $D \times \{q\}$ to x and $D \times \{q'\}$ to x'. Taking a minimal resolution Z of this map and contracting all curves (in fibres of $Z \longrightarrow D$) that are mapped to a point by $Z \longrightarrow X$, we obtain either that $R_{\mathfrak{p}}$ degenerates somewhere into a sum of two or more curves each of lower degree or that there is a morphism from a $\mathbf{P^1}$-bundle over D into $X_{\mathfrak{p}}$ that sends one section to x and another section to x'.

The latter case is impossible since it would imply negative-definite intersection matrix on the Neron-Severi group of the $\mathbf{P^1}$-bundle. So we must be able to find a rational curve of lower degree as long as $R_{\mathfrak{p}} \cdot (-K_X) > (\dim X+1)$.

(1.16) Step 6: In this last step, we must conclude the existence of a rational curve on the variety X of characteristic zero from the existence of the $R_p$ of bounded degree for almost all $p$. (The general case using $\mathfrak{p}$ in Spec $\mathcal{R}$ is analogous.)

**Principle:** If a homogeneous system of algebraic equations with integral coefficients has a nontrivial solution in $\mathbf{F}(p)$^ for infinitely many $p$ (for a Zariski dense subset of Spec $\mathcal{R}$), then it has a nontrivial solution in any algebraically closed field.

**Proof:** By elimination theory, the existence of a common solution to a system of equations is given by the vanishing of a series of determinants of matrices whose entries are polynomials (with integral coefficients) in the coefficients of the equations. A determinant vanishes if it vanishes mod $p$ for an infinite number of primes $p$.

In our situation, for most $p$ we have homogeneous forms

$$((g_p)_0, \ldots, (g_p)_n)$$

of degree $m(\dim X + 1)$ in $(t_0, t_1)$ giving the map

$$\mathbf{P}^1 \longrightarrow X \subseteq \mathbf{P}^n$$

such that

$$h_i((g_p)_0, \ldots, (g_p)_n) = 0$$

identically in $(t_0, t_1)$ for all $i$. This condition can be expressed as a system of equations in the coefficients of the $g_k$. Since this system has a solution for a Zariski dense subset of the primes $p$, it has a solution in any algebraically closed field by the above principle.

(1.17) Step 7: Finally, we should remark that Steps 2 and 5 allow the construction of a rational curve of degree $\leq (\dim X + 1)$ through *any* pre-given point of X. So, if $c_1(X)$ is positive, X must be covered by an algebraic family of rational curves of degree $\leq (\dim X + 1)$.

(1.18) References: Most of these results are due to Mori[M1]. (1.2) is due to Abhyankar [Ab, Prop.4]. (1.7) is taken from [Bℓ]. The existence of rational curves through any given point is implicit in [M1]; it was first noted explicitly in [Ko1].

**Lecture #2: Finding rational curves when $K_X$ is non-semi-positive**

(2.1) Now let's weaken our hypotheses about X in (1.8). Namely, from now on we only assume that, for some fixed f,

$$c_1(X)\cdot f(C) > 0,$$

rather than assuming the positivity of $c_1(X)$. We also fix a hyperplane section H of X. If

$$(f(C)\cdot c_1(X) - g(C)\cdot \dim X) > 0, \tag{*}$$

then C deforms with one point fixed. As before, this family must degenerate to

$$f'(C) + (\text{sum of rational curves}).$$

As before, to achieve (*), we pass to finite characteristic, and compose f with the m-th power of the Frobenius morphism. For $m \gg 0$, we are able to degenerate $p^m\cdot f(C_p)$ to

$$C_{p,m} + Z_{p,m}, \tag{**}$$

where $Z_{p,m}$ is a sum of rational curves. Notice that the ratio

$$(f(C_p)\cdot c_1(X_p))/(f(C_p)\cdot H_p) = M$$

is constant for almost all p and does not change if we replace f with its composition with a power of Frobenius. If

$$(C_{p,m}\cdot c_1(X_p) - g(C_p)\cdot \dim X_p) > 0,$$

we can move $C_{p,m}$ as before (*without* composing again with the Frobenius morphism). We iterate these moves. Each time the intersection number of $H_p$ with the component corresponding to $C_{p,m}$ goes down, so the process must stop. Thus we reach an equation (**) which is a degeneration of the original $p^m\cdot f(C_p)$ and which has

$$C_{p,m}\cdot c_1(X_p) \le g(C_p)\cdot \dim X_p.$$

Let

$$\begin{aligned} a &= C_{p,m}\cdot c_1(X_p)\\ b &= Z_{p,m}\cdot c_1(X_p)\\ c &= C_{p,m}\cdot H_p\\ d &= Z_{p,m}\cdot H_p. \end{aligned}$$

For large m, (c+d) is large, (a+b)/(c+d) = M, so (a+b) must be large. But a is bounded, so b must get large.

(2.2) **Lemma:** Suppose $c > 0$ and $d > 0$. Then

$$(a+b)/(c+d) \leq \max\{(a/c), (b/d)\}.$$

**Proof**: Suppose $a' = (a/c) \leq b' = (b/d)$. Put $d' = (d/c)$. Then

$$(a'+d'b')/(1+d') \leq b'.$$

(2.3) If $a/c < M$, then $b/d \geq M$, since otherwise we contradict (2.2). For large $m$, if $c$ gets large then indeed we eventually get

$$a/c < M.$$

But if $c$ stays bounded, then $d$ must get large and

$$(a+b)/(c+d)$$

must approach $b/d$. So, given any $\varepsilon > 0$, we can find an $m$ so that

$$(Z_{p,m} \cdot c_1(X_p))/(Z_{p,m} \cdot H_p) > M - \varepsilon.$$

Now the Lemma gives that for some irreducible component $E_p$ of $Z_{p,m}$ we also have the inequality

$$(***) \qquad (E_p \cdot c_1(X_p))/(E_p \cdot H_p) > M - \varepsilon.$$

(2.4) Suppose now that $(E_p \cdot c_1(X_p)) > (\dim X + 1)$. Then, as in (1.10), we can move the rational curve $E_p$ with two points fixed and the moving curve must degenerate somewhere into a sum of two or more distinct rational curves. We use (2.2) again to conclude that the inequality (***) must hold for at least one of the components $E'_p$ of the degeneration. If

$$(E'_p \cdot c_1(X_p)) > (\dim X + 1),$$

$E'_p$ moves and as above we find $E''_p$ for which (***) holds. This process cannot continue indefinitely, since at each step $E_p \cdot H_p$ goes down. So eventually we arrive at a curve (which we again call $E_p$) such that $0 < (E_p \cdot c_1(X_p)) \leq (\dim X + 1)$. So

$$0 < (E_p \cdot H_p) \leq (\dim X + 1)/(M - \varepsilon).$$

Since this bound is independent of $p$, we can reason as in (1.11) to conclude the existence of a rational curve $E$ on the complex projective manifold $X$. If $c_1(X) \cdot E > \dim X + 1$, we can apply (1.10) repeatedly until we find an $E$ with

$$c_1(X) \cdot E \leq \dim X + 1.$$

(2.5) **Remark:** This argument does not allow us to say anything about the position of the rational curves on X. A different argument, however, shows that, through any point of C there is a rational curve.

We can summarize our results in the following

(2.6) **Theorem:** Let X be a smooth projective variety, and let H be an ample divisor on X. Assume that there is a curve $C \subseteq X$ such that $C \cdot (-K_X) \geq \varepsilon(C \cdot H)$ for some $\varepsilon > 0$. Then there is a rational curve $E \subseteq X$ such that

$$(\dim X + 1) \geq E \cdot (-K_X) \geq \varepsilon(E \cdot H).$$

(2.7) References: All these results are in [M1].

## Lecture #3: Surface classification

(3.1) We will now begin to see what finding a rational curve has to do with classification theory of algebraic varieties. We begin by remarking that any algebraic curve X admits a metric of constant curvature, and that, for any Kähler manifold X, $c_1(X)$ is represented by the Ricci form associated to the curvature. Note that, for an algebraic manifold, $K_X = -c_1(X)$.

List:

$c_1(X) > 0 : X = \mathbf{CP}^1$
$c_1(X) = 0 : X = (\mathbf{C}/\text{lattice})$
$c_1(X) < 0 :$ many X.

(3.2) <u>Principle of classification of surfaces:</u>

Surfaces tend to be negatively curved in the sense that the divisor corresponding to $-c_1(X)$ tends to be nef, or even ample. Often using the fact that we can produce a rational curve on a surface X whenever $-c_1(X)$ is not nef, we can make a list of surfaces which are not negatively curved.

(3.3) There are three possible ways to describe the notion of negative curvature:

1) $T_X$ has a metric with negative Ricci curvature.

2) $\Lambda^{\dim X} T_X = \mathcal{O}(-K_X)$ has a metric with negative Ricci curvature. (This is equivalent to 1) by Yau's famous theorem.)

3) $c_1(X) \cdot C < 0$ for all curves C on X.

Notice that 2) always implies 3), but that to obtain 2) from 3) for surfaces, one must show that 3) implies that $(c_1(X))^2 > 0$ and so, by the Nakai-Moishezon criterion, $K_X$ is ample. The proof that

$$(c_1(X))^2 > 0$$

for surfaces for which 3) holds comes *a posteriori* using the classification theory of surfaces, and so is unsatisfactory in some sense.

(3.4) **Question:** Is there a manifold X for which 3) does not imply 2)?

(3.5) **Definition:** A divisor D on X is called **semi-negative** if

$$C \cdot D \leq 0$$

for all curves C on X.

(3.6) **Problem:** Suppose $c_1(X)$ is semi-negative. Does $\det T_X$ admit a metric with semi-negative curvature form?

(3.7) **Exercise:** Produce a line bundle L such that $c_1(L)\cdot C \leq 0$ for all curves C, yet no metric on L has curvature everywhere less than or equal to zero.

(3.8) So now let's start trying to classify surfaces according to the above principle. First assume $c_1(X)$ is not semi-negative. Then there is a curve C on X for which

$$(c_1(X)\cdot C) > 0.$$

So, by what we have done before, we can produce a rational curve E such that, for

$$f\colon E \longrightarrow X,$$

we have

$$0 < (c_1(X)\cdot f(E)) \leq 3 = \dim X + 1.$$

We need to assume a result which we will discuss next time (see (4.7)), namely, that we can take $C = f(E)$ to be "extremal," which roughly means that E generates an edge of the cone NE(X) of effective divisor classes on X.

Case 1: $C^2 < 0$.
So, from the formula

$$(*) \qquad C^2 + C\cdot K_X = 2g(C) - 2,$$

we see that the only possibility is

$$g(C) = 0 \text{ and } C^2 = -1.$$

So C is an exceptional curve of the first kind and we can blow it down to a smooth point. Since we decrease the second Betti number of X each time we do this, we can eventually assume that X has no extremal curves C with $C^2 < 0$.

Case 2: $C^2 = 0$.
So, by (*), $g(C) = 0$, and f is an embedding.
Since $(c_1(X)\cdot C) = 2$, f has at least a four-dimensional family of deformations (by the formula in (1.10)). But C has only a three-dimensional family of automorphisms, so C must actually move. So X is ruled, and the fact that C gives an *edge* of NE(X) means that all fibres of the ruling must be irreducible.

Case 3: $C^2 > 0$.
Next time (Corollary (4.4)), we show that this implies that E lies in the *interior* of NE(X) in the vector space spanned by NE(X). But E is also on an edge. Thus the Picard number of X is one. Let H be an indivisible ample divisor on X. Then $K_X = -aH$ for some $a > 0$.

For the rest of the argument we assume that we are over **C**. The result is true in general but the proof is harder.

By the Kodaira Vanishing Theorem,

$$H^{0,1}(X) = H^{0,2}(X) = 0.$$

Thus H generates $H^2(X; \mathbf{Z})$ modulo torsion, and so by Poincaré duality

$$H \cdot H = 1,$$

and $c_2(X) = 3$. By Noether's formula

$$c_1(X)^2 = 9 \text{ and } K_X = -3H.$$

By the Riemann-Roch formula,

$$\dim|H| = 2.$$

Since $H^2 = 1$, $|H|$ has no basepoints and so defines a morphism to $\mathbf{CP}^2$. This morphism has degree one and separates points, thus it is an isomorphism.

(3.9) Except for the above X, there only exist surfaces X with $c_1(X)$ semi-negative. We list known results about these:

Case 1: $c_1(X) \cdot C = 0$ for all curves C.

It is known that $\mathcal{O}(12K_X)$ is the trivial bundle in this case. Then it can be shown that X is either an abelian surface, a K3-surface, or a finite quotient of one of these two under a free action of a finite group. (If X comes from a K3 surface, then the group in question must be $\mathbf{Z}/2\mathbf{Z}$, since the Euler characteristic of the structure sheaf of a K3-surface is 2.) Some other cases exist in characteristic 2 and 3.

Case 2: $c_1(X) \cdot c_1(X) = 0$ but $c_1(X) \cdot C \neq 0$ for some curve C.

Then it can be shown that X maps to a curve D with elliptic fibres, and that some multiple of $c_1(X)$ is the pull-back of a negative divisor on D. Some other cases exist in characteristic 2 and 3.

Case 3: $c_1(X) \cdot c_1(X) > 0$.

Then it can be shown that, for $m \gg 0$, the divisor $-mc_1(X)$ defines a birational morphism into some projective space. If this contracts a curve, the curve is rational.

(3.10) References: These results are classical. See [GH] for curvature and related topics. Also see [BPV] and the references there for further results. For (3.3), see [Y].

## Lecture #4: The cone of curves, smooth case

(4.1) Our main goal today is to prove the Cone Theorem, which gives, among other things, the existence of the *extremal* rational curves which we used to help classify surfaces in Lecture #3. First, we will motivate things with some definitions and examples.

(4.2) Let X be a non-singular projective variety. Let C be an irreducible curve on X. We denote the homology class of C in $H_2(X;\mathbf{R})$ by [C]. Let

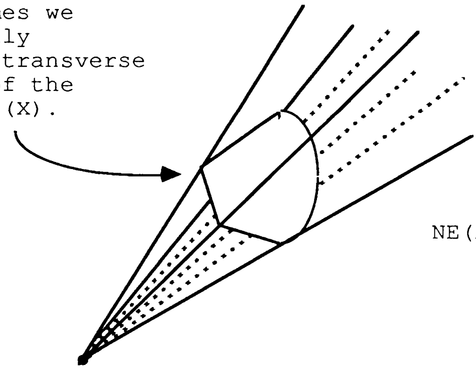
$$NE_{\mathbf{Q}}(X) \quad (\text{resp. } NE(X))$$

be the subset of $H_2(X;\mathbf{R})$ given by

$$\{\Sigma a_i[C_i]: C_i \text{ an irreducible proper curve on X}, \\ a_i \in \mathbf{Q} \ (\text{resp. } a_i \in \mathbf{R}), \text{ and } a_i \geq 0\}$$

Clearly $NE_{\mathbf{Q}}(X)$ is dense in NE(X).

Sometimes we will only draw a transverse slice of the cone NE(X).



$NE(X) \subset H_2(X;\mathbf{R})$

For any divisor D, let

$$D_{>0} = \{\xi: \xi\cdot D > 0\}$$

(similarly for $\geq 0$, $<0$, and $\leq 0$).

Next we work out some examples where X is a surface, H a hyperplane section. Then,

$$\langle NE(X)\rangle \text{ , the closure of } NE(X),$$

lies in $H_{\geq 0}$ and only its vertex **0** lies in $H^{\perp}$ (the real hyperplane annihilated by H).

(4.3) **Lemma:** If $D$ is a divisor on the surface $X$ with $D^2 > 0$, then either $|nD| \neq \emptyset$ or $|-nD| \neq \emptyset$ for $n >> 0$.

**Proof:** By the Riemann-Roch Theorem,

$$h^0(nD) - h^1(nD) + h^0(K_X - nD) = (n^2/2)D^2 - (n/2)D \cdot K_X + \chi(\mathcal{O}_X)$$

$$h^0(-nD) - h^1(-nD) + h^0(K_X + nD) = (n^2/2)D^2 + (n/2)D \cdot K_X + \chi(\mathcal{O}_X).$$

Letting $n$ get large, we notice that the right-hand-side of each equation gets big. But it cannot be true that both $h^0(K_X - nD)$ and $h^0(K_X + nD)$ get big, since the two divisors sum to a fixed linear system $2K_X$.

(4.4) **Corollary:** If $[D] \in \langle NE(X) \rangle$ and if $D^2 > 0$, then $[D]$ lies in $\langle NE(X) \rangle^o$, the interior of $\langle NE(X) \rangle$ in the vector space it spans in $H_2(X; \mathbf{R})$.

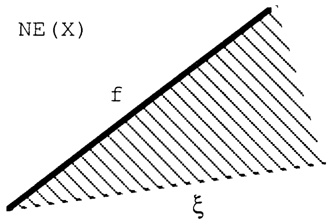
**Proof:** Pick $H$ ample. By (4.3), $H \cdot D > 0$. If $D' \in NE_{\mathbf{Q}}(X)$ is near $D$, then $D'^2 > 0$ and $-D' \cdot H < 0$. $mD'$ is an integral cycle for some $m > 0$, and so we can apply (4.3) to $mD'$ to obtain that $mD' \in NE(X)$ and hence $D' \in NE(X)$. Therefore $[D] \in \langle NE(X) \rangle^o$.

(4.5) **Lemma:** If $C$ is an irreducible curve on $X$ and $C^2 \leq 0$, then $[C] \in \partial NE(X)$. If $C^2 < 0$, $[C] \in$ (edge of $NE(X)$).

**Proof:** Suppose, for irreducible $D$, $D \cdot C < 0$. Then $D = C$. So $NE(X)$ is spanned by $[C]$ and $NE(X) \cap C_{\geq 0}$.

(4.6) Let's now look at our series of examples:

4.6.1) Suppose $X$ is a $\mathbf{P}^1$-bundle over a curve of genus at least 2. Then $NE(X)$ = cone in $\mathbf{R}^2$. Let $f$ be the homology class of the fibre, $\xi$ the other edge.



By (4.4), $\xi^2 \leq 0$. If $\xi^2 < 0$, take a sequence $D_n$ of effective curves converging to a point of $\mathbf{R}_{\geq 0}[\xi]$, and notice that, for $n >> 0$, $D_n{}^2 < 0$. There is an irreducible component $E_n$ of $D_n$ such that $E_n{}^2 < 0$ , hence by the Lemma just above, $E_n \in \mathbf{R}_{\geq 0}[\xi]$. If $\xi^2 = 0$, fix any irreducible D other than f. Then D and f span $H_2(X;\mathbf{R})$. Write

$$(xf + yD)^2 = 2xy(f \cdot D) + y^2(D \cdot D) = 0.$$

Then $\xi$ is a solution to $2x(f \cdot D) + y(D \cdot D) = 0$, so $\xi$ must have a rational slope, but its slope need not be represented by any effective **Q**-divisor. By the adjunction formula, $f \in (K_X)_{<0}$.

4.6.2) Let A be an abelian surface with an ample divisor H. Since the self-intersection of any curve on an abelian surface is non-negative, it follows from (4.3) that $\langle NE(X) \rangle$ is given by the conditions $D^2 \geq 0$ and $D \cdot H \geq 0$. If $\operatorname{rk} NS \geq 3$ (e.g. $A = E \times E$ for some elliptic curve E), then $\langle NE(X) \rangle$ is a "circular" cone.

4.6.3) Del Pezzo surfaces: Characterized by the condition $c_1(X)$ ample (positive).

We shall see that, in this case, either $X \approx \mathbf{P}^2$ or one can find rational curves $C_1, \ldots, C_r$ such that $C_i{}^2 \leq 0$ and

$$NE(X) = \mathbf{R}_{\geq 0}[C_1] + \ldots + \mathbf{R}_{\geq 0}[C_r].$$

So, in particular, $NE(X) = \langle NE(X) \rangle$, a cone over a finite polyhedron.

4.6.4) Let X' = $\mathbf{P}^2$ blown up at the 9 basepoints of a generic pencil of cubic curves. Choosing one of the 9 points as the zero section, we get an infinite group generated by the other 8 sections. So X' has infinitely many exceptional curves of the first kind. All of these deform under a generic deformation of X' (obtained by moving the 9 points into general position). By (4.5), each of these curves gives an edge of the cone NE(X). Now $-K_X$ is represented by the unique elliptic curve through the 9 points and $-K_X$ is semi-positive. (However, no multiple of $-K_X$ moves.) So NE(X) is not locally finite near $K_X{}^{\perp}$.

With these examples in mind, we are ready to state the first result of Mori for varieties of arbitrary dimension. The proof of the result in the smooth case is more geometric so we consider it first. The proof in the general case will be given in Lecture #11.

(4.7) **Cone Theorem:** Let X be a non-singular projective variety.

There exists on X a set of rational curves $C_i$, $i \in I$, with $0 < C_i \cdot (-K_X) \leq \dim X + 1$ such that:

1) $\langle NE(X) \rangle = \Sigma(\mathbf{R}_{\geq 0})[C_i] + (\langle NE(X) \rangle \cap (K_X)_{\geq 0})$.

(The $(\mathbf{R}_{\geq 0})[C_i]$, which, together with $(\langle NE(X) \rangle \cap (K_X)_{\geq 0})$, form a minimal generating set for $\langle NE(X) \rangle$, are called **extremal rays**.)

2) For any $\varepsilon > 0$ and ample divisor H, 1) gives

$$\langle NE(X) \rangle \cap (K_X + \varepsilon H)_{\leq 0} = (\langle NE(X) \rangle \cap (K_X + \varepsilon H)_{=0}) + \Sigma_{\text{finite}}(\mathbf{R}_{\geq 0})[C_j].$$

**Proof:** Recall that in Lecture #2 we showed that, if H is any ample divisor, and if C is an irreducible curve with $C \cdot K_X < 0$, then there exists a rational curve C' with $0 < C' \cdot (-K_X) \leq \dim X + 1$ and

$$\frac{C' \cdot (-K_X)}{C' \cdot H} \geq \frac{C \cdot (-K_X)}{C \cdot H} - \varepsilon$$

for any $\varepsilon > 0$. The numerator on the left-hand-side takes only finitely many values, so we can set $\varepsilon = 0$ in the inequality.

Now let $[C_i]$, $i \in I$, be the collection of classes of rational curves with

$$0 < C_i \cdot (-K_X) \leq \dim X + 1.$$

Let $\mathcal{N}$ be the cone generated by the $[C_i]$ and $\langle NE(X) \rangle \cap (K_X)_{\geq 0}$. Choose a rational divisor J such that

$$(\langle NE(X) \rangle \cap (K_X)_{\geq 0}) \subseteq J_{>0} \cup \{\mathbf{0}\}.$$

By the convexity of $\langle NE(X) \rangle$, the closed set

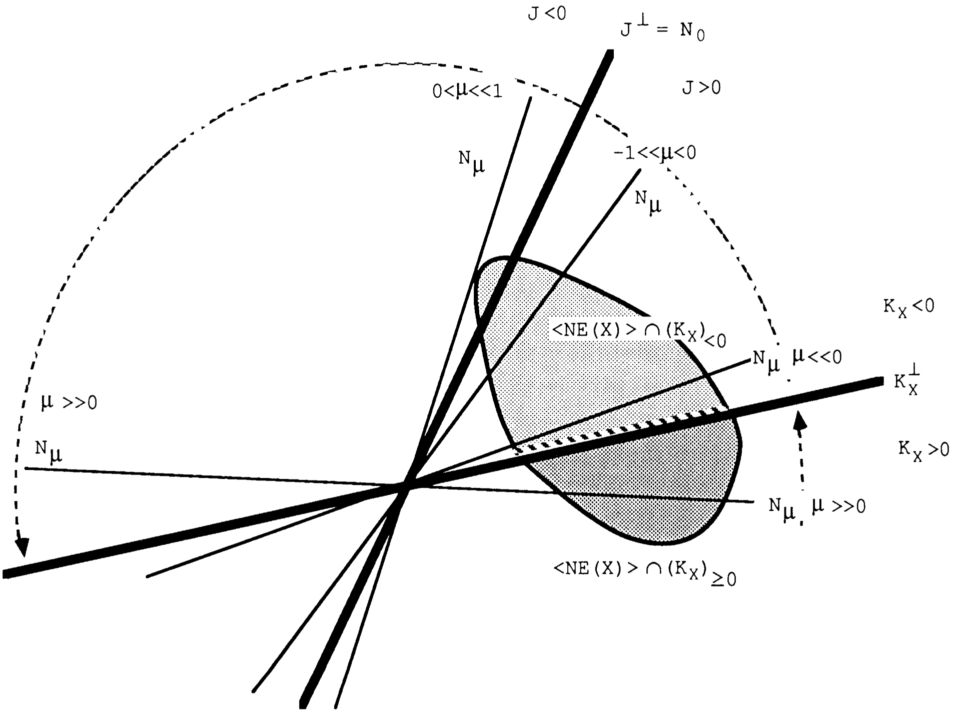
$$\{\mu: (J - \mu K_X)^{\perp} \cap (\langle NE(X) \rangle \cap (K_X)_{\geq 0}) \neq \{\mathbf{0}\}\}$$

is disjoint from the closed set

$$\{\mu: (J - \mu K_X)^{\perp} \cap (\langle NE(X) \rangle \cap (K_X)_{<0}) \neq \{\mathbf{0}\}\}.$$

We represent the various regions and subspaces we are considering, and the relationships between them, in the following diagram:

Rotation of $N_\mu = (J-\mu K_X)$ as $\mu$ changes:




Let $\mu_J$ be a positive rational number strictly between the two above sets. We will need

(4.8) **Kleiman's Criterion:** If X is a non-singular projective variety and D is any divisor, then D is ample if and only if

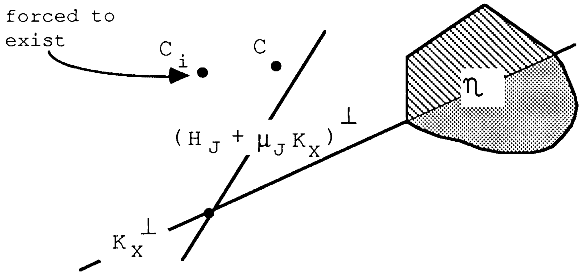
$$D_{>0} \supseteq \langle NE(X)\rangle - \{\mathbf{0}\}.$$

(4.9) By Kleiman's Criterion, $(J-\mu_J K_X)$ is ample, so we can set

$$H = H_J = (J-\mu_J K_X)$$

in the considerations at the beginning of the proof. Now suppose we have $[C] \notin \mathfrak{N}$. Then we can pick a rational J with $[C] \in (J_{<0})$ and

$$\mathfrak{N} \subseteq (J_{>0}).$$




We have seen that there is a $C_i$ with

$$\frac{C_i\cdot(-K_X)}{C_i\cdot H_J} \geq \frac{C\cdot(-K_X)}{C\cdot H_J}$$

by our previous considerations. But

$$(H_J+\mu_J K_X)\cdot C_i > 0 \text{ and } (H_J+\mu_J K_X)\cdot C < 0,$$

giving a contradiction. This gives 1) in the Cone Theorem.

The second statement is now immediate from the finiteness of the number of connected families of $C_i$ with

$$\dim X + 1 \geq C_i\cdot(-K_X) \geq \lambda(C_i\cdot H).$$

(4.10) Kleiman's Criterion tells us that a proper smooth algebraic variety X is projective if and only if $\langle NE(X)\rangle - \{\mathbf{0}\}$ lies in a proper half-space of $H_2(X;\mathbf{R})$, that is, if and only if $\langle NE(X)\rangle$ contains no straight lines. In the case that X was a surface, if C was a curve with $C^2 < 0$ and the variety Y obtained by contracting C to a point was smooth, then this says that Y must be projective.

If X is a projective threefold, here are some cases where C is an irreducible curve inside a smooth divisor D on X, and $D\cdot C < 0$:

Case 1: If $H_2(D;\mathbf{R})$ has one-dimensional image in $H_2(X;\mathbf{R})$, then just as in the Lemma for surfaces, C must lie on an edge of NE(X). Contraction of D corresponds to projection from this edge. So, if the variety Y obtained by contraction is smooth, it is projective.

Case 2: Suppose D is a smooth ruled surface with C as fibre and $D\cdot C = -1$. Then the contracted variety Y is smooth by Nakano's Criterion. So, if C is on an edge of NE(X), then, by Kleiman's Criterion, the contracted variety is projective.

(4.11) References: Example (4.6.4) is an old example of Nagata[Nag]. Kleiman's Criterion(4.8) can be found in [Kl]. (4.7) is in [M1].

## Lecture #5: Introduction to Mori's program

(5.1) **Example:** We begin letting

$$g: X \longrightarrow \mathbf{P}^2$$

be the blow-up of $\mathbf{P}^2$ at 12 points $P_1, \ldots, P_{12}$ on a smooth cubic plane curve D. Let C be the proper transform of the plane cubic. $C^2 = -3$, so that C can be blown down via an analytic morphism

$$f: X \longrightarrow Y$$

to an *analytic* surface Y. However Y cannot be projective if the 12 points are in general position. To see this, suppose M is any line bundle on Y. Then, $f^*M = L$ is given by a divisor

$$g^*\mathcal{O}_{\mathbf{P}^2}(b) + \Sigma a_i E_i$$

where $E_i$ is the exceptional curve above $P_i$. But the divisor

$$(g^*\mathcal{O}_{\mathbf{P}^2}(b) + \Sigma a_i E_i)\cdot C$$

must be linearly equivalent to 0 (denoted " $\approx 0$ ") on C. So we would have to have

$$\mathcal{O}_D(b) + \Sigma a_i P_i \approx 0$$

on D, which is clearly impossible for generic choice of the $P_i$.

However, if the $P_i$ are the points of intersection of a quartic curve Q with D, then the linear system determined by the proper transform of Q in X realizes $f: X \longrightarrow Y$ as a morphism into a projective space.

These examples show that there can be no *numerical* criterion for contractibility in the projective category. A major point of what follows is that for extremal rays such criteria can exist. The result is:

(5.2) **Theorem:** Let X be a non-singular projective variety. If R is an extremal ray, then there is a morphism

$$f: X \longrightarrow Y$$

onto a normal projective variety Y so that f contracts an irreducible curve D to a point if and only if [D] generates R. The morphism f is called the *extremal contraction* of the ray R

(A proof will be given in Lecture 11.)

(5.3) The theorem completely characterizes Y as a set. To get an idea of its projective structure, find a $\mathbf{Q}$-divisor L so that

$$[D]\cdot L = 0$$

and

$$\langle NE(X)\rangle - (\mathbf{R}_{\geq 0}[D])$$

lies in $L_{>0}$. By Kleiman's Criterion, $(mL-K_X)$ is ample for $m >> 0$.

So, by the Kodaira Vanishing Theorem,

$$H^i(X;mL) = 0$$

for $i > 0$. One uses this to show that $|mL|$ is basepoint-free for $m >> 0$. This linear system gives the morphism $f\colon X \longrightarrow \mathbf{P}^n$.

Also $mL - K_X$ is ample which implies that

$$(-K_X\cdot D) > 0$$

for all D lying in a fibre of f. We will later prove a vanishing theorem (8.8) which implies that therefore all the higher direct-image sheaves $R^if_*\mathcal{O}_X$ are zero.

(5.4) It is the vanishing of $R^1f_*\mathcal{O}_X$ which insures that the contractions take place in the projective category. Roughly this is because then $R^1f_*\mathcal{O}_X{}^*$ injects into $R^2f_*\mathbf{Z}$. Then, with Y as in (5.2), we use the exact sequence

$$\mathrm{Pic}Y \longrightarrow \mathrm{Pic}X \longrightarrow R^1f_*\mathcal{O}_X{}^*,$$

to see that the criterion for a line bundle on X to come from one on Y is numerical.

(5.5) We will denote the contraction morphisms constructed above as

$$\mathrm{cont}_R\colon X \longrightarrow Y.$$

For X a non-singular projective threefold, we will categorize the possibilities for $\mathrm{cont}_R$ according to the following types:

**Exceptional:**

If $\dim Y = 3$, then $f = \mathrm{cont}_R$ is birational and there are five types of local behavior near contracted curves:

E1) $\mathrm{Cont}_R$ is the blow-up of a smooth curve in the set of non-singular points on Y.

E2) $\mathrm{Cont}_R$ is the blow-up of a smooth point of Y.

E3) $\mathrm{Cont}_R$ is the blow-up of an ordinary double point of Y. Analytically, an ordinary double point is given locally by the equation

$$x^2+y^2+z^2+w^2 = 0.$$

E4) $\mathrm{Cont}_R$ is the blow-up of a point of Y which is locally analytically given by the equation

$$x^2+y^2+z^2+w^3 = 0.$$

E5) $\mathrm{Cont}_R$ blows down a smooth $\mathbf{CP}^2$ with normal bundle $\mathcal{O}(-2)$ to a point of multiplicity 4 on Y which is locally analytically the quotient of $\mathbf{C}^3$ by the involution

$$(x,y,z) \longrightarrow (-x,-y,-z).$$

**Conic:**

If $\dim Y = 2$, then $f = \mathrm{cont}_R$ is a fibration with fibres conic curves. (The generic fibre is, of course, smooth.)

C1) If $f$ has singular fibres, then $f$ is a "conic bundle".

C2) If $f$ has no singular fibres, $f$ is an étale $\mathbf{CP}^1$-bundle.

**Del Pezzo:**

If $\dim Y = 1$, the generic fibre of $\mathrm{cont}_R$ is a del Pezzo surface since the canonical divisor of the fibre, $-K_X|_{\text{fibre}}$, is ample.

**Fano:**

If $\dim Y = 0$, $-K_X$ is ample. X is a "Fano variety". By Kodaira Vanishing Theorem,

$$H^i(X;\mathcal{O}) = 0 \text{ for } i > 0,$$

so R generates $H_2(X;\mathbf{R})$.

(5.6) Now we are in a position to give a short summary of the aim of Mori's program. Let X be a smooth projective variety. If $K_X$ is not nef, then we can find a morphism, called the contraction morphism of an extremal ray or an **extremal contraction**,

$$f = \mathrm{cont}_R\colon X \longrightarrow Y.$$

In low dimensions we have the following basic cases:

5.6.1) $\dim X = 2$:

Then either

$\dim Y < \dim X$: In this case we have a complete structure theory for X.

or

$\dim Y = \dim X$: In this case Y is again smooth and
$$\mathrm{rk}\,NS(Y) < \mathrm{rk}\,NS(X).$$
Thus Y can be considered "simpler" than X.

In short, either we obtain a description of X or we can simplify its structure.

5.6.2) $\dim X = 3$:

Then either

$\dim Y < \dim X$: In this case we again have a nearly complete structure theory for X; in particular we obtain that X is covered by rational curves.

or

$\dim Y = \dim X$: In this case Y can unfortunately be singular (cases E3,E4,E5). Thus it is not clear that Y is any "simpler" than X.

(5.7) Thus we see that we have to put up with certain singularities in higher dimensions. We have to establish a suitable category of singularities to work with, and it is not at all clear *a priori* that a reasonable class can be found. It is *a priori* possible that contraction morphisms create worse and worse singularities. The correct class will be called "terminal" singularities. The definition is unimportant for the moment and will be given only later--for now we only note one defining property:

For terminal singularities, some multiple of $K_X$ is Cartier, thus it makes sense to talk about $K_X$ being nef.

Next we will have to prove the existence of the contraction morphisms in this wider class of "mildly singular" varieties:

(5.8) **Theorem:** Let X be a projective variety with only **Q**-factorial terminal singularities such that $K_X$ is not nef. Then there exists a morphism

$$f: X \longrightarrow Y$$

such that $-K_X$ is f-ample and one of the following holds:

a) dimX > dimY and f is a **Q**-Fano fibration.

b) f is birational and contracts a divisor (divisorial contraction).

c) f is birational and contracts a subvariety of codimension $\geq 2$ (small contraction).

(5.9) **Comments:**

Case a) of the theorem: This means that the general fiber of f is an algebraic variety where $-K_X$ is ample. Thus, at least in principle we reduce the problem of understanding X to understanding the lower dimensional variety Y and the fibres of f. Moreover these fibres are of very special kind--they are analogues of $\mathbf{CP}^1$ and of Del Pezzo surfaces.

Case b): In this case, Y again has terminal singularities and so we manage to stay inside the class of singularities we started with. Moreover,

$$\operatorname{rk} NS(Y) < \operatorname{rk} NS(X),$$

thus Y can be considered to be "simpler" than X.

Case c): This is a new case. It could never happen for surfaces for dimension reasons, and it did not happen for smooth threefolds X. In this case, Y can have a very bad singularity where no multiple of $K_Y$ is Cartier. Thus the expression "$K_Y$ is nef" does not even make sense. So we are led out of the class of varieties that we can control. In order to continue at this point, we have to introduce a new operation called a *flip*. This is the algebraic analogue of codimension-two surgery:

Instead of contracting some curves $\cup C_i \subseteq X$, we remove them, and then compactify

$$X - \cup C_i$$

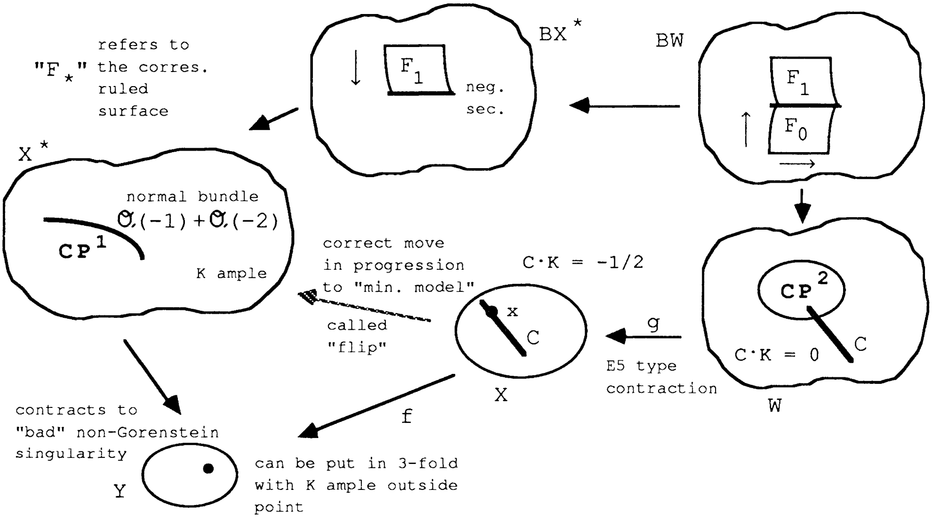
by adding another union of curves $\cup D_j$. (For the moment, it is not at all clear that this operation exists or that it is well-defined, let alone that it improves things.)

We first study an example of this situation. In the example, the (directed) flip will remove the curve $C = \mathbf{CP}^1$ from the singular variety $X$ and replace it with $D = \mathbf{CP}^1$ to achieve the "improved" variety $X^*$ (which in this case is non-singular). The process is most easily explained in reverse, as a sequence of blowing-ups

$$X^* \longleftarrow BX^* \longleftarrow BW$$

followed by a sequence of blowing-downs

$$BW \longrightarrow W \longrightarrow X:$$




We start with the threefold $X^*$ which contains a smooth rational curve whose normal bundle is $\mathcal{O}(-1) + \mathcal{O}(-2)$. Assume that this curve can be contracted to an algebraic variety $Y$.

If we blow up this curve, we get $BX^*$ which contains the ruled surface $F_1$ as the exceptional divisor. We can blow-up the negative section of this $F_1$ to get $BW$. The new exceptional surface is $F_0 = \mathbf{P}^1 \times \mathbf{P}^1$. This can be blown down in the other direction to obtain $W$. The exceptional curve of $F_1$ is blown down so this becomes a $\mathbf{CP}^2$. The image of $F_0$ in $W$ is a curve $C$ with normal bundle

$$\mathcal{O}(-1) + \mathcal{O}(-1),$$

in particular $C \cdot K_W = 0$.

The normal bundle of the $\mathbf{CP}^2$ can be computed as follows. It has to be $\mathcal{O}(k)$ and we need to compute $k$. We can do this by restricting to a line which does not intersect C. The pre-image of this line in BW is a section S of $F_1$ which does not intersect $F_0$. We can also look at the image of this section S' in $BX^*$. Thus we need to compute the restriction of the normal bundle of $F_1$ in $BX^*$ to a general section of $F_1$. This can be done easily and we obtain that $k = -2$.

Now the $\mathbf{CP}^2$ can be contracted, this is the case E5 in (5.5). X is locally a quotient at x, thus $K_X$ is only $\mathbf{Q}$-Cartier. If

$$g^* K_X \equiv K_W + a\mathbf{CP}^2$$

for some rational number $a$, then from the adjunction formula applied to $\mathbf{CP}^2$ we get that $a = 1/2$. Thus $C \cdot K_X = -1/2$.

One can see that C in X generates an extremal ray and that C is the only irreducible curve whose homology class is in that ray. Thus the corresponding contraction morphism contracts only C and leads to the very singular space Y.

(5.10) The operation that happens at the lower left corner of the above diagram can be formalized as follows:

(5.11) **Definition:** Let $f: X \longrightarrow Y$ be an extremal contraction such that the exceptional set E in X has codimension at least two. A variety $X^+$ together with a map

$$f^+: X^+ \longrightarrow Y$$

is called the **flip** of $f$ if $X^+$ has only $\mathbf{Q}$-factorial terminal singularities and $K_{X^+}$ is $f^+$-ample. By a slight abuse of terminology, the rational map $X \dashrightarrow X^+$ will be called a **flip**.

$$\begin{tikzcd} X \arrow[rr, dotted, "\text{isomorphism}", "\text{in codim 1}"'] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

$-K_X$ f-ample

only Q-factorial terminal singularities

$K_{X^+}$ $f^+$-ample

If we perform a flip, it is not clear that $X^+$ is any "simpler" than X. In the example above this happens since X is singular but $X^+$ is smooth. We will see that in general flips lead to simpler singularities.

(5.12) **Mori's program:**

Starting with an algebraic variety X, we perform a sequence of well-defined and understandable birational modifications, until we arrive at a variety Y (possibly with terminal singularities) such that either

i) Y has a fiber-space structure whose generic fiber is a $\mathbf{Q}$-Fano variety (in particular Y and X are covered by rational curves)

or

ii) $K_Y$ is nef.

(5.13) At the moment this program is complete only in dimension 2 and 3. Even there much remains to be done. The applicability of the program hinges on our ability to understand the process that creates Y, so that we can interpret structural properties of X in terms of those of Y. Furthermore we need to learn a lot about threefolds with K nef and about fiber spaces whose general fibers are Fano varieties. Even in the case in which the general fiber of the fiber space is $\mathbf{P}^1$, it is not known how to decide when two such fiber spaces are birational.

(5.14) Here we give some examples of extremal contractions in higher dimensions:

i) If X is a smooth projective variety and $X \supset Z$ is a smooth irreducible subvariety, then the inverse of the blowing-up

$$B_Z X \longrightarrow X$$

is an extremal contraction.

ii) Over $\mathbf{P}^n$ let V be the total space of the rank k vector bundle

$$\mathcal{O}(-1) + \ldots + \mathcal{O}(-1),$$

and let

$$V^{\wedge} = \mathbf{P}(\mathcal{O}(1) + \ldots + \mathcal{O}(1) + \mathcal{O}).$$

(Note: The Grothendieck convention for projectivization is used.)

If $k \leq n$, then the line in

$$\mathbf{P}^n \subseteq V \subseteq V^{\wedge}$$

generates an extremal ray in $V^{\wedge}$. The corresponding contraction morphism contracts $\mathbf{P}^n$ to a point and is an isomorphism outside $\mathbf{P}^n$. Thus, if $k \geq 2$, then the exceptional set is *not* a divisor. This gives such examples for $\dim V \geq 4$. No such examples exist for smooth threefolds.

iii) Let Y be the space of non-zero linear maps from

$$\mathbf{C}^{n+1} \longrightarrow \mathbf{C}^n$$

modulo constants.

$$Y = \mathbf{P}^{n(n+1)-1},$$

thus Y is smooth. Let X be the set of pairs (g,L) where $g \in Y$, and L is a one-dimensional subspace in the kernel of g. Let

$$f\colon X \longrightarrow Y$$

be the natural morphism. This f will turn out to be an extremal contraction. X has a natural morphism p onto

$$\mathbf{P}^n \text{ (= the set of one-dimensional subspaces in } \mathbf{C}^{n+1}),$$

given by

$$p(g,L) = L.$$

The fibers are all projective spaces of dimension $n^2-1$. Thus X is also smooth. Define

$$F = \{g\colon \operatorname{rk} g \leq n-1\},$$

and

$$E = \{(g,L)\colon \operatorname{rk} g \leq n-1\}.$$

The restriction of p to E exhibits E as a fiber bundle over $\mathbf{P}^n$ whose fiber over L is the projectivization of the set of singular maps

$$\mathbf{C}^{n+1}/L \longrightarrow \mathbf{C}^n,$$

thus E is irreducible. If $g \in F$, then $f^{-1}(g)$ is a projective space of dimension $(n - \operatorname{rk} g)$. Thus, for general $g \in F$, it is a $\mathbf{P}^1$. If $n \geq 2$, then there is a $g \in F$ such that

$$\operatorname{rk} g = n-2,$$

and so

$$f^{-1}(g) = \mathbf{P}^2.$$

This shows that f cannot be a smooth blow-up. In fact, one can see that F is singular at g iff

$$\operatorname{rk} g \leq n-2.$$

(5.15) **Comparison between surface case and threefold case:**

We have the following table of parallel results:

X a smooth projective **surface** :

1) The canonical ring
$$\bigoplus_{m\geq 0} H^0(X;\ mK_X)$$
is finitely generated.

2) $H^0(X;\ mK_X) = 0$ for all $m>0$ if and only if X is ruled.

3) If $f\colon X \longrightarrow Y$ is a birational morphism of smooth projective surfaces, then f is a succession of blow-downs.

4) Let (Z,p) be a germ of a surface singularity (not necessarily isolated). Then there exist projective birational morphisms f, g, and h:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

X smooth; Y only RDP's (rational double points)

$K_X$ is f-semi-ample, that is, there is a morphism (over Z)
$$F\colon X \longrightarrow Z\times \mathbf{P}^n$$
with $F^*\mathcal{O}(1) = mK_X$ for some $m>0$.

$K_Y$ is g-ample, that is, there is an imbedding (over Z)
$$G\colon Y \longrightarrow Z\times \mathbf{P}^n$$
with $G^*\mathcal{O}(1) = mK_Y$ for some $m>0$.

X is unique and is called the **minimal resolution** .

Y is unique and is called the **canonical resolution** .

X a smooth projective **threefold**:

1) The canonical ring
$$\bigoplus_{m\geq 0} H^0(X;\ mK_X)$$
is finitely generated.

2) $H^0(X;\ mK_X) = 0$ for all $m>0$ if and only if X is uniruled.

3) If $f\colon X \longrightarrow Y$ is a birational morphism of smooth projective threefolds, then f is a succession of divisorial contractions and flips.

4) Let (Z,p) be a germ of a threefold singularity (not necessarily isolated). Then there exist projective birational morphisms f, g, and h:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

X only Q-factorial terminal singularities (these are in codim 3); Y only canonical singularities (these are in codim 2)

$K_X$ is f-semi-ample

$K_Y$ is g-ample.

X is unique outside a union of rational curves and is called a **Q-factorial terminal modification** .

Y is unique and is called the **canonical modification** .

(5.16) References: Example (5.1) is due to Zariski [Z]. (5.5) is in [M1], while the example in (5.9) is in [F]. General references for Mori's program are [KMM], [Ko4] and [W]. Example (5.14.iii) was pointed out to us by L. Ein.

## Lecture #6: Singularities in the minimal model program

(6.1) Let $X$ be a variety of dimension > 1 such that $mK_X$ is Cartier. Suppose

$$f\colon Y \longrightarrow X$$

is a proper birational morphism from a normal variety $Y$. Let $e$ denote a generic point for a divisor $E$ which is exceptional for $f$.

If $E$ is locally defined (as a scheme) by $g = 0$, then locally

$$f^*(\text{generator of } \mathcal{O}(mK_X)) = g^{m\cdot a(E)}(dy_1\wedge\ldots\wedge dy_n)^{\otimes m}$$

for some rational number $a(E)$ such that $m\cdot a(E)$ is an integer, where the $y_i$ form a local coordinate system at $e$.

$a(E)$ is independent of $f$ and $Y$ in the sense that, for any

$$f'\colon Y' \longrightarrow X$$

such that $Y$ and $Y'$ are locally isomorphic (over $X$) at a generic point of $E$ (resp. $E'$),

$$a(E) = a(E').$$

If $f\colon Y \longrightarrow X$ is a proper birational morphism such that $K_Y$ is a line bundle (e.g. $Y$ is smooth), then $mK_Y$ is linearly equivalent to

$$f^*(mK_X) + \Sigma\, m\cdot a(E_i)\cdot E_i,$$

where the $E_i$ are the exceptional divisors. Using numerical equivalence, we can divide by $m$ and write

$$K_Y \equiv f^*(K_X) + \Sigma\, a(E_i)\cdot E_i.$$

(6.2) **Definition:** $a(E)$ is called the **discrepancy** of $X$ at $E$. The discrepancy of $X$ is given by

$$\operatorname{discrep}(X) = \inf\{a(E)\colon E \text{ exceptional for some } f\colon Y \longrightarrow X\}.$$

For example, if $X$ is smooth, $\operatorname{discrep}(X) = 1$.

(6.3) **Claim:** Either $\operatorname{discrep}(X) = -\infty$ or $-1 \leq \operatorname{discrep}(X) \leq 1$.

**Proof:** Blowing up a locus of codimension two which intersects the set of smooth points of X, one sees that

$$\operatorname{discrep}(X) \leq 1.$$

Next take a desingularization

$$f\colon Y \longrightarrow X$$

and E exceptional for f. Suppose $a(E) < -1$, so locally near general $s \in E$

$$K_Y \equiv f^*K_X - (1+c)E \text{ with } c > 0.$$

Let S be a generic codimension 2 locus through s which is contained in E. Let $Z = B_S Y$, and let

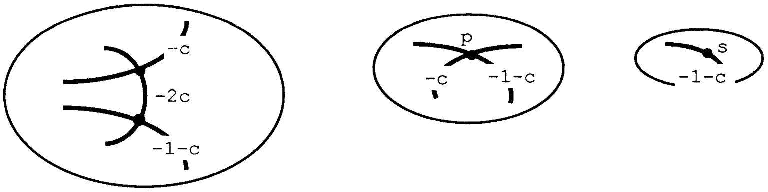
$$g\colon Z \longrightarrow Y$$

denote the blow-up of Y at S and $E_S$ the exceptional variety above S. Then

$$(*) \qquad \begin{aligned} K_Z &= g^*K_Y + E_S \\ &= g^*f^*K_X - (1+c)g^*E + E_S = g^*f^*K_X - (1+c)F - cE_S \end{aligned}$$

where F is the proper transform of E. Let P be a component of $F \cap E_S$. Then, if $W = B_P Z$, $E_P$ occurs in $K_W$ with multiplicity $-2c$.

The picture for surfaces is:




Repeat the blowing-up, this time at the point of intersection of the proper transform of F and $E_P$ to get a component with discrepancy $-3c$, etc.

(6.4) **Definition:** We say that X has

| | | | | |
|---|---|---|---|---|
| terminal | singularities | if | discrep(X) | $> 0$ |
| canonical | | | | $\geq 0$ |
| log-terminal | | | | $> -1$ |
| log-canonical | | | | $\geq -1$ |

(6.5) **Proposition:** Let $f\colon Y \longrightarrow X$ be a resolution of singularities. If

$$a(E) \geq c \text{ for some } 1 \geq c \geq 0,$$

for every $f$-exceptional divisor $E$, then

$$\operatorname{discrep}(X) \geq c.$$

If the exceptional set of $f$ is a simple-normal-crossing divisor and if

$$a(E) \geq c \text{ for some } 1 \geq c \geq -1,$$

for every $f$-exceptional divisor $E$, then

$$\operatorname{discrep}(X) \geq c.$$

**Proof:** By an analogous calculation to (*) above, $a(E_S) \geq a(E)$ for $S \subseteq E$. To compare with $a(E')$ at $e'$ on another desingularization

$$f'\colon Y' \longrightarrow X,$$

notice that there is a sequence of blow-ups $Y''$ of $Y$ with an exceptional divisor at a generic point of which $Y''$ is locally isomorphic to $(Y', e')$.

(6.6) **Lemma:** If $D$ is a generic hyperplane section of $X$,

$$\operatorname{discrep}(X) \leq \operatorname{discrep}(D).$$

**Proof:** This is a trivial application of the adjunction formula.

(6.7) **Proposition:** Let $g\colon X' \longrightarrow X$ be proper. Then

i) $(\deg g)(\operatorname{discrep}(X) + 1) \geq (\operatorname{discrep}(X') + 1)$;

ii) if $g$ is étale in codimension 1 on $X'$, then $\operatorname{discrep}(X') \geq \operatorname{discrep}(X)$.

**Proof:** The proof of i) follows from commutativity in the fibred-product diagram with exceptional divisors given below:

r = ramification index of E'/E

$$\begin{tikzcd}
e' \in E' \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & E \arrow[d, phantom, "\cap" rotate=-90] \\
Y' \arrow[r, "h"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\
X' \arrow[r, "g"] & X
\end{tikzcd}$$

$$r \leq \deg g$$

"$A \succ B$" means "$A-B$ effective"

Near $e'$:
$$\begin{aligned}
K_{Y'} &= f'^* K_{X'} + a(E')E' \\
&\succ f'^* g^* K_X + a(E')E' \\
&= h^* f^* K_X + a(E')E'
\end{aligned}$$

$$\begin{aligned}
K_{Y'} &= h^* K_Y + (r-1)E' \\
&= h^* f^* K_X + h^*(a(E)E) + (r-1)E' \\
&= h^* f^* K_X + (a(E)r + (r-1))E'
\end{aligned}$$

If $g$ is étale in codimension 1, then " " above becomes "=". This implies ii).

(6.8) **Definition:** We define an **index-one cover** as follows:

Assume X is a germ of a normal variety for which $K_X$ is **Q**-Cartier with index m. Then

$$\mathcal{O}(mK_X) \approx \mathcal{O}_X$$

so that the preimage X' of the section "1" under the m-th tensor power map

$$K_X \longrightarrow \mathcal{O}_X$$

has the property that $K_{X'} = \mathcal{O}_{X'}$. So X' has index one. X' is called the **index-one cover** of X (well-defined only up to analytic isomorphism).

Notice that X' is étale in codimension one over X, and that the discrepancy of an index one variety must be an integer. So, by (6.7):

(6.9) **Proposition:** A germ X is log-terminal if and only if it is a cyclic quotient of a canonical singularity via an action which is free in codimension one.

(6.10) **Proposition:** For surfaces X:

1) X has terminal singularities if and only if X is smooth;

2) a singularity of X is canonical if and only if it is **DV**, that is a **DuVal** singularity (also called a **rational double point**).

**Proof:** Let X be a surface germ. Suppose X has (at most) canonical singularities, and let $f: Y \longrightarrow X$ be a minimal resolution. Then

$$K_Y = f^*K_X + \Sigma a_i E_i$$

with all $a_i \geq 0$. If not all the $a_i$ are zero, there must be some $E_j$ such that $K_Y \cdot E_j < 0$ because

$$K_Y \cdot \Sigma a_i E_i = (\Sigma a_i E_i)^2 < 0.$$

But then, by the adjunction formula, $E_j$ must be smooth and rational with self-intersection -1, which contradicts the minimality of the resolution. So $K_Y = f^*K_X$. So, again by the adjunction formula, all $E_i$ are smooth and rational with self-intersection -2. The normal singularities with this property are exactly the DuVal singularities.

(6.11) **Proposition:** For a normal surface germ $(X,x)$, the following are equivalent:

1) $(X,x)$ is log-terminal,

2) $(X,x)$ is a quotient of $(\mathbf{C}^2,0)$ under the action of a finite group which is free in codimension 1,

3) $(X,x)$ is a quotient of $(\mathbf{C}^2,0)$ under the action of a finite group.

**Proof:** For any normal surface germ $X$ with $K_X$ $\mathbf{Q}$-Cartier let

$$g: X' \longrightarrow X$$

be its index one cover.

To see that 1) implies 2):

We saw in (6.7)that $X'$ is log-terminal if $X$ is. Since $K_{X'}$ is Cartier, $\text{discrep}(X') > -1$ and is an integer. So $X'$ is canonical. Thus, $X'$ is DV and therefore a quotient of $\mathbf{C}^2$ under a group which acts freely in codimension one. So $\mathbf{C}^2-\{0\}$ is the universal cover of $X-\{x\}$ and 2) is proved.

To see that 3) implies 1):

If $(X,x)$ is a quotient of $(\mathbf{C}^2,0)$ under the action of a finite group, then the inequality

$$(\deg g)(\text{discrep}(X) + 1) \geq (\text{discrep}(X') + 1)$$

shows that $(X,x)$ is log-terminal.

A somewhat more detailed analysis leads to:

(6.12) **Proposition:** A normal surface germ is log-canonical if and only if it is log-terminal or "simple elliptic" or a "cusp" or a quotient of one of these two latter types of singularity.

Using that $\text{discrep}(X) \leq \text{discrep}(H)$, where $H$ is a generic hyperplane section of $X$, and the characterization of terminal and canonical surface singularities, we obtain:

(6.13) **Corollary:** If $X$ has only canonical singularities, then $X$ is Gorenstein in codimension 2.

If $X$ has only terminal singularities, then $X$ is smooth in codimension two.

(6.14) **Theorem:** All log-terminal singularities are rational, that is, for some (any) resolution $f\colon Y \longrightarrow X$,

$$R^i f_* \mathcal{O}_Y = 0 \text{ for } i > 0.$$

**Outline of proof for threefold singularities:**

As we saw above, the index one cover X' has only canonical singularities, so we reduce to the case in which X has only canonical singularities and $K_X$ is Cartier.

First we blow up the one-dimensional singular set (if there is one). At a general point, this set is locally analytically isomorphic to the product of a disc and a DV surface singularity, so $K_X$ pulls back to $K_{X''}$, where X" is the blow-up. Let

$$f\colon Y \longrightarrow X'' \longrightarrow X$$

be a resolution, and write $K_Y = f^*K_X + S$ for some effective Cartier divisor S. By the above, S lies over a finite set in X. S is a hypersurface, so it is Gorenstein.

We check that $R^1 f_* \mathcal{O}_Y = 0$. Applying $f_*$ to the sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y(S) \longrightarrow \mathcal{O}_S(S) \longrightarrow 0,$$

we obtain an exact sequence

$$\ldots \longrightarrow H^0(\mathcal{O}_S(S)) \longrightarrow R^1 f_* \mathcal{O}_Y \longrightarrow R^1 f_* \mathcal{O}_Y(S) \longrightarrow \ldots.$$

If $\mathcal{J}$ is the ideal sheaf for S, then, by the Grauert-Riemenschneider Vanishing Theorem (8.8),

$$R^i f_* \mathcal{O}_Y(S) = R^i f_* \omega_Y = 0, \text{ for } i = 1,2,$$

and so, by the above sequence, also

$$H^2(\mathcal{O}_S(S)) = H^2(\omega_Y/\mathcal{J}\omega_Y) = 0.$$

Since $\omega_S = \mathcal{O}_S(2S)$, we also have by duality that $H^0(\mathcal{O}_S(S)) = 0$. Using the above sequence again, we see that $R^1 f_* \mathcal{O}_Y = 0$.

To see that $R^2 f_* \mathcal{O}_Y = 0$, a relative duality theorem (the Leray spectral sequence and Serre duality) gives that $R^2 f_* \mathcal{O}_Y$ is dual to $\mathcal{O}_X(K_X)/f_* \mathcal{O}_Y(K_Y)$. But this last sheaf is zero since, for X canonical, all sections of $K_X$ lift to sections of $K_Y$.

(6.15) **Corollary:** If X is canonical, local and $g\colon X' \longrightarrow X$ is the index one cover, then any flat deformation $\{X_s\}$ of X is covered by a deformation of X'.

**Idea of proof:** Let Z = {index > 1 points in X}. Then g restricts to a cyclic cover over X-Z. Using a Lefschetz-type argument, one shows that the fundamental group of $X_s - Z_s$ maps onto the fundamental group of X-Z via retraction to the central fibre.

(6.16) **Corollary:** If $\mathcal{X}/S$ is flat with fibres having canonical singularities,

$$\mathcal{O}_X(qK_X) = \mathcal{O}_X \otimes ((\omega_{\mathcal{X}/S}{}^{\otimes q})^{**})$$

and so the formation of $(\omega_{\mathcal{X}/S}{}^{\otimes q})^{**}$ commutes with base change.

**Idea of proof:** The assertion is local on $\mathcal{X}$. By (6.15), $f: \mathcal{X}' \longrightarrow \mathcal{X}$ commutes with base change as does $\omega_{\mathcal{X}'/S}$. Decompose

$$f_*\omega_{\mathcal{X}'/S} = (\omega_{\mathcal{X}/S})^{**} + (\omega_{\mathcal{X}/S}{}^2)^{**} + \dots + (\omega_{\mathcal{X}/S}{}^{m-1})^{**} + \underbrace{(\omega_{\mathcal{X}/S}{}^{m})^{**}}_{\text{locally free}}.$$

Structure of 3-dimensional canonical singularities:

(6.17) **Definition:** If $(Z^n, z)$ is a Gorenstein singularity, it is **elliptic** if, for some (any) resolution $f: Y \longrightarrow Z$, one has

i) $R^i f_* \mathcal{O}_Y = 0$ for $0 < i < n-1$, and

ii) $R^{n-1} f_* \mathcal{O}_Y = \mathbf{C}$.

By the same relative duality theorem used in (6.14), ii) is equivalent to

ii') $f_*\omega_Y = \mathfrak{m}_{z,Z}\omega_Z$.

(6.18) **Theorem:** If $(X,x)$ is a Gorenstein canonical singularity and $H$ is a generic hyperplane section through $x$, then $(H,x)$ is either rational or elliptic.

**Proof:** Let $f: Y \longrightarrow X$ be a resolution of $X$ which resolves $H$ and which is such that the scheme $f^{-1}(x)$ is a Cartier divisor $E$, and the line bundle $L = f^{-1}\mathfrak{m}_{x,X}$ is generated by global sections, so

$$f^*H = E + L.$$

Then $\omega_H = \omega_X(H)|_H$, so that, if $s$ locally generates $\omega_X$ and $h$ locally defines $H$, residue$(s/h)$ generates $\omega_H$. Next let $e$ define $E$ locally and let $\ell$ define $L$ locally. Then, if $a \in \mathfrak{m}_x$

$$f^*a \cdot f^*s/f^*h = (f^*a/e)\cdot(f^*s/\ell).$$

Here $f^*a/e$ is regular along $E$ and $f^*s/\ell \in \omega_Y(L)$. So, residue$(f^*a\cdot f^*s/f^*h) \in \Gamma(\omega_Y(L))$. Taking $f_*$ of this section, we get back to $a\cdot$residue$(s/h)$. So any section of $\mathfrak{m}_{x,H}\omega_H$ is a push-forward. So $f_*\omega_L = \mathfrak{m}_{x,H}\omega_H$ (elliptic) or $f_*\omega_L = \omega_H$ (rational).

We now give, without proof a series of results in dimension 3:

(6.19) **Proposition:** Let $(S,s)$ be an elliptic surface singularity.

1) If $\mathrm{mult}_s S \geq 3$, then the blow-up
$$g\colon B_s S = B \longrightarrow S$$
has only DV singularities, and
$$\omega_B = g^*\omega_S \otimes g^{-1}\mathfrak{m}_{s,S}.$$

2) If $\mathrm{mult}_s S = 2$, then some weighted blow-up
$$g\colon B \longrightarrow S$$
has only DV singularities, and
$$\omega_B = (g^*\omega_S \otimes g^{-1}\mathfrak{m}_{s,S})^{**}.$$

(6.20) **Corollary:** Let $(X,x)$ be a 3-dimensional canonical singularity such that, for generic hyperplane $H$ through $x$, $H$ is elliptic at $x$. Then:

1) if $\mathrm{mult}_x X \geq 3$, then the blow-up
$$g\colon B_x X = B \longrightarrow X$$
has only canonical singularities, and
$$\omega_B = g^*\omega_X.$$

2) If $\mathrm{mult}_x X = 2$, then some weighted blow-up
$$g\colon B \longrightarrow X$$
has only canonical singularities, and
$$\omega_B = g^*\omega_X.$$

(Roughly, this corollary is proved by running backwards through the proof of (6.18).)

(6.21) **Corollary:** Let $X$ be a Gorenstein threefold with only canonical singularities. Then there exists a proper and bimeromorphic morphism
$$g\colon X' \longrightarrow X$$
such that, for every $x' \in X'$, the generic hyperplane section $H$ through $x'$ has only rational singularities. So $H$ is Gorenstein and has only rational double points.

(6.22) **Definition:** A threefold singularity $(X,x)$ is called **compound DuVal** (cDV) if a generic hyperplane section through $x$ is a DV surface singularity.

So all cDV singularities are smooth or hypersurface double points (and so are also Gorenstein).

(6.23) **Theorem:** A threefold singularity is terminal and Gorenstein if and only if it is isolated cDV.

**Outline of proof:** Using (6.20), one direction of the proof becomes easy: We know that a terminal singularity must be isolated. If an isolated singularity has generic hyperplane section elliptic, the blow-up will have discrepancy zero and so cannot be terminal. Thus, if the singularity is terminal, the section must be DV. We will outline the proof of the other direction later (16.1).

By passing first to a Gorenstein cover and then taking quotients, after some computation one arrives at:

(6.24) **Theorem:** If X is a threefold with only canonical singularities, then there is a projective birational morphism $f\colon Y \longrightarrow X$ such that

$K_Y = f^*K_X$ (i.e. f is **crepant**)

and Y has only terminal singularities.

Inductive structure of canonical singularities:

(6.25) **Theorem:** If $X^3$ has only canonical singularities, there exists a sequence of morphisms:

$$Y = X_q \longrightarrow X_{q-1} \longrightarrow \dots \longrightarrow X_1 \longrightarrow X_0 = X$$

such that:

1) $X_i$ is $\mathbf{Q}$-factorial and canonical for $i \geq 1$;

2) $X_1 \longrightarrow X$ contracts only finitely many curves, and is an isomorphism if X is $\mathbf{Q}$-factorial;

3) for $i > 1$, $X_i \longrightarrow X_{i-1}$ contracts exactly one divisor and $NE(X_i/X_{i-1})$ has dimension one;

4) Y has only terminal singularities;

5) $K_Y$ is the pull-back of $K_X$, in fact, the $\mathbf{Q}$-Cartier canonical divisor of each $X_i$ is the pull-back of $K_X$.

(6.26) References: Terminal and canonical singularities were defined by Reid [R2]. The log-versions were introduced later in [Ka4]. (6.5-6.8) can be found in [R2]. (6.9) was noticed in [Ka4]. (6.12) is in [Ka1]. (6.14) was proved by Shepherd-Barron[S-B] in dimension three, and by Elkik[El] and by Flenner[Fl] in general. The proof given is due to Shepherd-Barron[R5]. (6.15-16) are in [Ko2]. (6.17-24) are all due to Reid[R1,R2]. (6.19) was also done by Laufer[L1]. (6.25) is in [Ka5].

**Lecture #7: Extensions of the mimimal model program**

We discuss three useful extensions of the minimal model program:

1) Relativization
2) Analytic case
3) Varieties with group actions

(7.1) <u>Relativization</u>

$$\begin{array}{ll} X & \text{(not nec. projective)} \\ \downarrow f & \text{(projective morphism)} \\ Y & \text{(not nec. compact or algebraic)} \end{array}$$

If X is projective, we define:

$N(X) = \{$group generated by irreducible curves modulo numerical equivalence$\}\otimes\mathbf{R}$

On the other hand, if $f$ is a projective morphism:

$$N(X/Y) = \frac{\{\mathbf{Z}\text{-module generated by irreducible } C \text{ such that } f(C) = \text{pt.}\}}{\{\text{cycles } Z \text{ such that } Z\cdot D = 0 \text{ for all Cartier divisors } D\}} \otimes \mathbf{R}$$

$NE(X/Y)$ = effective cone (defined as before)

The Cone Theorem and Contraction Theorem are just as in the absolute case (with the same proofs). In the technique used to prove the Cone Theorem, if the starting curve C has f(C) = point, then all curves produced go to the *same point* in Y.

If X above is a threefold which is smooth (or has only $\mathbf{Q}$-factorial terminal singularities), then successive contractions over Y must lead either to a minimal model over Y or a $\mathbf{Q}$-Fano fibration g', where

$$\begin{tikzcd} X' \arrow[rr, "g'"] \arrow[dr, "f'"'] & & Z' \arrow[dl, "h'"] \\ & Y & \end{tikzcd}$$

g' has connected fibres

$-K_{X'}$ is g'-ample

$\dim Z' < \dim X'$

In case $f$ is birational, then by a succession of divisorial contractions and directed flips, we arrive at

$$f'\colon X' \longrightarrow Y$$

with $K_{X'}$ f'-nef. This implies, analogously to the surface case, that $K_{X'}$ is f'-semi-ample (see Lecture 3).

"Factorization" of birational morphisms over Y follows from:

(7.2) **Proposition:** Let $g\colon Z \longrightarrow X$ be a birational proper morphism (over Y) of normal algebraic (or analytic) varieties such that $K_Z$ is $\mathbf{Q}$-Cartier and such that X has only $\mathbf{Q}$-factorial terminal singularities. Then, if $K_Z$ is g-nef, g is an isomorphism.

(7.3) <u>Analytic case</u>

The situation here which we can handle is $f\colon X \longrightarrow Y$, with Y an analytic space with some mild finiteness assumptions and f projective. The same results hold as in the relative case, because the required relative vanishing theorems are true in this situation. We will see these relative vanishing theorems in upcoming seminars.

(7.4) <u>Varieties with group actions</u>

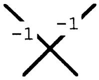
Suppose a projective variety X, smooth or with only $\mathbf{Q}$-factorial terminal singularities, is acted on by a finite group G. Then we have Cone and Contraction Theorems for $NE(X)^G$ in $N(X)^G$. The only difference is that the G-orbit of an extremal ray is an extremal face, since $K_X$ is G-invariant. So the Contraction Theorem involves contraction of G-invariant extremal faces.

There are applications in other settings, too. For example, suppose X is a surface defined over a field k. We achieve a minimal model over k by letting $G = \mathrm{Gal}(K/k)$, where K = algebraic closure of k. Although this is not a finite group, its action on the Neron-Severi group of $X_K$ factors through a finite group, so the construction of a G-minimal model proceeds as in the case of algebraically closed base field.

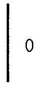
(7.5) In case X is a smooth complex projective surface with G-action, G a finite group, we proceed as before with the classification with some minor changes. A G-extremal ray is generated by a one-cycle of the form $C = \Sigma C_i$, where the $C_i$ are irreducible rational curves in a G-orbit.

1) If $C^2 < 0$, one easily sees that the $C_i$ must be smooth, mutually disjoint, each with self-intersection -1. Thus all the $C_i$ can be blown down to smooth points.

2) If $C^2 = 0$, then the connected components of C must have the form:




or




3) If $C^2 > 0$, then $N(X)^G = \mathbf{Z}$, and $-K_X$ is ample, so that X is a del Pezzo surface.

(7.6) **Theorem:** Suppose now X is in the class of projective G-threefolds with terminal, G**Q**-factorial singularities (i.e. every G-stable Weil divisor is **Q**-Cartier). Any such X is G-birational to:

1) a G-threefold Y in the same birational equivalence class with $K_Y$ nef,

or

2) a G-threefold Y in the same class which has a G-morphism f to a normal projective G-variety Z such that $-K_Y$ is f-ample and

$$\dim Z < \dim X.$$

(7.7) Finally, let's outline a proof (using the minimal model program) of Peternell's theorem that every smooth Moishezon threefold Z which is not projective contains a rational curve: (The original proof was done before the completion of Mori's program in dimension three. It required very skillful computations using only the existence and structure of extremal contractions on smooth threefolds.)

We begin by recalling that we can find a birational morphism

$$f\colon X \longrightarrow Z$$

where X is a smooth *projective* threefold. We apply the steps of the *absolute* minimal model program to X as long as the morphism to Z can be maintained. Then either

1) we arrive at X' minimal, in which case, by Proposition (7.2), X' would have to be isomorphic to Z (ruled out by assumption),

or

2) we come to an extremal contraction $f'\colon X' \longrightarrow X''$ such that the rational map $X'' \dashrightarrow Z$ is *not* a morphism. Since the latter map is not a morphism, by Zariski's Main Theorem at least one fibre of f' is not contracted to a point in Z. But the fibres of f' are covered by rational curves, thus Z must contain a rational curve.

(7.8) References: (7.3) was worked out by Nakayama[Nak]. The original proof of (7.7) is in [P]. The present proof is due to Kollár. The rest of the chapter is in [M3].

## Lecture #8: Vanishing theorems

(8.1) **Principle:** If the cohomology of a sheaf $\mathcal{F}$ comes from topological cohomology, then there is a Kodaira-type vanishing theorem.

By this we mean that, if L is an ample line bundle on X, then

$$H^i(X;\ \mathcal{F}\otimes L^{-1}) = 0, \quad i < \dim X.$$

(8.2) We illustrate this principle by using it to prove the classical case of the theorem, the case $\mathcal{F} = \mathcal{O}_X$, X a smooth projective variety:

8.2.1) Step 1:

A coherent sheaf $\mathcal{F}$ in the analytic topology is also a sheaf of abelian groups. Find a topologically constructible sheaf **F** and a natural map

$$\mathbf{F} \to \mathcal{F}$$

such that the induced map on cohomologies is surjective. (The coherent cohomology of a coherent analytic sheaf is the same as its cohomology as a sheaf of abelian groups.) For $\mathcal{F} = \mathcal{O}$, the constant sheaf $\mathbf{C}_X$ will suffice, since Hodge theory tells us that the natural mapping

$$H^i(X;\ \mathbf{C}_X) \longrightarrow H^i(X;\ \mathcal{O}_X),$$

induced by inclusion of sheaves, is surjective.

8.2.2) Step 2:

Suppose $L^m$ is very ample. Let $s$ be a general section. Then the zero set D of $s$ is a smooth, very ample divisor. Consider the diagram:

$$\begin{tikzcd}
Z = (\text{preimage } s(X) \text{ in } L) \subseteq \arrow[dr, "p", "\text{m to one cyclic branched covering}"'] & L \arrow[r, "\otimes m"] \arrow[d] & L^m \supseteq s(X) \arrow[dl, shift right] \\
D = (\text{branch locus of } p) \subseteq & X \arrow[ur, bend right, "s"'] &
\end{tikzcd}$$

By Hodge theory, the map
$$H^i(Z;\mathbf{C}_Z) \longrightarrow H^i(Z;\mathcal{O}_Z)$$
is surjective. Since the fibres of $p$ are zero-dimensional, there are no higher direct-image sheaves, so
$$H^i(X;p_*\mathbf{C}_Z) \longrightarrow H^i(X;p_*\mathcal{O}_Z)$$
is surjective. The action of $\mathbf{Z}/m\mathbf{Z}$ on Z decomposes this last morphism into a direct sum of morphisms on eigenspaces. The intersection pairing on
$$H^*(X;p_*\mathbf{C}_Z) = H^*(Z;\mathbf{C}_Z)$$
respects this decomposion into eigenspaces; Poincaré duality respects the decomposition also.

8.2.3) Step 3:

Let $\xi$ be the primitive m-th root of unity $e^{2\pi\sqrt{-1}/m}$. We can decompose
$$p_*\mathbf{C}_Z = \oplus\,\mathbf{C}[\xi^r],$$
where $\mathbf{C}[\xi^r]$ denotes the local system that has monodromy $\xi^r$ if we go around the divisor D once. If one denotes by $H^*(X;\,p_*\mathbf{C}_Z)[\xi^r]$ the $\xi^r$-eigenspace of the $\mathbf{Z}_m$ action on $H^*(X;\,p_*\mathbf{C}_Z)$, then we have
$$H^*(X;p_*\mathbf{C}_Z)[\xi^r] = H^*(X;\mathbf{C}[\xi^r]).$$
If $r\neq m$, then, for the inclusion $i\colon (X-D)\longrightarrow X$, the natural map
$$\mathbf{C}[\xi^r] \longrightarrow i_!(\mathbf{C}[\xi^r]|_{X-D})$$
is an isomorphism, where $i_!$ means the extension to X which has zero stalks at points of D. Thus
$$\begin{aligned} H^*(X;p_*\mathbf{C}_Z)[\xi^r] &= H^*(X;\mathbf{C}[\xi^r]) \\ &= H^*(X;i_!(\mathbf{C}[\xi^r]|_{X-D})) = H^*(X-D;\mathbf{C}[\xi^r]|_{X-D}). \end{aligned}$$

8.2.4) Step 4:

We are now ready to finish the argument. Since (Z-D) is affine, it has the homotopy type of a *real* n-dimensional CW-complex. ($n = \dim_{\mathbf{C}} X$.) So for $i<n$,
$$0 = H^{2n-i}(Z-D;\mathbf{C}) = H^{2n-i}(X-D;p_*\mathbf{C}_Z),$$

and, using the above identifications and duality,

$$0 = H^{2n-i}(X-D; \mathbf{C}[\xi^r]|_{X-D}) = H^{2n-i}(X; p_*\mathbf{C}_Z)[\xi^r]$$

which is dual to $H^i(X; p_*\mathbf{C}_Z)[\xi^r]$ for $i<n$ and $r \neq m$. So, by surjectivity,

$$H^i(X; p_*\mathcal{O}_Z)[\xi^r] = 0 \text{ for } i<n.$$

But $H^i(X; p_*\mathcal{O}_Z)[\xi^r] = H^i(X; p_*\mathcal{O}_Z[\xi^r])$. Also $p_*\mathcal{O}_Z[\xi^r] = L^{-r}$, since D is given locally in Z by $z^m = g$ for $z$ a function on L (i.e. a *section* of $L^{-1}$). This completes the proof.

Using the same basic construction, we obtain:

(8.3) **General Vanishing Theorem for Line Bundles:**
Let X be a smooth complex projective variety. Let L be a line bundle on X such that

$$c_1(L) = M + \Sigma a_i D_i$$

such that
1) M is a nef and big $\mathbf{Q}$-divisor,
2) $\Sigma D_i$ is a simple normal-crossing divisor,
3) $0 \leq a_i < 1$, and $a_i \in \mathbf{Q}$ for all $i$.

Then

$$H^i(X; L^{-1}) = 0 \text{ for } i < \dim X.$$

**Proof:** First we give the proof in the special case in which M is ample. The proof in this case is much simpler, and this is the main case that we will use. Choose a positive integer $m$ so that $M^{\otimes m}$ is Cartier and very ample and $m \cdot a_i$ is an integer for each $i$. Take a general divisor B from the linear system of $M^{\otimes m}$. Then B is smooth and meets the $D_i$ transversely. Also

$$D = B + \Sigma m a_i D_i$$

is the zero set of some section of $L^{\otimes m}$. Again we consider:

$$\begin{tikzcd}
Z = (\text{preimage } s(X) \text{ in } L) \subseteq \arrow[drr, "p", "\text{m to one cyclic branched covering}"'] & L \arrow[rr, "\otimes m"] \arrow[dr] & & L^m \supseteq s(X) \arrow[dl] \\
D = (\text{branch locus of } p) \subseteq & & X \arrow[ur, bend right, "s"'] &
\end{tikzcd}$$

The proof now goes just as in the special case we did previously, except that we have to take care of two problems. Let $Z^\wedge$ be the normalisation of $Z$. In general $Z^\wedge$ is singular but $Z^\wedge - p^{-1}(D)$ is still smooth and affine thus it has the homotopy type of a real n-dimensional complex. The singularities of $Z^\wedge$ are all quotient singularities, thus Poincaré duality holds with $\mathbf{Q}$ coefficients.

The other problem is that it is the sheaf

$$p_*\mathcal{O}_Z = \mathcal{O}_X + L^{-1} + \ldots L^{-(m-1)}$$

which obviously contains $L^{-1}$ as a direct summand. Thus it remains to be checked, that, under the inclusion

$$p_*\mathcal{O}_Z \subseteq (p^\wedge)_*\mathcal{O}_{Z^\wedge},$$

$L^{-1}$ goes to a summand. This is where condition 3) in the statement of the theorem enters.

Let $e(i) = ma_i$, and suppose that $D_i$ is locally defined by $f_i = 0$ and $B$ is locally defined by $g = 0$. Then $Z$ is given locally by the equation

$$z^m = g\cdot\Pi f_i^{e(i)}.$$

The r-th summand of $(p^\wedge)_*\mathcal{O}_{Z^\wedge}$ is locally generated by

$$(z^r/g^a\Pi f_i^{b(i)})$$

with m-th power in $\mathcal{O}_X$. So $a = 0$, and $r\cdot e(i) \geq m\cdot b(i)$, that is, $r\cdot a_i \geq b(i)$. When $r = 1$, this means that all $b(i) = 0$ by Condition 3) of the Theorem. Thus $L^{-1}$ is a summand of $(p^\wedge)_*\mathcal{O}_{Z^\wedge}$, and the theorem is proved when $M$ is ample.

The rest of the proof is somewhat technical. The reader who is interested mainly in the applications can skip the rest of this chapter. We need the following auxiliary results:

(8.4) **Corollary**: Let $X$ be a smooth variety and let $Z$ be a codimension $c$ smooth subvariety. Let

$$f\colon Y \longrightarrow X$$

be the blow-up of $Z$ in $X$, and let $E$ be the exceptional divisor. Then for $0 \leq i \leq c-1$ we have

i) $f_*\omega_Y(-iE) = \omega_X$ ;

and

ii) $R^jf_*\omega_Y(-iE) = 0$ for $j > 0$.

**Proof**: Since $\omega_Y = f^* \omega_X((c-1)E)$, the first assertion is trivial. The second one will be proved using the Theorem on Formal Functions. For simplicity of notation, we compute the case when Z is a point. Then $E = \mathbf{P}^{c-1}$, and $\omega_Y(-iE)|_E = \mathcal{O}_E(i+1-c) = \omega_E(i+1)$. Thus

$$H^j(E;\ \omega_Y(-iE)(-kE)|_E) = 0$$

for $k \geq 0$ and $j > 0$. If $\mathcal{O}_{kE}$ denotes the kth-order neighborhood of E then we have an exact sequence

$$0 \longrightarrow \omega_Y(-iE)(-kE)|_E \longrightarrow \omega_Y(-iE)\otimes\mathcal{O}_{(k+1)E} \longrightarrow \omega_Y(-iE)\otimes\mathcal{O}_{kE} \longrightarrow 0.$$

Thus the vanishing of $H^j(E;\ \omega_Y(-iE)(-kE)|_E)$ gives inductively the vanishing of $H^j(\omega_Y(-iE)\otimes\mathcal{O}_{kE})$ for every $k \geq 0$ and $j > 0$. This is what we wanted.

(8.5) **Definition**:

(i) Let X be a smooth algebraic variety, Z a subvariety and $D_i$ divisors. We say that Z and the $D_i$ **cross normally** if, for every point x of X, there is a local analytic coordinate system $(x_j)$ such that locally every $D_i$ passing through x is a coordinate hyperplane and, if Z passes through x, Z is the intersection of some coordinate hyperplanes. (Z may lie in some of the $D_i$.)

(ii) Given a birational morphism $g: Y \longrightarrow X$ between smooth varieties, a subvariety Z of Y and divisors $D_i$ on X we say that Z and the $D_i$ **cross normally** if Z, the proper transforms of the $D_i$ and the exceptional divisors of g cross normally on Y.

(8.6) **Corollary**: Let X be a smooth variety and let Z be a codimension c smooth subvariety. Let $f: Y \longrightarrow X$ be the blow-up of Z in X, and let E be the exceptional divisor. Let L, M and $D_i$ as in (8.3). Assume Z and $D_i$ cross normally. Let $D_i'$ be the proper transform of $D_i$. Then there is $0 \leq k \leq c-1$ such that for

$$f^*(\Sigma a_i D_i) - kE = \Sigma a_i D_i' + bE$$

we have

i) $0 \leq b < 1$,

and

ii) $H^j(Y;\ \omega_Y(-kE)\otimes f^*L) = H^j(X;\ \omega_X\otimes L)$.

**Proof**: Assume that $D_1, \dots, D_p$ are those divisors that contain Z. Since the $D_i$ intersect transversally, $p \leq c$. E appears in $f^*(\Sigma a_i D_i)$ with multiplicity $a_1 + \dots + a_p < c$. Now let

$$k = [a_1 + \dots + a_p],$$

where "[ ]" denotes "greatest integer in." Statement ii) now follows from (8.4) and the Leray spectral sequence.

(8.7) **Proof of (8.3)**: Pick any ample divisor H. For large k, $H^0(kM) >> H^0(kM|_H)$. Thus we can write $kM = H + B$, where B is effective, and so we can write for each positive integer N:

$$M \equiv N^{-1}(H + (N-k)M) + N^{-1}B,$$

where the first summand is ample and the second one is effective. Let $\varepsilon$ be such that $a_i/\varepsilon$ is integral for every i. Now choose a resolution $f: Y \longrightarrow X$ with exceptional divisor $\Sigma E_i$ and then N sufficiently large such that

i) f is a composition of blowing-ups with centers $Z_{i-1}$

$$f_i: Y_i \longrightarrow Y_{i-1}$$

such that $Z_{i-1}$ and $D_i$ cross normally.
$(X = Y_0$ and $Y = Y_n)$,

ii) $\Sigma E_i + f^*(B + \Sigma D_i)$ has simple normal crossings only,

iii) $f^*(N^{-1}(H + (N-k)M)) - \Sigma p_j E_j$ is ample for some $0 < p_j << \varepsilon$,

iv) every divisor in $f^* N^{-1} B + \Sigma p_j E_j$ appears with coefficient less than $\varepsilon$.

The trouble is that in $f^*(\Sigma D_i)$ the exceptional divisors can appear with coefficients larger than 1, therefore we cannot apply our vanishing for the pull-back. Quite miraculously the situation becomes tractable if we consider the dual form of vanishing.

Repeatedly applying (8.6) we get that there is a linear combination

$$\Sigma k_j E_j$$

such that

i) the $k_j$ are integers,

ii) in $f^* \Sigma a_i D_i - \Sigma k_j E_j$, every divisor appears with coefficient less than 1,

iii) $H^j(Y; \omega_Y(-\Sigma k_j E_j) \otimes f^*L) = H^j(X; \omega_X \otimes L)$.

Now we can look at

$$f^*L - \Sigma k_j E_j \equiv$$
$$\{f^*(N^{-1}(H+(N-k)M)) - \Sigma p_j E_j\} + f^*N^{-1}B + \Sigma p_j E_j + f^*\Sigma a_i D_i - \Sigma k_j E_j.$$

In $f^*\Sigma a_i D_i - \Sigma k_j E_j$, every divisor appears with coefficient less than 1, and so by the choice of $\varepsilon$ in fact with coefficient $\leq 1-\varepsilon$. Thus

$$f^*L - \Sigma k_j E_j$$

is written as the sum of an ample divisor and of a **Q**-divisor with normal crossings and coefficients less than 1. Thus by the already proved case

$$H^j(Y;\ \omega_Y \otimes f^*L(-\Sigma k_j E_j)) = 0 \text{ for } j > 0.$$

By iii) above, this gives that

$$H^j(X;\ \omega_X \otimes L) = 0,$$

which is the required vanishing.

(8.8) **Corollary**: Let $f: Y \longrightarrow X$ be a birational morphism, Y smooth. Assume that M is a nef line bundle on Y. Then, for $i > 0$,

$$R^i f_*(\omega_Y \otimes M) = 0.$$

In particular,

$$R^i f_* \omega_Y = 0.$$

**Proof**: Choose H ample on X. Apply (8.3) to $L = f^*H \otimes M$ on Y and then use the following:

(8.9) **Proposition**: Let $f: Y \longrightarrow X$ be a morphism and let $\mathcal{F}$ be a sheaf on Y. Then the following are equivalent:

i) $H^j(Y; \mathcal{F} \otimes f^*L) = 0$ for every L which is sufficiently ample on X,

ii) $R^j f_* \mathcal{F} = 0$.

**Proof**: Choose L such that $H^i(X; L \otimes R^k f_* \mathcal{F}) = 0$ for all $i > 0$ and $k$. Then the Leray spectral sequence degenerates at $E_2$. Thus

$$H^j(Y; \mathcal{F} \otimes f^*L) = H^0(X; L \otimes R^j f_* \mathcal{F}).$$

(8.10) References: The General Vanishing theorem was first proved by Miyaoka[Mi] for surfaces and by Kawamata[Ka2] and Viehweg[V] in general. The special case of (8.8) is due to Grauert-Riemenschneider[GR].

**Lecture #9: Introduction to the proof of the Cone Theorem**

In Lecture #4, we proved the Cone Theorem for smooth varieties. We now begin a sequence of theorems leading to the proof of the Cone Theorem in the general case. This proof is built on a very different set of ideas. Applied even in the smooth case, it gives results not accessible by the previous method; namely it proves that extremal rays can always be contracted. On the other hand, it gives little information about the curves that span an extremal ray. Also, this proof works only in characteristic 0. Before proceeding, we reformulate slightly the Vanishing Theorem proved in Lecture 8:

(9.1) Let Y be a non-singular complex projective variety. Let $\Sigma d_i D_i$ be a **Q**-divisor on Y, written as a sum of *distinct* prime divisors, and let L be a line bundle (or Cartier divisor). Let

$$D = L + \Sigma d_i D_i.$$

We define the **round-up**

$$\lceil D \rceil$$

of D to be the divisor

$$L + \Sigma e_i D_i,$$

where $e_i$ is the smallest integer $\geq d_i$.

(9.2) **Theorem:** Suppose that D as above is nef and big and that $\Sigma D_i$ has only simple normal crossings. Then

$$H^i(K_Y + \lceil D \rceil) = 0 \quad \text{for } i > 0.$$

We will prove four basic theorems finishing with the Cone Theorem:

(9.3) **Basepoint-free Theorem:**

Let X be a projective variety with only canonical singularities. Let D be a nef Cartier divisor such that

$$aD - K_X$$

is nef and big for some $a > 0$. Then $|bD|$ has no basepoints for all $b >> 0$.

(9.4) **Non-vanishing Theorem:**

Let X be a non-singular projective variety, D a nef Cartier divisor and G a **Q**-divisor with $\lceil G \rceil$ effective. Suppose

i) $aD + G - K_X$ is ample for some $a > 0$,

ii) the fractional part of G has only simple normal crossings.

Then, for all $m >> 0$,

$$H^0(X; mD + \lceil G \rceil) \neq 0.$$

(9.5) **Rationality Theorem:**

Let X be a projective variety with only canonical singularities such that $K_X$ is not nef. Let H be an ample Cartier divisor, and define

$$r = \max\{t \in \mathbf{R}: H + tK_X \text{ nef}\}.$$

Then r is a *rational number* of the form u/v where

$$0 < v \leq (\operatorname{index} X)(\dim X + 1).$$

(9.6) **Cone Theorem:**

Let X be a projective variety with only canonical singularities. Then

1) $\langle NE(X) \rangle = (\langle NE(X) \rangle \cap (K_X)_{\geq 0}) + \Sigma(\mathbf{R}_{\geq 0})[C_j]$

for a collection of curves $C_j$ with $K_X \cdot C_j < 0$.

(The sum has the property that the set of $C_j$ is minimal--no smaller set is sufficient to generate the cone. The $(\mathbf{R}_{\geq 0})[C_i]$ which, together with $(\langle NE(X) \rangle \cap (K_X)_{\geq 0})$, form a minimal generating set for $\langle NE(X) \rangle$, are called **extremal rays**.)

2) For any $\varepsilon > 0$ and ample divisor H, 1) gives

$$\langle NE(X) \rangle \cap (K_X + \varepsilon H)_{\leq 0} = (\langle NE(X) \rangle \cap (K_X + \varepsilon H)_{=0}) + \Sigma_{\text{finite}}(\mathbf{R}_{\geq 0})[C_j].$$

(9.7) The logical order of proof of these theorems is the following: Non-vanishing Theorem $\Rightarrow$ Basepoint Free $\Rightarrow$ Rationality $\Rightarrow$ Cone Theorem. However for better understanding we prove first Basepoint Freeness and then the Cone Theorem. The proofs of Non-vanishing and of Rationality utilize the same ideas, but they are technically more involved. These proofs will be presented at the end.

(9.8) The basic strategy for proving the Basepoint-free Theorem (as well as for proving the Non-vanishing and Rationality Theorems) is as follows. We work with resolutions $f: Y \longrightarrow X$, and with smooth divisors $F_j$ which are either fixed divisors of $|aD|$ or exceptional over X. We show that we can single out one $F_j$, call it F, and an effective sum A' of exceptional divisors so that

$$H^0(F;\ (b\cdot f^*D + A')|_F) \neq 0 \quad \text{(Non-van. Th.)}$$

and

$$H^1(Y;\ b\cdot f^*D + A' - F) = 0 \quad \text{(Van. Th.)},$$

for sufficiently large b. Since

$$\begin{array}{ccc} H^0(X;\ b\cdot D) & \longrightarrow & H^0(F;\ (b\cdot f^*D)|_F) \\ \downarrow = \text{ since A' effective} & & \downarrow \\ H^0(Y;\ b\cdot f^*D + A') & \longrightarrow & H^0(F;\ (b\cdot f^*D + A')|_F) \end{array}$$

this means that f(F) is not contained in the base locus of $|bD|$ by

$$H^0(Y;\ b\cdot f^*D + A') \longrightarrow \underset{\text{non-zero}}{H^0(F;\ (b\cdot f^*D + A')|_F)} \longrightarrow \underset{\text{zero}}{H^1(Y;\ b\cdot f^*D + A' - F)},$$

(although f(F) is contained in the base locus of $|aD|$). An iteration will then eliminate the base locus altogether for all sufficiently high multiple of D.

(9.9) So we will need to worry about the restriction of $\mathbf{Q}$-divisors and their round-ups to smooth hypersurfaces F of a non-singular Y. We only restrict divisors

$$D = L + \Sigma d_i D_i$$

where either $F \neq D_i$ for any i, or $F = D_j$ for some j for which $d_j$ is an integer. In the latter case, we absorb $d_j D_j$ into L before restricting. In either case, we only consider situations in which the sum of the remaining $F_i$ meets F in a simple normal crossing divisor. Then *round-up commutes with restriction.*

(9.10) References: The proofs of these four theorems are fairly interwoven in history. For smooth threefolds Mori[M1] obtained some special cases. The first general result for threefolds was obtained by Kawamata[Ka3], completed by Benveniste[B1] and Reid[R4]. Non-vanishing was done by Shokurov[Sh]. The Cone Theorem appears in [Ka4] and is completed in [Ko3].

**Lecture #10: Basepoint-free Theorem**

(10.1) Step 1: In this step, we establish that $|mD| \neq \emptyset$ for every $m \gg 0$. By our assumptions on X and D, we have as in (8.7) that

$$aD - K_X \equiv (\text{ample divisor}) + N^{-1}(\text{fixed effective divisor})$$

for $N \gg 0$. So we can construct some resolution

$$f\colon Y \longrightarrow X$$

which has a simple-normal-crossing divisor $\Sigma F_j$ such that

1) $K_Y \equiv f^*K_X + \Sigma a_j F_j$ with all $a_j \geq 0$,
2) $f^*(aD - K_X) - \Sigma p_j F_j$ is ample for some $a > 0$ and for suitable $0 < p_j \ll 1$.

On Y, we write a divisor

$$f^*(aD - K_X) - \Sigma p_j F_j = af^*D + (\Sigma a_j F_j - \Sigma p_j F_j) - (f^*K_X + \Sigma a_j F_j)$$

$$= af^*D + G - K_Y,$$

where $G = \Sigma(a_j - p_j)F_j$. By assumption, $\lceil G \rceil$ is an effective f-exceptional divisor ($a_j > 0$ only when $F_j$ is f-exceptional),

$$af^*D + G - K_Y$$

is ample, and

$$H^0(Y;\, mf^*D + \lceil G \rceil) = H^0(X;\, mD).$$

We can now apply Non-vanishing to get that

$$H^0(X;\, mD) > 0 \text{ for all } m \gg 0.$$

(10.2) Step 2: We let $c > 1$ and define

$$B(c) = \text{reduced base locus of } |cD|.$$

Clearly

$$B(c^a) \subseteq B(c^b)$$

for any positive integers $a > b$. Noetherian induction implies that the sequence $B(c^n)$ stabilizes, and we call the limit $B_c$. So either $B_c$ is non-empty for some $c$ or $B_c$ and $B_{c'}$ are empty for two relatively prime integers $c$ and $c'$. In the latter case, take $a$ and $b$ such that $B(c^a)$ and $B(c'^b)$ are empty, and use that every sufficiently large integer is a linear combination of $c^a$ and $c'^b$ with non-negative coefficients to conclude that $|mD|$ is basepoint-free for all $m \gg 0$. So we must show that the assumption that some $B_c$ is non-empty leads to a contradiction. We let $m = c^a$ such that $B_c = B(m)$ and assume that this set is non-empty.

Starting with the linear system obtained from the Non-vanishing Theorem, we can blow up further to obtain a new

$$f\colon Y \longrightarrow X$$

for which the conditions of Step 1 hold, and, for some $m > 0$,

$$f^*|mD| = |L| \text{(moving part)} + \Sigma r_j F_j \text{(fixed part)}$$

with $|L|$ basepoint-free. Therefore $\bigcup\{f(F_j): r_j > 0\}$ is the base locus of $|mD|$. Note that

$$|mD| \text{ is basepoint-free}$$

if and only if

$$f^*|mD| \text{ is basepoint-free}$$

if and only if

$$r_j = 0 \text{ for all } j.$$

We obtain the desired contradiction by finding some $F_j$ with $r_j > 0$ such that, for all $b \gg 0$, $f(F_j)$ is not contained in the base locus of $|bD|$.

(10.3) Step 3: For an integer $b > 0$ and a rational number $c > 0$ such that $b \geq cm+a$, we define divisors:

$$N(b,c) = bf^*D - K_Y + \Sigma(-cr_j+a_j-p_j)F_j$$

$$\equiv \underbrace{f^*(b-cm-a)D}_{\text{nef}} + \underbrace{c(f^*mD - \Sigma r_jF_j)}_{\text{basepoint-free}} + \underbrace{(f^*(aD-K_X) - \Sigma p_jF_j)}_{\text{ample}}.$$

Thus, $N(b,c)$ is ample.

Since $N(b,c)$ is ample for $b \geq cm+a$, we have, by the Vanishing Theorem,

$$H^1(Y; \lceil N(b,c)\rceil + K_Y) = 0$$

where

$$\lceil N(b,c)\rceil = bf^*D + \Sigma\lceil -cr_j+a_j-p_j\rceil F_j - K_Y.$$

(10.4) Step 4: $c$ and the $p_j$ can be so chosen that, for some $F = F_{j'}$, we have that

$$\Sigma(-cr_j+a_j-p_j)F_j = A - F,$$

where $\lceil A\rceil$ is effective and $A$ does not have $F$ as a component. In fact, we choose $c > 0$ so that

$$\min\{-cr_j + a_j - p_j\} = -1.$$

If this last condition does not single out a *unique* $j$, we wiggle the $p_j$ slightly to achieve the desired unicity. This $j$ will have $r_j > 0$ and

$$\lceil N(b,c) \rceil + K_Y = bf^*D + \lceil A \rceil - F.$$

Now Step 3 implies that

$$H^0(Y;\ b\cdot f^*D + \lceil A \rceil) \longrightarrow H^0(F;\ (b\cdot f^*D + \lceil A \rceil)|_F)$$

is a surjection for $b \geq cm+a$.

Note: If that if any $F_j$ appears in $\lceil A \rceil$, then $a_j > 0$, so $F_j$ is f-exceptional. Thus $\lceil A \rceil$ is f-exceptional.

(10.5) Step 5: Notice that

$$\begin{aligned} N(b,c)|_F &= (bf^*D + A - F - K_Y)|_F \\ &= (bf^*D + A)|_F - K_F. \end{aligned}$$

So we can apply the Non-vanishing Theorem on F to get

$$H^0(F;\ (b\cdot f^*D + \lceil A \rceil)|_F) \neq 0.$$

So $H^0(Y;\ b\cdot f^*D + \lceil A \rceil)$ has a section not vanishing on F. But, since $\lceil A \rceil$ is f-exceptional and effective,

$$H^0(Y;\ b\cdot f^*D + \lceil A \rceil) = H^0(X;\ b\cdot D) = H^0(Y;\ b\cdot f^*D).$$

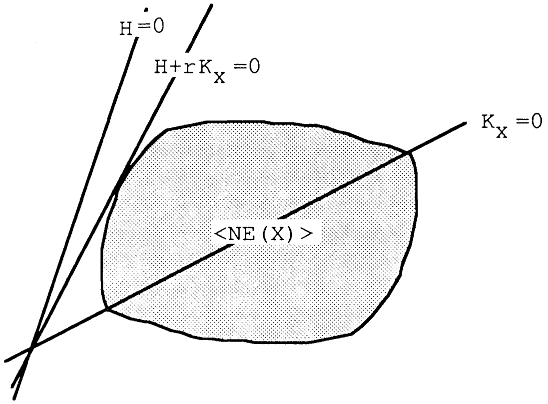
So, as in (9.8), f(F) is not contained in the base locus of $|b\cdot D|$ for all $b \gg 0$. This completes the proof of the Basepoint-free Theorem.

(10.6) Reference: This proof is taken almost verbatim from [R4].

**Lecture #11: The Cone Theorem**

(11.1) First we give an infomal explanation of the way the Rationality Theorem is used to get information about the cone of curves.

If (Picard no. of X) $\geq 2$ and H is ample, we have in N(X):




Since $r$ is rational by the Rationality Theorem, $m(H + rK_X)$ is Cartier for some $m > 0$. Note that $m(H + rK_X)$ is nef but not ample. Thus $(<NE(X)> \cap \{H+rK_X = 0\})$ is a "face" of $<NE(X)>$. Starting with various ample divisors, we get various faces of $<NE(X)>$. The proof of the Cone Theorem turns out to be a completely formal consequence of this observation. To be precise, the Cone Theorem follows immediately from the Rationality Theorem and the following abstract result:

(11.2) **Theorem:** Let $N_{\mathbf{Z}}$ be a free $\mathbf{Z}$-module of finite rank and $N_{\mathbf{R}}$ the base change to (tensor product over $\mathbf{Z}$ with) $\mathbf{R}$. Let $<NE>$ be a closed convex cone not containing a straight line. Let K be an element of the dual $\mathbf{Z}$-module $N_{\mathbf{Z}}{}^*$ such that $(K\cdot C) < 0$ for some $C \in <NE>$. Assume that there exists $a > 0$ such that, for all $H \in N_{\mathbf{Z}}{}^*$ with $H > 0$ on $<NE> - \{0\}$,

$$r = \max\{t \in \mathbf{R}\colon H + tK \geq 0 \text{ on } <NE>\}$$

is a rational number of the form $u/v$ such that $0 < v \leq a$. Then

$$<NE> = (<NE(X)> \cap K_{\geq 0}) + \Sigma(\mathbf{R}_{\geq 0})[\xi_i]$$

for a collection of $\xi_i \in N_{\mathbf{Z}}$ with $(\xi_i \cdot K) < 0$ such that the $(\mathbf{R}_{\geq 0})[\xi_i]$ don't accumulate in $K_{<0}$ (see 2) of (9.6)).

**Proof of Theorem (11.2) and the Cone Theorem:**

We may assume that $K_X$ is not nef.

(11.3) Step 1: Let L be any non-ample, nef divisor class such that $L^{\perp}$ does not meet $(\langle NE(X)\rangle \cap (K_X)_{\geq 0})$ except at 0. Define

$$F_L = L^{\perp} \cap \langle NE(X)\rangle.$$

Then, by Kleiman's criterion, $F_L \neq \{0\}$. Let H be an arbitrary ample Cartier divisor. For $\nu \in \mathbf{N}$, let $e = ((\operatorname{index} X)(\dim X + 1))!$ and

$$r_L(\nu, H) = \max\{t \in \mathbf{R}\colon \nu L + H + (t/e)K_X \text{ is nef}\}.$$

By the Rationality Theorem, $r_L(\nu, H)$ is a (non-negative) integer, and, since L is nef, $r_L(\nu, H)$ is a non-decreasing function of $\nu$. Now $r_L(\nu, H)$ stabilizes to a fixed $r_L(H)$ for $\nu \geq \nu_0$ since, if $\xi \in F_L$, then

$$r_L(\nu, H) \leq e(H\cdot\xi)/(-K_X\cdot\xi).$$

Also L and

$$\nu_0 eL + eH + r_L(H)K_X$$

are both non-ample nef divisors, so, putting

$$D(\nu L, H) = \nu eL + eH + r_L(H)K_X,$$

we have

$$0 \neq F_{D(\nu L, H)} \subseteq F_L \text{ for } \nu \geq \nu_0.$$

(11.4) Step 2: We claim that, if $\dim F_L > 1$, then we can find an ample H with

$$\dim F_{D(\nu L, H)} < \dim F_L.$$

To see this, choose ample divisors $H_i$ which give a basis for $N(X)^*$. If $\dim F_L > 1$, the equations

$$\nu L + H_i + (r_L(H_i)/e)K_X = 0$$

cannot all be identically satisfied on $F_L$ since they give independent conditions. Repeating the argument over successively smaller faces, we obtain that for every L there is an L' such that

$$F_L \supset F_{L'} \quad \text{and} \quad \dim F_{L'} = 1.$$

(11.5) Step 3: We claim that

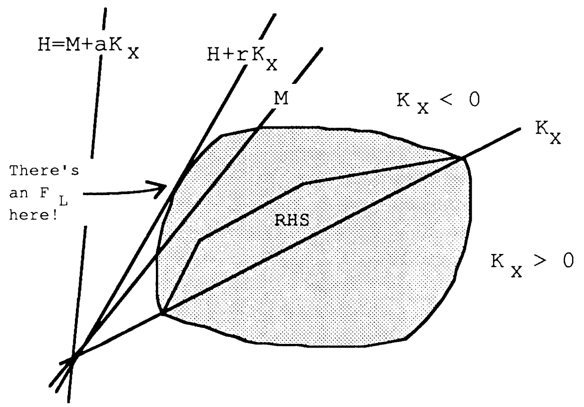
$$\langle NE(X)\rangle = (\langle NE(X)\rangle \cap (K_X)_{\geq 0}) + \langle \Sigma F_L \rangle,$$

where, as above, we sum over L such that $\dim F_L = 1$.
(Recall that "< >" means "closure".)

To prove this, assume that the right-hand-side of the claimed equality is smaller. Then there is a divisor so that the hyperplane

$$M = 0$$

misses the right-hand-side but not the left-hand-side:




The straightforward application of the Rationality Theorem to r in the above picture, followed by Step 2, gives a contradiction.

(11.6) Step 4: Next we show that the one-dimensional $F_L$ "don't accumulate" in $(K_X)_{<0}$. To see this, let

$$\{H(i)\}$$

be a set of ample Cartier divisors which, together with $K_X$, form a basis of $N(X)^*$. For each one-dimensional $F_L$ and i, take $\nu(i)$ such that

$$F_{D(\nu(i)L,H(i))} = F_L.$$

Then, for $\xi$ generating $F_L$ and for all i,

$$(*)\qquad (\xi\cdot H(i))/(\xi\cdot K_X) = (\text{integer})/e.$$

If the $F_L$ accumulated somewhere in $(K_X)_{<0}$, then the points of the projectivization

$$(N(X)-\{0\})/\mathbf{R}^*$$

of $N(X)$ to which they correspond would have to accumulate somewhere in the affine subset $U$ of $(N(X)-\{0\})/\mathbf{R}^*$ given by

$$K_X \neq 0.$$

But the equation (*) just above rules out that possibility, because

$$\xi \in U \longrightarrow ((\xi\cdot H(i))/(\xi\cdot K_X))_i$$

is an affine coordinate system.

(11.7) Step 5: Finally, for each one-dimensional $F_L$, the Rationality and Basepoint-free Theorems show that there exists a morphism contracting only $F_L$ and so

$$F_L = (\mathbf{R}_{\geq 0})[C]$$

for some curve C. So we now have

$$\langle NE(X)\rangle = (\langle NE(X)\rangle \cap (K_X)_{\geq 0}) + \Sigma(\mathbf{R}_{\geq 0})[C_j],$$

and the Cone Theorem is proved.

(11.8) References: This proof of the cone theorem is new. It grew out of conversations among J. Kollár, T. Luo, K. Matsuki and S. Mori.

## Lecture #12: Rationality Theorem

(12.1) **Proof of the Rationality Theorem:**

Step 1: Suppose $Y$ is a smooth projective variety, and suppose $\{D_i\}$ is a finite collection of Cartier divisors and $A$ is a fractional simple-normal-crossing divisor with $\lceil A \rceil$ effective. Consider the Poincaré polynomial

$$P(u_1, \ldots, u_k) = \chi(\Sigma u_i D_i + \lceil A \rceil).$$

Suppose that, for some values of the $u_i$, $\Sigma u_i D_i$ is nef and

$$\Sigma u_i D_i + A - K_Y \text{ is ample.}$$

Then, for all integers $m >> 0$,

$$\Sigma m u_i D_i + A - K_Y$$

is still ample so that

$$H^i(\Sigma m u_i D_i + \lceil A \rceil) = 0$$

for $i > 0$ by the Vanishing Theorem, and

$$\mathcal{O}(\Sigma m u_i D_i + \lceil A \rceil)$$

must have a section by the Non-vanishing Theorem. Therefore

$$\chi(\Sigma m u_i D_i + \lceil A \rceil) \neq 0.$$

Thus $P(u_1, \ldots, u_k)$ is not identically zero and its degree is less than or equal to $\dim Y$.

(12.2) Step 2:

**Claim:** Let $r \in \mathbf{R}$.

a) Let $P(x,y)$ be a non-trivial polynomial of degree $\leq n$, and assume that $P$ vanishes for all sufficiently large integral solutions of

$$0 < ay - rx < \varepsilon$$

for some fixed positive integer $a$ and positive $\varepsilon$. Then $r$ is rational.

b) Let $r$ be as in Part a). Then, in reduced form, $r$ has denominator

$$\leq a(n+1)/\varepsilon.$$

**Proof:** a) First assume $r$ irrational. Then an infinite number of integral points in the $(x,y)$-plane on each side of the line

$$ay - rx = 0$$

are closer than $\varepsilon/(n+2)$ to that line. So there is a large integral solution $(x',y')$ with

$$0 < ay' - rx' < \varepsilon/(n+2).$$

But then

$$(2x',2y'), \ldots, ((n+1)x',(n+1)y')$$

are also solutions by hypothesis. So

$$(y'x - x'y)$$

divides $P$, since $P$ and $(y'x - x'y)$ have $(n+1)$ common zeroes. Choose a smaller $\varepsilon$ and repeat the argument. Do this $n+1$ times to get a contradiction.

b) Now suppose $r = u/v$ (in lowest terms). For given $j$, let $(x',y')$ be a solution of

$$ay - rx = aj/v.$$

(Note that an integral solution exists for any $j$.) Then

$$a(y'+ku) - r(x'+akv) = aj/v$$

for all $k$. So, as above, if

$$aj/v < \varepsilon,$$

$(ay - rx) - (aj/v)$ must divide $P$. So we can have at most $n$ such values of $j$. Thus

$$a(n+1)/v \geq \varepsilon.$$

(12.3) Step 3: Let $\varepsilon$ be a positive number. Let $H$ be an ample Cartier divisor. Let $a \in \mathbf{Z}$ be such that $aK_X$ is also Cartier. Assume that $K_X$ is not nef and let

$$r = \max\{t \in \mathbf{R}\colon H + tK_X \text{ nef}\}.$$

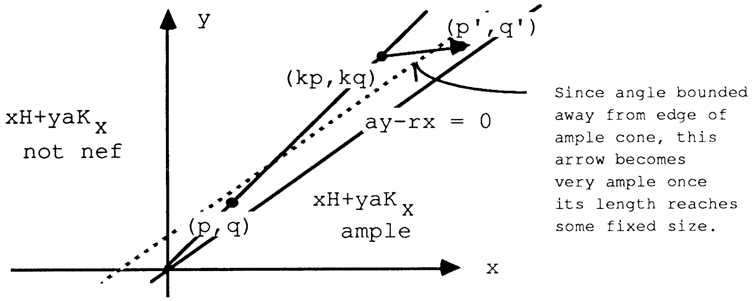
For each $(p,q)$, let

$$\Lambda(p,q) = \text{base locus (with reduced scheme structure) of the linear system } |pH+qaK_X| \text{ on } X.$$

By definition, $\Lambda(p,q) = X$ if $|pH+qaK_X| = \emptyset$.

(12.4) **Claim:** For $(p,q)$ sufficiently large and $0 < aq-rp < \varepsilon$, $\Lambda(p,q)$ is the same subset of $X$. We call this subset $\Lambda_0$.

**Proof**: Consider the following diagram of divisors on X:




The above diagram shows that

$$\Lambda(p',q') \subseteq \Lambda(p,q),$$

which proves the claim by the Noetherian condition on subvarieties.

(12.5) For $(p,q)$ as in (12.4), the linear system $|pH+qaK_X|$ *cannot* be basepoint-free on X since $pH+qaK_X$ is not nef. We let $\mathcal{J} \subseteq \mathbf{Z}\times\mathbf{Z}$ be the set of $(p,q)$ for which

$$0 < aq-rp < 1$$

and $\Lambda(p,q) = \Lambda_0$. Let us emphasize that $\mathcal{J}$ contains all sufficiently large $(p,q)$ with $0 < aq-rp < 1$.

(12.6) Step 4: Suppose X has only canonical singularities. Let

$$g\colon Y \longrightarrow X$$

be a resolution which is a composite of blow-ups of closed subvarieties such that the exceptional set is a divisor $\cup E_i$ with simple normal crossings. We can choose $\varepsilon_i > 0$ such that

$$-E = -\Sigma\, \varepsilon_i E_i$$

is g-ample. Let $A = \Sigma\, a_i E_i$ be an effective $\mathbf{Q}$-divisor such that $A \equiv K_Y - g^*K_X$. Let $D_1 = g^*H$ and $D_2 = g^*(aK_X)$.

Then we put

$$P(x,y) = \chi(xD_1+yD_2+\lceil A\rceil).$$

Since $D_1$ is nef and big, P is not identically zero by Riemann-Roch. Since A' is effective and g-exceptional,

$$H^0(Y; pD_1+qD_2+\lceil A\rceil) = H^0(X; pH+qaK_X).$$

(12.7) Step 5: Suppose now that the assertion of the Rationality Theorem that $r$ is rational is *false*. If

$$0 < ay-rx < 1,$$

then

$$xD_1+yD_2+A-K_Y$$

is numerically equivalent to the pull-back of the ample $\mathbf{Q}$-divisor

$$xH + (ay-1)K_X.$$

Thus, for some $1 >> \delta > 0$,

$$xD_1 + yD_2 + A - K_Y - \delta E$$

is ample and $\lceil A - \delta E \rceil = \lceil A \rceil$. Thus, by the Vanishing Theorem,

$$H^i(Y;\ xD_1+yD_2+\lceil A \rceil) = 0 \text{ for } i > 0.$$

By Step 2, there must exist arbitrarily large $(p,q)$ with $0 < aq-rp < 1$ for which

$$P(p,q) = h^0(Y;\ pD_1+qD_2+\lceil A \rceil) \neq 0,$$

since otherwise $P(x,y)$ would vanish "too often" implying that $r$ is *rational* for $X$ and $H$. Thus

$$|pH+qaK_X| \neq \emptyset$$

for all $(p,q) \in \mathcal{A}$. See (12.5).

(12.8) Step 6: For $(p,q) \in \mathcal{A}$, choose a resolution

$$f\colon Y \longrightarrow X$$

such that there exists a simple-normal-crossing divisor $\Sigma F_j$ with the following properties:

a) $K_Y = f^*K_X + \Sigma a_jF_j$ for $a_j$ non-negative and rational.

b) $f^*(pH+(qa-1)K_X) - \Sigma p_jF_j$ is ample for some sufficiently small, positive $p_j$.

(This is possible since $pH+(qa-1)K_X$ is ample.)

c) $|f^*(pH+qaK_X)| = |L|$ (basepoint-free part) $+ \Sigma r_jF_j$ (fixed part)

for some non-negative integers $r_j$.

(12.9) Step 7: Let $(p,q) \in \mathcal{A}$ be as chosen in (12.8). As before, we can choose rational $c > 0$ and $p_j > 0$ so that

$$\Sigma(-cr_j+a_j-p_j)F_j = A' - F$$

with $\lceil A' \rceil$ effective, $A'$ not involving $F$. By examining coefficients, we notice that $F$ maps into some component $B$ of the base locus $\Lambda(p,q)$ of $|pH+qaK_X|$. Define

$$N(p',q') = f^*(p'H+q'aK_X) + A' - F - K_Y$$

$$\equiv f^*((p'-(1+c)p)H + (q'-(1+c)q)aK_X) + f^*((1+c)pH + (1+c)qaK_X) + \Sigma(-cr_j+a_j-p_j)F_j - K_Y$$

$$\begin{aligned} \equiv\ & cL && \text{bp-free} \\ & + f^*((p'-(1+c)p)H + (q'-(1+c)q)aK_X) && \text{nef if } p',q' \text{ big enough} \\ & && \text{and } (q'-(1+c)q)a \le r(p'-(1+c)p) \\ & + f^*(pH+(qa-1)K_X) - \Sigma p_jF_j && \text{ample} \end{aligned}$$

Notice that if $p'$ and $q'$ are big enough and

$$aq'-rp' \le aq-rp,$$

then

$$(q'-(1+c)q)a \le r(p'-(1+c)p),$$

so $N(p',q')$ is ample. Thus, by the Vanishing Theorem, the map

$$H^0(Y;\ f^*(p'H+q'aK_X)+\lceil A' \rceil) \longrightarrow H^0(F;\ (f^*(p'H+q'aK_X)+\lceil A' \rceil)|_F)$$

is surjective.

(12.10) Step 8: By the adjunction formula, the restriction of the divisor

$$f^*(p'H+q'aK_X) + A' - F - K_Y$$

to $F$ is the divisor

$$(f^*(p'H+q'aK_X) + A')|_F - K_F.$$

As in Step 1, the Poincaré polynomial

$$\chi(F;\ (f^*(p'H+q'aK_X)+\lceil A' \rceil)|_F)$$

is not identically zero.

But, for $0 < aq'-rp' \leq aq-rp$,

$$(f^*(p'H+q'aK_X) + A')|_F - K_F$$

is ample, so, in this strip,

$$\chi(F;\ (f^*(p'H+q'aK_X)+\lceil A'\rceil)|_F) = h^0(F;\ (f^*(p'H+q'aK_X)+\lceil A'\rceil)|_F).$$

So, by Part a) in Step 2 applied to the Poincaré polynomial on F with $\varepsilon = aq-rp$, there must be arbitrarily large $(p',q')$ such that

$$0 < aq'-rp' \leq aq-rp$$

and

$$h^0(F;\ (f^*(p'H+q'aK_X)+\lceil A'\rceil)|_F) \neq 0.$$

(12.11) Step 9: We are now ready to derive a contradiction. By assumption $\Lambda(p,q) = \Lambda_0$. For $(p',q')$ as in Step 8

$$H^0(Y;\ f^*(p'H+q'aK_X)+\lceil A\rceil) \longrightarrow H^0(F;\ (f^*(p'H+q'aK_X)+\lceil A'\rceil)|_F) \neq 0$$

is surjective. Thus F is not a component of the base locus of $|f^*(p'H+q'aK_X) + \lceil A\rceil|$. Since $\lceil A\rceil$ is f-exceptional and effective,

$$H^0(Y;\ f^*(p'H+q'aK_X)+\lceil A\rceil) = H^0(X;\ p'H+q'aK_X),$$

and so, as in (9.8), this implies that $f(F)$ is not contained in $\Lambda(p',q')$. Thus $\Lambda(p',q')$ is a proper subset of $\Lambda(p,q) = \Lambda_0$, giving the desired contradiction.

(12.12) Step 10: So now we know that $r$ is rational. We next suppose that the assertion of the Rationality Theorem concerning the denominator of $r$ is *false*. We proceed to a contradiction in much the same way.

Using part b) of Step 2 with $\varepsilon = 1$, conclude as in Step 5 that there exist arbitrarily large $(p,q)$ with $0 < aq-rp < 1$ such that

$$P(p,q) = h^0(Y;\ pD_1+qD_2+\lceil A\rceil) \neq 0,$$

since otherwise $P(x,y)$ would vanish "too often". Thus

$$|pH+qaK_X| \neq \emptyset$$

for all $(p,q) \in \mathcal{L}$ by (12.5).

Choose $(p,q) \in \mathcal{A}$ such that $aq-rp$ is the maximum; say it is equal to $d/v$. Choose a resolution $f$ as in Step 6. In the strip

$$0 < aq'-rp' \leq d/v,$$

we have as before that

$$\chi(F; (f^*(p'H+q'aK_X)+\ulcorner A'\urcorner)|_F) = h^0(F; (f^*(p'H+q'aK_X)+\ulcorner A'\urcorner)|_F).$$

By part b) of Step 2, there exists $(p',q')$ in the strip

$$0 < aq'-rp' < 1 \quad \text{with } \varepsilon = 1$$

for which

$$h^0(F; (f^*(p'H+q'aK_X)+\ulcorner A'\urcorner)|_F) \neq 0.$$

But then

$$aq'-rp' \leq d/v = aq-rp$$

automatically. The desired contradiction is then derived as in Steps 7-9. This completes the proof of the Rationality Theorem.

(12.13) The use of the Poincaré polynomial in the proof of the Rationality Theorem is analogous to its use in proving a classical result about the divisibility of $K_X$:

Suppose a smooth projective variety X has dimension $n$ and $-K_X$ is ample. Suppose $mH = K_X$, thus $-H$ is ample. The Poincaré polynomial $\chi(\nu H)$ for H has at most $n$ zeros, so it is non-zero for some $1 \leq \nu \leq n+1$. However, in this range,

$$\chi(\nu H) = \pm h^n(\nu H) = \pm h^0(K_X-\nu H).$$

So $m \leq n+1$.

(12.14) References: The proof is from [Ka4] with simplifications and additions of [Ko3]. See also [KMM, 4.1].

## Lecture #13: Non-vanishing Theorem

(13.1) **Proof of the Non-vanishing Theorem:**

First notice that we can assume that D is not numerically trivial, since otherwise

$$h^0(X;\ mD+\lceil G\rceil) = \chi(mD+\lceil G\rceil) = \chi(\lceil G\rceil) = h^0(X;\lceil G\rceil) \neq 0,$$

so the assertion of the theorem is trivially satisfied.

(13.2) Now pick some simple point $x \in X$ which does not lie in the support of G. (We will blow up this point first in the construction of $f$ below.) We claim that we can pick positive integers $q_0 \geq a$ and $e(q)$ for each $q \geq q_0$ so that

i) $(e(qD+G-K_X) - K_X)$ is ample for all $e \geq e(q)$,

and

ii) for any $k > 0$ there is $e(q,k)$ such that for all $e \geq e(q,k)$ such that $e\cdot(qD+G-K_X)$ is Cartier, there is a divisor

$$M(q,e) \in |e\ (qD+G-K_X)|$$

with multiplicity $> ek\cdot\dim X$ at $x$.

To see that this is possible, let $d = \dim X$ and write

$$(qD+G-K_X)^d = ((q-a)D+aD+G-K_X)^d.$$

Since D is nef, $D+\varepsilon(\text{ample})$ is ample. Letting $\varepsilon \to 0$, we see that

$$D^{d'}\cdot(\text{any } d'\text{-dimensional subvariety}) \geq 0.$$

Thus

$$(qD+G-K_X)^d = ((q-a)D+aD+G-K_X)^d \geq (q-a)D\cdot(aD+G-K_X)^{d-1}.$$

There is some curve C so that $D\cdot C > 0$ and some $p$ such that $(p(aD+G-K_X))^{d-1}$ is represented by C plus an effective one-cycle. So $D\cdot(aD+G-K_X)^{d-1} > 0$. Thus the right-hand quantity goes to infinity with $q$. Then, by the Riemann-Roch formula and the Vanishing Theorems (cf. i) above and (9.2)),

$$h^0(e\cdot(qD+G-K_X)) \geq (1/d!)\ (qD+G-K_X)^d\cdot e^d + (\text{lower powers of } e).$$

On the other hand, the number of conditions on $M(q,e)$ that $x$ be a point of multiplicity $>$ dek on $M(q,e)$ is at most

$$(1/d!)(dk)^d\cdot e^d + (\text{lower powers of } e).$$

Since $(qD+G-K_X)^d \to \infty$ as $q \to \infty$, we have more sections than conditions. This proves the claim.

(13.3) **Lemma**: Let X and $G = \Sigma g_i G_i$ be as in (9.4). Let

$$f\colon Y \longrightarrow X$$

be any proper birational morphism with Y smooth and let

$$K_Y + f^*G \equiv f^*K_X + \Sigma b_j F_j,$$

where the $F_j$ are distinct. Let $\delta$ be a positive number. If $g_i > -1+\delta$ for every $i$, then also $b_j > -1+\delta$ for every $j$.

**Proof**: This is essentially the same as the second part of (6.5). It is sufficient to check this for one blow-up with smooth center. In this case it is an easy explicit calculation.

(13.4) With $d$ as above, let

$$f = f(q,e)\colon Y \longrightarrow X$$

be some resolution of the singularities of $M(q,e)$ with a simple-normal-crossing divisor $\Sigma F_j$ (not necessarily exceptional) in Y such that $f$ dominates the blow up $B_xX$ of $x \in X$ and

a) $K_Y + f^*G \equiv f^*K_X + \Sigma b_j F_j$,

where we note that $b_j > -1$ by (13.3),

b) for suitable $0 < p_j << 1$,

$$(1/2)f^*(aD+G-K_X) - \Sigma p_j F_j$$

is ample,

c) $f^*M(q,e) = \Sigma r_j F_j$ with $j = 0$ corresponding to the exceptional divisor of the blow-up of $x$.

(13.5) We define

$$N(b,c) = bf^*D + \Sigma(-cr_j + b_j - p_j)F_j - K_Y.$$

As before, we want to make $N(b,c)$ ample. We calculate

$$\begin{aligned} N(b,c) &= bf^*D + \Sigma(-cr_j+b_j-p_j)F_j - K_Y \\ &\equiv bf^*D - cef^*(qD+G-K_X) - \Sigma p_j F_j + f^*G - f^*K_X \\ &= (b-a)f^*D + (1-ce)f^*(aD+G-K_X) - \Sigma p_j F_j \end{aligned}$$

$$= \underbrace{(b-a)f^*D}_{\text{nef}} + \underbrace{(1/2 - ce)f^*(aD+G-K_X)}_{\text{nef}} + \underbrace{\{(1/2)f^*(aD+G-K_X) - \Sigma p_j F_j\}}_{\text{ample}}.$$

Now as long as $ce \le 1/2$ and $b \ge a$, $N(b,c)$ will be ample.

(13.6) Now choose $k = 2$ in (13.2) and pick

$$c = \min\{(1+b_j-p_j)/r_j\},$$

where minimum is taken over those $j$ such that $r_j > 0$. Then $c > 0$. As before, we wiggle the $p_j$ so that this minimum is achieved for only one value $j'$ of $j$ and set $F = F_{j'}$. By the choice of $F_0$,

$$b_0 = d-1, \text{ and } r_0 \geq dek,$$

and therefore

$$c \leq (1+(d-1)-p_0)/2de < 1/2e .$$

Thus, $ce < 1/2$, and so, for $b \geq a$, $N(b,c)$ will be ample.

(13.7) The rest of the story is as in the proofs of the Basepoint-free and Rationality Theorems. Write

$$N(b,c) = bf^*D + A - F - K_Y.$$

Note that the coefficient of $F_j$ in $A$ is $(-cr_j+b_j-p_j) \leq b_j$ and therefore $\lceil G\rceil - f_*(\lceil A\rceil)$ is effective. Thus we have

$$H^0(Y;\ bf^*D + \lceil A\rceil) \subseteq H^0(X;\ bD + \lceil G\rceil).$$

Since $N(b,c)$ is ample,

$$H^1(Y;\ bf^*D + \lceil A\rceil - F) = H^1(Y;\ bf^*D + \lceil A-F\rceil) = 0,$$

so $H^0(X;\ bD + \lceil G\rceil) \neq 0$ if we show that

$$H^0(F;\ (bf^*D + \lceil A\rceil)|_F) \neq 0.$$

This last inequality can be achieved by making an induction on $\dim X$. We can assume that we have already proved the Non-vanishing Theorem for varieties of dimension $< \dim X$. Applying the induction assumption to $F$, we complete the proof of the theorem.

(13.8) This completes the proof of the first step in Mori's program. If $X$ is a projective variety with canonical singularities and if $K_X$ is not nef, then one can find an extremal ray and the corresponding contraction morphism. The next step is to prove the existence of flips. So far this is known only in dimension three. The proof is too long and complicated to present in detail, but we will try to discuss some of the main points, frequently ignoring technical difficulties.

(13.9) References: The proof is from [Sh].

**Lecture #14: Introduction to flips**

(14.1) Today we return to the minimal model program in dimension three which we discussed in Lecture #5. The one part of the program that we have not yet examined was the step called *flip*, defined in (5.11). Two things must be shown:

1) Existence of flips.
2) Termination of flips.

We begin with a discussion of the latter.

(14.2) Recall that in the definition a threefold X with terminal singularities, we took a resolution

$$f\colon Y \longrightarrow X$$

and had

$$K_Y = f^*K_X + \Sigma a_i E_i,\ a_i > 0.$$

We define the **difficulty** of X, d(X), to be the number of the $a_i$'s such that $a_i < 1$. The difficulty is independent of the resolution Y. The point is that, under flips, the difficulty goes *down*, so that any sequence of flips must terminate:

(14.3) **Theorem:** If

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

$-K_X$ is f-ample, $K_{X^+}$ is $f^+$-ample

is a flip, then $d(X^+) < d(X)$.

**Proof:** Let

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "g^+"] & \\ X & & X^+ \end{tikzcd}$$

be a common resolution of X and $X^+$. Then

$$K_Y = g^*K_X + \Sigma a_i E_i \text{ and } K_Y = (g^+)^*K_{X^+} + \Sigma b_i E_i.$$

We take an integer r large and divisible enough so that $rK_{X^+}$ is Cartier and $f^+$-very-ample. Choose generic $D^+ \in |rK_{X^+}|$. Then, for the lift $D' = (g^+)^*D^+$,

$$D' + \Sigma rb_iE_i \in |rK_Y|.$$

If D denotes the image of D' in X, $D \in |rK_X|$ and $K_X$ is f-negative, so D must contain the union C of curves contracted by f. So $g^*D$ contains all the $E_i$ as components, and

$$D' + \Sigma rb_iE_i = rK_Y = g^*D + \Sigma ra_iE_i = D' + \Sigma c_iE_i + \Sigma ra_iE_i,$$

where $c_i > 0$ for every i.

So $a_i < b_i$ for each i. We can choose Y in such a way that it dominates the blow-up of $C^+$ in $X^+$ whose associated exceptional divisors will all have $b_i = 1$. So the difficulty decreases by at least one.

For the existence of flips, we have:

(14.4) **Flip Theorem:**
Let $f: X \longrightarrow Z$ be a proper birational morphism of normal threefolds such that X has only terminal singularities, f contracts no divisors, and such that $-K_X$ is f-ample. Then there exists a proper birational morphism $f^+: X^+ \longrightarrow Z$ such that $X^+$ has only terminal singularities, $f^+$ does not contract any divisors, and $K_{X^+}$ is $f^+$-ample:

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

$-K_X$ is f-ample $\qquad$ $K_{X^+}$ is $f^+$-ample

(14.5) An outline of the strategy of the proof is roughly as follows:

14.5.1) By working in the analytic category, we can contract the components of the curves contracted by f one at a time. We are thereby reduced to proving a "local" version of the Flip Theorem, that is, a version in which X is replaced by the germ of X along an irreducible curve C with $C \cdot K_X < 0$. This germ is also called an **extremal neighborhood**. (Any flip is a composition of these analytic flips, done one at a time.)

14.5.2) In the above situation, $R^1f_*\omega_X = 0$ and $R^1f_*\mathcal{O}_X = 0$. If X is smooth, then these are consequences of (8.8). Here we have singularities, and so (8.8) does not apply, but essentially the same proof goes through.

14.5.3) We claim that C must be smooth and rational. By 2), $R^1f_*\mathcal{O}_X = 0$. So, applying $f_*$ to the exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I} \longrightarrow 0,$$

we obtain that $H^1(\mathcal{O}_X/\mathcal{I}) = 0$ and $C = \mathbf{CP}^1$.

14.5.4) We claim that X must necessarily be singular along C:

Suppose X is non-singular. Since $C \cdot K_X < 0$,

$$(h^0(C;\ f^*T_V) - h^1(C;\ f^*T_V)) > 3$$

by the formula in (1.2), and C deforms, contradicting the fact that C is the whole exceptional set.

14.5.5) We next show that X can have no more than *two* singular points along C where the index is > 1 (see(6.8). We will present a purely topological argument to see this:

If a terminal singularity (U,p) of dimension three has index m, then

$$\pi_1(U-\{p\}) = \mathbf{Z}_m,$$

because (U,p) is the quotient of a hypersurface singularity by $\mathbf{Z}_m$. (Here U can be thought of as a suitable small neighborhood of p.)

We will analyze the local topology near the $\mathbf{CP}^1$. Suppose that we had three singular points of index > 1:

$$P, Q, R.$$

Denote the three indices by i, j, and k. Assume for simplicity that X has quotient singularities at P,Q,R and is smooth elsewhere. Then

$$(X-\{P,Q,R\})$$

has the homotopy type of $S^2$ with three little open discs removed, and then with three lens spaces

$$L_i,\ L_j,\ \text{and } L_k$$

sewn in at the respective holes. The essential case is the one in which the boundary of the hole is identified with a generator of $\pi_1$ of the corresponding lens space. Then

$$\pi_1(X-\{P,Q,R\}) = \langle\alpha,\beta,\gamma\rangle/\{\alpha\beta\gamma = 1,\ \alpha^i = 1,\ \beta^j = 1,\ \gamma^k = 1\}.$$

Algebra fact: This group has a finite quotient G in which
$\alpha$ has order i, $\beta$ has order j, and $\gamma$ has order k.

The kernel of the homomorphism from $\pi_1$ to G defines a finite Galois covering $X^{¤}$ of (X-{P,Q,R}). By filling in finitely many points over P, Q, and R, one completes $X^{¤}$ to a connected covering space X^ of X. But then X^ is smooth, and

$$C\hat{} \cdot K_{X\hat{}} < 0,$$

gives a contradiction as in 4) above.

(Note that, if we have only *two* singular points, the fundamental group in the above argument is usually trivial. The proof in the case of four or more singular points is very similar.)

14.5.6) Now we have the contraction

$$f\colon (X,C) \longrightarrow (Z,p).$$

Let $\mathcal{J}$ be a sheaf of ideals whose radical is the ideal sheaf $\mathcal{I}$ of C. Applying $f_*$ to the sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{J} \longrightarrow 0,$$

and to that sequence tensored with $\omega_X$, and using 14.5.2), we conclude that

$$(*) \qquad H^1(\mathcal{O}_C/\mathcal{J}) = 0 \text{ and } H^1(\omega_X/\mathcal{J}\omega_X) = 0.$$

We already saw an important consequence of this vanishing result in 3). We will see that these vanishings impose very strong restrictions on the possible singularities and on the global structure of the extremal neighborhood. Here we derive two such results that will be needed in the sequel. Let again $\mathcal{I}$ be the ideal sheaf of the curve C.

14.5.7) $\qquad \omega_X/\mathcal{I}\omega_X = \mathcal{O}(-1) + (\text{torsion sheaf})$:

From 6) we know that the $H^1$ of this sheaf is zero, thus the degree of the torsion free part is at least -1. On the other hand we have a natural map

$$\beta\colon (\omega_X/\mathcal{I}\omega_X)^{\otimes m} \longrightarrow \mathcal{O}_C(mK_X)$$

which is generically injective. The line bundle on the right has negative degree thus $\deg(\omega_X/\mathcal{I}\omega_X) < 0$.

As a corollary of this argument we also obtain that

$$-1 \leq C\cdot K_X < 0.$$

14.5.8) $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}(a) + \mathcal{O}(b) + (\text{torsion sheaf})$ with $a,b \geq -1$:

In the long cohomology sequence associated to

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}/\mathcal{I}^2 \longrightarrow \mathcal{O}/\mathcal{I} \longrightarrow 0,$$

note that $H^0(\mathcal{O}/\mathcal{I}^2) \to H^0(\mathcal{O}/\mathcal{I})$ is onto and $H^1(\mathcal{O}/\mathcal{I}^2) = 0$. Thus

$$H^1(\mathcal{I}/\mathcal{I}^2) = 0.$$

14.5.9) The main part of the proof of the existence of flips consists of an intricate and technical analysis in which we are able to construct a Weil divisor E in $|-2K_X|$ such that the double cover

$$p\colon W \longrightarrow X$$

induced by E has only canonical singularities. This W has only canonical singularities and

$$K_W \equiv p^*K_X + (1/2)p^*E$$

is trivial.

On W, we are in a situation in which we can do a **flop**, which is described by the following diagram:

$$\begin{tikzcd} W \arrow[rr, dashed] \arrow[dr, "f"'] & & W^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

some $-D$ is f-ample

$K_W$ is f-trivial

$D^+$, the proper transform of D, is $f^+$-ample

$K_{W^+}$ is $f^+$-trivial

where D is some divisor. Again there is an existence theorem for flops and a termination theorem for sequences of flops.

We obtain the desired flip of the irreducible curve C by taking as $X^+$ the quotient of $W^+$ under the involution induced by the involution on W.

(14.6) In most cases, we will be able to find a divisor $D \in |-K_X|$ such that D has only DuVal singularities. Following Reid, such a D is called a *DuVal elephant*. It is conjectured that a DuVal elephant always exists. Using the explicit description of terminal singularities, it is easy to see that the existence of a DuVal elephant implies the existence of the above double cover

$$W \longrightarrow X.$$

To get an idea why generic Weil divisors in $|-K_X|$ should have only DuVal singularities, we look at the case in which the singularities of X are all ordinary, that is, they are all cyclic quotient terminal singularities. These are all of the form

$$\mathbf{C}^3/\mu_r$$

where the generator $\xi$ of the group $\mu_r$ of r-th roots of unity acts by the rule

$$(x, y, z) \longrightarrow (\xi x,\ \xi^{-1}y,\ \xi^a z)$$

with a prime to r.

On $\mathbf{C}^3$ above a cyclic quotient singularity,

$$-K_{\mathbf{C}^3} = \mathcal{O}(dx\wedge dy\wedge dz)^{-1}.$$

If we let $\omega = dx\wedge dy\wedge dz$ , then the action of $\xi$ on $\omega$ is given by

$$\omega \longrightarrow \xi^a\omega.$$

So the section $z/\omega$ descends to give (locally) a Weil divisor D for $-K_X$.

The divisor D is given as a quotient singularity by the action

$$(x,y) \longrightarrow (\xi x, \xi^{-1}y)$$

on $\mathbf{C}^2$, and hence is a DuVal singularity, embedding into $\mathbf{C}^3$ via the map $(xy,\ x^r,\ y^r)$.

(14.7) References: (14.3) is due to Shokurov[Sh], (14.4) to Mori[M3], (14.5.5) to [M3] and also to Benveniste[B2]. The idea of taking double covers appears in Kawamata[Ka5].

**Lecture #15: Singularities on an extremal neighborhood**

(15.1) The aim of today's lecture is to elaborate on the part (14.5.9) of the proof of the Flip Theorem (14.4), namely we try to outline the local classification of the points occurring on extremal neighborhoods. We cover all the important techniques that are contained in sections 2-7 of Mori's paper. Thus by reviewing some definitions and theorems in those sections, the reader should be able to proceed to the last two sections, which are the real core of the article.

(15.2) Let X be the extremal neighborhood containing a single extremal rational curve C and let p be a point of C. We intend to give a classification of the triplets (X,C,p). For illustration, assume that X has a quotient singularity at p. As we saw, C is a smooth curve and therefore one might think that knowing (X,p) uniquely determines the triplet up to isomorphism. This is however far from being true. Before we give some examples, we set up the notation that will be used to describe the situation.

(15.3) **Notation:** Let $\mathbf{Z}_m$ be the cyclic group of order m.

Fix a primitive m-th root of unity $\zeta$. Assume that $\mathbf{Z}_m$ acts on $\mathbf{C}^n$ linearly, and that the coordinate functions are eigenfunctions of this action, that is

$$1 \in \mathbf{Z}_m \text{ acts on } x_i \text{ as } 1(x_i) = \zeta^{a(i)} x_i .$$

In this case we say that $\mathbf{Z}_m$ acts on $\mathbf{C}^n$ with weights

$$(a(1), \ldots, a(n)).$$

Similarly, if f is a polynomial function on $\mathbf{C}^n$ which is an eigenfunction of this action, then we say that the group acts with a certain weight on f. We denote the weight of f by wt(f).

(15.4) **Example:** Let $\mathbf{Z}_m$ act on $\mathbf{C}^3$ with weights (1,a,m-a) on the coordinates $(x_1,x_2,x_3)$. Let

$$V \subseteq \mathbf{C}^3$$

be the monomial curve given as the image of the map

$$t \longrightarrow (t^{km+1}, t^a, t^{m-a}).$$

Then $\mathbf{C}^3/\mathbf{Z}_m$ is a terminal singularity, and $V/\mathbf{Z}_m$ is a smooth curve germ inside this singularity.

If we go back to the problem of finding a good member D of $-K_X$, we see that $\{x_1=0\}$ descends to such a good member at least locally. Assume now that the above singularity is the only one on an extremal neighborhood X. One way to find a good member of $-K_X$ is to use D which is transversal to the curve C, so that, in a small enough neighborhood of C, it will be a global divisor. D is a member of $|-K_X|$ if it has the correct intersection with the curve C. In our case, one can easily obtain that the intersection number is

$$D\cdot C = k+(1/m),$$

whereas we know that

$$-1 \leq C\cdot K_X < 0.$$

Thus we must have $k=0$ to have any chance at all.

This shows that we have to analyse the location of C near the singularities of X very carefully.

(15.5) **Proposition**: Assume that $\mathbf{Z}_m$ acts on $\mathbf{C}^n$ with weights $(a_i)$. Let $V \subseteq \mathbf{C}^n$ be an irreducible curve germ which is $\mathbf{Z}_m$-invariant. Assume that $V/\mathbf{Z}_m$ is smooth. Then, after a suitable $\mathbf{Z}_m$-invariant coordinate change, V becomes monomial; namely, it will be the image of

$$t \to (t^{b(i)})$$

for some $(b(i))$.

**Proof**: We can assume that $\mathbf{Z}_m$ acts faithfully on V. $\mathbf{Z}_m$ acts on the normalization $V^\wedge$ of V. We let $t$ be a local parameter on $V^\wedge$ which is an eigenfunction. Then the ring of $\mathbf{Z}_m$-invariant functions on $V^\wedge$ is generated by $t^m$. Since $V/\mathbf{Z}_m$ is smooth,

$$V^\wedge/\mathbf{Z}_m \longrightarrow V/\mathbf{Z}_m$$

is an isomorphism. Therefore every $\mathbf{Z}_m$-invariant regular function on $V^\wedge$ is also regular on V. For every $i$, we can write

$$x_i = t^{b(i)} g_i(t)$$

where $g_i$ is $\mathbf{Z}_m$-invariant with non-zero constant term. Since $g_i$ is $\mathbf{Z}_m$-invariant, it is the restriction of an invertible $\mathbf{Z}_m$-invariant function $h_i$ on $\mathbf{C}^n$. Now we can introduce new coordinates by the rule

$$y_i = x_i \cdot \sqrt[m]{h_i}.$$

In this new coordinate system, V is obviously monomial.

(15.6) **Notation**: Let

$$(X,C,p)$$

be the neighborhood of a point in an extremal neighborhood. The index-one cover (constructed in (6.8)) will be denoted by

$$(X^{\#},C^{\#},p^{\#}).$$

Thus the group $\mathbf{Z}_m$ acts on this cover and the quotient is $(X,C,p)$. In general it is not true that $C^{\#}$ is irreducible, but for the purpose of this lecture we will always assume this. No new ideas are needed to handle the more general case.

As we saw, every three-dimensional terminal singularity is the quotient of a smooth point or of a hypersurface double point. Thus we can always assume that $(X^{\#},C^{\#},p^{\#})$ is embedded in $\mathbf{C}^4$ in which it is defined by an equation

$$\Phi = 0,$$

where $\Phi$ defines either a smooth point or a double point at the origin.

By the above considerations, we can choose a coordinate system on $\mathbf{C}^4 \supset X^{\#}$ such that $C^{\#}$ becomes a monomial curve. If $f$ is any regular function on $X^{\#}$, then by

$$\operatorname{ord} f$$

we denote the order of vanishing of $f$ on the normalization of $C^{\#}$. The values $\operatorname{ord} f$ form a semigroup, which is denoted by

$$\operatorname{ord} C^{\#}.$$

If $\operatorname{ord} x_i = a_i$, then this semigroup is generated by the $a_i$'s. If $(a_i-m)$ is in $\operatorname{ord} C^{\#}$, then we can write down a monomial $M$ in the $x_i$'s which has the order $(a_i-m)$ and introduce the new coordinate $x_i-M$. Thus we may always assume that $(a_i-m)$ is not in $\operatorname{ord} C^{\#}$.

Note that $\operatorname{ord} x_i$ depends only on $C^{\#}$, whereas the choice of the weight of a function depends on the choice of a generator of $\mathbf{Z}_m$. We can clearly choose the generator in such a way that

$$\operatorname{ord} x_i \equiv \operatorname{wt} x_i \pmod{m}$$

holds for every $i$. We shall always assume in the sequel that such a choice was made

(15.7) **Definition:** A coordinate system which satisfies the above conditions will be called a normalized $\ell$-coordinate system.

(15.8) Threefold terminal singularities are very special quotients of smooth or double points, and a complete list is known. We ignore finitely many exceptions, and look only at the main series where we can choose the order of the coordinates $x_i$ in such a way that the following conditions are satisfied:

$$a_2 + a_3 \equiv 0 \pmod m, \quad (a_1a_2a_3, m) = 1, \quad a_4 \equiv 0 \pmod m,$$
$$\mathrm{wt}(\Phi) \equiv 0 \pmod m.$$

Note that since C is smooth and $C^\# \longrightarrow C$ has degree $m$, we have $\mathrm{ord}\, C^\# \ni m$; thus, as we noted above, $a_4 = m$.

Next we define two of the simplest local invariants invented by Mori to measure the effect of the singularity $(X,C,p)$ on the extremal neighborhood.

(15.9) **Definition**: i) Given a triplet $(X,C,p)$, let $m$ be the index of X at $p$. As we saw, there is a natural map

$$\beta\colon\ (\omega_X/\mathcal{I}\omega_X)^{\otimes m} \longrightarrow \mathcal{O}_C(mK_X)$$

and we define $w_p = m^{-1}\cdot(\text{length coker } \beta)$. (In Mori's original notation, this is $w_p(0)$.)

ii) We can define natural maps

$$\mathcal{I}/\mathcal{I}^2 \times \mathcal{I}/\mathcal{I}^2 \times \omega_C \longrightarrow \omega_X \otimes \mathcal{O}_C \longrightarrow \omega_X/\mathcal{I}\omega_X \longrightarrow \mathrm{gr}^0(\omega_X)$$

given as

$$x \times y \times z\,du \to z\,dx \wedge dy \wedge du$$

where $\mathrm{gr}^0(\omega_X)$ is the locally free part of $\omega_X/\mathcal{I}\omega_X$. This in turn defines a homomorphism

$$\alpha\colon\ \Lambda^2(\mathcal{I}/\mathcal{I}^2) \otimes \omega_C \longrightarrow \mathrm{gr}^0(\omega_X).$$

Now let

$$i_p = \text{length coker}\, \alpha.$$

(This is $i_p(1)$ of Mori.)

The results of Lecture #14 now imply:

(15.10) **Proposition**: i) $\sum w_p < 1$;
ii) $\sum i_p \le 3$.

**Proof:** The first statement follows from (14.5.7) since

$$m \cdot \sum w_p = -m\cdot \deg \operatorname{gr}^0(\omega_X) + \deg \mathcal{O}_C(mK_X).$$

The second part follows from the definition using (14.5.8).

This result shows that the local invariants of the singularities combine to give a global invariant, and that we have some restrictions on the possible singularities of an extremal neighborhood. Therefore we set out to compute or at least estimate the above invariants for the triplets $(X,C,p)$.

Computation of $w_p$:

(15.11) $X^{\#}$ is a hypersurface singularity given by equation $\Phi$, thus

$$\sigma = (\partial\Phi/\partial x_1)^{-1}dx_2\wedge dx_3\wedge dx_4 = \operatorname{Res}\,\Phi^{-1}dx_1\wedge dx_2\wedge dx_3\wedge dx_4$$

is a local generator of the dualizing sheaf of $X^{\#}$, where Res is the Poincaré residue map. Clearly $\sigma$ is a $\mathbf{Z}_m$-eigenvector with

$$\operatorname{wt}(\sigma) \equiv \sum a_i \pmod m.$$

Thus $\sigma^m$ is invariant, and it descends to give a local generator of $\mathcal{O}_C(mK_X)$. In order to get a local generator of $\operatorname{gr}^0(\omega)$, we have to look for a section of the dualizing sheaf of $X^{\#}$ which is invariant. We can look for one of the form $M\sigma$, where $M$ is a monomial. Then $M^m\sigma^m$ is a local generator of $\operatorname{gr}^0(\omega)^m$; so

$$w_p = m^{-1} \dim(\mathcal{O}_C(mK_X)/M^m\sigma^m\mathcal{O}_C(mK_X))$$

Therefore we get that

$$w_p = m^{-1}\operatorname{ord} M.$$

If we denote by $\hat{a}$ the remainder of the integer $a \pmod m$, then for the series of singularities we are considering,

$$\operatorname{wt}(\sigma) = \hat{a}_1.$$

Thus, for the above monomial $M$, we have

$$\operatorname{ord} M + \hat{a}_1 \equiv 0 \pmod m.$$

If we take into account that $w_p < 1$, then we get an equation of the form

$$\sum b_i a_i + \hat{a}_1 = m.$$

Therefore $b_4 = 0$, and one of $b_2$ and $b_3$ is also zero, say the latter one. Thus we have

$$b_1 a_1 + b_2 a_2 + \hat{a}_1 = m.$$

This already shows that $a_1$ or $a_2$ is less than $m$, thus the curve $C^{\#}$ is not arbitrarily complicated.

Computation of $i_p$:

(15.12) We already have a local generator of $\mathrm{gr}^0(\omega)$, namely $M\sigma$. If $t$ is a local parameter on the normalization of $C\#$ which is a $\mathbf{Z}_m$-eigenvector, then $dt^m = dx_4$ is a local generator of $\omega_C$. Let $\mathcal{J}\#$ denote the ideal of $C\#$ in $\mathbf{C}^4$ and $\mathcal{J}\#_{\{0\}}$ the set of $\mathbf{Z}_m$-invariant functions in $\mathcal{J}\#$. (For any sheaf $\mathcal{F}$ with $\mathbf{Z}_m$-action, $\mathcal{F}_{\{0\}}$ denotes the subsheaf of $\mathbf{Z}_m$-invariant sections of $\mathcal{F}$.) Local generators of the locally free part of $\mathcal{I}/\mathcal{I}^2$ lift back to elements $f$ and $g$ of $\mathcal{J}\#_{\{0\}}$, thus $f\wedge g\wedge dt^m$ is a local generator of $\Lambda^2(\mathcal{I}/\mathcal{I}^2)\otimes\omega_C$.

We can see the relationship between $M\sigma$ and the image of $f\wedge g\wedge dt^m$ in $\mathrm{gr}^0(\omega)$ as follows:

$$
\begin{aligned}
df\wedge dg\wedge dx_4 &= \mathrm{Res}\ \Phi^{-1}\ d\Phi\wedge df\wedge dg\wedge dx_4\\
&= \mathrm{Res}\ \Phi^{-1}\partial(\Phi,f,g)/\partial(x_1,x_2,x_3)\ dx_1\wedge dx_2\wedge dx_3\wedge dx_4\\
&= M^{-1}\,\partial(\Phi,f,g)/\partial(x_1,x_2,x_3)\ M\sigma,
\end{aligned}
$$

where $\partial(\ ,\ ,\ )/\partial(\ ,\ ,\ )$ is the Jacobian determinant. Thus

$$i_P = m^{-1}(-\mathrm{ord}\,M + \mathrm{ord}\,\partial(\Phi,f,g)/\partial(x_1,x_2,x_3)).$$

In the case we are considering, $\Phi$ is also an element of $\mathcal{J}\#_{\{0\}}$, thus we can further simplify to obtain

$$m\cdot i_P \geq -\mathrm{ord}\,M + \mathrm{ord}\,\partial(h,f,g)/\partial(x_1,x_2,x_3)$$

where $f,g,h$ generate the locally free part of $\mathcal{J}\#_{\{0\}}/\mathcal{J}\#^2_{\{0\}}$. It is an easy exercise to see that this does not depend on the choice of $f,g$ and $h$.

Now we are ready to derive the main result of this section:

(15.13) **Theorem:** Given $(X,C,p)$ where $(X,p)$ is a three-dimensional terminal singularity and $C$ is the germ of a smooth curve through $p$ with $w_p < 1$ and $i_p \leq 3$, then either

i) $\mathrm{ord}\,C^\#$ is generated by two elements, i.e. $C^\#$ is planar;

or

ii) $3 \in \mathrm{ord}\,C^\#$, so $\mathrm{mult}_{p^\#}C^\# \leq 3$.

(15.14) **Remark:** i) This theorem says in particular that, if $(X,C,p)$ appears on an extremal neighborhood, then the singularity of $C^{\#}$ is not too complicated. We will prove this only for the main series of singularities, although the statement is true in general. The proof in the additional cases is very easy.

ii) In fact, $\operatorname{ord} C^{\#}$ is always generated by two elements if $(X,C,p)$ appears on an extremal neighborhood, but the proof of this would require the consideration of a new invariant.

(15.15) **Proof:** We already noted that $w_p < 1$ implies

$$b_1a_1 + b_2a_2 + \hat{a}_1 = m.$$

If $a_1 < m$, then this reduces to

$$(b_1+1)a_1 + b_2a_2 = m.$$

We claim that in this case $\operatorname{ord} C^{\#}$ is generated by $a_1$ and $a_2$. Indeed, since $a_4 = m$, $a_4$ is a linear combination of $a_1$ and $a_2$. Since $a_2 + a_3 \equiv 0 \pmod m$, for some $c \geq 0$ we can write

$$a_3 = (b_1+1)a_1 + (b_2-1)a_2 + cm.$$

Thus $a_3$ is also a linear combination of $a_1$ and $a_2$, provided $b_2 > 0$. If $b_2 = 0$, then $a_1$ divides $m$. Since $(m,a_1) = 1$, this implies $a_1 = 1$, and in this case $\operatorname{ord} C^{\#}$ is generated by 1.

Therefore we are left with the case when $a_1 > m$. Note that, in this case, the identity

$$b_1a_1 + b_2a_2 + \hat{a}_1 = m$$

reduces to

$$ba_2 + \hat{a}_1 = m.$$

We can also write

$$a_1 = cm + \hat{a}_1 \text{ (for some } c > 0)$$

and

$$a_3 = km - a_2 \text{ (for some } k > 0).$$

Note also that $a_2 < m$ and $a_4 = m$. We want to prove that these conditions, together with $i_p \leq 3$, imply that $a_2$ or $a_3$ is at most 3. We consider the formula for $i_p$:

$$m \cdot i_p \geq -\operatorname{ord} M + \operatorname{ord} \partial(h,f,g)/\partial(x_1,x_2,x_3)$$

where $f,g,h$ generate the locally free part of $\mathcal{J}^{\#}\{0\}/\mathcal{J}^{\#2}\{0\}$.

Since $\mathcal{J}^\#$ is the set of invariant elements in the ideal of a monomial curve and

$$a_4 = m,$$

a moments reflection shows that it has a linear basis consisting of elements of the form

$$x_4{}^e(N - x_4{}^{(\operatorname{ord} N)/m})$$

where N is a monomial in the variables $x_1, x_2, x_3$ such that m divides $\operatorname{ord} N$. We can pick monomials in the variables $x_1, x_2, x_3$, which we call F, G, and H, such that

$$f = F - x_4{}^{(\operatorname{ord} F)/m}, \quad g = G - x_4{}^{(\operatorname{ord} G)/m}, \quad h = H - x_4{}^{(\operatorname{ord} H)/m}$$

generate the locally free part of $\mathcal{J}^\#\{0\}/\mathcal{J}^{\#2}\{0\}$.

It is clear that

$$\operatorname{ord} \partial(h,f,g)/\partial(x_1,x_2,x_3) = \operatorname{ord} F + \operatorname{ord} G + \operatorname{ord} H - a_1 - a_2 - a_3.$$

Now the formula for $i_P$ becomes

$$m(c+k+4) \geq \min\{\operatorname{ord} F + \operatorname{ord} G + \operatorname{ord} H\colon \text{F, G, and H are monomials in the variables } x_1,x_2,x_3 \text{ whose order is divisible by m and such that none of them divides the other}\}.$$

Thus we have to search for such monomials of low order. Since the order of $x_2x_3$ is divisible by m, we only have to consider monomials of the following forms:

$x_2x_3$, which has order $k\cdot m$;

$x_1{}^e x_2{}^d$, where the smallest order is $\operatorname{ord} x_1x_2{}^b = (c+1)m$;

$x_1{}^e x_3{}^d$, which all have fairly large order;

$x_2{}^m$ (resp $x_3{}^m$), which has order $ma_2$ (resp. $ma_3$).

If one spends fifteen minutes computing the orders of the various terms one will see that if $\min\{a_i\} \geq 3$, then the only way to satisfy the above inequality is to pick

$$x_2x_3,\ x_1x_2{}^b \text{ and } x_2{}^m \text{ (resp. } x_3{}^m)$$

for F,G H. We also must necessarily have that $a_2$ or $a_3$ is at most three. If $\operatorname{ord} C^\# \ni 2$, then $\operatorname{ord} C^\#$ is generated by 2 and the smallest odd element in it. Thus again we end up in case i). Otherwise $\operatorname{ord} C^\# \ni 3$. This was what we had to prove.

(15.16) Mori has to consider infinitely many local invariants. They are used partly to get more restrictions on the individual singularities on an extremal neighborhood, partly to detect the interrelation of different singularities on the same neighborhood. The inequalities $\sum w_P < 1$ and $\sum i_P \leq 3$ are the simplest examples of the latter. The first inequality shows, for example, that there can be at most one index-two point on an extremal neighborhood; the second can be used to give a proof that there can be at most three singular points on an extremal neighborhood.

(15.17) References: The classification of three dimensional terminal singularities is due to Reid[R3], Danilov[D], Morrison-Stevens[MS] and Mori[M2]. See [R5] for a good overview. All the rest is taken from [M3] with minor simplifications.

## Lecture #16: Small resolutions of terminal singularities

Today we will discuss in greater detail the characterization of terminal Gorenstein singularities of threefolds, their small resolutions, and their relation to flops. Flops are much easier to understand than flips; still the emerging picture is very similar. First we complete the proof of (6.23).

(16.1) **Theorem:** A threefold Gorenstein singularity is terminal if and only if it is an isolated cDV point.

**Outline of proof:** One direction was discussed in (6.23). Suppose now that $(X,x)$ is an isolated cDV point which is not smooth. Let

$$f\colon B \longrightarrow X$$

denote the blow-up of $X$ at $x$ and let

$E$ = exceptional locus = projectivized tangent cone of $X$. Then, since $x$ is a double point, the adjunction formula gives

$$K_B = f^*K_X + E.$$

We claim that $B$ has only rational singularities. If we show this, we will be done. Indeed, rationality implies that, if

$$g\colon Y \longrightarrow B$$

is a resolution, then $g_*\omega_Y = \omega_B$. Therefore, since some section of $K_Y$ pushes forward to a section of $K^B$ vanishing on $E$,

$$K_Y = g^*f^*K_X + E' + F$$

where $E'$ means proper transform, and $F$ involves *every* exceptional divisor of $g$ since they all lie over $E$.

To see that $B$ has only rational singularities, we reason as follows. Since $(X,x)$ is cDV, there are local analytic coordinates such that $X$ is given by the equation

$$(*) \qquad p(x,y,z) + tq(x,y,z,t) = 0,$$

where $p(x,y,z)=0$ is a rational double point and the (generic) $H$ is given by $t=0$. One then forms a flat family over the $\varepsilon$-line by replacing $t$ with $\varepsilon t$ in $(*)$. By equi-multiplicity, the blow-up of the line $\{(0,\varepsilon)\}$ is flat. At $\varepsilon=0$, an explicit analysis of possible equations shows that all singularities of the blow-up are rational. All fibres for $\varepsilon\neq 0$ are isomorphic, and rationality is an open condition, so nearby singularities must be rational as well.

Small resolutions of terminal singularities

(16.2) **Proposition:** Let X be a normal threefold singularity, and let f: $Y^3 \longrightarrow X^3$ be a proper morphism which contracts only finitely many curves. Assume that Y has only canonical singularities and that $K_Y$ is f-trivial. Then

1) Y terminal implies X terminal;
2) Y Gorenstein implies X Gorenstein.

**Proof:** Choose H on X so that $mK_Y + f^*H$ is a nef Cartier divisor, and $(m-1)K_Y + f^*H$ is nef and big. The Basepoint-free Theorem (9.3) holds under these hypotheses, so that

$$n(mK_Y + f^*H)$$

is basepoint-free for $n \gg 0$. Using this for some large n and n+1, we conclude that $mK_Y$ must be the pull-back of a line bundle on X. Since there are no exceptional divisors, this line bundle must be $mK_X$. Both conclusions now follow immediately.

(16.3) **Corollary:** Let f: $Y^3 \longrightarrow X^3$ be a morphism of compact threefolds which contracts only an irreducible curve C. Suppose that Y is smooth and $C \cdot K_Y = 0$. Then $C = \mathbf{CP}^1$ and

$N_{C/Y} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$.

**Proof:** By (16.2) X has only terminal, hence only rational singularities, so $R^1f_*\mathcal{O}_Y = 0$. As we have seen in (14.5.6), this implies that $H^1(\mathcal{O}_C) = 0$ so that $C = \mathbf{CP}^1$. In the same way $H^1(\mathcal{O}/\mathcal{I}^2) = 0$, where $\mathcal{I}$ is the ideal sheaf of C. Thus

$$H^1(\mathcal{I}/\mathcal{I}^2) = 0,$$

from which the second conclusion follows since $N_{C/Y} = \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a + b = -C \cdot K_Y - 2$.

(16.4) **Proposition:** Let f: $Y^3 \longrightarrow X^3$ be a small contraction where Y is smooth and X has only cDV singularities. Then

1) $K_Y = f^*K_X$;
2) if H is generic through the singular points of X, then $f^*H$ is normal and

$$f^*H \longrightarrow H$$

is a partial resolution.

**Proof:** The first assertion is immediate since $K_Y$ and $f^*K_X$ are both line bundles and they agree in codimension one. Now

$$K_{f^*H} = K_Y + f^*H|_{f^*H} = f^*(K_X + H|_H) = f^*K_H .$$

If $g: H' \longrightarrow f^*H$ is the normalization, then

$$\omega_{H'} = (\text{conductor ideal})\, g^*\omega_{f^*H}.$$

On the other hand, $\omega_{H'} \supseteq (fg)^*\omega_{f^*H}$, since H is a DuVal singularity. Therefore $f^*H$ is normal. Let $h: H'' \longrightarrow f^*H$ be the minimal resolution. Then $h^*\omega_{f^*H} \supseteq \omega_{H''}$ (a property of the minimal resolution of any normal Gorenstein surface singularity). On the other hand, $\omega_{H''} \supseteq h^*\omega_{f^*H} = h^*f^*\omega_H$, since H is a DuVal singularity. Hence they are equal and $f^*H$ has only DuVal singularities.

(16.5) Partial resolutions of DV singularities and their deformations give a way to construct examples of small contractions. We begin with a partial resolution of a DV singularity

$$f: H' \longrightarrow H$$

which contracts a single (smooth rational) curve C to a point $x \in H$. We smooth H' via a deformation with smooth total space Y. It turns out that f extends to a map (also denoted by f)

$$f: Y \longrightarrow X,$$

where X is a deformation of H. Now X may be singular along a curve, but, by "openness of versality," there is always a deformation of H' so that the deformation of H in X is smooth.

(16.6) **Theorem:** Suppose C contracts to an isolated singular point $x \in X$. Then the following are equivalent:

1) C has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-3)$;

2) $f^{-1}\mathfrak{m}_{x,X}$ does not generate the ideal of C in X at a generic point of C;

3) $f^{-1}\mathfrak{m}_{x,H'}$ does not generate the ideal of C in H' at a generic point of C.

**Proof:** Let $\mathcal{I}$ denote the ideal of C in Y. If $N_{C/Y}$ is

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \quad \text{or} \quad \mathcal{O} \oplus \mathcal{O}(-2),$$

then

$$\mathcal{I}/\mathcal{I}^2 = \mathcal{O}(1) \oplus \mathcal{O}(1) \quad \text{or} \quad \mathcal{O} \oplus \mathcal{O}(2),$$

so that

$$H^1(C, \mathcal{I}^n/\mathcal{I}^{n+1}) = H^1(C, S^n(\mathcal{I}/\mathcal{I}^2)) = 0,$$

and therefore

$$H^0(\mathcal{O}/\mathcal{I}^{n+1}) \longrightarrow H^0(\mathcal{O}/\mathcal{I}^n)$$

is surjective for all n.

So we obtain two formal functions defining C. By the theorem on formal functions, this means that there are two functions defined in a neighborhood of C in Y which generate $\mathcal{I}$ at a generic point of C. But, since X is by definition normal, these functions are pull-backs of elements of $\mathfrak{m}_{x,X}$. So 2) implies 1).

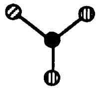
2) and 3) are equivalent since H' itself is defined by the pull-back of an element on $\mathfrak{m}_{x,X}$. Finally, if C has normal bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(-3),$$

the ideal of C in $\mathcal{O}/\mathcal{I}^2$ is not even generated by $f^{-1}\mathfrak{m}_{x,X}$.

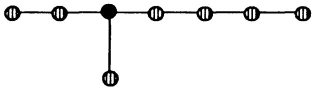
(16.7) Notice that the above proof gives an invariant of (1,-3) curves, namely the length of $\mathcal{O}_Y/f^{-1}\mathfrak{m}_{x,X}$. Some examples:

16.7.1) $D_4$-singularity on H with the partially shaded curves contracted on H':



length = 2

16.7.2) $E_8$-singularity on H with the partially shaded curves contracted on H':



length = 6

<u>Another look at flops</u>:

(16.8) Suppose we have a small contraction $f\colon Y \longrightarrow X$ of threefolds resulting in a Gorenstein terminal singularity (X,x) with $f^{-1}(x)$ irreducible. Then the singularity is cDV and so can be written in terms of appropriate coordinates as

$$x^2 + q(y,z,t) = 0.$$

Then $C\cdot K_Y = 0$. Suppose we have a Weil divisor D with $C\cdot D < 0$. Form the involution $\iota$ over a ball in $\mathbf{C}^3$ given by

$$(x,y,z,t) \longrightarrow (-x,y,z,t)$$

and the fibred product

$$\begin{tikzcd} Y^+ \arrow[r, "\iota'"] \arrow[d, "f^+"'] & Y \arrow[d, "f"] \\ X \arrow[r, "\iota"] & X \end{tikzcd}$$

Define

$D^+ = (\iota')^{-1}(D)$.

$f^+(D^+) \equiv -f(D)$

since

$f(D) + \iota f(D) \equiv 0$.

The rational map $(f^+)^{-1} \circ f\colon Y \dashrightarrow Y^+$ (not $(\iota')^{-1}$) *over* X is the D-flop. The flop

$$\begin{tikzcd} Y \arrow[rr, dashed] \arrow[dr, "f"'] & & Y^+ \arrow[dl, "f^+"] \\ & X & \end{tikzcd}$$

is an isomorphism outside C (resp. $(\iota')^{-1}(C)$).

(16.9) If $f\colon Y \longrightarrow X$ is a small contraction and X has threefold terminal singularities (not necessarily Gorenstein), then we can take the index one cover of X , apply the above construction to the covering and take the quotient again. This will give the flop of $f\colon Y \longrightarrow X$.

(16.10) References: (16.1) is due to Reid[R2], this proof is from [KS]. (16.3) is due to Laufer[L2]. (16.4) is again in [R2]. The existence of flops for threefolds with terminal singularities is due to Reid[R3]. (16.6-7) are due to Kollár. The proof given in (16.8-9) is due to Mori.

[Autumn '88: Recently J. Stevens ("On canonical singularities as total spaces of deformations," preprint, Hamburg) proved that if a hyperplane section of an isolated Gorenstein singularity is rational then the singularity is terminal. He also proved that, if $mK_X$ is Cartier and the general member of $|-K_X|$ is rational, then X is canonical.]

## Lecture #17: Kähler structures on locally symmetric spaces

Today we will look at an entirely different aspect of the Hodge theory of Kähler manifolds, namely, a relation between Hodge theory and harmonic maps. A possible relation between this and other things we have studied in the seminar will come from the study of period mappings of families of subvarieties of an algebraic manifold. The set-up is as follows:

(17.1) **Definitions:** Let G be a semi-simple Lie group with no compact factor, and let K be a maximal compact subgroup of G. Examples are

17.1.1) $G = SL(n,\mathbf{R})$ and $K = SO(n)$,

17.1.2) $G = SO(p,q)$ and $K = SO(p) \times SO(q)$.

The Cartan involution on the Lie algebra $\mathfrak{g}$ of G gives a decomposition into +1 and -1 eigenspaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Since the involution normalizes K, it induces an involution on $Y = G/K$ which acts as -1 on the tangent space $\mathfrak{p}$ to Y at {K}. Via conjugation, one obtains, for each $y \in Y$, an involution fixing y and acting as -1 on the tangent space at y. The Killing form on $\mathfrak{g}$ decomposes into the sum of a negative-definite form on $\mathfrak{k}$ and a positive-definite form on $\mathfrak{p}$, giving Y an invariant metric so that the involutions mentioned above are all isometries. Thus Y is called a *symmetric space*. For the Lie bracket we have

$$[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k} \text{ and } [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}.$$

The curvature tensor at {K} is given by

$$R(X,Y)Z = -[[X,Y],Z].$$

In example 1) above, the Cartan involution is simply *minus transpose*, so that $\mathfrak{p}$ is the collection of symmetric n×n matrices of trace zero, and Y is the set of positive definite matrices of determinant one. (This is just the fact that every invertible matrix has a unique "polar" decomposition into a product of a positive-definite matrix and an orthogonal matrix.)

In example 2), the Cartan decomposition is given by

$$\begin{pmatrix} \text{skew} & A \\ {}^tA & \text{skew} \end{pmatrix} = \begin{pmatrix} \text{skew} & 0 \\ 0 & \text{skew} \end{pmatrix} + \begin{pmatrix} 0 & A \\ {}^tA & 0 \end{pmatrix}$$

For the theorem which follows, we need to assume that $Y = G/K$ is "of non-compact type", that is, $G$ and $K$ are as described above. In this situation, all sectional curvatures on $G/K$ are non-positive. Finally we must assume that $Y$ is *not* Hermitian symmetric.

(17.2) **Theorem:** Suppose that $Y$ is as above, that $\Gamma$ is a discrete subgroup of $G$ which acts freely (on the left) on $Y$ such that $\Gamma\backslash Y$ is compact, and that

$$f\colon M \longrightarrow \Gamma\backslash Y$$

is a continuous mapping from a Kähler manifold $M$. Then $f$ is homotopic to a non-surjective map, or, what is equivalent in case $\dim M = \dim Y$, the fundamental cycle of $\Gamma\backslash Y$ is not in the image of $H_*(M)$.

(17.3) In what follows, we want to give some idea of how this theorem is proved. First notice that an immediate corollary of the theorem is that $\Gamma\backslash Y$ itself cannot have a Kähler structure. In fact we make the stronger conjecture:

(17.4) **Conjecture:** If $G/K$ and $\Gamma$ are as above, then $\Gamma$ cannot be the fundamental group of any compact Kähler manifold. (The conjecture is true if $G = SO(n,1)$, $n > 2$. See [CT].)

(17.5) Note that (17.1.2) above is closely related to another example in which $\Gamma\backslash Y$ is the period space arising from the polarized Hodge structure on the primitive second cohomology of algebraic surfaces:

17.5.1) $$G = SO(2p,q) \text{ and } K = U(p)\times SO(q),$$
$$\Gamma = SO(2p,q) \cap GL(2p+q,\mathbf{Z}).$$

Here $G/K$ is a complex manifold since it can be realized as a locally closed subvariety of the variety of $(p,p+q)$-flags $(F^2, F^1)$ in $V^{\mathbf{C}}$, where $V$ is a $\mathbf{R}$-vector space with a non-degenerate symmetric bilinear form of signature $(2p,q)$. However $\Gamma\backslash Y$ is not compact.

(17.6) An example in which $\Gamma$ is co-compact is given by replacing G in (17.5.1) by the orthogonal group of the quadratic form

$$Q = |x|^2 - \sqrt{2}|y|^2$$

on $\mathbf{R}^{2p+q}$, and replacing $\Gamma$ by

$$SO(Q) \cap GL(2p+q, (\text{ring of integers of } \mathbf{Q}(\sqrt{2}))).$$

If $\sigma$ denotes conjugation in $\mathbf{Q}(\sqrt{2})$, then $\Gamma$ has discrete image in $SO(Q) \times SO(Q)$ under the map

$$\gamma \longrightarrow (\gamma, \gamma^{\sigma}).$$

It can be shown that the image of $\Gamma$ is co-compact and so we get the desired co-compactness by the map induced by projecting onto the first factor. If it is not true that

$$p = 2 \text{ or } q = 2,$$

we conclude that the complex manifold $\Gamma\backslash Y$ does not admit a Kähler structure, even though it is "pseudo-Kähler" (that is, it has a natural *indefinite* metric whose Kähler form is closed).

(17.7) **Outline of a proof of Theorem(7.2):**

17.7.1) The first ingredient is a theorem of Eells and Sampson which says that every continuous map from a compact Riemannian manifold to a compact Riemannian manifold with non-positive sectional curvature is homotopic to a *harmonic* map. (A map

$$\phi: M \longrightarrow N$$

is harmonic if it is a local minimum of the *energy function*

$$\int_M |d\phi|^2 \, dv_M = E(\phi)$$

where the norm is induced from the metric on N.) So from now on we can assume that f in the statement of the theorem is harmonic (and then we no longer need assume $\Gamma\backslash Y$ compact).

17.7.2) The second ingredient is another theorem of Sampson for

$$f: M(\text{compact Kähler}) \longrightarrow \Gamma\backslash Y.$$

Namely, the differential df of f takes the holomorphic tangent space $T_{1,0}(M)|_x$ at a point x into the complexified tangent space to $\Gamma\backslash Y$ at f(x). This latter vector space can be identified with $\mathfrak{p}^{\mathbf{C}}$ via the left G-action on Y. Sampson's result is that the image of $T_{1,0}(M)|_x$ must lie in an *abelian* subspace of $\mathfrak{p}^{\mathbf{C}}$, that is,

$$[df, df] \equiv 0.$$

(In this result, it is allowed that $\Gamma\backslash Y$ have Euclidean factors.) Sampson's theorem is proved using Bochner-type identities--we will give the proof in a later seminar.)

17.7.3) The final ingredient comes by measuring the size of abelian subspaces of $\mathfrak{p}^{\mathbf{C}}$:

**Theorem:** Assume that $\mathfrak{g}$ has no factor isomorphic to $\mathfrak{sl}(2,\mathbf{R})$. If $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{p}^{\mathbf{C}}$, then

$$\dim_{\mathbf{C}} \mathfrak{a} \leq (1/2)\cdot \dim_{\mathbf{C}} \mathfrak{p}^{\mathbf{C}}.$$

Furthermore, equality holds only in the case in which $Y = G/K$ is Hermitian symmetric, and $\mathfrak{a}$ corresponds to the (1,0)-tangent space of one of the standard Hermitian symmetric structures on $\Gamma\backslash Y$.

(17.8) References: For a general introduction to symmetric spaces see [H]. (17.2) is due to Carlson-Toledo[CT]. (17.7.1) is in [ES], (17.7.2) in [Sa] and (17.7.3) in [CT].

**Lecture #18: Proof of Sampson's theorem**

Today we will prove the theorem of Sampson that is used in (17.7.2).

(18.1) **Notation:** Given

$$f\colon M \longrightarrow N = \Gamma\backslash G/K,$$

let "T( )" denote the "tangent bundle". We consider the bundle $f^*T(N)^{\mathbf{C}}$ on M, with metric induced from the Riemannian metric on N. This metric induces a connection

$$\nabla\colon \Gamma(f^*T(N)^{\mathbf{C}}) \longrightarrow \Gamma(T^*(M)\otimes f^*T(N)^{\mathbf{C}}).$$

Let $\nabla = \nabla' + \nabla''$ be the decomposition of $\nabla$ given by the decomposition

$$T^*(M) = T^{1,0}(M) + T^{0,1}(M).$$

The curvature tensor R is given by

$$-R(X,Y)s = \nabla_X \circ \nabla_Y(s) - \nabla_Y \circ \nabla_X(s) - \nabla_{[X,Y]}(s).$$

(18.2) **Theorem:** If M and N are as above, then $f$ is harmonic if and only if

i) for $X, Y \in T_{1,0}(M)$, then $R(X,Y) \equiv 0$, (so also for $X, Y \in T_{0,1}(M)$, $R(X,Y) \equiv 0$);

ii) $df\colon T_{1,0}(M) \longrightarrow f^*T(N)^{\mathbf{C}}$ is a holomorphic mapping of holomorphic vector bundles where the holomorphic structure on $f^*T(N)^{\mathbf{C}}$ is one such that $\nabla''$ becomes the $\bar{\partial}$-operator. (Such a holomorphic structure exists by i).)

**Proof:** The Euler-Lagrange equation for the above-defined energy function E of a harmonic map $f$ is gotten as follows:

$$|df|^2 = \mathrm{tr}({}^t(df)\cdot df)$$

so the variational formula for a local minimum is

$$\tau(f)_x = \mathrm{tr}(\nabla df)_x = \Sigma \nabla_{X(i)} df(X(i))|_x = 0$$

for $x \in X$ and an orthogonal basis $\{X(i)\}$ of $T(M)_x$. (Recall that df is a section of the bundle $T^*(M) \otimes f^*T(N)$ with connection induced by the Riemannian connections on $T^*(M)$ and $f^*T(N)$. Intuitively, the energy function $E(\phi)$ is measuring how far a mapping $\phi$ is from being an isometry.)

(18.3) We will get the result we want by covariant differentiations of $f^*(g_N)$, where $g_N$ denotes the metric on N. Since M is Kähler, these covariant differentiations will respect the decomposition of $f^*(g_N)$ into types and, more precisely, we will get the desired result by following the summand of $f^*(g_N)$ of type (2,0) through a commutative diagram of covariant differentiations:

"$\lfloor$" denotes the contraction induced by the metric on $T^*(M)$

$$\begin{tikzcd}
\Gamma(S^2T^*M) \arrow[d, "\nabla"] & & \\
\Gamma(T^*M\otimes S^2T^*M) \arrow[r, "\nabla"] \arrow[d, "\lfloor"] & \Gamma(T^*M\otimes T^*M\otimes S^2T^*M) \arrow[dr, "\lfloor\otimes\lfloor"] & \\
\Gamma(T^*M) \arrow[r, "\nabla"] \arrow[rr, bend right, "d^*"'] & \Gamma(T^*M\otimes T^*M) \arrow[r, "\lfloor"] & \Gamma(\mathbf{R})
\end{tikzcd}$$

Here $\nabla$ means the covariant differentiation induced from the Kähler metric on M.

We apply the composition of maps in the diagram to $f^*(g_N)$, for f harmonic. We will obtain an expression of the form

$$\|\nabla df\|^2 + \mathrm{Ricci}_M(\ldots) - R_N(\ldots)$$

where the "(...)" means an expression in the Ricci curvature of M and an expression in the curvature of $f^*T(N)$ respectively. Since this expression is in the image of $d^*$, it will have to integrate to zero over M. But, since there are terms with opposite signs, we don't get much information from this fact.

(18.4) However, if we apply the composition of the maps in the diagram to the (0,2)-component of $f^*(g_N)$, the term involving the Ricci curvature of M drops out and we obtain an expression

$$\|\nabla''d'f\|^2 - \Sigma\Sigma\langle R(Z(i)'',Z(j)'')df(Z(i)''),\ df(Z(j)'')\rangle$$

where " denotes the (0,n)-component and d'f is the restriction of df to $T_{1,0}(M)$. Again this expression must vanish for harmonic f when we integrate over M. The vanishing of the first term corresponds to the second assertion of Sampson's theorem and the vanishing of the second term corresponds to the first assertion. We now compute, first for the entire tensor $f^*(g_N)$:

$$\nabla_Z f^*(g_N)(X,Y) = d_Z(f^*(g_N)(X,Y)) - f^*(g_N)(\nabla_Z X,Y) - f^*(g_N)(X,\nabla_Z Y)$$

$$= <\nabla_Z(df(X)),\ df(Y)> + <df(X),\ \nabla_Z(df(Y))> - f^*(g_N)(\nabla_Z X,Y) - f^*(g_N)(X,\nabla_Z Y)$$

$$= <(\nabla_Z df)(X),\ df(Y)> + <df(X),\ (\nabla_Z df)(Y)>$$

(Warning: $\nabla_Z$ sometimes means the connection on $S^2T^*(M)$, sometimes the connection on $f^*T(N)$, sometimes the connection on $T^*(M)\otimes f^*T(N)$. Decide by looking at the vector $\nabla_Z$ operates on!)

Therefore

$$\nabla_W\nabla_Z f^*(g_N)(X,Y) = <\nabla_W\nabla_Z df(X),\ df(Y)> + <\nabla_W df(X),\ \nabla_Z df(Y)> + <\nabla_Z df(X),\ \nabla_W df(Y)> + <df(X),\ \nabla_W\nabla_Z df(Y)>.$$

Using normal coordinates at a point $p$ and an orthonormal basis $\{X(i)\}$ for $T(m)$ such that $[X(i),X(j)] = 0$ at $p$, we use the above formula and the Euler-Lagrange formula

$$\Sigma\, \nabla_{X(i)} df(X(i)) = 0$$

for harmonic maps to compute the image at $p$ of $f^*(g_N)$ under the composition of maps in the diagram given in (18.3):

$$\Sigma\Sigma\, \nabla_{X(j)}\nabla_{X(i)} f^*(g_N)(X(i),X(j))$$

$$= \Sigma\Sigma <\nabla_{X(j)} df(X(i)),\ \nabla_{X(i)} df(X(j))> + \Sigma\Sigma <df(X(i)),\ \nabla_{X(j)}\nabla_{X(i)} df(X(j))>$$

$$= \|\nabla df\|^2 + \Sigma\Sigma <df(X(i)),\ \nabla_{X(j)}\nabla_{X(i)} df(X(j))> - \Sigma\Sigma <df(X(i)),\ \nabla_{X(i)}\nabla_{X(j)} df(X(j))>$$

$$= \|\nabla df\|^2 - \Sigma\Sigma <df(X(i)),\ R^{\otimes}(X(j),X(i))df(X(j))>$$

where $R^{\otimes}$ here is the curvature of the connection on the bundle

$$T^*(M)\otimes f^*T(N).$$

But the curvature of a tensor product of two bundles with the tensor-product connection satisfies a Leibniz rule so that we finally get:

$$\Sigma\Sigma\, \nabla_{X(j)}\nabla_{X(i)} f^*(g_N)(X(i),X(j)) = \|\nabla df\|^2 + \Sigma\Sigma <df(X(i)),\ df(R_M(X(j),X(i))X(j))> - \Sigma\Sigma <dfX(i),\ R_N(dfX(j),dfX(i))(dfX(j))>.$$

(The change in sign on the second term comes in the passage from cotangent to tangent bundle.)

(18.5) Now since M is a Kähler manifold, covariant differentiation $\nabla_Z$ respects types in $S^2T^*(M)$. We replace the orthonormal basis $\{X(i)\}$ with a standard Hermitian basis $\{Z(i)',Z(i)''\}$ for

$$T_{(1,0)}(M)+T_{(0,1)}.$$

Applying the composition in (18.3) to the (0,2)-component of $f^*g_N$, we get

$$\begin{aligned}\Sigma\Sigma &<\nabla_{Z(j)''}df(Z(i)'),\ \nabla_{Z(i)''}df(Z(j)')> \\ &+ \Sigma\Sigma <df(Z(i)''),\ df(R_M(Z(j)'',Z(i)'')Z(j)'')> \\ &- \Sigma\Sigma <dfZ(i)'',\ R_N(dfZ(j)'',dfZ(i)'')(dfZ(j)'')>.\end{aligned}$$

The Kähler identities for $R_M$ imply that the term involving $R_M$ vanishes. Applying the definition of $R_N$, the above expression becomes

$$\begin{aligned}\Sigma\Sigma &<\nabla_{Z(j)''}df(Z(i)'),\ \nabla_{Z(i)''}df(Z(j)')> \\ &+ \Sigma\Sigma <dfZ(i)'',\ [[dfZ(j)'',dfZ(i)''],dfZ(j)'']>.\end{aligned}$$

Now using the identity for the Killing form

$$<[X,Y],Z> + <Y,[X,Z]> = 0,$$

the above expression becomes

$$\begin{aligned}\Sigma\Sigma &<\nabla_{Z(j)''}df(Z(i)'),\ \nabla_{Z(i)''}df(Z(j)')> \\ &+ \Sigma\Sigma <[dfZ(j)'',dfZ(i)''],[[dfZ(j)'',dfZ(i)'']]>\end{aligned}$$

as desired.

(18.6) References: This proof is a reformulation of the original one in [Sa]. Finding a Bochner formula not involving the Ricci tensor of M was first accomplished by Siu[Si].

## Lecture #19: Abelian subalgebras of Lie algebras

Finally, we want to discuss the proof of the last step in the program presented in Lecture #17:

(19.1) **Theorem:** Assume that $\mathfrak{g}$ is a semi-simple real Lie algebra. Let $\mathfrak{p}$ be the -1 eigenspace of the Cartan involution (see (17.1)). If W is an abelian subalgebra of $\mathfrak{p}^{\mathbf{C}}$, then

$$\dim_{\mathbf{C}} W \leq (1/2)\dim_{\mathbf{R}} \mathfrak{p}.$$

Furthermore, if $\mathfrak{g}$ has no sl(2,**R**) factor, then equality holds only in the case in which $\mathfrak{g}$ is the Lie algebra of infinitesimal isometries of an Hermitian symmetric space and W is the (1,0)-tangent space to a natural symmetric complex structure.

Notice that for the sake of simplicity, we will only treat the case in which $\mathfrak{g}$ is simple. (The general proof is essentially the same.)

The steps in the proof are:

(19.2) Suppose that W is a maximal abelian subspace of $\mathfrak{p}^{\mathbf{C}}$. We first reduce to proving the case $W \cap W^{-} = 0$. Suppose that

$$\mathfrak{a} = W \cap W^{-} \neq 0,$$

where "$^{-}$" denotes "conjugate". Then

$$\mathfrak{a} \subseteq \mathfrak{t} \subseteq \mathfrak{p},$$

where $\mathfrak{t}$ is the tangent space to a maximal flat subspace of G/K.

(19.3) From the theory of roots for a real semi-simple Lie group, the action of $\mathfrak{t}$ on

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is as follows:

There is a finite set of roots $\{\alpha\}$ and $X_\alpha \in \mathfrak{k}$, $Y_\alpha \in \mathfrak{p}$ such that

$$[X, X_\alpha] = \alpha(X) Y_\alpha, \qquad [X, Y_\alpha] = \alpha(X) X_\alpha$$

for all $X \in \mathfrak{t}$.

(19.4) Let $\mathfrak{k}_\alpha$ denote the one-dimensional space generated by $X_\alpha$ and let $\mathfrak{p}_\alpha$ denote the one-dimensional space generated by $Y_\alpha$. Then

$$\mathfrak{p} = \mathfrak{t} + \Sigma \mathfrak{p}_\alpha.$$

Also, if we let $\{\beta\} \subseteq \{\alpha\}$ be the collection of roots which vanish on $\mathfrak{a}$. Since the roots generate the dual space of $\mathfrak{a}$,

$$\#\{\beta\} + \dim \mathfrak{a} \leq \#\{\alpha\}.$$

(19.5) Using that W was chosen to be maximal, one shows that the subspace $\mathfrak{p}'$ orthogonal to $\mathfrak{a}$ in $(\mathfrak{t} + \Sigma \mathfrak{p}_\beta)$ with respect to the Killing form is again a symmetric space of the same (non-compact) type because it is closed under [[ , ], ]. Since $(\mathfrak{t} + \Sigma \mathfrak{p}_\beta)$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{p}$,

$$W \subseteq (\mathfrak{t} + \Sigma \mathfrak{p}_\beta)^{\mathbf{C}}$$

so that $W = \mathfrak{a} \oplus W'$, where $W' = (\mathfrak{p}' \cap W)$. Notice that

$$W' \cap (W')^- = 0.$$

Suppose we know that

$$\dim_{\mathbf{C}} W' \leq (1/2)\dim_{\mathbf{R}} \mathfrak{p}'.$$

Then, since the codimension of $\mathfrak{p}'$ in $\mathfrak{p}$ is at least twice $\dim_{\mathbf{R}} \mathfrak{a}$, we conclude

$$\dim_{\mathbf{C}} W \leq (1/2)\dim_{\mathbf{R}} \mathfrak{p}.$$

Notice also that equality is only possible if $\mathfrak{a} = 0$ in the first place.

(19.6) Since we may now assume that $W \cap W^- = 0$, the inequality

$$\dim_{\mathbf{C}} W \leq (1/2)\dim_{\mathbf{R}} \mathfrak{p}$$

is automatic. We need only show that equality implies that G/K is Hermitian symmetric with $W = \mathfrak{p}_{(1,0)}$ or $W = \mathfrak{p}_{(0,1)}$. The conditions

$$W \cap W^- = 0 \quad \text{and} \quad W \oplus W^- = \mathfrak{p}^{\mathbf{C}}$$

mean that W induces a complex structure J on $\mathfrak{p}$.

(19.7) Equivalent conditions which imply that G/K is Hermitian symmetric are:

i) $J \in K$, that is, J is induced by and element of K under the adjoint representation,
ii) J is an isometry for the Killing form,
iii) W is K-invariant under the adjoint representation,
iv) W is isotropic for the Killing form.

(19.8) We will complete the proof by showing that, if rank(G/K) > 1, then J is an isometry, and, if rank(G/K) = 1, W is isotropic for the Killing form.

19.8.1) rank(G/K) > 1: Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Then one sees easily that $J(\mathfrak{t})$ must also be abelian. But K operates transitively on the set of maximal abelian subalgebras of $\mathfrak{p}$, so there must be an element $k \in K$ so that Ad(k)J takes $\mathfrak{t}$ to itself. Again one shows that Ad(k)J must permute the "singular" hyperplanes of $\mathfrak{t}$ given by the roots. Since, by irreducibility, there are

$$\dim \mathfrak{t} + 1$$

of these in general position, Ad(k)J must be a multiple of the identity on $\mathfrak{t}$. One then shows that this implies that Ad(k)J is a multiple of the identity on all of $\mathfrak{p}$. So

$$\langle JX, JY \rangle = m\langle X, Y \rangle$$

for all X and Y in $\mathfrak{p}$. But $J^2 = -1$. So m = 1.

19.8.2) rank(G/K) = 1: We will prove that W is isotropic for $\langle\ ,\ \rangle$. Recall that G/K has rank one if and only if K operates transitively on

$$S(\mathfrak{p}) = \{X \in \mathfrak{p} : \langle X, X \rangle = 1\}.$$

One then shows that this implies that $K^{\mathbf{C}}$ operates transitively on

$$\{X \in \mathfrak{p}^{\mathbf{C}} : \langle X, X \rangle = 1\}.$$

So, if $\langle X, X \rangle \neq 0$, then the $K^{\mathbf{C}}$-orbit of X has codimension one in $\mathfrak{p}^{\mathbf{C}}$. Suppose now that $X \in W$. Let $\mathfrak{c}$ denote the centralizer of X in $\mathfrak{p}^{\mathbf{C}}$. Then $Y \in \mathfrak{c}$ if and only if $Y \perp [\mathfrak{k}^{\mathbf{C}}, X]$ since

$$\langle [X,Y], Z \rangle = -\langle Y, [X,Z] \rangle.$$

Since we are in the case in which $\dim W \geq 2$, the codimension of $[\mathfrak{k}^{\mathbf{C}}, W]$ in $\mathfrak{p}^{\mathbf{C}}$ must be $\geq 2$, so that the codimension of the $K^{\mathbf{C}}$-orbit of X in $\mathfrak{p}^{\mathbf{C}}$ must be $\geq 2$, so that $\langle X, X \rangle = 0$.

Notice that the above result implies the following:

(19.9) **Rigidity Theorem of Siu:**
If G/K is hermitian symmetric, irreducible, and not the hyperbolic plane, and if M is compact, Kähler, and if

$$f\colon M \longrightarrow N = \Gamma\backslash G/K$$

is harmonic, and if, at some $x \in M$, $\mathrm{rank}_x f = \dim N$, then $f$ is either holomorphic or anti-holomorphic.

**Outline of proof:** By Sampson's result, df is a holomorphic bundle map and so is of maximal rank off a proper complex analytic subvariety M'. Above we showed that $\mathfrak{p}_{(1,0)}$ and $\mathfrak{p}_{(1,0)}$ are the only maximal rank abelian subspaces of $\mathfrak{p}^{\mathbf{C}}$, so d"f must map to one of these. Extend the map over M' by analytic continuation.

(19.10) References: (19.1) is proved in [CT] and (19.9) in [Si].

## Lecture #20: Maximal variations of Hodge structures

Today we will discuss a result on "variation of Hodge structures" which is closely related to the results on harmonic maps described in Lectures 17-19.

(20.1) The geometric model for a variation of Hodge structure comes from an analytic family $\{X_s: s \in S\}$ of Kähler manifolds. After framing $H^*(X_s; \mathbf{Z})$ locally, the Hodge decomposition

$$H^k(X_s) = \sum_{p+q=k} H^{p,q}$$

gives a continuously varying direct-sum-decomposition of a *fixed* complex vector space $H \approx H^k(X_s)$. Alternatively, the decreasing filtration

$$F^p = \sum_{p' \geq p} H^{p',k-p'}$$

gives a holomorphically varying family of subspaces of H. As we shall see below, this realizes $\{F^p(X_s)\}$ locally as a holomorphic map of S into a product of Grassmann varieties. The image will lie in a locally closed complex analytic subvariety D of the product of Grassmannians. D is a complex manifold and a homogeneous space.

(20.2) Rather than define things in generality, we illustrate this construction for polarized Hodge structures of weight two. Given a complex vector space H of dimension 2p+q with an integral structure which has an integral-valued symmetric bilinear form $\langle\ ,\ \rangle$ of signature (2p,q), we define D to be the space of all filtrations

$$\{F^0 = H,\ F^1,\ F^2,\ F^3 = 0\}$$

with $\dim F^1 = p+q$, $\dim F^2 = p$, such that with respect to $\langle\ ,\ \rangle$ we have:

20.2.1) $(F^p)^{\perp} = F^{3-p}$,

20.2.2) the subspaces

$$H^{p,2-p} = F^p \cap (F^{2-p})^-$$

give a direct-sum-decomposition of H (where "$^-$" means "conjugate"),

20.2.3) $-\langle i^{p-q}(\ ),(\ )^- \rangle$ is a positive definite Hermitian form on $H^{p,q}$ whenever p+q = 2.

(20.3) Picking a reference Hodge structure $H \in D$, D becomes the homogeneous space

$$SO(2p,q)/U(p)\times SO(q) = G/V.$$

The complexified Lie algebra $\mathfrak{g}^{\mathbf{C}}$ of G has a direct sum decomposition

$$\mathfrak{g}^{\mathbf{C}} = \oplus\, \mathfrak{g}^{-p,p}$$

where $\mathfrak{g}^{-p,p}$ is the subspace of elements of the Lie algebra which takes each $H^{p',q'}$ to $H^{p'-p,q'+p}$.

If we take the sum $\oplus\, \mathfrak{g}^{-p,p}$ only over positive p, we obtain the holomorphic tangent space to D, which we will denote as $\mathfrak{g}^{-}$. If we frame H by taking a unitary basis for $H^{2,0}$, the conjugate basis for $H^{0,2}$, and an orthonormal basis for $H^{1,1}$, we can write:

$$< , > \quad = \quad \begin{array}{c c} \begin{array}{ccc} p & q & p \end{array} & \\ \begin{pmatrix} 0 & 0 & I \\ 0 & -I & 0 \\ I & 0 & 0 \end{pmatrix} & \begin{array}{c} p \\ q \\ p \end{array} \end{array}$$

$$\mathfrak{g}^{-} \quad = \quad \left\{ \begin{array}{c c} \begin{array}{ccc} p & q & p \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ Y & Z & 0 \end{pmatrix} & \begin{array}{c} p \\ q \\ p \end{array} \end{array} : \begin{array}{l} Z = {}^{t}X \\ Y \text{ skew} \end{array} \right\}$$

$$\mathfrak{g}^{(-2,2)} \qquad \mathfrak{g}^{(-1,1)}$$

(20.4) Griffiths showed that a family of surfaces $\{X_s\}_{s\in S}$ induces locally an analytic map

$$f: S \longrightarrow D,$$

called the period mapping, and that this mapping is horizontal, that is,

$$dF^p/ds \subseteq F^{p-1}.$$

Calculating at any reference point H, we see that f is horizontal at H if and only if df takes values in the subspace

$$\mathfrak{g}^{-1,1}.$$

(20.5) **Definition**: A (local) **variation of** (polarized, weight two) **Hodge structures** is a horizontal analytic map

$$f: S \longrightarrow D$$

where S is any complex manifold (not necessarily the parameter space of a family of surfaces).

From the above matrix presentation of $\mathfrak{g}^-$ we see that horizontality of $f$ is automatic if and only if $p = 1$ (in which case D is Hermitian symmetric).

We wish to address the following question:

<u>How large can the rank of df be?</u>

To answer this, we begin with the following observation:

(20.6) **Lemma**: The image $df(T_o{}^{1,0}S)$ in the holomorphic tangent space of $D$ can be identified with a subspace

$$\mathfrak{a} \subseteq \mathfrak{g}^{-1,1}$$

which satisfies $[\mathfrak{a},\mathfrak{a}] \subseteq \mathfrak{g}^{-1,1}$.

One would like to say "this follows from the integrability condition on vector fields tangent to an integral submanifold". However, one must distinguish between Lie bracket of vectorfields on $D$ and Lie bracket of left-invariant vector fields on the group, and one must also *choose* the identification to be used. We must therefore give an argument:

**Proof**: Let

$$\xi = \begin{bmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ Y & {}^tX & 0 \end{bmatrix}$$

be an element of $\mathfrak{g}^-$, and consider the map which sends $\xi$ to $e^\xi F_o{}^*$ where $F_o{}^*$ is a reference filtration. This map defines a local coordinate system on a neighborhood $W$ of the reference filtration, and the map $n$ which sends $e^\xi F_o{}^*$ to $e^\xi$ defines a lifting of $W$ into the group. Let $\omega = n^{-1}dn$ be the associated Maurer-Cartan form, and set

$$\mathfrak{a} = f^*\omega(T_o{}^{1,0}S).$$

By construction, $\omega$ is a form with values in $\mathfrak{g}^{-1,1}$, i.e., in the space of matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & {}^tX & 0 \end{bmatrix} .$$

Now pull the integrability condition $d\omega - \omega\wedge\omega = 0$ back via $f$ and evaluate on a pair of tangent vector fields $U$ and $V$ to get

$$U(f^*\omega(V)) - V(f^*\omega(U)) - f^*\omega([U,V]) - [f^*\omega(U), f^*\omega(V)] = 0 .$$

Since the first three terms lie in $\mathfrak{g}^{-1,1}$, so must the last. But this is the assertion to be proved.

With this result in hand we can establish a fundamental property of the subspaces $\mathfrak{a}$ :

(20.7) **Lemma:** If $\mathfrak{a}$ is the tangent space to a variation of Hodge structure, identified with a subspace of $\mathfrak{g}^{-1,1}$ as above, then

$$[\mathfrak{a},\mathfrak{a}] = 0.$$

In other words, the tangent spaces to variations of Hodge structure are abelian.

**Proof:** For formal reasons one has

$$[\mathfrak{a},\mathfrak{a}] \subseteq \mathfrak{g}^{-2,2} .$$

By the previous lemma,

$$[\mathfrak{a},\mathfrak{a}] \subseteq \mathfrak{g}^{-1,1}.$$

But $\mathfrak{g}^{-1,1}$ and $\mathfrak{g}^{-2,2}$ are complementary, so $[\mathfrak{a},\mathfrak{a}] = 0$, as required.

**Remark:** The condition $[\mathfrak{a},\mathfrak{a}] = 0$ is inspired by, but slightly stronger than, the analogous condition for infinitesimal variations of Hodge structure.

Let us now draw the consequences of this last result. If one writes a horizontal tangent vector as

$$N(X) \quad = \quad \begin{array}{cc} \begin{array}{ccc} p & q & p \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & {}^tX & 0 \end{pmatrix} & \begin{array}{c} p \\ q \\ p \end{array} \end{array}$$

then the condition that $\mathfrak{a}$ be abelian becomes

$$(*) \qquad {}^tX \cdot X' - {}^tX' \cdot X = 0,$$

whenever $N(X), N(X') \in \mathfrak{a}$. We therefore consider the following:

(20.8) **Lemma:** Let $\mathfrak{a}$ be a space of $q \times p$ matrices satisfying (*). If $p > 1$, then

$$\dim \mathfrak{a} \leq (1/2)pq.$$

**Proof:** Let $\{e_i\}$ be the standard basis of $\mathbf{C}^p$, let $\{f_j\}$ be the standard basis of $\mathbf{C}^q$, let ( , ) be the complex bilinear form given by the rule $f_i \cdot f_j = \delta_{ij}$. Define

$$\mathfrak{a}_j = \{X \in \mathfrak{a} \colon X(e_i) = 0 \text{ for all } i \leq j\}$$

with $\alpha_0 = \{\text{all } q \times p \text{ matrices}\}$. Then $\{\mathfrak{a}_j \colon 0 \leq j \leq p\}$ is a decreasing filtration of $\mathfrak{a}$. We also define the subspaces

$$S_j = \mathfrak{a}_j / \mathfrak{a}_{j+1} = \mathfrak{a}_j(e_{j+1}) \subseteq \mathbf{C}^q.$$

Then we have

$$\alpha \cong \oplus\, \mathfrak{a}_i / \mathfrak{a}_{i+1} \cong \oplus\, S_i.$$

To conclude, we observe the following:

20.9.1) $S_i \perp S_j$ for $i < j$, since

$${}^tX(e_{i+1}) \cdot Y(e_{j+1}) = {}^tY(e_{i+1}) \cdot X(e_{j+1}) = 0$$

for $X \in \mathfrak{a}_i$, $Y \in \mathfrak{a}_j$.

20.9.2) If $S_1, \ldots, S_k$ is a collection of mutually perpendicular subspaces of $\mathbf{C}^q$ and $k > 1$, then $\Sigma s_j \le (1/2)kq$, where $s_j = \dim S_j$.

To see 20.9.2), notice that if $i < j$, then

$$s_i + s_j \le q,$$

since $S_i \subseteq S_j^{\perp}$ and $\dim S_j + \dim S_j^{\perp} = q$. Consequently

$$\sum_{i<j} (s_i + s_j) \le (1/2)k(k-1)q$$

and also

$$\sum_{i<j} (s_i + s_j) = (k-1)\sum s_j ,$$

from which the lemma follows.

We have therefore established the following:

(20.10) **Theorem:** Let D be a period domain for polarized weight-two Hodge structures. Let

$$f\colon S \longrightarrow D$$

be a local variation of Hodge structures. Then

$$\operatorname{rank} f \le (1/2)h^{2,0}h^{1,1}.$$

If $h^{1,1}$ is even and $h^{2,0} > 2$, the above bound is sharp, as we will show below. However, not all variations are contained in variations of rank

$$(1/2)h^{2,0}h^{1,1}.$$

For example:

(20.11) **Theorem:** With three exceptions, variations of Hodge structures coming from hypersurfaces of dimension

$$n \ge 2$$

are maximal.

(For surfaces of degree $d$ in $\mathbf{CP}^3$,

$$(1/2)h^{2,0}h^{1,1}$$

grows like $d^6$ whereas the variation dimension grows like $d^3$.)

(20.12) To see sharpness, let $q = 2q'$. Let $V$ be a maximal totally isotropic subspace of $\mathbf{C}^q$ with respect to $( \, , \, )$. Then $\dim V = q'$. For example, $V$ might have basis

$$\{(1,i,0,\ldots,0),(0,0,1,i,0,\ldots,0),\text{etc.}\}.$$

Then

$$\mathbf{C}^q = V + V^-,$$

and $\{N(X): X \in \mathrm{Hom}(\mathbf{C}^p, V)\}$ defines an abelian subspace of $\mathfrak{g}^{-1,1}$. In fact, the corresponding variation of Hodge structure is easily seen to be induced from the group homomorphism

$$SU(p,q') \longrightarrow SO(2p,2q').$$

Moreover, all maximal-dimensional variations are of this form in the case $h^{1,1}$ even and $h^{2,0} > 2$.

(20.13) The dimension bound for variations of Hodge structures can be seen as an analogue to the bounds on the dimension of harmonic maps from a Kähler manifold to a symmetric space of non-compact type given in Lectures #17-19, and the sharpness result can be seen as an analogue of Siu's Rigidity Theorem. In fact, we can give more substance to this analogy as follows:

(20.14) Let $D = G/V$ where $V$ is compact. Find a maximal compact subgroup $K$ containing $V$. Let

$$D_0 = G/K.$$

The group

$$\Gamma \leq G$$

of integral-valued $< \, , \, >$-isometries is a discrete subgroup, and we have

$$\pi: \Gamma\backslash D \longrightarrow \Gamma\backslash D_0$$

with fibre $K/V$.

(20.15) **Theorem:** If $M$ is a complex manifold and

$$f: M \longrightarrow \Gamma\backslash D$$

is a variation of polarized weight-two Hodge structures, then $\pi \circ f$ is harmonic.

Sometimes the converse holds. For example, if $D_0$ is quaternionic hyperbolic space, then all harmonic maps to $\Gamma\backslash D_0$ of rank greater than two lift to variations of Hodge structures.

(20.16) **Remark:** Sharp bounds on the rank of a variation of Hodge structure in arbitrary weight have recently been obtained in joint work of Carlson, Kasparian and Toledo. For weight two, these results give an improved (and sharp) bound for the case of $h^{1,1}$ odd:

$$\text{rank } f \leq (1/2)h^{2,0}(h^{1,1} - 1) + 1 \ ,$$

where $h^{2,0} > 1$.

(20.17) References: (20.10) is due to Carlson [Ca]. (20.11) is in [CD]. (20.15) is in [CT].

## Lecture #21: Subvarieties of generic hypersurfaces

(21.1) We work over an arbitrary algebraically-closed base field. We consider generically finite morphisms

$$f: \quad \mathbf{X} \longrightarrow \mathbf{V} \subseteq \mathbf{Y}$$

of a projective manifold **X** into a subvariety **V** of an ambient projective variety **Y**. We require that **V** and **Y** be smooth at points of $f(\mathbf{X})$. The normal sheaf, whose sections measure first-order deformations of $f$ which leave the target space **V** fixed, is given by the formula

$$N_{f,\mathbf{V}} = f^*T_{\mathbf{V}}/T_{\mathbf{X}} .$$

Typical of the estimates we obtain is the case in which **X** is a rational curve and **V** is a generic hypersurface of degree $m$ in $\mathbf{P}^n$. If $\tau$ denotes the length of the torsion subsheaf of $N_{f,\mathbf{V}}$, and let

$$c = \text{rank of } N_{f,\mathbf{V}}/(\text{image of } H^0(N_{f,\mathbf{V}} \otimes \mathcal{O}_{\mathbf{X}})).$$

Then

$$c \geq (m - (n+1)) + ((2+\tau)/(\deg f)).$$

So, the more positive the canonical bundle of **V** is, the harder it is to find rational curves on **V**.

(21.2) We begin by developing these ideas in a general setting. We assume that we are in a situation in which the normal sheaf $N_{f,\mathbf{Y}}$ to $f$ in the ambient space **Y** has enough sections to generate $f^*N_{\mathbf{V},\mathbf{Y}}$. In this situation, we have a surjective morphism of locally free sheaves

$$\psi: \; H^0(N_{f,\mathbf{Y}}) \otimes \mathcal{O}_{\mathbf{X}} \longrightarrow f^*N_{\mathbf{V},\mathbf{Y}}$$

induced by the natural map of normal sheaves. Let $\mathcal{K}$ denote the kernel of this map. Then $\mathcal{K}$ is a locally free sheaf on **X**. Furthermore, we have the natural morphism of exact sequences of sheaves:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & H^0(N_{f,\mathbf{Y}}) \otimes \mathcal{O}_{\mathbf{X}} & \longrightarrow & f^*N_{\mathbf{V},\mathbf{Y}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & N_{f,\mathbf{V}} & \longrightarrow & N_{f,\mathbf{Y}} & \longrightarrow & f^*N_{\mathbf{V},\mathbf{Y}} & \longrightarrow & 0
\end{array}$$

Intuitively, $\mathcal{K}$ cuts out the directions in $N_{f,\mathbf{V}}$ taken by points "left behind" in $\mathbf{V}$ as $f$ moves in $\mathbf{Y}$.

Let us denote

$$\mathcal{L} = \det f^*N_{\mathbf{V},\mathbf{Y}}.$$

(21.3) **Lemma:** $\mathcal{K}\otimes\mathcal{L}$ is generated by global sections.

**Proof:** Given a vector $\sigma(x)$ in the geometric fibre of $\mathcal{K}$ at a point $x \in \mathbf{X}$, $\sigma(x)$ determines a unique section $\tau_0$ of $H^0(N_{f,\mathbf{Y}})\otimes\mathcal{O}_{\mathbf{X}}$ which has the value $\sigma(x)$ at $x$. Choose sections $\tau_i$ such that $\psi(\tau_i)$, $i = 1,\ldots,r$, generate the geometric fibre of $f^*N_{\mathbf{V},\mathbf{Y}}$ at $x$. The section required by the lemma is

$$\sum_{i=0}^{r} (-1)^i \det(\psi(\tau_0)\ldots\psi(\tau_{i-1})\ \psi(\tau_{i+1})\ldots\psi(\tau_r))\,\tau_i$$

(21.4) **Lemma:** Let $\mathcal{S}$ be the image of $\mathcal{K}$ in $N_{f,\mathbf{V}}$. The sequence

$$0 \longrightarrow N_{f,\mathbf{V}}/\mathcal{S} \longrightarrow N_{f,\mathbf{Y}}/\mathcal{S} \longrightarrow f^*N_{\mathbf{V},\mathbf{Y}} \longrightarrow 0$$

is split.

**Proof:** The map $\psi$ is surjective. The result now follows immediately from the commutative diagram in (21.2).

(21.5) More generally, suppose we have a transverse intersection

$$\mathbf{V} = \mathbf{V}_1 \cdot \mathbf{V}_2 \cdot \ \ldots \cdot\ \mathbf{V}_s$$

for $s$ varieties in a projective variety $\mathbf{W}$, and

$$f:\ \mathbf{X} \longrightarrow \mathbf{V} \subseteq \mathbf{W}$$

with $\mathbf{V}$, the $\mathbf{V}_i$, and $\mathbf{W}$ all smooth along $f(\mathbf{X})$. We let

$$\mathbf{Y}_i = \bigcap_{j\neq i} \mathbf{V}_j$$

and require that, for each $i$, the mapping

$$\psi_i\colon H^0(N_{f,\mathbf{Y}_i}) \otimes \mathcal{O}_{\mathbf{X}} \longrightarrow f^*N_{\mathbf{V},\mathbf{Y}_i} = f^*N_{\mathbf{V}_i,\mathbf{W}}$$

be surjective. Then as above there is a sheaf $\mathcal{K}_i$ for each $i = 1,\ldots,r$, and a diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus \mathcal{K}_i & \longrightarrow & \oplus H^0(N_{f,\mathbf{Y}_i}) \otimes \mathcal{O}_{\mathbf{X}} & \longrightarrow & \oplus f^*N_{\mathbf{V},\mathbf{Y}_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & N_{f,\mathbf{V}} & \longrightarrow & N_{f,\mathbf{W}} & \longrightarrow & f^*N_{\mathbf{V},\mathbf{W}} & \longrightarrow & 0
\end{array}$$

We let

$$\mathcal{L}_i = \det f^*N_{\mathbf{V},\mathbf{Y}_i} = \det f^*N_{\mathbf{V}_i,\mathbf{W}}$$

and suppose that $\mathcal{L}$ is a line bundle on $\mathbf{X}$ such that, for each $i$ and each $x \in \mathbf{X}$, we have morphisms

$$\mathcal{L}_i \longrightarrow \mathcal{L}$$

which are surjective at $x$. Then, as in Lemma(21.3) above, we conclude that $\mathcal{K}_i \otimes \mathcal{L}$ is generated by global sections. Furthermore, letting $\mathcal{S} = \oplus \mathcal{S}_i$, we again have that the sequence in Lemma(21.4) splits.

(21.6) Let $\mathcal{S}_0$ be the subsheaf of $N_{f,\mathbf{V}}$ generated by its global sections. Clearly $\mathcal{S} \supseteq \mathcal{S}_0$. By the adjunction formula

$$\begin{aligned} f^*c_1(\mathbf{W}) &= c_1(\mathbf{X}) + c_1(N_{f,\mathbf{W}}) \\ &= c_1(\mathbf{X}) + f^*c_1(N_{\mathbf{V},\mathbf{W}}) + c_1(N_{f,\mathbf{V}}/\mathcal{S}) + c_1(\mathcal{S}/\mathcal{S}_0) + c_1(\mathcal{S}_0) \end{aligned}$$

so that

$$f^*c_1(\mathbf{V}) = c_1(\mathbf{X}) + c_1(N_{f,\mathbf{V}}/\mathcal{S}) + c_1(\mathcal{S}/\mathcal{S}_0) + c_1(\mathcal{S}_0).$$

But $\mathcal{S}_0$ is generated by global sections, as is $(\mathcal{S}/\mathcal{S}_0)\otimes\mathcal{L}$, and $N_{f,\mathbf{V}}/\mathcal{S}$ is a quotient of $N_{f,\mathbf{W}}$. In what follows, we will apply the above equality in case $N_{f,\mathbf{W}}$ is "semi-positive" to conclude a *lower* bound on the rank of $\mathcal{S}/\mathcal{S}_0$ and therefore an *upper* bound on the rank of $\mathcal{S}_0$.

(21.7) As an example of the use of the formula in (21.6), suppose that $\mathbf{X}$ is a curve, and that

$$\tau = \text{length of torsion subsheaf of } N_{f,\mathbf{V}}.$$

As in [C2], we define a sheaf on $\mathbf{X}$ to be **semi-positive** if it has no quotients of negative degree.

(21.8) **Theorem:** If $\mathbf{X}$ is a curve and

$$f: \mathbf{X} \longrightarrow \mathbf{V} \subseteq \mathbf{W}$$

is as in (21.5), if $N_{f,\mathbf{W}}$ is semi-positive, and if $\mathcal{L}$ is a "basepoint-free multiple" of each $\mathcal{L}_i$, then

$$\operatorname{rank}(\mathcal{B}/\mathcal{B}_0)(\deg \mathcal{L}) \geq (\deg f^*K_{\mathbf{V}}) + (2-2g) + \tau,$$

where $K_{\mathbf{V}}$ denotes the canonical bundle of $\mathbf{V}$.

**Proof:** The theorem results from writing down the formula in (21.6). In this case $c_1(\mathbf{X}) = 2-2g$ where $g$ = genus $\mathbf{X}$. Also

$$c_1(\mathcal{B}_0) \geq \tau,$$

and, by Lemma(21.4), $c_1(N_{f,\mathbf{V}}/\mathcal{B}) \geq 0$.

(21.9) To give an example of the use of this theorem, we restrict further to the case in which $\mathbf{X}$ is a rational curve, $\mathbf{W}$ is a generic hypersurface of degree $m$ in $\mathbf{P}^{n+m}$, and $\mathbf{V}$ is cut out in $\mathbf{W}$ by a generic linear space of dimension $n$. (The semi-positivity of $N_{f,\mathbf{W}}$ is shown in [C2].) Since $\mathbf{X}$ is rational, we have the formula

$$B_{f,\mathbf{V}} = \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_{n-2})$$

for the "locally free part" $B_{f,\mathbf{V}}$ of $N_{f,\mathbf{V}}$, where

$$\Sigma a_j = -(\deg f^*K_{\mathbf{V}}) - 2.$$

The semi-positivity of $N_{f,\mathbf{V}}/\mathcal{B}$ implies that the injection

$$(\text{loc. free part } \mathcal{B}/\mathcal{B}_0) \longrightarrow \Sigma\{\mathcal{O}(a_j) : a_j<0\}$$

cannot project to zero on any factor. So, for example, we have

$$a_j \geq -(\deg \mathcal{L}), \quad j = 1,\ldots,r.$$

(21.10) Suppose now that we are studying rational curves on generic hypersurfaces of degree $m$ in $\mathbf{P}^n$. Theorem(21.8) tells us that

$$\operatorname{rank}(\mathcal{B}/\mathcal{B}_0) \geq (m - (n+1)) + ((2+\tau)/(\deg f)).$$

So, in particular, to have any rational curves at all, $m$ must be less than or equal to $2n-2$.

(21.11) Finally, we give a lemma which indicates how the existence of a rational curve of degree $d$ on a generic hypersurface of degree $m$ in $\mathbf{P}^n$ influences the distribution of rational curves of degree $d$ on generic hypersurfaces of degree $m$ in higher dimensional projective spaces.

**Lemma:** Suppose **V** is a generic hypersurface in $\mathbf{P^n}$ of degree $m \geq (n+1)$:

a) If $m \geq (2n-1)$, then **V** contains no rational curves.

b) If $m = (2n-2)$ and **V** contains a rational curve of degree $d$, then the generic hypersurface **Z** in $\mathbf{P^m}$ of degree $m$ is covered by deformations of that rational curve, each of which span (at most) a $\mathbf{P^n}$.

c) If $m = (2n-2)-k$ and **V** admits a family of rational curves of degree $d$, covering a subvariety of dimension $(k+1)$, then the generic hypersurface of degree $m$ in $\mathbf{P^m}$ is covered by deformations of that family of rational curves, each of which span (at most) a $\mathbf{P^n}$.

**Proof:** Suppose **V** admits a rational curve:

$$f: \mathbf{X} \longrightarrow \mathbf{V}$$

Let **W** be a generic hypersurface of degree $m$ in $\mathbf{P^{m+n}}$. As in [C2], $N_{f,\mathbf{W}}$ is semi-positive. The fact that $f$ deforms with every deformation of the linear section **V** in **W**, says that, in all the above consideration, we can replace

$$H^0(N_{f,\mathbf{Y}_i})$$

with a subspace $R_i$ of sections arising from deformations of the pair **V** in $\mathbf{Y}_i$, that is, by a vector space obtained by picking a deformation of $f$ compatible with each geometric deformation of **V** in $\mathbf{Y}_i$. The formula in (21.6) says in this case that

$$\mathrm{rank}(\mathcal{S}/\mathcal{S}_0) \geq (m-(n+1)) + (2+\tau)/(\deg f).$$

Since $\mathrm{rank}(\mathcal{S}/\mathcal{S}_0) \leq n-2$, we know that $m \leq 2n-2$, and, if $m = 2n-2$, we must have

$$(\deg f) \geq (2+\tau) \text{ and } \mathrm{rank}(\mathcal{S}/\mathcal{S}_0) = n-2.$$

To finish b), we express **V** as the intersection of hyperplane sections $\mathbf{V}_i$ of **W**, and write $f^*N_{\mathbf{V},\mathbf{W}}$ as a direct sum as in (21.5). By general position and the genericity of **V** and **W**, for every $(n-2)$ values of the index $i$, the corresponding subsheaves $\mathcal{S}_i$ of $N_{f,\mathbf{V}}$ must generate a subsheaf $\mathcal{S}'$ of rank $n-2$. This says that, for the generic hypersurface **Z** of degree $2n-2$ in $\mathbf{P^{2n-2}}$,

$$N_{f,\mathbf{Z}}$$

is generically generated by global sections coming from the geometric deformations of the pair $(f,\mathbf{V})$ in **W**. But these deformations comes from deformations of n-dimensional linear subspaces of $\mathbf{P^{n+m}}$ which lie in $\mathbf{P^{2n-2}}$ to first order. Since $(f,\mathbf{V})$ and the deformations are generic, they can be taken to lie in $\mathbf{P^{2n-2}}$ to all orders, and so come from geometric deformations of $(f,\mathbf{V})$ in **Z**.

The proof of c) is the same--by hypothesis

$$\operatorname{rank}(\mathcal{S}/\mathcal{S}_0) \leq (n-2-k),$$

so, by the above formula, equality must hold. Again, every choice of $(n-2-k)$ values of the index $i$ must give $\mathcal{S}_i$'s which together generate a subsheaf of $\mathcal{S}/\mathcal{S}_0$ of maximal rank.

(21.12) Reference: Most of these results in the case of embedded submanifolds appear in [C2]. The generalization to the singular case is due to Clemens.

**Lecture #22: Conjectures about curves on generic quintic threefolds**

Today, we will outline a series of conjectures about threefolds V with $K_V$ trivial. The prototype will be the quintic hypersurface in $\mathbf{CP}^4$. Our starting point will be:

(22.1) **Conjecture:** The generic quintic hypersurface in $\mathbf{CP}^4$ admits only a finite number of rational curves of every positive degree.

**Remark:** S. Katz has shown that there exist isolated rational curves of each degree on a generic quintic threefold. He has also shown that the Conjecture is true for low degrees, and has counted the conics (609,250). It was known classically that there are 2875 lines on a generic quintic.

We wish to discuss a (conjectural) corollary of the above Conjecture. In what follows, let V denote a non-singular quintic threefold:

(22.2) **Conjecture**: If V is generic, then V cannot be covered by elliptic curves.

(22.3) As a warm-up to a discussion of these conjectures, let us recall that no complex projective threefold V with $K_V$ trivial can be covered by rational curves. This is clear from the adjunction formula, but we present another method of proof which will be useful later:

**Proof:** Suppose that V can indeed be covered by rational curves. Then we have the following diagram:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "q"] \arrow[d, "p"'] & V \\ \mathcal{F} & \end{tikzcd}$$

(proper flat morphism whose fibers are unions of rational curves)

(dominant morphism)

(smooth projective surface)

If we let H be a generic hyperplane section of V, then by taking fibred product over p with $q^{-1}(H)$ and resolving any singularities of the resulting parameter surface $q^{-1}(H)$, we can assume in the above diagram that the fibration p has a section s such that

$$q(s(\mathcal{F})) = H.$$

Now, the cup-product pairing is non-degenerate on $H^3(V;\mathbf{Q})$ and $q$ is a generically finite morphism, so the natural map

$$q^*\colon H^3(V;\mathbf{Q}) \longrightarrow H^3(\mathfrak{L};\mathbf{Q})$$

is injective since cup product is non-degenerate on its image. We analyze $H^3(\mathfrak{L};\mathbf{Q})$ using the Leray spectral sequence for $p$. Since all fibres are unions of rational curves, $R^1p_*\mathbf{Q} = 0$. Also the image of $H^3(\mathfrak{F};R^0p_*\mathbf{Q})$ in $H^3(\mathfrak{L};\mathbf{Q})$ intersects $q^*H^3(V;\mathbf{Q})$ only in $\{0\}$, since the image of $H^3(\mathfrak{F};R^0p_*\mathbf{Q})$ restricts isomorphically onto

$$H^3(s(\mathfrak{F});\mathbf{Q})$$

whereas $q^*H^3(V;\mathbf{Q})$ restricts to $0$ because $H^3(H;\mathbf{Q}) = 0$. Thus all of $q^*H^3(V;\mathbf{Q})$ is generated by $H^1(\mathfrak{F};R^2p_*\mathbf{Q})$. But this implies by duality that the mapping

$$q_*p^*\colon H_1(\mathfrak{F};\mathbf{Q}) \longrightarrow H_3(V;\mathbf{Q})$$

is surjective. This contradicts the fact that the image of this last map is annihilated by $H^{3,0}(V) \neq 0$.

Next we check:

(22.4) **Proposition:** The generic quintic hypersurface can be covered by curves of genus 2.

**Proof:** The Grassmann variety of plane sections of

$$V \subseteq \mathbf{CP}^4$$

has dimension 6. For each fixed plane P and a generic set of 4 points $p_j \in P$, the set of quintics tangent to P at each $p_j$ is a linear space codimension 12 in the set of all quintics. So the set of pairs (P,V) with P four-times tangent to V has codimension

$$12 - (4\cdot 2) = 4.$$

If we can show that the is *some* four-tangent pair (P,V) has the property that P only moves in a two-dimensional family when V is fixed, then we have shown that the generic V admits a two-parameter family of plane quintic curves with four nodes, i.e., a two-parameter family of curves of genus 2. For example, let V be given by the equation

$$F(X_0,\ldots,X_4) = f(X_0,X_1,X_2) + X_3\cdot g(X_0,X_1,X_2) + X_4\cdot h(X_0,X_1,X_2) = 0$$

where $f$ gives a plane quintic with 4 nodes $p_1,\ldots,p_4$, and $g$ and $h$ are generically chosen plane quartics which vanish at the $p_j$.

Deformations of the plane

$$X_3 = X_4 = 0$$

are given by

$$X_3 = \alpha(X_0,X_1,X_2) \qquad X_4 = \beta(X_0,X_1,X_2)$$

and it is immediate to check that the condition on tangency of the deformations reduces to the statement that the the plane curves

$$f(X_0,X_1,X_2) + \alpha(X_0,X_1,X_2)\cdot g(X_0,X_1,X_2) + \beta(X_0,X_1,X_2)\cdot h(X_0,X_1,X_2) = 0$$

which have four nodes is of (local) codimension 4. So generic quintics V admit a two-parameter family of (plane) curves of genus 2. If the generic family were to lie in a divisor D on V, then the dual mapping from D to pencils of hyperplanes in $\mathbf{P}^4$ would be 4-1, which is impossible since, if the dual mapping is finite, double dual is the original variety. Thus the family covers V.

(22.5) Finally, we turn to Conjecture(22.2). The derivation from Conjecture(22.1) proceeds as follows:

Step 1) Assume that V can be covered by elliptic curves. Then, as above, there exists a diagram:

$$\begin{tikzcd} \mathfrak{E} \arrow[r, "q"] \arrow[d, "p"'] & V \\ \mathfrak{F} & \end{tikzcd}$$

(proper flat morphism with generic fibre elliptic)

(dominant morphism)

(smooth projective surface)

Again, by base extension, we can assume that p has a section s whose image maps to a generic hyperplane section H of V.

We can assume that $\mathfrak{F}$ has been blown up sufficiently that the modular map to the compactification of the moduli space $\mathcal{M}_1$ of curves of genus one

$$j: \mathfrak{F} \longrightarrow (\mathcal{M}_1{}^\wedge) \approx \mathbf{CP}^1$$

is a morphism.

Step 2) Using Conjecture(22.1), we can assume that we have chosen H so that it intersects each of the countable collection of rational curves on V *transversally*. This means that $(q \circ s)$ maps the divisor $j^{-1}(\infty)$ to a zero-dimensional set in V, so that, if j is not constant, just as in Lecture #1, we would have a "disappearing curve"

$$(q \circ s)(j^{-1}(t))$$

as t goes to infinity. Thus j must be constant.

Now there are two ways to continue further. One is global and the other is local. We start with the global one.

Step 3) Let $U \subseteq \mathbf{P}^N$ be the subset parametrizing quintic hypersurfaces with at worst ordinary nodes as singularities. The complement of U has codimension at least 2. Now if the general quintic is covered by elliptic curves then there is a family covering the universal quintic over U. As we saw, the generic quintic is covered by copies of the same elliptic curve and this elliptic curve can vary with the quintic. If it indeed varies then on a codimension one subset of U, it degenerates to a rational curve. Hence a (possibly nodal) quintic hypersurface would be covered by rational curves. This however contradicts (22.3), since ordinary nodes do not effect the adjunction-theoretic argument. So all quintic threefolds are covered be the same elliptic curve.

Step 4) Let $\mathcal{V}/U$ be the universal quintic and let

$$q\colon (\mathcal{F}/U)\times E \longrightarrow \mathcal{V}/U$$

be the covering family of elliptic curves where $\mathcal{F}/U$ is generically a family of surfaces. We can blow up $(\mathcal{F}/U)\times E$ suitably to get

$$Z \longrightarrow (\mathcal{F}/U)\times E,$$

where Z admits a regular map onto $\mathcal{V}/U$. Let

$$U' \subseteq U$$

be the open set above which the maps

$$g\colon Z \longrightarrow U \qquad \text{and} \qquad h\colon \mathcal{V} \longrightarrow U$$

are smooth. Thus we have two variations of Hodge structures over U' and a natural injection:

$$R^3h_*\mathbf{C}_{\mathcal{V}} \longrightarrow R^3g_*\mathbf{C}_Z$$

The variation of Hodge structures $R^3g_*\mathbf{C}_Z$ splits as a direct sum of variations of Hodge structures as follows.

We get one component coming from $(\mathcal{F}/U)\times E$. This is a weight two variation, namely $R^2p_*\mathbf{C}_{\mathcal{F}}$ (coming from $H^2$ of the surfaces in $\mathcal{F}/U$)

tensored with the constant variation of $H^1(E;\mathbf{C})$.

The other components come from the blowing up process that created Z.

In each fiber we blow up a point or a smooth curve one at a time. The first one leaves $H^3$ unchanged and the second one changes it with the Jacobian of the blown-up curve. Thus these give weight three variations in which there are only two non-trivial Hodge subbundles.

Step 5) The monodromy representation on $R^3h_*\mathbf{C}_{\mathcal{V}}$ is irreducible; thus $R^3h_*\mathbf{C}_{\mathcal{V}}$ maps into one of the above summands of $R^3g_*\mathbf{C}_Z$. By the above considerations, the only possibility is that we have an injection

$$R^3h_*\mathbf{C}_{\mathcal{V}} \longrightarrow R^2p_*\mathbf{C}_{\mathfrak{F}} \otimes H^1(\mathfrak{E};\mathbf{C}).$$

This is impossible, since the left-hand-side has a degeneration with $W^6$ of the weight filtration non-zero, but obviously the right-hand-side can not have such a degeneration. This completes the proof.

The more local approach is the following:

Step 3') By further base extension, we can achieve a dominant rational map

$$q:\ \mathfrak{F}\times E \longrightarrow V.$$

If $q$ is in fact a morphism, then, as before, $H^3(V;\mathbf{Q})$ injects into

$$H^2(\mathfrak{F})\otimes H^1(E) + H^1(\mathfrak{F})\otimes H^2(E).$$

(Again use $H^3(H)=0$ to eliminate $H^3(\mathfrak{F})\otimes H^0(E)$.)

Step 4') It cannot be that $q^*H^3(V;\mathbf{Q})$ lies entirely in $H^2(\mathfrak{F})\otimes H^1(E)$, since, as before, the latter has type $(2,1)+(1,2)$. In fact, these two subspaces can only intersect in $\{0\}$ since the cup-product pairing is non-degenerate on $q^*H^3(V;\mathbf{Q})$. (In case $q$ is not everywhere defined, this statement must be modified, but the argument proceeds in essentially the same way, so we will continue to assume $q$ is a morphism.)

Step 5') We let V vary over the projective space $\mathfrak{P}$ of all quintics in $\mathbf{CP}^4$. Then for each V we have an elliptic curve $E_V$. Let $\mathfrak{D}$ be a divisor on $\mathfrak{P}$ along which the modulus of the elliptic curve $E_V$ is constant. Therefore first and second derivatives of the period mapping along $\mathfrak{D}$ send $H^{3,0}(V)$ into $H^1(E)^{\perp}$. Let $S_N{}^d$ denote the set of homogeneous forms of degree $d$ in $n$ variables. Via Griffiths' theory of residues for hypersurfaces, this fact about first and second derivatives along $\mathfrak{D}$ gives rise to a hyperplane

$$H\subseteq S_5{}^5$$

such that

1) H contains $\partial F/\partial X_j$, $j=0,\ldots,4$, for a generic quintic form F,
2) $H\cdot H$ lies in a hyperplane of $S_5{}^{10}$.

But this is impossible because of the following lemmas:

(22.6) **Lemma:** Suppose a proper subspace $W \subsetneq S_N{}^d$ has no common zeroes. Suppose $k = \operatorname{codim}(W, S_N{}^d) \leq d+1$. Then

$$S_N{}^k \cdot W = S_N{}^{k+d}.$$

(22.7) **Lemma:** Let $d = N$, and let $H$ be a hyperplane in $S_N{}^N$ such that the conductor $W = [H:S_N{}^1] \subseteq S_N{}^{N-1}$ has no common zeroes. Then

$$H \cdot H = S_N{}^{2N}.$$

**Proof:** $W = [H:S_N{}^1] = \cap\{[H:P]: P \in S_N{}^1\} \subseteq S_N{}^{N-1}$.
So $\operatorname{codim} W \leq N$. Thus, by Lemma (22.6), $S_N{}^N \cdot W = S_N{}^{2N-1}$. Therefore, again using Lemma (22.6),

$$H \cdot H \supseteq W \cdot S_N{}^1 \cdot H = W \cdot S_N{}^{N+1} = S_N{}^{2N}.$$

(22.8) References: S. Katz's results appear in [Kat]. The conjecture (22.1) appeared first in [C1]. The (conjectural) Corollary (22.2) of (22.1) evolved in discussions involving H. Clemens, J. Kollár and S. Mori. The alternative local approach to the end of the proof was pointed out to us by C. Voisin. Lemma (22.6) appears as a special case of Theorem 2.16 in [G] and Lemma (22.7) is due to Voisin. We are grateful to her for allowing us to use her unpublished results. Griffiths' theory of residues for hypersurfaces appears in [CGGH].

**Lecture #23: Submanifolds of generic complete intersections in Grassmannians**

(23.1) Today we will give a generalization of the results for curves on hypersurfaces given in Lecture #21. The situation is as follows:

$V$ = $(n+1)$-dimensional complex vector space
$G$ = Grassmann variety of $r$-dimensional quotient spaces of $V$
$X \subseteq G$ a generic complete intersection of type $(m_1, \ldots, m_k)$

We will let

$H_X$ = irreducible open subset of the Hilbert scheme of $X$ parametrizing smooth irreducible subvarieties of $X$ of some given type.

$Z_X = \{(Z,x) : Z \in H_X,\ x \in Z\}$.

$$\begin{tikzcd}
Z \arrow[d] & \subseteq & Z_X \arrow[r, "F"] \arrow[d] & X \subseteq G \\
z & \in & H_X & = p^{-1}(X)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{Z} \arrow[r, "F"] \arrow[d] & G \\
\mathcal{H} \arrow[d, "p"'] & \\
X \in \mathcal{S} &
\end{tikzcd}$$

parameter space of complete intersections of the given type

(23.2) **Theorem:** Let $m = \Sigma m_j$. Let $m_0$ be the least integer $s$ such that

$$h^0(K_Z \otimes \mathcal{O}_Z(s)) \neq 0.$$

Then

a) $N_{Z/X} \otimes \mathcal{O}_Z(1)$ is generated by global sections;

b) $\operatorname{codim}_X F(Z_X) \geq m + m_0 - n - 1$.

(23.3) **Corollary:** a) If $m \geq \dim X + n + 1$, then every such $Z$ is of general type.

b) If $m \geq \dim X + n$, then every such $Z$ has non-zero geometric genus.

(The Corollary follows since, for example, if $Z$ is not of general type, $h^0(K_Z \otimes \mathcal{O}_Z(-1)) = 0$.)

For example, we might have $G = \mathbf{CP}^n$, and X a generic hypersurface of degree $m \geq 2n-1$. Then X contains no rational curve.

(23.4) **Proof of theorem:** The proof will begin with the construction of the *Koszul resolution* which resolves the ideal of the graph of a morphism

$$f: Z \longrightarrow G$$

into the Grassmann variety G:

Given $f: Z \longrightarrow G$

$$\Gamma = (\text{graph } f) \subseteq Z \times G$$

$$\begin{tikzcd} & Z \times G \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ Z & & G \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[dr, "\text{universal sub-bundle}"'] & V \times G \arrow[r] \arrow[d] & Q \arrow[dl, "\text{universal quotient}"] \\ & G & \end{tikzcd}$$

$$\begin{tikzcd} S_1 \arrow[r] \arrow[dr] & V \times Z \arrow[r] \arrow[d] & Q_1 \arrow[dl] \\ & Z & \end{tikzcd} \quad \overset{f^*}{\dashleftarrow\!\dashrightarrow} \quad G$$

then, putting

$$\mathcal{E} = \pi^* S_1 \otimes p^* Q^*$$

we obtain the resolution

$$\ldots \longrightarrow \Lambda^2 \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{Z\times G} \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0.$$

$$\sigma \otimes \xi \longrightarrow \xi(\sigma)$$

$$\sigma \otimes \xi \wedge \sigma' \otimes \xi' \longrightarrow \xi(\sigma)\,\sigma' \otimes \xi' - \xi'(\sigma')\,\sigma \otimes \xi$$

We will apply this construction in the case of

$$f: Z \longrightarrow \mathbf{P}(V) = \mathbf{P}.$$

Here the exact sequence

$$(\#) \qquad 0 \longrightarrow \Omega_{\mathbf{P}}{}^1 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow 0$$

gives that

$$S = \Omega_{\mathbf{P}}{}^1(1) \quad \text{and} \quad Q = \mathcal{O}_{\mathbf{P}}(1).$$

We take the above Koszul resolution above and tensor it with $\mathcal{O}_{\mathbf{P}}(m)$. We then apply $\pi_*$ to pass from a complex of sheaves on $Z \times \mathbf{P}$ to a complex of sheaves on Z. Since the higher direct-image sheaves are given by

$$R^i\pi_* = H^i(\mathcal{O}_{\mathbf{P}}(m-k) \otimes \Lambda^k(f^*\Omega_{\mathbf{P}}{}^1(1))),$$

which are zero for $i > 0$, exactness is preserved. We obtain:

$$\ldots \longrightarrow H^0(\mathcal{O}_{\mathbf{P}}(m-1)) \otimes f^*\Omega_{\mathbf{P}}{}^1(1) \xrightarrow{(*)} H^0(\mathcal{O}_{\mathbf{P}}(m)) \otimes \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(m) \longrightarrow 0.$$

Let $\mathcal{F}$ be the sheaf which is the image of the arrow to the left of (*) and the kernel to the arrow to the right of (*).
Now $\Omega_{\mathbf{P}}{}^1(2)$ is generated by the global sections

$$X_i dX_j - X_j dX_i,$$

so:

1) $\mathcal{F}\otimes\mathcal{O}_Z(1)$ is generated by global sections.

Suppose that $f\colon Z \longrightarrow \mathbf{P}$ factors through $X \subseteq \mathbf{P}$ and that

$$H^0(\mathcal{O}_{\mathbf{P}}(m)) \longrightarrow H^0(\mathcal{O}_X(m))$$

is surjective. Let $\mathfrak{U}$ denote the kernel of

$$H^0(\mathcal{O}_X(m))\otimes\mathcal{O}_Z \longrightarrow \mathcal{O}_Z(m).$$

Then the Snake Lemma implies

2) $\mathfrak{U}\otimes\mathcal{O}_Z(1)$ is generated by global sections.

In fact, using a lemma of Lazarsfeld which we will prove next time, we can do a little better in the case in which Z is a curve. We achieve this by examining $f^*\Omega_{\mathbf{P}}{}^1(1)$ a little more closely in case $f\colon Z \longrightarrow \mathbf{P}$ is generically injective. Let $\mathcal{F}$ be as above, and let

$d$ = degree f

$n_0$ = dimension of linear subspace of $\mathbf{P}$ spanned by f(Z).

(23.5) **Lemma**: There is a line bundle $\mathcal{L}$ of degree $(d-n_0+1)$ such that

$$\mathcal{F}\otimes\mathcal{L}$$

is semi-positive.

**Proof**: By (#) one has

$$f^*\Omega_{\mathbf{P}}{}^1(1) = (n-n_0)\mathcal{O}_Z \oplus \mathcal{M}.$$

Lazarsfeld's lemma then says that, for $(n_0-1)$ general points $p_j$ on Z, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_Z(\Sigma p_j)\otimes\mathcal{O}_Z(-1) \longrightarrow \mathcal{M} \longrightarrow \oplus\,\mathcal{O}_Z(-p_j) \longrightarrow 0.$$

Let $\mathcal{L} = \mathcal{O}_Z(-\Sigma p_j)\otimes\mathcal{O}_Z(1)$. Since the sections of $\mathcal{L}$ have no base points, $\mathcal{L}\otimes\mathcal{O}(-p_j)$ has a section for each j. So $\mathcal{M}\otimes\mathcal{L}$ sits in an exact sequence whose extremes are semi-positive.

(23.6) **Lemma:** $K_G = \mathcal{O}_G(-n-1)$.

**Proof:** Tensor the sequence

$$0 \longrightarrow S \longrightarrow V \otimes \mathcal{O}_G \longrightarrow Q \longrightarrow 0$$

with $Q^*$. Since $\Omega_G{}^1 = S \otimes Q^*$ and $Q \otimes Q^*$ is self-dual and so has trivial determinant, $K_G = \Lambda^{r(n+1)}(V \otimes Q^*)$. Now use that

$$\Lambda^r Q^* = \mathcal{O}_G(-1).$$

(23.7) We now finish the proof of Theorem(23.2) announced at the beginning of this lecture. Our situation is:

$$\begin{tikzcd} Z \arrow[d] & \subseteq & Z_X \arrow[r, "F"] \arrow[d] & X \subseteq G \\ z & \in & H_X = p^{-1}(X) & \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{Z} \arrow[r, "F"] \arrow[d, "h"] & G \\ \mathcal{H} \arrow[d, "p"] & \\ X \in \mathcal{A} & \end{tikzcd}$$

parameter space of complete intersections of the given type

We begin with the diagram:

$$\begin{tikzcd} 0 \arrow[r] & T_Z \arrow[r] \arrow[d, "="] & T_{\mathfrak{Z}}|_Z \arrow[r] \arrow[d, "dF"] & h^*T_{\mathcal{H}}|_Z \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & T_Z \arrow[r] & T_G|_Z \arrow[r] & N_{Z/G} \arrow[r] \arrow[d, "\psi"] & 0 \\ & & & N_{X/G}|_Z & \end{tikzcd}$$

Notice that the composition $\psi\phi$ is exactly a direct sum of maps

$$H^0(\mathcal{O}_X(m_j)) \otimes \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(m_j)$$

considered above, if we denote the kernel of this composition by $\mathfrak{A}$, we obtain the diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{A} \arrow[r] \arrow[d] & h^*T_{\mathcal{H}}|_Z \arrow[r] \arrow[d] & N_{X/G}|_Z \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & N_{Z/X} \arrow[r] & N_{Z/G} \arrow[r] & N_{X/G}|_Z \arrow[r] & 0 \end{tikzcd}$$

where, by 2) above, $\mathfrak{A} \otimes \mathcal{O}_Z(1)$ is generated by global sections. Thus, as in the two Lemmas from Lecture #21, $N_{Z/X} \otimes \mathcal{O}_Z(1)$ is also generated by global sections, giving a) of the theorem.

(23.8) Next we consider the map

$$h^* T_{H_X}|_Z \longrightarrow N_{Z/X}$$

and let $E_1$ and $E_2$ respectively denote its image and coimage modulo torsion. If $e_i = \operatorname{rank} E_i$, then $e_2 = \operatorname{codim}_X F(Z_X)$, the integer we need to estimate for b) of the theorem. Outside a subset of codimension two we have

$$\Lambda^{e_1+e_2} N_{Z/X} \approx \Lambda^{e_1} E_1 \otimes \Lambda^{e_2} E_2$$

with

$$\Lambda^{e_1+e_2} N_{Z/X} \approx \Lambda^{e_1} E_1 \otimes \Lambda^{e_2} E_2 \approx \mathcal{O}(D) \otimes \mathcal{O}(D')(-e_2)$$

where D and D' are effective divisors. On the other hand

$$\Lambda^{e_1+e_2} N_{Z/X} \approx K_X^{-1} \otimes K_Z \approx \mathcal{O}(-m+n+1) \otimes K_Z$$

so, if $m_0$ is the least integer such that $h^0(K_Z \otimes \mathcal{O}(m_0)) \neq 0$, then

$$m_0 \leq e_2 - m + n + 1.$$

So the proof is complete.

(23.9) Since the earlier Lemma showed that a somewhat "less positive" bundle $\mathcal{F} \otimes \mathcal{L}$ is generated by global sections in the case $Z = C$, an imbedded curve of degree $d$, we get a correspondingly sharper estimate in this case:

(23.10) **Theorem**: Let C be a smooth curve on a generic X. Then

$$\operatorname{codim}_X F(C_X) \geq (1/(d-n_0+1))[(2-2g)+(m-n+1)d],$$

where, as before, $n_0$ = dimension of linear span of C.

(23.11) Lastly, for curves of "small" degree $d \leq \min\{m_j\} + n_0 - 1$, we show that the Hilbert scheme $H_C$ is smooth at C when $H^1(N_{C/G}) = 0$. (Note: This condition is always satisfied for rational curves.)

**Proof**: We must show that $H^1(N_{C/X}) = 0$. But this will follow immediately from the normal bundle sequence if we can show that the map

$$H^0(N_{C/G}) \longrightarrow H^0(N_{X/G}|C)$$

is surjective. Since C lies on generic X, $H^0(N_{C/G})$ maps onto the image of $H^0(N_{X/G}) = \oplus H^0(\mathcal{O}_X(m_j))$ in $H^0(N_{X/G}|C)$. But, by a theorem of Gruson-Lazarsfeld-Peskine which we will prove tomorrow, the maps

$$H^0(\mathcal{O}_X(m_j)) \longrightarrow H^0(\mathcal{O}_C(m_j))$$

are surjective whenever $m_j \geq d - n_0 + 1$.

(23.12) Reference: These results appear in [E].

**Lecture #24: A theorem of Gruson-Lazarsfeld-Peskine and a lemma of Lazarsfeld**

Today, we want to look at the proof of:

(24.1) **Theorem:** Let $C \subseteq \mathbf{P}^n$ be a smooth curve of degree $d$ which does not lie in a hyperplane. Then

$$H^0(\mathbf{P}^n; \mathcal{O}(a)) \longrightarrow H^0(C; \mathcal{O}(a))$$

is surjective if $a \geq d-n+1$.

**Proof:** Let $L^{n-3}$ be a generically chosen linear subspace of dimension $n-3$ in $\mathbf{P}^n$. Let $\mathbf{P}^\wedge$ be the blow-up of $\mathbf{P}^n$ along L. Then $\mathbf{P}^\wedge$ is a projective space bundle over $\mathbf{P}^2$, in fact,

$$\mathbf{P}^\wedge = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(1) \oplus (n-2)\mathcal{O}_{\mathbf{P}^2}).$$

We have

$$\begin{array}{ccccc} C & \subseteq & \mathbf{P}^\wedge & \xrightarrow{f} & \mathbf{P}^2 \\ & & \Big\downarrow h & & \\ C & \subseteq & \mathbf{P}^n & & \end{array}$$

We define the bundles

$$\mathcal{O}_{\mathbf{P}^\wedge}(a,b) = h^*\mathcal{O}_{\mathbf{P}^n}(a) \otimes f^*\mathcal{O}_{\mathbf{P}^2}(b).$$

Then, for example, $\mathcal{O}_{\mathbf{P}^\wedge}(1,0)$ is the tautological line bundle, so that $f_*\mathcal{O}_{\mathbf{P}^\wedge}(1,0) = \mathcal{O}_{\mathbf{P}^2}(1) \oplus (n-2)\mathcal{O}_{\mathbf{P}^2}$. Consider the sequence

$$0 \longrightarrow \mathcal{I}_C(1,0) \longrightarrow \mathcal{O}_{\mathbf{P}^\wedge}(1,0) \longrightarrow \mathcal{O}_C(1,0) \longrightarrow 0.$$

Notice that $\mathcal{O}_C(1,0) = \mathcal{O}_C(0,1)$ since C does not meet L. We apply $f_*$. By the projection formula, we obtain

$$(*) \qquad 0 \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{O}_{\mathbf{P}^2}(1) \oplus (n-2)\mathcal{O}_{\mathbf{P}^2} \longrightarrow f_*\mathcal{O}_C \otimes \mathcal{O}_{\mathbf{P}^2}(1) \longrightarrow 0$$

since $R^1\mathcal{E}(1) = 0$ because no fibre of $f$ contains more than two points of C. Also, by writing down a local basis for $f_*\mathcal{I}_C(1,0)$ explicitly, one sees that that sheaf is locally free. By tracing through the definition, one sees that the surjection in (*) is given by

$$(a, (a_3, \ldots, a_n)) \longrightarrow (a + a_3X_3 + \ldots + a_nX_n),$$

so that, to prove the theorem, it suffices to show that

$$H^1(\mathcal{E}(b)) = 0$$

for $b$ above the given bound.

(24.2) We begin with

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}^2} \oplus (n-2)\mathcal{O}_{\mathbf{P}^2}(-1) \longrightarrow f_*\mathcal{O}_C \longrightarrow 0$$

from (*) above. We have

$$\operatorname{rank}\mathcal{E} = (n-1) \text{ and } \det\mathcal{E} = \mathcal{O}_{\mathbf{P}^2}(n-2) \otimes \mathcal{O}_{\mathbf{P}^2}(-d).$$

Next we resolve

$$M = \oplus_a H^0(f_*\mathcal{O}_C(a))$$

as a module over $S = k[X_0, X_1, X_2]$:

$$\ldots \longrightarrow S + S(-1)^{r-2} + T_1' \longrightarrow M \longrightarrow 0$$

where $r+1$ is the dimension of $H^0(\mathcal{O}_C(1))$. (The first factor $S$ goes onto the constants and the linear span of $X_0, X_1, X_2$.) Putting $T_1 = T_1' + S(-1)^{r-n}$, we get a diagram of coherent sheaves on $\mathbf{P}^2$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-1)^{n-2} \arrow[r] \arrow[d] & f_*\mathcal{O}_C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{T}_2 \arrow[r] \arrow[d] & \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-1)^{n-2} \oplus \mathcal{T}_1 \arrow[r] \arrow[d] & f_*\mathcal{O}_C \arrow[r] & 0 \\
 & \mathcal{T}_1 \arrow[d] \arrow[r, phantom, "="] & \mathcal{T}_1 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

So, in particular, $\mathcal{T}_2$ is locally free and, by construction,

$$H^1(\mathcal{T}_2(a)) = 0 \quad \text{for all } a.$$

This means that $\mathcal{T}_2$ must be a sum of line bundles, since, restricted to a line, the bundle splits, and the induced isomophism from a sum of $\mathcal{O}(n_i)$'s to $\mathcal{T}_2|_{\text{line}}$ must come from a morphism on all of $\mathbf{P}^2$ which is an isomorphism off a set of codimension $\geq 2$.

(24.3) We are reduced to analyzing the kernel of the epimorphism

$$\mathcal{T}_2 \longrightarrow \mathcal{T}_1$$

of sums of line bundles in the diagram above. There is a standard tool for analyzing the kernel of an epimorphism

$$\phi\colon \mathcal{A} \longrightarrow \mathcal{B}$$

of vector bundles (of ranks $a$ and $b$ respectively) over a variety $X$. It is the *Eagon-Northcott complex*, another form of Koszul resolution:

$$\to \Lambda^{b+3}\mathfrak{A}\otimes S^2\mathfrak{B}^* \longrightarrow \Lambda^{b+2}\mathfrak{A}\otimes S^1\mathfrak{B}^* \longrightarrow \Lambda^{b+1}\mathfrak{A} \longrightarrow \mathfrak{A}\otimes\det\mathfrak{B} \longrightarrow \mathfrak{B}\otimes\det\mathfrak{B} \to 0$$

$$\alpha_1\wedge\ldots\wedge\alpha_{b+1} \to \Sigma(-1)^j\alpha_j D_j$$

where $D_j$ is the determinant of the $\phi(\alpha_k)$ for $k \neq j$. To see exactness, we reason as follows:

Let $\mathcal{O}(1)$ be the hyperplane bundle for $f: \mathbf{P}(\mathfrak{B}) \longrightarrow X$. The canonical morphism $f^*\mathfrak{A} \longrightarrow f^*\mathfrak{B} \longrightarrow \mathcal{O}(1)$ induces a Koszul resolution

$$(*) \ \ldots \longrightarrow \Lambda^3 f^*\mathfrak{A} \otimes \mathcal{O}(-2) \longrightarrow \Lambda^2 f^*\mathfrak{A} \otimes \mathcal{O}(-1) \longrightarrow f^*\mathfrak{A} \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Apply $f_*$ and notice that the $R^i f_*$ vanish except for $i=0$ (for $f^*\mathfrak{A}$ and $\mathcal{O}(1)$) and for $i=b$ (for $\Lambda^{j+1}f^*\mathfrak{A}\otimes\mathcal{O}(-j)$ when $j \geq b$). Also, by the projection formula and Serre duality,

$$\begin{aligned} R^b f_*(\Lambda^{j+1}f^*\mathfrak{A}\otimes\mathcal{O}(-j)) &= \Lambda^{j+1}\mathfrak{A}\otimes R^b f_*\mathcal{O}(-j) \\ &= \Lambda^{j+1}\mathfrak{A}\otimes(f_*\mathcal{O}(j)\otimes\omega_{\mathbf{P}(\mathfrak{B})/X})^* \\ &= \Lambda^{j+1}\mathfrak{A}\otimes(f_*\mathcal{O}(j-b)\otimes\det\mathfrak{B})^* \end{aligned}$$

So by looking at the spectral sequence associated to the double complex given by an injective resolution of (*), we obtain the Eagon-Northcott complex.

(24.4) Using the Eagon-Northcott complex to resolve $\mathfrak{E}$, we obtain:

$$\ldots \longrightarrow \Lambda^{t'+2}\mathfrak{J}_2 \otimes \mathfrak{J}_1^* \otimes (\Lambda^{t'}\mathfrak{J}_1)^* \xrightarrow{(\#)} \Lambda^{t'+1}\mathfrak{J}_2 \otimes (\Lambda^{t'}\mathfrak{J}_1)^* \longrightarrow \mathfrak{E} \longrightarrow 0.$$

Let $\mathfrak{K}$ denote the kernel of the map (#). Then, since the $\mathfrak{J}_i$ are sums of line bundles, we have injections

$$H^1(\mathfrak{E}(b)) \longrightarrow H^2(\mathfrak{K}(b))$$

for all b. Also, by dimension, we have surjections

$$H^2(\Lambda^{t'+2}\mathfrak{J}_2 \otimes \mathfrak{J}_1^* \otimes (\Lambda^{t'}\mathfrak{J}_1)^*(b)) \longrightarrow H^2(\mathfrak{K}(b)$$

for all b. So, we will be finished if we can show that

$$H^2(\Lambda^{t'+2}\mathfrak{J}_2 \otimes \mathfrak{J}_1^* \otimes (\Lambda^{t'}\mathfrak{J}_1)^*(b)) = 0$$

for $b \geq d-n+1$.

(24.5) To do this, write

$$\mathcal{J}_1 = \oplus \mathcal{O}(a_i),\ i=1,\ldots,t',$$
$$\mathcal{J}_2 = \oplus \mathcal{O}(b_j),\ j=1,\ldots,t'',$$

and notice that, by construction,

$$a_i \leq -1, \text{ for each } i,$$
$$b_j \leq -2, \text{ for each } j.$$

Notice that

$$\Lambda^{t''}\mathcal{J}_2 \otimes (\Lambda^{t'}\mathcal{J}_1)^* = \Lambda^e \mathcal{E} = \mathcal{O}(-n+2-d).$$

Also $t''-t' = \operatorname{rank} \mathcal{E} = n-1$. The rest is elementary arithmetic--if

$$b \geq d-n+1,$$

one computes that the degree of every summand of

$$\Lambda^{t'+2}\mathcal{J}_2 \otimes \mathcal{J}_1{}^* \otimes (\Lambda^{t'}\mathcal{J}_1)^*(b)$$

is greater than or equal to -2, so $H^2 = 0$.

We should remark that the bound in the theorem is sharp. A rational curve of degree $d$ in $\mathbf{P}^{d-1}$ gives the required example. Also the theorem still holds if one only assumes that the curve C is reduced and irreducible, but the proof is more complicated.

(24.6) We also need to prove the lemma of Lazarsfeld used last time:

**Lemma:** Suppose an irreducible curve $C_0$ spans a projective space $\mathbf{P} = \mathbf{P}^n$. Let

$$\mathcal{M} = f^*\Omega_{\mathbf{P}}{}^1(1).$$

Then, for (n-1) general points $p_j$ on $C_0$, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_C(\Sigma p_j) \otimes \mathcal{O}_C(-1) \longrightarrow \mathcal{M} \longrightarrow \oplus \mathcal{O}_C(-p_j) \longrightarrow 0.$$

**Proof:** Let C be the normalization of $C_0$, and let $D = \Sigma p_j$. Let $\mathcal{L}$ denote the pull-back of $\mathcal{O}_{\mathbf{P}}(1)$ to C. We have the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_C{}^{\oplus n+1} \longrightarrow \mathcal{L} \longrightarrow 0.$$

Choose a linear subspace L of $\mathbf{P}$ of dimension (n-2) meeting $C_0$ in exactly the points $p_j$. Project $C_0$ to $\mathbf{P}^1$ with center L.
The lemma follows from the resulting diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{L}^{-1}(D) \arrow[r] \arrow[d] & \mathcal{O}^{\oplus 2} \arrow[r] \arrow[d] & \mathcal{L}(-D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & V \otimes \mathcal{O}_C \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus \mathcal{O}(-p_j) \arrow[r] \arrow[d] & \mathcal{O}^{\oplus (n-1)} \arrow[r] \arrow[d] & \mathcal{L}|_D \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(24.7) We end today with another example of the usefulness of the above lemma. Suppose that $\mathcal{L}$ is some line bundle on a smooth curve C such that $d = \deg \mathcal{L} \geq g(C)$. Assume

$$h^0(\mathcal{L}) = r+1 \text{ and } h^1(\mathcal{L}) = \delta > 0.$$

Suppose we want an upper bound on the local dimension of $W^r{}_d$, the set of line bundles of degree $d$ on C with index of speciality at least $\delta$. Let $V = H^0(\mathcal{L})$. Tensor the sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow V \otimes \mathcal{O}_C \longrightarrow \mathcal{L} \longrightarrow 0$$

with $K_C \otimes \mathcal{L}^{-1}$ and take global sections to obtain

$$0 \longrightarrow H^0(\mathcal{M} \otimes K_C \otimes \mathcal{L}^{-1}) \longrightarrow H^0(\mathcal{L}) \otimes H^0(K_C \otimes \mathcal{L}^{-1}) \longrightarrow H^0(K_C).$$

The last map above is called the *Petri map*, and its image is the annihilator of the tangent space of $W^r{}_d$ at $\mathcal{L}$. So we can get the desired result by estimating the dimension of $H^0(\mathcal{M} \otimes K_C \otimes \mathcal{L}^{-1})$. To do this, we tensor the sequence

$$0 \longrightarrow \mathcal{O}_C(\Sigma p_j) \otimes \mathcal{O}_C(-1) \longrightarrow \mathcal{M} \longrightarrow \oplus \mathcal{O}_C(-p_j) \longrightarrow 0$$

with $K_C \otimes \mathcal{L}^{-1}$ and take global sections to obtain

$$0 \longrightarrow H^0(K_C \otimes \mathcal{L}^{-2}(D)) \longrightarrow H^0(\mathcal{M} \otimes K_C \otimes \mathcal{L}^{-1}) \longrightarrow \oplus H^0(K_C \otimes \mathcal{L}^{-1}(-p_j)).$$

Since $d \geq g$, we get

$$h^0(\mathcal{M} \otimes K_C \otimes \mathcal{L}^{-1}) \leq (r-1)(\delta-1).$$

So the annihilator of the tangent space to $W^r{}_d$ at $\mathcal{L}$ has

$$\text{dimension} \geq (r+1)\delta - (r-1)(\delta+1).$$

(24.8) There is a variant of Lazarsfeld's lemma for vector bundles $\mathfrak{E}$ with

$$0 \longrightarrow \mathfrak{M} \longrightarrow V \otimes \mathcal{O}_C \longrightarrow \mathfrak{E} \longrightarrow 0$$

for which

$$f: \mathbf{P}(\mathfrak{E}) \longrightarrow \mathbf{P}(V)$$

is generically injective. Let m = rank $\mathfrak{M}$. By a similar argument to the above, one obtains

$$0 \longrightarrow \mathcal{O}_C(\Sigma p_j) \otimes \det \mathfrak{E}^{-1} \longrightarrow \mathfrak{M} \longrightarrow \oplus \mathcal{O}_C(-p_j) \longrightarrow 0.$$

Applying this to the "first-order jet bundle" associated to the line bundle $\mathfrak{L}$ considered above, one achieves an upper bound on the local dimension of $\mathcal{W}^r{}_d$, the space of pairs $(C, \mathfrak{L})$, $\mathfrak{L} \in W^r{}_d$.

(24.9) References: The theorem of Gruson-Lazarsfeld-Peskine appears in [GLP]. Lazarsfeld's lemma appears in [GL].

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Ce travail comprend vingt-quatre conférences qui ont fait partie d'un séminaire d'été sur la géométrie complexe des variétés de dimension plus élevée qu'un. Le séminaire a eu lieu à l'Université d'Utah pendant les mois de juillet et août 1987. Les seize premières conférences fournissent une introduction au programme de Mori sur la recherche des modèles minimaux pour des variétés projectives complexes de dimension au moins trois. Le thème central est l'étude de variétés sur lesquelles la classe canonique n'est pas numériquement effective. Les conférences dix-sept à vingt étudient la géométrie de l'application des périodes, et, plus généralement, des applications harmoniques des variétés de Kähler compactes dans certaines variétés localement symétriques. Les quatre dernières conférences étudient l'existence et les propriétés des courbes de genre petit sur des variétés projectives avec classe canonique suffisamment ample.