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POINT SETS IN PROJECTIVE SPACES AND THETA FUNCTIONS

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TO HILARY AND NATASHA

Table of Contents

Introduction.

The purpose of these notes is to re-introduce some of the work of A. Coble [Co 1] in a language that a modern mathematician can easily understand. There is a well-known relationship between the theory of invariants of finite sets of points in a projective line and the theory of hyperelliptic curves. The book of Coble gives an account of the theory which generalizes this relationship to point sets in projective spaces of higher dimension and non-hyperelliptic curves. Though some aspects of this theory were known before Coble (see for example [Fr 1], [Fr 2], [Ka], [Sch 1, [Sch 2], [Sch 3]), his exposition is by far the most complete and conceptually motivated. In recent years the book of Coble was saved from oblivion and the number of references to it grew substantially. This prompted us to serve the mathematical community by giving a modern account of his theory.

The contents of these notes is the following. In Chapters I and II we give a development of the general theory of projective invariants for ordered point sets. We use a presentation of these invariants by certain tableaux, along with the straightening algorithm to describe the structure of the ring they form. Following a now standard approach to the theory of invariants [Mu 1], we construct the moduli spaces P_n^m for the projective equivalence classes of sets of m ordered points in a n-dimensional projective space $\mathbb{P}_n$ and provide a description of the stable and semi-stable ones. A rather complete discussion of the "most special" point sets is given. These are the point sets which are parametrized by the spaces P_n^m . The chapters conclude with some examples that illustrate how the general techniques work for specific cases. Note that we are able to discern the structure of the moduli spaces in these examples without too much effort, whereas Coble had to devise rather complicated and ingenious methods to reveal the same information.

Chapter III is concerned with the classical concept of association, which is a form of duality between the spaces P_n^m and P_{m-n-2}^m. It is difficult to trace out the origins of this concept, but it was refined and used extensively by Coble **[Co 5]**. Our approach is to show that association arises from an isomorphism between the coordinate rings of the respective moduli spaces, which is based on the notion of duality between tableaux. In the case m = 2n+2 the notion of association leads to the notion of self-association. We provide a criterion, essentially due to Coble, for a stable set to be self-associated. This condition is closely related to questions of independence of point sets with respect to the linear system of quadrics through them which is extensively studied in modern and classical works on algebraic curves. After various geometric properties of associated sets, we prove the rationality of the moduli space S_n that parametrizes projective equivalence classes of ordered self-associated point sets.

In Chapter IV we extend the invariant theory of points sets to the case where some of the points are considered to be infinitely near. Following a construction from **[K1]** we construct the variety parametrizing such point sets, and then consider the extension of the action of the projective linear group on this variety. We use some recent results from **[Rei]** to derive explicit criterion of stability of infinitely near points sets. This allows to construct the spaces $\hat{P}_n^m$ which are extensions of the spaces P_n^m, and birational morphisms $\hat{P}_n^m \to P_n^m$.

In Chapter V we begin to consider point sets from a different point of view. Blowing-up such a set gives a certain rational variety, which we call a generalized Del Pezzo variety. The order on the set equips this variety with an additional structure. This additional structure is interpreted as a certain marking in the 1-codimensional and 1-dimensional components of the Chow ring of this variety. The varieties $\hat{P}_n^m$ can be interpreted as certain moduli varieties of marked generalized Del Pezzo varieties. Here the most interesting part of Coble's theory enters into the discussion. This is the notion of root systems and their Weyl groups. The discovery that Coble was aware of some of these notions even in the case of infinite root systems, a long time before Cartan's work, and it goes without saying, before the work of V. Kac and R. Moody, was the main motivation for the first author to study his work. The theory of Del Pezzo surfaces and surface singularities is known to have a relationship with this theory. A modern account of this can be found for example in **[Ma]**, **[De]**, **[Pi]**. An earlier exposition of this is due to P. Du Val **[DV 1–DV 4]** who apparently was not aware of Coble's work. A new result of this chapter is a partial description of roots for certain root systems in hyperbolic vector spaces. The notion of roots corresponds to the classical notion of a discriminant condition on a point set which substitutes the condition for a binary form to have a multiple root.

In Chapter VI we develop the notion of the Cremona action on the point sets. It was observed by Coble and S. Kantor (in the case n = 2), and later by P. Du Val **[DuV 3]**, **[DuV 4]** that certain types of Cremona transformations of the projective space act birationally on projective equivalence classes of point sets. More precisely they give a representation of a certain Weyl group $W_{n,m}$ in the group of birational automorphisms of P_n^m. Much effort was applied to give a rigorous exposition of this beautiful theory. The kernel of the Cremona representation of $W_{n,m}$ can be identified with a subgroup of pseudo-automorphisms (i.e. birational automorphisms which are isomorphisms in codimension 1) of the blowing-up of a point set represented by a generic point of P_n^m. In the case n = 2, the kernel is the full automorphism group, and we prove, following Coble, that this group is trivial if $m \geq 9$. A modern proof of this result, also based on Coble's ideas, was given in **[Gi]** (m = 9) and **[Hir]**.

In Chapter VII we discuss all special cases where the Weyl group $W_{n,m}$ is finite, and compute the kernel of the Cremona representation. This leads to a beautiful interpretation of certain elements of the center of the Weyl groups as certain types of Cremona transformations in the projective space. We refer to a recent paper of P. Du Val **[DuV 4]**, where, again without mentioning Coble's work, a nice account of this is given.

Starting from Chapter VIII we study the relationship between point sets and theta functions. The existence of this relationship in the case n = 2 and 3 goes back to Frobenius, Schottky and Wirtinger. Much of Coble's book is devoted to an exposition of Schottky's results. In the case n = 1, this relationship is based on the observation that 2g+2 points in $\mathbb{P}_1$ define a hyperelliptic curve of genus g, and its ordering equips the Jacobian variety of this curve with a level 2 structure. The invariants of points can be translated into the language of theta functions. An explicit formula of this kind which relates the fourth powers of theta constants and coordinate functions of the varieties P_1^{2g+2} is due to R. Thomae. In many aspects we follow here an exposition of D. Mumford **[Mu 2]** of the theory of hyperelliptic curves and their theta functions.

The last Chapter IX is the longest one. Here we give an account of classical work on extension of the theory of the previous chapter to the case of curves of genus 3. Coble's contribution to this is a clear understanding that seven points in the plane not only define a curve of genus 3 but also equip it with a level 2 structure in such a way that the natural action of $Sp(6,\mathbb{F}_2)$ on these structures is equivalent to the Cremona action on the variety P^7_2. This approach gives a nice geometrical insight into the structure of the moduli space of principally polarized abelian varieties of dimension 3 with level 2 structure, and also its Satake compactification and its Igusa's blow-up.

Besides our primary goal to advertize Coble's book we tried to take the reader on an exciting journey where he meets the most fascinating objects of classical algebraic geometry such as the sets of 27 lines on a cubic surface and 28 bitangents to a plane quartic, a Segre's cubic primal and its dual quartic 3-fold, Kummer surfaces and their generalization for dimension 3, Cayley dianode surfaces, nets of quadrics, Cremona transformations, theta functions, the theory of invariants and many others.

These notes are based on a course of lectures of the first author at the University of Paris-Orsay, in Winter of 1987, and on the thesis of the second author. The first author would like to thank Professor Arnaud Beauville for the invitation and for his hospitality. He also expresses his gratitude to all participants in the course for their interest and patience. We are both thankful to all mathematicians who shared our enthusiasm toward Coble's work and to classical algebraic geometry in general. We are particularly indebted to Francois Cossec and Bert van Geemen for numerous helpful discussions on different aspects of Coble's work. Our special thanks go to the referee for his special effort to improve substantially the presentation of this work.

I. CROSS-RATIO FUNCTIONS.

Throughout this chapter and later on we will use the following notations:

$\Bbbk$ = an algebraically closed field of characteristic $p \geq 0$;

$\mathbb{P}_n$ = the n-dimensional projective space over $\Bbbk$;

$\mathbb{P}_n^m = (\mathbb{P}_n)^m = \mathbb{P}_n \times \ldots \times \mathbb{P}_n$, m times;

$\pi_i : \mathbb{P}_n^m \to \mathbb{P}_n$ = the i-th projection;

$G = PGL(n+1, \Bbbk) = Aut(\mathbb{P}_n)$;

$\sigma : G \times \mathbb{P}_n^m \to \mathbb{P}_n^m$ = the morphism of the diagonal action:

$$\sigma(g, (x^1, \ldots, x^m)) = (g(x^1), \ldots, g(x^m)), \quad g \in G, \quad (x^1, \ldots, x^m) \in \mathbb{P}_n^m ;$$

$p_1 : G \times \mathbb{P}_n^m \to G$, $p_2 : G \times \mathbb{P}_n^m \to \mathbb{P}_n^m$ = the projections;

$\mathfrak{L} = \bigotimes_{i=1}^{m} \pi_i^*(\mathcal{O}_{\mathbb{P}_n}(l))$, where l is the smallest positive integer satisfying the equality

$$lm = w(n+1)$$

for some integer w.

1. <u>The variety P_n^m (first definition)</u>.

Recall that a G-linearization of a sheaf $\mathfrak{F}$ on an algebraic variety X with an action $\sigma : G \times X \to X$ is an isomorphism:

$$\sigma^*(\mathfrak{F}) \xrightarrow{\sim} p_2^*(\mathfrak{F}),$$

where $p_2 : G \times X \to X$ is the second projection (see **[Mu 1]**, Chapter 1).

Proposition 1. $\mathfrak{L}$ admits a unique G-linearization.

<u>Proof</u>. Since G does not admit nontrivial characters, it is enough to construct one G-linearization of $\mathfrak{L}$ ([Mu 1], Chapter 1, Proposition 1.4). We may view G as an

open subset of $\mathbb{P}_{n^2+2n}$, the complement of which is the determinantal hypersurface of degree n+1. This implies that

$$\mathcal{O}_G(1)^{\otimes n+1} \cong \mathcal{O}_G(n+1) \cong \mathcal{O}_G .$$

Since G acts linearly on each factor $\mathbb{P}_n$ of $\mathbb{P}_n^m$, we have a natural isomorphism

$$(\pi_i \circ \sigma)^*(\mathcal{O}_{\mathbb{P}_n}(1)) \cong p_1^*\mathcal{O}_G(1) \otimes (\pi_i \circ p_2)^*(\mathcal{O}_{\mathbb{P}_n}(1)).$$

Thus

$$\sigma^*(\mathcal{L}) = \sigma^*(\bigotimes_{i=1}^{m} \pi_i^*(\mathcal{O}_{\mathbb{P}_n}(1))) = \bigotimes_{i=1}^{m} (\pi_i \circ \sigma)^*(\mathcal{O}_{\mathbb{P}_n}(1)) \cong$$
$$\cong \bigotimes_{i=1}^{m} p_1^*\mathcal{O}_G(1) \otimes (\pi_i \circ p_2)^*(\mathcal{O}_{\mathbb{P}_n}(1)) \cong$$
$$\cong p_1^*\mathcal{O}_G(m1) \otimes p_2^*\mathcal{L} = p_1^*\mathcal{O}_G(w(n+1)) \otimes p_2^*\mathcal{L} \cong$$
$$\cong p_1^*\mathcal{O}_G \otimes p_2^*\mathcal{L} \cong p_2^*\mathcal{L}.$$

Recall that for every G-linearized sheaf $\mathcal{F}$ on a G-variety X there is a natural linear representation of G in the space $\Gamma(X,\mathcal{F})$. It is derived from the composition:

$$\Gamma(X,\mathcal{F}) \xrightarrow{\sigma^*} \Gamma(G\times X, \sigma^*\mathcal{F}) \to \Gamma(G\times X, p_2^*\mathcal{F}) \to \Gamma(G,\mathcal{O}_G) \otimes \Gamma(X,\mathcal{F}),$$

where the second arrow is defined by the linearization of $\mathcal{F}$ and the third one is defined by the Künneth formula. Viewing every $g \in G(k)$ as a homomorphism of k-algebras $\Gamma(G,\mathcal{O}_G) \to k$, we let

$$\rho(g)(s) = (g \otimes 1)(\sigma^*(s)), \qquad g \in G(k), s \in \Gamma(X,\mathcal{F}).$$

As usual, $\Gamma(X,\mathcal{F})^G$ will denote the subspace of G-invariant sections.

Returning to our situation ($X = \mathbb{P}_n^m$, $\mathcal{F} = \mathcal{L}$), let us set

$$R_n^m = \bigoplus_{k=0}^{\infty} \Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes k})^G,$$

where we equip $\mathcal{L}^{\otimes k}$ with the G-linearization that is the k-th tensor product of the G-linearization of $\mathcal{L}$. Since $\mathcal{L}$ is an ample invertible sheaf on $\mathbb{P}_n^m$, the graded k-algebra

$$\bigoplus_{k=0}^{\infty} \Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes k})$$

is of finite type. The group G acts on this algebra by automorphisms of graded algebras, and

$$R_n^m = (\bigoplus_{k=0}^{\infty} \Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes k}))^G$$

is the subalgebra of G-invariant elements graded by

$$(R^m_n)_k = \Gamma(\mathbb{P}^m_n, \mathcal{L}^{\otimes k})^G.$$

Since G is a reductive algebraic group, R^m_n is of finite type over k ([Mu 1]). Thus we can set

$$P^m_n = \mathrm{Proj}(R^m_n)$$

to obtain a projective algebraic variety over k. This is the principal object of our notes. In the next chapter we will interpret P^m_n as a certain quotient of an open subset of $\mathbb{P}^m_n$ by G.

2. Standard monomials.

Let $\mathbb{P}_n = \mathbb{P}(V)$ for a linear n+1-dimensional space V over k, i.e. k-points of $\mathbb{P}_n$ are lines in V. We have

$$\Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(k)) \cong \mathrm{Sym}^k(V^*),$$

where

$$\mathrm{Sym}(V^*) = \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(V^*)$$

is the graded symmetric algebra of the dual vector space V*. Thus, by the Künneth formula,

$$(R^m_n)_k = \Gamma(\mathbb{P}^m_n, \mathcal{L}^{\otimes k})^G = \Gamma(\mathbb{P}^m_n, \bigotimes_{i=1}^{m} \pi_i{}^*\mathcal{O}_{\mathbb{P}_n}(kl))^G \cong ((\mathrm{Sym}^{kl}(V^*))^{\otimes m})^G.$$

The linear representation of G in $\Gamma(\mathbb{P}^m_n, \mathcal{L}^{\otimes k})$ is the m-th tensor product of its natural representation in the space of homogeneous polynomial functions on V of degree kl. Applying the First Fundamental Theorem of Invariant Theory ([Di-C]) we obtain:

Proposition 2. Consider an element of $(R^m_n)_k$ as a function $\mu(v^1,\dots,v^m)$ on V^m which is a homogeneous polynomial of degree kl in each variable. Then the functions

$$\mu_\tau(v^1,\dots,v^m) = \prod_{1\le i\le wk} \det(v^{\tau_{i1}},\dots,v^{\tau_{in+1}})$$

span $(R^m_n)_k$, where $\tau_{ij} \in \{1,2,\dots,m\}$ and each $a \in \{1,2,\dots,m\}$ occurs exactly kl times among the τ_{ij}'s.

A matrix

$$\tau = \begin{bmatrix} \tau_{11} & \cdots\cdots & \tau_{1n+1} \\ \cdots & \cdots\cdots & \cdots \\ \tau_{wk\,1} & \cdots\cdots & \tau_{wk\,n+1} \end{bmatrix}$$

is said to be a <u>tableau</u> and the corresponding function μ_τ is said to be the <u>monomial</u> that belongs to τ. The number wk is the <u>weight</u> of τ or μ_τ and the number kl is its <u>degree</u>.

We assume that $\tau_{ij} \neq \tau_{ij'}$ for $j \neq j'$ and each i. Otherwise the corresponding monomial is zero.

A tableau τ is said to be <u>standard</u> if

$$\tau_{ij} < \tau_{ij+1} \qquad \text{for each } i = 1,\dots,wk,\ j = 1,\dots,n,$$

$$\tau_{ij} \leq \tau_{i+1j} \qquad \text{for each } i = 1,\dots,wk-1,\ j = 1,\dots,n+1.$$

A <u>standard monomial</u> is a monomial that belongs to a standard tableau.

Theorem 1. The standard monomials of degree kl and weight wk form a basis of $(R_n^m)_k$.

<u>Proof</u>. We will prove only that standard monomials span the linear space $(R_n^m)_k$ and refer to **[DeC-P]** for the proof of their linear independence.

Here is an algorithm (the "straightening algorithm") that allows us to write any monomial as a linear combination of standard ones.

Suppose μ_τ is not standard. We permute the entries of each row of τ so that in the new tableau τ' all rows are in strictly increasing order. Then

$$\mu_\tau = \pm\mu_{\tau'}.$$

Next we permute the rows of τ' so that in the obtained tableau τ''

$$\tau''_{i1} \leq \tau''_{i+11} \qquad \text{for each } i = 1,\dots,wk.$$

Continue permuting the rows so that if $\tau_{ij} = \tau_{i+1j}$ then $\tau_{ij+1} \leq \tau_{i+1j+1}$. Note that these permutations do not change the monomial. We call the monomial obtained so far semi-standard.

The rest of the algorithm proceeds by induction on the lexigraphic order of tableaux defined by setting $\tau < \tau'$ if

$$(\tau_{11},\dots,\tau_{1n+1},\tau_{21},\dots,\tau_{wk\,n+1}) < (\tau_{11}',\dots,\tau_{1n+1}',\tau_{21}',\dots,\tau_{wk\,n+1}')$$

with respect to the lexigraphic order.

Suppose that μ_τ is not yet standard and let i_0 be such that

$$\tau_{i_0 j_0} > \tau_{i_0+1 j_0}$$

for some j_0.

Consider the increasing sequences

$$S_1 = (s_1, \ldots, s_{j_0}),\ S_2 = (s_{j_0+1}, \ldots, s_{n+2}),\ S = (S_1, S_2),$$

where

$$s_K = \tau_{i_0+1 K} \quad \text{if } K \le j_0,$$

$$= \tau_{i_0 K-1} \quad \text{if } K > j_0.$$

For example, if

$$\tau = \begin{bmatrix} 1\,2\,6 \\ 1\,4\,5 \\ 2\,3\,6 \\ 3\,4\,5 \end{bmatrix}$$

we have $(i_0, j_0) = (2,2)$, and

$$S_1 = (23),\ S_2 = (45),\ S = (2345).$$

Let $A \subset \Sigma_{n+2}$ be the subset of the permutation group Σ_{n+2} such that $\sigma \epsilon A$ iff $(s_{\sigma(1)}, \ldots, s_{\sigma(j_0)})$ and $(s_{\sigma(j_0+1)}, \ldots, s_{\sigma(n+2)})$ are increasing subsequences of S. We set for every $\sigma \epsilon A$

$$\tau_\sigma' = (\tau_{i_0,1}, \ldots, \tau_{i_0, j_0-1}, s_{\sigma(j_0+1)}, \ldots, s_{\sigma(n+2)}),$$

$$\tau_\sigma'' = (s_{\sigma(1)}, \ldots, s_{\sigma(j_0)}, \tau_{i_0+1 j_0+1}, \ldots, \tau_{i_0+1, n+1}).$$

In our example

$$A = \{(1234), (1324), (1423), (2314), (2413), (3412)\},$$

$$\tau_\sigma' = \{(1,4,5), (1,3,5), (1,3,4), (1,2,5), (1,2,4), (1,2,3)\},$$

$$\tau_\sigma'' = \{(2,3,6), (2,4,6), (2,5,6), (3,4,6), (3,5,6), (4,5,6)\}.$$

For every sequence $\gamma = (i_1, \ldots, i_{n+1})$ of numbers from $\{1, \ldots, m\}$ we will consider the determinant

$$(\gamma) = (i_1 \ldots i_{n+1}) = \det(v^{i_1}, \ldots, v^{i_{n+1}}) \epsilon (V^*)^{\otimes n+1}$$

as a section of $\pi_{i_1}^* \mathcal{O}_{\mathbb{P}_n}(1) \otimes \ldots \otimes \pi_{i_{n+1}}^* \mathcal{O}_{\mathbb{P}_n}(1)$. For example,

$$\mu_\tau = \prod_{i=1}^{wK} (\tau_i),$$

where τ_i is the i-th row of τ ($\prod$ is really the tensor product).

Then

$$\sum_{\sigma\in A} \mathrm{sgn}(\sigma)(\tau_{\sigma}'\tau_{\sigma}'') \in \Gamma(\mathbb{P}_n^m, \pi_{s_1}^*(\mathcal{O}_{\mathbb{P}_n}(1))\otimes\ldots\otimes\pi_{s_{n+2}}^*\mathcal{O}_{\mathbb{P}_n}(1)) \cong (V^*)^{\otimes n+2}$$

is skew-symmetric and hence is identically zero (use that $\dim V = n+1$). Therefore we can write

$$(\tau_{i_0})(\tau_{i_0+1}) = - \sum_{\sigma\in A'} \mathrm{sgn}(\sigma)(\tau_{\sigma}')(\tau_{\sigma}''),$$

where $A' = A\setminus\{id\}$.

Let $\tau(\sigma)'$ denote the tableau that is obtained from τ by replacing τ_{i_0} with τ_{σ}', and replacing τ_{i_0+1} with τ_{σ}''. Let $\tau(\sigma)$ be obtained from $\tau(\sigma)'$ by rearranging the rows in increasing order. Then

$$\mu_{\tau} = - \sum_{\sigma\in A'} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau(\sigma)')\mu_{\tau(\sigma)},$$

where $\mu_{\tau(\sigma)} = \mathrm{sgn}(\tau(\sigma)')\mu_{\tau(\sigma)'}$. It is obvious that

$$\tau(\sigma) < \tau$$

for every $\sigma\in A'$. Thus we can continue our algorithm until we express μ_{τ} as a linear combination of standard monomials.

In our example we find

$$\mu_{\begin{bmatrix}126\\145\\236\\345\end{bmatrix}} = \mu_{\begin{bmatrix}126\\135\\246\\345\end{bmatrix}} - \mu_{\begin{bmatrix}126\\134\\256\\345\end{bmatrix}} - \mu_{\begin{bmatrix}126\\125\\346\\345\end{bmatrix}} + \mu_{\begin{bmatrix}126\\124\\356\\345\end{bmatrix}} - \mu_{\begin{bmatrix}126\\123\\456\\345\end{bmatrix}}$$

and so on.

Remark 1. It would be interesting to find a general formula for $\dim(R_n^m)_k$. In the simplest case when $1 = k = 1$, i.e. $m = w(n+1)$, our tableaux (resp. standard tableaux) are equal to the Young tableaux (resp. standard tableaux) from the representation theory of symmetric groups corresponding to the partition $(n+1,\ldots,n+1)$ of m. Applying the "hook-formula" we obtain

$$\dim(R_n^m)_1 = \frac{m!n!(n-1)!\ldots2!}{(n+w)!(n+w-1)!\ldots(w-1)!}.$$

In general,

$$\dim \ (R_n^m)_\kappa = K_{\lambda,\mu}$$

is the Kostka number ([McD 1]) corresponding to the pair of partitions of $\kappa m l$

$$\lambda = (\kappa w, \dots, \kappa w), \ \mu = (\kappa l, \dots, \kappa l).$$

Remark 2. Since all standard monomials are equal to zero if $m \le n$, we see that for such m and n, all the spaces P_n^m are empty. Similarly, if $m = n+1$ then all standard monomials are powers of of $\mu_{[1,2,\dots,n+1]}$. Hence P_n^m is a one-point set.

3. Examples.

Example 1 (n=1,m=4).

Here $w = 2$, $l = 1$. A standard tableau τ of degree κ must look like

$$\begin{bmatrix} a_1^1 \ a_2^2 \\ a_2^1 \ a_3^2 \\ a_3^1 \ a_4^2 \end{bmatrix}$$

where a_i^j is a column vector that consists entirely of the integer i in the j-th column of τ. Let $|a_i^j|$ denote the height of a_i^j. It is clear that

$$|a_1^1| = |a_4^2| = \kappa, \ |a_2^1| + |a_2^2| = |a_3^1| + |a_3^2| = \kappa,$$

$$|a_1^1| + |a_2^1| + |a_3^1| = |a_2^2| + |a_3^2| + |a_4^2| = 2\kappa.$$

This shows that the standard tableau τ is completely determined by the number $a = |a_2^1|$ that satisfies

$$0 \le a \le \kappa.$$

In particular

$$\dim \ (R_1^4)_\kappa = \kappa+1.$$

Let

$$t_0 = \mu_{\begin{bmatrix} 12 \\ 34 \end{bmatrix}}, \ t_1 = \mu_{\begin{bmatrix} 13 \\ 24 \end{bmatrix}}$$

Then

$$t_0^i t_1^{\kappa-i} = \mu \begin{bmatrix} a_1^1 a_2^2 \\ a_2^1 a_3^2 \\ a_3^1 a_4^2 \end{bmatrix}$$

where $a = |a_2^1| = \kappa - i$.

Therefore

$$R_1^4 \cong \kappa[t_0, t_1]$$

and

$$P_1^4 \cong \mathbb{P}_1 .$$

The field of rational functions of P_1^4 is generated by the function

$$t_0/t_1 = \frac{(12)(34)}{(13)(24)} = \frac{(a_1-a_2)(a_3-a_4)}{(a_1-a_3)(a_2-a_4)},$$

where $(x^1, x^2, x^3, x^4) \in \mathbb{P}_1^4$ is represented by the vectors $(a_i, 1)$ from V. This function is known as the <u>double cross ratio</u> of 4 points in $\mathbb{P}_1$.

Example 2 (n=1, m=6). In this case 1 = 1 and w = 3.

A standard tableau of degree κ and weight 3κ looks like

$$\begin{bmatrix} a_1^1 & a_2^2 \\ a_2^1 & a_3^2 \\ a_3^1 & a_4^2 \\ a_4^1 & a_5^2 \\ a_5^1 & a_6^2 \end{bmatrix}$$

Let

$$|a_2^1| = i_2,\ |a_3^1| = i_3,\ |a_4^1| = i_4.$$

They satisfy

$$0 \le i_2, i_3, i_4 \le \kappa,\ 2i_2 + i_3 \ge \kappa,$$

$$\kappa \le i_2 + i_3 + i_4 \le 2\kappa,\ 2i_2 + 2i_3 + i_4 \ge 2\kappa.$$

Setting

$$x = i_2,\ y = i_2 + i_3,\ z = i_2 + i_3 + i_4,$$

we obtain that our tableau is completely determined by a vector (x,y,z) satisfying

$$0 \le x \le k,\ 0 \le y-x \le k,\ 0 \le z-y \le k,$$

$$x+y \ge k,\ y+z \ge 2k,\ k \le z \le 2k.$$

When $0 \le y \le k$ these inequalities are equivalent to

$$y \ge x \ge k-y,\ 2k-y \le z \le y+k.$$

This gives

$$\sum_{y=0}^{k} (2y-k+1)^2$$

solutions.

When $2k \ge y \ge k$ we have $y \le z \le 2k$, which gives

$$\sum_{y=k}^{2k} (2k-y+1)^2$$

solutions.

Summing up by using the well-known formulae for the sum of consecutive integers and for the sum of their squares, we find

$$\dim(R_1^6)_k = \tfrac{1}{2}(k^3+3k^2+4k)+1.$$

The Poincare function of the graded ring R_1^6 is

$$\sum_{k=0}^{\infty} (\tfrac{1}{2}(k^3+3k^2+4k)+1)t^k = (1-t^3)/(1-t)^5.$$

This suggests that P_1^6 is isomorphic to a cubic hypersurface in $\mathbb{P}_4$. This is true.

Let

$$t_0 = (12)(34)(56),\ t_1 = (13)(24)(56),\ t_2 = (12)(35)(46),$$

$$t_3 = (13)(25)(46),\ t_4 = (14)(25)(36)$$

be a basis of $(R_1^6)_1$ corresponding to standard monomials of degree 1 In the notation of p. 11. For every $(i,j) \neq (0,3),(0,4)$ the product $t_i t_j$ is a standard monomial from $(R_1^6)_2$. Applying the straightening algorithm, we find

$$t_0t_3 = (12)(13)(25)(34)(46)(56) = -(12)(13)(23)(45)(46)(56)$$

$$+(12)(13)(24)(35)(46)(56) = -y_1+t_1t_2,$$

where

$$y_1 = (12)(13)(23)(45)(46)(56)$$

is a standard monomial from $(R_1^6)_2$.

Similarly,

$$t_0t_4 = (12)(14)(25)(34)(36)(56) = (12)(14)(24)(35)(36)(56)-$$
$$-(12)(13)(24)(35)(46)(56)+(12)(13)(24)(34)(56)(56)+$$
$$+(12)(12)(34)(35)(46)(56)-(12)(12)(34)(34)(56)(56) =$$
$$= y_2-t_1t_2+t_0t_1+t_0t_2-t_0^2,$$

where

$$y_2 = (12)(14)(24)(35)(36)(56)$$

is a standard monomial from $(R_1^6)_2$.

We see that all standard monomials from $(R_1^6)_2$ can be expressed as linear combinations of the products of standard monomials from $(R^6{}_1)_1$. In fact every standard monomial can be written in this way. By using the coordinates (i_2,i_3,i_4,k) for the monomials and avoiding products of monomials that are not standard, we may write the general standard monomial as follows:

$$\mu_{(a,b,c,k)} = t_0^{2k-2a-b-c}t_1^{k-c}t_2^{c}y_2^{a+b-k}, \qquad \text{if } a+b \geq k;$$
$$= t_0^{2k-2a-b-c}t_1^{2a+b-k}t_2^{2a+2b+c-2k}y_1^{k-b-a}, \qquad \text{if } a+b \leq k;$$

whenever $2k\geq 2a+b+c$, and:

$$\mu_{(a,b,c,k)} = t_1^{2k-a-b-c}t_2^{k-a}t_3^{a+c-k}t_4^{a+b-k}, \qquad \text{if } a+b \geq k,\ a+c \geq k;$$
$$= t_1^{k-b}t_2^{c}t_3^{2a+b+c-2k}y_2^{k-c-a}, \qquad \text{if } a+c \leq k;$$
$$= t_1^{k-c}t_2^{b}t_3^{2a+b+c-2k}y_1^{k-b-a}, \qquad \text{if } a+b \leq k,$$

whenever $2k\leq 2a+b+c$. Note how the inequalities on x,y, and z help to keep the exponents positive.

It is immediately verified that

$$t_3y_2 = t_1t_2t_4.$$

This shows that the natural homomorphism of graded algebras

$$k[T_0,\dots,T_4]/(T_1T_2T_4-T_3(T_0T_4+T_1T_2-T_0T_1-T_0T_2+T_0^2)) \to R_1^6,$$
$$T_i \to t_i,\ i = 0,\dots,4,$$

is surjective. Comparing the Poincare functions we find that it is bijective. Therefore

$$P_1^6 \cong V_3 \subset \mathbb{P}_4,$$

where V_3 is a cubic hypersurface given by the equation:

$$T_1T_2T_4-T_3T_0T_4+T_3T_1T_2+T_3T_0T_1+T_3T_0T_2-T_3T_0^2 = 0.$$

The permutation group Σ_6 acts linearly in R_1^6 by acting on tableaux. Computing its character we find that this representation is isomorphic to the 5-dimensional irreducible representation of Σ_6 associated to the partition (2,2,2) (cf. Remark 1). In fact our realization of this representation is a special case of MacDonald's construction of irreducible representations of Weyl groups (**[McD 2]**). In new coordinates:

$$Z_0 = 2t_0-t_1-t_2+t_3+t_4,\ Z_1 = t_1-t_2-t_3+t_4,\ Z_2 = -t_1+t_2-t_3+t_4,$$

$$Z_3 = t_1+t_2-t_3-t_4,\ Z_4 = -t_1-t_2+t_3-t_4,\ Z_5 = -2t_0+t_1+t_2+t_3-t_4$$

our cubic hypersurface V_3 is defined by the equations:

$$\Sigma Z_i = 0,\ \Sigma Z_i^3 = 0,$$

where Σ_6 acts by permutations (see **[Co 1]**, p.114). In this form V_3 is known in the classical literature as the <u>Segre cubic primal</u> (see **[S-R]**, Chapter VIII, **[Ba]**). It contains 10 nodes and 15 planes. Note that by a theorem of Bertini, V_3 is the unique (up to an isomorphism) cubic hypersurface in $\mathbb{P}_4$ with the maximal possible number of nodes (cf. **[Ka1]**).

Example 3 (n=2,m=6). In this case l=1 and w=2.

A standard tableau of degree k and weight 2k looks like

$$\begin{bmatrix} a_1^1 & a_2^2 & a_3^3 \\ a_2^1 & a_3^2 & a_4^3 \\ a_3^1 & a_4^2 & a_5^3 \\ a_4^1 & a_5^2 & a_6^3 \end{bmatrix}$$

Let

$$i_3 = |a_3^1|,\ i_4 = |a_4^1|,\ j_3 = |a_3^3|,\ j_4 = |a_4^3|.$$

These integers must satisfy the following inequalities:

$$0 \le i_3, i_4, j_3, j_4 \le k,\ i_3+i_4 \le k,\ i_3+j_3 \le k,\ i_4+j_4 \le k,\ 2i_4+i_3-j_3 \le k,$$

$$2j_3+j_4-i_4 \le k,\ j_3 \le i_3+i_4,\ j_3+j_4 \le i_4.$$

Set

$$x = i_4,\ y = i_3+i_4,\ z = j_3,\ u = j_3+j_4\ .$$

Then a standard tableau is completely determined by the integers x, y, z, and u satisfying the following system of inequalities:

$$0 \le x \le y \le k\ ,\ 0 \le z \le u \le k,$$

$$z \le y \le z-x+k\ ,\ z \le y \le x-z+k,$$

$$x \le u \le z-x+k\ ,\ x \le u \le x-z+k\ .$$

After lengthy calculations we find

$$\dim\ (R_2^6)_k = \frac{1}{12}(k^4+6k^3+17k^2+24k)+1.$$

The Poincare series of R_2^6 is

$$\dim\ (R_2^6)_k = \sum_{k=0}^{\infty}\frac{1}{12}(k^4+6k^3+17k^2+24k)+1)t^k =$$

$$= \frac{1}{12}\sum_{k=0}^{\infty}(k(k+1)(k+2)(k+3)t^k + \frac{1}{2}\sum_{k=0}^{\infty}k(k+1)t^k + \sum_{k=0}^{\infty}kt^k + \sum_{k=0}^{\infty}t^k =$$

$$= \frac{2t}{(t-1)^5} - \frac{t}{(t-1)^3} + \frac{t}{(t-1)^2} - \frac{1}{(t-1)} = \frac{(1-t^4)}{(1-t)^5(1-t^2)}.$$

This suggests that R_2^6 is generated by 5 elements of degree 1 and one element of degree 2 with a basic relation of degree 4. In another words, P_2^6 is isomorphic to a hypersurface of degree 4 in the weighted projective space $\mathbf{P}(1,1,1,1,1,2)$. This is true. We have 5 standard monomials of degree 1:

$$t_0 = (123)(456),\ t_1 = (124)(356),\ t_2 = (125)(346),$$

$$t_3 = (134)(256),\ t_4 = (135)(246).$$

For every pair $(i,j) \neq (2,3), (3,2)$ the product t_it_j is a standard monomial of degree 2. The remaining two standard monomials of degree 2 are:

$$y_1 = (123)(145)(246)(356),\ y_2 = (124)(135)(236)(456).$$

Furthermore, the monomials $t_0,\dots,t_4,y_1$, and y_2 generate the graded algebra R_2^6, as the following formulae show:

$$\mu(x,y,z,u,k) = t_0^x t_1^{y-z} t_3^{u-y} t_4^{k-u-t+x} y_1^{z-x}, \qquad \text{if } x \le z \le y \le u;$$

$$= t_0^x t_1^{u-z} t_2^{y-u} t_4^{k-y-z+x} y_1^{z-x}, \qquad \text{if } x \le z \le u \le y;$$

$$= t_0^z t_1^{y-x} t_3^{u-y} t_4^{k-u-x+z} y_2^{x-z}, \qquad \text{if } z \le x \le y \le u;$$

$$= t_0^{z}t_1^{u-x}t_2^{y-u}t_4^{k-y-x+z}y_2^{x-z}, \qquad \text{if } z \le x \le u \le y.$$

We find by the straightening algorithm

$$t_2t_3 = -y_1-y_2+t_1t_4+t_0t_1+t_0t_4-t_0t_2-t_0t_3-t_0^2,$$

$$y_1y_2 = t_0t_1t_4(-t_0+t_1-t_2-t_3+t_4).$$

Thus, if $\mathrm{char}(k) \neq 2$, by setting

$$t_5 = y_1-y_2,$$

we obtain

$$t_5^2 = (y_1+y_2)^2-4y_1y_2 = F_4(t_0,\dots,t_4),$$

where

$$F_4=(-t_2t_3+t_1t_4+t_0t_1+t_0t_4-t_0t_2-t_0t_3-t_0^2)^2-4t_0t_1t_4(-t_0+t_1-t_2-t_3+t_4).$$

Together with the computation of the Poincare series this implies that

$$R_2^6 \cong k[T_0,\dots,T_5]/(T_5^2-F_4(T_0,\dots,T_4)).$$

In other words, P_2^6 is isomorphic to a hypersurface of degree 4 in $\mathbb{P}(1^5,2)$ given by the equation

$$T_5^2 - F_4(T_0,\dots,T_4) = 0.$$

If $\mathrm{char}(k) = 2$, we find similarly that P_2^6 is isomorphic to the hypersurface

$$T_5^2+T_5(T_2T_3+T_1T_4+T_0T_1+T_0T_4+T_0T_2+T_0T_3+T_0^2)+T_0T_1T_4(T_0+T_1+T_2+T_3+T_4) = 0$$

in $\mathbb{P}(1^5,2)$.

The inclusion $k[t_0,\dots,t_4] \hookrightarrow R_2^6$ realizes P_2^6 as a separable double cover of $\mathbb{P}_4$ branched along a hypersurface V_4 of degree 4 (resp. along a quadric Q_2 if $p = 2$) given by the equation

$$F_4(T_0,\dots,T_4) = 0$$

$$(\text{resp. } T_2T_3+T_1T_4+T_0T_1+T_0T_4+T_0T_2+T_0T_3+T_0^2 = 0).$$

The points over the branch divisor satisfy

$$y_1-y_2 = (123)(145)(246)(356)-(124)(135)(236)(456) = 0.$$

If we fix first 5 points $(x^1,\dots,x^5)\in P_2^1$ and let x^6 vary, we see that this equation represents a curve of degree 2 that passes through $x^1,\dots,x^5$. Thus y_1-y_2 vanishes on the sets of six points that lie on a curve of degree 2.

Remark 3. Assume $k = \mathbb{C}$. By a change of variables

$$X_0 = T_0,\ X_1 = T_1,\ X_2 = T_4,\ X_3 = -T_2-T_0,\ X_4 = -T_3-T_0$$

we transform the equation of the hypersurface V_4 to the form:

$$(X_0X_1+X_0X_2+X_1X_2-X_3X_4)^2-4X_0X_1X_2(X_0+X_1+X_2+X_3+X_4) = 0.$$

This equation can be found in **[Ig 3]**, where it is shown that the corresponding variety is isomorphic to the Baily-Satake compactification $\mathfrak{A}_2(2)$ of the moduli space of abelian surfaces with level 2 structure. In other words

$$R^6_2/(t_5) \cong M(\Gamma_2(2)),$$

where $M(\Gamma_2(2))$ is the graded ring of modular forms with respect to the 2-level congruence subgroup $\Gamma_2(2)$ of the Siegel modular group $\Gamma_2 = Sp(4,\mathbb{Z})$ (see more about this in Chapter 8).

Note that the Segre cubic primal V_3 and the quartic 3-fold V_4 are dual hypersurfaces in $\mathbb{P}_4$. The easiest way to see this is as follows. Let

$$s : \mathbb{P}_1^6 \to \mathbb{P}_2^6$$

be the 6-th Cartesian power of the Veronese map

$$v_2 \colon \mathbb{P}_1 \to \mathbb{P}_2,\ (t_0,t_1) \to (t_0^2, t_0t_1, t_1^2).$$

Under this map

$$s^*(\mathcal{L}) = \mathcal{L}^{\otimes 2},$$

where, abusing the notation, we denote by the same letter our standard sheaves for both spaces $\mathbb{P}_1^6$ and $\mathbb{P}_2^6$. Let μ_τ be a standard monomial on $\mathbb{P}_2^6$, say $\mu_\tau = (123)(456)$. Then we immediately verify that

$$s^*(\mu_\tau) = (12)(13)(23)(45)(46)(56) \in (R_1^6)_2,$$

and, in the notation of Example 2, is equal to $y_1 = -t_0t_3+t_1t_2$. Now note that

$$y_1 = \frac{\partial F_3}{\partial t_4},$$

where V_3 is given by the equation $F_3 = 0$. Similarly, we find that s^* maps other standard monomials to the elements of $(R_1^6)_2$ which are equal to linear independent combinations of the partials of the cubic form F_3. This shows that the image of V_3 under the birational map given by the partials is isomorpic to V_4. This proves the assertion.

We will call the quartic threefold V_4 the <u>level 2 modular quartic 3-fold</u>. The reader is referred to **[vdG]** and **[Ba]** for further information about this 3-fold.

II. GEOMETRIC INVARIANT THEORY.

In this chapter we will show that the spaces P_n^m from the previous chapter are certain quotient spaces of some open subset of $\mathbb{P}_n^m$. Most of the notions and the results that we introduce here can be found in [Mu 1].

1. $\underline{P_n^m}$ (second definition).

Let G be a reductive algebraic group (e.g. G = PGL(n+1)) that acts regularly on an algebraic variety X. Let $\mathcal{L}$ be a G-linearized ample invertible sheaf on X. A point $x \in X$ is said to be semi-stable (with respect to $\mathcal{L}$) if there exists a G-invariant section of some positive tensor power of $\mathcal{L}$ such that $s(x) \neq 0$. A semi-stable point is stable if G acts with closed orbits in $X_s = \{x \in X: s(x) \neq 0\}$ and the stabilizer group $G_x = \{g \in G: g \cdot x = x\}$ is finite.

We denote by $X^{ss}(\mathcal{L})$ (resp. $X^s(\mathcal{L})$) the subset of semi-stable (resp. stable) points of X. Both of these subsets are open G-invariant subsets of X. The usefulness of them is explained by the following:

Proposition 1. Assume that X is proper. Then the categorical quotient $X^{ss}(\mathcal{L})/G$ exists and there is an isomorphism

$$X^{ss}(\mathcal{L})/G \cong \mathrm{Proj}(\bigoplus_{K=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes K})^G).$$

Moreover, the open subset $X^s(\mathcal{L})/G$ of $X^{ss}(\mathcal{L})/G$ is a geometric quotient of $X^s(\mathcal{L})$.

Recall that a categorical quotient X/G is an algebraic variety together with a surjective morphism $\pi: X \to X/G$ which is G-equivariant, where G acts identically on X/G, and is universal with respect to this property. A geometric quotient is a categorical quotient the fibres of which are the orbits of G in X.

Corollary.

$$P_n^m \cong (\mathbb{P}_n^m)^{ss}(\mathcal{L})/G \ .$$

The idea of the proof of Proposition 1 is very simple. First we note that $X \cong \mathrm{Proj}(\bigoplus_{K=0}^{\infty} \Gamma(X,\mathcal{L}^{\otimes K}))$ because X is proper and $\mathcal{L}$ is ample. Let

$$A_X = \bigoplus_{K=0}^{\infty} \Gamma(X,\mathcal{L}^{\otimes K}),\ C_X = \mathrm{Spec}\ A_X \ .$$

The group G acts on A_X and on C_X and $\mathrm{Spec}(A_X)^G$ is a categorical quotient C_X/G ([Mu 1], Theorem.1.1). Let $o \in C_X/G$ be the point defined by the maximal ideal $\bigoplus_{K>0} \Gamma(X,\mathcal{L}^{\otimes K})$ of $(A_X)^G$. Then its pre-image in C_X is the set of all points which define non-semi-stable points in X. Thus the projection $C_X \to C_X/G$ induces a morphism $X^{ss}(\mathcal{L}) \to \mathrm{Proj}((A_X)^G)$. It is easy to verify that it is a categorical quotient of $X^{ss}(\mathcal{L})$ by G.

We will denote

$$\Phi: (\mathbb{P}_n^m)^{ss} \to P_n^m$$

the canonical projection of the categorical quotient. We set

$$\mathcal{D} = (\mathbb{P}_n^m)^{ss} \setminus (\mathbb{P}_n^m)^{s},$$

$$\bar{\mathcal{D}} = \Phi(\mathcal{D}).$$

The projection

$$\Phi: (\mathbb{P}_n^m)^{s} \to P_n^m \setminus \bar{\mathcal{D}}$$

is the geometric quotient.

2. A criterion of semi-stability.

To describe the set of semi-stable point sets we use the following numerical criterion of Hilbert-Mumford. Let $\lambda: \mathbb{k}^* \to G$ be a one-parameter subgroup of G . For every closed point $x \in X$ we define the map

$$\mu_x : \mathbb{k}^* \to X,\ \ \alpha \to \lambda(\alpha)\cdot x .$$

Assume X is proper. Then μ_x extends uniquely to a morphism:

$$\bar{\mu}_x : \mathbb{A}_1 \to X$$

and defines the point

$$\bar{\mu}_x(0) := \lim_{\alpha \to 0} \lambda(\alpha)\cdot x.$$

Clearly this point is fixed under the action of $\lambda(k^*)$ and the restriction of $\mathcal{L}$ to it defines a k^*-linearized invertible sheaf on it. As such it is completely determined by a character

$$\chi(\lambda,x) \colon K^* \to K^*,$$

and the latter, in turn, is defined by the integer $r(\lambda,x)$ such that

$$\chi(\lambda,x)(\alpha) = \alpha^{r(\lambda,x)} \qquad \text{for each } \alpha \in k^*.$$

Proposition 2. Assume X is proper. Then

$$x \in X^{ss}(\mathcal{L}) \text{ iff } \quad r(\lambda,x) \le 0 \quad \text{for all } \lambda: k^* \to G,$$

$$x \in X^{s}(\mathcal{L}) \text{ iff } \quad r(\lambda,x) < 0 \quad \text{for all } \lambda: k^* \to G.$$

Now we are ready to make the analysis of semi-stable points in $\mathbb{P}_n^m$.

Theorem 1. Let $x = (x^1,\dots,x^m) \in \mathbb{P}_n^m$. Then $x \in (\mathbb{P}_n^m)^{ss}(\mathcal{L})$ if and only if for any proper subset $\{i_1,\dots,i_k\}$ of $\{1,\dots,m\}$

$$\dim \langle x_{i_1},\dots,x_{i_k}\rangle + 1 \ge k(n+1)/m,$$

where $\langle\ \rangle$ denotes the projective span of a finite set of points in $\mathbb{P}_n$. Moreover, x is stable if and only if strict inequalities hold.

<u>Proof</u>. (cf. [Mu 1]). Let $\lambda: k^* \to G$ be a 1-parameter subgroup of G. Choose homogeneous coordinates in $\mathbb{P}_n$ in such a way that the action of $\lambda(k^*)$ is diagonalized:

$$\lambda(\alpha)(t_0,\dots,t_n) = (\alpha^{r_0}t_0,\dots,\alpha^{r_n}t_n)$$

for some integers r_i. We may also assume that

$$(*) \qquad r_0 \ge r_1 \ge \dots \ge r_n, \ \sum_{i=1}^{n} r_i = 0, \ r_0 > 0.$$

Let

$$X = \begin{bmatrix} t_0^{(1)} & \dots & t_0^{(m)} \\ \dots & \dots & \dots \\ t_n^{(1)} & \dots & t_n^{(m)} \end{bmatrix}$$

be the matrix whose columns are the projective coordinates of the points $x^1,\dots,x^m$. For every $I = \{i_1,\dots,i_m\}$, $i_j \in \{0,\dots,n\}$ we denote by X_I the monomial $t_{i_1}^{(1)}\dots t_{i_m}^{(m)}$. The monomials X_I are the coordinates of points of $\mathbf{P}_n^m$ in the Segre embedding given by the sheaf $\bigotimes_{i=1}^{m} \mathcal{O}_{\mathbf{P}_n}(1)$. For every $\mathbf{I} = (I(1),\dots,I(l))$ the products

$$X_{\mathbf{I}} = \prod_{i=1}^{l} X_{I(i)}$$

are the coordinates of points of $\mathbf{P}_n^m$ in the Segre-Veronese embedding given by the sheaf $\bigotimes_{i=1}^{m} \mathcal{O}_{\mathbf{P}_n}(l)$. A 1-parameter subgroup $\lambda:\mathbb{k}^* \to G$ acts on these coordinates via:

$$\lambda(\alpha)(X_{\mathbf{I}}) = \alpha^{N(\mathbf{I})}X_{\mathbf{I}},$$

where

$$N(\mathbf{I}) = \sum_{i=0}^{n} n_i r_i,$$

and where n_i is the number of times that i appears in $I(1),\dots,I(l)$.

By Proposition 2 we have to look for the points $(x^1,\dots,x^m)$ such that

$$(**) \qquad \min_{\mathbf{I}}\{N(\mathbf{I}) : X_{\mathbf{I}} \neq 0\} \leq 0 \text{ (resp. } < 0).$$

Permuting the points $x^1,\dots,x^m$ we may assume that the matrix X of their coordinates has the following form:

$$\begin{bmatrix} * & \dots & * & * & \dots & * & \dots & * & \dots \\ 0 & \dots & 0 & * & \dots & * & \dots & * & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & * & \dots \end{bmatrix},$$
$$\underbrace{\quad}_{K_0} \qquad \underbrace{\quad}_{K_1} \qquad \underbrace{\quad}_{K_n}$$

where the bottom most entry in each column that is indicated by a "*" is non-zero. Obviously the minimum $N(\mathbf{I})$ occurs when

$$I(i) = (\underbrace{0,\dots,0}_{K_0},\underbrace{1,\dots,1}_{K_1},\dots,\underbrace{n,\dots,n}_{K_n})$$

$$l\sum_{i=0}^{n} r_i \kappa_i \ .$$

Now note that every vector $r = (r_0,\dots,r_n)$ satisfying (*) can be written as a linear combination of the vectors

$$r_d = (r_{d,1},\dots,r_{d,n}) = (\underbrace{n-d,\dots,n-d}_{d+1},-(d+1),\dots,-(d+1))$$

$d = 0,\dots,n$, with positive coefficients. This shows that it is enough to check (**) for each λ defined by $r = r_d$ for some $d < n$. We find that

$$N(l)/l = \sum_{i=0}^{n} r_{d,i}\kappa_i = (n-d)\sum_{i=0}^{d}\kappa_i - (d+1)\sum_{i=d+1}^{n}\kappa_i =$$

$$= (n-d)\sum_{i=0}^{d}\kappa_i - (d+1)(m-\sum_{i=0}^{d}\kappa_i) =$$

$$= (n+1)\sum_{i=0}^{d}\kappa_i - m(d+1).$$

Thus (**) holds if and only if

$$\sum_{i=0}^{d}\kappa_i \le m(d+1)/(n+1) \qquad \text{for } d = 0,\dots,n-1.$$

It remains to observe that the maximal number of points among the x^{i}'s which span a projective subspace of dimension $\le d$ is equal to $\sum_{i=0}^{d}\kappa_i$. Thus (**) holds if and only if the condition of the theorem is satisfied. This proves the theorem.

Corollary.

$$(\mathbb{P}_n^m)^{ss} = (\mathbb{P}_n^m)^s \Leftrightarrow m \text{ and } n+1 \text{ are coprime.}$$

In particular, P_n^m is nonsingular in this case.

Remark 1. Assume $m \le n$. Then

$$\dim \langle x_1,\dots,x_{m-1}\rangle + 1 \le m-1 < (m-1)(m+1)/m \le (m-1)(n+1)/m < n+1.$$

This shows that

$$(\mathbb{P}_n^m)^{ss} = \emptyset \quad \text{if } m \le n.$$

This agrees with Remark 2 from n°2 of Chapter I.

Similarly, we see that

$$(\mathbb{P}_n^m)^{s} = \emptyset \quad \text{if } m = n+1.$$

3. Most special point sets.

Here we describe the image $\bar{\mathcal{D}}$ of

$$\mathcal{D} = (\mathbb{P}_n^m)^{ss} \setminus (\mathbb{P}_n^m)^s$$

in P_n^m.

Let $x = (x^1,\dots,x^m) \in \mathcal{D}$. Then x contains $\kappa < m$ points which span a subspace of dimension

$$d_1 = \frac{\kappa(n+1)}{m} - 1.$$

Choose such a subset $\{x^{i_1},\dots,x^{i_{\kappa_1}}\}$ with minimal possible d_1. Let $\{x^{i_1},\dots,x^{i_{\kappa_1}},\dots,x^{i_{\kappa_1+\kappa_2}}\}$ be the subset containing $\{x^{i_1},\dots,x^{i_{\kappa_1}}\}$ and spanning a subspace of minimal possible dimension $d_1+d_2-1 \geq d_1$ that satisfies $(\kappa_1+\kappa_2)(n+1) = m(d_1+d_2)$. Continuing in this way we will be able to find an element $g \in G$ and a permutation σ of the set $(x^1,\dots,x^m)$ such that the coordinate matrix of the point set $\sigma \cdot g \cdot x$ has the form:

$$\begin{bmatrix} I_{d_1} & X_1^1 & 0 & X_1^2 & \dots & 0 & X_1^j \\ 0 & 0 & I_{d_2} & X_2^2 & \dots & 0 & X_2^j \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & I_{d_j} & X_j^j \end{bmatrix}$$

where $\kappa_i = md_i/(n+1)$, X_i^j is a $d_i \times (\kappa_j - d_j)$-matrix, $i \leq j$, with no column zero when $i = j$, and I_{d_j} is the identity matrix of order d_j. We say that x is of type $(d_1,\dots,d_j)$ if its coordinate matrix is of the above form. We extend this definition by assigning the type $(n+1)$ for every stable point set. Clearly $n+1 = d_1+d_2+\dots+d_j$, i.e. $(d_1,\dots,d_j)$ is a partition of n+1. Evidently the type of x is not defined uniquely.

Let us see how the 1-parameter subgroup $\lambda_d : k^* \to G$ defined by the vector r_d from the proof of Theorem 1 acts on x. Observe that

$$\lim_{\alpha \to 0} \lambda_d(\alpha) \cdot x \in (\mathbb{P}_n^m)^{ss}$$

if and only if

$$d = D_t := \sum_{i=1}^{t} d_i$$

for some $t \leq j$. In this case the specialization is a point set defined by the matrix

$$\begin{bmatrix} I_{d_1} & X_1^1 & \dots\dots & 0 & X_1^t & 0 & 0 & \dots\dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots\dots & I_{d_t} & X_t^t & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots\dots\dots & 0 & 0 & I_{d_{t+1}} & X_{t+1}^{t+1} & \dots & 0 & X_{t+1}^j \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots\dots & 0 & 0 & 0 & 0 & \dots & I_{d_j} & X_j^j \end{bmatrix}$$

The stabilizer of this point set contains the subgroup of transformations of the form

$$\begin{bmatrix} \alpha I_{D_t} & 0 \\ 0 & \beta I_{D_j - D_t} \end{bmatrix}.$$

Therefore the orbit of such point set is of dimension smaller than the dimension of the general fibre of the projection Φ: $(\mathbb{P}_n^m)^{ss} \to P_n^m$.

We say that a point set is <u>special</u> if its stabilizer is of positive dimension. The orbit of such a point set is called special too. It is clear that every special orbit is contained in the closure of an orbit of some non-special point set.

Applying all the 1-parameter subgroups $\lambda_{D_l}(k^*)$, we can specialize x further to obtain a point set with coordinate matrix of the form:

$$\begin{bmatrix} I_{d_1} & X_1^1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & I_{d_2} & X_2^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & I_{d_j} & X_j^j \end{bmatrix}.$$

This will be called a <u>most special point set of type</u> $(d_1,\dots,d_j)$. It is easy to see that a most special point set cannot be specialized further. Its orbit is closed and is of dimension:

$$\dim G - \dim G_x = n(n+2)-j+1.$$

Every fibre of Φ: $(\mathbb{P}_n^m)^{ss} \to P_n^m$ over $\mathcal{B}$ contains the orbit of some most special point set (a <u>most special orbit</u>). We extend this definition by also calling the orbits of stable point sets most special.

Let $\mathbf{d} = (d_1,\dots,d_j)$ be a partition of n+1. We call it <u>admissible with respect to m</u>

if $\kappa_i = d_i m/(n+1)$ is an integer for each $i = 1,\dots,j$. The partition $d = (n+1)$ is always admissible and is called trivial. For each admissible partition of n+1 with respect to m, let $L_1,\dots,L_j$ be disjoint subspaces of $\mathbb{P}_n$ of dimension $d_1-1,\dots,d_j-1$ respectively, and let $U_d(L_1,\dots,L_j)$ be the image of the natural map

$$(L_1^{\kappa_1})^s \times \dots \times (L_j^{\kappa_j})^s \to \mathbb{P}_n^m$$

and U_d be the union of all subsets $U_d(L_1,\dots,L_j)$ and their images under permutations of the factors. It follows from the above discussion that U_d is equal to the union of most special orbits of type **d**. It is easy to see that

$$\dim\Phi(U_d) = \sum_{i=1}^{j} \dim((L_1^{\kappa_i})^s/PGL(d_i)) =$$

$$= \sum_{i=1}^{j} (d_i-1)(\kappa_i-d_i-1) = j-m+(\sum_{i=1}^{j} d_i^2)(m-n-1)/(n+1).$$

Note that $\Phi(U_d)$ consists of several components permuted under the natural action of Σ_m in $(\mathbb{P}_n^m)^{ss}$. Moreover

$$U_{(n+1)} = (\mathbb{P}_n^m)^s \setminus \mathcal{D} = \bigcup_{d \neq (n+1)} U_d.$$

Note that a non-trivial admissible partition of n+1 with respect to m exists if and only if m is not coprime to n+1. This agrees with the Corollary to Theorem 1.

Theorem 2. P_n^m is a normal rational variety of dimension $n(m-n-2)$ if $m \geq n+2$ and dimension zero if $m = n+1$. Its singular locus is contained in $\bar{\mathcal{D}}$.

Proof. It is well known that the ring $\bigoplus_{\kappa=0}^{\infty} \Gamma(\mathbb{P}_n^m, \mathcal{L}^{\otimes\kappa})$ is normal (it follows from the fact that the Segre and Veronese varieties are projectively normal). By a standard argument this implies that the ring of invariants R_n^m is normal, and

$$P_n^m = \mathrm{Proj}(R_n^m)$$

is normal. We know that, if $m \geq n+2$,

$$\dim P_n^m = \dim \Phi((\mathbb{P}_n^m)^s) = \dim \Phi(U_{(n+1)}) = n(m-n-2),$$

and P_n^m is a point if $m = n+1$.

The assertion about the singularities of P_n^m and its rationality follows from a stronger result asserting that $P_n^m \setminus \bar{\mathcal{D}}$ is covered by open subsets each of which is isomorphic to an open $U \subset \mathbb{A}_{n(m-n-2)}$. To see this we note that a point set $x = (x^1,\dots,x^m) \in (\mathbb{P}_n^m)^s$ cannot be separated by two disjoint linear subspaces. That is, there do not exist disjoint linear projective subspaces L' and L'' of $\mathbb{P}_n$ such that every x_i lies in either L' or L''. In fact, if this happens, after a permutation of the points, the coordinate matrix of x looks like

$$\begin{bmatrix} X_1 0 \\ 0 X_2 \end{bmatrix}.$$

This easily implies that $\dim G_x > 0$, and hence x is not stable. Thus we can choose $n+1$ points x^i which are not in one hyperplane, say $x^1,\dots,x^{n+1}$. Without loss of generality we may assume that the coordinate matrix of $x^1,\dots,x^{n+1}$ is equal to the identity matrix I_{n+1}. Now for each k between 2 and $m-n$ we let S_k' be the set of integers i such that the points $x^1,\dots,\hat{x}^i,\dots,x^{n+1},x^{n+k}$ span $\mathbb{P}_n$. In other words,

$$S_k' = \{i\in\{0,\dots,n\}:\ x_i^{(n+k)} \neq 0\},$$

where $x^{n+k} = (x_0^{(n+k)},\dots,x_n^{(n+k)})$. It is obvious that $\{0,1,\dots,n\}$ cannot be separated into two disjoint subsets I' and I'' such that every S_k' is contained in I' or I''. Thus we can find a suitable set of subsets $S_k \subset S_k'$ such that

(i) $\cup S_k = \{0,\dots,n\}$;

(ii) $S_i\cap(S_{i-1}\cup\dots\cup S_2)$ consists of one integer, for $3 \le i \le m-n$.

Let U be the open subset of $\mathbb{P}_n^m$ defined by

$$\mathbb{P}_n = \langle x^1,\dots,\hat{x}^i,\dots,x^{n+1},x^{n+k}\rangle \text{ for all } i\in S_i,\ k = 2,\dots,m-n,$$

$$\mathbb{P}_n = \langle x^1,\dots,x^{n+1},\rangle.$$

There exists a unique $g\in G$ such that for every $x\in U$ the coordinate matrix of $g\cdot x$ has the following form:

$$[I_{n+1}\ X],$$

where for each k, the k-th column of X has 1 as the entries in the rows whose indices are from S_k. For example, if $\{S_2,\dots,S_{m-n}\} = \{\{0,\dots,n\},\{n\},\dots,\{n\}\}$, the coordinate matrix of $g\cdot x$ must look like

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 & * & \dots & * \\ 0 & 1 & 0 & \dots & 0 & 1 & * & \dots & * \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

To see this we observe that after reducing the points $x^1,\dots,x^{n+1}$ to the points $(1,0,\dots,0),\dots,(0,\dots,0,1)$ by a suitable $g\in G$, there are still non-trivial transformations left in G which fix $x^1,\dots,x^{n+1}$. They are the homotheties

$$(t_0,\dots,t_n) \to (\lambda_0 t_0,\dots,\lambda_n t_n), \qquad \lambda_0\dots\lambda_n = 1.$$

Thus we may use them to normalize the j-th coordinate of x_{n+2}, $j\in S_2$. Then we normalize the j-th coordinate of x^{n+3} for $j\in S_2\cap S_3$ by a projective factor. Next use again the homotheties to normalize the remaining i-th coordinates of x^{n+3} for $i\in S_3$, and so on. Clearly this defines g uniquely and defines a G-equivariant isomorphism

$$G\times\mathbf{A}_{n(m-n-2)} \cong U,$$

where G acts on G by left multiplication and identically on the affine space. Of course, the affine space $\mathbf{A}_{n(m-n-2)}$ is the space of all non-normalized coordinates of the points from $g\cdot x$. This shows that $(\mathbb{P}^m_n)^s$ is covered by the invariant open subsets $U\cap(\mathbb{P}^m_n)^s$ whose quotients are open in $\mathbf{A}_{n(m-n-2)}$. This proves the assertion. Moreover it shows that the projection

$$\Phi: (\mathbb{P}^m_n)^s \to P^m_n\backslash\mathfrak{D}$$

is a principal fibre bundle of G over $P^m_n\backslash\mathfrak{D}$ in the sense of **[Mu 1]**, Definition 0.10.

Remark 2. It is convenient to use $\mathbf{P}_{n(m-n-2)}$ as a birational model of P^m_n in such a way that the factor projection Φ identifies the set $U_0 \subset \mathbf{P}^m_n$ of points with the coordinate matrix:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 & t_0^{(n+3)} & \dots & t_0^{(m)} \\ 0 & 1 & 0 & \dots & 0 & 1 & t_1^{(n+3)} & \dots & t_1^{(m)} \\ 0 & \dots & 0 & 1 & 0 & 1 & t_{n-1}^{(n+3)} & \dots & t_{n-1}^{(m)} \\ 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

with the open subset $\{(t_0,\dots,t_{n(m-n-2)})\in\mathbf{P}_{n(m-n-2)}: t_{n(m-n-2)} = 1\}$ by assigning to the point set $x = (x^1,\dots,x^m)$ the point

$$(t_0^{(n+3)}, t_1^{(n+3)},\dots,t_{n-1}^{(n+3)},\dots,t_0^{(m)},\dots,t_{n-1}^{(m)},1).$$

Remark 3. One can also show that P^m_n is a Cohen-Macaulay variety with rational singularities. This follows from general properties of the orbit spaces under reductive groups or from the fact that P^m_n is a toroidal embedding (M.Hochster).

4. Examples.

Example 1 (n=1, m is odd).

In this case

$$(\mathbb{P}_1^m)^s = (\mathbb{P}_1^m)^{ss}$$

and P_1^m is a nonsingular rational variety of dimension m-3.

For instance, if m =5, P_1^5 is isomorphic to a Del Pezzo surface of degree 5, that is, a surface obtained by blowing up 4 points in the projective plane.

Example 2 (n=1,m=2κ is even).

A point set $x = \{x^1,\ldots,x^{2\kappa}\}$ belongs to $\mathcal{B} = (\mathbb{P}_1^m)^{ss} \setminus (\mathbb{P}_1^m)^s$ if and only if exactly κ of the of the points x^i coincide. The fibre of $\Phi^{-1}(\Phi(x))$ for such a point set x contains a most special orbit of type (1,1). It consists of point sets x such that $x^{i_1} = \ldots = x^{i_\kappa}$, $x^{j_1} = \ldots = x^{j_\kappa}$ for the complementary subsets $\{i_1,\ldots,i_\kappa\}$, $\{j_1,\ldots,j_\kappa\}$ of $\{1,\ldots,m\}$. The subavriety $\mathcal{B}$ is the union of $\frac{1}{2}\binom{m}{\kappa}$ points, each of which is a singular point of P_n^m. For instance, if m = 6, P_1^6 is isomorphic to the Segre cubic primal V_3 with $10 = \frac{1}{2}\binom{6}{3}$ nodes. Their coordinates are (±1,±1,±1,±1,±1,±1), where exactly half of them are positive. Here we assume that the equation of the cubic is taken in its Σ_6-invariant form (cf. p.17).

Example 3 (n = 2, m = 6).

There are three different partitions of n+1 = 3: **d** = (3),(1,2), and (1,1,1). All of them are admissible with respect to 6. The first corresponds to stable points. The second one corresponds to most special orbits of point sets $x = (x^1,\ldots,x^6)$, where two of the x^i's coincide and the four remaining points lie on a line disjoint from the two coinciding points. The images of these orbits define $15 = \binom{6}{2}$ one-dimensional components of $\mathcal{B}$. We denote them by l_{ij}. Each of them is isomorphic to $P_1^4 \setminus \mathcal{B} = \mathbb{P}_1 \setminus \{0,1,\infty\}$.

The third partition (1,1,1) corresponds to most special orbits of point sets which contain 3 disjoint pairs of coinciding points $\{x_i,x_j\}$, $\{x_\kappa,x_l\}$, and $\{x_m,x_n\}$. The images of these orbits give

$$15 = \frac{1}{3!}\binom{6}{2}\binom{4}{2}$$

points in $\mathcal{B}$. We denote them by $x_{ij,\kappa l,mn}$. It is easy to verify that each line l_{ij}

contains in its closure exactly 3 points $x_{ij,kl,mn}$. Moreover, each point $x_{ij,kl,mn}$ is contained in the closure of the 3 lines l_{ij}, l_{kl}, and l_{mn}.

Note that most special point sets lie on a curve of degree 2. Hence

$$\bar{\mathcal{S}} \subset V_4 = (t_5 = 0) \subset P_2^6.$$

The lines l_{ij} are the double lines of the level 2 modular quartic 3-fold V_4. The points $x_{ij,kl,mn}$ are the triple points of V_4. The union of the singular lines $\bar{\mathcal{S}}$ is the boundary of $\bar{\mathfrak{a}}_2(2)$ and is described for example in **[vdG]**.

III. ASSOCIATED POINT SETS.

In this chapter we describe a duality between point sets known in classical literature as <u>association</u>. It establishes an isomorphism of algebraic varieties:

$$a_{m,n}: P_n^m \to P_{m-n-2}^m.$$

satisfying

$$a_{m,m-n-2} = a_{m,n}^{-1}.$$

From now on we will always assume that

$$m \geq n+2.$$

1. <u>The association</u>.

Let $x \in \mathbb{P}_n^m$ and X be its matrix of projective coordinates. A point set $y \in \mathbb{P}_{m-n-2}^m$ is said to be <u>associated</u> to x if its coordinate matrix Y satisfies

$$X \cdot \Lambda \cdot {}^tY = 0$$

for some diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_m)$ with all $\lambda_i \neq 0$. Note that the relation " q is associated to p" is symmetric and is preserved under the G-actions.

Another way of viewing the definition of association is to consider X and ${}^tY$ as linear transformations

$$X: K^m \to K^{n+1}, \quad {}^tY: K^{m-n-1} \to K^m.$$

Then $x \in \mathbb{P}_n^m$ and $y \in \mathbb{P}_{m-n-2}^m$ are associated if and only if they have coordinate matrices X and Y, respectively, such that the following sequence is exact:

$$0 \to K^{m-n-1} \xrightarrow{{}^tY} K^m \xrightarrow{X} K^{n+1} \to 0.$$

Note also that for every permutation $\sigma \in \Sigma_m$ the point sets $x = (x^1,\dots,x^m)$ and

$y = (y^1,\dots,y^m)$ are associated if and only if $\sigma x = (x^{\sigma(1)},\dots,x^{\sigma(m)})$ and $\sigma y = (y^{\sigma(1)},\dots,y^{\sigma(m)})$ are associated.

We will see later that a point set associated to a stable point set x is stable and its G-orbit is determined uniquely. Thus there is an isomorphism:

$$a_{m,n}: (\mathbb{P}_n^m)^s/PGL(n+1) \to (\mathbb{P}_{m-n-2}^m)^s/PGL(m-n-1).$$

Example 1. Let $x = (x^1,\dots,x^4)$ be a stable point set in $\mathbb{P}_1^4$. A point set $y \in \mathbb{P}_1^4$ is associated to x if and only if it is projectively equivalent to x. Let us verify this. By the stability criterion, x consists of distinct points. Replacing x by a projectively equivalent point set we may assume that the coordinate matrix X of x has the form:

$$\begin{bmatrix} 1&0&1&1\\ 0&1&1&a \end{bmatrix}.$$

Assume that $y = (y^1,\dots,y^4)$ is associated to x. If $y^1 = y^2$, we can choose a coordinate matrix of y in the form

$$\begin{bmatrix} 1&1&b&d\\ 0&0&c&e \end{bmatrix}$$

and find nonzero $\lambda_1, \lambda_2, \lambda_3$, and λ_4 satisfying

$$\lambda_1+\lambda_3 b+\lambda_4 d = 0,$$

$$\lambda_3 c+\lambda_4 e = 0,$$

$$\lambda_2+\lambda_3 b+\lambda_4 ad = 0,$$

$$\lambda_3 c+\lambda_4 ae = 0.$$

Computing the determinant of the coefficent matrix of this sytem of linear equations in λ_i, we find that $a = 1$. This contradiction shows that $y^1 \neq y^2$. Similarly we verify that $y^i \neq y^j$ for any $i \neq j$. Thus y must be stable. Then, applying a projective transformation to y we may assume that its coordinate matrix has the form

$$\begin{bmatrix} 1&0&1&1\\ 0&1&1&b \end{bmatrix}$$

and obtain the following system of equations for the λ_i's:

$$\lambda_1+\lambda_3+\lambda_4 = 0,$$
$$\lambda_3+\lambda_4 b= 0,$$
$$\lambda_3+\lambda_4 a= 0,$$
$$\lambda_2+\lambda_3+\lambda_4 ab = 0.$$

This implies that $a = b$, i.e. $x = y$. Moreover, the system can be solved in this case by taking $\lambda_4 = 1$, $\lambda_3 = -a$, $\lambda_2 = a(1-a)$, $\lambda_1 = a-1$, all non zero

Note that the assertion fails if x is not stable. For example, if $x^1 = x^2 \neq x^3 \neq x^4$, we can find an associated set y to x with $y^1 \neq y^2 \neq y^3 = y^4$.

The main algebraic property of associated point sets is that the complementary minors of maximal order in their suitable coordinate matrices are proportional (see Theorem 1 below). This implies that the images of associated points in the spaces P^m_n and P^m_{m-n-2} are equal after we establish a certain isomorphism between the graded algebras R^m_n and R^m_{m-n-2}. Let us do first the latter.

Let $\tau = (\tau_{i,j})$ be a tableau of degree kl and weight wk. We denote its rows by $\tau_i = (\tau_{i\,1},\dots,\tau_{i\,n+1})$, $i = 1,\dots,wk$. We view it as an ordered subset of $\{1,\dots,m\}$.

Define the <u>associated tableau</u> $A(\tau)$ by

$$A(\tau)_i = \{1,\dots,m\}\setminus\tau_{wk-i+1}$$

reordered in the increasing order. Clearly $A(\tau)$ is of weight wk and of degree kl' with $l' = w-l$. If

$$\tau_{wk-i+1} = (\tau_{wk-i+1\,1},\dots,\tau_{wk-i+1\,n+1}),$$

we have

$$A(\tau)_i = (1,\dots,\tau_{wk-i+1\,1}-1,\tau_{wk-i+1\,1}+1,\dots,\tau_{wk-i+1\,n+1}-1,\tau_{wk-i+1\,n+1}+1,\dots,m),$$
$$A(\tau)_{i+1} = (1,\dots,\tau_{wk-i\,1}-1,\tau_{wk-i\,1}+1,\dots,\tau_{wk-i\,n+1}-1,\tau_{wk-i\,n+1}+1,\dots,m),$$

hence $A(\tau)$ is standard as soon as τ is.

Note that A is involutive, i.e.

$$A^2 = \text{identity}.$$

The association $\tau \to A(\tau)$ extends to the corresponding monomials

$$A(\mu_\tau) = \mu_{A(\tau)}$$

and by linearity to an isomorphism of graded vector spaces

$$A_{n,m}: R_n^m \cong R_{m-n-2}^m.$$

Theorem 1. $A_{n,m}$ is an isomorphism of graded algebras

$$A_{n,m}: R_n^m \cong R_{m-n-2}^m.$$

Proof. We have to verify that

$$A(\mu_\tau)A(\mu_{\tau'}) = A(\mu_\tau\mu_{\tau'}).$$

Note that the R.H.S. can be defined by writing the product as a sum of standard monomials and extending A by linearity. The L.H.S. can be written also as a sum of standard monomials. Thus we achieve our goal if we can show that the steps in the algorithm of straightening monomials are the same for both μ_τ and $\mu_{A(\tau)}$. This verification is rather tedious and we skip it (see **[Or1]**).

Corollary. The isomorphism $A_{n,m}: R_n^m \cong R_{m-n-2}^m$ induces an isomorphism

$$a_{n,m}: P_n^m \cong P_{m-n-2}^m.$$

We will call this isomorphism the association isomorphism.

Example 2. Let n = 1, m = 4. Under the association

$$\mu\begin{bmatrix}12\\34\end{bmatrix} \to \mu\begin{bmatrix}12\\34\end{bmatrix}$$

$$\mu\begin{bmatrix}13\\24\end{bmatrix} \to \mu\begin{bmatrix}13\\24\end{bmatrix}$$

Thus $A_{1,4}$: $R_1^4 \cong R_1^4$ is the identity isomorphism, hence $a_{1,4}$: $P_1^4 \cong P_1^4$ is the identity isomorphism. This is in accord with Example 1.

Example 3. Let n = 1, m = 6. We use the notation of Example 2 from Chapter 1. Under the association of standard monomials:

$$t_0 = (12)(34)(56) \to (1234)(1256)(3456) = z_0,$$

$$t_1 = (13)(24)(56) \to (1234)(1356)(2456) = z_1,$$

$$t_2 = (12)(35)(46) \to (1235)(1246)(3456) = z_2,$$

$$t_3 \; = \; (13)(25)(46) \; \rightarrow \; (1235)(1346)(2356) \; = \; z_3,$$

$$t_4 \; = \; (14)(25)(36) \; \rightarrow \; (1235)(1346)(2356) \; = \; z_4.$$

Both of the varieties P_1^6 and P_3^6 are isomorphic to the same cubic hypersurface in $\mathbb{P}_4$ (the Segre cubic primal). Note that if we fix 5 points in general position among $(y^1,\dots,y^6)\in P_3^6$ and let the other vary, say y^6, the functions z_i represent quadrics in $\mathbb{P}_3$ passing through the points $y^1,\dots,y^5$. These quadrics map $\mathbb{P}_3$ birationally onto a cubic hypersurface in $\mathbb{P}_4$ isomorphic to the Segre cubic primal V_3.

Example 4 (n = 2, m = 6).

In the notation of Example 3 from Chapter 1, we find that under the association isomorphism $A_{2,6}$: $R_2^6 \rightrightarrows R_2^6$

$$t_i \rightarrow t_i \;, \; i = 0,\dots,4,$$

$$y_1 \; = \; (123)(145)(246))(356) \; \rightarrow \; y_2=(124)(135)(236)(456).$$

Thus the association involution

$$a_{2,6}\colon \; P_2^6 \; \rightarrow P_2^6$$

is the cover involution of the projection $P_2^6 \rightarrow \mathbb{P}_4$. In particular, its locus of fixed points is the divisor parametrizing point sets lying on a curve of degree 2. If $p \neq 2$, it is isomorphic to the level 2 modular quartic 3-fold V_4.

Theorem 2. Let $x\in(\mathbb{P}_n^m)^{ss}$ and $y\in(\mathbb{P}_{m-n-2}^m)^{ss}$ be associated point sets. Then

$$a_{n,m}(\Phi(x)) \; = \; \Phi(y).$$

The proof will follow from the next two lemmas.

Lemma 1. Let $x\in\mathbb{P}_n^m$ and $y\in\mathbb{P}_{m-n-2}^m$ be associated point sets. One can choose the coordinate matrices X and Y of x and y, respectively, such that

$$X_{i_1\cdots i_{n+1}}Y_{j_1\cdots j_{m-n-1}} \pm X_{i_1'\cdots i_{n+1}'}Y_{j_1'\cdots j_{m-n-1}'} = 0,$$

where $\{i_1,\dots,i_{n+1}\},\{j_1,\dots,j_{m-n-1}\}$ and $\{i_1',\dots,i_{n+1}'\},\{j_1',\dots,j_{m-n-1}'\}$ are any two pairs of complementary subsets of $\{1,\dots,m\}$, and $X_{i_1\cdots i_{n+1}}$, $Y_{j_1\cdots j_{m-n-1}},\dots$ are the corresponding minors of X and Y composed of the columns indexed by these subsets. Moreover, the sign + must be taken if and only if $\{i_1,\dots,i_{n+1}\}$ and $\{i_1',\dots,i_{n+1}'\}$ differ by an odd number of entries.

<u>Proof</u>. Obviously we may assume that the sets $\{i_1,\dots,i_{n+1}\}$ and $\{i_1',\dots,i_{n+1}'\}$ differ only in one element. Also, after reindexing, we may assume that

$$\{i_1,\dots,i_{n+1}\} = \{m-n-1, m-n+1,\dots,m\},$$

$$\{i_1',\dots,i_{n+1}'\} = \{m-n, m-n+1,\dots,m\}.$$

For every $1 \le i \le m-n$, we expand the minor $X_{i,m-n+1,\dots,m}$ along the first column to obtain

$$X_{i,m-n+1,\cdots,m} = \sum_{k=0}^{n} M_k x_k^{(i)}.$$

Similarly we have for every $m-n-1 \le i \le m$

$$Y_{1,\dots,m-n-2,i} = \sum_{j=0}^{m-n-2} N_j y_j^{(i)}.$$

Choose X and Y in such a way that

$$X\cdot {}^tY = 0.$$

Then

$$0 = \sum_{i=1}^{m} x_k^{(i)} y_j^{(i)}, \qquad k = 0,\dots,n,\ j = 0,\dots,m-n-2.$$

implies

$$0 = \sum_{k=0}^{n} \sum_{j=0}^{m-n-2} \sum_{i=1}^{m} M_k x_k^{(i)} y_j^{(i)} N_j =$$

$$= \sum_{i=1}^{m} X_{i,m-n+1,\cdots,m} Y_{1,\dots,m-n-2,i}.$$

In this sum only the terms corresponding to $i = m-n-1$ and $m-n$ are non zero. Thus we obtain

$$0 = X_{m-n-1,m-n+1,\cdots,m} Y_{1,\dots,m-n-2,m-n-1} + X_{m-n,m-n+1,\cdots,m} Y_{1,\dots,m-n-2,m-n}.$$

Lemma 2. Let $x \in (\mathbb{P}_n^m)^{ss}$ and $y \in (\mathbb{P}_{m-n-2}^m)^{ss}$ be associated point sets. Assume that R_n^m is generated by $(R_n^m)_k$. Let μ_τ be a monomial of weight wk and degree kl that does not vanish on x. Then the monomial $\mu_{A(\tau)}$ does not vanish on y, and for any other monomial τ' weight wk and degree kl we have

$$(\mu_{\tau'}/\mu_\tau)(x) = (\mu_{A(\tau')}/\mu_{A(\tau)})(y).$$

<u>Proof</u>. Clearly we may assume that τ and τ' are both semi-standard. Note that any

semi-standard tableau may be obtained from a fixed one by a series of the operations $\tau \to \tau'$, where

$$\tau_{i_0}' = (\tau_{i_0,1},\dots,\tau_{i_1,j_1},\dots,\tau_{i_0,n+1}),$$
$$\tau_{i_1}' = (\tau_{i_1,1},\dots,\tau_{i_0,j_0},\dots,\tau_{i_1,n+1}),$$
$$\tau_i' = \tau_i\ ,\ i \neq i_0, i_1,$$

and τ_{i_1,j_1} appears in slot j_0, τ_{i_0,j_0} appears in slot j_1, followed by row straightening. Thus it suffices to show that

$$(*)\qquad \frac{\mu_{\tau'}(x)}{\mu_\tau(x)} = \frac{\mu_{A(\tau')}(y)}{\mu_{A(\tau)}(y)}$$

for such pairs τ and τ'. Note that, if $\mu_\tau(x) \neq 0$, there exists τ' such that $\mu_{A(\tau')}(y) \neq 0$ because $y \in (\mathbb{P}^m_{m-n-2})^{ss}$ and $(R^m_{m-n-2})_K$ generates R^m_{m-n-2}. This shows that $\mu_{A(\tau)}(y) \neq 0$ and checks our first assertion.

We have in the notation of Lemma 1:

$$\frac{\mu_{\tau'}(x)}{\mu_\tau(x)} = \frac{(\mu_{\tau_{i_0}}' \cdot \mu_{\tau_{i_1}}' \cdot \prod_{i \neq i_0, i_1} \mu_{\tau_i}')(x)}{(\mu_{\tau_{i_0}} \mu_{\tau_{i_1}} \prod_{i \neq i_0, i_1} \mu_{\tau_i})(x)} =$$

$$= \frac{X_{\tau'_{i_0 1} \cdots \tau'_{i_0 n+1}} X_{\tau'_{i_1 1} \cdots \tau'_{i_1 n+1}}}{X_{\tau_{i_0 1} \cdots \tau_{i_0 n+1}} X_{\tau_{i_1 1} \cdots \tau_{i_1 n+1}}} \prod_{i \neq i_0, i_1} \frac{\mu_{\tau_i}'(x)}{\mu_{\tau_i}(x)} =$$

$$= \frac{Y_{A(\tau')_{wk-i_0+1\,1} \cdots A(\tau')_{wk-i_1+1\,n+1}} Y_{A(\tau')_{wk-i_1+1\,1} \cdots A(\tau')_{wk-i_1+1\,n+1}}}{Y_{A(\tau)_{wk-i_0+1\,1} \cdots A(\tau)_{wk-i_0+1\,n+1}} Y_{A(\tau)_{wk-i_1+1\,1} \cdots A(\tau)_{wk-i_1+1\,n+1}}} \prod_{i \neq wk-i_0, wk-i_1} \frac{\mu_{A(\tau')_i}(y)}{\mu_{A(\tau)_i}(y)} =$$

$$= \frac{\mu_{A(\tau')}(y)}{\mu_{A(\tau)}(y)}\ .$$

Proposition 1. Let $x \in (\mathbb{P}^m_n)^{ss}$ be of type $(d_1,\dots,d_j)$. There exists an associated point set $y \in (\mathbb{P}^m_{m-n-2})^{ss}$ of type $(d_1',\dots,d_j')$, where

$$d_i' = \kappa_i - d_i = d_i(m-n-1)/(n+1)).$$

<u>Proof</u>. After the reordering the points of x, we may assume that the coordinate matrix of x is of the form

$$\begin{bmatrix} I_{d_1} & 0 & 0 & \dots & 0 & X_1^1 & X_1^2 & \dots & X_1^j \\ 0 & I_{d_2} & 0 & \dots & 0 & 0 & X_2^2 & \dots & X_2^j \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_{d_j} & 0 & \dots & 0 & X_j^j \end{bmatrix}$$
$$d_1' \; d_2' \dots d_j',$$

where the numbers at the bottom of the matrix indicate the number of columns in the submatrices X_j^i. We can find a matrix Y of order $m\times(m-n-1)$ satisfying $X\cdot{}^tY = 0$ and having the form

$$\begin{bmatrix} Y_j^1 & Y_j^2 & \dots & Y_j^j & 0 & \dots & 0 & 0 & I_{d_j'} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ Y_2^1 & Y_2^2 & \dots & 0 & 0 & I_{d_2'} & 0 & \dots & 0 \\ Y_1^1 & 0 & \dots & 0 & I_{d_1'} & 0 & 0 & \dots & 0 \end{bmatrix}$$
$$d_1 \; d_2 \dots d_j$$

where

$${}^tY_j^i + X_i^j = 0$$

This shows that Y is the coordinate matrix of an associated point set which is of type $(d_1',\dots,d_j')$.

Corollary 1.

$$a_{n,m}(P_n^m\setminus\mathcal{D}) = P_{m-n-2}^m\setminus\mathcal{D}.$$

Corollary 2. For every $x\in(\mathbb{P}_n^m)^s$ there exists a unique associated point set y up to projective equivalence. Moreover $y\in(\mathbb{P}_{m-n-2}^m)^s$.

Remark 1. In the notation of Remark 2 from Chapter II, let $x\in U_0\subset\mathbb{P}_n^m$ be identified with a point

$$(x_0^{(n+3)},x_1^{(n+3)},\dots,x_{n-1}^{(n+3)},\dots,x_0^{(m)},\dots,x_{n-1}^{(m)},1)$$

of $\mathbb{P}_{n(m-n-2)}$. Similarly, we may identify the subset

$V_0 = \{y\in\mathbb{P}_{m-n-2}^m : y^{(k)}=(0,\dots,1,\dots,0),\ k=n+2,\dots,m,\ y^{(n+1)}=(1,\dots,1),\ y_{m-n-2}^{(n)},\dots,y_{m-n-2}^{(1)}\neq 0\}$

with $\mathbb{P}_{n(m-n-2)}$ by sending $y\in V_0$ to the point

$$z(y) = (y_{m-n-3}^{(1)}, \dots, y_{m-n-3}^{(n)}, \dots, y_0^{(1)}, \dots, y_0^{(n)}, 1).$$

Then it is verified by direct computation that x is associated to y if and only if the corresponding points in $\mathbb{P}_{n(m-n-2)}$ are equal.

2. Geometric properties of associated point sets.

So far our definition of associated point sets was purely algebraic. In this section we look at the association from the geometric point of view. We will restrict ourselves to general point sets. A point set $x \in \mathbb{P}_n^m$ is said to be general if any subset of $k \le n+1$ points spans a $(k-1)$-dimensional linear projective space. We denote the subset of general point sets by $(\mathbb{P}_n^m)^{gen}$. It is clear that

$$(\mathbb{P}_n^m)^{gen} \subset (\mathbb{P}_n^m)^s.$$

Proposition 2. If $x=(x^1,\dots,x^{n+3}) \in (\mathbb{P}_n^{n+3})^{gen}$ is associated to $y \in (\mathbb{P}_1^{n+3})^{gen}$ then there exists a unique isomorphism from $\mathbb{P}_1$ onto a rational normal curve R_n which sends y to x (preserving the order).

Proof. Recall that a rational normal curve R_n is the image of $\mathbb{P}_1$ under a map given by the complete linear system $|\mathcal{O}_{\mathbb{P}_1}(n)|$. Counting constants we check that any set of n+3 points in $\mathbb{P}_n$ lies on a rational normal curve. The uniqueness of such a curve is clear in the case $n = 2$. The general case is reduced to this case by projecting the curve to $\mathbb{P}_2$ from a subset of n-2 points (see **[G-H]**). Let us verify the other assertion of the proposition. Note that a linear parametrization on R_n is given by a pencil of hyperplanes through any (n-1)-secant (n-2)-plane, for example the (n-2)-plane L_{n-2} that is spanned by the points $x^5,\dots,x^{n+3}$. Let

$$M(u,u') = \langle x^5,\dots,x^{n+3},u,u'\rangle$$

denote the bilinear form that vanishes whenever the line $\langle u,u'\rangle$ intersects the (n-2)-plane L_{n-2}. $M(x,u')$ is then the equation of a hyperplane on the n points $x, x^5,\dots,x^{n+2}$, unless $x \in L_{n-2}$, in which case it is identically zero. Thus M may be considered as a rational map from $\mathbb{P}_n$ onto the pencil inside the dual projective space $\mathbb{P}_n^*$ spanned by the hyperplanes through L_{n-2}. It follows that M may be represented by a $2\times(n+1)$ matrix $M = (M_{i,j})$, so that, once a basis $\{h(u')_1, h(u')_2\}$ is chosen for the pencil, we have

$$M(u,u') = \sum_{j=1}^{n+1}(h(u')_1M_{1j}u_j+h(u')_2M_{2j}u_j).$$

Now apply M to the left side of the equation $X\cdot\Lambda\cdot{}^tY = 0$ for the association of x and y, and use $M(x^i,u') = 0$ for $i = 5,\dots,n+3$ to obtain

$$\sum_{K=1}^{4}\sum_{r=1}^{n+1}M_{i,r}x_r^{(K)}\lambda_K y_j^{(K)} = 0.$$

The 2×4 matrix obtained from M•X by throwing away the zero columns can be considered as the matrix of coordinates for the projection of $x^1,\dots,x^4$ onto a line from L_{n-2}. Thus the above equality says that this projection of $(x^1,\dots,x^4)$ is associated to $(y^1,\dots,y^4)$. By Example 1 they are projectively equivalent. Since this is true for any choice of four points, the entire sets are projectively equivalent to each other (when x is considered as a subset of $R_n \cong \mathbb{P}_1$), and we are done.

In a similar way we can prove:

Proposition 3. Let $x\in(\mathbb{P}_n^m)^{gen}$ be associated to $y\in(\mathbb{P}_{m-n-2}^m)^{gen}$, and $(\bar{x}^{i_n},\dots,\bar{x}^{i_m})$ be a point set on $\mathbb{P}_1$ obtained by projecting an ordered subset $(x^{i_n},\dots,x^{i_m})$ of x from the (n-2)-plane spanned by $x^1,\dots,x^{n-1}$. Then there exists a unique isomorphism from $\mathbb{P}_1$ to a rational normal curve in $\mathbb{P}_{m-n-2}$ which sends $(\bar{x}^{i_n},\dots,\bar{x}^{i_m})$ to the ordered subset $(y^{i_n},\dots y^{i_m})$ of y.

We shall call a rational normal curve R_n a <u>basic rational normal curve</u> if it passes through a fixed general set of n+2 points (a <u>basis</u> of $\mathbb{P}_n$) which is assumed to be fixed from now on. The set of basic R_n has dimension n-1 and may be used as a model of P_1^{n+2} since the projective equivalence classes determined on each of the R_n by the n+2 points in the basis are distinct. Thus, to construct a point set y that is associated to a given point set x, we proceed as follows. First choose $y^1,\dots,y^{m-n}$ to be a basis for $\mathbb{P}_{m-n-2}$ and set, for $i = m-n+1,\dots,m$:

$$L_i = \langle x^{m-n+1},\dots,\hat{x}^i,\dots,x^m\rangle \subset \mathbb{P}_n.$$

Then find the unique basic rational normal curve R^i in $\mathbb{P}_{m-n-2}$ for which there exists an isomorphism to $\mathbb{P}_1$ which maps the basis of $\mathbb{P}_{m-n-2}$ to the projection of $x^1,\dots,x^{m-n}$ from L_i onto $\mathbb{P}_1$. Finally set $y^i\in R^i$ equal to the point which is mapped to the projection of x^i.

We may generalize the technique used to obtain the previous two propositions to provide one more property of associated sets.

Proposition 4. If $x\in(\mathbb{P}_n^m)^{gen}$ is associated to $y\in(\mathbb{P}_{m-n-2}^m)^{gen}$ then the projection of the point set $(x^1,\dots,x^{m-\kappa})$ from the plane L spanned by $x^{m-\kappa+1},\dots,x^m$ to $\mathbb{P}_{n-\kappa}$ is associated to the point set $(y^1,\dots,y^{m-\kappa})$.

Proof. Use

$$M(u,u_1',\dots,u_{n-\kappa}') = \langle u_1',\dots,u_{n-\kappa}',u,x^{m-\kappa+1},\dots,x^m\rangle$$

to denote the multilinear form that, for fixed u, vanishes when the (n-κ-1)-plane $\langle u_1',\dots,u_{n-\kappa}'\rangle$ intersects the subspace spanned by L and u. As such, M may be viewed as a map from $\mathbb{P}_n$ to the (n-κ)-dimensional linear subspace of the Grassmannian $G(n-\kappa,n+1)^* \cong G(\kappa+1,n+1)$ given by the κ-planes that pass through L. Represent M with an $(n-\kappa+1)\times(n+1)$ matrix and proceed as in Proposition 2.

3. Self-associated point sets.

Assume

$$m = 2n+2.$$

Then

$$a_n := a_{n,m} : P_n^m \to P_n^m$$

is an involution.

We set

$$S_n = \{x\in P_n^{2n+2}: a_n(x) = x\}.$$

A point set $x\in(\mathbb{P}_n^{2n+2})^{ss}$ is said to be self-associated if $\Phi(x)\in S_n$. It follows from Corollary 2 in the previous section that a stable point set is self-associated if and only if it is associated to itself.

We have already seen in Example 4 that $x\in(\mathbb{P}_2^6)^{ss}$ is self-associated if and only if it lies on a curve of degree 2, and

$$S_2 \cong V_4,$$

the level 2 modular quartic 3-fold. Note also that

$$S_1 = P_1^4.$$

We will generalize these two examples by proving a theorem of Coble that asserts that a sufficiently general point set is self-associated if and only if it imposes "one less condition on quadrics". The latter means the following.

Let Z be a 0-dimensional closed subscheme of $\mathbb{P}_n$, $n > 1$, with the ideal sheaf $\mathcal{J}_Z$. The exact sequence

$$0 \to \mathcal{J}_Z \to \mathcal{O}_{\mathbb{P}_n} \to \mathcal{O}_Z \to 0$$

defines, after twisting by $\mathcal{O}_{\mathbb{P}_n}(k)$ and passing to cohomology, an exact sequence

$$0 \to H^0(\mathbb{P}_n,\mathcal{J}_Z(k)) \to H^0(\mathbb{P}_n,\mathcal{O}_{\mathbb{P}_n}(k)) \to H^0(Z,\mathcal{O}_Z(k)) \to H^1(\mathbb{P}_n,\mathcal{J}_Z(k)) \to 0\ ,$$

where the middle map

$$r_k : H^0(\mathbb{P}_n,\mathcal{O}_{\mathbb{P}_n}(k)) \to H^0(Z,\mathcal{O}_Z(k))$$

is interpreted as the restriction of a homogeneous form of degree k to the subscheme Z. Its kernel consists of hypersurfaces of degree k that vanish at Z. We set

$$\delta(Z,k) = \dim \operatorname{Coker}(r_k) = \dim H^1(\mathbb{P}_n,\mathcal{J}_Z(k)).$$

Clearly

$$\dim H^0(\mathbb{P}_n,\mathcal{J}_Z(k)) = \dim H^0(\mathbb{P}_n,\mathcal{O}_{\mathbb{P}_n}(k)) - l(Z) + \delta(Z,k),$$

where

$$l(Z) = \dim H^0(Z,\mathcal{O}_Z)$$

is the length of Z.

If $Z = Z_{red}$, $l(Z) = \#\mathrm{Supp}(Z)$, and we expect that each point from Z imposes one condition on a hypersurface of degree k to pass through it. This shows that $\delta(Z,k)$ is the number of "extra" linearly independent hypersurfaces passing through Z.

We apply this to our situation where $Z = \{x^1,\dots,x^m\}$ is equal to $\{x\}$, and is considered as a reduced subscheme. We assume that all the points x^i's are distinct.

Remark 2. Note that $H^0(\{x\},\mathcal{O}_{\{x\}}(1)) \cong k^m$ and $H^1(\{x\},\mathcal{J}_{\{x\}}(1)) \cong k^{m-n-1}$ if $x^1,\dots,x^m$ span $\mathbb{P}_n$. The points x^i define an ordered subset in $\mathbb{P}(H^0(\{x\},\mathcal{O}_{\{x\}}(1)))$ and their projections to $\mathbb{P}(H^1(\{x\},\mathcal{J}_{\{x\}}(1)))$ define an associated set of points (see **[Ty 2]**).

We start with the following generalization of a lemma from **[Sh]**:

Lemma 3. Let $Z = \{x^1,\ldots,x^{2n+1}\}$ be a set of 2n+1 distinct points in $\mathbb{P}_n$, $n > 1$, such that any subset of $2\kappa+2$ points spans a linear projective space of dimension $> \kappa$. Then:

$$\delta(Z,2) = 0,$$

or equivalently, the dimension of the linear system of quadrics passing through Z is equal to $\frac{1}{2}n(n-1)-1$.

Proof. We prove this by induction on n. For $n = 2$ we have to show that $\{x^1,\ldots,x^5\}$ is not contained in the base locus of a pencil of conics. If it does, the pencil must contain a fixed line l and consist of reducible conics $l+l'$, where l' belongs to a pencil of lines through a point y. Thus either four of the x^i's lie on l, or two of them coincide with y. Both of these cases are excluded by the assumption of the lemma.

Assume now that the lemma is true for the sets Z of 2m+1 points in $\mathbb{P}_m$ for all $2 \le m < n$. Note that for any subset $Z' \subset Z$ we have

$$\delta(Z',2) \le \delta(Z,2) = 0$$

since completing Z' to Z will reduce the dimension of the linear system of quadrics through Z' by at most $\#Z-\#Z'$.

Let S be a set of 2n+1 points in $\mathbb{P}_n$ satisfying the assumption of the lemma. We can write

$$S = S_1 \amalg S_2,$$

where S_1 consists of some $s \ge n$ points spanning a hyperplane H. We have

$$\dim H^0(\mathbb{P}_n,\mathcal{I}_S(2)) \le \dim H^0(H,\mathcal{I}_{S_1}(2))+\dim H^0(\mathbb{P}_n,\mathcal{I}_{S_2}(1))$$

by restricting quadrics to H.

Applying the inductive assumption to $Z = S_1 \subset H \cong \mathbb{P}_{n-1}$, we have $\delta(S_1,2) = 0$. Hence

$$\text{R.H.S} \le (\tfrac{1}{2}n(n+1)-s) + n+1 -(2n+1-s)+\delta(S_2,1) = \tfrac{1}{2}n(n-1)+\delta(S_2,1),$$

and

$$\delta(S,2) = \dim H^0(\mathbb{P}_n,\mathcal{I}_S(2)) -\tfrac{1}{2}(n+2)(n+1)+2n+1 \le$$
$$\le \tfrac{1}{2}n(n-1)-\tfrac{1}{2}(n+2)(n+1)+2n+1+\delta(S_2,1) = \delta(S_2,1).$$

Evidently,

$$\delta(S_2,1) = 0 \text{ iff the set } S_2 \text{ is linearly independent.}$$

Thus we are done in the case where S_2 spans a subspace of dimension 2n−s. If it does not, we choose the separation $S = S_1 \amalg S_2$ differently. Take a subset $S_0 \subset S$

which consists of n points that span H. Then $S_2 \cup (S_1 \setminus S_0)$ consists of n+1 linearly dependent points. This shows that we can find a hyperplane H' that contains this set. Now set

$$S_1' = S_2 \cup (S_1 \setminus S_0),\ S_2' = S_0$$

and replace H by H'. For this new decomposition of S we have $\delta(S_2',1) = 0$ and, repeating the argument, we prove the lemma.

Theorem 3. A stable point set $x = (x^1,\ldots,x^{2n+2}) \in P_n^{2n+2}$ with $x^i \neq x^j$ for all $i \neq j$ is self-associated if and only if

$$\delta(\{x\},2) = 1.$$

<u>Proof</u>. For every subset $\{x^{i_1},\ldots,x^{i_{2k+2}}\}$, $k < n$, we have by the stability criterion:

$$1+\dim\langle x^{i_1},\ldots,x^{i_{2k+2}}\rangle > (2k+2)(n+1)/(2n+2) = k+1.$$

This shows that every subset of 2n+1 points in $\{x^{i_1},\ldots,x^{i_{2k+2}}\}$ satisfies the assumption of the previous lemma. Hence

$$\delta(\{x\},2) \leq 1,$$

and it suffices to prove that $\delta(\{x\},2) \geq 1$, i.e.

$$\dim H^0(\mathbb{P}_n,\mathcal{O}_{\{x\}}(2)) \geq \tfrac{1}{2}(n+2)(n+1)-2n-2+1 = \tfrac{1}{2}n(n-1).$$

Let $X = (x_j^{(i)})$ be the coordinate matrix of x. Then x is self-associated if and only if

$$X \cdot \Lambda \cdot {}^tX = 0$$

for some $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_{2n+2})$ with all $\lambda_i \neq 0$. This condition is equivalent to the condition that $(\lambda_1,\ldots,\lambda_{2n+2})$ is a solution of the system of $\frac{1}{2}(n+2)(n+1)$ linear equations

$$(*) \qquad C \cdot \lambda = 0,$$

where

$$C = (x_k^{(j)} x_r^{(j)}) = (c_{(k,r),j}).$$

If x is self-associated

$$\mathrm{rk}(C) \leq 2n+1$$

and there exists a matrix Q of maximal rank such that

$$Q \cdot C = 0.$$

For every column C_j of C and row Q_i of Q the equality

$$Q_i \bullet C_j = 0$$

expresses the condition that a quadric whose coefficients are the entries of Q_i passes through the points $x_1, \ldots x_{2n+2}$. Thus we obtain

$$rk(Q) = \tfrac{1}{2}(n+2)(n+1) - rk(C) \geq \tfrac{1}{2}(n+2)(n+1) - 2n - 1 = \tfrac{1}{2}n(n-1)$$

linearly independent quadrics passing through $\{x\}$. Conversely, if $Q \bullet C = 0$ as above, we have $rk(C) \leq 2n+1$ and there exists a nonzero vector $\lambda = (\lambda_1, \ldots, \lambda_{2n+2})$ satisfying (*). Hence

$$X \bullet \Lambda \bullet {}^tX = 0,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2n+2})$. The only problem is that some of the λ_i's may be zero. Suppose this happens for some λ_j. Then the matrix obtained from C by deleting the j-th column is of rank $\leq 2n$. This implies that $\{x\} \setminus x^j$ lies on $\tfrac{1}{2}n(n-1)+1$ linearly independent quadrics, i.e. $\delta(\{x\} \setminus x^j, 2) \geq 1$. This contradicts Lemma 3.

Remark 3. The assumption of stability of x is essential. For example, a semi-stable point set in $\mathbb{P}_4$ that consists of 6 coplanar points and 4 collinear points lies on a linear system of quadrics of dimension 5 , but it is not self-associated unless the six coplanar points lie on a conic. If that is the case, the dimension of the linear system of quadrics will jump to 6. In the other direction, a semi-stable point set in $\mathbb{P}_3$ that consists of two sets of four collinear points is self-associated, but there is a linear system of quadrics of dimension 3 that contains them. We do not know a clear cut geometrical statement for the general set in $\mathcal{B}$ to be self-associated. However, if we use the block-diagonal coordinate matrix for a most special point set x of type $(d_1, \ldots, d_j)$, it is easy to see that x is self-associated if and only if each of the subsets of $2d_i+2$ points that span a subspace of dimension d_i are self-associated when considered as a point set in $\mathbb{P}_{d_i}$.

Remark 4. One can strengthen a little the assertion of Lemma 3 by assuming in it that Z spans $\mathbb{P}_n$ and that every subset of $2k+2$ points in Z spans a linear subspace L of dimension $\geq k$, as long as we assume that such a subset is not self-associated in L in the case when the equality holds (cf. **[ACGH]**, Exercise F-1 on p.199).

Example 5. Let $C \subset \mathbb{P}_{g-1}$ be a canonical non-trigonal curve of genus $g \geq 3$. A general hyperplane section H cuts C in $2g-2$ points. The corresponding ordered point set in $\mathbb{P}_{g-2} \cong H$ is self-associated. This follows from Theorem 3 and a well-known property of the linear system of quadrics containing a canonical curve ([G-H], p.528).

Example 6. A point set $x = (x^1,\dots,x^8) \in (\mathbb{P}_3^8)^s$ with $x^i \neq x^j, i \neq j$, is self-associated if and only if it is contained in the base-set of a net of 3 quadrics. Generically (we will make this more precise later in Chapter IX), it is equal to the base-set. By Proposition 4 the point set $y = (y^1,\dots,y^7)$ in $\mathbb{P}_2$ obtained by projecting $x^1,\dots,x^7$ from x^8 is associated to the point set $(x^1,\dots,x^7)$. Conversely, let $(y^1,\dots,y^7) \in \mathbb{P}_2^7$ be associated to $(x^1,\dots,x^7) \in \mathbb{P}_3^7$. The linear system of quadrics containing $x^1,\dots,x^7$ is 2-dimensional and, again generically, contains one more base point x^8. The point set $(x^1,\dots,x^8)$ is self-associated. This establishes a natural birational map

$$P_2^7 \dashrightarrow S_8.$$

We will return to this example in Chapter IX.

Let us give another geomeric characterization of general self-associated point sets.

Lemma 4. Given a hyperplane $H \subset \mathbb{P}_n$ not containing any of the basis points, there is a nonsingular quadric Q_H such that any basic rational normal curve that is tangent to H is tangent along $Q_H \cap H$.

Proof. The unique basic rational normal curve that passes through a given point $p = (t_0,\dots,t_n) \in \mathbb{P}_n$ not on any linear space spanned by a proper subset of the basis can be constructed as the image of $\rho: \mathbb{P}_1 \to \mathbb{P}_n$ given by:

$$\rho(\tau_0,\tau_1) \quad (\prod_{i=0}^{n}(\tau_0-\tau_1 t_i))(\frac{t_0}{\tau_0-\tau_1 t_0},\dots,\frac{t_n}{\tau_0-\tau_1 t_n}) =$$

$$= (\rho_0(\tau_0,\tau_1),\dots,\rho_n(\tau_0,\tau_1)),$$

where, if we let $\sigma_{ij}(t)$ denote the j-th elementary symmetric function in the n variables $t_0,\dots,\hat{t}_i,\dots,t_n$ multiplied by $(-1)^j$,

$$\rho_i(\tau_0,\tau_1) = t_i \sum_{j=0}^{n} \sigma_{ij}(t)\tau_1^{\,j}\tau_0^{\,n-j}.$$

Note that

$$\rho(1,0) = p, \ \rho(0,1) = (1,\dots,1), \ \rho(t_i,1) = (0,\dots,1,\dots,0).$$

Let H be given by $\Sigma a_i x_i = 0$, and suppose that R_n is tangent to H at p. Define R_n by using the coordinates of p so that we have:

$$(\sum_i a_i \partial p_i/\partial\tau)(1,0) = -\sum_i a_i t_i \sigma_{ij}(t) = -\sum_{i\neq j} a_i t_i t_j = \sum_j a_j t_j^2 = 0,$$

where $\tau = \tau_1/\tau_0$. Thus the quadric that we are seeking is

$$Q_H(x) = \sum_i a_i x_i^2,$$

and its nonsingularity follows from $a_i \neq 0$ for all i, due to the condition on H.
We will call the quadric Q_H constructed in the previous lemma the <u>basic quadric</u> with respect to H.

Lemma 5. Let H be a general hyperplane in $\mathbb{P}_n$. For every basic rational normal curve R_n the points $H\cap R_n$ are mutually polar with respect to the basic quadric Q_H.

<u>Proof</u>. Let R_n be defined as the unique basic rational curve passing through a point $p = (t_0,\dots,t_n)\in H$, and let $p' = (t_0',\dots,t_n') = p(1,\tau)\in H\cap R_n$. We must show that

$$Q_H(p,p') = \Sigma a_i t_i t_i' = 0,$$

where H is given by an equation $\Sigma a_i x_i = 0$. Let $t^{I_j} = t_{i_1}\dots t_{i_j}$ denote the monomial indexed by an increasing sequence $I_j=(i_1,\dots,i_j)$ where $i_k\in\{0,\dots,n\}$ (also, let $t^{I_0} = 1$ for $I_0 = \emptyset$). We can write the coordinates of p' in the form:

$$t_i' = t_i\sum_{j=0}^{n}\sigma_{ij}(t)\tau^j = t_i\sum_{j=0}^{n}\sum_{I_j}\sum_{i\notin I_j}(-1)^j t^{I_j}\tau^j.$$

By using $\Sigma a_i t_i = 0$, we may write $\Sigma a_i t_i' = 0$ as:

$$0 = \sum_{j=0}^{n}\sum_{I_j}\sum_{i\notin I_j}(-1)^j a_i t_i t^{I_j}\tau^j = -\sum_{j=1}^{n}\sum_{I_j}\sum_{i\notin I_j}(-1)^j a_i t_i t^{I_j}\tau^j = \tau\sum_{j=1}^{n-1}\sum_{I_j}\sum_{i\notin I_j}(-1)^j a_i t_i (t_i t^{I_j})\tau^j.$$

Since $t_i t^{I_n} = t_0 t_1\dots t_n$ for $i\notin I_n$, we obtain

$$\sum_i a_i t_i (t_i t^{I_n}) = 0.$$

By adding this to the last equation above, we obtain that:

$$0 = \tau\sum_{j=0}^{n}\sum_{I_j}\sum_{i\notin I_j}(-1)^j a_i t_i (t_i t^{I_j})\tau^j =$$

$$= \tau \sum_{i=0}^{n} a_i a_i t_i t_i' = \tau Q_H(t,t').$$

Lemma 6. Let $\rho:\mathbb{P}_1 \to \mathbb{P}_n$ be a morphism whose image is a rational normal curve R_n, and let $\pi_L:(\mathbb{P}_n\backslash L) \to \mathbb{P}_1$ be the projection from an (n-2)-plane L. If the set of n+2 points $(x^1,\dots,x^{n+2})$ in $\mathbb{P}_1$ is projectively equivalent to the point set $((\pi_L\circ\rho)(x^1),\dots,(\pi_L\circ\rho)(x^{n+2}))$ then $L\cap R_n$ consists of n-1 points and $\pi_L:R_n \to \mathbb{P}_1$ is an isomorphism.

Proof. Let $\{H_t\}_{t\in\mathbb{P}^1}$ be a pencil of hyperplanes through L and s be a parameter for $\mathbb{P}_1$ adjusted so that the parameter s_i of x^i is equal to t_i where $\rho(x^i)\in H_{t_i}$. The points in $R_n\cap H_{t_i}$ satisfy an equation $a_t(s) = 0$ that has degree n in s and is linear in t. The restituted form $a_s(s)$ of degree n+1 has the n+2 roots s_i and hence is identically zero. The form $a_t(s)$ must then have s-t as a factor, and so by writing

$$a_t(s) = (s-t)b(s)$$

for some form b(s) of degree n-1 we see that the n-1 roots of b(s) represent base points of the pencil $\{H_t\}_{t\in\mathbb{P}^1}$ on R_n.

Proposition 5. A general set of 2n+2 points in $\mathbb{P}_n$ is self-associated if and only if, when n+2 of its points are used as a basis, the remaining n points are all mutually polar with respect to the basic quadric Q_H, where H is the hyperplane that they span.

Proof. Let $x = (x^1,\dots,x^{2n+2})$ be a self-associated general point set in $\mathbb{P}_n$. Choose the first n+2 points to be a basis and let R_n be the unique basic rational normal curve that contains them along with x^{n+3}. By Proposition 4, the point set on $\mathbb{P}^1$ obtained by projecting $x^1,\dots,x^{n+3}$ onto a line from the (n-2)-plane $L = \langle x^{n+4},\dots,x^{2n+2}\rangle$ is projectively equivalent to the point set $(x^1,\dots,x^{n+3})$ on $R_n \cong \mathbb{P}^1$ By the previous lemma, L intersects R_n in n-1 points and the projection is an isomorphism of R_n to $\mathbb{P}_1$. Let $H = \langle x^{n+3},\dots,x^{2n+2}\rangle$ be the hyperplane containing L and the point x^{n+3}. By Lemma 5, the points $H\cap R_n$ are mutually polar with respect to Q_H. Thus L is polar to x^{n+3} with respect to Q_H. By repeating this argument for each of the last n points in x we find that they all lie in the same hyperplane H, and that each point x^i is determined as the intersection:

$$x^i = H\cap\Big(\bigcap_{j=n+3,\, j\neq i}^{2n+2} Q_H(x^j)\Big),$$

where $Q_H(x_j)$ denotes the polar hyperplane of x^j with respect to Q_H. The converse follows easily from the construction of the associated point set given in the previous section.

Theorem 4. The variety S_n of projective equivalence classes of self-associated point sets in $\mathbb{P}_n$ is a rational variety of dimension $\frac{1}{2}n(n+1)$.

Proof. We will show that the open set of projective equivalence classes of general self-associated point sets is isomorphic to the open set of full flags $\{L_0 \subset \ldots \subset L_{n-1} \subset \mathbb{P}_n\}$ that do not contain any point in the basis. First, given a self-associated point set x, we construct the flag by sending the first n+2 points in x to a basis and by setting

$$L_i = \langle x^{n+3}, \ldots, x^{n+3+i} \rangle .$$

Conversely, given a flag, we shall use the polarity Q_H for $H = L_{n-1}$ as prescribed by the previous proposition to construct a self-associated set. Once again, set the first n+2 points of x equal to the basis. Define:

$$x^{n+3} = L_0, \text{ and } x^{n+3+i} = Q_H(L_{i-1}) \cap L_i \text{ for } i = 1, \ldots, n-1,$$

where $Q_H(L_i)$ denotes the polar (n-1-i)-plane to the i-plane L_i with respect to the basic quadric Q_H. Note that the construction forces each of the last n points to be mutually polar, hence the previous proposition gives us that the point set x is self-associated. Also note that L_i contains $x^{n+3}, \ldots, x^{n+3+i}$ so that the two constructions are inverse to each other. Since the flag variety is rational of dimension $\frac{1}{2}n(n+1)$, we are done.

Remark 5. Applying the previous theorem in the case n = 2 we obtain the proof of the rationality of the level 2 modular quartic 3-fold V_4 without using that it is dual to the Segre cubic primal.

Remark 6. We will see later that, when $n \le 3$, there is a natural birational isomorphism between the varieties S_n and the moduli space $\mathcal{A}_n(2)$ of principally polarized abelian varieties of dimension n with level 2 structure. Note that

$$\dim S_n = \dim \mathcal{A}_n(2)$$

for all $n \ge 1$, however, the rationality of S_n and the non-unirationality of $\mathcal{A}_n$ for

$n \geq 9$ (**[Ta]**) implies that such an isomorphism does not exist for $n \geq 9$. However one may find an interesting correspondence between these varieties.

IV. BLOWING-UPS OF POINT SETS.

Let $x = (x^1,\dots,x^m) \in \mathbb{P}_n^m$ be a point set. There is a natural variety associated to it. Namely, we consider $\{x\} = \{x^1,\dots,x^m\}$ as a 0-dimensional closed subscheme of $\mathbb{P}_n$ and blow it up. Of course, this is well-defined only if x consists of distinct points. To define the blowing-up variety $V(x)$ for a general point set x we have to enlarge our original notion of a point set assuming that some of the points are infinitely near. In this chapter we will define simultaneously, following S. Kleiman [Kl1], the variety parametrizing infinitely near point sets and the blowing-ups of such sets.

1. Infinitely near point sets.

Let Z be a smooth algebraic variety of dimension $n>1$, $z \in Z$ be a closed point, and $Z' = Z(z)$ be the blowing-up of z. Recall that $Z(z)$ is defined uniquely (up to isomorphism) by the properties:

(i) there exists a proper birational morphism $\pi: Z(z) \to Z$ that is an isomorphism over $Z \setminus \{z\}$;

(ii) there is a natural isomorphism

$$\pi^{-1}(z) \xrightarrow{\sim} \mathbb{P}(T(Z)_z) \cong \mathbb{P}_{n-1},$$

where $T(Z)_z$ is the tangent space of Z at z.

A closed point $z' \in Z(z)$ lying in $E = \pi^{-1}(z)$ is called an <u>infinitely near point of order 1 to z</u>. It is denoted by

$$z' \to z.$$

An infinitely near point of order κ to z is defined by induction as an infinitely near point of order 1 to an infinitely near point of order $\kappa-1$ to z. It is denoted by

$$z^{(\kappa)} \to \dots \to z^{(1)} \to z.$$

Let Z^m denote the Cartesian product of m copies of Z. For every subset I of $\{1,\dots,m\}$ with $\# I \geq 2$ we denote

$$\Delta(m)_I = \{(z^1,\dots,z^m)\in Z^m : z_i = z_j \text{ for all } i,j\in I\},$$

$$\Delta(m)_K = \cup_{\# I=K}\Delta(m)_I,\ \Delta(m) = \Delta(m)_2,$$

$$U(m)_K = Z^m\setminus\Delta(m)_K,\ U(m) = U(m)_2,$$

$$\pi_i: Z^m \to Z, \text{ the } i\text{-th projection},$$

$$\pi^m = \pi_1\times\dots\times\pi_{m-1}: Z^m \to Z^{m-1}.$$

Theorem 1. For every $m \geq 1$ there exists a proper birational morphism of smooth varieties

$$b_m : \hat{Z}^m \to Z^m$$

satisfying the following properties:

(i) the restriction of b_m over U(m) is an isomorphism;

(ii) b_m is a composition of blowing-ups with smooth centers;

(iii) if $m\geq 2$ there exists a smooth proper morphism

$$\hat{\pi}^m: \hat{Z}^m \to \hat{Z}^{m-1}$$

such that the fibre $(\hat{\pi}^m)^{-1}(z)$ over $z\in\hat{Z}^{m-1}$ is isomorphic to the blowing-up of z considered as a closed point on the fibre $(\hat{\pi}^{m-1})^{-1}(\hat{\pi}^{m-1}(z))$;

(iv) the diagram

$$\begin{array}{ccc} \hat{Z}^m & \xrightarrow{b_m} & Z^m \\ \hat{\pi}^m\downarrow & & \downarrow\pi^m \\ \hat{Z}^{m-1} & \xrightarrow{b_{m-1}} & Z^{m-1} \end{array}$$

commutes;

(v) If m = 1

$$\hat{Z}^1 = Z^1 = Z,$$

$$(\hat{\pi}^2)^{-1}(z) = Z(z) = \text{blowing-up of } z\in Z.$$

<u>Proof</u>. Let $\hat{Z}^0$ be a single point, $\hat{Z}^1 = Z$, $\hat{\pi}^1:\hat{Z}^1 \to \hat{Z}^0$. Then for each $i > 1$ define inductively a $Y=\hat{Z}^{i-1}$-variety $\hat{\pi}^i:\hat{Z}^i \to \hat{Z}^{i-1}$ as follows. By assumption, $\hat{Z}^{i-1}$ is a $V=\hat{Z}^{i-2}$-variety. Define $\hat{Z}^i$ as the blowing-up of the diagonal of $Y\times_V Y$, and the morphism

$$\hat{\pi}^i:\hat{Z}^i \to \hat{Z}^{i-1}$$

as the composition of the blowing-up morphism with the projection of the fibred product to the first factor. Define the projections b_i: $\hat{Z}^i \to Z^i$ by induction as follows. Let b_1 be the identity. Assume that b_{i-1}:$\hat{Z}^{i-1} \to Z^{i-1}$ is defined. The composition of the two projections $Y \times_Y Y \to Y = \hat{Z}^{i-1}$ with b_{i-1} define two projections to Z^{i-1}, hence $2i-2$ projections p_k and q_k to Z, $k = 1,\dots,i-1$. Since $p_k = q_k$ for $k=1,\dots,i-2$, we obtain i projections $p_1,\dots,p_{i-1},q_{i-1}$ to Z . Let b_i be the composition of the blowing-up morphism $\hat{Z}^i \to Y \times_Y Y$ with the product $Y \times_Y Y \to Z^i$ of these projections. Since we only blow up smooth projective varieties along smooth centers, all the varieties $\hat{Z}^i$ are smooth and the morphisms $\hat{\pi}^i$ are proper and birational. We only sketch the proofs of properties (i) $\to$ (v) stated in the theorem, leaving the details to the reader. Only (ii) and (iii) do not follow immediately from the construction. To see (iii) we use the definition of the tangent space of a variety Z at a point $z \in Z$ as the fibre of the inverse transform of the normal sheaf of the diagonal of $Z \times Z$ under the diagonal map $Z \to Z \times Z$. To see (ii) we use induction on m. By construction b_2: $\hat{Z}^2 \to Z^2$ is the blowing-up of Δ_{12}. Assume b_{m-1}: $\hat{Z}^{m-1} \to Z^{m-1}$ is a composition of blowing-ups with smooth centers. The morphism

$$\varphi_0 = \hat{\pi}^{m-1} \times 1 \colon X_0 = \hat{Z}^{m-1} \times Z \to Z^m = Z^{m-1} \times Z$$

is a composition of blowing-ups with smooth centers. Then one easily checks that the morphism b_m: $\hat{Z}^m \to Z^m$ is equal to the composition:

$$\hat{Z}^m = X_m \xrightarrow{\varphi_m} X_{m-1} \to \dots \to X_1 \xrightarrow{\varphi_1} X_0 \xrightarrow{\varphi_0} Z^m,$$

where φ_1: $X_1 \to X_0$ is the blowing-up of $\varphi_0^{-1}(\Delta(m)_{1m})$, φ_2: $X_2 \to X_1$ is the blowing-up of $(\varphi_0 \circ \varphi_1)^{-1}(\Delta(m)_{2m})$, and so on. It is easy to see that

$$X_{m-1} \cong \hat{Z}^{m-1} \times_{\hat{Z}^{m-2}} \hat{Z}^{m-1},$$

and φ_m is the blowing-up of the diagonal isomorphic to $(\varphi_0 \circ \dots \circ \varphi_{m-1})^{-1}(\Delta(m)_{m-1m})$.

It is natural to view every closed point of $\hat{Z}^m$ as a m-tuple $\hat{z} = (\hat{z}^1,\dots,\hat{z}^m)$, where each point $\hat{z}^i$ is either a point of Z or an infinitely near point to some $\hat{z}^j$ with $j < i$. We usually drop the hat over a point from Z. In this notation the morphism b_m: $\hat{Z}^m \to Z^m$ sends $\hat{z} = (\hat{z}^1,\dots,\hat{z}^m)$ to $z = (z^1,\dots,z^m)$, where $z^i \in Z$ and $\hat{z}^i$ is either equal to z^i, or is infinitely near to some z^j, $j < i$. The projection $\hat{\pi}^m$ is the map:

$$(\hat{z}^1,\dots,\hat{z}^m) \to (\hat{z}^1,\dots,\hat{z}^{m-1}).$$

We define the <u>blowing-up variety</u> $Z(\hat{z})$ of $\hat{z} = (\hat{z}^1,\dots,\hat{z}^m) \in \hat{Z}^m$ by setting

$$Z(\hat{z}) = (\hat{\pi}^{m+1})^{-1}(\hat{z}) \subset \hat{Z}^{m+1}.$$

Thus the blowing-up variety $Z(\hat{z})$ comes with a natural birational morphism:

$$\sigma(\hat{z}) \colon Z(\hat{z}) \to Z$$

which is the composition of the morphisms

$$Z((\hat{z}^1,\dots,\hat{z}^m)) \overset{\sigma(\hat{z})_m}{\to} Z((\hat{z}^1,\dots,\hat{z}^{m-1}) \overset{\sigma(\hat{z})_{m-1}}{\to} \dots \overset{\sigma(\hat{z})_2}{\to} Z((\hat{z}^1)) \overset{\sigma(\hat{z})_1}{\to} Z,$$

where

$$\sigma(\hat{z})_i := Z((\hat{z}^1,\dots,\hat{z}^i)) \to Z((\hat{z}^1,\dots,\hat{z}^{i-1}))$$

is the blowing up of the point $(\hat{z}^1,\dots,\hat{z}^i) \in Z((\hat{z}^1,\dots,\hat{z}^{i-1})) \subset \hat{Z}^i$. Note that it is natural to identify each $\hat{z}^m$ with the point $(\hat{z}^1,\dots,\hat{z}^m)$ considered as a point of $Z((\hat{z}^1,\dots,\hat{z}^{m-1}))$.

Let

$$\hat{\Delta}(m)_I = b_m^{-1}(\Delta(m)_I),$$

$$\hat{\Delta}(m)_K = b_m^{-1}(\Delta(m)_K), \ \hat{\Delta}(m) = b_m^{-1}(\Delta(m)),$$

$$\hat{U}(m)_K = b_m^{-1}(Z^m \setminus \Delta(m)_K), \ \hat{U}(m) = b_m^{-1}(U(m)).$$

It is easy to see that $\hat{\Delta}(m)_{m-1m}$ is the exceptional divisor of the blowing-up of the diagonal of $\hat{Z}^{m-1} \times_{\hat{Z}^{m-2}} \hat{Z}^{m-1}$, and

$$\hat{\Delta}(m)_{ij} = p_1^{-1}(\hat{\Delta}(m-1)_{ij}) \text{ for } 1 \le i < j \le m-1,$$

$$\hat{\Delta}(m)_{im} = p_2^{-1}(\hat{\Delta}(m-1)_{im-1} \le i \le m-2.$$

For example, if $m = 3$, $\hat{\Delta}(3)_{123}$ consists of points of type

$$(z_1, z_2 \to z_1, z_3 \to z_1) \quad \text{or} \quad (z_1, z_2 \to z_1, z_3 \to z_2 \to z_1),$$

$\hat{\Delta}(3)_{12} \setminus \hat{\Delta}(3)_{123}$ consists of points of type

$$(z_1, z_2 \to z_1, z_3),$$

and so on.

The following proposition follows immediately from the above description of $\hat{\Delta}(m)$:

Proposition 1. $\hat{\Delta}(m)$ is a hypersurface in $\hat{Z}^m$. Its irreducible components are the hypersurfaces $\hat{\Delta}(m)_I$ with $\#I = 2$.

Remark 1. Note that the "diagonals" $\hat{\Delta}_{ij}(m)$ are not isomorphic subvarieties of $\hat{Z}^m$ contrary to the case of Z^m. However, their intersections with the open subset $\hat{U}(m)_3$ are isomorphic. This open subset can be characterized as the maximal subset of $\hat{Z}^m$ which the natural action of the permutation group Σ_m on Z^m extends to.

Remark 2. The analog for the space $\hat{Z}^m$ for the variety of unordered point sets $Z^{(m)} = Z^m/\Sigma_m$ is the Hilbert scheme $\mathrm{Hilb}_m(Z)$ of 0-dimensional closed subschemes of length m in Z. It is known **[Fo]** that the canonical cycle map:

$$\mathrm{Hilb}_m(Z) \to Z^{(m)}$$

is a resolution of singularities. Let $\mathrm{Hilb}_m(Z)^c$ be the open subscheme of $\mathrm{Hilb}_m(Z)$ that parametrizes 0-dimensional subschemes whose ideal is, locally at each point of their support, of the form $(t_1^r, t_2, \ldots, t_n)$, where $(t_1, \ldots, t_n)$ is a suitable system of local parameters. According to **[Ra]** there is a natural rational map

$$\hat{Z}^m \dashrightarrow \mathrm{Hilb}_m(Z)^c \times_{Z^{(m)}} Z^m$$

whose restriction to $\hat{U}(m)_3$ is an isomorphism onto its image.

Remark 3. We refer to **[Ha 1]**, where the variety $\hat{Z}^m$ is defined as representing a certain functor of families of point sets.

Returning to our situation where $Z = \mathbb{P}_n$, we have defined the space

$$\hat{\mathbb{P}}_n^m = \hat{Z}^m,$$

the birational morphism

$$b_m: \hat{\mathbb{P}}_n^m \to \mathbb{P}_n^m,$$

and the projection

$$\hat{\pi}^m: \hat{\mathbb{P}}_n^m \to \hat{\mathbb{P}}_n^{m-1},$$

satisfying the properties stated in Theorem 1.

2. Analysis of stability in $\hat{Z}^m$.

Let

$$\mu: G \times Z \to Z$$

be an action of an algebraic group G on Z, and

$$\mu^m : G \times Z^m \to Z^m$$

be the corresponding diagonal action on Z^m. We want to extend this action to an action

$$\hat{\mu}^m : G \times \hat{Z}^m \to \hat{Z}^m$$

such that the following diagrams commute

$$\begin{array}{ccc} G \times \hat{Z}^m & \xrightarrow{\hat{\mu}^m} & \hat{Z}^m \\ 1\times\hat{\pi}^m \downarrow & & \downarrow \hat{\pi}^m \\ G \times \hat{Z}^{m-1} & \xrightarrow{\hat{\mu}^{m-1}} & \hat{Z}^{m-1} \end{array} \qquad \begin{array}{ccc} G \times \hat{Z}^m & \xrightarrow{\hat{\mu}^m} & \hat{Z}^m \\ 1\times b_m \downarrow & & \downarrow b_m \\ G \times Z^m & \xrightarrow{\mu^m} & Z^m \end{array}$$

This can be done step by step by using that each $\hat{Z}^{l+1}$ is obtained from $\hat{Z}^l$ by blowing up the diagonal of $\hat{Z}^l \times_{\hat{Z}^{l-1}} \hat{Z}^l$ and that the extension of the action $\hat{\mu}^l$ to the fibred product leaves the diagonal invariant.

To be a little more explicit, we denote by T_m the relative tangent bundle of the morphism $\hat{\pi}^m : \hat{Z}^m \to \hat{Z}^{m-1}$. Let $\hat{z} = (\hat{z}^1,\dots,\hat{z}^m) \in \hat{Z}^m$. If $\hat{z}^m$ is infinitely near of order 1 to some $\hat{z}^l$, then it belongs to some fibre of

$$\mathbb{P}(T_m|(\hat{\pi}^m)^{-1}(\hat{z}^1,\dots,\hat{z}^{m-1})) \cong \mathbb{P}(T(Z((\hat{z}^1,\dots,\hat{z}^{m-1}))))$$

and we verify that

$$\hat{\mu}^m(g,(\hat{z}^1,\dots,\hat{z}^m)) = (\hat{\mu}^{m-1}(g,(\hat{z}^1,\dots,\hat{z}^{m-1})), dg(\hat{z}^m)),$$

where dg is the differential of the map

$$\hat{\mu}^m(g) : (\hat{\pi}^m)^{-1}(\hat{z}^1,\dots,\hat{z}^{m-1}) \to (\hat{\pi}^m)^{-1}(\hat{\mu}^{m-1}(g)(\hat{z}^1,\dots,\hat{z}^{m-1})).$$

If $\hat{z}^m = z^m \in Z$ we have as usual

$$\hat{\mu}^m(g,(z^1,\dots,z^m)) = (\hat{\mu}^{m-1}(g,(z^1,\dots,z^{m-1})), g(z^m)).$$

To find stable points in $\hat{Z}^m$ we need to study the functorial behavior of stability under G-equivariant maps. The following result is the first step toward this problem:

Proposition 2. Let G be a reductive group acting on algebraic varieties X and Y, and $f:X \to Y$ be a G-equivariant morphism. Let $\mathcal{L}$ (resp. $\mathcal{M}$) be a G-linearized invertible sheaf on Y (resp. on X). Assume that $\mathcal{L}$ is ample and $\mathcal{M}$ is f-ample. Then for sufficiently large n the sheaf $f^*(\mathcal{L}^{\otimes n})\otimes\mathcal{M}$ is ample, and

$$X^s(f^*(\mathcal{L}^{\otimes n})\otimes\mathcal{M}) \supset f^{-1}(Y^s(\mathcal{L}))$$

Proof. See ([**Mu 1**], Proposition 2.18).

More precise results about the behaviour of stability under blowing-ups were recently obtained in [**Ki 2**] and in a thesis of Z. Reichstein [**Rei**]. We state Reichstein's results without proof.

Proposition 3. In the notation of the previous proposition

$$X^{ss}(f^*(\mathcal{L}^{\otimes n})\otimes\mathcal{M}) \subset f^{-1}(Y^{ss}(\mathcal{L}))$$

for sufficiently large n.

Next we assume that $f: X \to Y$ is the blowing-up of a G-invariant closed subscheme C of Y. The action of G on Y extends naturally to an action on X. Denote the exceptional divisor of f by E. Fix a very ample G-linearized invertible sheaf $\mathcal{L}$ on Y, and let

$$\hat{\mathcal{L}}_\kappa = f^*(\mathcal{L}^{\otimes\kappa})\otimes\mathcal{O}_X(-E).$$

Then $\hat{\mathcal{L}}_\kappa$ is a very ample G-linearized invertible sheaf on X if κ is sufficiently large. Let $p: Y^{ss} \to Y^{ss}/G$ be the quotient map and let $\tilde{C} = p^{-1}(p(C\cap Y^{ss}))$. For every subvariety Z of Y we denote by Z' its proper inverse transform under f, that is, the closure of $f^{-1}(Z\setminus(C\cap Z))$ in X.

Proposition 4. Assume X and C are smooth. Then, for sufficiently large κ, the open subsets $X(\hat{\mathcal{L}}_\kappa)^{ss}$ and $X(\hat{\mathcal{L}}_\kappa)^s$ are independent of κ and

(i) $X(\hat{\mathcal{L}}_\kappa)^{ss} = f^{-1}(Y(\mathcal{L})^{ss})\setminus\tilde{C}'$;

(ii) $X(\hat{\mathcal{L}}_\kappa)^s = X(\hat{\mathcal{L}}_\kappa)^{ss}\setminus(Y(\mathcal{L})^{ss}\setminus Y(\mathcal{L})^s)'$.

We want to apply these results to the case where $f:X \to Y$ is the map $b_m:\hat{Z}^m \to Z^m$. By Theorem 1, the morphism b_m is a composition

$$Y_\kappa = \hat{Z}^m \xrightarrow{f_\kappa} Y_{\kappa-1} \xrightarrow{f_{\kappa-1}} \dots \xrightarrow{f_1} Y_0 = Z^m$$

of blowing-ups with smooth centers. It follows from the proof of this assertion that each such blowing-up is G-equivariant, and that its center is isomorphic to a certain proper inverse transform of some $\Delta(m)_{ij}$. Choose a G-linearized ample invertible sheaf $\mathcal{L}_0$ on $Y_0 = Z^m$, then define a similar sheaf $\mathcal{L}_1 = f_1^*(\mathcal{L}_0^{\otimes n})\otimes \mathcal{O}_{Y_1}(-E_1)$, where n is sufficiently large and E_1 is the exceptional divisor of f_1, and proceed in this way until we obtain a sequence of G-linearized ample invertible sheaves $\mathcal{L}_i$ at each Y_i. Each of them defines the subset Y_i^{ss} (Y_i^s) of semi-stable (stable) points. Let

$$\Phi_i: Y_i^{ss} \to Y_i^{ss}/G$$

be the corresponding quotient projection. Set

$$\mathcal{D}_i = Y_i^{ss} \setminus Y_i^s,$$

$$\tilde{C}_i = \Phi_i^{-1}(\Phi_i(C_i \cap Y_i^{ss})),$$

where C_i is the center of the blowing-up f_{i+1}. Applying Propositions 3 and 4 to each f_i, we obtain that

$$Y_{i+1}^{ss} = f_i^{-1}(Y_i^{ss})\setminus \tilde{C}_i',$$

$$Y_{i+1}^{s} = Y_{i+1}^{ss}\setminus \mathcal{D}_i',$$

where "prime" denotes the proper inverse transform. In particular, we have

$$Y_{i+1}^{ss} \subset f_i^{-1}(Y_i^{ss}),$$

$$Y_{i+1}^{s} \supset f_i^{-1}(Y_i^{s}).$$

Since

$$\tilde{C}_{i+1}\cap\tilde{C}_i' = \emptyset,$$

we have

$$Y_{i+2}^{ss} = f_{i+1}^{-1}(f_i^{-1}(Y_i^{ss}))\setminus\tilde{C}_{i+1}'\setminus\tilde{C}_i'',$$

where $\tilde{C}_i''$ is the proper inverse transform of $\tilde{C}_i$ under $f_i \circ f_{i+1}$. Starting with $i = 0$, and climbing up to $i = \kappa$, we use the previous properties to obtain the following:

Theorem 2. Let $\tilde{C}'$ (resp. $\mathcal{D}_0'$) be the proper inverse transform of $\tilde{C} = \Phi_0^{-1}(\Phi_0(\Delta(m))$ (resp. of $\mathcal{D}_0$) under b_m. There exists a G-linearized ample invertible sheaf $\hat{\mathcal{L}}_0$ on $\hat{Z}^m$ such that

(i) $\hat{Z}^m(\hat{\mathcal{L}}_0)^{ss} = b_m^{-1}(Z^m(\mathcal{L}_0)^{ss})\setminus\tilde{C}'$,

(ii) $\hat{Z}^m(\hat{\mathcal{L}}_0)^{s} = \hat{Z}^m(\hat{\mathcal{L}}_0)^{ss}\setminus\mathcal{D}_0'$.

In particular,

$$\hat{Z}^m(\hat{\mathcal{L}}_0)^{ss} \subset b_m^{-1}(Z^m(\mathcal{L}_0)^{ss}),$$

$$\hat{Z}^m(\hat{\mathcal{L}}_0)^{s} \supset b_m^{-1}(Z^m(\mathcal{L}_0)^{s}).$$

Returning to our situation where $Z = \mathbb{P}_n$, $G = PGL(n+1)$, we take for $\mathcal{L}_0$ our standard sheaf $\mathcal{L}$, and obtain a G-linearized ample invertible sheaf $\hat{\mathcal{L}}$ which allows us to define the open subsets of $\hat{\mathbb{P}}^m_n$:

$$(\hat{\mathbb{P}}^m_n)^{ss}, (\hat{\mathbb{P}}^m_n)^{s},$$

the quotient

$$\hat{P}^m_n = (\hat{\mathbb{P}}^m_n)^{ss}/G,$$

the projection

$$\hat{\Phi}: (\hat{\mathbb{P}}^m_n)^{ss} \to \hat{P}^m_n,$$

and the morphism

$$\bar{b}_m: \hat{P}^m_n \to P^m_n$$

such that the following diagram is commutative:

$$\begin{array}{ccc} (\hat{\mathbb{P}}^m_n)^{ss} & \xrightarrow{b_m} & (\mathbb{P}^m_n)^{ss} \\ \hat{\Phi}\downarrow & & \Phi\downarrow \\ \hat{P}^m_n & \xrightarrow{\bar{b}_m} & P^m_n \end{array}$$

Note that this diagram is Cartesian if $(\mathbb{P}^m_n)^{ss} = (\mathbb{P}^m_n)^{s}$ (see **[Mu]**, Definition 0.7 and Theorem 1.10)

Corollary 1. Assume $n > 2$. The following properties are equivalent

(i) $(\mathbb{P}^m_n)^{ss} = (\mathbb{P}^m_n)^{s}$;

(ii) $(\hat{\mathbb{P}}^m_n)^{ss} = (\hat{\mathbb{P}}^m_n)^{s}$,

(iii) m and n+1 are coprime.

Proof. (i) $\Rightarrow$ (ii) Follows from Theorem 2.

(ii) $\Rightarrow$ (i) By Theorem 2, (ii) implies that $\mathcal{D} = (\mathbb{P}^m_n)^{ss} \setminus (\mathbb{P}^m_n)^{s}$ is contained in $\tilde{C} = \Phi^{-1}(\Phi(\Delta(m) \cap (\mathbb{P}^m_n)^{ss}))$. It follows from the description of $\bar{\mathcal{D}} = \Phi(\mathcal{D})$ given in section 3 of Chapter II that in this case, for every non-trivial admissible partition $(d_1,\dots,d_j)$ of n+1 with respect to m, one of the d_i's is equal to 1. This implies that n+1 divides m, hence every partition of n+1 is admissible. Since $n > 2$, (2,n-1) is a partition of n+1 which does not contain 1. This contradiction shows that n+1 admits only trivial admissible partition, hence $\mathcal{D} = \emptyset$ and (ii) holds.

(i) ⇔ (iii) This is Corollary of Theorem 1 from Chapter II.

Corollary 2. Assume $n = 2$, $m \geq 4$. Then

$$(\hat{\mathbb{P}}_2^m)^{ss} = (\hat{\mathbb{P}}_2^m)^s.$$

In particular, the morphism

$$\sigma_m : \hat{P}_2^m \to P_2^m$$

is a resolution of singularities.

Proof. Since every non-trivial admissible partition of 3 contains 1, we obtain, in the notation from the proof above,

$$\mathcal{D} \subset \tilde{\mathcal{C}}.$$

By Theorem 2 this implies that $(\hat{\mathbb{P}}_2^m)^{ss} = (\hat{\mathbb{P}}_2^m)^s$, hence the projection

$$\Phi : (\hat{\mathbb{P}}_2^m)^{ss} \to \hat{P}_2^m$$

is a geometric quotient. It is easy to see that for each point set $\hat{x}$ in $(\hat{\mathbb{P}}_2^m)^{ss}$ the stabiliser group $G_{\hat{x}}$ is trivial. Applying Luna's slice theorem [Lu], we obtain that the quotient space $\hat{P}_2^m$ is smooth. Obviously, the morphism σ_m is birational and proper.

Remark 4. We refer to [Ish], where the analog of the space $\hat{P}_n^m$ for unordered point sets is discussed in the case $n = 2$ and $m \leq 8$.

V. GENERALIZED DEL PEZZO VARIETIES.

In this chapter we study the rational varieties obtained by blowing up a point set $\hat{x}\in\hat{\mathbb{P}}_n^m$.

1. The Neron-Severi bilattice.

A generalized Del Pezzo variety (gDP-variety) of type (n,m) is an algebraic variety V isomorphic to a blowing-up $V(\hat{x})$ of some point set $\hat{x}\in\hat{\mathbb{P}}_n^m$. A blowing-down structure of type (n,m) is a pair (V,σ), where V is a gDP-variety of type (n,m) and σ is a sequence of birational morphisms

$$V = V_m \xrightarrow{\sigma_m} V_{m-1} \xrightarrow{\sigma_{m-1}} \dots \xrightarrow{\sigma_2} V_1 \xrightarrow{\sigma_1} V_0 = \mathbb{P}_n,$$

where each $\sigma_i\colon V_i \to V_{i-1}$ is a blowing-up of a closed point. Two blowing-down structures (V,σ) and (V',σ') are isomorphic if there exist isomorphisms

$$\varphi_i\colon V_i \to V_i'$$

such that

$$\sigma_i'\circ\varphi_i = \varphi_{i-1}\circ\sigma_i,$$

$i = 0,\dots,m$. We say that (V,σ) is relatively isomorphic to (V',σ') if φ_0 = identity.

Thus, by definition, $\hat{\mathbb{P}}_n^m$ parametrizes the relative isomorphism classes of blowing-down structures of type (n,m). The projection

$$\hat{\pi}^{m+1}\colon\hat{\mathbb{P}}_n^{m+1} \to \hat{\mathbb{P}}_n^m$$

is a universal family (see the corresponding functorial statement in [Ha 1]). The varieties

$$\hat{P}_n^{m\circ} = (\hat{\mathbb{P}}_n^m)^s/G \subset \hat{P}_n^m$$

are the coarse moduli varieties of isomorphism classes of blowing-down structures of type (n,m) corresponding to stable point sets $\hat{x}$.

Now note that the action of Σ_m on $\hat{\mathbb{P}}_n^m$, whenever it is defined, changes the blowing-down structure but leaves the gDP-variety itself unchanged. Also note that, if $n > 2$, the blowing-down structure of a gDP-variety is defined uniquely up to isomorphism and up to the Σ_m-action. Indeed, assume for simplicity that $V \cong V(x) \cong V(x')$ for some $x, x' \in \hat{\mathbb{P}}_n^m \setminus \hat{\Delta}(m) = \mathbb{P}_n^m \setminus \Delta(m)$. Let $E_1, \ldots, E_m$ be the disjoint exceptional divisors in $V(x)$, and let $E_1', \ldots, E_m'$ be the same for $V(x')$. Assume that some E_j intersects two different E_i' and E_k'. Since each exceptional divisor is isomorphic to $\mathbb{P}_{n-1}$ and $n>2$, $E_i' \cap E_k' \neq \emptyset$. This contradicts the assumption that all E_i' are disjoint. Therefore we may assume that each E_i intersects at most one E_j'. But then $E_i = E_j'$, otherwise E_i is a proper inverse transform of a hypersurface in $\mathbb{P}_n$ with respect to the blowing-down structure defined by the point set x', hence it is numerically effective. This shows that $\{E_1, \ldots, E_m\} = \{E_1', \ldots, E_m'\}$ as sets of divisors.

However, we obtain many different blowing-down structures if we consider varieties up to pseudo-isomorphism.

Recall that a <u>pseudo-isomorphism</u> of smooth algebraic varieties is a birational map:

$$f: X \dashrightarrow Y$$

that induces an isomorphism in codimension 1, that is, an isomorphism of open subsets whose complements are of codimension ≥ 2. Note that every pseudo-isomorphism of surfaces is an isomorphism, as follows from the theorem of decomposition of birational maps of surfaces.

Let X be a smooth algebraic variety of dimension n,

$$A(X) = \bigoplus_{i=0}^{n} A^i(X)$$

be its Chow ring of algebraic cycles modulo algebraic equivalence graded by codimension. We set

$$N^1(X) = A^1(X)/\equiv,$$

$$N_1(X) = A^{n-1}(X)/\equiv,$$

where $\equiv$ denotes numerical equivalence. We denote by $[\gamma]$ the numerical class of a cycle γ.

The pair

$$N(X) = (N^1(X), N_1(X))$$

is called the <u>Neron-Severi bilattice</u> of X. It is a pair of free abelian groups of finite rank equipped with a pairing:

$$N^1(X) \times N_1(X) \to \mathbf{Z}, \quad (\gamma^1,\gamma_1) \mapsto \gamma^1\bullet\gamma_1,$$

defined by the intersection of cycles.

Lemma 1. Let X be a smooth complete variety of dimension $n > 1$, $\sigma : X' \to X$ be a blowing-up of its closed point x, $E = \sigma^{-1}(x) \cong \mathbb{P}_{n-1}$ be the exceptional divisor, l be a line in E. Then

$$N^1(X') = \sigma^*(N^1(X)) + \mathbf{Z}[E],$$

$$N_1(X') = \sigma^*(N_1(X)) + \mathbf{Z}[l],$$

$$\sigma^*(\gamma^1)\bullet\sigma^*(\gamma_1) = \gamma^1\bullet\gamma_1 \text{ for any } (\gamma^1,\gamma_1)\in N(X),$$

$$\sigma^*(\gamma^1)\bullet[l] = [E]\bullet\sigma^*(\gamma_1) = 0 \text{ for any } (\gamma^1,\gamma_1)\in N(X),$$

$$[E]\bullet[l] = -1.$$

<u>Proof</u>. This is well-known and is left to the reader.

Applying this lemma to the blowing-down structure

$$V = V_m \xrightarrow{\sigma_m} V_{m-1} \xrightarrow{\sigma_{m-1}} \dots \xrightarrow{\sigma_2} V_1 \xrightarrow{\sigma_1} V_0 = \mathbb{P}_n,$$

on a gDP-variety V, we obtain:

Proposition 1. Let V be a gDP-variety of type (n,m) and

$$V = V_m \xrightarrow{\sigma_m} V_{m-1} \xrightarrow{\sigma_{m-1}} \dots \xrightarrow{\sigma_2} V_1 \xrightarrow{\sigma_1} V_0 = \mathbb{P}_n,$$

be a blowing-down structure. Then

$$N^1(V) = \mathbf{Z}h_0+\mathbf{Z}h_1+\dots+\mathbf{Z}h_m,$$

$$N_1(V) = \mathbf{Z}l_0+\mathbf{Z}l_1+\dots+\mathbf{Z}l_m,$$

where

$$h_0 = [(\sigma_1\circ\dots\sigma_m)^{-1}(H)], \text{ H is a hyperplane in } \mathbb{P}_n,$$

$$h_i = [(\sigma_i\circ\dots\circ\sigma_m)^{-1}(x^i)], \ i = 1,\dots,m,$$

$$l_0 = [(\sigma_1\circ\dots\sigma_m)^{-1}(l)], \text{ l is a line in } \mathbb{P}_n,$$

$$l_i = [(\sigma_{i+1}\circ\ldots\circ\sigma_m)^{-1}(l_i)],\ l_i \text{ is a line in } \sigma_i^{-1}(x^i) \cong \mathbb{P}_{n-1},\ i = 1,\ldots,m.$$

Moreover

$$h_0\cdot l_0 = 1,\ h_i\cdot l_i = -1,\ i \neq 0,\ h_i\cdot h_j = 0,\ i \neq j.$$

Let

$$\sigma: X' \to X$$

be a blowing up of a closed point x on a smooth variety X, and Z be a hypersurface in X such that x is a κ-multiple point of Z. We have

$$\sigma^{-1}([Z]) = Z'+\kappa\sigma^{-1}(x),$$

where Z' is the proper inverse transform of V'. For every infinitely near point $x'\to x$ of order 1 we define the <u>multiplicity</u> $\mathrm{mult}_{x'}(Z)$ <u>of Z at x'</u> as the multiplicity of Z' at x'. Proceeding by induction we can define the multiplicity of Z at an infinitely near point of arbitrary order. Thus, in the above notation, if Z is a hypersurface of degree κ_0 in $\mathbb{P}_n$ with $\mathrm{mult}_{x^i}(Z) = \kappa_i$, and Z' its proper inverse transform in $V((x^1,\ldots,x^m))$, then

$$[Z'] = \kappa_0 h_0 - \kappa_1 h_1 - \ldots - \kappa_m h_m.$$

A similar result is true for the class of the proper inverse transform of a curve in $\mathbb{P}_n$.

Proposition 2. Let $V \cong V(\hat{x})$ be a gDP-variety of type (n,m) and $[K_V]\in N^1(V)$ be its canonical class K_V modulo numerical equivalence. In the above notation

$$[K_V] = -(n+1)h_0+(n-1)(h_1+\ldots+h_m).$$

Remark 1. Recall that a <u>Del Pezzo surface</u> is usually defined as a nonsingular rational surface V with ample anti-canonical class $-K_V$ (cf. **[Ma]**). It is easy to prove that each such surface is isomorphic to a gDP variety of type (2,m) with m $\leq$ 8 obtained by blowing up a point set $\hat{x}\in\hat{\mathbb{P}}_2^m$ satisfying the following conditions:
(i) $\hat{x}$ does not contain infinitely near points;
(ii) no 3 points from $\hat{x}$ are collinear;
(iii) no 6 points from $\hat{x}$ lie on a conic;
(iv) if m = 8, $\hat{x}$ does not lie on a cubic with a singular point at one of the points from $\hat{x}$.

In the terminology of **[De]**, this means that $\hat{x}$ is in <u>"general position"</u>. We will later interpret these conditions by saying that $\hat{x}$ is an <u>unnodal point set</u>. If $m \le 6$, the anti-canonical linear system of V maps it isomorphically onto a nonsingular surface of degree $d = 9-m$ in $\mathbb{P}_d$. In general, the number $d = 9-m$ is called the degree of V. We extend the definition by defining a <u>nodal</u>(or <u>degenerate</u>) <u>Del Pezzo surface of degree d</u> by requiring that $-K_V$ is not ample but <u>almost ample</u> in the sense that for large m the linear system $|-mK_V|$ is base-point-free and defines a birational morphism onto a normal surface. This will include gDP varieties of type (2,m) with $m \le 8$ which are obtained by blowing up a point set $\hat{x} \in \hat{\mathbb{P}}_2^m$ satisfying the following conditions:

(i) $\hat{x}$ does not contain two different points which are infinitely near of order 1 to the same point;

(ii)' no 4 points from $\hat{x}$ are collinear;

(iii)' no 7 points from $\hat{x}$ lie on a conic.

In the terminology of **[De]** this means that $\hat{x}$ is in <u>"almost general position"</u>. If $m \le 6$, the image of V under the map given by the linear system $|-K_V|$ is a normal surface $\bar{V}$ of degree $d = 9-m$ in $\mathbb{P}_d$ with double rational singularities. We will call the latter surface an <u>anti-canonical Del Pezzo surface of degree d.</u> Its minimal resolution of singularities is a degenerate Del Pezzo surface of degree d. We will return to a description of Del Pezzo surfaces in Chapter VII.

2. <u>Geometric markings of gDP-varieties</u>.

Among various concepts related with the word <u>lattice</u> we use one that means a free abelian group of finite rank L equipped with a symmetric bilinear form

$$L \times L \to \mathbf{Z}, \qquad (v,v') \to v \cdot v'.$$

Tensoring L by $\mathbb{R}$ defines a quadratic form on the real vector space $L_{\mathbb{R}}$. We apply the usual terminology of the latter to L. Thus we can speak about the signature, rank, etc. of L. For our purposes we need a slightly more general concept of a <u>bilattice</u>. We define it to be a pair (L_1,L_2) of free abelian groups of finite rank equipped with a bilinear form

$$L_1 \times L_2 \to \mathbf{Z}, \qquad (v_1,v_2) \to v_1 \cdot v_2.$$

A lattice L is considered as a bilattice (L,L). One naturally defines a <u>morphism</u> of bilattices

$$\varphi = (\varphi_1,\varphi_2) : (L_1,L_2) \to (L_1',L_2')$$

as a pair of homomorphisms of abelian groups $\varphi_i:L_i \to L_i'$ satisfying

$$\varphi_1(v_1)\cdot\varphi_2(v_2) = v_1\cdot v_2 \qquad \text{for any } v_1\in L_1, v_2\in L_2.$$

Every bilattice (L_1,L_2) admits natural morphisms to the bilattice (L_i,L_i^*), where

$$L_i^* = \mathrm{Hom}_{\mathbf{Z}}(L_i,\mathbf{Z})$$

is the dual abelian group, and $x\cdot x^* = x^*(x)$ for every $x\in L_i$, $x^*\in L_i^*$, $i = 1,2$. A bilattice is said to be <u>unimodular</u> if these morphisms are isomorphisms.

Our main example of a lattice is the <u>standard hyperbolic lattice</u> of rank m+1

$$H_m = \mathbf{Z}e_0+\mathbf{Z}e_1+\ldots+\mathbf{Z}e_m,$$

where

$$e_0\cdot e_0 = 1,\ e_i\cdot e_i = -1,\ i \neq 0,\ e_i\cdot e_j = 0,\ i \neq j.$$

The Neron-Severi bilattice $N(X)$ of a smooth complete variety X gives an example of a bilattice. Similarly, the <u>homology bilattices</u>

$$(H^i(X,\mathbf{Z})/\mathrm{Tors},\ H_i(X,\mathbf{Z})/\mathrm{Tors})$$

are examples of unimodular bilattices.

Proposition 3. Let V be a gDP-variety. In the notation of Proposition 1 the maps

$$\varphi^1: H_m \to N^1(V), \qquad e_i \to h_i,$$

$$\varphi_1: H_m \to N_1(V), \qquad e_i \to l_i,$$

define an isomorphism of bilattices

$$\varphi = (\varphi^1,\varphi_1) : H_m \to N(V).$$

In particular, $N(V)$ is unimodular.

Let $L = (L_1,L_2)$ be a bilattice. We define a <u>L-marking</u> of a smooth complete variety X as an isomorphism of bilattices:

$$\varphi: L \to N(X).$$

An <u>L-marked variety</u> X is a pair (X,φ), where φ is a L-marking. An isomorphism of L-markings (or of L-marked varieties) is an isomorphism $f: X \to Y$ such that

$$f^*\circ\varphi' = \varphi.$$

Lemma 2. Let $f\colon X \dashrightarrow X'$ be a pseudo-isomorphism of smooth complete varieties. Assume that the Neron-Severi bilattices of X and X' are unimodular. Then there exists a natural isomorphism of these bilattices:

$$f^*\colon N(X') \to N(X).$$

Proof. The pseudo-isomorphism f defines an isomorphism $f'\colon U \to U'$ of open subsets whose complements are of codimension ≥ 2. Then we have a composition of isomorphisms of groups:

$$A^1(X') \xrightarrow{r'} A^1(U') \xrightarrow{f^*} A^1(U) \xrightarrow{r^{-1}} A^1(X),$$

where $r\colon A^1(X) \to A^1(U)$ and $r'\colon A^1(X') \to A^1(U')$ are the restriction homomorphisms. The composition $A^1(X') \to A^1(X)$ induces an isomorphism

$$(f^*)^1\colon N^1(X') \to N^1(X).$$

Since $N(X')$ and $N(X)$ are unimodular, the groups $N_1(X)$ and $N_1(X')$ can be identified with the dual groups $N^1(X)^*$ and $N^1(X')^*$ respectively. This allows us to set

$$(f^*)_1 = {}^t((f^*)^1)^{-1},$$

to obtain that the pair $f^* = ((f^*)^1,(f^*)_1)$ is an isomorphism of bilattices.

The previous lemma allows us to define a pseudo-isomorphism of L-marked varieties (X,φ) and (X',φ') as a pseudo-isomorphism $f\colon X \dashrightarrow X'$ such that $f^*\circ\varphi' = \varphi$.

Remark 2. The assertion of the previous lemma is probably true without the assumption of the unimodularity of the Neron-Severi bilattices. It can be verified for example in the case $\dim X = 3$ by applying Danilov's theorem on the factorization of small birational morphisms (**[Da]**).

Here comes our main definition:

A strict geometric marking of a gDP-variety V of type (n,m) is an H_m-marking

$$\varphi = (\varphi^1,\varphi_1)\colon (H_m, H_m) \to (N^1(V), N_1(V))$$

defined via a blowing-down structure on X by

$$\varphi^1(e_i) = h_i,\ i = 0,\dots,m,$$

$$\varphi_1(e_i) = l_i, \ i = 0,\dots,m,$$

in the notation of Proposition 1. A geometric marking of V is an H_m-marking pseudo-isomorphic to a strict geometric marking.

Lemma 3. Let f: $\mathbb{P}_n \dashrightarrow \mathbb{P}_n$ be a pseudo-isomorphism. Then f extends to an isomorphism $\bar{f}$: $\mathbb{P}_n \to \mathbb{P}_n$.

Proof. The rational map f is given by a linear system W of hypersurfaces of some degree d > 0. Since f^* induces an isomorphism $A^1(\mathbb{P}_n) \to A^1(\mathbb{P}_n)$, both groups being isomorphic to $\mathbf{Z}$, we obtain that d = 1. Thus f is given by a linear system of hyperplanes, hence is a projective isomorphism.

Corollary. Two strict geometric markings of gDP-varieties of type (n,m) are pseudo-isomorphic if and only if they are isomorphic.

Proof. Induction on m. If m = 0 this is asserted in Lemma 3. Let

$$V = V_m \xrightarrow{\sigma_m} V_{m-1} \xrightarrow{\sigma_{m-1}} \dots \xrightarrow{\sigma_2} V_1 \xrightarrow{\sigma_1} V_0 = \mathbb{P}_n,$$

$$V' = V'_m \xrightarrow{\sigma'_m} V'_{m-1} \xrightarrow{\sigma'_{m-1}} \dots \xrightarrow{\sigma'_2} V'_1 \xrightarrow{\sigma'_1} V'_0 = \mathbb{P}_n,$$

be two blowing-down structures corresponding to the given pseudo-isomorphic strict geometric markings of gDP-varieties V and V'. Obviously they define pseudo-isomorphic strict geometric markings of gDP-varieties V_{m-1} and V'_{m-1} of type (n,m−1). By induction they are isomorphic. The corresponding isomorphism f': $V_{m-1} \to V'_{m-1}$ sends the image of the exceptional divisor of σ_m to the image of the exceptional divisor of σ'_m and hence lifts to an isomorphism f: V → V'. Obviously, it defines the needed isomorphism.

From the previous definitions and results we obtain:

Theorem 1. There is a natural bijective correspondence between the set of G-orbits in $\mathbb{P}_n^m$ and the set of pseudo-isomorphism classes of geometrically marked gDP-varieties of type (n,m).

3. The Weyl groups $W_{n,m}$.

We are looking for a group that acts on the G-orbits in $\hat{\mathbb{P}}_n^m$ by acting on the geometric markings. A natural candidate for this group is the isometry group $O(H_m)$ of the lattice H_m. It certainly acts on H_m-markings φ: $H_m \to N(X)$ by composing them with isometries σ: $H_m \to H_m$. However, there is no reason to expect that this action preserves the subset of geometric markings. Thus we are led to look for a suitable subgroup of $O(H_m)$ which will consist of isometries (= automorphisms) of H_m preserving the set of geometric markings. It turns out that the right subgroup is the Weyl group of a certain natural root basis in H_m.

Let us recall the necessary definitions (cf. **[Lo 2]**).

A root basis in a bilattice $L = (L_1,L_2)$ is a pair $(B,\check{B})$ of subsets of L_1 and L_2, respectively, together with a bijection $B \to \check{B}$, $\alpha \to \check{\alpha}$, satisfying:

(i) $\alpha \cdot \check{\alpha} = -2$;

(ii) $\alpha \cdot \check{\beta} \geq 0$ for any $\alpha,\beta \in B$, $\alpha \neq \beta$.

A root basis is said to be symmetric if the following additional property holds:

(iii) $\alpha \cdot \check{\beta} = \beta \cdot \check{\alpha}$ for any $\alpha,\beta \in B$.

For every $\alpha \in B$ the formulae

$$s_\alpha: x_1 \to x_1 + (x_1 \cdot \check{\alpha})\alpha, \qquad \text{for any } x_1 \in L_1,$$

$$\check{s}_\alpha: x_2 \to x_2 + (x_2 \cdot \alpha)\check{\alpha}, \qquad \text{for any } x_2 \in L_2$$

define linear involutions of L_1 and L_2 respectively, called simple reflections. The subgroup of $GL(L_1)$ (resp. of $GL(L_2)$) generated by such transformations is denoted by W_B (resp. $W_{\check{B}}$). The map $s_\alpha \to \check{s}_\alpha$ extends to an isomorphism:

$$W_B \to W_{\check{B}}, \quad w \to \check{w}.$$

Each of these groups is called the Weyl group of the root basis $(B,\check{B})$. We will denote it by W if no confusion arises.

For any $w \in W_B$ we have

$$w(x_1) \cdot \check{w}(x_2) = x_1 \cdot x_2, \qquad \text{for any } x_1 \in L_1, x_2 \in L_2.$$

This shows that W is isomorphic to a subgroup of the isometry group $O(L)$.

An element of a W_B-orbit of B in L_1 (resp. of $\check{B}$ in L_2) is called a B-root (resp. $\check{B}$-root). The set of such elements is denoted by R_B (resp. $R_{\check{B}}$). An element of B (resp. $\check{B}$) is said to be a <u>simple B-root</u> (resp. <u>simple $\check{B}$-root</u>) The bijection $\alpha \to \check{\alpha}$ between simple B-roots and simple $\check{B}$-roots extends naturally to a bijection $\alpha \to \check{\alpha}$ from R_B to $R_{\check{B}}$:

$$w(\alpha) \to \check{w}(\check{\alpha}) \text{ for any } \alpha \in B,\ w \in W_B.$$

A B-root α is called <u>positive</u> (resp. <u>negative</u>) if it can be written as a linear combination of simple B-roots with integral non-negative (resp. non-positive) coefficients. Let R_B^+ (resp. R_B^-) denote the set of positive (resp. negative) B-roots. It can be shown (see **[Kac]**, **[Lo 2]**) that

$$R_B = R_B^+ \sqcup R_B^-,$$

$$R_B^- = \{-\alpha\colon \alpha \in R_B^+\}.$$

Similar definitions and corresponding properties hold for $\check{B}$-roots.

We denote by $Q(B,\check{B}) = (Q(B),Q(\check{B}))$ the sub-bilattice of (L_1,L_2) spanned by the subsets B and $\check{B}$ of L_1 and L_2, respectively.

For every root basis $(B,\check{B})$ one can define its <u>Dynkin diagram</u> (oriented graph) $\Gamma(B,\check{B})$ by assigning to every simple B-root $\alpha \in B$ a vertex α and joining two distinct vertices α and β by $\alpha \cdot \check{\beta}$ arcs ending at β. If the root basis is symmetric we forget about the orientation of $\Gamma(B,\check{B})$. It is easy to see that in this case all B-roots (resp. $\check{B}$-roots) are W-equivalent if and only if $\Gamma(B,\check{B})$ is connected and all vertices are joined by at most one arc.

The generating set $S = \{s_i = s_{\alpha_i}\}_{i=0,\dots,m}$ of W_B satisfies the relations:

$s_i^2 = 1$, $(s_i \cdot s_j)^3 = 1$ if α_i and α_j are connected in $\Gamma(B,\check{B})$ by one arc,

$(s_i \cdot s_j)^2 = 1$ if α_i and α_j are not connected in $\Gamma(B,\check{B})$,

$s_i \cdot s_j$ is of infinite order otherwise.

One can show that these relations are the basic relations for the s_i's and that the pair (W,S) is a Coxeter group (see **[Kac]**).

Returning to our situation when L is equal to H_m , $m \geq n+1 \geq 3$, we define a <u>canonical root basis of type n</u> >1 in H_m by setting:

$$B_n = \{\alpha_0,\dots,\alpha_{m-1}\},\ \check{B}_n = \{\check{\alpha}_0,\dots,\check{\alpha}_{m-1}\},$$

where

$$\alpha_0 = e_0-e_1-\dots-e_{n+1},\ \alpha_i = e_i-e_{i+1},\ i = 1,\dots,m-1,$$

$$\check{\alpha}_0 = (n-1)e_0-e_1-\dots-e_{n+1},\quad \check{\alpha}_i = \alpha_i = e_i-e_{i+1},\quad i = 1,\dots,m-1.$$

It is a symmetric root basis. Denote its corresponding Weyl group by $W_{n,m}$.

Let

$$k_{n,m} = (n+1)e_0-e_1-\dots-e_m,$$

$$K_{n,m} = (n+1)e_0-(n-1)(e_1+\dots+e_m).$$

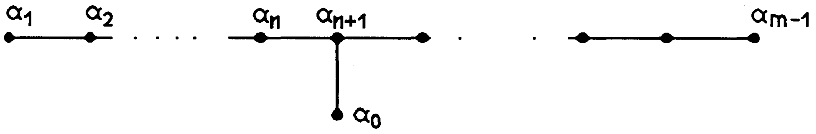
Then

$$Q(B_n) = (\mathbf{Z}k_{n,m})^{\perp} = \{v\in H_m : v\bullet k_{n,m} = 0\},$$

$$Q(\check{B}_n) = (\mathbf{Z}K_{n,m})^{\perp} = \{v\in H_m : v\bullet K_{n,m} = 0\}.$$

Note that the subgroup of $W_{n,m}$ generated by the simple reflections s_{α_i}, $i\neq 0$, is isomorphic to the permutation group Σ_m. It is the Weyl group of the root basis $(B\setminus\{\alpha_0\},\check{B}\setminus\{\check{\alpha}_0\})$. It acts on the set $\{e_1,\dots,e_m\}$ by permutations. The simple reflection s_{α_i} acts as the transposition $(i,i+1)$.

The Dynkin diagram of a canonical root basis of type n in H_m looks as follows:



If $n = 2$, $m = 3,\dots,8$, we recognize the familiar Dynkin diagrams of root systems of finite-dimensional simple Lie algebras of type

$$A_1+A_2,\ A_4,\ D_5,\ E_6,\ E_7 \text{ and } E_8,$$

of rank $3,\dots,8$, respectively.

If $m = n+2$ (resp. $n+3$), we obtain the Dynkin diagram of type A_m (resp. D_m). If $n = 3$, $m = 7$, we get the diagram of type E_7. In all other cases we have the Dynkin diagram of type $T_{2,n+1,m-n-1}$ of an infinite-dimensional simple Kac-Moody algebra (see **[Kac]**).

We say that a canonical basis in H_m is of <u>finite type</u> if

$$(n,m)\in\{(2,3),(2,4),(2,5),(2,6),(2,7),(2,8),(3,7),(n,n+2),(n,n+3)\}$$

It is easy to see that a canonical basis $(B,\check{B})$ is of finite type if and only if the set of B-roots R_B is finite, or if and only if the Weyl group W_B is finite. We use the notation $W(A_n)$, $W(D_n)$, $W(E_n)$ to denote the group $W_{n,m}$ in these cases.

Remark 3. By Proposition 2, for every strict geometric marking $\varphi:H_m \to N(V)$ we have that

$$\varphi^1(K_{n,m}) = -K_V.$$

Since the canonical class is invariant under a pseudo-isomorphism, the same is true for any geometric marking of a gDP-variety. What is the geometric significance of $\varphi_1(\kappa_{n,m})$ expressing the virtual class of a normal elliptic curve through m points?

The next proposition gives us a partial description of B_n-roots in H_m:

Proposition 4. Let $\alpha = a_0e_0-a_1e_1-\ldots-a_m e_m$ be a positive B_n-root in H_m. Then

(i) $(n+1)a_0-a_1-\ldots-a_m = \alpha\cdot\kappa_{n,m} = 0;$

(ii) $(n-1)a_0{}^2-a_1{}^2-\ldots-a_m{}^2 = -2;$

(iii) $a_0 \geq 0$, and if $a_0= 0$, then $\alpha = e_i-e_j$ for some $1 \leq i < j \leq m;$

Assume $a_0 > 0$. Then

(iv) $(n-1)a_0 < a_{i_1}+\ldots+a_{i_{n+1}}$ if $a_{i_1} \geq a_{i_2} \geq \ldots \geq a_{i_m}, i_j\in\{1,\ldots,m\};$

(v) $a_i \geq 0$ for $i = 1,\ldots,m;$

(vi) $(n-1)a_0 \geq a_{i_1}+\ldots+a_{i_n}$ if $a_0>1$, $a_{i_1} \geq a_{i_2} \geq \ldots \geq a_{i_m}$.

Proof. (i) Note that for every simple B_n-root α_i we have

$$\alpha_i\cdot((n+1)e_0-e_1-\ldots-e_m) = 0.$$

Therefore the action of $W_{\check{B}}$ in H_m leaves the vector

$$\kappa_{n,m} = (n+1)e_0-e_1-\ldots-e_m$$

invariant. Hence for every $w\in W$ and $\alpha_i\in B_n$

$$w(\alpha_i)\cdot\kappa_{n,m} = \alpha_i\cdot\check{w}^{-1}(\kappa_{n,m}) = \alpha_i\cdot\kappa_{n,m} = 0.$$

If $\alpha = w(\alpha_i)\in R_{B_n}$ this equality is equivalent to (i).

(ii) Let $\alpha = w(\alpha_i)\in R_{B_n}$. Then

$$\alpha\cdot\check{\alpha} = w(\alpha_i)\cdot\check{w}(\check{\alpha}_i) = \alpha_i\cdot\check{\alpha}_i = -2.$$

This implies (ii).

(iii) In the decomposition of α as a sum of simple roots the root $\alpha_0=e_0-e_1-\ldots-e_{n+1}$ enters with nonnegative coefficient. This implies that $a_0 \geq 0$. If $a_0 = 0$, we apply (i)

and (ii) to obtain that $\alpha = e_i - e_j$ for some $i \neq j$. Since α is positive, we obtain $i < j$.
(iv) Applying elements $w \in \Sigma_m$ to α we may assume that $(i_1,\dots,i_m) = (1,\dots,m)$. Assume the opposite inequality holds. Then

$$\alpha \cdot \check{\alpha}_0 = (n-1)a_0 - a_1 - \dots - a_{n+1} \geq 0 \text{ and}$$
$$\alpha \cdot \check{\alpha}_i = a_i - a_{i+1} \geq 0,\ i = 1,\dots,m,$$

This means that

$$\alpha \in C(B_n) = \{v \in H_m : v \cdot \check{\alpha}_i \geq 0 \text{ for all } \check{\alpha}_i \in \check{B}_n\}.$$

Now we use the following property of $C(B_n)$ (called the fundamental chamber):

(*) Let $w \in W$ be written as a product of a minimal number of simple reflections (which is well-defined and is denoted by $l(w)$), then for every $\alpha_i \in B_n$

$$w(C(B_n)) \subset \{v \in H_m : v \cdot \check{\alpha}_i \leq 0\} \text{ if and only if } l(s_{\alpha_i} \circ w) < l(w).$$

(see **[Bo]** Chapter V, §4, n°4).

Let us argue by induction on $k = l(w)$ that, for any $w \in W$,

(**) $$w(\alpha) = \alpha + c_0\alpha_0 + \dots + c_{m-1}\alpha_{m-1}, \text{ with all } c_i \geq 0.$$

If $k = 1$, $w = s_{\alpha_i} \in B_n$ for some i and $\alpha \cdot \alpha_i \geq 0$ for every $\alpha_i \in B_n$, hence

$$w(\alpha) = s_{\alpha_i}(\alpha) = \alpha + (\alpha \cdot \check{\alpha}_i)\alpha_i = \alpha + c_i\alpha_i \text{ for some } c_i \geq 0.$$

If $k > 1$, we write $w = s_{\alpha_i} \cdot w'$ with $l(w') = l(w) - 1$ and apply (*) to obtain

$$0 \geq w(\alpha) \cdot \check{\alpha}_i = s_{\alpha_i} \circ w'(\alpha) \cdot \check{\alpha}_i = w'(\alpha) \cdot s_{\alpha_i}(\check{\alpha}_i) = -w'(\alpha) \cdot \check{\alpha}_i.$$

By induction

$$w'(\alpha) = \alpha + c_0'\alpha_0 + \dots + c_{m-1}'\alpha_{m-1}, \text{ with } c_i' \geq 0,$$

hence we have

$$w(\alpha) = s_{\alpha_i} \circ w'(\alpha) = w'(\alpha) + (w'(\alpha) \cdot \check{\alpha}_i)\alpha_i = \alpha + c_0'\alpha_0 + \dots + c_{m-1}'\alpha_{m-1} + c'\alpha_i$$

for some $c_i', c' \geq 0$. Now choose w so that $\alpha_1 = w(\alpha)$ and observe that the equality

$$\alpha = \alpha_1 - (c_0'\alpha_0 + \dots + c_{m-1}'\alpha_{m-1} + c'\alpha_i)$$

contradicts the assumption that α is positive, unless $\alpha = \alpha_1$ in which case (iv) holds.
(v) We use induction on a_0. The assertion is true for $a_0 = 1$. Indeed subtracting equality (ii) from equality (i) we obtain

$$\sum_{i=1}^{m} a_i(a_i - 1) = 0.$$

This implies that each a_i is equal to 0 or 1. Assume that the assertion is true for all α with $1 < a_0 = \alpha\cdot f_0 < N$. We may assume, as in the previous case, that $a_1 \geq \ldots \geq a_m$. By (iv)

$$a = \alpha\cdot\check{\alpha}_0 = (n-1)a_0 - \sum_{i=1}^{n+1} a_i < 0,$$

and

$$\alpha' = s_{\alpha_0}(\alpha) = \alpha + a\alpha_0 = (a_0+a)e_0 - \sum_{i=1}^{n+1}(a_i+a)e_i - \sum_{i=n+1}^{m} a_i e_i.$$

It is known that for every simple root α_i we have

$$\sigma_{\alpha_i}(R_{B_n}^{+}\setminus\{\alpha_i\}) \subset R_{B_n}^{+}$$

([**Kac**], Lemma 1.3). This shows that $a_0+a \geq 0$. If $a_0+a > 0$, we are done by induction.

Finally, if $a_0+a = 0$, we apply (iii) to obtain

$$-\sum_{i=1}^{n+1}(a_i+a)e_i - \sum_{i=n+1}^{m} a_i e_i = \pm(e_j - e_k)$$

for $1 \leq j < k \leq m$. Thus at least n-1 coefficients a_i+a, $1 \leq i \leq n+1$, must be equal to zero. Equivalently, at least n-1 coefficients a_i, $1 \leq i \leq n+1$, must be equal to a_0. By examining (i) and (ii) we deduce that $a_0 \leq 1$ again.

(vi) Assume the contrary. Applying s_{α_0} to $\alpha \neq \alpha_0$, we find as above that

$$\alpha' = s_{\alpha_0}(\alpha) = a_0'e_0 - \sum_{i=n+1}^{m} a_i' e_i \in R_{B_n}^{+},$$

where

$$a_{n+1}' = (n-1)a_0 - \sum_{i=1}^{n} a_i < 0, \; a_{n+2}' = a_{n+2}.$$

By (v)

$$a_0' = na_0 - \sum_{i=1}^{n+1} a_i = 0,$$

and, by (iii),

$$a_{n+3}' = 0, \; a_{n+2} = 0 \text{ or } 1.$$

Applying (i), we obtain

$$0 = (n+1)a_0 - \sum_{i=1}^{m} a_i = a_0 - a_{n+2}.$$

Thus $a_0 \leq 1$, which is excluded by the assumption.

Corollary. Let $\alpha = a_0e_0 - a_1e_1 - \ldots - a_me_m$ be a positive $\check{B}_n$-root in H_m. Then $a_0 = (n-1)a_0'$ for some integer a_0', and

(i) $(n+1)a_0 - (n-1)(a_1+\ldots+a_m) = \alpha\cdot K_{n,m} = 0$;

(ii) $a_0^2 - (n-1)(a_1^2+\ldots+a_m^2) = -2(n-1)$;

(iii) $a_0 \geq 0$, and, if $a_0 = 0$, $\alpha = e_i - e_j$ for some $1 \leq i < j \leq m$;

Assume $a_0 > 0$. Then

(iv) $a_0 < a_1 + \ldots + a_{n+1}$ if $a_1 \geq a_2 \geq \ldots \geq a_m$;

(v) $a_0 \geq 0$, and $a_1, \ldots, a_m \geq 0$;

(vi) $a_0 \geq a_1 + \ldots + a_n$ if $a_0 > 1$, $a_1 \geq a_2 \geq \ldots \geq a_m$.

Proof. Use that the bijection $\alpha \rightarrow \check{\alpha}$ between R_{B_n} and $R_{\check{B}_n}$ is given by the formula:

$$a_0 e_0 \ - a_1 e_1 - \ldots - a_m e_m \ \rightarrow \ a_0(n-1)e_0 \ - a_1 e_1 - \ldots - a_m e_m .$$

Remark 4. We will see in Proposition 6 that in case $n = 2$, $m \leq 10$, properties (i) and (ii) alone imply that α is a B_2-root. However, already if $n = 2$, $m = 11$, it is not true that every vector $\alpha = a_0 e_0 - a_1 e_1 - \ldots - a_m e_m$ satisfying (i)-(vi) is a B-root. For example

$$\alpha \ = \ 7e_0 - 3e_1 - 3e_2 - 3e_3 - 3e_4 - 3e_5 - e_6 - e_7 - e_8 - e_9 - e_{10} - e_{11}$$

satisfies (i) - (vi) but is not a B_2-root. To see this we apply s_{α_0} to α to obtain

$$\alpha' = s_{\alpha_0}(\alpha) \ = \ 5e_0 - e_1 - e_2 - e_3 - 3e_4 - 3e_5 - e_6 - e_7 - e_8 - e_9 - e_{10} - e_{11},$$

which does not satisfy (vi). Therefore α' is not a B_2-root, hence α is not a B_2-root.

The following well-known result shows that in the case $n = 2$ properties (i), (ii) and (v) imply (iv).

Proposition 5 (Noether's inequality). Let $\alpha = a_0 e_0 - a_1 e_1 - \ldots - a_m e_m \in H_m$ satisfy the following properties:

(i) $a_0 > 0$, $a_1 \geq \ldots \geq a_m \geq 0$;

(ii) $a = a_0^2 - a_1^2 - \ldots - a_m^2 \in \{-2, -1, 0, 1\}$;

(iii) $3a_0 - a_1 - \ldots - a_m = a + 2$;

Then either

$$a_0 < a_1 + a_2 + a_3,$$

or $\alpha = e_0 - e_1$, or $\alpha = e_0$.

Proof (following **[Coo]**). We have

$$0 \leq a_3(a_4 + \ldots + a_m) - (a_4^2 + \ldots + a_m^2) =$$

$$= a_3(3a_0 - a_1 - a_2 - a_3 - a - 2) - (a_0^2 - a - a_1^2 - a_2^2 - a_3^2) =$$

$$= a_1^2+a_2^2-a_3(a_1+a_2)-(a_0-3a_3)a_0+a-a_3(a+2).$$

Assume

$$a_0 \geq a_1+a_2+a_3.$$

Then

$$0 \leq a_1^2+a_2^2-a_3(a_1+a_2)-(a_0-3a_3)a_0+a-a_3(a+2) \leq$$
$$\leq a_1^2+a_2^2-a_3(a_1+a_2)-(a_1+a_2-2a_3)(a_1+a_2+a_3)+a-a_3(a+2) =$$
$$= a_1^2+a_2^2-a_3(a_1+a_2) - (a_1+a_2)^2+a_3(a_1+a_2)+2a_3^2+a-a_3(a+2) =$$
$$= 2(a_3^2-a_1a_2)+a-a_3(a+2) .$$

This is only possible if $a_2 = a_3 = 0$, which implies $\alpha = e_0$ if $a = 1$, and $\alpha = e_0-e_1$ if $a = 0$.

Proposition 6. Let $(B,\check{B})$ be a canonical root basis of type 2 in H_m. Assume that $m \leq 10$. Then

$$R_B = R_{\check{B}} = \{v\epsilon H_m:\ v\bullet v = -2,\ v\bullet K_{2,m} = 0\}.$$

Proof. By Proposition 4 the set R_B is a subset of the R.H.S.. Assume $m \leq 8$. Then $(B,\check{B})$ is of finite type and the set R_B is a root system (in the sense of **[Bo]**) in the space

$$E_B = \{v\epsilon H_m\otimes\mathbb{R}:\ v\bullet K_{2,m} = 0\}.$$

In this case the result is well-known. We recall its proof. Let v be a vector from the R.H.S. Since B spans E_B we can write

$$v = \beta_1+...+\beta_s$$

for some (not necessary simple) roots β_i. Choose such a representation with minimal s. We have (**[Bo]**, Chapter VI, **§3**, Theorem.1),

$$\beta_i\bullet\beta_j \leq 0$$

unless $\beta_i = -\beta_j$, or $\beta_i+\beta_j\epsilon R_B$. By the minimality of s neither of these cases occurs. Thus

$$v^2 \leq \beta_1\bullet\beta_1+...+\beta_s\bullet\beta_s = -2s.$$

Since $v^2 = -2$ this implies $s = 1$, hence $v\epsilon R_B$.

Assume $m = 9$. Then

$$K_{2,9}\bullet K_{2,9} = 0,$$

hence for any $v = a_0e_0-a_1e_1-\dots-a_9e_9\in R_B$

$$v' = v-a_9K_{2,9} = (a_0-3)e_0-(a_1-1)e_1-\dots-(a_8-1)e_8$$

satisfies

$$v'\cdot v' = -2,\ v'\cdot K_{2,8} = 0.$$

By the previous case $v'\in R_{B'} \subset R_B$, where $B' = B\setminus\{\alpha_8\}$. In particular

$$\beta = K_{2,9}-\alpha_8 \in R_B.$$

Now

$$s_\beta\circ s_{\alpha_8}(v) = s_\beta(v+(v\cdot\alpha_8)\alpha_8) = s_\beta(v+a_9\alpha_8) = v+(\beta\cdot v)\beta+a_9(\alpha_8+(\alpha_8\cdot\beta)\beta) =$$

$$= v-(\alpha_8\cdot v)\beta+a_9(\alpha_8+2\beta) = v-a_9\beta+a_9K_{2,9}+a_9\beta = v+a_9K_{2,9} = v'.$$

Since v' is a B-root, $v' \in w(B)$ for some $w\in W$, hence

$$v \in ((s_\beta\circ s_{\alpha_8})^{-1}\circ w)(B)$$

is a B-root.

Assume $m = 10$. Then $(B,\check{B})$ is a hyperbolic (or crystallographic) root basis in the sense that W_B is of finite index in the isometry group $O(Q_B)$ of its root lattice. By **[Vi]**, every reflection $s_\gamma = x \to x+(x\cdot\gamma)\gamma$, $\gamma\in Q_B$, $\gamma^2=-2$, is conjugate in W_B to some reflection s_α, $\alpha\in R_B$. This implies the assertion.

Remark 5. As we saw in the previous Remark, Proposition 6 cannot be extended to the case $m > 10$.

4. <u>Discriminant conditions</u>.

Let $\varphi:H_m \to N(V)$ be a geometric marking of a gDP-variety V of type (n,m) and $(B,\check{B})$ be a canonical root basis of type n in H_m. We set

$$R_B(\varphi)^+ = \{\alpha\in R_B:\ \varphi^1(\alpha) \text{ is effective}\},$$

$$R_{\check{B}}(\varphi)^+ = \{\alpha\in R_{\check{B}}:\ \varphi_1(\alpha) \text{ is effective}\}.$$

If $x\in\hat{P}^m_n$ and $\varphi_x:H_m \to N(V(x))$ is the corresponding strict geometric marking we define:

$$R_B(x)^+ = R_B(\varphi_x)^+,$$

$$R_{\check{B}}(x)^+ = R_{\check{B}}(\varphi_x)^+.$$

The elements of the set $R_B(\varphi)^+$ (resp. $R_{\check{B}}(\varphi)^+$) are called <u>effective (or nodal)</u>

B-roots (resp. effective $\check{B}$-roots) with respect to the geometric marking φ. Also, the elements of $R_B(x)^+$ are said to be the discriminant conditions on the point set x. A point set x (resp. a geometric marking φ) is said to be unnodal if $R_B(x)^+ = \emptyset$ (resp. $R_B(\varphi) = \emptyset$).

Note that

$$R_B(\varphi)^+ \subset R_B{}^+.$$

We say that a gDP-variety V is unnodal if all of its geometric marking are unnodal.

Proposition 7. Assume that a gDP-surface V admits an unnodal geometric marking. Then V is unnodal.

Proof. As follows from Theorem 2 of the next Chapter, for every two geometric markings $\varphi:H_m \to N(V)$, $\varphi:H_m \to N(V)$ there exists $w \in W_{2,m}$ such that

$$\varphi = \varphi \circ w.$$

Thus

$$\alpha \in R_B(\varphi)^+ \Leftrightarrow w(\alpha) \in R_B(\varphi)^+.$$

This proves the proposition.

Corollary. Let V be a gDP-surface. Assume $m \le 8$. The following properties are equivalent:

(i) V is unnodal for some geometric marking $\varphi:H_m \to N(V)$;

(ii) V is a Del Pezzo surface (see Remark 1);

(iii) the anti-canonical model

$$\bar{V} = \mathrm{Proj}(\bigoplus_{r=0}^{\infty} \Gamma(V, \mathcal{O}_V(-rK_V)))$$

is a nonsingular surface.

Proof. (i) $\Leftrightarrow$ (ii) We have to show that to be unnodal is equivalent to satisfying the following properties:

(a) $\hat{x}$ does not contain infinitely near points;

(b) no 3 points from $\hat{x}$ are collinear;

(c) no 6 points from $\hat{x}$ lie on a conic;

(d) if $m = 8$, $\hat{x}$ does not lie on a cubic with a singular point at one of the points from $\hat{x}$.

Using Proposition 4, we can easily find all positive B-roots with respect to a canonical root basis in H_m of type 2, $m \le 8$. They are:

$$\alpha = e_i - e_j,\ 1 \le i < j \le m,$$

$$\alpha = e_0 - e_i - e_j - e_k,\ i \ne j \ne k,\ i,j,k \ne 0,$$

$$\alpha = 2e_0 - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5} - e_{i_6},\ 1 \le i_k < i_s \le m \text{ if } k < s,$$

$$\alpha = 3e_0 - e_1 - \ldots - e_8 - e_i,\ 1 \le i \le 8.$$

This obviously proves the assertion.

(ii) $\Leftrightarrow$ (iii). This is well-known. See **[De]**, **[Ma]**.

Proposition 8. Let V be a Del Pezzo surface. An H_m-marking $\varphi: H_m \to N(V)$ is geometric if and only if $\varphi(K_{2,m}) = -K_V$.

Proof. The condition is obviously necessary. Let

$$\varphi(e_i) = h_i,\ i = 1,\ldots,m.$$

Assume $\varphi(K_{2,m}) = -K_V$. Then

$$h_i^2 = -1,\ h_i \bullet K_V = -1.$$

By Riemann-Roch

$$h^0(h_i) + h^0(K_V - h_i) \ge 1.$$

Since

$$(K_V - h_i)\bullet(-K_V) = -K_V^2 - 1 < 0,$$

and $-K_V$ is ample, $h^0(K_V - h_i) = 0$. Thus e_i is effective. Since $-K_V$ is ample and $h_i \bullet (-K_V) = 1$, $h_i = [E_i]$ for some irreducible curve, which must be an exceptional curve of the first kind. This shows that $\{E_1,\ldots,E_m\}$ is the set of m disjoint exceptional curves of the first kind. It can be blown down to define a geometric marking of V equal to φ.

5. Exceptional configurations.

An ordered sequence $(v_1,\ldots,v_r)$ of vectors from H_m satisfying $v_i \bullet v_i = -1$ is called an exceptional r-sequence (cf. **[De]**).

We denote the set of exceptional r-sequences by $\mathcal{E}_m(r)$. Let $\mathcal{E}_{n,m}(r)'$ denote the $W_{n,m}$-orbit of the sequence $(e_1,\ldots,e_r)$. Clearly $e_i \in \mathcal{E}_{n,m}(1)'$ for all $i > 0$.

Let Z be a nonsingular complete variety. An ordered sequence of elements

$(h_1,\dots,h_r)$ from $N^1(Z)$ is called an <u>exceptional r-configuration</u> if Z is isomorphic to the blowing-up $Z'(x)$ of some nonsingular variety Z' at some point set $x = (x^1,\dots,x^r)\in Z'^r$, and each h_i is equal to the class of the inverse image of x^i under the blowing-up map $Z \rightarrow Z'(x^1,\dots,x^{i-1})$.

We denote the set of exceptional r-configurations on Z by $\mathcal{E}_r(Z)$. For example, if $\varphi:H_m \rightarrow N(V)$ is a strict geometric marking of a gDP-variety V, the sequence $(\varphi(e_1),\dots,\varphi(e_m))$ is an exceptional m-configuration on V.

Lemma 4. Assume $m \le 9$, $n = 2$. Then $W_{2,m}$ acts transitively on the set $\mathcal{E}_m(r)$ if $r \ne m-1$, and $W_{2,m}$ has 2 orbits in $\mathcal{E}_m(r)$ represented by the sequences $(e_1,\dots,e_{m-1})$ and $(e_0-e_1-e_2,e_3,\dots,e_m)$ if $r = m-1$.

<u>Proof</u>. See **[De]**, **[Ma]**.

Proposition 9. Let $\varphi:H_m \rightarrow N(V)$ be a geometric marking of a gDP-surface V. Assume that $|-K_V|$ contains an irreducible divisor. Then

$$\mathcal{E}_1(V) = \varphi(\mathcal{E}_{2,m}(1)').$$

<u>Proof</u>. See **[Lo 1]**.

Corollary. Assume $m \le 9$ and $|-K_V|$ contains an irreducible divisor. Then

$$\mathcal{E}_1(V) = \varphi(\mathcal{E}_m(1)).$$

Proposition 10. Let V be a Del Pezzo surface. Then $\mathcal{E}_1(V)$ is equal to the set of exceptional curves of the first kind on V, and for every geometric marking $\varphi:H_m \rightarrow N(V)$

$$\mathcal{E}_m(V) = \varphi(\mathcal{E}_{2,m}(m)').$$

<u>Proof</u>. Since $m \le 8$, by Lemma 4,

$$\mathcal{E}_m(V) \subset \varphi(\mathcal{E}_m(m)) = \varphi(\mathcal{E}_{2,m}(m)').$$

By Riemann-Roch, for every $h\in\varphi(\mathcal{E}_1(m))$,

$$h^0(h)+h^0(K_V-h) \ge 1.$$

Since $-K_V$ is ample and $(-K_V)\cdot(K_V-h) = -K_V^2-1 < 0$, $h^0(K_V-h) = 0$, and $h^0(h) > 0$. Let

$$h = [E_1+\dots+E_r],$$

where E_i are irreducible curves. Intersecting both sides with $-K_V$, we obtain that

$$1 = \Sigma(-K_V)\bullet E_i.$$

This implies that $r = 1$, and hence h is the class of an exceptional curve of the first kind. This shows that

$$\varphi(\mathcal{E}_m(m)) \subset \mathcal{E}_m(V)$$

and we are done.

VI. CREMONA ACTION.

Let Σ_m be the permutation group on m letters. It acts naturally on the varieties P_n^m via its natural action on $\mathbb{P}_n^m$. In this chapter we will see that this action can be extended to a birational action of the Weyl group $W_{n,m}$. Roughly speaking, this action arises from applying to the point sets certain types of Cremona transformations of $\mathbb{P}_n$.

1. The Cremona representation of the Weyl group $W_{n,m}$.

Recall that a standard Cremona transformation T_0 in $\mathbb{P}_n$ is a birational transformation of $\mathbb{P}_n$ defined by the formula:

$$T_0\colon\ (t_0,\dots,t_n) \to (t_1\dots t_n,\dots,t_0\dots \hat{t}_i\dots t_n,\dots,t_0\dots t_{n-1}).$$

The linear system of hypersurfaces defining T_0 consists of hypersurfaces of degree n that pass through the points $x^i = (0,\dots,1,\dots 0)$ with multiplicity $n-1$. The choice of the basis in this linear system is determined by the property:

$$T_0^2 = \text{identity}.$$

Note that T_0 is defined everywhere except at the points x^i which are transformed to the hyperplanes:

$$H_i = \{(t_0,\dots,t_n)\in\mathbb{P}_n\colon\ t_i = 0\}.$$

Lemma 1. There exists a commutative diagram of birational maps:

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ f\downarrow & & f\downarrow \\ \mathbb{P}_n & \overset{T_0}{\dashrightarrow} & \mathbb{P}_n, \end{array}$$

where g is an isomorphism, and f is the composition

$$Y = Y_{n-1} \xrightarrow{f_{n-1}} Y_{n-2} \to \dots \to Y_1 \xrightarrow{f_1} Y_0 = \mathbb{P}_n,$$

where

$f_k{:}Y_k \to Y_{k-1}$ is a blowing up of the proper transforms of the subspaces

$$H_{i_1} \cap \dots \cap H_{i_{n+1-k}},\ 0 \le i_1 < \dots < i_{n+1-k} \le n+1, \text{ under } f_{k-1}\ (f_0 = id),$$

and

$$(f \circ g)(f^{-1}(H_{i_1} \cap \dots \cap H_{i_k}) = H_{j_1} \cap \dots \cap H_{j_{n+1-k}},$$

where $\{i_1,\dots,i_k\}$ and $\{j_1,\dots,j_{n+1-k}\}$ are complementary subsets of $\{1,\dots,n+1\}$.

Moreover, under some identification of $f^{-1}(H_{i_1} \cap \dots \cap H_{i_n})$ and H_{j_1} with $\mathbb{P}_{n-1}$, the rational map

$$f \circ g \circ f_{n-1}^{-1}{:}\ \mathbb{P}_{n-1} \dashrightarrow \mathbb{P}_{n-1}$$

is a standard Cremona transformation.

Proof. Left to the reader.

Corollary. Let σ: $V(x) \to \mathbb{P}_n$ be the blowing-up of the point set $x = (x^1,\dots,x^{n+1})$, φ_x: $H_{n+1} \to N(V(x))$ be the corresponding strict geometric marking. Then there exist a pseudo-isomorphism $h{:}V(x) \dashrightarrow V(x)$ and commutive diagrams:

$$\begin{array}{ccc} V(x) & \overset{h}{\dashrightarrow} & V(x) \\ \downarrow\sigma & & \downarrow\sigma \\ \mathbb{P}_n & \overset{T_0}{\dashrightarrow} & \mathbb{P}_n \end{array} \qquad \begin{array}{ccc} H_{n+1} & \overset{\varphi_x}{\rightrightarrows} & N(V(x)) \\ \downarrow s_{\alpha_0} & & \downarrow h^* \\ H_{n+1} & \overset{\varphi_x}{\rightrightarrows} & N(V(x)). \end{array}$$

Proof. It follows from the previous lemma that T_0 induces a rational map

$$h = (f_{n-1} \circ \dots \circ f_2) \circ g \circ (f_{n-1} \circ \dots \circ f_2)^{-1}{:}\ Y_1 = V(x) \dashrightarrow Y_1$$

which is a pseudo-isomorphism that sends the strict geometric marking $\varphi_x{:}H_{n+1} \to N(Y_1)$ of Y_1 to the geometric marking $\varphi = h^* \circ \varphi_x$: $H_{n+1} \to N(Y_1)$ defined by

$$\varphi^1(e_0) = n\varphi^1(e_0) - (n-1)(\varphi^1(e_1) + \dots + \varphi^1(e_{n+1})),$$

$$\varphi^1(e_i) = \varphi^1(e_0) - \varphi^1(e_1) + \dots + \varphi^1(e_{n+1}) + \varphi^1(e_i),\ i = 1,\dots,n+1,$$

$$\varphi_1(e_0) = n\varphi_1(e_0) - (\varphi_1(e_1) + \dots + \varphi_1(e_{n+1})),$$

$$\varphi_1(e_i) = (n-1)\varphi_1(e_0) - (\varphi_1(e_1) + \dots + \varphi_1(e_{n+1})) + \varphi_1(e_i),\ i = 1,\dots,n+1.$$

Recalling the action of the simple reflection s_{α_0}, this yields:

$$\varphi = \varphi_x \circ s_{\alpha_0}.$$

Thus in its natural action on the set of markings $\varphi\colon H_{n+1} \to N(V)$, the reflection s_{α_0} transforms a strict geometric marking $\varphi = \varphi_x$ of V defined by a point set $x = (x^1,\dots,x^{n+1})$ into a geometric marking $\varphi = h^*\circ\varphi$. Similarly, a simple reflection s_{α_i} transforms φ_x into φ_y, where

$$y = (x^1,\dots,x^{i-1},x^{i+1},x^i,x^{i+2},\dots,x^{n+1}).$$

This suggests that the whole group $W_{n,m}$ acts on the set of pseudo-isomorphism classes of geometric markings of any gDP-variety V of type (n,m).

Proposition 1. Let $x = (x^1,\dots,x^m)\in\mathbb{P}_n^m$. Assume that all points x^i are distinct and the first n+1 points span $\mathbb{P}_n$. Then for every i = 0,...,m-1, there exist a point set $y = (y^1,\dots,y^m)\in\mathbb{P}_n^m$, a pseudo-isomorphism $f\colon V(x) \dashrightarrow V(y)$, a birational transformation $T_i\colon \mathbb{P}_n \dashrightarrow \mathbb{P}_n$, and commutative diagrams:

$$\begin{array}{ccc} V(x) & \overset{f}{\dashrightarrow} & V(y) \\ \downarrow\sigma & & \downarrow\sigma' \\ \mathbb{P}_n & \overset{T_i}{\dashrightarrow} & \mathbb{P}_n \end{array} \qquad\qquad \begin{array}{ccc} H_{n+1} & \overset{\varphi_y}{\rightrightarrows} & N(V(y)) \\ \downarrow s_{\alpha_i} & & \downarrow f^* \\ H_{n+1} & \overset{\varphi_x}{\rightrightarrows} & N(V(x)). \end{array}$$

Proof. Let

$$\sigma\colon V(x) = V_m \to V_{m-1} \to \dots \to V_1 \to \mathbb{P}_n$$

be the corresponding blowing-down structure on V(x). Assume first that $i \neq 0$. Choose a projective transformation T_i of $\mathbb{P}_n$ which permutes the points x^i and x^{i+1} and sends the the remaining points x^j to some points y^j. Define the point set y by

$$y = (y^1,\dots,y^{i-1},x^i,x^{i+1},y^{i+2},\dots,y^m).$$

The composition $T_i\circ\sigma$ maps the exceptional divisors $E_1,\dots,E_m$ of σ to the points $y^1,\dots,y^m$ of y respectively. Let $(V(y),\sigma')$ be the blowing-down sructure corresponding to y. The composition $T_i\circ\sigma\colon V(x) \to \mathbb{P}_n$ blows up the same set y. By the uniqueness of the blowing-up, there exists an isomorphism $f\colon V(x) \to V(y)$ defining the first diagram in the statement. It is clear that

$$f^*([\sigma'^{-1}(y^j)]) = [\sigma^{-1}(x^j)] \text{ if } j \neq i,i+1,$$

$$f^*([\sigma'^{-1}(y^i)]) = [\sigma^{-1}(x^{i+1})]),$$

$$f^*([\sigma'^{-1}(y^{i+1})]) = [\sigma^{-1}(x^i)]).$$

This immediately implies that

$$\varphi_x \circ s_{\alpha_i} = f^* \circ \varphi_y,$$

proving our assertion.

Next let $i = 0$. Since $x^1,\dots,x^{n+1}$ span $\mathbb{P}_n$, we may replace x by a projectively equivalent set to assume that $x^i = (0,\dots,1,\dots,0)$, $i = 1,\dots,n+1$. Let

$$f':V_{n+1} \dashrightarrow V_{n+1}$$

be the pseudo-isomorphism defined in the previous Corollary. It extends to a pseudo-isomorphism

$$f: V(x) \dashrightarrow V(y),$$

where

$$y = T_0(x) = (x^1,\dots,x^{n+1},T_0(x^{n+2}),\dots,T_0(x^m)).$$

It is immediately verified, using the Corollary, that $\varphi_x \circ s_{\alpha_i} = f^* \circ \varphi_y$. This proves the proposition.

Using the previous proposition we can apply any product of simple reflections to a geometric marking to obtain another geometric marking, provided that at every step the resulting point set $x = (x^1,\dots,x^m)$ satisfies:

(i) x does not contain infinitely near points;

(ii) $\langle x^1,\dots,x^{n+1}\rangle = \mathbb{P}_n$.

This can be stated in terms of the strict geometric marking

$$\varphi_x:H_m \to N(V(x))$$

by saying that all simple B-roots α_i are not effective.

Observe that

$$w(R_B(\varphi)^+) = R_B(\varphi \circ w^{-1})^+ \quad \text{for any } w \in W_B.$$

This shows that we can apply every $w \in W_B = W_{n,m}$ to x if

$$R_B(x)^+ = \emptyset,$$

i.e. if x is unnodal in the sense of the previous Chapter.

Hence we are led to the study of the orbits in the set

$$(\hat{\mathbb{P}}_n^m)^{un} = \hat{\mathbb{P}}_n^m \setminus Z,$$

where

$$Z = \bigcup_{\alpha \in R_B} Z(\alpha),$$

$$Z(\alpha) = \{x \in \hat{\mathbb{P}}^m_n : \alpha \in R_B(x)^+\}.$$

Note that for any $i > j > 0$, $\alpha = e_j - e_i = \alpha_i + \ldots + \alpha_{j-1} \in R_B$, and

$$Z(\alpha) = \hat{\Delta}_{ij}(m).$$

This shows that

$$(\hat{\mathbb{P}}^m_n)^{un} \subset \hat{\mathbb{P}}^m_n \setminus \hat{\Delta}(m) \cong \mathbb{P}^m_n \setminus \Delta(m),$$

thus it allows us to use

$$(\hat{\mathbb{P}}^m_n)^{un}, (\mathbb{P}^m_n)^{un}$$

to denote the same set. Taking $\alpha = e_0 - e_{i_1} - \ldots - e_{i_{n+1}}$, we obtain that

$$Z(\alpha) = \{\hat{x} \in \hat{\mathbb{P}}^m_n : \text{no } n+1 \text{ points lie in a hyperplane}\}.$$

It follows from the criteron of stability of point sets that

$$(\hat{\mathbb{P}}^m_n)^{un} \subset (\mathbb{P}^m_n \setminus \Delta(m))^s.$$

Set

$$(P^m_n)^{un} = \Phi((\mathbb{P}^m_n)^{un}) \subset P^m_n \setminus \bar{\mathfrak{D}}.$$

Note that, when the number of roots is infinite, $(\mathbb{P}^m_n)^{un}$ is neither open nor closed.

Let us see first that $(\mathbb{P}^m_n)^{un}$ is not empty.

Theorem 1. For every B-root α the subset

$$Z(\alpha) = \{x \in \hat{\mathbb{P}}^m_n : \alpha \in R_B(x)^+\}.$$

is a closed subset of $\hat{\mathbb{P}}^m_n$. Moreover, its restriction to $(\hat{\mathbb{P}}^m_n \setminus \hat{\Delta}(m))^s$ is an irreducible hypersurface.

Proof. Let

$$\alpha = a_0 e_0 - a_1 e_1 - \ldots - a_m e_m \in R_B.$$

Assume $\varphi_x(\alpha) \geq 0$. Since $h_0 = \varphi_x(e_0)$ is numerically effective,

$$\varphi_x(\alpha) \cdot h_0 = (a_0 h_0 - a_1 h_1 - \ldots - a_m h_m) \cdot h_0 = a_0 \geq 0.$$

If $a_0 = 0$, $\alpha = e_i - e_j$ for some $i, j > 0$ (Proposition 4 of Chapter V). Hence

$$\varphi_x(\alpha) = h_i - h_j \geq 0 \quad \text{iff } x^j \text{ is infinitely near to } x^i,$$

and $Z(\alpha) = \hat{\Delta}_{ij}(m)$ in this case. By Proposition 1 of Chapter IV, it is a hypersurface.

Assume $a_0 > 0$. By Proposition 4 of Chapter V

$$a_i \geq 0, \; i = 1, \ldots, m.$$

Assume that x does not contain infinitely near points. Then

$$\varphi_x(\alpha) = a_0h_0-a_1h_1-\ldots-a_mh_m$$

is the class of an effective divisor D if and only if there exists a hypersurface in $\mathbb{P}_n$ of degree a_0 that passes through the points x^i with multiplicity $\geq a_i$. In this case

$$D = D'+\kappa_1E_1+\ldots+\kappa_mE_m,$$

where D' is the proper inverse transform of the hypersurface. The existence of such a hypersurface is expressed by algebraic equations in the coordinates of the points x^i. This proves that $Z(\alpha)\cap(\hat{\mathbb{P}}_n^m\setminus\hat{\Delta}(m))$ is a closed subset of $\hat{\mathbb{P}}_n^m\setminus\hat{\Delta}(m)$.

Assume $x\in\hat{\Delta}(m)$. For simplicity we also assume that $x\notin\hat{\Delta}_I(m)$ with $\#I > 2$ and leave the general case to the reader. Without loss of generality we may take x in $\hat{\Delta}_{12}(m)$. If $\kappa = \sup\{-a_1+a_2,0\}$, any effective divisor with class $\varphi_x(\alpha)$ contains $\kappa(E_1-E_2)$, where $\varphi_x(e_i) = [E_i]$. Thus $\varphi_x(\alpha) \geq 0$ if and only if there exists a hypersurface in $\mathbb{P}_n$ of degree a_0 passing through x^1 with multiplicity $\geq a_1+\kappa$, passing through the infinitely near point $x^2 \to x^1$ with multipliicty $\geq a_2$, and passing through the remaining points with multiplicity $\geq a_i$, $i > 2$. This is expressed by algebraic conditions on the coordinates of the x^i's.

So far we have only shown that each $Z(\alpha)$ is a closed subset in $\hat{\mathbb{P}}_n^m$. It remains to prove that its codimension is 1 at every point of $\hat{\mathbb{P}}_n^m\setminus\hat{\Delta}(m)$. Obviously, this is true for simple roots $\alpha_i\in B$. Indeed, we have seen this already for $i > 0$, and

$$Z(\alpha_0) = \{x\in\hat{\mathbb{P}}_n^m : (12\ldots n+1)(x) = 0\},$$

where

$$(12\ldots n+1) = \det((x_j^{(i)}))\in\Gamma(\hat{\mathbb{P}}_n^m, b_m^*(\bigotimes_{i=1}^{n+1}\pi_i^*\mathcal{O}_{\mathbb{P}_n}(1))) .$$

Evidently each $Z(\alpha)$ is G-invariant. Let

$$\bar{Z}(\alpha) = \bar{b}_m\circ\hat{\Phi}(\hat{U}(m)^s\cap Z(\alpha)) \subset \Phi(U(m)^s) \subset P_n^m.$$

The assertion will follow if we show that each $\bar{Z}(\alpha)$ is an irreducible hypersurface in $\Phi(U(m)^s)$. Assume $\alpha\notin B$ and write

$$\alpha_0 = s_1\circ\ldots\circ s_\kappa(\alpha)$$

for some simple reflections $s_j = s_{\alpha_{i_j}}$, $j = 1,\ldots,\kappa$. Then

$$s_\kappa: \bar{Z}(\alpha)\setminus\bar{Z}(\alpha_{i_\kappa}) \to \bar{Z}(s_\kappa(\alpha))\setminus\bar{Z}(\alpha_{i_\kappa})$$

is an isomorphism. By induction on κ, we may assume that $\bar{Z}(s_\kappa(\alpha))\setminus\bar{Z}(\alpha_{i_\kappa})$ is open and dense in $\bar{Z}(s_\kappa(\alpha))$, and $\bar{Z}(s_\kappa(\alpha))$ is a hypersurface. Thus $\bar{Z}(\alpha)$ is a hypersurface.

Corollary. Assume that the canonical root system of type n in H_m is of finite type. Then $(\mathbb{P}^m_n)^{un}$ (resp. $(P^m_n)^{un}$) is an open Zariski subset of $\mathbb{P}^m_n$ (resp. P^m_n).

Remark 1. If $n = 2$, one can prove that each $Z(\alpha)\cap\hat{U}(m)$ is a hypersurface by "counting constants". In fact, the condition that $\alpha = a_0e_0-a_1e_1-\ldots-a_me_m\in B$, $a_0 > 0$, is effective imposes $\frac{1}{2}\sum_{i=1}^{m} a_i(a_i+1)$ linear equations on the $\frac{1}{2}(a_0+1)(a_0+2)$ coefficients in the equation of a plane curve of degree a_0. By Proposition 4 from Chapter 5, these numbers are equal.

Let $w = s_{\alpha_{i_1}}\circ\ldots\circ s_{\alpha_{i_k}}\in W_{n,m}$. It follows from the above discussion that w acts regularly on the open dense subset

$$\bar{U}(w) = \Phi(U(w)),$$

where

$$U(w) = ((\mathbb{P}^m_n \setminus\Delta(m))\setminus Z(\alpha_{i_k})\setminus Z(s_{\alpha_{i_k}}(\alpha_{i_{k-1}}))\setminus\ldots\setminus Z(s_{\alpha_{i_k}}\circ\ldots\circ s_{\alpha_{i_2}}(\alpha_{i_1})))^s.$$

The restriction of w to the generic point of $\bar{U}(w)$ is a k-automorphism of the field of rational functions on P^m_n. This defines a birational action of $W_{n,m}$ on P^m_n:

$$cr_{n,m}: W_{n,m} \to Bir(P^m_n) \cong Cr(n(m-n-2)),$$

where $Cr(k)$ denotes the <u>Cremona group</u> in dimension k, i.e. the group of birational transformations of $\mathbb{P}_k$. We will call this action the <u>Cremona representation (or action)</u> of $W_{n,m}$.

If $(B,\check{B})$ is of finite type, R_B and $W_{n,m}$ are finite, and $W_{n,m}$ acts biregularly on the open set $(P^m_n)^{un}$. In general, $W_{n,m}$ does not act regularly on any open subset of P^m_n.

The next result shows, at least in the case $n = 2$, that the Weyl group acts transitively on the set of unnodal geometric markings of the same gDP-variety.

Theorem 2. Let $\varphi:H_m \to N(V)$ and $\varphi:H_m \to N(V)$ be two geometric markings of a gDP-surface. Then there exists $w\in W_{2,m}$ such that

$$\varphi = \varphi\circ w.$$

<u>Proof</u>. Let

$$\varphi(e_i) = h_i, \quad i = 0,\ldots,m,$$

$$\varphi(e_0) = a_0h_0-a_1h_1-\dots-a_mh_m = \varphi(a_0e_0-a_1e_1-\dots-a_me_m).$$

Since $\varphi(e_0)$ is numerically effective,

$$a_0 = \varphi(e_0)\bullet h_0 > 0,\ a_i = \varphi(e_0)\bullet h_i \geq 0\ ,\ i > 0.$$

Set

$$v = a_0e_0-a_1e_1-\dots-a_me_m,$$

so that

$$\varphi(v) = \varphi(e_0).$$

Suppose we show that there exists an element $w \in W_{2,m}$ such that

$$w(v) = e_0.$$

Then

$$w^{-1}\circ\varphi^{-1}\circ\varphi(e_0) = e_0,$$

hence

$$w^{-1}\circ\varphi^{-1}\circ\varphi(e_i) = e_{\sigma(i)},\ i = 1,\dots,m,$$

for some permutation σ of $\{1,\dots,m\}$. Replacing w by $w\circ\sigma$, we may assume that

$$w^{-1}\circ\varphi^{-1}\circ\varphi(e_i) = e_i,\ i = 1,\dots,m.$$

This certainly implies that

$$\varphi = \varphi\circ w.$$

To show that such a w exists we assume first that φ is unnodal. By assumption, $R_B(\varphi)^+ = \emptyset$. Hence

$$R_B(\varphi\circ w)^+ = \emptyset \quad \text{for any } w\in W_{2,m}.$$

Thus for every $w\in W_{2,m}$ the composition $\varphi\circ w$ is an unnodal geometric marking. Obviously, $\varphi(v) = \varphi(e_0)$ is represented by an irreducible curve. Thus there exists an irreducible plane curve of degree a_0 with a_i-multiple points at the x^{i}'s. Applying an element of Σ_m we may assume that

$$a_1 \geq a_2 \geq \dots \geq a_m \geq 0.$$

This implies that $\varphi(e_0)$ satisfies the assumptions of Noether's inequality, and

$$a = a_0-a_1-a_2-a_3 < 0$$

unless $v = e_0$, in which case we are done. If $v \neq e_0$ we apply s_{α_0} to v to obtain

$$w(v) = v' = (a_0+a)e_0-(a_1+a)e_1-(a_2+a)e_2-(a_3+a)e_3-a_4e_4-\dots-a_me_m.$$

Since $\varphi\circ s_{\alpha_0}$ is a geometric marking, $\varphi(s_{\alpha_0}(v))$ is the class of a numerically effective divisor. Hence

$$0 < a_0+a < a_0 \ , \ a_i+a \geq 0, \ i = 1, 2, 3.$$

Proceeding in this way, we decrease the coefficent at e_0 until we reach the case

$$w(v) = e_0$$

for some $w \in W_{2,m}$.

Assume φ is any geometric marking. Let x be a generic point set, that is, the generic point of $\mathbb{P}_2^m$. Let

$$D \in Pic(V(x))$$

represent the class $\varphi_x(v)$. We know that

$$D^2 = 1, \ D \cdot K_{V(x)} = -3.$$

Since

$$(K_{V(x)}-D) \cdot \varphi_x(e_0) = -3-a_0 < 0,$$

it follows that $h^0(K_{V(x)}-D) = 0$. By Riemann-Roch

$$h^0(D) \geq 3$$

and we may assume that $D \geq 0$. Specializing x to the point set $\bar{x}$ representing the geometric marking φ we obtain that D specializes to an element of the irreducible linear system $|\varphi(v)| = |\varphi(e_0)|$ on $V(\bar{x})$. Thus we can choose D to be irreducible. This easily implies that the linear system $|D|$ is of dimension 2 and defines a birational morphism $V(x) \to \mathbb{P}_2$. Thus there exists a geometric marking φ' of $V(x)$ such that

$$[D] = \varphi'(e_0).$$

By Theorem 1, x is unnodal. Thus we are in the previous situation and can find $w \in W_{2,m}$ for which $w(v) = f_0$. This completes the proof of the theorem.

Remark 2. The previous theorem is essentially due to M. Nagata (**[Na]**, Corollary on p. 283).

Remark 3. We do not know whether $W_{n,m}$ acts transitively on the set of pseudo-isomorphism classes of unnodal geometric markings for $n > 2$. To prove this we would need an analog of Noether's inequality

$$(n-1)a_0 < a_1+\ldots+a_{n+1}$$

for vectors $a_0e_0-a_1e_1-\ldots-a_me_m$ satisfying properties (i),(ii),(iv) and (vi) of

Proposition 4 of Chapter V. This question is closely related to the following one. A birational transformation

$$T: \mathbb{P}_n \dashrightarrow \mathbb{P}_n$$

is said to be punctual (cf. **[DuV 3]**) if there exists a commutative diagram

$$\begin{array}{ccc} V & \overset{f}{\dashrightarrow} & V' \\ \downarrow\sigma & & \downarrow\sigma' \\ \mathbb{P}_n & \overset{T}{\dashrightarrow} & \mathbb{P}_n, \end{array}$$

where $\sigma: V \to \mathbb{P}_n$ and $\sigma': V' \to \mathbb{P}_n$ are blowing-ups of point sets from $\hat{\mathbb{P}}_n^m$, and f is a pseudo-isomorphism. An example of a punctual transformation is a standard Cremona transformation of $\mathbb{P}_n$. Since

$$rk(N^1(V)) = rk(N^1(V')),$$

V and V' are gDP-varieties of the same type. One can prove that all punctual transformations form a subgroup

$$Punct(n) \subset Cr(n)$$

of the Cremona group $Cr(n) = Bir(\mathbb{P}_n)$. This subgroup contains a subgroup $Cr_{reg}(n)$ generated by the standard Cremona transformation and projective automorphisms. Elements of this group are called regular Cremona transformations. By Noether's factorization theorem (see **[AS]**, **[Coo]**)

$$Cr(2) = Punct(2) = Cr_{reg}(2).$$

It is hinted in **[Co 3]** and **[DuV 3]** that

$$Punct(n) = Cr_{reg}(n)$$

for all n. However, we were not able to find a proof of this result. Note that Cr(n) is "much bigger" than Punct(n) for $n > 2$.

Theorem 3. Assume $m \le 8$ and $char(k) = 0$. Then the quotient space

$$(P_2^m)^{un}/W_{2,m}$$

is an algebraic variety isomorphic to the coarse moduli variety $\mathfrak{M}_{DP}(m)$ of Del Pezzo surfaces of degree 9-m.

Proof. First let us recall a construction of the latter space. If $m = 4$, $\mathfrak{M}_{DP}(4)$ is a one-point set. If $m = 5$, the anticanonical linear system $|-K_V|$ maps the surface V isomorphically onto the intersection of two quadrics in $\mathbb{P}_4$. In this case $\mathfrak{M}_{DP}(5)$ can be realized as an appropriate quotient of an open subset in the Grassmann variety of pencils of quadrics in $\mathbb{P}_4$. If m = 6, $|-K_V|$ maps V isomorphically onto a nonsingular cubic surface in $\mathbb{P}_3$. In this case $\mathcal{M}_{DP}(6)$ is constructed by standard methods of geometric invariant theory. If m = 7, $|-K_V|$ defines a double cover of degree 2 onto $\mathbb{P}_2$ branched along a nonsingular quartic curve. Thus $\mathcal{M}_{DP}(7)$ is isomorphic to a certain quotient of an open subset of the space of quartic curves. Finally, if m = 8, $|-2K_V|$ defines a double cover onto a singular quadratic cone Q in $\mathbb{P}_3$ and ramifies along a curve of degree 6 cut out on Q by a cubic. The construction of $\mathcal{M}_{DP}(8)$ in this case is similar to the previous case.

Let

$$(P_2^m)^{un} \to \mathcal{M}_{DP}(m)$$

be the map defined by forgetting the blowing-down structure. It follows from Theorem 2 and Proposition 7 of Chapter 5 that this map factors through the quotient by the finite group $W_{2,m}$ and defines a bijective map

$$(\mathbb{P}_2^m)^{un}/W_{2,m} \to \mathcal{M}_{DP}(m).$$

Since both spaces are normal algebraic varieties, the assertion follows from Zariski's Main Theorem.

Remark 4. We believe that the birational action of the finite Weyl groups $W_{n,m}$ on P_n^m can be extended to a biregular action on $\hat{P}_n^m$. The quotient $\hat{P}_2^m/W_{2,m}$ ($m \le 8$) would be a certain compactification of the moduli space $\mathcal{M}_{DP}(m)$.

2. Explicit formulae.

Let us give explicit formulae for the action of simple reflections s_{α_i} on P_n^m via the Cremona representation $cr_{n,m}$. We use a birational model of P_n^m introduced in Remark 3 of Chapter 2. According to this we assume that a point set $x = (x^1,\dots,x^m)$ is normalized by a projective transformation to satisfy:

$$x^1 = (1,0,\dots,0),\dots,\ x^{n+1} = (0,\dots,0,1),\ x^{n+2} = (1,\dots,1,1),\ x_n^{(i)} = 1,\ i > n+2,$$

and then identify x with the point

$$(x_0^{(n+3)},\dots,x_{n-1}^{(n+3)},x_0^{(n+4)},\dots,x_{n-1}^{(n+4)},\dots,x_0^{(m)}\dots,x_{n-1}^{(m)})$$

of $\mathbb{A}_{n(m-n-2)}$.

Assume $w = s_{\alpha_i}$, i=1,...,n-1. Then w switches the points x^i and x^{i+1} and leaves the remaining points unchanged. Applying the projective transformation

$$(x_0,\dots,x_i,x_{i+1},\dots,x_n) \rightarrow ((x_0,\dots,x_{i+1},x_i,\dots,x_n),$$

we see that $w(x)$ is G-equivalent to a point set x' defined by the projective coordinates

$$(y_0^{(n+3)},\dots,y_{n-1}^{(n+3)},y_0^{(n+4)},\dots,y_{n-1}^{(n+4)},\dots,y_0^{(m)},\dots,y_{n-1}^{(m)}),$$

where

$$y_i^{(j)} = x_{i+1}^{(j)},\ y_{i+1}^{(j)} = x_i^{(j)},\ y_k^{(j)} = x_k^{(j)}\ k \neq i,i+1,\ j = n+3,\dots,m.$$

Assume $w = s_{\alpha_n}$. Then w switches x^n with x^{n+1} and leaves other points unchanged. Using the projective transformation

$$(x_0,\dots,x_n) \rightarrow (x_0,x_1,\dots,x_{n-2},x_n,x_{n-1}),$$

we find that $w(x)$ is G-equivalent to a point set x' defined by the projective coordinates

$$\left(\frac{x_0^{(n+3)}}{x_{n-1}^{(n+3)}},\dots,\frac{x_{n-2}^{(n+3)}}{x_{n-1}^{(n+3)}},\frac{1}{x_{n-1}^{(n+3)}},\dots,\frac{x_0^{(m)}}{x_{n-1}^{(m)}},\dots,\frac{x_{n-2}^{(m)}}{x_{n-1}^{(m)}},\frac{1}{x_{n-1}^{(m)}}\right).$$

Note that in this case w is not defined everywhere on U_0.

Assume $w = s_{\alpha_{n+1}}$. It switches x^{n+1} and x^{n+2}. Using the projective transformation

$$(x_0,\dots,x_n) \rightarrow (x_n-x_0,\dots,x_n-x_{n-1},x_n),$$

we find that $w(x)$ is G-equivalent to a point set x' defined by the coordinates

$$(1-x_0^{(n+3)},\dots,1-x_{n-1}^{(n+3)},\dots,1-x_0^{(m)}\dots,1-x_{n-1}^{(m)}).$$

Assume $w = s_{\alpha_{n+2}}$. In this case we easily find that $w(x)$ has the coordinates

$$\left(\frac{1}{x_0^{(n+3)}},\dots,\frac{1}{x_{n-1}^{(n+3)}},\dots,\frac{x_0^{(n+4)}}{x_0^{(n+3)}},\dots,\frac{x_{n-1}^{(n+4)}}{x_{n-1}^{(n+3)}},\dots,\frac{x_{n-1}^{(m)}}{x_{n-1}^{(n+3)}}\right).$$

Assume $w = s_{\alpha_i}$, $i > n+2$. This is the simplest case. The point set remains normalized after applying w, and $w(x)$ has the coordinates

$$(x_0^{(n+3)},\dots,x_{n-1}^{(n+3)},\dots,x_0^{(i+1)},\dots,x_{n-1}^{(i+1)},x_0^{(i)},\dots,x_{n-1}^{(i)},\dots,x_0^{(m)},\dots,x_{n-1}^{(m)}).$$

Finally let $w = s_{\alpha_0}$. The point set x is transformed under $cr_{m,n}$ to the point set $y = (y^1,\dots,y^m)$, where

$$y^i = x^i,\ i = 1,\dots,n+2,$$

$$y^i = (x_1^{(i)}\dots x_n^{(i)},x_0^{(i)}x_2^{(i)}\dots x_n^{(i)},\dots,x_0^{(i)}x_1^{(i)}\dots x_{n-1}^{(i)}),\ i > n+2.$$

Normalizing the last coordinate of every y^i with $i > n+2$, we obtain that the coordinates of the point set $w(x)$ are equal to

$$\left(\frac{1}{x_0^{(n+3)}}, \ldots, \frac{1}{x_{n-1}^{(n+3)}}, \ldots, \frac{1}{x_0^{(m)}}, \ldots, \frac{1}{x_{n-1}^{(m)}}\right).$$

We see again that s_{α_0} is not defined everywhere.

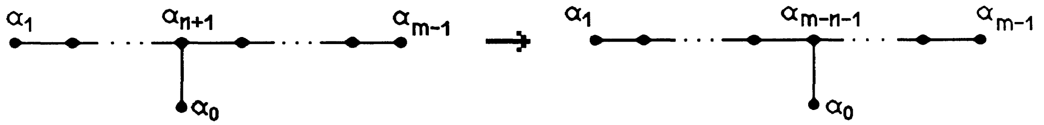
3. Cremona action and association.

Recall that the association map is an isomorphism:

$$a_{n,m}: P_n^m \to P_{m-n-2}^m,$$

defined in Chapter 3. In this section we will show that this isomorphism is compatible with the Cremona action.

First, let us observe that the Dynkin diagrams of the canonical root bases in H_m of type n and m-n-2 are obtained from each other by the permutation $\sigma = (1,m-1)(2,m-2)\ldots([\frac{1}{2}m],[m-[\frac{1}{2}m])$ of the vertices $\alpha_1,\ldots,\alpha_{m-1}$ corresponding to simple roots $\alpha_1,\ldots,\alpha_{m-1}$:



This observation implies that there is an isomorphism $W_{n,m} \to W_{m-n-2,m}$ defined by

$$s_{\alpha_0} \to s_{\alpha_0'} \quad s_{\alpha_i} \to s_{\alpha'_{\sigma(i)}}, \ i > 0.$$

Composing it with the inner automorphism of $W_{m-n-2,m}$

$$w \to \sigma\circ w\circ\sigma^{-1}, \qquad w\in W_{m-n-2,m},$$

where σ is considered as an element $(1,m)(2,m-1)\ldots([\frac{1}{2}m],m-[\frac{1}{2}m]+1)$ of $\Sigma_m \subset W_{m-n-2,m}$, we obtain an isomorphism

$$\tau: W_{n,m} \xrightarrow{\sim} W_{m-n-2,m}.$$

By definition

$$\tau(s_{\alpha_0}) = \sigma\circ s_{\alpha_0'}\circ\sigma^{-1}, \qquad \tau(s_{\alpha_i}) = s_{\alpha_i'}, \ i \neq 0.$$

Theorem 4. Let a_η be the restriction of $a_{n,m}$ to the generic point. Then for any $w\in W_{n,m}$

$$cr_{n,m-n-2}(\tau(w))\circ a_\eta = a_\eta\circ cr_{n,m}(w).$$

Proof. Let U_0 be the locally closed subset of $\mathbb{P}_n^m$ defined by the conditions:

$$x^1 = (1,0,\dots,0),\dots,\ x^{n+1} = (0,\dots,0,1),\ x^{n+2} = (1,\dots,1,1),\ x_n^{(i)} = 1,\ i > n+2.$$

Similarly, let V_0 denote the locally closed subset of $\mathbb{P}^m_{m-n-2}$ defined by the conditions:

$$y^m = (1,0,\dots,0),\dots,\ y^{n+2} = (0,\dots,0,1),\ y^{n+1} = (1,\dots,1),\ y^{(i)}_{m-n-2} = 1,\ i \le n.$$

As in the previous section we make a birational identification of U_0 (resp. V_0) with P_n^m (resp. P^m_{m-n-2}). Both of these sets are also identified with a subset of affine space $\mathbb{A}_{n(m-n-2)}$ by assigning to a point set $x = (x^1,\dots,x^m)\epsilon U_0$ (resp. $y = (y^1,\dots,y^m)\epsilon V_0$) the point

$$z(x) = (x_0^{(n+3)},\dots,x_{n-1}^{(n+3)},\dots,x_0^{(m)},\dots,x_{n-1}^{(m)})$$

(resp.

$$z(y) = (y^{(1)}_{m-n-3},\dots,y^{(n)}_{m-n-3},\dots,y_0^{(1)},\dots,y_0^{(n)})).$$

We have already noted in Remark 1 of Chapter III that

$$a_{n,m}(\Phi(x)) = \Phi(y) \Leftrightarrow z(x) = z(y).$$

We will identify point sets x and y with the points z(x) and z(y).

To verify the theorem it suffices to show that

$$z(s_{\alpha_i})(\Phi(x)) = z(\tau(s_{\alpha_i})(\Phi(y))) \text{ if } z(x) = z(y),$$

for each simple reflection s_{α_i}. The needed formulae are given in the previous section. Let $i = 0$. Then

$$s_{\alpha_0}(\Phi(x)) = (\frac{1}{x_0^{(n+3)}},\dots,\frac{1}{x_{n-1}^{(n+3)}},\dots,\frac{1}{x_0^{(m)}},\dots,\frac{1}{x_{n-1}^{(m)}}).$$

We know that $\tau(s_{\alpha_0}) = s_\alpha$, where $\alpha = e_0-e_{n+2}-\dots-e_m$. This shows that

$$z(\tau(s_{\alpha_0})(\Phi(y))) = (\frac{1}{y^{(1)}_{m-n-3}},\dots,\frac{1}{y^{(n)}_{m-n-3}},\dots,\frac{1}{y_0^{(1)}},\dots,\frac{1}{y_0^{(n)}}),$$

hence

$$z(s_{\alpha_0}(\Phi(x))) = z(\tau(s_{\alpha_0})(\Phi(y))) \text{ if } z(x) = z(y).$$

Let $i = 1,\dots,n-1$. Assume $i = 1$, since the other cases can be treated similarly. Then

$$s_{\alpha_1}(\Phi(x)) = (x_1^{(n+3)},x_0^{(n+3)},\dots,x_{n-1}^{(n+3)},\dots,x_1^{(m)},x_0^{(m)},\dots,x_{n-1}^{(m)}),$$

$$\tau(s_{\alpha_1})(\Phi(y)) = s_{\alpha_1}(\Phi(y)) = (y^{(2)}_{m-n-3},y^{(1)}_{m-n-3},\dots,y^{(n)}_{m-n-3},\dots,y_0^{(2)},y_0^{(1)}\dots,y_0^{(n)}),$$

hence

$$z(s_{\alpha_i}(\Phi(x))) = z(\tau(s_{\alpha_i})(\Phi(y))) \text{ if } z(x) = z(y).$$

Let $i = n$. Then

$$s_{\alpha_n}(\Phi(x)) = \left(\frac{x_0^{(n+3)}}{x_{n-1}^{(n+3)}}, \ldots, \frac{x_{n-2}^{(n+3)}}{x_{n-1}^{(n+3)}}, \frac{1}{x_{n-1}^{(n+3)}}, \ldots, \frac{x_0^{(m)}}{x_{n-1}^{(m)}}, \ldots, \frac{x_{n-1}^{(m)}}{x_{n-1}^{(m)}}, \frac{1}{x_{n-1}^{(m)}}\right).$$

The corresponding transformation $\tau(s_{\alpha_n})$ interchanges $y^n = (y_0^{(n)}, \ldots, y_{m-n-2}^{(n)})$ with $y^{(n+1)} = (1, \ldots, 1)$. Thus

$$z(\tau(s_{\alpha_n})(\Phi(y))) = \left(\frac{y_{m-n-3}^{(1)}}{y_{m-n-3}^{(n)}}, \ldots, \frac{y_{m-n-3}^{(n-1)}}{y_{m-n-3}^{(n)}}, \frac{1}{y_{m-n-3}^{(n)}}, \ldots, \frac{y_0^{(1)}}{y_0^{(n)}}, \ldots, \frac{y_0^{(n-1)}}{y_0^{(n)}}, \frac{1}{y_0^{(n)}}\right),$$

hence it is equal to $z(s_{\alpha_n}(\Phi(x))$.

Let $i = n+1$. Then

$$z(s_{\alpha_{n+1}}(\Phi(x))) = (1-x_0^{(n+3)}, \ldots, 1-x_{n-1}^{(n+3)}, \ldots, 1-x_0^{(m)} \ldots, 1-x_{n-1}^{(m)}),$$

and

$$z(\tau(s_{\alpha_{n+1}})(\Phi(y))) = (1-y_{m-n-3}^{(1)}, \ldots, 1-y_{m-n-3}^{(n)}, \ldots, 1-y_0^{(1)}, \ldots, 1-y_0^{(n)}).$$

This again verifies the assertion.

Let $i = n+2$. Then

$$z(s_{\alpha_{n+2}}(\Phi(x))) = \left(\frac{1}{x_0^{(n+3)}}, \ldots, \frac{1}{x_{n-1}^{(n+3)}}, \ldots, \frac{x_0^{(m)}}{x_0^{(n+3)}}, \ldots, \frac{x_{n-1}^{(m)}}{x_{n-1}^{(n+3)}}\right),$$

and

$$z(\tau(s_{\alpha_{n+2}})(\Phi(y))) = \left(\frac{1}{y_{m-n-3}^{(1)}}, \ldots, \frac{1}{y_{m-n-3}^{(n)}}, \ldots, \frac{y_0^{(1)}}{y_{m-n-3}^{(1)}}, \ldots, \frac{y_0^{(n)}}{y_{m-n-3}^{(n)}}\right).$$

This verifies the assertion.

Let $i > n+2$. Then

$$z(s_{\alpha_i}(\Phi(x))) = (x_0^{(n+3)}, \ldots, x_{n-1}^{(n+3)}, \ldots, x_0^{(i+1)}, \ldots, x_{n-1}^{(i+1)}, x_0^{(i)}, \ldots, x_{n-1}^{(i)} \ldots, x_0^{(m)}, \ldots, x_{n-1}^{(m)}),$$

and

$$z(\tau(s_{\alpha_i})(\Phi(y))) = (y_{m-n-3}^{(1)}, \ldots, y_{m-n-3}^{(n)}, \ldots, y_{i+1}^{(1)}, \ldots, y_{i+1}^{(n)}, y_i^{(1)}, \ldots, y_i^{(n)}, \ldots, y_0^{(1)}, \ldots, y_0^{(n)}).$$

hence

$$z(s_{\alpha_i}(\Phi(x))) = z(\tau(s_{\alpha_i})(\Phi(y))) \text{ if } z(x) = z(y).$$

This proves the theorem.

4. Pseudo-automorphisms of gDP-varieties.

Here we study the kernel of the Cremona representation

$$cr_{n,m}: W_{n,m} \rightarrow \mathrm{Bir}(P_n^m).$$

First let us show that any element of the kernel can be interpreted as a pseudo-automorphism of a generic gDP-variety of type (n,m).

Lemma 2. Let $w \in W_{n,m}$ and $\bar{x} \in \bar{U}(w)$. Then $cr_{n,m}(w)(\bar{x}) = \bar{x}$ if and only if for any $x \in \Phi^{-1}(\bar{x}) \in \mathbb{P}_n^m$ there exists a pseudo-automorphism $g: V(x) \dashrightarrow V(x)$ such that

$$\varphi_x \circ w \circ \varphi_x^{-1} = g^*$$

as isometries of $N(V(x))$.

Proof. This follows from the definition of pseudo-isomorphic geometric markings and Theorem 1 of Chapter V.

Theorem 5. There exists an injective homomorphism of groups

$$\mathrm{Ker}(cr_{n,m}) \rightarrow \mathrm{Psaut}(V(\eta)/k(\eta)),$$

where η is the generic point of P_n^m. Moreover for any finite subgroup A of $\mathrm{Ker}(cr_{n,m})$ one can find a Zariski open subset $U \subset \mathbb{P}_n^m$ such that for every $\hat{x} \in U$ the group $\mathrm{Psaut}(V(\hat{x}))$ of pseudo-automorphisms of $V(\hat{x})$ contains a subgroup isomorphic to A.

Proof. The Weyl group $W_{n,m}$ is naturally embedded into $W_{n,m+1}$ as a subgroup generated by the simple reflections s_{α_i}, $i = 0,\ldots,m-1$. Restricting $cr_{n,m+1}$ to this subgroup we obtain a homomorphism

$$cr_{n,m+1}': W_{n,m} \rightarrow \mathrm{Bir}(P_n^{m+1}).$$

Let

$$\hat{\pi} = \hat{\pi}^m : \hat{\mathbb{P}}_n^{m+1} \rightarrow \hat{\mathbb{P}}_n^m,$$

$$b_m : \hat{\mathbb{P}}_n^m \rightarrow \mathbb{P}_n^m$$

be the projections defined in Chapter IV. For every $w \in W_{n,m}$ we set

$$\hat{U}(w) = b_m^{-1}(U(w)) \subset (\hat{\mathbb{P}}_n^m)^s$$

$$\hat{U}(w)' = \hat{\pi}^{-1}(\hat{U}(w))^s \subset (\hat{\mathbb{P}}_n^{m+1})^s$$

$$\bar{U}(w)' = \hat{\Phi}(\hat{U}(w)') \subset \hat{P}_n^{m+1},$$

where $U(w) \subset (\mathbb{P}_n^m)^s$ is defined above.

It is easy to see that $cr_{n,m+1}'$ is defined on $\bar{U}(w)'$ and that there is a commutative diagram

$$\begin{array}{ccc} \bar{U}(w)' & \xrightarrow{cr_{n,m+1}'} & \bar{U}(w)' \\ \bar{\pi}\downarrow & & \bar{\pi}\downarrow \\ \hat{\Phi}(\hat{U}(w)) & \xrightarrow{cr_{n,m}} & \hat{\Phi}(\hat{U}(w)), \end{array}$$

where $\bar{\pi}$ is induced by the projection $\hat{\pi}$: $\hat{U}(w)' \to \hat{U}(w)$ by passing to the quotient. The fibres of $\bar{\pi}$ are isomorphic to the fibres of $\hat{\pi}$. If

$$x = (x^1,\dots,x^m) \in U(w),$$

$$\hat{x} = (\hat{x}^1,\dots,\hat{x}^m) \in b_m^{-1}(x) \in \hat{U}(w),$$

$$(\hat{x}^1,\dots,\hat{x}^m,\hat{x}^{m+1}) \in \hat{\pi}^{-1}(\hat{x}) \subset \hat{U}(w)',$$

then

$$\hat{\pi}^{-1}(\hat{x}) \cong V(\hat{x})' = \{\hat{x}^{m+1} \in V(\hat{x}) : (\hat{x}^1,\dots,\hat{x}^m,\hat{x}^{m+1}) \in (\hat{\mathbb{P}}_n^{m+1})^s\} \subset V(\hat{x}).$$

By Theorem 2 of Chapter IV we know that

$$V(\hat{x})' \supset b_{m+1}^{-1}(V(\hat{x})''),$$

where

$$V(\hat{x})'' = \{x^{m+1} \in \mathbb{P}_n : (x^1,\dots,x^m,x^{m+1}) \in (\mathbb{P}_n^{m+1})^s\}.$$

We may change $U(w)$ to a smaller open subset to assume that all $x \in U(w)$ are in general position in sense of Chapter III. Then the stability criterion from Chapter II implies that

$$\operatorname{codim}(\mathbb{P}_n, \mathbb{P}_n \setminus V(\hat{x})'') \geq 2.$$

Moreover, by Theorem 2 of Chapter IV, a generic point set of every $\hat{\Delta}_{ij}$ is always stable in $\hat{\mathbb{P}}_n^{m+1}$. This implies that

$$\operatorname{codim}(V(\hat{x}), V(\hat{x}) \setminus V(\hat{x})') \geq 2.$$

Thus, replacing $U(w)$ by a smaller open subset on which $cr_{n,m}(w)$ is invertible, we obtain that $cr_{n,m+1}'(w)$ induces a pseudo-isomorphism

$$V(\hat{x})' = \bar{\pi}^{-1}(\Phi(\hat{x})) \dashrightarrow \bar{\pi}^{-1}(cr_{n,m}(w)(\Phi(\hat{x}))), \qquad \hat{x} \in \hat{U}(w).$$

In particular, if $w \in \operatorname{Ker}(cr_{n,m})$, $cr_{n,m+1}'(w)$ defines a pseudo-automorphism of each $V(\hat{x})$, where $\hat{x} \in \hat{U}(w)$. Obviously, if $w \neq 1$, this pseudo-automorphism is not trivial (because it defines a non-trivial isometry of $NS(V(\hat{x}))$. If A is a finite subgroup of $\operatorname{Ker}(cr_{n,m})$, we obtain an injective homomorphism

$$cr'_{n,m+1}: A \to Psaut(V(\hat{x})), \quad \hat{x} \in \bigcap_{w \in A} \hat{U}(w).$$

Finally we can restrict the action of $cr_{n,m+1}'(w)$ over the generic point of $U(w)$ and thus obtain the first assertion of the theorem.

Lemma 3. Let V be a gDP-surface and $\varphi: H_m \to N(V)$ be a geometric marking. The image of the homomorphism

$$\varphi^{\#}: Aut(V) \to O(H_m),\ g \to \varphi^{-1} \circ g^{*} \circ \varphi$$

is contained in $W_{2,m}$.

Proof. Obviously

$$g^{*} \circ \varphi = \varphi \circ \varphi^{\#}(g)$$

is a geometric marking. By Theorem 2 there exists $w \in W_{2,m}$ such that

$$g^{*} \circ \varphi = \varphi \circ w,$$

i.e.

$$w = \varphi^{-1} \circ g^{*} \circ \varphi = \varphi^{\#}(g) \in W_{2,m}.$$

Corollary. Assume that k is uncountable. Then there exists a non-empty subset $U \subset \mathbb{P}_2^m$. the complement of which is the union of a countable number (finite if $m \leq 8$) of closed subsets, such that

$$Aut(V(x))^{\#} \cong Ker(cr_{2,m})$$

for every $x \in U$. Here

$$Aut(V(x))^{\#} = Aut(V(x)) \text{ if } m \geq 4,$$

$$Aut(V(x))^{\#} = Aut(V(x))/H_x \text{ if } m < 4,$$

where H_x is the subgroup of PGL(3) that fixes the point set x.

Proof. By the previous lemma we have a natural homomorphism

$$\varphi_x^{\#}: Aut(V(x)) \to W_{2,m} \subset O(H_m),\ x \in \mathbb{P}_2^m.$$

By definition of the Cremona action for each $w \in Im(\varphi_x^{\#})$, the G-orbit of x is a fixed point of $cr_{2,m}(w)$. Now for every $w \in W_{2,m} \setminus Ker(cr_{2,m})$ the set of fixed points of $cr_{2,m}(w)$ on $U(w)$ is a proper closed subset. Thus for some open subset V of $\bigcap_{w \in W} U(w)$

$$Im(\varphi_x^{\#}) = Ker(cr_{2,m}) \text{ for all } x \in V.$$

It remains to use that $\mathrm{Ker}(\varphi_x^{*})$ is isomorphic to H_x, and that, if $m \geq 4$, H_x is trivial for all x which belong to a certain open Zariski subset.

Theorem 6. The Cremona representation

$$cr_{2,m}: W_{2,m} \to \mathrm{Bir}(P_2^m)$$

is injective if $m \geq 9$.

Proof. Let E_2^m be the subvariety of $\Phi(U(m)^s) \subset P_2^m$ parametrizing the orbits of stable point sets that lie on a plane cubic curve. For any $w \in W_{2,m}$ the generic point of E_2^m belongs to $U(w)$ and is fixed under the map $cr_{2,m}(w)$: $U(w) \to w(U(w))$. In fact, the condition $\bar{x} \in E_2^m$ means that $-K_{V(x)} \geq 0$ for every $x \in \Phi^{-1}(\bar{x})$, and hence is independent of a choice of a geometric marking. This implies that for any $w \in \mathrm{Ker}(cr_{2,m})$ there exists a dense subset $E(w) \subset E_2^m$ such that

$$\mathrm{Aut}(V(x)) \neq \{1\}, \quad \text{for any } x \in E(w).$$

Let us show that this is contradictory if $m \geq 9$. We may assume that x lies on a nonsingular cubic C. Since $m \geq 9$, we can also assume that such a cubic is unique. Then a non-trivial automorphism g of $V(x)$ preserves C and induces an automorphism $\bar{g}$ of C. We may obviously assume that C does not have complex multiplications. In this case it suffices to verify the assertion when $\bar{g}$ is either a translation

$$x \to x \oplus a, \quad a \in C,$$

or the inversion

$$x \to -x.$$

Let ε be defined as equal to 1 in the first case and equal to -1 in the second case. Let

$$tr: \mathrm{Pic}(V(x)) \to \mathrm{Pic}(C)$$

be the restriction map. In the usual notation

$$h_i = \varphi_x(e_i), \ i = 0,\dots,m,$$

where $\varphi_x: H_m \to \mathrm{Pic}(V(x))$ is the geometric marking defined by the point set x. Let

$$d = [a-o] \ (\varepsilon = 1), \ = 2[o] \ (\varepsilon = -1),$$

where o is an inflection point of C taken to be the zero of the group law on C, and [ ] denotes the divisor class. We have

$$tr((g^*(h_i)-\varepsilon h_i) = d, \ i = 1,\dots,m,$$

$$tr(g^*(h_0)-\varepsilon h_0) = 3d.$$

Now note that the restriction homomorphism tr is injective. Indeed, if $tr(x) = 0$ for some $x = \Sigma a_i h_i$, then we obtain

$$[3a_0 o + \Sigma a_i x^i] = 0,$$

i.e.

$$\bigoplus_i a_i x^i = 0$$

in the group law of C. Taking x general enough we avoid this possibility.

It follows that the divisor class

$$D = g^*(h_i)-\varepsilon h_i$$

is independent of $i = 1,\dots,m$, and

$$tr(D) = d.$$

Recall that $C \in |-K_{V(x)}|$. Therefore

$$D \bullet K_{V(x)} = \varepsilon - 1 \in 2\mathbf{Z},$$

and thus

$$D^2 = -2\varepsilon\lambda$$

for some integer λ. We have also

$$-1 = g^*(h_i)^2 = (\varepsilon h_i + D)^2 = -1+D^2+2\varepsilon(h_i \bullet D),$$

$$1 = g^*(h_0)^2 = (\varepsilon h_0+3D)^2 = 1+6\varepsilon(h_0 \bullet D)+9D^2.$$

This implies

$$h_i \bullet D = \lambda, \qquad h_0 \bullet D = 3\lambda.$$

Hence

$$tr(\lambda[C]) = tr(\lambda(3h_0 - \Sigma h_i)) = tr(D)$$

and

$$D = \lambda[C], \ \lambda \neq 0.$$

Now it is easy to finish the proof. We have

$$D \bullet K_{V(x)} = \varepsilon - 1 = \lambda(m-9),$$

$$D^2 = -2\varepsilon\lambda = \lambda^2(9-m).$$

This is absurd if $m \neq 10$ or 11. If $m = 10$ (resp. 11), $\varepsilon = -1$, $\lambda = -2$ (resp. -1). However,

$D = g^*(h_i)+h_i$ is effective and cannot be equal to $\lambda[C]$. This shows that g^* is the identity and hence g leaves the points x^i fixed. Since x is a generic point set in E_2^m, this implies that g is the identity.

Corollary. Assume that k is uncountable and $m \geq 9$. Then there exists a non-empty subset $U \subset \mathbb{P}_2^m$ the complement of which is the union of a countable number of closed subsets such that

$$\mathrm{Aut}(V(x)) = \{1\}$$

for every $x \in U$.

In the next chapter we will describe $\mathrm{Ker}(c_{2,m})$ for $m \leq 8$.

Remark 5. Theorem 6 is due to A. Coble **[Co 2]**. The lacking point in Coble's proof is the justification of the reduction to the case of point sets lying on a cubic curve. It has been mended in **[Hi]**. The application of this theorem to automorphisms of rational surfaces was first noticed in **[Do]**, see also **[Gi]** (m = 9), **[Hi]**, **[Ko]**.

5. Special subvarieties of $\mathcal{P}_n^m$.

In this section we give some examples of subvarieties V of $\mathcal{P}_n^m$ whose generic point belongs to the domain of definition of each $cr_{n,m}(w)$, $w \in W_{n,m}$, and is fixed under the Cremona action. We will call such subvarieties special. The most interesting among them are those for which the Weyl group $W_{n,m}$ is infinite but the induced Cremona representation:

$$W_{n,m} \rightarrow \mathrm{Bir}(V)$$

factors through a finite quotient. All examples known to us of such special varieties will be presented here.

We have already used one such special subvariety. It consists of point sets in $\mathbb{P}_2$ lying on an irreducible cubic curve (see the proof of Theorem 6). We refer to **[Ha 1]**, **[Ha 2]**, **[Lo 1]** for some special properties of these point sets. Let E(m) denote the subvariety of $\mathcal{P}_2^m$ corresponding to point sets lying on an irreducible cubic curve. These are all stable by Theorem 1 of Chapter II. Note that "$\hat{x}$ lies on a plane curve" means that each point in $\hat{x}$ lies in the proper inverse transform under the blowing-down map $V(\hat{x}) \to \mathbb{P}_2$. We know that the restriction of $cr_{2,m}$ to E(m) is a faithful representation

$$cr_{2,m} : W_{2,m} \to \mathrm{Bir}(E(m)).$$

It may happen however that the restriction of the Cremona action to an invariant subvariety of $\hat{P}^m_n$ factors through a finite group. Here we give some examples when this occurs. Assume n = 2 and m = 9. A point set $\hat{x} \in \hat{P}^9_2$ is said to be an <u>Halphen point set</u> if the blowing-up surface $V = V(\hat{x})$ has a structure of a minimal rational elliptic surface (an <u>Halphen surface</u>). It follows easily from the theory of elliptic surfaces that this is equivalent to the condition that $|-rK_V|$ is an irreducible pencil for some $r > 0$. The number r is called the <u>index</u> of x (resp. of V(x)). The image of this pencil on $\mathbb{P}_2$ is a pencil of curves of degree 3r with nine r-multiple points at the points $\hat{x}^i$ (<u>an Halphen pencil</u>, see **[Ha]**, **[Gi]**, [C-D 2]).

Let $\mathcal{H}a(r)$ be the subvariety of $\hat{P}^9_2$ which parametrizes the projective equivalence classes of stable Halphen point sets of index r. Since the action of the Weyl group $W = W_{2,9}$ in $N^1(V(\hat{x}))$ preserves the canonical class, the generic point of $\mathcal{H}a(r)$ belongs to every U(w), $w \in W$, and is fixed under the Cremona action $cr_{2,9}$. This defines a birational action

$$cr_{2,9}(r) : W \to \mathrm{Bir}(\mathcal{H}a(r)).$$

Its kernel is isomorphic to the automorphism group of a generic Halphen surface of index r.

To state a result about $\mathrm{Ker}(cr_{2,9}(r))$ we recall some well-known facts about the Weyl group $W_{2,9}$. Let $Q_9 = Q(B_2)$ be the root lattice of the root system of type 2 in H_9. Its radical is spanned by the vector

$$K_{2,9} = 3e_0 - e_1 - \ldots - e_9 = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$$

and

$$Q_9/\mathbf{Z}K_{2,9} \cong Q_8,$$

where Q_8 is the root lattice of type E_8 for $W_{2,8} \cong W(E_8)$. Using the projection map $Q_9 \to Q_9/\mathbf{Z}K_{2,9}$, the lattice Q_8 can be identified with the sublattice of Q_9 spanned by the first 8 simple roots $\alpha_0, \ldots, \alpha_7$. The Weyl group $W_{2,9}$ preserves $K_{2,9}$ and acts naturally on Q_8. Let $a : W_{2,9} \to O(Q_8) = W_{2,8}$ be the corresponding restriction homomorphism.

Lemma 4. There is an exact sequence of groups:

$$1 \to Q_8 \xrightarrow{i} W_{2,9} \xrightarrow{a} W_{2,8} \to 1,$$

where for every root $\alpha \in Q_8$

$$i(\alpha)(v) = v + (\beta \cdot v)K_{2,9} + (v \cdot K_{2,9})\alpha = s_\alpha \circ s_\beta(v), \text{ for any } v \in H_9,$$

where $\beta = K_{2,9} - \alpha$ is easily checked to be a root.

Theorem 7. Let x be a generic Halphen point set and $\varphi_x : H_9 \to N^1(V)$ be the corresponding geometric marking. Under the natural injection $\mathrm{Aut}(V(x)) \hookrightarrow W_{2,9}$, $g \to \varphi_x^{-1} \circ g^* \circ \varphi_x$, there is an exact sequence:

$$1 \to rQ_8 \xrightarrow{i} \mathrm{Aut}(V(x)) \xrightarrow{a} \{\pm 1\} \to 1.$$

In particular $cr_{2,9}(r)(W_{2,9})$ acts on $\mathcal{H}a(r)$ via its finite quotient isomorphic to the group G given by the extension

$$1 \to (\mathbf{Z}/r\mathbf{Z}) \to G \to W(E_8)/\{\pm 1\} \to 1.$$

Proof. See **[Co 1]**, S52 ($r \leq 2$), **[Gi]**.

Remark 6. An Halphen surface $V(\hat{x})$ of index 1 is a jacobian elliptic surface. This means that its elliptic fibration has a section. After fixing a section, the set of sections is equipped with a structure of an abelian group of finite rank equal to 8 if x is generic. The group of sections acts on $V(\hat{x})$ by translations (see **[AS]**, **[C-D 2]**). The corresponding subgroup of $\mathrm{Aut}(V(x))$ is the image of $Q_8 \cong \mathbf{Z}^8$ in $W_{2,9}$. The element $-1 \in \mathrm{Aut}(V(x))/i(Q_8)$ corresponds to the automorphism of $V(x)$ which induces the homomorphism $z \to -z$ on each nonsingular fibre of the elliptic fibration.

Assume now that $n = 2$, $m = 10$. A point set $x \in \hat{\mathbb{P}}_2^{10}$ (resp. its blowing-up $V(x)$) is said to be a Coble point set (resp. a Coble surface) if $|-2K_V|$ contains an irreducible curve. It is easy to see that this curve is a smooth rational curve C with $C^2 = -4$. Its image in $\mathbb{P}_2$ is a sextic with double points at each $\hat{x}^i$.

Let $\mathcal{C} \subset \hat{P}_2^{10}$ be the variety parametrizing projective equivalence classes of stable Coble point sets. As in the above example, the generic point of $\mathcal{C}$ is invariant under the Cremona action $cr_{2,10}$. This defines a birational action

$$cr_{2,10}' : W_{2,10} \to \mathrm{Bir}(\mathcal{C}).$$

Its kernel is isomorphic to the automorphism group of a generic Coble surface.

Theorem 8. Let x be a generic Coble point set and φ_x: $H_{10} \to N^1(V(x))$ be the corresponding geometric marking. Under the natural injection $Aut(V(x)) \hookrightarrow W_{2,10}$, $g \to \varphi_x^{-1} \circ g^* \circ \varphi_x$

$$Aut(V(x)) \cong W_{2,10}(2) = \{w \in W_{2,10}: w(x) \equiv x \bmod 2Q(R_B)\}.$$

In particular, $cr_{2,10}'$ acts on $\mathfrak{C}$ via its finite quotient isomorphic to $W_{2,10}/W_{2,10}(2)$.

Proof. See **[Co 1]**,S52, **[Co 5]**, **[Do 2]**, **[C-D 3]**.

Next, let n = 3 and m = 8. A point set $\hat{x} \in \hat{\mathbb{P}}_3^8 = \hat{\mathbb{P}}_3^8$ is called a Cayley octad (cf. **[Ca]**) if it is the base-set of the proper inverse transform of in $V(\check{x})$ of a net of quadrics in $\mathbb{P}^3$. One easily verifies that every Cayley octad is a stable point set with no three collinear points, and no five coplanar points. Its image in $\hat{\mathbb{P}}_3^8$ is a semi-stable point set with no three coinciding points. We denote by $\mathfrak{CO}$ the subvariety of $\hat{P}_3^8$ parametrizing the orbits of Cayley octads. The projection of $\mathfrak{CO}$ to P_3^8 defines a birational map:

$$\mathfrak{CO} \overset{\sim}{\dashrightarrow} S_8$$

onto the variety of the orbits of self-associated point sets. It follows from Theorem 4 that the generic point of the variety $S_8 \subset P_3^8$ is invariant with respect to the Cremona action $cr_{3,8}$. Let

$$cr_{3,8}': W_{3,8} \to Bir(S_8)$$

be the restriction of $cr_{3,8}$ to S_8.

To state a result about $Ker(cr_{3,8}')$ we use some facts about the Weyl group $W_{3,8}$ which are similar to those from above about the group $W_{2,9}$. We have

$$\tfrac{1}{2}K_{3,8} = 2e_0-e_1-\ldots-e_8 = 2\alpha_0+\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7 \in Q_B.$$

$$\kappa_{3,8} = 4e_0-2(e_1-\ldots-e_8) = 2\check{\alpha}_0+\check{\alpha}_1+2\check{\alpha}_2+3\check{\alpha}_3+4\check{\alpha}_4+3\check{\alpha}_5+2\check{\alpha}_6+\check{\alpha}_7 \in Q_{\check{B}}.$$

It is easy to check that the map

$$(\alpha_i,\check{\alpha}_j) \to (\alpha_i \bmod \mathbf{Z}\tfrac{1}{2}K_{3,8}, \ \check{\alpha}_j \bmod \mathbf{Z}\kappa_{3,8}), \ i,j = 0,\ldots,6$$

defines an isomorphism of bilattices:

$$Q_7 \cong (Q(B)/\mathbf{Z}\tfrac{1}{2}K_{3,8}, Q(\check{B})/\mathbf{Z}\kappa_{3,8}),$$

where Q_7 is the root lattice of type 2 in H_7 (the root system of type E_7).

Lemma 5. There is an exact sequence of groups:

$$1 \to Q_7 \xrightarrow{i} W_{3,8} \xrightarrow{a} W(E_7) \to 1,$$

where for every root $\alpha \in R_B$ in Q_7

$$i(\alpha)(v) = v + \tfrac{1}{2}(v \cdot \check{\alpha})K_{3,8} + \tfrac{1}{2}(v \cdot K_{3,8})\check{\beta} = s_\alpha \circ s_\beta(v), \text{ for any } v \in H_9,$$

where $\beta = \frac{1}{2}K_{3,8} - \alpha$ is easily checked to be a root.

Theorem 9. Let $x \in P_3^8$ be a generic Cayley octad and φ_x: $H_9 \to N(V(x))$ be the corresponding geometric marking. Under the natural injection $\mathrm{Psaut}(V(x)) \hookrightarrow W_{2,8}$, $g \to \varphi_x^{-1} \circ g^* \circ \varphi_x$ there is a subgroup $\mathrm{Psaut}(V(x))'$ of $\mathrm{Psaut}(V(x))$ given by an exact sequence:

$$1 \to Q_7 \xrightarrow{i} \mathrm{Psaut}(V(x))' \xrightarrow{a} \{\pm 1\} \to 1.$$

In particular $cr_{3,8}'(W_{2,8})$ acts on S_8 via its finite quotient isomorphic to $W(E_7)/\{\pm 1\} \cong Sp(6, \mathbb{F}_2)$.

<u>Proof</u>. This is similar to the proof of Theorem 7 ($r = 1$). We know that a generic point $x \in S_8$ is the base set of a net of quadrics in $\mathbb{P}_3$. This net defines a morphism

$$\pi: V(x) \to \mathbb{P}_2$$

which is an elliptic fibration. Fixing a section of π corresponding to one of the exceptional divisors of σ: $V(x) \to \mathbb{P}_3$, we equip the generic fibre of π with a structure of an abelian variety A of dimension 1 over the field K of rational functions on $\mathbb{P}_2$. Its group of rational points $A(K)$ is isomorphic to the subgroup of $N^1(V(x))$ of the divisor classes whose restriction to the generic fibre is a divisor of degree 0. It is easy to see that

$$A(K) \cong \mathbb{Z}^7,$$

and is generated by the classes $\varphi_x(\alpha_i)$, $i = 0,\dots,6$. A direct calculation shows that the pseudo-automorphisms of $V(x)$ induced by the translation automorphisms t_a, $a \in A(K)$, of A is isomorphic to the group Q_7. The inversion automorphism $x \to -x$ of A is mapped to -1 under the map $W_{3,8} \to W(E_7)$.

For another more direct proof see **[Co 1]**, S53.

Finally, we assume that $n = 3$, $m = 10$. A point set $x \in \mathbb{P}_3^{10} \setminus \Delta(10)$ is said to be a <u>Cayley decad</u> if there exists a web W of quadrics in P_3 such that x is the set of double points of the Hessian surface of this web. Recall that the latter is defined as the subvariety of W parametrizing singular quadrics. We refer to (**[Ca]**, **[Co 1]**,

[Co 5], [Cos], [C-D 3]) for the beautiful geometry of such Hessian surfaces (called Cayley symmetroids). Let $\mathcal{C}\mathcal{D}$ be the subavariety of P_3^{10} corresponding to stable Cayley decads. It is easy to see that its generic point is fixed under the Cremona action $cr_{3,10}$ and defines a birational action:

$$cr_{3,10}': W_{3,10} \to \mathrm{Bir}(\mathcal{C}\mathcal{D}).$$

Let

$$W_{3,10}(2) = \{w \in W_{3,10}: W(x) \equiv x \bmod 2Q(B) \text{ for any } x \in Q(B)\}.$$

It is known (see **[Co 6], [Gri 1]**) that the factor group $W_{3,10}/W_{3,10}(2)$ is given by an extension

$$1 \to (\mathbb{Z}/2\mathbb{Z})^8 \to W_{3,10}/W_{3,10}(2) \to Sp(8,\mathbb{F}_2) \to 1.$$

Let $\bar{W}_{3,10}$ be the inverse image of $(\mathbb{Z}/2\mathbb{Z})^8$ in $W_{3,10}$ under the projection $W_{3,10} \to W_{3,10}/W_{3,10}(2)$.

Theorem 10. Let x be a generic Cayley decad and $\varphi_x: H_{10} \to N(V(x))$ be the corresponding geometric marking. Under the natural injection $\mathrm{Psaut}(V(x)) \hookrightarrow W_{2,9}$, $g \to \varphi_x^{-1} \circ g^* \circ \varphi_x$, we have

$$\bar{W}_{3,10} \subset \mathrm{Psaut}(V(x))'.$$

In particular, the homomorphism $cr_{3,10}'$ factors via its finite quotient isomorphic to $Sp(8,\mathbb{F}_2)$.

Proof. See **[Co 1]**, S53, **[Co 5], [C-D 1], [C-D 3]**.

Remark 7. We refer to **[Pi]** for applications of the Cremona action to simultaneous resolutions of singularities of rational double points.

VII. EXAMPLES.

Here we review in examples everything we have learned so far.

1. Point sets in $\mathbb{P}_1$.

In this case we complete our definition of the Cremona action by setting

$$W_{1,m} = \Sigma_m$$

and by defining

$$cr_{1,m}: W_{1,m} \to Bir(P_1^m)$$

to be the action of the permutation group via its natural action on $\mathbb{P}_1^m$. Note that we have an analog of Theorem 5 from Chapter VI:

$$cr_{1,m}(w)(\bar{x}) = \bar{x} \Leftrightarrow w(x) = g(x) \text{ for some } g \in PGL(2)$$

for any representative x of $\bar{x} \in P_1^m$.

The spaces P_1^m are defined for $m \geq 3$. Its open subset $\Phi((\mathbb{P}_1^m)^s)$ parametrizes the G-orbits of point sets where strictly less than $\frac{1}{2}m$ points coincide; it is equal to P_1^m when m is odd.

We have

$$P_1^3 = \{\text{point}\},$$

$$Ker(cr_{1,3}) \cong \Sigma_3,$$

$$P_1^4 \cong \mathbb{P}_1,$$

$$Ker(cr_{1,4}) \cong (\mathbb{Z}/2)^2 \subset \Sigma_4.$$

The quotient group is

$$\Sigma_4/Ker(cr_{1,4}) \cong SL(2,\mathbb{F}_2),$$

and the nontrivial elements in $Ker(cr_{1,4})$ are the permutations

$$(12)(34),\quad (13)(24),\quad (14)(23).$$

For example, if $x^1 = (1,0)$, $x^2 = (0,1)$, $x^3 = (1,1)$, $x^4 = (1,a)$, $a \neq 0,1$,

$$(x^1,x^2,x^3,x^4) \overset{(12)(34)}{\rightarrow} (x^2,x^1,x^4,x^3) \overset{g}{\rightarrow} (x^1,x^2,x^3,x^4),$$

where

$$g\colon (t_0,t_1) \rightarrow (at_1,t_0).$$

The analog of Theorem 6 from Chapter VI says that

$$\mathrm{Ker}(cr_{1,m}) = \{1\} \text{ if } m > 4.$$

This is almost trivial because the only non-trivial normal subgroups of Σ_m are the alternating subgroup A_m and Σ_m itself. Let us identify Σ_{m-1} with the subgroup of Σ_m that fixes the subset $\{m\}$. Then , if $A_m \subset \mathrm{Ker}(cr_{1,m})$,

$$A_{m-1} = A_m \cap \Sigma_{m-1} \subset \mathrm{Ker}(cr_{1,m-1}),$$

which is contradictory for $m = 5$.

The explicit formulae for the Cremona action can be applied in our case also. Our final remark is that

$$P_1^m / W_{1,m} \cong \mathbb{P}(\mathrm{Sym}^m(V^*))^{ss}/PGL(2) = (\mathbb{P}_m)^{ss}/PGL(2)$$

The approach to the invariant theory of binary forms via the Cremona action $cr_{1,m}$ is due to E. Moore **[Mo]**. We refer to **[Re]** and **[Sz]** for some interesting details in the case $m = 5$.

2. <u>Point sets in $\mathbb{P}_2$ ($m \leq 5$)</u>.

Although the spaces P_2^m are defined only for $m \geq 3$ we make some remarks starting from the case $m = 1$. All root bases B_n considered below are of canonical type 2 in H_m ($m \geq 3$). Their Dynkin diagram is of type $A_2 \times A_1$ ($m = 3$), A_4 ($m = 4$), D_5 ($m = 5$). We denote by

$$(\hat{\mathbb{P}}_2^m)^{pg} \subset \hat{\mathbb{P}}_2^m$$

the subset of point sets in "almost general position" in the sense of Remark 1 of Chapter V. This is the set of points $\hat{x}$ such that the gDP-surface $V(\hat{x})$ is either a Del Pezzo surface or a nodal Del Pezzo surface.

<u>$m = 1$</u>. There is one orbit of $G = PGL(3)$ in $\hat{\mathbb{P}}_2^1 = \mathbb{P}_2^1$. For every $x \in \mathbb{P}_2^1$, the blowing-up $V(x)$ is a Del Pezzo surface of degree 8. It is isomorphic to the minimal ruled

surface $\mathbb{F}_1$, and its anti-canonical model is a nonsingular surface of degree 8 in $\mathbb{P}_8$. The automorphism group of V(x) acts identically on N(V(x)). There is only one geometric marking of V(x), and one exceptional curve of the first kind.

<u>m = 2</u>. $\hat{\mathbb{P}}_2^2$ is the union of two G-orbits: $\hat{\Delta}(2)$ and its complement. For every $\hat{x}\in\hat{\mathbb{P}}_2^2\setminus\hat{\Delta}(2)$, the blowing-up surface $V(\hat{x})$ is a Del Pezzo surface of degree 7. Its anti-canonical model is a nonsingular surface of degree 7 in $\mathbb{P}_7$. If $\hat{x}\in\hat{\Delta}(2)$, $V(\hat{x})$ is a nodal Del Pezzo surface, its anti-canonical model is a surface of degree 7 with one ordinary double point. Every Del Pezzo surface $V(\hat{x})$ of degree 7 has exactly two geometric markings. They differ by an automorphism of $V(\hat{x})$ that is induced by a projective transformation of $\mathbb{P}_2$. There are three exceptional curves of the first kind on $V(\hat{x})$. In the nodal case $V(\hat{x})$ has only one geometric marking, and two exceptional curves of the first kind.

<u>m = 3</u>. Here

$$(\mathbb{P}_2^3)^s = (\hat{\mathbb{P}}_2^3)^s = \emptyset,$$

$$(\mathbb{P}_2^3)^{ss} = b_3((\hat{\mathbb{P}}_2^3)^s) = (\mathbb{P}_2^3)^{gen}$$

consists of non-collinear point sets with no coinciding points. We have

$$P_2^3 = \hat{P}_2^3 = \{\text{point}\}.$$

The set of positive roots consists of four elements:

$$e_0-e_1-e_2-e_3,\ \ e_1-e_2,\ \ e_1-e_3,\ \ e_2-e_3.$$

This shows that

$$(\mathbb{P}_2^3)^{un} = (\hat{\mathbb{P}}_2^3)^{ss}.$$

The set

$$\mathbb{P}_2^3 \setminus (\hat{\mathbb{P}}_2^3)^{pg} \neq \emptyset$$

consists of point sets of the form:

$$(x^1, x^2\to x^1, x^3\to x^1).$$

Obviously,

$$\mathrm{Ker}(cr_{2,3}) = W_{2,3} \cong \Sigma_3\times\mathbb{Z}/2.$$

The homomorphism $\mathrm{Aut}(V(x)) \to W_{2,3}$ is surjective. Its kernel is isomorphic to the torus k^{*2}.

Every Del Pezzo surface V(x), $x\in(\hat{\mathbb{P}}_2^3)^{un}$, has 6 exceptional curves of the first kind. If φ: $H_3 \to N(V(x))$ is a geometric marking of $V(x)$, then the classes of the exceptional curves are the images of the following vectors:

$$e_0-e_1-e_2,\ e_0-e_2-e_3,\ e_0-e_1-e_3,\ e_1,e_2,e_3.$$

The number of geometric markings on a Del Pezzo surface of degree 6 is equal to the order of the Weyl group $W_{2,3}$, which is 12.

We leave to the reader the study of geometric markings on $V(x)$ for $\hat{x} \notin (\hat{\mathbb{P}}_2^3)^{ss}$.

<u>m = 4</u>. We have

$$(\mathbb{P}_2^4)^{ss} = (\mathbb{P}_2^4)^s = (\mathbb{P}_2^4)^{gen}$$

consisting of point sets with no coinciding points and no three points on a line,

$$(\hat{\mathbb{P}}_2^4)^{ss} = (\hat{\mathbb{P}}_2^4)^s = b_4^{-1}((\mathbb{P}_2^4)^s),$$

and

$$P_2^4 = \hat{P}_2^4 = \{\text{point}\}.$$

There are 10 positive roots in R_B. They are given by

$$e_0-e_i-e_j-e_k,\ 1 \le i < j < k \le 4,$$

$$e_i-e_j,\qquad 1 \le i<j \le 4.$$

This implies that

$$(\hat{\mathbb{P}}_2^4)^{un} = (\hat{\mathbb{P}}_2^4)^{ss}.$$

Thus for every $x \in (\hat{\mathbb{P}}_2^4)^{ss}$, its blowing-up is a Del Pezzo surface of degree 5. Its anti-canonical model is isomorphic to a nonsingular surface of degree 5 in $\mathbb{P}_5$, which is defined by five equations of degree 2. We have

$$\mathrm{Aut}(V(x)) \cong \mathrm{Ker}(cr_{2,4}) = W_{2,4} \cong \Sigma_5.$$

The number of geometric markings of a Del Pezzo surface of degree 5 is equal to the order of the Weyl group $W_{2,4}$, which is 120. Any two geometric markings of $V(x)$ differ by an automorphism of $V(x)$.

A Del Pezzo surface of degree 5 has 10 exceptional curves of the first kind. Their classes are the images under φ_x of the following vectors in H_4:

$$e_0-e_i-e_j,\ 1 \le i < j \le 4,\ e_i,\ i = 1,\dots,4.$$

A curious remark is that

$$V(x) \cong P_1^5 \text{ for any } x \in (\mathbb{P}_2^4)^{ss}.$$

This can be seen in various ways. One of them is as follows. We know from Chapter II that P_1^5 is a nonsingular rational surface. The projection $\mathbb{P}_1^5 \to \mathbb{P}_1^4$ induces a map

$$\pi: P_1^5 \to P_1^4.$$

Its general fibre is isomorphic to $\mathbb{P}_1$, and its 3 degenerate fibres are composed of two irreducible components intersecting transversally. They lie over the boundary $\mathcal{B} \subset P_1^4$. One of the components lies over the orbit in $\mathbb{P}_1^4$ of a point set in which two points coincide, the second one over the orbit of point sets in which the complementary pair of points coincide. The intersection point lies over the orbit of the point sets in which two complementary pairs of point coincide. There are four disjoint sections of π. They are defined by the maps:

$$(x^1,x^2,x^3,x^4) \to (x^1,x^2,x^3,x^4,x^i).$$

A standard argument shows that the images of these sections can be blown down to an unnodal point set from $\mathbb{P}_1^4$. Finally note that the action of Σ_5 on $V(x)$ corresponds to the action of Σ_5 on P_1^5 via permutation of the factors of $\mathbb{P}_1^5$.

<u>m = 5</u>. We have

$$(\mathbb{P}_2^5)^{ss} = (\mathbb{P}_2^5)^s$$

consisting of point sets where no points coincide and at most three points are collinear.

$$(\hat{\mathbb{P}}_2^5)^{ss} = (\hat{\mathbb{P}}_2^5)^s = b_5^{-1}((\mathbb{P}_2^5)^s).$$

By association

$$\hat{P}_2^5 \cong P_2^5 \cong P_1^5,$$

and, by the previous remark, is isomorphic to a Del Pezzo surface of degree 5.

There are 20 positive roots in R_B. They are given by

$$e_0-e_i-e_j-e_k, \quad 1 \le i < j < k \le 5,$$

$$e_i-e_j, \qquad 1 \le i<j \le 5.$$

Thus

$$(\mathbb{P}_2^5)^{un} = (\mathbb{P}_2^5)^{ss} \setminus (\cup Z_{ijk}),$$

where

$$Z_{ijk} = Z(e_0-e_i-e_j-e_k).$$

The anti-canonical Del Pezzo surface is a complete intersection of two quadrics in $\mathbb{P}_4$. If it is nonsingular, the corresponding quadrics can be given by diagonalized equations:

$$\sum_{i=1}^{5} z_i^2 = \sum_{i=1}^{5} \lambda_i z_i^2 = 0,$$

where all λ_i are distinct. The group $(\mathbb{Z}/2)^4$ acts on $V(x)$ by automorphisms

$$(z_0,\ldots,z_4) \to (\pm z_0,\ldots,\pm z_4).$$

For every $x\in(\hat{\mathbb{P}}_2^5)^s$, its blowing-up is a Del Pezzo surface of degree 4. Under the homomorphism

$$Ker(cr_{2,5}) \to Aut(V(x)),$$

the subgroup $(\mathbb{Z}/2)^4 \subset Aut(V(x))$ corresponds to the subgroup H of $W_{2,5} \cong W(D_5)$ generated by the products of two reflections $s_\alpha \circ s_\beta$, where $\alpha = e_0-e_i-e_j-e_k$, $\beta = e_r-e_s$, $\{i,j,k\}\cap\{r,s\} = \emptyset$.

There is an isomorphism:

$$W_{2,5}/H \cong \Sigma_5,$$

as is predicted by the association isomorphism, and $cr_{2,5}$ defines a faithful representation

$$cr_{2,5}: \Sigma_5 \to Bir(P_2^5)$$

that is induced by the permutation of factors of $\mathbb{P}_2^5$.

There are 16 exceptional curves of the first kind on a Del Pezzo surface of degree 4. Their classes correspond to the following vectors in H_5:

$$e_0-e_i-e_j,\ 1 \le i<j \le 5,$$

$$e_i,\ i = 1,\ldots,5,$$

$$2e_0-e_1-\ldots-e_5.$$

The number of geometric markings on a Del Pezzo surface of degree 4 is equal to the order of the Weyl group $W_{2,5}$ which is $2^4 \cdot 5! = 1920$.

All possible singularities of an anti-canonical nodal Del Pezzo surface are known. For every nodal Del Pezzo surface $V(\hat{x})$, the set of nodal roots defines a root basis B^n in H_m. Its Dynkin diagram $\Gamma(B^n)$ is a connected sum of Dynkin diagrams isomorphic to subdiagrams of $\Gamma(B)$. Its root lattice is isomorphic to a sublattice of Q(B). Each connected component of $\Gamma(B^n)$ is equal to the intersection graph of irreducible components of the exceptional locus of one of the singular points of the anti-canonical model of $V(\hat{x})$. This allows one to classify all possible configurations of singular points of an anti-canonical Del Pezzo surface. We refer to **[DuV 1]**, **[Ti]**, **[Ur]** for the corresponding lists.

3. <u>Cubic surfaces (n = 2, m = 6).</u>

By the stability criterion from Chapter II, we have

$(\mathbb{P}_2^6)^{ss} = \{x = (x^1,\dots,x^6)\in\mathbb{P}_2^6$: no 3 points coincide and no 5 points are collinear$\}$,

$(\mathbb{P}_2^6)^{s} = \{x = (x^1,\dots,x^6)\in\mathbb{P}_2^6$: no 2 points coincide and no 4 points are collinear$\}$.

By Theorem 2 from Chapter IV, we have

$(\hat{\mathbb{P}}_2^6)^{ss} = (\hat{\mathbb{P}}_2^6)^{s} = \{\hat{x} = (\hat{x}^1,\dots,\hat{x}^6)\in\hat{\mathbb{P}}_2^6$: $\hat{x}\notin\hat{\Delta}(6)_3$ and no 4 points are collinear$\}$.

The variety P_2^6 is a normal rational variety of dimension 4 isomorphic to a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1,1,1,1,1,2)$ (see its equation in Chapter I). The morphism

$$\bar{b}_6: \hat{P}_2^6 \to P_2^6$$

is a resolution of singularities. Its fiber over a nonsingular point of the singular locus $\mathcal{S}$ of P_2^6 is isomorphic to a nonsingular quadric.

Note that

$$(\hat{\mathbb{P}}_2^6)^{ss} \subset (\hat{\mathbb{P}}_2^6)^{pg}.$$

For every $\hat{x}\in(\hat{\mathbb{P}}_2^6)^{pg}$ the anti-canonical linear system maps $V(\hat{x})$ onto a cubic surface $\bar{V}$ in $\mathbb{P}_3$ with at most rational double points. It is an anti-canonical Del Pezzo surface of degree 3. Those which have at most nodes as singularities correspond to point sets from $(\hat{\mathbb{P}}_2^6)^{ss}$. The variety $\hat{P}_2^6$ is a natural compactification of the coarse moduli variety of nonsingular cubic surfaces with a geometric marking.

There are 36 positive roots in H_6 with respect to a canonical root basis of type 2:

$e_0-e_i-e_j-e_k$, $1 \le i < j < k \le 6$,

e_i-e_j, $1 \le i<j \le 6$,

$2e_0-e_1-\ldots-e_6$.

Each of them can be realized as a discriminant condition for $V(\hat{x})$, $\hat{x}\in(\hat{\mathbb{P}}_2^6)^s$. Thus $(\hat{\mathbb{P}}_2^6)^{un}$ consists of point sets $\hat{x}\notin\hat{\Delta}(6)$, $\hat{x}$ does not lie on a conic, and no three points from $\hat{x}$ are collinear. The boundary

$$\hat{P}_2^6\setminus(\hat{P}_2^6)^{un}$$

consists of 36 irreducible hypersurfaces, each of which corresponds to one discrimimant condition. Note that the permutation group Σ_6 acts biregularly on $\hat{P}_2^6$ (see Remark 2 in Chapter IV). The quotient space

$$\hat{P}_2^6/\Sigma_6$$

is a compactification of the moduli space of Del Pezzo surfaces of degree 3 together with a contraction sheaf(cf. **[Ish]**). Unfortunately, the birational action of the whole group $W_{2,6} \cong W(E_6)$ does not extend to a biregular action on $\hat{P}_2^6$. We believe that it can be biregularly extended to the variety $\hat{\hat{P}}_2^6$ obtained from $\hat{P}_2^6$ by blowing up all intersections of discriminant hypersurfaces. If this is true, the quotient variety would be a natural compactification of the moduli space of nonsingular cubic surfaces.

The Weyl group $W_{2,6} \cong W(E_6)$ is of order 51840. This is also the number of geometric markings of a nonsingular cubic surface. The Weyl group is "almost" simple. The only non-trivial proper normal subgroup of W is the subgroup W' of index 2 generated by the products of pairs of simple reflections.

We have

$$\mathrm{Ker}(cr_{2,6}) = \{1\}.$$

In fact, the only other possibility is $W' \subset \mathrm{Ker}(cr_{2,6})$. In this case W' contains the alternating group A_5 that acts by permutations of the first 5 points. This implies $\mathrm{Ker}(cr_{2,5}) \supset A_5$, which is impossible.

Every nonsingular cubic surface V contains 27 lines, which are exceptional curves of the first kind. Their classes are the images of the following vectors:

$e_0-e_i-e_j$, $1 \le i<j \le 6$,

e_i, $i = 1,\ldots,6$,

$2e_0-e_1-\ldots-e_6+e_i$, $i = 1,\ldots,6$.

Each geometric marking $\varphi: H_6 \to N(V)$ defines a <u>double-sixer</u> of lines. This is a set of 12 lines the classes of which are given by

$$h_i = \varphi(e_i), \text{ and } h_i' = \varphi(2e_0-e_1-\ldots-e_6+e_i), \; i = 1,\ldots,6 .$$

They are determined by the property:

$$h_i \bullet h_j = h_i' \bullet h_j' = 0, \; h_i \bullet h_j' = 1, \; 1 \le i < j \le 6,$$

$$h_i \bullet h_i' = 0, \; i = 1,\ldots,6 .$$

Conversely, every double-sixer $\{l_1,\ldots,l_6;l_1',\ldots,l_6'\}$ defines a pair (φ_1,φ_2) of geometric markings which satisfy:

$$\varphi_1(e_i) = [l_i], \; \varphi_2(e_i) = [l_i'], \quad i = 1,\ldots,6 .$$

We will see a little later that, if $\varphi_1 = \varphi_x$, $\varphi_2 = \varphi_y$ for some $x;y \in \mathbb{P}_2^6$, then x is associated to y. Also note that

$$\varphi_1 = \varphi_2 \circ s_\alpha,$$

where

$$\alpha := \alpha_{max} = 2e_0-e_1-\ldots-e_6 \text{ ("the maximal root").}$$

The Weyl group W acts transitively on the set of 27 lines and can be defined as the group of bijections of this set preserving incidence relations. It acts transitively on the set of double-sixers. The isotropy subgroup of a double-sixer is a subgroup $\Sigma_6 \times (s_\alpha)$. Thus the number of different double-sixers is 36, which is also the number of positive roots with respect to the root system of type E_6.

Finally, let us show that the association map

$$a_{2,6}: P_2^6 \to P_2^6$$

restricted to the subset of point sets not lying on a conic coincides with $cr_{2,6}(s_{\alpha_{max}})$. This shows that the geometric markings corresponding to the same double-sixer are associated.

Let

$$x = (x^1,\ldots,x^6) \in (\mathbb{P}_2^6)^{ss} \setminus Z(\alpha_{max}) \subset \mathbb{P}_2^6 \setminus \Delta(6),$$

and x^7, x^8, x^9 be three non-collinear points that together with the first 6 points form the base set of a pencil of cubic curves. Replacing x by a projectively equivalent set, we may assume that

$$x^7 = (1,0,0), \; x^8 = (0,1,0), \; x^9 = (0,0,1).$$

Let T_0 be the standard Cremona transformation. Let us verify that

$$x' = (T_0(x^1),\dots,T_0(x^6)) \text{ is associated to } x.$$

Let $X = (x_j^{(i)})$ be a coordinate matrix of x. We have to show that

$$\begin{bmatrix} x_0^{(1)} x_0^{(2)} x_0^{(3)} x_0^{(4)} x_0^{(5)} x_0^{(6)} \\ x_1^{(1)} x_1^{(2)} x_1^{(3)} x_1^{(4)} x_1^{(5)} x_1^{(6)} \\ x_2^{(1)} x_2^{(2)} x_2^{(3)} x_2^{(4)} x_2^{(5)} x_2^{(6)} \end{bmatrix} \cdot \Lambda \cdot \begin{bmatrix} x_1^{(1)}x_2^{(1)} & x_0^{(1)}x_2^{(1)} & x_0^{(1)}x_1^{(1)} \\ x_1^{(2)}x_2^{(2)} & x_0^{(2)}x_2^{(2)} & x_0^{(2)}x_1^{(2)} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ x_1^{(5)}x_2^{(5)} & x_0^{(5)}x_2^{(5)} & x_0^{(5)}x_1^{(5)} \\ x_1^{(6)}x_2^{(6)} & x_0^{(6)}x_2^{(6)} & x_0^{(6)}x_1^{(6)} \end{bmatrix} = 0$$

for some $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_6)$, $\lambda_i \neq 0$. Expanding the product we obtain the following system of 7 linear equations in 6 unknowns λ_i:

$$\sum_{i=0}^{6} \lambda_i x_0^{(i)} x_1^{(i)} x_2^{(i)} = 0,$$

$$\sum_{i=0}^{6} \lambda_i x_j^{(i)2} x_k^{(i)} = 0, \quad 0 \le j < k \le 2,$$

$$\sum_{i=0}^{6} \lambda_i x_j^{(i)} x_k^{(i)2} = 0, \quad 0 \le j < k \le 2.$$

Let

$$\sum_{0 \le i \le j \le k \le 2} a_{ijk} t_i t_j t_k = 0$$

be the equation of a generic cubic passing through the points $x^1,\dots,x^9$. Obviously

$$a_{000} = a_{111} = a_{222} = 0,$$

and we observe that the coefficient matrix of our system of linear equations is equal to the transpose of the coefficient matrix of the system of 6 linear equations:

$$\sum_{0 \le i \le j \le k \le 2} a_{ijk} x_i^{(n)} x_j^{(n)} x_k^{(n)} = 0, \ 1 \le n \le 6,$$

in 7 unknowns a_{ijk}, $(i,j,k) \neq (0,0,0), (1,1,1), (2,2,2)$. By the choice of x^7, x^8, x^9, the space of solutions of this system is of dimension 2. Thus our original system has a non-trivial solution $(\lambda_1,\dots,\lambda_6)$. If $\lambda_i = 0$ for some i, then the points $x^1,\dots,\hat{x}^i,\dots,x^9$ lie on a net of cubics. It is easy to see that this is impossible. This proves that x is associated to $x' = T_0(x)$.

Let $\bar{x} = (x^1,\dots,x^9)$ and $\bar{V} = V(\bar{x})$. The pencil of cubic curves passing through $\bar{x}$ defines the structure of a minimal elliptic surface on $\bar{V}$. We have

$$\varphi_{\bar{x}} \circ s_\alpha = \varphi_{\bar{x}'},$$

where $\bar{x}' = (x',x^7,x^8,x^9)$, $\alpha = e_0-e_7-e_8-e_9$. On the other hand, we have

$$\varphi_{\bar{x}} \circ s_\beta = \varphi_{\bar{x}''},$$

where $\beta = 2e_0-e_1-\dots-e_6 \in H_9$, and

$$\bar{x}'' = cr_{2,6}(s_{\alpha_{max}})(x,x^7,x^8,x^9).$$

This shows that

$$\varphi_{\bar{x}''} = \varphi_{\bar{x}'}\circ(s_\alpha \circ s_\beta).$$

Since $\alpha+\beta = K_{2,9}$, it follows from Theorem 7 of Chapter VI that

$$s_\alpha \circ s_\beta = \varphi_{\bar{x}}^{-1}\circ g^* \circ \varphi_{\bar{x}}$$

for some $g \in \mathrm{Aut}(\bar{V})$ (inducing a translation on the generic fibre of the elliptic fibration of $\bar{V}$). This implies that $\bar{x}'$ is projectively equivalent to $\bar{x}''$. Hence

$$a_{2,6}(\Phi(x)) = cr_{2,6}(x) = \Phi(x') .$$

Remark 1. The following group theoretical argument gives an indication why the previous result should be true. Let us identify $W = W(E_6)$ with a subgroup of $\mathrm{Bir}(P_2^6)$ by means of the Cremona representation $cr_{2,6}$. Then the association automorphism $a: P_2^6 \to P_2^6$ defines an automorphism of W

$$w \to a^{-1}\circ w \circ a.$$

It is known that every automorphism of $W(E_6)$ is inner. Thus

$$a^{-1}\circ w\circ a = w_0^{-1}\circ w\circ w_0$$

for some $w_0 \in W$. Since a preserves the subvariety of point sets lying on a conic, w_0 must be equal to $s_{\alpha_{max}}$. Unfortunately it does not yet prove that $a \in W$.

4. Del Pezzo surfaces of degree 2 .

Assume $n = 2$ and $m = 7$. In this case

$$(\mathbb{P}_2^7)^{ss} = (\mathbb{P}_2^7)^s$$

consists of point sets such that at most two points coincide and at most 4 lie on a line.

By Theorem 2 of Chapter IV,

$$(\hat{\mathbb{P}}_2^7)^{ss} = (\hat{\mathbb{P}}_2^7)^s = b_7^{-1}((\mathbb{P}_2^7)^s).$$

The morphism

$$\bar{b}_7: \hat{P}_2^7 \to P_2^7$$

is a birational morphism of nonsingular varieties of dimension 6. It is an isomorphism over $\Phi(U(7)^s)$, its restriction over the subvariety of orbits of point sets with exactly two coinciding points is a $\mathbb{P}_1$-bundle.

For every $\hat{x}\in(\hat{\mathbb{P}}_2^7)^s$ with at most 3 collinear points, the blowing-up surface $V(\hat{x})$ is a (nodal) Del Pezzo surface of degree 2. The anti-canonical linear system defines a morphism of degree 2

$$\pi: V(\hat{x}) \to \mathbb{P}_2$$

that factors through a birational morphism $V(\hat{x}) \to V'$ onto a surface V' with at most double rational points as singularities and a double cover $\pi': V' \to \mathbb{P}_2$. If $char(k) \neq 2$ the branch curve B of π' is of degree 4 (see **[De]**). Conversely, for every plane quartic curve B such that the double cover V' of $\mathbb{P}_2$ branched along B has at most rational double points as singularities, a minimal resolution of singularities of V' is a Del Pezzo surface of degree 2. All possible configurations of singularities of V', and hence of B, are known (**[DuV 1]**, **[Ti]**, **[Ur]**).

There are 63 positive roots in H_7 given by:

$$e_0-e_i-e_j-e_k, \quad 1 \le i < j < k \le 7,$$

$$e_i-e_j, \qquad 1 \le i<j \le 7,$$

$$2e_0-e_1-\ldots-e_7+e_i, \quad i = 1,\ldots,7.$$

Each of them can be realized as a discriminant condition for $V(\hat{x})$, $\hat{x}\in(\mathbb{P}_2^7)^s$. A point set $\hat{x}$ is unnodal if and only if $\hat{x}\in\hat{U}(7) = U(7)$, no conics pass through 6 points from $\hat{x}$, and no three points from $\hat{x}$ are collinear. Nonsingular plane quartics correspond to unnodal point sets.

The Weyl group $W_{2,7} \cong W(E_7)$ is of order $2903040 = 2^{10}\cdot3^4\cdot5\cdot7$. This is also the number of geometric markings of an unnodal Del Pezzo surface of degree 2. The Weyl group has a non-trivial center. It is generated by an element w_0 of order 2 that can be characterized as the "longest" element of W, that is, its length with respect to the set of Coxeter generators is maximal. The element w_0 acts on the root lattice Q(B) as -1. If $\varphi: H_7 \to N(V)$ is a geometric marking of a Del Pezzo surface of degree 2, then

$$\varphi \circ w_0 \circ \varphi^{-1} = g^*,$$

where g is the involution of V defined by the covering transformation of $\pi: V \to \mathbb{P}_2$. Indeed the L.H.S. preserves $K_V = \varphi(-K_{2,7})$ and acts as -1 on the orthogonal complement $L = (\mathbf{Z}K_V)_{N(V)}^{\perp} = \varphi(Q(B))$. Since π is given by $|-K_V|$, for every $x \in N(V)$

$$g^*(x)+x = mK_V \in \mathbf{Z}K_V = \pi^*(\mathrm{Pic}(\mathbb{P}_2))$$

for some $m \in \mathbf{Z}$. Intersecting both sides with K_V, we obtain

$$(g^*(x)+x)\bullet K_V = g^*(x)\bullet g^*(K_V) + x\bullet K_V = 2x\bullet K_V = mK_V \bullet K_V = 2m.$$

This implies that $g^* = \varphi \circ w_0 \circ \varphi^{-1} = -1$ on the orthogonal complement of K_V.

Let

$$\bar{Q}(B) = Q(B)/2Q(B) \cong \mathbb{F}_2^7.$$

We equip it with the symmetric bilinear form defined by

$$(\bar{\alpha},\bar{\beta}) = \alpha\bullet\check{\beta} \bmod 2$$

for every $\bar{\alpha} = \alpha+2Q(B)$, $\alpha \in R_B$. The vector

$$\bar{r} = \bar{\alpha}_0+\bar{\alpha}_4+\bar{\alpha}_6$$

spans the radical of this bilinear form, and

$$\bar{Q}(B)' = \bar{Q}(B)/(\bar{r}) \cong \mathbb{F}_2^6$$

has a structure of a symplectic space over $\mathbb{F}_2$. The Weyl group acts naturally on $\bar{Q}(B)$ and $\bar{Q}(B)'$, and we have an exact sequence:

$$1 \to (w_0) \to W(E_7) \to Sp(6,\mathbb{F}_2) \to 1.$$

This shows that the Cremona representation $cr_{2,7}$ factors through a homomorphism

$$\bar{cr}_{2,7}:\ Sp(6,\mathbb{F}_2) \to \mathrm{Bir}(P_2^7).$$

The group $Sp(6,\mathbb{F}_2)$ is simple. As in the case $m = 6$, this easily implies that $\bar{cr}_{2,7}$ is injective, and hence

$$\mathrm{Ker}(cr_{2,7}) = (w_0) \cong \mathbf{Z}/2.$$

It is easy to verify that

$$w_0(e_0) = 8e_0-3e_1-\ldots-3e_7,$$

$$w_0(e_i) = 3e_0-e_1-\ \ldots\ -e_7+e_i,\ i = 1,\ldots,7.$$

The Cremona transformation corresponding to w_0 is the so-called <u>Geiser involution</u>. It is given by the linear system of plane curves of degree 8 with triple points at $x^1,\ldots,x^7$ (see **[S-R]**, Chapter VII, 8.1).

A Del Pezzo surface of degree 2 has 56 exceptional curves of the first kind. Their classes are equal to

$$e_0-e_i-e_j,\ 1 \le i<j \le 7,$$

$$e_i,\ i = 1,\ldots,7,$$

$$2e_0-e_1-\ldots-e_7+e_i,\ 1 \le i<j \le 7,$$

$$3e_0-e_1-\ldots-e_7-e_i,\ i = 1,\ldots,7.$$

Under the covering involution g they are divided into 28 pairs, each of which is mapped to a bitangent of the branch quartic B.

5. Del Pezzo surfaces of degree 1.

This is very similar to the previous case. We have

$$(\mathbb{P}_2^8)^{ss} = (\mathbb{P}_2^8)^s$$

consisting of point sets in which at most two points coincide and at most 5 lie on a line.

By Theorem 2 of Chapter IV,

$$(\hat{\mathbb{P}}_2^8)^{ss} = (\hat{\mathbb{P}}_2^8)^s = b_8^{-1}((\mathbb{P}_2^8)^s).$$

The morphism

$$b_8\colon \hat{\mathbb{P}}_2^8 \to \mathbb{P}_2^8$$

is a birational morphism of nonsingular rational varieties of dimension 8. It is a $\mathbb{P}_1$-bundle over the subvariety of the orbits of point sets with exactly two coinciding points.

For every $\hat{x}\in(\hat{\mathbb{P}}_2^8)^s$ with at most 3 collinear points, the blowing-up surface $V(\hat{x})$ is a (nodal)Del Pezzo surface of degree 1. The anti-bicanonical linear system $|-2K_{V(x)}|$ defines a morphism of degree 2

$$\pi\colon V(\hat{x}) \to \mathbb{P}_3$$

that factors through a birational morphism $V(\hat{x}) \to V'$ onto a surface V' with at most double rational points as singularities and a double cover $\pi'\colon V' \to C$, where C is an irreducible singular quadric. If char$(\Bbbk) \neq 2$ the branch curve B of π' belongs to $|\mathcal{O}_C(3)|$ and does not pass through the singular point of C (see **[Del]**). Conversely, for every curve $B\in|\mathcal{O}_C(3)|$ such that the double cover V' of C branched along B has at most rational double points as singularities, a minimal nonsingular model of V'

is a Del Pezzo surface of degree 1. All possible configurations of singularities of V', and hence of B, are known (**[DuV 1], [Ti], [Ur]**).

There are 120 positive roots in H_8 given by:

$$e_0-e_i-e_j-e_k,\ 1 \le i < j < k \le 8,$$

$$e_i-e_j, \qquad 1 \le i < j \le 8,$$

$$2e_0-e_1-\ldots-e_7+e_i,+e_j\ ,1 \le i < j \le 8,$$

$$3e_0-e_1-\ldots-e_8-e_i,\ i = 1,\ldots,8.$$

Each of them can be realized as a discriminant condition for $V(\hat{x})$, $\hat{x}\epsilon(\hat{\mathbb{P}}_2^8)^s$. Note that a nonsingular B is a canonical model of a nonsingular curve of genus 4 with a vanishing theta constant. All such curves correspond to unnodal point sets.

The Weyl group $W_{2,8} \cong W(E_8)$ is of order $696729600 = 2^{14}\cdot3^5\cdot5^2\cdot7$. This is also the number of geometric markings of a Del Pezzo surface of degree 1. The Weyl group has a non-trivial center. It is generated by an element w_0 of order 2 that can be characterized as the "longest" element of W, that is, its length with respect to the set of Coxeter generators is maximal. The element w_0 acts on the root lattice Q(B) as -1. If φ: $H_8 \to N(V)$ is a geometric marking of a Del Pezzo surface of degree 1, then

$$\varphi\circ w_0\circ\varphi^{-1} = g^*,$$

where g is the involution of V defined by the covering transformation of $\pi:V \to C$. This can be proved in the same way as in the previous case.

Let

$$\bar{Q}(B) = Q(B)/2Q(B) \cong \mathbb{F}_2^8.$$

We equip it with the symmetric bilinear form defined by

$$(\bar{\alpha},\bar{\beta}) = \alpha\cdot\check{\beta} \bmod 2$$

for every $\bar{\alpha} = \alpha+2Q(B)$, $\alpha\epsilon R_B$. This form is non-degenerate and is associated to the quadratic form $q:\bar{Q}(B) \to \mathbb{F}_2$ defined by

$$q(\bar{x}) = \tfrac{1}{2}x\cdot x \bmod 2$$

for any $\bar{x} = x+2Q(B)\epsilon\bar{Q}(B)$. This quadratic form is of Witt index 1 and its orthogonal group is denoted by $O^+(8,\mathbb{F}_2)$. The Weyl group acts naturally on $\bar{Q}(B)$ preserving q and there is an exact sequence:

$$1 \to (w_0) \to W(E_8) \to O^+(8,\mathbb{F}_2) \to 1.$$

This shows that the Cremona representation $cr_{2,8}$ factors through a homomorphism

$$\mathrm{cr}_{2,8}: O^+(8,\mathbb{F}_2) \to \mathrm{Bir}(P_2^8).$$

The group $O^+(8,\mathbb{F}_2)$ contains a simple subgroup of index 2. As in the previous case this immediately implies that

$$\mathrm{Ker}(\mathrm{cr}_{2,8}) = (w_0) \cong \mathbb{Z}/2.$$

It is easy to verify that

$$w_0(e_0) = 17e_0-6e_1-\ldots-6e_8,$$

$$w_0(e_i) = 6e_0-2e_1- \ldots -2e_8-e_i, \ i =1,\ldots,8.$$

The Cremona transformation corresponding to w_0 is the so-called <u>Bertini involution</u>. It is given by the linear system of plane curves of degree 17 with sextuple points at $x^1,\ldots,x^8$ (see **[S-R]**, Chapter VII, 8.2).

A Del Pezzo surface of degree 1 has 240 exceptional curves of the first kind. Their classes correspond to the following vectors in H_8:

$$e_i, \ i = 1,\ldots,8,$$

$$e_0-e_i-e_j, \ 1 \le i < j \le 8,$$

$$2e_0-e_1-\ldots-e_8+e_i+e_j+e_k, \ 1 \le i < j < k \le 8,$$

$$3e_0-e_1-\ldots-e_8-e_i+e_j, \ i,j = 1,\ldots,8, \ i \ne j,$$

$$4e_0-e_1-\ldots-e_8-e_i-e_j-e_k, \ 1 \le i < j < k \le 8,$$

$$5e_0-2e_1-\ldots-2e_8+e_i+e_j, \ 1 \le i < j \le 8,$$

$$6e_0-2e_1- \ldots-2e_8-e_i, \ i = 1,\ldots,8.$$

Under the covering involution g they are divided into 120 pairs, each of which is equal to the inverse image under π of a tritangent plane to the branch curve B.

The anti-canonical linear system $|-K_{V(x)}|$ of a Del Pezzo surface V(x) of degree 1 is composed of a pencil and has one base point x^9. Blowing it up, we obtain a gDP-surface $V(\bar{x})$, where $\bar{x} = (x^1,\ldots,x^8,x^9)$. The linear system $|-K_{V(\bar{x})}|$ is base-point-free and defines an elliptic fibration

$$f:\bar{V} = V(\bar{x}) \to \mathbb{P}_1.$$

The exceptional curve of the first kind blown-up from x^9 can be taken as the zero section of the group scheme $\bar{V}^{\#}$ = V\{singular points of fibres of f}. The inversion automorphism of $V^{\#}$ extends to an automorphism of $\bar{V}$ and defines a Cremona transformation of $\mathbb{P}_2$. If $\bar{V}$ is unnodal (equivalently, V is unnodal), this is the Bertini involution defined above (see Theorem 7 of Chapter VI).

6. Point sets in $\mathbb{P}_3$.

We start with m = 4, the first case where P_3^m is defined, and leave the cases m ≤ 3 to the reader.

If m = 4,

$$(\mathbb{P}_3^4)^{ss} = (\mathbb{P}_3^4)^s,$$

$$(\hat{\mathbb{P}}_3^4)^{ss} = (\hat{\mathbb{P}}_3^4)^s = b_4^{-1}((\mathbb{P}_3^4)^s),$$

$$\hat{P}_3^4 = P_3^4 = \{\text{point}\},$$

$$\mathrm{Ker}(cr_{3,4}) = W_{3,4} \cong W(A_3)\times W(A_1) \cong \Sigma_4\times \mathbb{Z}/2.$$

If m = 5,

$$(\mathbb{P}_3^5)^{ss} = (\mathbb{P}_3^5)^s,$$

$$(\hat{\mathbb{P}}_3^5)^{ss} = (\hat{\mathbb{P}}_3^5)^s = b_5^{-1}((\mathbb{P}_3^5)^s),$$

$$\hat{P}_3^5 = P_3^5 = \{\text{point}\},$$

$$\mathrm{Ker}(cr_{3,5}) = W_{3,5} \cong W(A_5) \cong \Sigma_6.$$

If m = 6,

$(\hat{\mathbb{P}}_3^6)^{ss} = (\mathbb{P}_3^6)^{ss} = \{x\in\mathbb{P}_3^6\setminus\Delta(6)$: at most 3 points are collinear and at most 4 are coplanar$\}$,

$(\hat{\mathbb{P}}_3^6)^{s} = (\mathbb{P}_3^6)^{s} = \{x\in\mathbb{P}_3^6\setminus\Delta(6)$: no 3 points are collinear and at most 4 are coplanar$\}$,

$\hat{P}_3^6 \cong P_3^6 \cong P_1^6$ by association,

$$\mathrm{Ker}(cr_{3,6}) \cong (\mathbb{Z}/2)^5 \subset W_{3,6} \cong W(D_6) \cong (\mathbb{Z}/2)^5\rtimes\Sigma_6.$$

The elements of $\mathrm{Ker}(cr_{3,6})$ can be described as follows. Let

$$w_1 = s_{\alpha_0}\circ s_{\alpha_5}.$$

Then $cr_{3,6}(w_1)$ acts by the standard Cremona transformation T_0 composed with the transposition (56). We may assume that $x^5 = (1,1,1,1)$, $x^6 = (a,b,c,d)$, $a,b,c,d \neq 0$. Let

$$cr_{3,6}(w_1) = x' = (x^1,x^2,x^3,x^4,y^5,y^6),$$

where $y^5 = (bcd,acd,abd,abc)$, $y^6 = x^5$. Let A be the projective transformation

$$(t_0,t_1,t_2,t_3) \to (at_0,bt_1,ct_2,dt_3).$$

Then

$$A(x') = x,$$

hence

$$w_1 \in \mathrm{Ker}(cr_{3,6}).$$

We find that for every $\sigma \in \Sigma_6 \subset W$

$$w_\sigma = \sigma \circ w_1 \circ \sigma^{-1} = s_{\sigma(\alpha_0)} \circ s_{\sigma(\alpha_5)} \in \mathrm{Ker}(cr_{3,6}).$$

Since $\alpha_0 \cdot \alpha_5 = 0$, the simple reflections s_{α_0} and s_{α_5} commute. Therefore, every w_σ is of order 2. Let us show that the elements w_σ commute. Note that

$$\sigma(\alpha_0) = 2e_0 - e_{\sigma(1)} - e_{\sigma(2)} - e_{\sigma(3)} - e_{\sigma(5)},$$

$$\sigma(\alpha_5) = e_{\sigma(5)} - e_{\sigma(6)}.$$

Let $\sigma, \sigma' \in \Sigma_6$, $\sigma \neq \sigma'$. Then

$$\#\sigma(\{5,6\}) \cap \sigma'(\{5,6\}) = 0,\ 1,\ \text{or } 2.$$

In the first case

$$\sigma(\alpha_0) \cdot \sigma'(\alpha_0) = 0,$$

$$\sigma(\alpha_5) \cdot \sigma'(\alpha_5) = 0,$$

hence

$$w_\sigma \circ w_{\sigma'} = w_{\sigma'} \circ w_\sigma.$$

In the second case

$$\sigma(\alpha_0) \cdot \sigma'(\alpha_0) = -1,$$

$$\sigma(\alpha_5) \cdot \sigma'(\alpha_5) = \pm 1.$$

One easily verifies that

$$w_\sigma \circ w_{\sigma'} = w_{\sigma'} \circ w_\sigma = w_{\sigma''},$$

where

$$\{\sigma''(5), \sigma''(6)\} \cap \{\sigma(5), \sigma(6)\} = \{\sigma''(5), \sigma''(6)\} \cap \{\sigma'(5), \sigma'(6)\} \neq$$

$$\neq \{\sigma(5), \sigma(6)\} \cap \{\sigma'(5), \sigma'(6)\}.$$

In the third case

$$w_\sigma = w_{\sigma'}.$$

Thus all the w_σ's commute. They generate a subgroup H of $W_{3,6}$ isomorphic to $(\mathbb{Z}/2)^5$.

There is a distinguished element w_0 in $\mathrm{Ker}(cr_{3,6})$. It defines a non-trivial pseudo-automorphism g_0 of the gDP-variety $V(x)$, $x\in(\mathbb{P}_3^6)^{gen}$, equal to the covering transformation of the rational map of degree 2

$$\pi: V(x) \dashrightarrow \mathbb{P}_3$$

given by the linear system of quadrics passing through $x^1,\dots,x^6$. Note that π blows down to points each of the proper inverse transforms of 15 lines joining pairs of points x^i's. It also blows down the proper inverse transform of the unique rational normal curve passing through all of the x^i's. The ramification surface $X \subset V(x)$ of π is a minimal nonsingular model of the Weddle surface, a quartic surface in $\mathbb{P}_3$ with nodes at the x^i's, defined as the locus of nodes of quadrics passing through the x^i's. The branch surface $Y \subset \mathbb{P}_3$ of π is a Kummer surface, birationally isomorphic to the Weddle surface X. Its 16 nodes are the images of the 15 lines and the rational normal curve from above (see **[S-R]**, Chapter VIII, 2.3).

Since $\mathrm{Ker}(cr_{3,6})$ is abelian, each of its elements commute with w_0, and hence leaves the ramification locus of π invariant. This shows that $\mathrm{Ker}(cr_{3,6})$ maps to the automorphism group of the Weddle surface with w_0 mapped to the identity. The image is a subgroup of $\mathrm{Aut}(X)$ isomorphic to $(\mathbb{Z}/2)^4$.

Let $\varphi: H_6 \to N(V(x))$ be a geometric marking of $V(x)$. Let us compute

$$w_0 = \varphi^{-1}\circ g_0{}^*\circ\varphi \in W_{3,6} .$$

We know that

$$g_0{}^*(K_{V(x)}) = K_{V(x)}.$$

For every $i = 1,\dots,6$, let Q_i be the quadric passing through the points $x^1,\dots,x^6$ and having a node at the point x^i, and let E_i be the exceptional divisor blown-up from it. Then

$$E_i+Q_i\in |-\tfrac{1}{2}K_{V(x)}|,$$

and hence

$$g_0{}^*([E_i]) = [Q_i],\ i = 1,\dots,6.$$

Thus

$$\varphi^{-1}\circ g_0{}^*\circ\varphi(e_i) = 2e_0-e_1-\dots-e_6-e_i,\ i = 1,\dots,6.$$

Similarly we observe that the union of the planes $\langle x^1,x^2,x^3\rangle$ and $\langle x^4,x^5,x^6\rangle$ belongs to $|-\tfrac{1}{2}K_{V(x)}|$. Hence

$$\varphi^{-1}\circ g_0{}^*\circ\varphi(e_0-e_1-e_2-e_3) = e_0-e_4-e_5-e_6.$$

Together, this yields

$$\varphi^{-1}\circ g_0{}^*\circ\varphi(e_0) \;=\; 7e_0-4e_1-\ldots-4e_6.$$

An easy calculation shows that

$$w_0 = w_\sigma\circ w_{\sigma'}\circ w_{\sigma''},$$

where

$$\sigma(\{5,6\})\cap\sigma'(\{5,6\}) \;=\; \sigma(\{5,6\})\cap\sigma''(\{5,6\}) \;=\; \sigma'(\{5,6\})\cap\sigma''(\{5,6\}) \;=\; \emptyset.$$

Finally we note that the representation

$$\overline{cr}_{3,6}\colon\ \Sigma_6 \;\rightarrow\; \mathrm{Bir}(P_3^6)$$

through which $cr_{3,6}$ factors is induced by permutations of the factors of $\mathbb{P}_3^6$.

Assume $m = 7$. Then

$$(\mathbb{P}_3^7)^{ss} = (\mathbb{P}_3^7)^{s} = (\hat{\mathbb{P}}_3^7)^{ss} = (\hat{\mathbb{P}}_3^7)^{s}$$

consists of point sets $x = (x^1,\ldots x^7)\notin\Delta(7)$ with no more than 3 points lying in a line and no more than 5 points lying in a plane. By association,

$$\hat{P}_3^7 \;\cong P_3^7 \;\cong P_2^7,$$

and the Cremona representation

$$cr_{3,7}\colon\ W_{3,7} \;\cong\; W(E_7) \;\rightarrow\; \mathrm{Bir}(P_3^7)$$

is isomorphic (twisted by an involution τ of $W_{3,7}$) to the Cremona representation

$$cr_{2,7}\colon\ W_{2,7} \;\cong\; W(E_7) \;\rightarrow\; \mathrm{Bir}(P_2^7).$$

In particular, we obtain

$$\mathrm{Ker}(cr_{3,7}) \;=\; (w_0) \;\cong\; \mathbf{Z}/2.$$

Let $x\in(\mathbb{P}_3^7)^{un}\subset(\mathbb{P}_3^7)^s$ and $\varphi\colon H_7 \;\rightarrow\; N(V(x))$ be the corresponding strictly geometric marking. Then

$$\varphi\circ w_0\circ\varphi^{-1} \;=\; g_0{}^*$$

for some pseudo-automorphism g_0 of $V(x)$. The action of g_0 is similar to the one of the Bertini involution. One easily checks that

$$\bar{V}(x) \;\; =\mathrm{Proj}(\bigoplus_{r=0}^{\infty} H^0(V(x),\mathcal{O}_{V(x)}(-\tfrac{1}{2}rK_{V(x)})))$$

is isomorphic to the weighted projective space $\mathbb{P}(1,1,1,2)$ (that is, the cone over the Veronese surface in $\mathbb{P}^5$). The canonical map

$$\pi: V(x) \to \bar{V}(x)$$

is a rational map of degree 2. The pseudo-automorphism g_0 is the covering birational transformation of π. Its fixed locus is a proper transform of a certain sextic surface in $\mathbb{P}_3$ (the <u>Cayley dianode surface</u> **[Ca]**). We will return to this surface later in Chapter 9.

One finds

$$w_0(e_0) = 15e_0-4e_1-\ldots-4e_7,$$

$$w_0(e_i) = 8e_0-2e_1-\ldots-2e_7-e_i, \quad i = 1,\ldots,7.$$

Thus g_0 is given by the Cremona transformation defined by the linear system of surfaces of degree 15 passing through $x^1,\ldots,x^7$ with multiplicity ≥ 4.

The linear system of quadrics through $x^1,\ldots,x^7$ is two-dimensional. Adding to x its 8-th base point x^8, we obtain a point set $\bar{x} = (x^1,\ldots,x^8)$, the blow-up of which $\bar{V} = V(\bar{x})$ admits an elliptic fibration

$$f: \bar{V} \to \mathbb{P}_2.$$

It is defined by $|-\frac{1}{2}K_{\bar{V}}|$, the proper transform of the net of quadrics. The open subset $V^{\#} = \bar{V}\setminus\{$singular points of fibres of f$\}$ has a structure of a group scheme over $\mathbb{P}_2$, with the image of the zero section equal to the exceptional divisor blown up from the point x^8. The inversion automorphism of $V^{\#}$ extends to the pseudo-automorphism g_0 of $\bar{V}$.

Starting from $m = 8$, the Weyl group $W_{3,m}$ becomes infinite. We do not know whether $cr_{3,m}$ is injective.

7. <u>Point sets in $\mathbb{P}_4$</u>.

As in the previous section we start with $m = 5$, the first case where P_4^m is defined, and leave the cases $m \leq 4$ to the reader.

If $m = 5$,

$$(\hat{\mathbb{P}}_4^5)^{ss} \cong (\mathbb{P}_4^5)^{ss} = (\mathbb{P}_4^5)^{gen},$$

$$(\hat{\mathbb{P}}_4^5)^{s} \cong (\mathbb{P}_4^5)^{ss} = \emptyset,$$

$$\hat{P}_4^5 = P_4^5 = \{\text{point}\},$$

$$\mathrm{Ker}(cr_{4,5}) = W_{4,5} \cong W(A_4)\times W(A_1) \cong \Sigma_5\times\mathbf{Z}/2.$$

If $m = 6$,

$$(\hat{\mathbb{P}}_4^6)^{ss} \cong (\mathbb{P}_4^6)^{ss} = (\mathbb{P}_4^6)^{s} = (\mathbb{P}_4^6)^{s} = (\mathbb{P}_4^5)^{gen},$$

$$\hat{P}_4^6 \cong P_4^6 = \{\text{point}\},$$

$$\mathrm{Ker}(cr_{4,6}) = W_{4,6} \cong W(A_6) \cong \Sigma_7.$$

If $m = 7$,

$$(\hat{\mathbb{P}}_4^7)^{ss} = (\hat{\mathbb{P}}_4^7)^{ss} \cong (\mathbb{P}_4^7)^{ss} = (\mathbb{P}_4^7)^{s} = (\mathbb{P}_4^7)^{gen},$$

$$\hat{P}_4^7 \cong P_4^7 \cong P_1^7 \text{ by association},$$

$$\mathrm{Ker}(cr_{4,7}) \cong (\mathbb{Z}/2)^6 \subset W_{4,7} \cong W(D_7) \cong (\mathbb{Z}/2)^6 \rtimes \Sigma_7.$$

The argument is similar to the case $n = 3$, $m = 6$. However in this case there are not any distinguished involutions in $\mathrm{Ker}(cr_{4,7})$.

The last case when $W_{4,m}$ is finite is the case $m = 8$. Here

$$(\hat{\mathbb{P}}_4^8)^{ss} = (\hat{\mathbb{P}}_4^8)^{s} \cong (\mathbb{P}_4^8)^{ss} = (\mathbb{P}_4^8)^{s}$$

and consists of point sets $x \in \mathbb{P}_4^8 \setminus \Delta(8)$ with no more than 3 collinear points and no more than 4 coplanar points.

By association

$$\hat{P}_4^8 \cong P_4^8 \cong P_2^8,$$

$$\mathrm{Ker}(cr_{4,8}) \cong \mathrm{Ker}(cr_{2,8}) = (w_0) \subset W_{4,8} \cong W(E_8).$$

We do not know any nice geometric description of the involution w_0 considered as a Cremona transformation of $\mathbb{P}_4$ (cf. **[DuV 4]**).

VIII. POINT SETS IN $\mathbb{P}_1$ AND HYPERELLIPTIC CURVES.

With this chapter we begin our discussion about the relationship between theta functions and point sets in $\mathbb{P}_n$.

1. Theta functions.

Let us recall some facts from the theory of theta functions which we will need in this and the next chapter (see **[Ig 1]**). We will use the following notations:

$\mathfrak{H}_g = \{\tau \in M_g(\mathbb{C}) : {}^t\tau = \tau,\ \mathrm{Im}(\tau) > 0\}$, the Siegel half space,

$\Gamma_g = Sp(2g,\mathbb{Z}) = \{M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL(2g,\mathbb{Z}) : {}^tM \cdot \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix} \cdot M = \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix}\}$, the Siegel modular group,

$\Gamma_g(n) = \{M \in \Gamma_g : M \equiv I_{2g} \bmod n\}$, the level n congruence subgroup,

$\Lambda_\tau = \{\tau \cdot m + n : m, n \in \mathbb{Z}^g\} \subset \mathbb{C}^g$, $\tau \in \mathfrak{H}_g$,

$A_\tau = \mathbb{C}^g / \Lambda_\tau$, an abelian variety of dimension g.

The group Λ_τ acts on $\mathbb{C}^g \times \mathbb{C}$ by

$$w: (z,t) \to (z+w, e_\tau(z,w)t),$$

where $w = \tau \cdot m + n \in \Lambda_\tau$, and

$$e_\tau(z,w) = \exp -\pi i({}^tm \cdot \tau \cdot m + 2\,{}^tm \cdot z).$$

The quotient space

$$L_\tau = \mathbb{C}^g \times \mathbb{C} / \Lambda_\tau$$

has a natural structure of a line bundle on A_τ.

The pair (A_τ, L_τ) is a principally polarized abelian variety (ppav), i.e. a pair consisting of an abelian variety A and an algebraic equivalence class of an ample line bundle L on it with dim $H^0(A,L) = 1$. The latter is equivalent to the property that the map

$$\eta_L : A \to \mathrm{Pic}(A)^\circ = \hat{A}, \qquad a \to t_a^*(L) \otimes L^{-1}$$

is an isomorphism.

The group Γ_g acts properly discontinuously and holomorphically on $\mathfrak{H}_g$ by the formula

$$M : \tau \to M_\tau \ := \ (A\tau+B)(C\tau+D)^{-1}.$$

One easily checks the commutativity of the following diagram:

$$\begin{array}{ccc} \mathbf{Z}^{2g} & \xrightarrow{(m,n)\to\tau m+n} & \mathbf{C}^g \\ {}^t\begin{bmatrix} A B \\ C D \end{bmatrix} \downarrow\uparrow \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} & & {}^t(C\tau+D)\uparrow\downarrow(M\tau)C+A \\ \mathbf{Z}^{2g} & \xrightarrow{(m,n)\to M\tau m+n} & \mathbf{C}^g \end{array}$$

where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g$, and each pair of the vertical arrows consists of a map and its inverse. This diagram shows that the map

$$\varepsilon_M : z \bmod \Lambda_\tau \to {}^t(C\cdot\tau+D)^{-1}z \bmod \Lambda_{M\tau}$$

defines an isomorphism of abelian varieties

$$\varepsilon_M : A_\tau \to A_{M\tau}$$

and of invertible sheaves

$$\varepsilon_M^*(L_{M\tau}) \to L_\tau \, .$$

In other words ε_M defines an isomorphism

$$(A_\tau, L_\tau) \cong (A_{M\tau}, L_{M\tau})$$

of principally polarized abelian varieties.

This allows us to define an isomorphism of complex varieties:

$$\mathfrak{H}_g/\Gamma_g \cong \mathfrak{A}_g,$$

where the latter stands for the coarse moduli variety of isomorphism classes of ppav of dimension g.

The factor space $\mathfrak{H}_g/\Gamma_g$ is an algebraic variety. It is isomorphic to an open Zariski subset of

$$\bar{\mathfrak{a}}_g = \mathrm{Proj}(M(\Gamma_g)),$$

where for every subgroup Γ of Γ_g

$$M(\Gamma)_k = \{f \in \mathcal{O}(\mathcal{H}_g) : f(M\tau) = \det(C\cdot\tau+D)^k f(\tau) \text{ for any } M \in \Gamma\}$$

$$M(\Gamma) = \bigoplus_{k=0}^{\infty} M(\Gamma)_k,$$

is the space of Siegel modular forms of weight k with respect to Γ, and the graded algebra of Siegel modular forms with respect to Γ, respectively. The space $\bar{\mathfrak{a}}_g$ is called the Satake compactification of $\mathfrak{a}_g$.

For every integer n and an abelian group G we denote by [n] the homomorphism of multiplication by n.

Let

$${}_2A_\tau = \mathrm{Ker}[2] = \tfrac{1}{2}\Lambda_\tau/\Lambda_\tau.$$

A natural homomorphism

$$\mathbf{Z}^{2g} \to {}_2A_\tau, \quad (m,n) \to \tfrac{1}{2}\tau\cdot m+\tfrac{1}{2}n \bmod \Lambda_\tau$$

factors through an isomorphism

$$\varphi_\tau : \mathbb{F}_2^{2g} \to {}_2A_\tau.$$

Both groups have a structure of a symplectic vector space over $\mathbb{F}_2$. The first one is defined by the bilinear form

$$e_2 : (x,y) \to {}^t x\cdot \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix}\cdot y = {}^t x_1 y_2 + {}^t x_2 y_1,$$

where $x = (x_1,x_2)$, $y = (y_1,y_2)$, $x_i, y_i \in \mathbb{F}_2^g$. The second one is the Weil pairing, defined by the formula:

$$e_2^\tau(x,y) = \log(\chi_y(x)),$$

where

$$\chi_y : {}_2A_\tau \to \mu_2 = \{\pm 1\}$$

is the homomorphism obtained via the identification of the point $\eta_L(y) \in {}_2\hat{A}_\tau$ with an element of

$$\mathrm{Ker}(\hat{A} \xrightarrow{[2]^*} \hat{A}) \cong \mathrm{Char}(\mathrm{Ker}(A \xrightarrow{[2]} A)) = \mathrm{Hom}({}_2\hat{A}_\tau, \mu_2)$$

and $\log: \mu_2 \to \mathbb{F}_2$ is an isomorphism of groups.

One verifies that

$$\varphi_\tau: \mathbb{F}_2^{2g} \to {}_2A_\tau$$

is an isomorphism of symplectic spaces.

The triple

$$(A_\tau, L_\tau, \varphi_\tau)$$

is a <u>ppav with a level 2 structure</u>.

The action of Γ_g on $\mathfrak{H}_g$ changes the triple $(A_\tau, L_\tau, \varphi_\tau)$ to the triple

$$(A_{M\tau}, L_{M\tau}, \varphi_{M\tau}),$$

where

$$\varphi_{M\tau} = \xi_M^\bullet \varphi_\tau \circ \bar{M},$$

and

$$\bar{M} = M \bmod 2 \in Sp(2g, \mathbb{F}_2)$$

is an automorphism of the symplectic space $\mathbb{F}_2^{2g}$.

In particular, we see that $M \in \Gamma_g(2)$ defines an isomorphism

$$\bar{\xi}_M: (A_\tau, L_\tau, \varphi_\tau) \cong (A_{M\tau}, L_{M\tau}, \varphi_{M\tau})$$

of principally polarized abelian varieties with level 2 structure. In this way one obtains an isomorphism

$$\mathfrak{H}_g / \Gamma_g(2) \to \mathfrak{A}_g(2),$$

where $\mathfrak{A}_g(2)$ is the coarse moduli variety of isomorphism classes of principally polarized abelian varieties of dimension g with level 2 structure.

Note that

$$Sp(2g, \mathbb{F}_2) = \Gamma_g / \Gamma_g(2)$$

acts naturally on $\mathfrak{A}_g(2)$ with the quotient variety isomorphic to $\mathfrak{A}_g$. A compactification of $\mathfrak{A}_g(2)$ is given by

$$\bar{\mathfrak{A}}_g(2) = \mathrm{Proj}(M(\Gamma_g(2))).$$

This is the <u>Satake compactification</u> of $\mathfrak{A}_g(2)$.

Recall that, for every line bundle L on A_τ, its holomorphic sections can be viewed as holomorphic functions f(z) on $\mathbb{C}^g$ satisfying

$$f(z+w) = e^L(z,w)f(z) \text{ for any } w\in\Lambda_\tau, \text{ and any } z\in\mathbb{C}^g,$$

where e^L is the <u>automorphy factor</u> defining L (i.e.

$$L \cong \mathbb{C}^g \times \mathbb{C}/\Lambda_\tau,$$

where $w\in\Lambda_\tau$ sends $(z,t)\in\mathbb{C}^g\times\mathbb{C}$ to $(z+w,e^L(z,w)t)$.

In our case L_τ has a unique, up to a scalar factor, section on A_τ. The corresponding holomorphic function can be given by the following infinite series:

$$\vartheta(z;\tau) = \sum_{m\in\mathbb{Z}^g} \exp\pi i({}^tm\cdot\tau\cdot m+2\,{}^tm\cdot z).$$

This is called the <u>Riemann theta function</u> of A_τ. The corresponding automorphy factor is $e_\tau(z,w)$ defined above.

For every $a = \frac{1}{2}\tau\cdot x+\frac{1}{2}y\in\mathbb{C}^g$, $x,y\in\mathbb{R}^g$, the translation $t_{\bar a}: A_\tau \to A_\tau$ by the element $\bar a = a \bmod \Lambda_\tau$ defines a line bundle $t_{\bar a}^*(L_\tau)$ algebraically equivalent to L_τ. One of its sections (defined uniquely up to a scalar factor) can be given by the series

$$\vartheta\begin{bmatrix}x\\y\end{bmatrix}(z;\tau) = \sum_{m\in\mathbb{Z}^g} \exp\pi i({}^t(m+\tfrac{1}{2}x)\cdot\tau\cdot(m+\tfrac{1}{2}x)+2\,{}^t(m+\tfrac{1}{2}x)\cdot(z+\tfrac{1}{2}y))$$

that relates to the function $\theta(z;\tau)$ by

$$\vartheta(z+a;\tau) = \vartheta\begin{bmatrix}x\\y\end{bmatrix}(z;\tau)\exp-\pi i(\tfrac{1}{4}{}^tx\cdot\tau\cdot x+{}^tx\cdot(z+\tfrac{1}{2}y)).$$

The function $\vartheta\begin{bmatrix}x\\y\end{bmatrix}(z;\tau)$ is called a <u>theta function with characteristic</u> $\begin{bmatrix}x\\y\end{bmatrix}$.

We will be using mostly theta functions with <u>half integral characteristic</u>

$$\vartheta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(z;\tau) = \sum_{m\in\mathbb{Z}^g} \exp\pi i({}^t(m+\tfrac{1}{2}\varepsilon)\cdot\tau\cdot(m+\tfrac{1}{2}\varepsilon)+2\,{}^t(m+\tfrac{1}{2}\varepsilon)\cdot(z+\tfrac{1}{2}\varepsilon'))$$

where ε, ε' have values in the set $\{0,1\}$.

Observe that

$$\vartheta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(-z;\tau) = (-1)^{{}^t\varepsilon\cdot\varepsilon'}\vartheta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(z;\tau).$$

This easily implies that we have

$$2^{g-1}(2^g+1) \text{ even theta functions } \vartheta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z;\tau), \text{ and}$$

$$2^{g-1}(2^g-1) \text{ odd theta functions } \vartheta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z;\tau).$$

As a function of the parameter $\tau\in\mathcal{H}_g$, the theta function $\vartheta\begin{bmatrix}x\\ y\end{bmatrix}(z;\tau)$ satisfies the following functional equation:

$$\vartheta\begin{bmatrix}x'\\ y'\end{bmatrix}(z';\tau') = \zeta\exp\pi i({}^tz\cdot(C\tau+D)^{-1}\cdot z)\det(C\cdot\tau+D)^{1/2}\vartheta\begin{bmatrix}x\\ y\end{bmatrix}(z;\tau),$$

where

$$(z',\tau') = ({}^t(C\cdot\tau+D)^{-1}z, M\tau),\ M = \begin{bmatrix}AB\\ CD\end{bmatrix}\in\Gamma_g,$$

$$(x',y') = (D\cdot x-C\cdot y, -B\cdot x+A\cdot y)+((C\cdot{}^tD)_\circ,(A\cdot{}^tB)_\circ),$$

$\zeta\in\mathbb{C}$ depends on (x,y) and M only,

and for every matrix X we denote by $X_\circ$ the vector of its diagonal elements.

If $\begin{bmatrix}x\\ y\end{bmatrix}$ is a half integral characteristic and $M\in\Gamma_g(2)$, the constant ζ satisfies

$$\zeta^4 = 1,$$

which implies that

$$\vartheta\begin{bmatrix}x\\ y\end{bmatrix}(\tau)^4 = \vartheta\begin{bmatrix}x\\ y\end{bmatrix}(0;\tau)^4\in M(\Gamma_g(2))_2.$$

The values $\vartheta\begin{bmatrix}x\\ y\end{bmatrix}(0;\tau)$ of theta functions with half integral characteristic at zero are called <u>theta constants</u>.

2. Jacobian varieties and theta characteristics.

Let C be a nonsingular projective algebraic curve over $\mathbb{C}$ of genus $g > 0$. Recall the definition of its Jacobian variety (see **[G-H]**).

Let

$$\{\gamma_1,\dots,\gamma_{2g}\}$$

be a symplectic basis of the first homology group $H_1(C,\mathbb{Z})$, i.e.

$$\gamma_i \cdot \gamma_j = 0, \ |i-j| \neq g, \ i,j = 1,\ldots,2g,$$

$$\gamma_i \cdot \gamma_{i+g} = -\gamma_i \cdot \gamma_{i+g} = 1, \ i = 1,\ldots,g,$$

with respect to the intersection form on $H_1(C,\mathbf{Z})$.

There exists a basis

$$\{\omega_1,\ldots,\omega_g\}$$

of the space $H^0(C,\Omega_C^1)$ of holomorphic differentials on C satisfying:

$$\int_{\gamma_j} \omega_i = 0, \ i,j = 1,\ldots,g, \ i \neq j,$$

$$\int_{\gamma_i} \omega_i = 1, \ i = 1,\ldots,g.$$

The matrix

$$\tau(C) = (\int_{\gamma_{j+g}} \omega_i)_{1 \leq i,j \leq g}$$

belongs to $\mathcal{H}_g$ and is called the <u>period matrix</u> of C.

The <u>Jacobian variety</u> of C is defined as the abelian variety

$$\mathrm{Jac}(C) = A_{\tau(C)} = \mathbb{C}^g / \Lambda_{\tau(C)}.$$

It can be identified with the component $\mathrm{Pic}^0(C)$ of the <u>Picard scheme</u> Pic(C) of C parametrizing divisor classes of degree 0 . This is done by means of the <u>Abel-Jacobi map</u>:

$$\mathrm{aj}: \mathrm{Pic}^0(C) \to \mathrm{Jac}(C), \ D \to \int_D \bar{\omega} = (\int_D \omega_1,\ldots,\int_D \omega_g) \bmod \Lambda_\tau.$$

The choice of a point $c_0 \in C$ allows us to define an isomorphism

$$\mathrm{Pic}^n(C) \to \mathrm{Pic}^0(C), \ D \to D - nc_0,$$

whose composition with the Abel-Jacobi map defines an isomorphism

$$\mathrm{aj}_{c_0}: \mathrm{Pic}^n(C) \to \mathrm{Jac}(C).$$

Let $C^{(n)}$ be the n-th symmetric product of C parametrizing effective divisors of degree n on C. There is a canonical map

$$\mu_n: C^{(n)} \to \mathrm{Pic}^n(C)$$

whose fibre over a divisor class $D \in Pic^n(C)$ is equal to the complete linear system $|D|$. The image of μ_n is a closed subvariety W^n of $Pic^n(C)$. In the special case $n = g-1$ that we need, W_{g-1} is a hypersurface. The fundamental theorem of Riemann says that

$$aj_{c_0}(W_{g-1}) + \kappa(c_0,\tau) = \Theta := \{z \in \mathbb{C}^g : \vartheta(z,\tau) = 0\}/\Lambda_\tau$$

for some point $\kappa(c_0,\tau)$ in A_τ (the Riemann constant). Moreover, if

$$W_{g-1}^r = \{D \in W_{g-1}: \dim |D| \geq r\},$$

then

$$aj_{c_0}(W_{g-1}^r) + \kappa(c_0,\tau) = Sing^r(\Theta) = \{z \in \Theta: mult_z(\Theta) \geq r+1\}.$$

The Riemann theorem asserts also that

$$D(c_0,\tau) = aj_{c_0}^{-1}(\kappa(c_0,\tau)) \in Th(C),$$

where

$$Th(C) = \{D \in Pic(C): 2D = K_C\}$$

is the set of theta characteristics on C. Note that this set has a natural structure of an affine space over ${}_2Pic(C) = {}_2Jac(C)$, and hence consists of 2^{2g} elements.

For every $D \in Th(C)$, we have

$$h^0(D) = mult_{aj_{c_0}(D)+\kappa(c_0,\tau)}\Theta = mult_0(\Theta + \eta_D),$$

where

$$\eta_D = aj_{c_0}(D) + \kappa(c_0,\tau) \in {}_2Jac(C).$$

Use τ to define a level 2 structure

$$\varphi_\tau : \mathbb{F}_2^{2g} \to {}_2Jac(C).$$

Let

$$\varphi_\tau(\varepsilon,\varepsilon') = \eta_D,$$

where we identify elements of $\mathbb{F}_2^{2g}$ with binary vectors. Then

$$h^0(D) = mult_0(\Theta+\eta_D) = mult_0\vartheta(z+\eta_D;\tau) = mult_0\vartheta\begin{bmatrix}\varepsilon \\ \varepsilon'\end{bmatrix}(z;\tau).$$

This implies that

$$h^0(D) \equiv {}^t\varepsilon \cdot \varepsilon' \mod 2.$$

A theta characteristic D is called <u>even</u> (<u>odd</u>) if $h^0(D)$ is even (odd). We see that there are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd theta characteristics on C. Thus a choice of the period matrix τ of C and a point $c_0 \epsilon C$ allows us to make a bijective correspondence between the set of theta characteristics and the set of theta functions with half integral characteristics in such a way that even theta characteristics correspond to even theta functions.

Also observe that for every $\eta = \varphi_\tau(\alpha,\alpha')\epsilon_2 Jac(C)$

$$h^0(D+\eta)+h^0(D) \equiv \mathrm{mult}_0 \vartheta\begin{bmatrix}\varepsilon+\alpha\\ \varepsilon'+\alpha'\end{bmatrix}(z;\tau)+\mathrm{mult}_0\vartheta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z;\tau) \equiv$$

$$\equiv {}^t(\varepsilon+\alpha)\cdot(\varepsilon'+\alpha')+{}^t\varepsilon\cdot\varepsilon' \equiv {}^t\alpha\cdot\varepsilon'+{}^t\varepsilon\cdot\alpha'+{}^t\alpha\cdot\alpha' \equiv$$

$$\equiv \sum_{i=1}^{g} \alpha_i^2\varepsilon_i'+\alpha_i'^2\varepsilon_i+\alpha_i\alpha_i' \mod 2.$$

This shows that the function

$$\eta \to h^0(D+\eta)+h^0(D) \mod 2$$

is a quadratic form on ${}_2Jac(C) \cong \mathbb{F}_2^{2g}$. Under this correspondence, even and odd theta characteristics define even and odd quadratic forms

$$q_D: {}_2Jac(C) \to \mathbb{F}_2,$$

distinguished from one another by the property that

$$\#q^{-1}(0) = 2^{g-1}(2^g+1)$$

for even quadratic forms, while

$$\#q^{-1}(0) = 2^{g-1}(2^g-1)$$

for odd quadratic forms. The orthogonal group of an even (resp. odd) quadratic form is isomorphic to the orthogonal group

$$O^+(2g,\mathbb{F}_2) \quad (\text{resp. } O^-(2g,\mathbb{F}_2))$$

of the quadratic form

$$\sum_{i=1}^{g} x_ix_{i+g} \qquad (\text{ resp. } \sum_{i=1}^{g-1} x_ix_{i+g}+x_g^2+x_{2g}^2)$$

on $\mathbb{F}_2^{2g}$. Note that

$$[Sp(2g,F_2):O^+(2g,\mathbb{F}_2)] = 2^{g-1}(2^g+1),$$

$$[Sp(2g,F_2):O^-(2g,\mathbb{F}_2)] = 2^{g-1}(2^g-1).$$

The bilinear form associated to q_D is equal to the Weil pairing

$$e_\tau: {}_2Jac(C) \times {}_2Jac(C) \to \mathbb{F}_2$$

defined by the principal polarization L_τ of Jac(C). Thus the map $D \to q_D$ is a bijection

$$Th(C) \to S^2({}_2Jac(C))_0$$

between the set Th(C) of theta characteristics on C and the set of quadratic forms on ${}_2Jac(C)$ with associated bilinear form equal to e_τ.

3. Hyperelliptic curves.

Let C be a hyperelliptic curve of genus $g > 1$ over $\mathbb{C}$. By definition C has a unique linear system g^1_2 of degree 2 and dimension 1 that defines a double cover

$$\pi: C \to \mathbb{P}_1$$

ramified at $2g+2$ points $c_1,\ldots,c_{2g+2}$ of C. Let $x^1,\ldots,x^{2g+2}$ be their projections to $\mathbb{P}_1$, the branch points of π. In this section we will show that an ordered set of branch points defines a level 2 structure on Jac(C), and in this way the variety $(P_2^{2g+2})^{un}$ becomes isomorphic to an irreducible component of the moduli variety of hyperelliptic Jacobians with level 2 structure.

We begin with a convenient notation for points of order 2 in Jac(C) (see [Mu 2]). For every subset $T \subset B = \{1,\ldots,2g+2\}$ let

$$e_T = \sum_{i\in T} c_i - \#T c_{2g+2} \in Div(C).$$

Then

$$2e_T = \pi^*(\sum_{i\in T} x^i - \#T x^{2g+2}) \sim 0,$$

where $\sim$ denotes linear equivalence of divisors. Hence

$$e_T \in {}_2Pic(C).$$

Note that

$$\sum_{i\in B} c_i \sim (2g+2)c_{2g+2},$$

and for every integer a, we have

$$\sum_{i\in T} c_i + ac_{2g+2} - (\#T+a)c_{2g+2} = \sum_{i\in T} c_i - \#Tc_{2g+2}.$$

This shows that

$$e_T \sim e_{\bar{T}}$$

where $\bar{T} = B\setminus T$, and

$$e_T \sim e_{T'}$$

for some T' with $\#T' \equiv 0 \bmod 2$.

Let $\mathbb{F}_2^B$ be the set of subsets of B (or functions $B \to \mathbb{F}_2$) equipped with the structure of a vector space over $\mathbb{F}_2$ with the addtition law:

$$T+T' = T\cup T'\setminus(T\cap T').$$

It carries also a symmetric bilinear form defined by

$$(T,T') \to \#T\cap T' \bmod 2.$$

The restriction of this bilinear form to the subspace $(\mathbb{F}_2^B)_{ev}$ spanned by subsets of even cardinality is degenerate, the radical being equal to $\{\emptyset,B\}$. Let

$$E_g \subset \mathbb{F}_2^B/\{\emptyset,B\}$$

denote the factor space of this subspace by the radical. Its elements are subsets of B of even cardinality modulo $T \sim B\setminus T$.

Note that the symmetric group Σ_{2g+2} acts naturally on E_g and preserves the symplectic form. This gives a natural inclusion:

$$\Sigma_{2g+2} \hookrightarrow Sp(2g,\mathbb{F}_2)$$

well known to group theorists.

Lemma 1. The map

$$e: E_g \to {}_2Jac(C),\ T \mapsto e_T$$

is an isomorphism of linear spaces.

Proof. Easy (cf. **[Mu 2]**).

To define the period matrix of C we choose a special symplectic basis $\{\gamma_1,\dots,\gamma_{2g}\}$ of $H_1(C,\mathbb{Z})$. We view C as a two-sheeted cover of the Riemann sphere. Each class γ_i, $i \le g$, is represented by a path which goes from c_{2i-1} to c_{2i} along one sheet of C and returns from c_{2i} to c_{2i-1} along the other sheet. Each class γ_i, $i>g$, is represented by a path which goes from c_{2i} to c_{2g+1} along one sheet and returns from c_{2g+1} to c_{2i} along the other sheet. We call such a basis a <u>branch point basis</u>.

Let $\{\omega_1,\dots,\omega_g\}$ be a basis in $H^0(C,\Omega_C^1)$ normalized in the usual way with respect to a branch point basis. The corresponding period matrix

$$\tau(C) = \Big(\int_{\gamma_j} \omega_j \Big)_{i,j=1,\dots,g}$$

will be called a <u>branch point period matrix</u> of C.

Lemma 2. Let $\tau(C)$ be a branch point period matrix of C, and

$$\varphi_{\tau(C)}: \mathbb{F}_2^{2g} \to {}_2\mathrm{Jac}(C)$$

be the corresponding level 2 structure on $\mathrm{Jac}(C) = \mathbb{C}^g/\Lambda_{\tau(C)}$. There exists an isomorphism of symplectic spaces

$$I: E_g \to \mathbb{F}_2^{2g}$$

such that the composition

$$\varphi_{\tau(C)} \circ I : E_g \to {}_2\mathrm{Jac}(C)$$

is equal to the map e defined in Lemma 1. In particular e is an isomorphism of symplectic spaces. The map I is uniquely defined by the property:

$$I(\{c_{2i-1},c_{2i}\}) = (e_i,0), \quad i = 1,\dots,g,$$

$$I(\{c_{2i},\dots,c_{2g+1}\}) = (0,e_i), \quad i = 1,\dots,g.$$

<u>Proof</u>. It is immediately verified that the subsets

$$T_i = \{c_{2i-1},c_{2i}\}, \quad T_{i+g} = \{c_{2i},\dots,c_{2g+1}\}, \quad i = 1,\dots,g,$$

form a symplectic basis in E_g. Thus we can define I by sending this basis to the standard symplectic basis of $\mathbb{F}_2^{2g}$. The assertion will follow if we verify that under the Abel-Jacobi map

$$aj(e_{T_i}) = \varphi_{\tau(C)}(e_i), \quad i = 1,\dots,2g.$$

Note that each ω_k reverses its sign when one switches the sheets of C. Hence

$$\tfrac{1}{2}\delta_{ij} = \tfrac{1}{2}\int_{\gamma_i}\omega_j = \int_{c_{2i}}^{c_{2i-1}}\omega_j = \int_{c_{2g+2}}^{c_{2i-1}}\omega_j - \int_{c_{2g+2}}^{c_{2i}}\omega_j =$$

$$= \int_{c_{2g+2}}^{c_{2i-1}} \omega_j + \int_{c_{2g+2}}^{c_{2i}} \omega_j - 2 \int_{c_{2g+2}}^{c_{2i}} \omega_j$$

Since

$$2\left(\int_{c_{2g+2}}^{c_{2i}} \omega_1, \ldots, \int_{c_{2g+2}}^{c_{2i}} \omega_g \right) = aj(2c_{2i} - 2c_{2g+2}) = 0,$$

we obtain

$$aj(e_{T_i}) = aj(c_{2i-1} + c_{2i} - 2c_{2g+1}) = \int_{c_{2g+2}}^{c_{2i-1}} \bar{\omega} + \int_{c_{2g+2}}^{c_{2i}} \bar{\omega} = \tfrac{1}{2} e_i \mod \Lambda_{\tau(C)},\ i = 1, \ldots, g.$$

Similarly, we check that

$$aj(e_{T_i}) = \tfrac{1}{2}\tau_i \bullet e_{i-g},\ i = g+1, \ldots, 2g$$

and prove the lemma.

Recall that by the Torelli theorem the map

$$C \to Jac(C)$$

defines a closed embedding

$$T: \mathfrak{M}_g \hookrightarrow \mathfrak{A}_g$$

of the coarse moduli variety of nonsingular projective curves of genus g. We denote by

$$\mathcal{H}yp_g \subset \mathfrak{A}_g$$

the image of the subvariety of $\mathfrak{M}_g$ parametrizing isomorphism classes of hyperelliptic curves. The inverse image of $\mathcal{H}yp_g$ under the projection $\mathfrak{A}_g(2) \to \mathfrak{A}_g$ is denoted by $\mathcal{H}yp_g(2)$. It is a coarse moduli variety of Jacobians of hyperelliptic curves with level 2 structure.

Theorem 1. Let $(P_1^{2g+2})^{un} = (\mathbb{P}_1^{2g+2} \setminus \Delta)/PGL(2)$. There is a natural isomorphism

$$(P_1^{2g+2})^{un} \cong \mathcal{H}yp_g(2)^\circ,$$

where $\mathcal{H}yp_g(2)^\circ$ is an irreducible component of $\mathcal{H}yp_g(2)$. This isomorphism associates to a point set $x = (x^1, \ldots, x^{2g+2})$ the isomorphism class of the Jacobian variety of the hyperelliptic curve C(x) obtained as a double cover of $\mathbb{P}_1$ branched over $x^1,\dots,x^{2g+2}$ equipped with the level 2 structure defined by the branch point period matrix of C(x).

Proof. The map

$$(\mathbb{P}_1^{2g+2})^{un}\setminus\Delta \to \mathcal{H}yp_g(2),\ x \to (\mathrm{Jac}(C(x)),\varphi_{\tau(C(x))})$$

factors through $(P_1^{2g+2})^{un}$ and defines a map

$$i: (P_1^{2g+2})^{un} \to \mathcal{H}yp_g(2).$$

Let C be a hyperelliptic curve of genus g. It defines, uniquely up to projective equivalence, the set of ramification points $\{c_1,\dots,c_{2g+2}\}$ of its double cover onto $\mathbb{P}_1$. We have to show that its order is determined uniquely by the level 2 structure $\varphi: \mathbb{F}_2^{2g} \to {}_2\mathrm{Jac}(C)$ defined by a branch point period matrix $\tau(C)$. Let

$$\{\varepsilon_1,\dots,\varepsilon_{2g}\}$$

be the image under φ of the standard symplectic basis in $\mathbb{F}_2^{2g}$. Let $l: E_g \to \mathbb{F}_2^{2g}$ be the map defined in Lemma 2. Then the order of ramification points can be reconstructed by setting

$$c_{2i} = l^{-1}(\varepsilon_i)\cap l^{-1}(\varepsilon_{i+g}),\ i = 1,\dots,g,$$

$$c_{2i-1} = l^{-1}(\varepsilon_i)\setminus\{c_{2i}\},\ i = 1,\dots,g,$$

The projection of the ordered set of ramification points to $\mathbb{P}_1$ defines the point set x such that $i(x) = (\mathrm{Jac}(C),\varphi)$. We have a natural isomorphism

$$\bar{i}:(P_1^{2g+2})^{un}/\Sigma_{2g+2} \cong \mathcal{H}yp_g$$

such that the diagram

$$\begin{array}{ccc} (P_1^{2g+2})^{un} & \xrightarrow{i} & \mathcal{H}yp_g(2) \\ p\downarrow & & p'\downarrow \\ (P_1^{2g+2})^{un}/\Sigma_{2g+2} & \xrightarrow{\bar{i}} & \mathcal{H}yp_g \end{array}$$

is commutative. Since the projections p and p' are finite morphisms, the morphism i is finite. As we showed above its degree is 1. Therefore i is a closed embedding onto an irreducible component of $\mathcal{H}yp_g(2)$.

Corollary. The number of irreducible components of $\mathcal{H}yp_g(2)$ is equal to

$$\frac{\#\mathrm{Sp}(2g,\mathbb{F}_2)}{\#\Sigma_{2g+2}} = \frac{2^{g^2}(2^{2g}-1)(2^{2g-2}-1)\dots(2^2-1)}{(2g+2)!}$$

In particular, $\mathcal{H}yp_g(2)$ is irreducible for $g \leq 2$ and is isomorphic to $(P_1^{2g+2})^{un}$.

Proof. The group $Sp(2g,\mathbb{F}_2)$ of covering transformations of $\mathcal{H}yp_g(2) \to \mathcal{H}yp_g$ acts transitively on the set of irreducible components of $\mathcal{H}yp_g(2)$. The stabilizer of the component $\mathcal{H}yp_g(2)^\circ$ contains the subgroup Σ_{2g+2}. It is known that this subgroup is a maximal proper subgroup of $Sp(2g,\mathbb{F}_2)$. Thus the stabilizer is equal to this subgroup. This proves the assertion.

Remark 1. We do not know whether $\mathcal{H}yp_g(2)$ is smooth, or equivalently, if $\mathcal{H}yp_g(2)^\circ$ is a connected component of $\mathcal{H}yp_g(2)$. This would follow from the smoothness of the hyperelliptic locus in the Siegel half space $\mathcal{H}_g$.

4. Theta characteristics on hyperelliptic curves.

Following **[Mu 2]** we give a very convenient notation for theta characteristics on hyperelliptic curves that is similar to the notation for points of order 2 on its Jacobians given in the previous section. We keep the notation from that section.

Let

$$Q_g \subset \mathbb{F}_2^B/\{\emptyset,B\}$$

be the subset represented by subsets S of $B = \{1,\ldots,2g+2\}$ with

$$\#S \equiv g+1 \bmod 2 .$$

It has a natural structure of an affine space over E_g with respect to the addition in $\mathbb{F}_2^B$.

We have the following analog of Lemma 2 from the previous section:

Lemma 3. Let C be a hyperelliptic curve with ramification points $c_1,\ldots,c_{2g+2}$. For every subset S of B with $\#S \equiv g+1 \bmod 2$ define

$$f_S = \sum_{i\in S} c_i + (g-1-\#S)c_{2g+2}$$

Then

$$f_S \in Th(C),$$

$$f_S = f_{S'} \text{ iff } S = S' \text{ or } S = B\setminus S',$$

and the map $S \to f_S$ defines a bijection

$$f: Q_g \to Th(C)$$

such that the pair

$$(f,e): (Q_g, E_g) \to (Th(C), {}_2Jac(C))$$

is an isomorphism of affine spaces.

Proof. Left to the reader (cf. **[Mu 2]**).

Lemma 4. Let q_0: $(\varepsilon,\varepsilon') \to {}^t\varepsilon\cdot\varepsilon'$ be the standard quadratic form on $\mathbb{F}_2^{2g}$, b_0 be its associated bilinear form, and

$$l^*(q_0): E_g \to \mathbb{F}_2\ ,\ T \to q_0(l(T)),$$

be its pull-back to E_g. Then

$$l^*(q_0)(T) = \tfrac{1}{2}(\#(T+U)-g-1) \bmod 2 = \tfrac{1}{2}\#T+\#T\cap U \bmod 2,$$

where $U = \{1,3,\dots,2g+1\}$ is the subset of odd numbers in B.

Moreover, for every quadratic form q on $\mathbb{F}_2^{2g}$ with associated bilinear form equal to b_0 there exists a unique element $S\in Q_g$ such that

$$l^*(q)(T) = \tfrac{1}{2}\#T+\#(T\cap S) \bmod 2 = \tfrac{1}{2}(\#(T+S)-\#S) \bmod 2.$$

Proof. We check first that

$$T \to \tfrac{1}{2}(\#(T+U)-g-1) \bmod 2$$

is a quadratic form on E_g with associated bilinear form equal to

$$(T,T') \to \#T\cap T' \bmod 2.$$

Then we verify that the values $l^*(q_0)(T)$ and $\tfrac{1}{2}(\#(T+U)-g-1) \bmod 2$ are equal for T belonging to a symplectic basis of E_g. We leave this to the reader.

Let $S^2(\mathbb{F}_2^{2g})_0$ be the set of quadratic forms associated to the bilinear form b_0. To establish the second assertion we observe that every $q\in S^2(\mathbb{F}_2^{2g})_0$ can be uniquely written in the form

$$q = q_0+l^2.$$

where $l\in(\mathbb{F}_2^{2g})^*$ is a linear function on $\mathbb{F}_2^{2g}$. Identifying l with an element η of $\mathbb{F}_2^{2g}$ by means of the standard symplectic form on $\mathbb{F}_2^{2g}$, we verify that

$$l^*(q)(T) \equiv l^*(q_0)(T)+\#l^{-1}(\eta)\cap T \equiv \tfrac{1}{2}\#T+\#(T\cap U)+\#T\cap l^{-1}(\eta) \equiv$$

$$\equiv \#T+\#(T\cap(U+l^{-1}(\eta))) \equiv \tfrac{1}{2}\#T+\#(T\cap S) \equiv \tfrac{1}{2}(\#(T+S)-\#S) \mod 2,$$

where $S = U+l^{-1}(\eta)$.

We will identify Q_g with $S^2(E_g)_0$ by viewing each element $S\in Q_g$ as the quadratic form

$$T \to \tfrac{1}{2}(\#(T+S)-\#S) \mod 2$$

on E_g.

The analog of lemma 2 is the following:

Lemma 5. Let

$$S^2(E_g)_0 = Q_g \to S^2(\mathbb{F}_2^{2g})_0 \to Th(C) = S^2({}_2Jac(C))$$

be the sequence of the bijections obtained from the sequence

$$E_g \to \mathbb{F}_2^{2g} \to {}_2Jac(C)$$

by applying functors $(l^{-1})^*$ and $(\varphi_{\tau(C)}{}^{-1})^*$. Then the composition

$$Q_g \to Th(C)$$

coincides with the map f.

Proof. Easy verification (cf. **[Mu 2]**).

Our final observation is that each set Q_g, $S^2(\mathbb{F}_2^{2g})$, and Th(C) contains some distinguished elements. They are

$$U,\ \emptyset\ (g \text{ odd}),\ \{c_{2g+2}\}\ (g \text{ even}) \text{ in } Q_g,$$

$$q_0 \text{ in } S^2(\mathbb{F}_2^{2g}),$$

$$(g-1)c_{2g+2},\ (g-1)c_{2g+2}+\kappa(c_{2g+2},\tau(C)) \text{ in } Th(C)$$

(note that the Riemann constant in Jac(C) is a point of order 2 if C is hyperelliptic). We leave to the reader to verify that

$$(l^{-1})^*(U) = q_0,$$

$$(g-1)c_{2g+2} = f_\emptyset \text{ if } g \text{ odd},$$

$$= f\{c_{2g+2}\} \text{ if } g \text{ even},$$

$$f_U = (g-1)c_{2g+2} + \kappa(c_{2g+2},\tau(C)).$$

In particular this explicitly computes the Riemann constant

$$\kappa(c_{2g+2},\tau(C)) = f_U - f_\emptyset = e_U, \ g \text{ odd},$$
$$= f_U - f\{c_{2g+2}\}, \ g \text{ even}.$$

Under $\psi_{\tau(C)}^{-1}$ this corresponds to the characteristic

$$\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} = \begin{bmatrix} 111\ldots11 \\ 101\ldots01 \end{bmatrix}, \ g \text{ odd},$$
$$= \begin{bmatrix} 111\ldots1 \\ 010\ldots1 \end{bmatrix}, \ g \text{ even}$$

(see [Mu 2], p.3.82 and p.3.99).

Summarizing we get the following commutative diagram:

$$\begin{array}{ccccc}
 & & e & & \\
E_g & \xrightarrow{\ I\ } & \mathbb{F}_2^{2g} & \xrightarrow{\psi_{\tau(C)}} & {}_2\mathrm{Jac}(C) \\
T\mapsto T+U \downarrow & & \downarrow \alpha\mapsto\alpha^2+q_0 & & \downarrow \eta\mapsto\kappa(c_0,\tau(C))+\eta \\
Q_g & \xrightarrow{(I^{-1})^*} & S^2(\mathbb{F}_2^{2g})_0 & \xrightarrow{(\psi_{\tau(C)}^{-1})^*} & \mathrm{Th}(C) \\
 & & f & &
\end{array}$$

where the dotted arrows can be considered only if C is hyperelliptic, $\tau(C)$ is its branch point period matrix, and $c_0 = c_{2g+2}$.

Lemma 6. For every $E \in Q_g$, $\#S \le g+1$,

$$h^0(f_S) \equiv \tfrac{1}{2}(g+1-\#S) \bmod 2.$$

In particular,

$$f_S \text{ is even iff } \#S \equiv g+1 \bmod 4,$$

$$f_S \text{ is odd iff } \#S \equiv g-1 \bmod 4.$$

Proof. Since

$$\sum_{i\in B} c_i - (2g+1)c_{2g+2} = (y)$$

for some rational function y on C, we have

$$f_S = \sum_{i\in S} c_i + (g-1-\#S)c_{2g+2} \sim (g-1+\#S)c_{2g+2} - \sum_{i\in S} c_i.$$

Recall that

$$\mathbb{C}(C) = \mathbb{C}(x,y),\ x = \pi^*(t),\ y^2\in\mathbb{C}(x)$$

for some rational function t on $\mathbb{P}_1$. Since $g-1+\#S \le 2g$, and y has a pole at c_{2g+2} of order $2g+1$, every rational function on C with only a pole at c_{2g+2} of order $\le 2g$ must be equal to $\pi^*(p(t))$ for some polynomial $p(t)$ of degree $\le g$. Thus

$$H^0(C,\mathcal{O}_C(f_S)) \cong \text{space of polynomials in } t \text{ of degree } \le\tfrac{1}{2}(g+1+\#T)$$
$$\text{with zeroes at all } \pi(c_i),\ i\in S.$$

The dimension of this space is equal to $\frac{1}{2}(g+1-\#S)$.

The indexing of the set ${}_2Jac(C)$ (resp.$Th(C)$) by the set E_g(resp.Q_g) can also be used for non-hyperelliptic curves C. We can set

$$\eta_T = \varphi_{\tau(C)}(I(T))\in {}_2Jac(C),\quad T\in E_g,$$
$$D_S = (\varphi_{\tau(C)}{}^{-1})^*((I^{-1})^*)(S) \in Th(C),\ S\in Q_g,$$

for every curve C. This indexing depends only on the choice of the period matrix $\tau(C)$. The same choice allows us to define a bijection

$$D_S \to \vartheta_S(z;\tau) := \vartheta[I(S+U)](z;\tau)$$

between theta characteristics and theta functions with half integral characteristic. Note that for hyperelliptic curves C

$$\eta_T = e_T,\ D_S = f_S.$$

Corollary 1. Let C be a nonsingular projective curve of genus g. Then

$$h^0(D_S) \equiv \tfrac{1}{2}(g+1-\#S) \bmod 2.$$

In particular,

$$D_S \text{ is even iff } \#S \equiv g+1 \bmod 4,$$
$$D_S \text{ is odd iff } \#S \equiv g-1 \bmod 4.$$

<u>Proof</u>. We know that this is true for hyperelliptic curves. It remains to use that the parity of theta characteristics remains constant in a family ([Mu 4]).

Corollary 2. Assume C is hyperelliptic. Then

$$h^0(D_S) > 0 \text{ iff } \#S \equiv g-1 \bmod 4 \ (D_S \text{ is odd}), \text{ or}$$

$$\#S \equiv g+1 \bmod 4 \ (D_S \text{ is even}), \text{ and } \#S \neq g+1.$$

In particular, C has $2^{g-1}(2^g+1) - \frac{(2g+2)!}{2(g+1)!^2}$ vanishing theta constants $\vartheta_S(0;\tau)$ (i.e. $\vartheta_S(0;\tau) = 0$ for so many even S's, namely when $\#S \neq g+1$).

Remark 2. According to a theorem from **[Mu 2]** the last property characterizes hyperelliptic curves. That is, C is hyperelliptic iff $\vartheta_S(0;\tau(C)) = 0$ for all $S\epsilon Q_g$ such that $\#S \equiv g+1 \bmod 4$, $\#S \neq g+1$.

Summarizing the above, we obtain for every $S\epsilon Q_g$:

D_S is even $\Leftrightarrow$ $\#S \equiv g+1 \bmod 4$,

D_S is odd $\Leftrightarrow$ $\#S \equiv g-1 \bmod 4$,

$h^0(D_S) = \text{mult}_0\theta_S(z;\tau(C))$,

$q_{D_S}(\eta_T) = 0 \Leftrightarrow h^0(D_S+\eta_T)+h^0(D_S) \equiv 0 \bmod 2 \Leftrightarrow \frac{1}{2}\#T+\#T\cap S \equiv 0 \bmod 2$,

$\vartheta_S(\eta_T;\tau(C)) = 0 \Leftrightarrow \frac{1}{2}\#T+\#S\cap T \equiv 1 \bmod 2$ if D_S is even and $h^0(D_S) = 0$,

$\vartheta_S(\eta_T;\tau(C)) = 0 \Leftrightarrow \frac{1}{2}\#T+\#S\cap T \equiv 0 \bmod 2$ if D_S is odd .

Note that for a generic (in the sense of the moduli space) curve C no even theta characteristic is effective. If one of them is effective, a curve is called a <u>curve with a vanishing theta constant</u>. By the above this happens if and only if

$$\vartheta_S(0;\tau(C)) = 0 \text{ for some } S\epsilon Q_g \text{ with } \#S \equiv g+1 \bmod 4.$$

It never happens for $g \leq 2$ and happens for $g = 3$ if and only if C is hyperelliptic.

Examples. <u>$g = 1$</u>:

3 even "thetas": $\vartheta_{12} = \vartheta_{34} = \vartheta[10]$ that vanishes at $\eta_{12} = \eta_{34}$,

$\vartheta_{13} = \vartheta_{24} = \vartheta[00]$ that vanishes at $\eta_{13} = \eta_{24}$,

$\vartheta_{14} = \vartheta_{23} = \vartheta[01]$ that vanishes at $\eta_{14} = \eta_{23}$,

1 odd theta: $\vartheta_\emptyset = \vartheta_{1234} = \vartheta[11]$ that vanishes at $0 = \eta_\emptyset$.

<u>$g = 2$</u>:

10 even thetas: $\vartheta_{123} = \vartheta_{456} = \vartheta\begin{bmatrix}01\\10\end{bmatrix}$ that vanishes at $\eta_{12},\eta_{23},\eta_{13},\eta_{45},\eta_{46},\eta_{56}$,

$\vartheta_{124} = \vartheta_{356} = \vartheta\begin{bmatrix}00\\10\end{bmatrix}$ that vanishes at $\eta_{12}, \eta_{24}, \eta_{14}, \eta_{45}, \eta_{46}, \eta_{56}$,

$\vartheta_{125} = \vartheta_{346} = \vartheta\begin{bmatrix}00\\11\end{bmatrix}$ that vanishes at $\eta_{12}, \eta_{25}, \eta_{15}, \eta_{34}, \eta_{36}, \eta_{46}$,

$\vartheta_{126} = \vartheta_{345} = \vartheta\begin{bmatrix}11\\11\end{bmatrix}$ that vanishes at $\eta_{12}, \eta_{16}, \eta_{26}, \eta_{34}, \eta_{35}, \eta_{45}$,

$\vartheta_{234} = \vartheta_{156} = \vartheta\begin{bmatrix}10\\01\end{bmatrix}$ that vanishes at $\eta_{23}, \eta_{35}, \eta_{25}, \eta_{14}, \eta_{46}, \eta_{16}$,

$\vartheta_{235} = \vartheta_{146} = \vartheta\begin{bmatrix}10\\00\end{bmatrix}$ that vanishes at $\eta_{12}, \eta_{24}, \eta_{14}, \eta_{45}, \eta_{46}, \eta_{56}$,

$\vartheta_{236} = \vartheta_{145} = \vartheta\begin{bmatrix}01\\00\end{bmatrix}$ that vanishes at $\eta_{23}, \eta_{26}, \eta_{36}, \eta_{14}, \eta_{45}, \eta_{15}$,

$\vartheta_{245} = \vartheta_{136} = \vartheta\begin{bmatrix}11\\00\end{bmatrix}$ that vanishes at $\eta_{24}, \eta_{25}, \eta_{13}, \eta_{45}, \eta_{16}, \eta_{36}$,

$\vartheta_{246} = \vartheta_{135} = \vartheta\begin{bmatrix}00\\00\end{bmatrix}$ that vanishes at $\eta_{26}, \eta_{24}, \eta_{13}, \eta_{35}, \eta_{46}, \eta_{15}$,

$\vartheta_{256} = \vartheta_{134} = \vartheta\begin{bmatrix}00\\01\end{bmatrix}$ that vanishes at $\eta_{25}, \eta_{26}, \eta_{13}, \eta_{14}, \eta_{34}, \eta_{56}$,

6 odd thetas: $\vartheta_1 = \vartheta_{23456} = \vartheta\begin{bmatrix}01\\01\end{bmatrix}$ that vanishes at $\eta_\emptyset, \eta_{12}, \eta_{13}, \eta_{14}, \eta_{15}, \eta_{16}$,

$\vartheta_2 = \vartheta_{13456} = \vartheta\begin{bmatrix}11\\01\end{bmatrix}$ that vanishes at $\eta_\emptyset, \eta_{12}, \eta_{23}, \eta_{24}, \eta_{25}, \eta_{26}$,

$\vartheta_3 = \vartheta_{12456} = \vartheta\begin{bmatrix}11\\10\end{bmatrix}$ that vanishes at $\eta_\emptyset, \eta_{13}, \eta_{23}, \eta_{34}, \eta_{35}, \eta_{36}$,

$\vartheta_4 = \vartheta_{12356} = \vartheta\begin{bmatrix}10\\10\end{bmatrix}$ that vanishes at $\eta_\emptyset, \eta_{14}, \eta_{24}, \eta_{34}, \eta_{45}, \eta_{46}$,

$\vartheta_5 = \vartheta_{12346} = \vartheta\begin{bmatrix}10\\11\end{bmatrix}$ that vanishes at $\eta_\emptyset, \eta_{15}, \eta_{35}, \eta_{45}, \eta_{25}, \eta_{56}$,

$\vartheta_6 = \vartheta_{12345} = \vartheta\begin{bmatrix}01\\11\end{bmatrix}$ that vanishes at $\eta_\emptyset, \eta_{16}, , \eta_{26}, \eta_{36}, \eta_{46}, \eta_{56}$

<u>g = 3</u>.

36 even thetas of type $\vartheta_\emptyset$, ϑ_{ijkl},

28 odd thetas of type ϑ_{ij}.

<u>g = 4</u>.

136 even thetas of type ϑ_i and ϑ_{ijklm} ,

120 odd thetas of type ϑ_{ijk}.

5. Thomae's theorem.

This theorem establishes a relationship between cross-ratio functions on P_1^{2g+2} and fourth powers of theta constants.

First we observe that a hyperelliptic curve C has in general exactly $\frac{1}{2}\binom{2g+2}{g+1}$ non-vanishing theta constants $\vartheta_S(0;\tau(C))$, $\#S = g+1$. Each subset $S = \{i_1,\dots,i_{g+1}\}$ of $\{1,\dots,2g+2\}$ of cardinality $g+1$ defines a tableau

$$\tau_S = \begin{bmatrix} i_1 \dots i_{g+1} \\ j_1 \dots j_{g+1} \end{bmatrix}$$

where $\{j_1,\dots,j_{g+1}\}$ is the complementary subset, and hence the corresponding monomial

$$\mu_S = (i_1\dots i_{g+1})(j_1\dots j_{g+1}) \in (R_1^{2g+2})_1.$$

Let

$$v_g^{2g+2} : P_1^{2g+2} \to P_g^{2g+2}$$

be the map induced by the Veronese map $v_g: \mathbb{P}_1 \to \mathbb{P}_g$. Note that its image lies in the variety S_g of self-associated point sets.

A remarkable result of Thomae **[Th]** asserts that under the embedding

$$(P_1^{2g+2})^{un} \to \mathcal{H}yp_g(2) \hookrightarrow \mathcal{A}_g(2)$$

established in Theorem 1, the pull-back of the $\vartheta_S(0;\tau)^4$'s (which are modular forms of weight 2 with respect to $\Gamma_g(2)$) are proportional to the pull-backs of the monomial μ_σ under the Veronese morphism v_g^{2g+2}.

This follows immediately from the following:

Theorem 2 (R. Thomae). Let $x^1,\dots,x^{2g+2}$ be the branch points of a hyperelliptic curve C. Assume that they are $(1,a_i)$, $i = 1,\dots,2g+2$. Then

$$\vartheta_S(0;\tau(C))^4 = c\prod_{i<j,\,i,j\in T+U}(a_i-a_j)\prod_{i<j,\,i,j\notin T+U}(a_i-a_j)$$

for some constant c independent of S.

Proof. See **[Fay]**.

Corollary 1. There is a commutative diagram:

$$\begin{array}{ccc} (P_1^{2g+2})^{un} & \cong & \mathcal{H}yp_g(2)^{\circ} \\ {\scriptstyle v_g^{2g+2}} \downarrow & & \downarrow \\ P_g^{2g+2} & \to & \mathbb{P}_N \subset \mathbb{P}_{\binom{2g+2}{g+1}/2}, \end{array}$$

where $N = \binom{2g+2}{g+1}/(g+2)$, the right vertical arrow is given by $(\vartheta_S(0;\tau)^4)_{\#S=g+1}$, and the lower horizontal arrow is given by $(\mu_S)_{\#S=g+1}$.

Proof. We have only to justify the value of N. This follows immediately from the "hook formula"

$$\dim(P_1^{2g+2})_1 = \binom{2g+2}{g+1}/(g+2).$$

Remark 3. We have already noticed in Chapter 1 that the space $(R_1^{2g+2})_1$ is an irreducible representation of the symmetric group Σ_{2g+2} corresponding to the partition (2,...,2) of 2g+2. The Thomae formulae show that this representation can be realized in the subspace of the space of modular forms $M(\Gamma_g(2))_2$ spanned by the fourth powers of theta constants $\vartheta_S(0;\tau)^4$, $\#S = g+1$.

Remark 4. We get some linear relations between $\vartheta_S(0;\tau)^4$ coming from the straightening algorithm for monomials.

Remark 5. We shall see in the next Chapter that even theta constants $\vartheta_S(0;\tau)^4$ span an irreducible representation $T_g \subset M(\Gamma_g(2))_2$ of $Sp(2g,\mathbb{F}_2)$ of dimension $\frac{1}{3}(2^g+1)(2^{g-1}+3)$. Restricting the theta constants to the hyperelliptic locus $\mathcal{H}yp_g(2)$, we obtain only $\binom{2g+2}{g+1}/(g+2)$ linearly independent functions. This shows that there are

$$N_g = \frac{1}{3}(2^g+1)(2^{g-1}+3)-\binom{2g+2}{g+1}/(g+2)$$

linearly independent even theta constants vanishing on the hyperelliptic locus. For example, $N_3 = 1$, which agrees with the fact that the codimension of the hyperelliptic locus is 1 in case g = 3. For g = 4 we have $N_4 = 26$, very far away from the codimension. See in this connection **[Ac]**.

6. Elliptic curves.

We already know from Chapters 1 and 2 that

$$\mathcal{A}_1(2) \cong (P_1^4)^{un} \cong \mathbb{P}_1\setminus\{0,1,\infty\}.$$

The three deleted points correspond to the orbits of point sets with coinciding points. The map $\mathcal{A}_1(2) \to \mathbb{P}_1 \subset \mathbb{P}_2$ is given by three fourth powers of odd thetas ϑ_{12}^4 ϑ_{13}^4, and ϑ_{14}^4 satisfying the relation

$$\vartheta_{14}^4 = \vartheta_{13}^4 - \vartheta_{12}^4.$$

This relation corresponds to the relation between monomials

$$\mu\begin{bmatrix}1\,4\\2\,3\end{bmatrix} = \mu\begin{bmatrix}1\,3\\2\,4\end{bmatrix} - \mu\begin{bmatrix}1\,2\\3\,4\end{bmatrix}$$

coming from the straightening algorithm.

The group Σ_4 acts on $\mathcal{A}_1(2)$ and P_1^4 via its quotient group

$$Sp(2,\mathbb{F}_2) \cong \Sigma_3 \cong \Sigma_4/G,$$

where $G \cong \mathbf{Z}/2\oplus\mathbf{Z}/2$ is generated by the permutations (12)(34) and (13)(24). The boundary $\mathcal{B} = \{0,1,\infty\} \subset P_1^4$ forms an orbit under the action of Σ_3 and there are natural isomorphisms of the quotient spaces

$$\mathcal{A}_1 = \mathcal{A}_1(2)/Sp(2,\mathbb{F}_2) \cong (P_1^4)^{un}/\Sigma_4 \cong \mathbb{P}_1\setminus\{\infty\}.$$

The compactification $\bar{\mathcal{A}}_1(2)$ (resp. $\bar{\mathcal{A}}_1$) of $\mathcal{A}_1(2)$ (resp. $\mathcal{A}_1$) is identified with $P_1^4 \cong \mathbb{P}_1$ (resp. $P_1^4/\Sigma_4 \cong \mathbb{P}_1$).

Let

$$\pi\colon \mathcal{A}_1(2)^1 \to \mathcal{A}_1(2)$$

be the universal elliptic curve. It parametrizes the isomorphism classes of elliptic curves with level 2 structure with a marked point on the curve. The morphism π defines the structure of an abelian scheme on $\mathcal{A}_1(2)^1$. Let

$$\tau\colon \mathcal{A}_1(2)^1 \to \mathcal{A}_1(2)^1$$

be the inversion involution $x \to -x$, and

$$\mathcal{K}um_1(2) = \mathcal{A}_1(2)^1/(\tau)$$

be the corresponding quotient space.

The next result gives a modular interpretation of the space P_1^5 (isomorphic to a Del Pezzo surface of degree 5).

Theorem 3. There is a natural isomorphism

$$f:\mathcal{H}um_1(2) \to (P_1^5)^{un}.$$

Moreover, f extends to an isomorphism of compactifications

$$\bar{f}:\overline{\mathcal{H}um_1(2)} \to P_1^5.$$

The projection $\mathbb{P}_1^5 \to \mathbb{P}_1^4$ defines a morphism $P_1^5 \to P_1^4$ such that the diagram

$$\begin{array}{ccc} \overline{\mathcal{H}um_1(2)} & \overset{\bar{f}}{\simeq} & P_1^5 \\ \downarrow & & \downarrow \\ \bar{\mathcal{A}}_1(2) & \simeq & P_1^4 \end{array}$$

is commutative.

Proof. We leave this as an exercise. The reader is referred to Chapter 7 for the geometry of the map $P_1^5 \to P_1^4$.

7. Abelian surfaces.

We know from Chapters 1 and 2 that

$$(P_1^6)^{un} \cong \mathcal{H}yp_2(2) \subset \mathcal{A}_2(2).$$

It is known (see **[Ig 3]**) that the algebra of modular forms $M(\Gamma_2(2))$ is generated by the fourth powers of theta constants $\vartheta_S(0,\tau)^4$. Applying Thomae's theorem, we obtain an isomorphism of graded algebras

$$R_2^6/(t_5) \simeq M(\Gamma_2(2))$$

and of their projective spectra

$$\bar{\mathcal{A}}_2(2) \simeq S_6 \subset P_2^6.$$

Under the Veronese morphism

$$v_2^6: P_1^6 \to S_6$$

the hyperelliptic locus $\mathcal{H}yp_2(2)$ is identified with the moduli space S_6 of self-associated point sets in $\mathbb{P}_2$. This immediately implies

Theorem 4. There is a natural isomorphism

$$(S_6)^{un} = S_6 \cap (P_2^6)^{un} \to \mathcal{H}yp_2(2)$$

which extends to an isomorphism

$$S_6 \to \bar{\mathfrak{a}}_2(2).$$

Recall from Chapter 1 that the variety S_6 is isomorphic to the level 2 modular quartic 3-fold $V_4 \subset \mathbb{P}_4$.

Let $\overline{\mathcal{H}yp_2(2)}$ denote the closure of $\mathcal{H}yp_2(2)$ in $\bar{\mathfrak{a}}_2(2)$.

Theorem 5. Let $\bar{\mathfrak{a}}_2(2)$ be identified with S_6 and V_4. Then
(i) $\bar{\mathcal{D}} = \bar{\mathfrak{a}}_2(2)\setminus\mathfrak{a}_2(2) = \mathrm{Sing}(V_4)$, and is equal to the union of 15 lines.
(ii) $\bar{\mathfrak{a}}_2(2)\setminus\overline{\mathcal{H}yp_2(2)}$ is the union of 10 irreducible components, each of them is isomorphic to a nonsingular quadric.

Proof. (i) Recall from Chapter 2 that the image $\mathcal{D}$ of semi-stable non-stable point sets in P_2^6 is contained in S_6. We know also that its complement in P_2^6 is nonsingular. Since P_2^6 is isomorphic to the double cover of $\mathbb{P}_4$ ramified along S_6, this implies that $S_6\setminus\bar{\mathcal{D}}$ is nonsingular. On the other hand we have checked in Chapter 2 that $\bar{\mathcal{D}}$ consists of 15 double lines of V_4. This shows that

$$\bar{\mathcal{D}} = \mathrm{Sing}(\bar{\mathfrak{a}}_2(2)).$$

It is known that $\Gamma_2(2)$ does not have torsion elements and hence acts freely on the Siegel half space. This implies that $\mathfrak{a}_2(2)$ is nonsingular. Moreover, it is known that the boundary $\bar{\mathfrak{a}}_2(2)\setminus\mathfrak{a}_2(2)$ is equal to the set of singular points of $\bar{\mathfrak{a}}_2(2)$. This implies that

$$\mathfrak{a}_2(2) = S_6\setminus\bar{\mathcal{D}}.$$

(ii) We know from Chapter 2 that

$$S_6\setminus(S_6)^{un} = \bar{\mathfrak{a}}_2(2)\setminus\mathcal{H}yp_2(2),$$

and is equal to the union of hypersurfaces parametrizing the images in S_6 of point sets with 3 collinear points. Each such hypersurface is given by the equation

$$\mu_\tau = 0,$$

for some tableau of the form

$$\tau = \begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix}$$

The number of such tableaux is 10. In the embedding $S_6 \cong V_4 \hookrightarrow \mathbb{P}_4$ these hypersurfaces are hyperplane sections. We claim that each of these hyperplane sections is a quadric taken with multiplicity 2. The group Σ_6 acts on V_4 and permutes these 10 hyperplane sections. Thus it is enough to verify this assertion for one of them. Note that the coordinate functions correspond to standard monomials. Using the equation of V_4 given at the end of Chapter 1, we find that

$$\{T_0 = 0\} \cap V_4 = \{(-T_2T_3+T_1T_4)^2 = 0\}.$$

This proves the assertion.

Remark 6. The irreducible components of $\bar{\mathfrak{a}}_2(2)\setminus\mathfrak{H}yp_2(2)$ are called Humbert surfaces. We refer to **[vdG]** for more information about these surfaces. Note also that the equation of their union in $\bar{\mathfrak{a}}_2(2)$ is given by the product of the squares of 10 even theta constants. As follows from Thomae's theorem and Chapter 1, under the birational map $v_2^6:P_1^6 \dashrightarrow S_6 \cong \bar{\mathfrak{a}}_2(2)$ the inverse image of this product is equal (up to a constant factor) to the product

$$\prod_{1\le i<j\le 6} (i,j)^2$$

This function is Σ_6-invariant and coincides with the discriminant of a homogeneous binary form of degree 6. We refer to **[Gr]** for another proof of this result which also gives the value of the multiplicative constant.

Next we describe a resolution of singularities of $V_4 = \bar{\mathfrak{a}}_2(2)$. We leave to the reader the verification of the following:

Theorem 6. Let $V_3 \cong P_1^6 \subset \mathbb{P}_4$ be the Segre cubic primal, and π: $\tilde{V}_3 \to V_3$ be the resolution of its 10 nodes obtained by blowing up the corresponding points in $\mathbb{P}_4$. The rational map $\mathbb{P}_3 \dashrightarrow \mathbb{P}_4$ given by the partials of the equation of V_3 extends to a birational morphism

$$f: \tilde{V}_3 \to V_4$$

which is a resolution of singularities of V_4. The image under f of the exceptional locus of π is equal to the complement of the hyperelliptic locus in V_4. The exceptional locus of f is equal to the proper inverse transform under π of the nodal locus $P_1^6\setminus(P_1^6)^{un}$. The latter consists of the union of 15 surfaces E_i isomorphic to $P_1^5 \cong \overline{\mathfrak{X}u}\, m_1(2)$. The induced map f: $E_i \to f(E_i) = l_i$ corresponds to the natural projection $\overline{\mathfrak{X}u}\, m_1(2)\to \bar{\mathfrak{a}}_1(2) \cong \mathbb{P}_1$.

Remark 7. The resolution of singularities of $\bar{\mathfrak{a}}_2(2)$ described in the previous theorem is a special case of Igusa's blowing-up of $\bar{\mathfrak{a}}_g(2)$ **[Ig 2]** defined for every $g \geq 2$.

Remark 8. Note that there exists another "small" resolution of singularities of V_3. It is given by the map

$$\varphi|-\tfrac{1}{2}K_{V(x)}| \to V_3,$$

where $x \in (\mathbb{P}_3^5)^{gen}$. In other terms it is obtained by the linear system of quadrics through a generic set of 5 points $x^1,\dots,x^5$ in $\mathbb{P}_3$ (see **[S-R]**). The exceptional locus of this resolution consists of proper transforms of the ten lines $\langle x^i,x^j\rangle$. The nodal locus $\mathcal{D}$ in $V_3 = P_1^6$ is equal to the image of the union of 5 planes blown up from the points x^i and the ten planes $\langle x^i,x^j,x^k\rangle$.

Remark 9. We refer the reader to **[Co 4]**, **[SB]** for the beautiful geometry of the modular variety $\bar{\mathfrak{a}}_2(3)$ parametrizing abelian surfaces with level 3 structure.

IX. CURVES OF GENUS 3.

The relationship between point sets in $\mathbb{P}_1$ and moduli spaces of hyperelliptic curves can be extended a few steps further. In this chapter, by two different methods, we construct an isomorphism

$$\mathfrak{M}_3(2)\setminus\mathcal{H}yp_3(2) \to (P_2^7)^{un},$$

$(C,\varphi) \to$ geometrically marked Del Pezzo surface of degree 1 which is a double cover of $\mathbb{P}_2$ branched along C,

where $\mathfrak{M}_3(2)$ is the moduli space of curves of genus 3 with a level 2 structure on its Jacobian variety. We shall show that it extends to a birational morphism:

$$\mathfrak{M}_3(2) \to P_2^7.$$

1. Level 2 structures on the Jacobian variety of a curve of genus 3.

As in the case of hyperelliptic curves, it is possible to give a geometric interpretation of a level 2 structure on Jac(C), where C is a nonsingular projective curve of genus 3.

Recall from Chapter VII that every C as above can be realized as the ramification curve of a finite cover of degree 2, π: $V \to \mathbb{P}_2$, where V is a uniquely defined (up to isomorphism) Del Pezzo surface V of degree 2, and π is given by the linear system $|-K_V|$. It follows from the formula for the canonical class of a double cover that

$$C\in|-2K_V|.$$

Let

$$Pic(V)_0 = (\mathbf{Z}K_V)^{\perp}_{Pic(V)} = (\mathbf{Z}[C])^{\perp}_{Pic(V)}$$

and

$$r: \operatorname{Pic}(V)_0 \to \operatorname{Pic}(C)$$

be the restriction map.

Lemma 1. For every $D \in \operatorname{Pic}(V)_0$

$$r(2D) = 0.$$

Proof. We know from Lecture 7 that the covering transformation g_0 of π acts on $\operatorname{Pic}(V)_0$ as -1. Thus for every $D \in \operatorname{Pic}(V)_0$ we have

$$r(D) = g_0^*(r(D)) = r(g_0^*(D)) = r(-D) = -r(D),$$

that is,

$$2r(D) = r(2D) = 0.$$

Denote

$$\bar{N}(V)_0 = \operatorname{Pic}(V)_0/2\operatorname{Pic}(V)_0$$

and let

$$\bar{r}: \bar{N}(V)_0 \to {}_2\operatorname{Pic}(C)$$

be the homomorphism induced by r. The intersection form on the lattice $N(V) = \operatorname{Pic}(V)$ defines by reduction mod 2 a symmetric bilinear form on $\bar{N}(V)_0$. If

$$\varphi: H_7 \to N(V)$$

is a geometric marking of V, then φ induces an isometry

$$\bar{\varphi}: \bar{Q}_B = Q_B/2Q_B \to \bar{N}(V)_0,$$

where Q_B is the root lattice of type 2 in H_7. We easily find that the radical of $\bar{Q}_B$ is spanned by the vector

$$\bar{v}_0 = \alpha_0+\alpha_4+\alpha_6 \mod 2Q_B.$$

Thus

$$\bar{\varphi}': \bar{Q}_B' = \bar{Q}_B/\mathrm{Rad} \to \bar{N}(V)_0' = \bar{N}(V)_0/\mathrm{Rad}$$

is an isomorphism of symplectic spaces of dimension 6 over $\mathbb{F}_2$.

Lemma 2. The homomorphism

$$\bar{r}: \bar{N}(V)_0 \to {}_2\operatorname{Pic}(C)$$

factors through an isomorphism

$$\bar{r}': \bar{N}(V)_0' \to {}_2\mathrm{Pic}(C) = {}_2\mathrm{Jac}(C)$$

of symplectic spaces.

<u>Proof</u>. It is enough to show that $\bar{r}$ is compatible with the corresponding symmetric bilinear forms. Observe first that

$$\mathrm{Pic}(V)/2\mathrm{Pic}(V) \cong H^2(V,\mu_2),$$

$${}_2\mathrm{Pic}(V) \cong H^1(V,\mu_2)$$

as follows from the Kummer exact sequence, and

$$\bar{N}(V)_0 \cong H^2(V,\mu_2)_0 = \mathrm{Ker}(1+g_0^*:H^2(V,\mu_2) \to H^2(V,\mu_2)).$$

Note that the symmetric bilinear form on $\mathrm{Pic}(V)/2\mathrm{Pic}(V)$ induced by the intersection form (resp. the Weyl pairing on ${}_2\mathrm{Pic}(C) = {}_2\mathrm{Jac}(C)$) corresponds to the usual multiplication in the cohomology defined by the cup-product. Now we observe that the map $\bar{r}$ is equal to the map

$$H^2(V,\mu_2)_0 \to H^1(C,\mu_2)$$

coming from the Smith exact sequence for the involution g_0 (see **[Br]**):

$$\to H^2(\mathbb{P}_2,C,\mu_2) \to H^2(V,\mu_2) \xrightarrow{\beta_2} H^2(C,\mu_2)\oplus H^2(\mathbb{P}_2,\pi(C),\mu_2) \xrightarrow{\gamma_2} H^3(\mathbb{P}_2,\pi(C),\mu_2).$$

We use that the kernel of the component

$$H^2(V,\mu_2) \to H^2(C,\mu_2)$$

of β_2 is equal to $H^2(V,\mu_2)_0$, the component

$$H^2(C,\mu_2) \to H^3(\mathbb{P}_2,\pi(C),\mu_2)$$

of γ_2 is equal to the coboundary homomorphism from the exact sequence of the pair $(\mathbb{P}_2,\pi(C))$, and the image of $H^2(V,\mu_2)_0$ in $H^2(\mathbb{P}_2,\pi(C),\mu_2)$ under β_2 is equal to $H^1(C,\mu_2)$. The latter is identified with a subspace of $H^2(\mathbb{P}_2,\pi(C),\mu_2)$ by means of the exact sequence of the pair $(\mathbb{P}_2,\pi(C))$. It remains to use that the Smith exact sequence is compatible with the cup-product.

Remark 1. The fact that the radical $\{0,\bar{v}_0\}$ of $\bar{N}(V)_0$ goes to zero under $\bar{r}$ can be seen without using that $\bar{r}$ is compatible with the bilinear forms. In fact,

$$v_0 = \varphi(\alpha_0+\alpha_4+\alpha_6) = \varphi(e_0-e_1-e_2-e_3+e_4-e_5+e_6-e_7) =$$

$$= \varphi(3e_0-e_1-\ldots-e_7)-\varphi(2e_0-2e_4-2e_6) = -K_V \quad -2\varphi(e_0-e_4-e_6).$$

Note that by adjunction

$$\omega_C = \mathcal{O}_V(K_V+C)\otimes\mathcal{O}_C = \mathcal{O}_V(-K_V)\otimes\mathcal{O}_C.$$

On the other hand, $\pi(\varphi(e_0-e_4-e_6))$ is a bitangent to $\pi(C) \cong C$ and hence cuts out on C an odd theta characteristic. Thus v_0 goes to zero under the restriction homomorphism $\mathrm{Pic}(V) \to \mathrm{Pic}(C)$.

Corollary. Let $\varphi: H_7 \to N(V)$ be a geometric marking of V, $Q_B = (\mathbb{Z}K_{2,7})^\perp$ be the root lattice of type 2, $\bar{Q}_B = Q_B/2Q_B$ be equipped with the symmetric bilinear form induced by the lattice structure on Q_B, $\bar{Q}_B{}' = \bar{Q}_B/\mathrm{Rad}$, and

$$\bar{\varphi}': \bar{Q}_B{}' \cong \mathbb{F}_2^6 \to \bar{N}(V)_0{}'$$

be the induced isomorphism. The composition of $\bar{\varphi}'$ with the isomorphism $\bar{r}'$ from the previous lemma defines a level 2 structure on the curve C.

Lemma 3. Let L and M be two lattices isomorphic to the root lattice Q_B of the root system of type E_7, $\bar{L}$ and $\bar{M}$ be their reductions mod 2, $\bar{L}' = \bar{L}/\mathrm{Rad}$ and $\bar{M}'=\bar{M}/\mathrm{Rad}$ be the corresponding symplectic spaces over $\mathbb{F}_2$. Then the canonical map

$$\gamma: \mathrm{Isom}(L,M) \to \mathrm{Isom}(\bar{L}',\bar{M}')$$

between the corresponding sets of isometries is surjective and $\gamma(\alpha) = \gamma(\beta)$ if and only if $\alpha = \pm\beta$.

<u>Proof</u>. By fixing an isomorphism $L \cong M$ we may assume that $L = M$. Then it suffices to show that the canonical map $O(L) \to O(\bar{L}')$ is bijective. But

$$O(L) \cong W(E_7),\ O(\bar{L}') \cong Sp(6,\mathbb{F}_2),$$

and we have already observed in Chapter VII that our map $O(L)\to O(\bar{L}')$ corresponds to a surjection $W(E_7) \to Sp(6,\mathbb{F}_2)$ with kernel $\{\pm 1\}$. This proves the lemma.

Let

$$\mathfrak{M}_g \text{ (resp. } \mathfrak{M}_g(2))$$

denote the subvariety of $\mathfrak{A}_g$ (resp. $\mathfrak{A}_g(2)$) parametrizing the Jacobian varieties of nonsingular projective curves of genus g (resp. with level 2 structure). Note that by the Torelli theorem the first of these varieties is isomorphic to the coarse moduli variety of nonsingular projective curves of genus g.

Theorem 1. There is a natural isomorphism of algebraic varieties:

$$f:(P_2^7)^{un} \cong \mathfrak{M}_3(2)\backslash \mathfrak{H}yp_3(2).$$

It maps a geometrically marked unnodal Del Pezzo surface (V,φ) representing a point of $(P_2^7)^{un}$ to the isomorphism class $(Jac(C),\alpha)$, where C is the ramification curve of the map $V \to \mathbb{P}_2$ given by $|-K_V|$ and $\alpha = \bar{r}'\circ\bar{\varphi}'$.

<u>Proof</u>. We have already defined the morphism f. Let us define its inverse. Given a nonsingular nonhyperelliptic projective curve C of genus 3, we can construct a Del Pezzo surface V of degree 2 as the double cover of $\mathbb{P}_2$ branched along the canonical image of C. It remains to show that a level 2 structure α: $\mathbb{F}_2^6 \to {}_2Jac(C)$ defines a geometric marking of V. Composing α with the homomorphism $\bar{r}'^{-1}$, we obtain a symplectic isomorphism

$$\bar{\varphi}': \mathbf{F}_2^6 \to \bar{N}(V)_0'.$$

By Lemma 3 this isomorphism induces an isomorphism of lattices

$$\varphi': (\mathbf{Z}K_{2,7})^{\perp}_{H_7} \to (\mathbf{Z}K_V)^{\perp}_{N(V)} = Pic(V)_0$$

which is defined uniquely up to composing with ± 1. We can extend $\pm\varphi'$ to isomorphisms

$$(\pm\varphi,-1): (\mathbf{Z}K_{2,7})^{\perp}_{H_7}\oplus \mathbf{Z}K_{2,7} \to (\mathbf{Z}K_V)^{\perp}_{H_7}\oplus \mathbf{Z}K_V.$$

Finally we can extend these isomorphisms to isomorphisms:

$$\varphi^{\pm}: H_7 \to Pic(V)$$

satisfying

$$\varphi^{\pm}(K_{2,7}) = -K_V.$$

This follows easily from the fact that the canonical homomorphism

$$\{\sigma\in O(H_7):\sigma(K_{2,7}) = K_{2,7}\} \to O((\mathbf{Z}K_{2,7})^{\perp}_{H_7}) = W(E_7)$$

is an isomorphism. Indeed, it is surjective because every simple reflection in $W(E_7)$ is in the image, and it is injective because $(\mathbf{Z}K_{2,7})^{\perp}_{H_7}\oplus \mathbf{Z}K_{2,7}$ is of finite index in H_7. Note that

$$\varphi^{+} = \varphi^{-}\circ g_0^{*},$$

where g_0 is the covering transformation of $\pi:V \to \mathbb{P}_2$. We finish the proof by using Proposition 8 from Chapter V where it was shown that every H_7-marking of a Del Pezzo surface V that sends $K_{2,7}$ to $-K_V$ is geometric.

2. Aronhold sets of bitangents to a quartic plane curve.

In this section, following **[vG 1]**, we explain another way to reconstruct a level 2 structure on the Jacobian of a curve of genus 3 from a point set in P_2^7.

Proposition 1. Let C be the curve of genus 3 associated to a Del Pezzo surface V of degree 2. Under the anti-canonical map π: $V \to \mathbb{P}_2$ the image of every exceptional curve of the first kind on V is equal to a bitangent to C. For every bitangent to C its inverse transform under π is equal to the union of two exceptional curves which are conjugate under the covering involution g_0 of π.

Proof. Let E be an exceptional curve of the first kind on V. Then $|-K_V-E|$ consists of an exceptional curve of the first kind E' equal to $g_0(E)$. This shows that the image of E (and E') is equal to a line l(E) which intersects the branch curve C at two points. Therefore $\pi(E)$ is a bitangent. Conversely, the double cover π splits over any bitangent to C. Its inverse image under π is the union of two nonsingular rational curves E and E', each intersecting the ramification curve at two points. Since the ramification curve of π belongs to $|-2K_V|$, we have $E \cdot K_V = E' \cdot K_V = -1$. Thus E and E' are exceptional curves of the first kind.

Let ℓ be a bitangent to C, $\ell \cap C = 2p+2q$ for some points p and q. The divisor $D(\ell) = p+q$ is a theta characteristic on C. Since $h^0(D(\ell)) = 1$, it is an odd theta characteristic. Conversely, each odd theta characteristic D is linearly equivalent to the divisor p+q for some points p and q such that 2p+2q is cut out by a line. This shows that the set of bitangents can be identified with the set of odd theta characteristics on C. Note that the equation of the bitangent corresponding to an odd theta characteristic can be given by the linear term of the Taylor expansion of the corresponding theta function at the origin (see **[Fr 1]**).

Corollary. There is a natural 2-to-1 map from the set of exceptional curves of the first kind on V and the set of odd theta characteristics on the curve C.

Another way to see the odd theta characteristics on a nonsingular curve C of genus 3 is furnished by the Steinerian embeddings of C. We recall what this means (see **[Be]**, **[Ty 1]**).

Let $\mathfrak{N} \subset \mathbb{P}(\Gamma(\mathbb{P}_3,\mathcal{O}_{P^3}(2)))$ denote a net of quadrics in $\mathbb{P}_3$ whose base locus consists of 8 distinct points. The Hessian curve of $\mathfrak{N}$ is the closed subset $H(\mathfrak{N})$ of

$\mathfrak{N}$ parametrizing singular quadrics. It is isomorphic to a plane quartic curve $C \subset \mathfrak{N} \cong \mathbb{P}_2$ given by the equation:

$$\det(t_0A_0+t_1A_1+t_2A_2) = 0,$$

where $\{{}^txA_0x = 0, {}^txA_1x = 0, {}^txA_2x = 0\}$ is a basis of $\mathfrak{N}$. The <u>Steinerian curve</u> of $\mathfrak{N}$ is a subset $S(\mathfrak{N})$ of $\mathbb{P}_3$ parametrizing the set of singular points of quadrics from $\mathfrak{N}$. It has a structure of a closed subscheme of $\mathbb{P}_3$ given by the vanishing of the 3x3-minors of the 4x3-matrix

$$[A_0x \ \ A_1x \ \ A_2x].$$

The net $\mathfrak{N}$ is called <u>regular</u> if $H(\mathfrak{N})$ is smooth. This implies that the base locus of $\mathfrak{N}$ consists of 8 distinct points. In this case $S(\mathfrak{N})$ is a smooth curve of degree 6, and the map $t \to \mathrm{Sing}(Q(t))$ is an isomorphism given by the linear system $|K_C+\theta|$ for some even theta characteristic θ. The correspondence $\mathfrak{N} \to (C,\theta)$ establishes a bijective map between the classes of regular nets of quadrics in $\mathbb{P}_3$ modulo projective equivalence and the isomorphism classes of smooth curves of genus 3 with a fixed even theta characteristic.

Let $x^1,\dots,x^8$ be the base points of a regular net of quadrics $\mathfrak{N}$. The pencil $\ell_{ij} \subset \mathfrak{N}$ of quadrics from $\mathfrak{N}$ passing through the line $\langle x^i,x^j\rangle$ contains exactly two singular quadrics with nodes at some points c_i and c_j which are the singular points of the base curve of the pencil. Thus the line $\langle x^i,x^j\rangle$ is a chord of the Steinerian curve joining the two points c_i and c_j. In the plane $\mathfrak{N}$, the line ℓ_{ij} intersects the Hessian curve C at two points. Hence it is a bitangent to C, and, under the isomorphism $C \to S(\mathfrak{N})$, the two points c_i and c_j go to an odd theta characteristic θ_{ij} defined by the bitangent ℓ_{ij}. Since there are 28 odd theta characteristics for C and 28 pairs of points x^i, x^j, we can account for all odd theta characteristics in this way. Notice that the subscript notation θ_{ij} agrees with the subscript notation for odd theta functions $\vartheta_{ij}(z;\tau)$ used in the previous Chapter.

In what follows we shall assume that distinct letters represent distinct values for indices, unless it is mentioned otherwise.

Lemma 4. For every $i = 1,\dots,8$

$$\theta_{i,jkr}: = \theta_{ij}+\theta_{ik}-\theta_{ir} \sim \theta_{ij}+\theta_{ik}+\theta_{ir}-K_C$$

is an even theta characteristic on C.

<u>Proof</u>. Suppose that, to the contrary, $\theta_{i,jkr}$ is odd and equal to θ_{mn} for some m and n. Since the chords of $N(\mathfrak{N})$ that connect the points x^i, x^j, and x^k lie in a plane, and since a plane cuts out the divisor $K_C+\theta$, we have the relation:

$$\theta_{ij}+\theta_{ik}+\theta_{jk} \sim \theta+K_C.$$

This implies that

$$\theta_{jk}+\theta_{ir}+\theta_{mn} \sim \theta+K_C,$$

and hence, that the chords $\langle x^j,x^k\rangle$ and $\langle x^i,x^r\rangle$ lie in the same plane. The point of intersection of these chords must be a base point of two different pencils of quadrics in $\mathfrak{N}$, therefore a base point of the whole net $\mathfrak{N}$. Obviously this is absurd.

Remark 2. One can easily show that it is possible to suppress the isolation of an index in the previous notation $\theta_{i,jkr}$ for the even theta characteristic $\theta_{ij}+\theta_{ik}-\theta_{ir}$ and write it simply by θ_{ijkr}. Then we verify that $\theta_{ijkr}=\theta_{lmnp}$ for complementary sets of indices. In this way θ_{ijkr} and θ account for all even theta characteristics. We again see the agreement of the notation for even theta characteristics θ, θ_{ijkr} with the one given in Chapter VIII.

The following result follows immediately from the previous discussion and results from Chapter III:

Proposition 2. Let $x^1,\ldots,x^8$ be an ordered point set in $\mathbb{P}_3$ which consists of base points of a regular net $\mathfrak{N}$ of quadrics (a <u>regular Cayley octad</u>). The projection of this set from the point x^8 defines a point set $y = (y^1,\ldots,y^7)$ in $\mathbb{P}_2$ which is associated to $(x^1,\ldots,x^7)$. The image of the Steinerian curve of $\mathfrak{N}$ under this projection is a sextic with seven double points at $y^1,\ldots,y^7$. The proper transform of this sextic in $V(y)$ is the ramification curve C of the anti-canonical double cover π: $V(y) \to \mathbb{P}_2$ of the Del Pezzo surface of degree 2. The images of the exceptional curves $E_1,\ldots,E_7$ blown up from the points $y^1,\ldots,y^7$ are the seven bitangents to $\pi(C)$ corresponding to the seven odd theta characteristics θ_{8i} defined by the chords $\langle x^i,x^j\rangle$.

Definition. An <u>Aronhold set</u> is an ordered set of seven odd theta characteristics θ_i $(1\le i\le 7)$ such that $\theta_i+\theta_j-\theta_k$ is an even theta characteristic for all triples (i,j,k) of different indices.

Proposition 3. Let C be a smooth projective curve of genus 3. There is a natural bijection between the set of symplectic isomorphisms $\varphi\colon \mathbb{F}_2^6 \to {}_2\mathrm{Jac}(C)$ and the set of Aronhold sets.

Proof. Let $\{\theta_1,\ldots,\theta_7\}$ be an Aronhold set. Define $\eta_i \in {}_2\mathrm{Jac}(C)$ by

$$\eta_i = \theta_i - \theta_1 \text{ , } i = 2,\ldots,7.$$

Let $e\colon {}_2\mathrm{Jac}(C) \times {}_2\mathrm{Jac}(C) \to \mathbb{F}_2$ be the canonical symplectic form on ${}_2\mathrm{Jac}(C)$. As we saw in the previous Chapter, for every theta characteristic θ, the function

$$\eta \to h^0(\eta+\theta)+h^0(\theta) \bmod 2$$

is a quadratic form on ${}_2\mathrm{Jac}(C)$ with associated bilinear form equal to e. Thus

$$e(\eta_i,\eta_j) = h^0(\eta_i+\theta)+h^0(\eta_j+\theta)+h^0(\eta_i+\eta_j+\theta)+h^0(\theta) \bmod 2.$$

Setting $\theta = \theta_1$, we obtain for $i \neq j$

$$e(\eta_i,\eta_j) = h^0(\theta_i)+h^0(\theta_j)+h^0(\theta_i+\theta_j-\theta_1)+h^0(\theta_1) = 1 \bmod 2$$

because θ_i, θ_j, and θ_1 are odd theta characteristics but $\theta_i+\theta_j-\theta_1$ is an even one. This implies that $\{\eta_1,\ldots,\eta_6\}$ is a basis in ${}_2\mathrm{Jac}(C)$, since $\eta = \Sigma a_i\eta_i = 0$ implies that $e(\eta,\eta_i) = 0$ for all $i = 1,\ldots,6$, contradicting $e(\eta_i,\eta_j) = 1$ for $i \neq j$. Now define a symplectic basis in ${}_2\mathrm{Jac}(C)$ by

$$\varepsilon_1 = \eta_1,\ \varepsilon_2 = \eta_2+\eta_3,\ \varepsilon_3 = \eta_4+\eta_5,$$

$$\varepsilon_4 = \eta_1+\ldots+\eta_6,\ \varepsilon_5 = \eta_3+\eta_4+\eta_5+\eta_6,\ \varepsilon_6 = \eta_5+\eta_6.$$

To obtain an Aronhold set from a symplectic basis $\{\varepsilon_1,\ldots,\varepsilon_6\}$ we first reconstruct the points $\eta_1,\ldots,\eta_6$ from the above system of linear equations. Then we define a quadratic form q on ${}_2\mathrm{Jac}(C)$ by

$$q(\Sigma a_i\eta_i) = \Sigma a_i a_j.$$

It is immediately verified that the associated bilinear form of q is equal to e, and that q vanishes at 28 points of the form $\Sigma a_i\eta_i$ having exactly 1,2,5 or all 6 zero coefficients. Thus q defines an odd theta characteristic θ_1. Adding to this quadratic form the linear form $e(\ ,\eta_i)$ we obtain 6 more odd theta characteristics θ_i, $i = 2,\ldots,7$. We leave it to the reader to verify that the seven odd theta characteristics $(\theta_1,\theta_2,\ldots,\theta_7)$ form an Aronhold set which defines the symplectic basis $\{e_1,\ldots,e_6\}$ we started with.

Corollary. Let C be a smooth non-hyperelliptic projective curve of genus 3, and V be the Del Pezzo surface of degree 2 corresponding to C (the double cover of $\mathbb{P}_2$

branched along the canonical model of C). There are natural bijections between the following sets of 36•8! = $\#Sp(6,\mathbb{F}_2)$ elements:

(i) the set of Aronhold sets;

(ii) the set of level 2 structures on Jac(C);

(iii) the set of isomorphism classes of geometric markings of V;

(iv) the set of exceptional 7-configurations on V;

(v) the set of projective equivalence classes of self-associated point sets from S_8 such that the Hessian curve of the corresponding net of quadrics is isomorphic to C.

This gives another proof of Theorem 1 from the previous section.

We define the subvariety

$$(S_8)^{reg} \subset \hat{P}_3^8$$

of regular Cayley octads, i.e. ordered base-sets of regular nets of quadrics. Clearly

$$(S_8)^{reg} \subset \hat{S}_8 \subset \hat{P}_3^8,$$

where $\hat{S}_8$ is the variety of all Cayley octads. The projection b_8: $\hat{P}_3^8 \to P_3^8$ defines an open embedding:

$$(S_8)^{reg} \hookrightarrow S_8.$$

Lemma 5. $\Phi^{-1}((S_8)^{reg})$ consists of self-associated point sets $x = (x^1,\dots,x^8) \in \mathbb{P}_3^8$ satisfying the following conditions:

(i) all x^i are distinct;

(ii) every 4 points in x span $\mathbb{P}_3$;

(iii) x is not contained in a rational normal cubic curve.

<u>Proof</u>. Let $x \in (S_8)^{reg}$. We know already that (i) is satisfied. If x contains 4 coplanar points, the net $\mathfrak{N}(x)$ of quadrics through x cuts out a net of conics through these 4 points, unless one of the quadrics contains the plane. In the former case, we find 3 collinear points among the four coplanar points. Then the line joining them is contained in the base-set of $\mathfrak{N}(x)$. This contradicts the regularity of $\mathfrak{N}(x)$. In the latter case, $\mathfrak{N}(x)$ contains a quadric of corank 2. It is known that, together with (i), this implies that the Hessian curve has a singular point. This contradicts the regularity of $\mathfrak{N}(x)$. Thus (ii) is satisfied. If x is contained in a rational normal

curve, then this curve is contained in the base-locus of $\mathfrak{N}(x)$. Again this is contradictory. Thus (iii) is satisfied.

Conversely, assume (i) - (iii) are satisfied. First of all, $\mathfrak{N}(x)$ does not contain reducible quadrics. Otherwise, one of the irreducible components of such a quadric contains at least 4 points from x. Let B be the base-scheme of the net $\mathfrak{N}(x)$. It follows from the previous remark that each irreducible component of B is of dimension ≤ 1. Since two quadrics from $\mathfrak{N}(x)$ intersect along a curve of degree 4, each one-dimensional irreducible component of B is a curve of degree ≤ 3. By (iii), the case of degree 3 is impossible. If B contains a conic, the plane containing this conic is contained in a pencil of quadrics from $\mathfrak{N}(x)$. This is impossible by our first remark. Assume B contains a line ℓ. By (ii), ℓ contains at most 2 points from x. Let π be the plane containing ℓ and a point x^i from x not on ℓ. Each quadric from $\mathfrak{N}(x)$ cuts out in π the line ℓ and a line ℓ' passing through x^i. Since the lines ℓ' form at most a pencil, there exists a quadric in $\mathfrak{N}(x)$ which contains the plane π. As we saw above this is impossible. Thus the base scheme B of $\mathfrak{N}(x)$ is 0-dimensional. Condition (i) tells us that it is smooth. Then the Hessian curve of $\mathfrak{N}(x)$ is singular if and only if $\mathfrak{N}(x)$ contains a reducible quadric. Since the latter is impossible, $\mathfrak{N}(x)$ is regular.

Corollary 1.

$$\Phi^{-1}((S_8)^{reg}) \subset (\mathbb{P}_3^8)^s.$$

Corollary 2.

$$(S_8)^{reg} = S_8 \cap (P_3^8)^{un}.$$

Proof. A canonical root system in H_8 of type 3 is isomorphic to an affine root system of type E_7. As in the case of an affine root system of type E_8 isomorphic to a canonical root system of type 2 in H_9, this allows us to find the set of positive roots R_B^+ (see Chapter V). We obtain that this set consists of vectors:

$$\alpha(i,j) = e_i-e_j, \quad 1\leq i<j\leq 8,$$

$$\alpha(i,j;m) = e_i-e_j+m(2e_0-e_1-\ldots-e_8), \quad m > 0$$

$$\alpha(i,j,k,l;m) = e_0-e_i-e_j-e_k-e_l+m(2e_0-e_1-\ldots-e_8), \quad 1\leq i<j<k<l\leq 8, \quad m \geq 0.$$

Conditions (i) and (ii) from Lemma 5 are equivalent to the non-effectiveness of $\varphi_x^{1}(\alpha(i,j;0))$ and $\varphi_x^{1}(\alpha(i,j,k,l;0))$, where $\varphi_x:H_8 \to N(V(x))$ is the geometric marking corresponding to x. Let $x\in\Phi^{-1}((S_8\cap(P_3^8)^{un})$. To show that x is a regular Cayley octad, we have to verify condition (iii) of Lemma 5. By Theorem 3 of Chapter III, x is contained in the base-set B of a net $\mathfrak{N}$ of quadrics. Assume x lies on a rational normal cubic curve C. Then $C \subset B$, and for every point x^i from x, each quadric from $\mathfrak{N}$ contains the tangent line to C at x^i in its tangent plane at x^i. This implies that there exists a quadric Q from $\mathfrak{N}$ with a singular point at x^1. Let $[Q']$ be the class of the proper transform of this quadric in V(x), and $[E_2]$ be the class of the exceptional divisor blown up from x^2. Then

$$\varphi_x(\alpha(2,1;1)) = \varphi_x(2e_0-2e_1-e_3-\dots-e_8) = [Q']+[E_2]$$

is effective, hence x is nodal. This contradiction proves that

$$S_8 \cap (P_3^8)^{un} \subset (S_8)^{reg}.$$

To prove the reversed inclusion we have to show a regular Cayley octad is an unnodal point set. We already know that the roots $\alpha(i,j)$ and $\alpha(i,j,k,l;1)$ are not nodal with respect to the geometric marking φ_x. Suppose

$$\varphi_x{}^1(\alpha(i,j;m)) = [D]$$

for some $m \geq 1$ and an effective divisor D on V(x). Let

$$f: V(x) \dashrightarrow \mathbb{P}_2$$

be a rational map given by the net $\mathfrak{N}(x)$ of quadrics containing x. Since the base locus of $\mathfrak{N}(x)$ consists only of the point set x, f is a morphism. Then its fibres are quartic curves representing the class $(\varphi_x)_1(4e_0-e_1-\dots-e_8)$. Since

$$\alpha(i,j;m)\bullet(4e_0-e_1-\dots-e_8) = 0,$$

$D = f^{-1}(B)$ for some curve B. In particular, D intersects a general quadric from the net in $k = \deg(B)$ quartic curves with class $(\varphi_x)_1(4e_0-e_1-\dots-e_8)$. This implies that $k = m$, and one of the quartic curves must have a singular point at x^j. Since every such curve is a base curve of a pencil of quadrics from the net, one of the quadrics from this pencil must have a singular point at x^j. This contradicts condition (i). Suppose now that

$$\varphi_x{}^1(\alpha(i,j,k,l;m)) = [D]$$

for some $m \geq 1$ and an effective divisor D on V(x). A similar argument shows that D intersects a general quadric from $\mathfrak{N}(x)$ in m quartic curves and a conic passing through x^i, x^j, x^k, and x^l. This implies that these points lie on a plane. This contradicts condition (ii) of Lemma 5.

Let

$$f:(S_8)^{reg} \to \mathcal{A}_3(2)$$

be the morphism given by associating to a regular Cayley octad $(x^1,\dots,x^8)$ the Jacobian variety of the Hessian curve of the net $\mathfrak{N}(x)$ of quadrics through x and the level 2 structure defined by the Aronhold set cut out on the Steinerian curve by the chords $\langle x^i,x^8\rangle$.

Theorem 2. The projection $(x^1,\dots,x^8) \to (y^1,\dots,y^7)$ from $\mathbb{P}_3$ to $\mathbb{P}_2$ with center at x^8 induces an isomorphism p: $(S_8)^{reg} \to (P_2^7)^{un}$. Composition of this morphism with the morphism $(P_2^7)^{un} \to \mathfrak{M}_3(2)\backslash Hyp_3(2) \subset \mathcal{A}_3(2)$ from Theorem 1 is equal to $f:(S_8)^{reg} \to \mathcal{A}_3(2)$. In particular f induces an isomorphism:

$$(S_8)^{reg} \cong \mathfrak{M}_3(2)\backslash Hyp_3(2).$$

Finally, note the following curious fact.

Proposition 4. Let S be the Steinerian curve of a regular net of quadrics $\mathfrak{N}$ with base points $x^1,\dots,x^8$, embedded into $\mathbb{P}_3$ by the linear system $|K_C+\theta|$, where C is the Hessian curve of $\mathfrak{N}$ and θ is an even theta characteristic on C. Then the Steinerian curve of the net corresponding to C and the even theta characteristic θ_{ijkr} is obtained from S by the standard Cremona transformation with fundamental points at x^i,x^j,x^k,x^r or at the complementary set of points x^m,x^n,x^p,x^q.

<u>Proof</u>. The standard Cremona transformation is given by mapping $\mathbb{P}_3$ to itself via the linear system $|3H-2x^i-2x^j-2x^k-2x^r|$ of cubic surfaces with nodes at x^i,x^j,x^k,x^r Since the chords that connect these nodes belong to the base locus of this linear system, and since the hyperplane divisor H, when restricted to S, cuts out the linear system of the divisor, we have:

$$3(K_S+\theta)-(\theta_{ij}+\theta_{ik}+\theta_{ir}+\theta_{jr}+\theta_{jk}+\theta_{kr}) \sim$$

$$\sim 4K_S+\theta-(K_S+\theta+\theta_{ir}+\theta_{jr}+\theta_{kr}) \sim 3K_S-(\theta_{ijkr}+K_S) \sim K_S+\theta_{ijkr}.$$

Since $\theta_{ijkr} = \theta_{mnpq}$, this proves the proposition.

Remark 3. The previous proposition gives another proof that the Cremona action of $W_{3,8}$ on S_8 factors through $Sp(6,\mathbb{F}_2)$ (see Chapter VII). In fact, let $A_{ijkr} \in W_{3,8}$ be the reflection with respect to the root $e_0-e_i-e_j-e_k-e_r$. It is known that the kernel G of the canonical surjection $W_{3,8} \to Sp(6,\mathbb{F}_2)$ is a minimal normal subgroup containing the product $A_{1234}\cdot A_{5678}$. It follows from the proposition that this product induces the identity transformation of S_8. Hence G is contained in the kernel of the Cremona representation of $W_{3,8}$ in $Bir(S_8)$.

3. The varieties S_8 and $\mathfrak{M}_3(2)$.

In this section we want to extend an isomorphism from Theorem 2 to take into account hyperelliptic curves of genus 3. For this we construct a morphism from $\mathfrak{M}_3(2)$ to S_8 whose restriction to $\mathfrak{M}_3(2)\setminus\mathfrak{H}yp_3(2)$ is the inverse of the isomorphism $f:(S_8)^{reg} \to \mathfrak{M}_3(2)\setminus\mathfrak{H}yp_3(2)$ constructed in Theorem 2.

Let C be a nonsingular curve of genus 3 and $\theta_1,\dots,\theta_7$ an Aronhold set of odd theta characteristics on C. Denote by θ_0 the even theta characteristic which is not equal to any of the 35 even theta characteristics $\theta_{ijk} = \theta_i+\theta_j-\theta_k$. Let

$$\theta_{ij} = \theta_i+\theta_j-\theta_0$$

be the remaining odd theta characteristics, and

$$f_{ijk} = \theta_{ij}+\theta_{kj}+\theta_{ik}\in|K_C+\theta_0|$$

for $1\le i<j<k\le 7$.

The linear system $|K_C+\theta_0|$ maps C into $\mathbb{P}_3 = |K_C+\theta_0|^*$. This map is an embedding if and only if $h^0(\theta_0) = 0$. If $h^0(\theta_0) = 2$, then C is hyperelliptic and its image is a rational normal cubic. We will identify the divisor f_{ijk} with the corresponding plane F_{ijk} in $\mathbb{P}_3$. For every fixed pair of indices i,j the planes F_{ijk} belong to the pencil

$$l_{ij} = \theta_{ij}+|\theta_{kj}+\theta_{ik}| = \theta_{ij}+|\theta_j+\theta_i| \subset |K_C+\theta_0|.$$

This defines 21 lines in $\mathbb{P}_3$. All planes F_{ijk} with the same index i belong to a two-dimensional linear system. Let x^i be the common intersection point of all such planes. This defines 7 ordered distinct points $x^1,\dots,x^7$ in $\mathbb{P}_3$ such that each line l_{ij} passes through x^i and x^j. If C is not hyperelliptic, we easily see that the net of quadrics through $x^1,\dots,x^7$ has the curve C as its Hessian curve and the image of C under the map given by $|K_C+\theta_0|$ as its Steinerian curve. Let x^8 be the eighth base point of this net. Then the map

$$(C,(\theta_1,\dots,\theta_7))\in\mathfrak{M}_3(2)\setminus\mathfrak{H}yp_3(2) \quad\to\quad (x^1,\dots,x^8)\in(S_8)^{reg}$$

is the inverse of the map $f:(S_8)^{reg} \to \mathfrak{M}_3(2)\setminus\mathfrak{H}yp_3(2)$ from Theorem 2.

Now let us see what happens if C is a hyperelliptic curve. Assume first that

$|\theta_0| = g_2^1$ on C. Then the image of C by the linear system $|K_C+\theta_0|$ is a rational normal curve $\bar{C}$ in $\mathbb{P}_3$. The plane F_{ijk} cuts this curve at 3 points. It follows from our description of odd theta characteristics on hyperelliptic curves that these three points are necessarily the branch points of $C \to \bar{C}$. Thus the points $x^1,\dots,x^7$ are branch points, and the lines l_{ij} are the 21 chords joining them by pairs. Together with the eighth branch point x^8 we obtain a well-defined (up to projective equivalence) self-associated point set $(x^1,\dots,x^8)$. Since every odd theta characteristic θ_{ij} corresponds to a divisor p_i+p_j, where $\{p_i,p_j\}$ is a pair of ramification points of $C \to \bar{C}$, we see that the only odd theta characteristics p_i+p_8 unaccounted for must correspond to the odd theta characteristics θ_i from the Aronhold set. This gives an identification of an Aronhold set with 7 ordered branch points. Also we see that the level 2 structure defined by an Aronhold set in Proposition 3 coincides in this case with a branch-point level 2 structure on hyperelliptic curves defined in Chapter VIII. This extends the map

$$\mathfrak{M}_3(2)\setminus\mathfrak{H}yp_3(2) \to S_8$$

to the irreducible component of the hyperelliptic locus corresponding to branch--point level 2 structures. Note that a point set $(x^1,\dots,x^8)$ lying on a rational normal curve is nodal, since for every point x^i there exists a cone with a node at x^i.

Assume now that $h^0(\theta_0) = 0$. We use our old notation for theta characteristics on hyperelliptic curves. In this notation $\theta_i = \theta_{ij}$ for some $i,j\in\{1,\dots,8\}$, $i \neq j$, and $\theta_0 = \theta_{ijkr}$ for some subset $\{i,j,k,r\}$ of 4 elements from $\{1,\dots,8\}$. An example of an Aronhold set is a set

$$(\theta_1,\dots,\theta_7) = (\theta_{12},\theta_{34},\theta_{15},\theta_{25},\theta_{36},\theta_{37},\theta_{38})$$

with $\theta_0 = \theta_{1235}$. It contains 3 theta characteristics $\theta_{12},\theta_{15},\theta_{25}$ which define the even theta characteristic $\theta_{12}+\theta_{15}-\theta_{25}$ with $h^0(\theta_{12}+\theta_{15}-\theta_{25}) = 2$. The remaining theta characteristics are defined by a choice of a number 3 different from 1,2,5 and takiing θ_{3k} for all k different from 1,2,5 and 3. Permuting {1,2,3,5} and {4,6,7,8} separately and switching {1,2,3,5} with {4,6,7,8}, we obtain all Aronhold sets with $\theta_0 = \theta_{1235}$. Permuting all the numbers $\{1,\dots,8\}$ we obtain 35 sets of Aronhold sets with fixed θ_0. This shows that all Aronhold sets with $\theta_0=\theta_{ijkr}$ are accounted for in this way. Thus, without loss of generality we may assume that an Aronhold set is given as above.

Let $p_1,\dots,p_8$ be the eight ramification points of the g_2^1 on C. We observe that

$$f_{123} = f_{124} = f_{234} = p_1+p_2+p_3+2p_4+p_5,$$

$$f_{125} = f_{156} = f_{167} = f_{126} = f_{127} = f_{156} = f_{157} = f_{167} = p_4+2p_5+p_6+p_7+p_8.$$

This implies that the points x^1,x^3,x^4 (resp. x^2,x^5,x^6,x^7) are on a line l_1 (resp. l_2). We define the eighth point x^8 as the unique point on l_1 such that the double ratio of x^1,x^3,x^4,x^8 is equal to the double ratio of x^2,x^5,x^6,x^7. This gives us a well-defined self-associated point set $x = (x^1,\dots,x^8)$. Note that in this case x is not stable but semi-stable and belongs to one of the 35 two-dimensional boundary components of $\mathcal{S}\cap S^8$ determined by admissible partitions $\mathbf{d} = (2,2)$ of $n+1 = 4$. Each such component is naturally birationally isomorphic to $\mathbb{P}_1\times\mathbb{P}_1$, and the self-associated point set x constructed above belongs to the diagonal of the boundary component.

Summarizing, we have proven the following:

Theorem 3. There is a morphism

$$\varphi: \mathcal{M}_3(2) \to S_8$$

satisfying the following properties:

(i) Its restriction to $\mathcal{M}_3(2)\setminus\mathcal{H}yp_3(2)$ defines a map to $(S_8)^{reg}$ which is the inverse to the map f from Theorem 2.

(ii) Its restriction to the irreducible component $\mathcal{H}yp_3(2)^\circ$ of $\mathcal{H}yp_3(2)$ corresponding to branch point level 2 structures is the composition of the map $\mathcal{H}yp_3(2)^\circ \to (P_1^8)^{un}$, which is the inverse of the isomorphism constructed in Chapter VIII, and the Veronese map $v_3^8 : (P_1^8)^{un} \to S_8$.

(iii) It blows down the 35 remaining irreducible components of $\mathcal{H}yp_3(2)$ to the diagonals of the 2-dimensional boundary components of S_8.

Remark 4. It is known that every point of the boundary $\bar{\mathcal{A}}_3(2)\setminus\mathcal{A}_3(2)$ of $\bar{\mathcal{A}}_3(2)$ represents an isomorphism class of an abelian variety of dimension 1 or 2 with a level 2 structure. By **[Ig 2]** the blow-up of the boundary is a nonsingular algebraic variety $\hat{\mathcal{A}}_3(2)$ together with a projection $\hat{\mathcal{A}}_3(2) \to \bar{\mathcal{A}}_3(2)$ the fibre of which, over a point representing an abelian variety of dimension 2, is isomorphic to the variety itself. It is quite natural to guess that the birational map

$$\varphi^{-1}: S_8 \to \mathcal{M}_3(2) \subset \bar{\mathcal{A}}_3(2)$$

is "essentially" the Igusa blow-up. First, we have to replace S_8 by its proper inverse transform S_8' in $\hat{P}_3^8$ (containing but not equal to the variety $\hat{S}_8$ of the orbits of Cayley octads) to include the case of point sets $(x^1,\dots,x^8)$ with $x^i = x^j$ for some $i \neq j$, and then blow up further the diagonals of the boundary components parametrizing non-stable point sets in $\hat{P}_3^8$. The so obtained variety should coincide with $\hat{\mathcal{A}}_3(2)$. Recall that we have 64 different discriminant hypersurfaces $Z(\alpha)$ in S_8'. They correspond to

$28 = \binom{8}{2}$ components $Z_{ij} = Z(e_i-e_j)$, $1\le i<j\le 8$ parametrizing point sets with $x^j \to x^i$,

$35 = \frac{1}{2}\binom{8}{4}$ components $Z_{ijkr} = Z(e_0-e_i-e_j-e_k-e_r)$ parametrizing point sets with coplanar points x^i, x^j, x^r, and x^k, $1\le i<j<k<r\le 8$.

1 component Z_0 parametrizing point sets on a rational normal curve.

Note that since we consider only self-associated point sets

$$Z_{ijkr} = Z_{mnop}$$

for complementary sets of indices. Also note that the discriminant conditions

$$\alpha_{ij} = 2e_0-e_1-e_2-e_3-e_4-e_5-e_6-e_7-e_8-e_i+e_j$$

are reduced to one of the previous conditions or to the condition that x lies on a rational normal curve. It follows from the proof of Corollary 2 to Lemma 5 that all discriminant conditions are accounted for.

For every discriminant hypersurface Z different from Z_0 there is a natural birational map $Z(\alpha) \to \mathcal{M}_2(2)$ whose fibre over a point [C] representing a curve of genus 2 with a level 2 structure is isomorphic to Jac(C). For example, let $Z(\alpha) = Z_{78}$. Let $\hat{x}$ be a generic point set from Z_{78}. Then the net $\mathcal{N}(\hat{x})$ of quadrics through $\hat{x}$ contains a singular quadric Q with a node at x^7. The Hessian curve C of $\mathcal{N}(\hat{x})$ is a plane quartic curve with a node z(Q) representing Q. The lines $\langle x^k, x^7\rangle$, $k=1,\dots,6$, define 6 bitangents l_i of C passing through the node. They define an ordered set of ramification points of the g_1^2 on the normalization $\bar{C}$ of C. Thus the map $\hat{x} \to (\bar{C},(l_1,\dots,l_6))$ defines a birational map from Z_{78} to $\mathcal{A}_2(2)$. Note also that $\hat{x}$ defines canonically 2 points on $\bar{C}$ corresponding to the branches of the node of C. Conversely, given a plane nodal quartic curve and its 6 bitangents from the node, we can reconstruct a unique net of quadrics which will define a point set $\hat{x}$. Note that a nodal plane quartic C is the image of a hyperelliptic curve $\bar{C}$ of genus 2 under a map given by the linear system $|K_{\bar{C}}+p+q|$ for some points $p,q\in\bar{C}$, corresponding to the branches of the node of C. This shows that the fibres of the birational map $Z_{78} \to \mathcal{A}_2(2)$ are naturally isomorphic to the symmetric square $\bar{C}^{(2)}$ of $\bar{C}$, which in its turn is birationally isomorphic to $\mathrm{Jac}(\bar{C})$. If $Z(\alpha) \ne Z_{78}$, we define the map $Z(\alpha) \to \mathcal{A}_2(2)$, as the composition $Z_{78} \to \mathcal{A}_2(2)$ and the birational isomorphism $Z(\alpha) \to Z_{78}$ defined by the Cremona action of $Sp(6,\mathbb{F}_2)$ on S_8'.

4. Theta structures.

Here we recall some more definitions from the theory of theta functions. We refer to **[Mu 3, Mu 5]** for the details.

Let (A,L) be a ppav of dimension g such that L satisfies:

$$[-1]^*(L) \cong L.$$

For every $\eta \in {}_2A$ we have an isomorphism

$$t_\eta^*(L^2) \cong L^2.$$

The (level 2) theta group of L is the group

$$G(L) = \{(\eta,\varphi) | \varphi: t_\eta^*(L^2) \xrightarrow{\sim} L^2,\ \eta \in {}_2A\}$$

with multiplication law:

$$(\eta,\varphi)\circ(\eta',\varphi') = (\eta+\eta', \varphi \circ t_\eta^*(\varphi')).$$

We have a natural central extension of groups:

$$1 \to \mathbb{C}^* \to G(L) \to {}_2A \to 1,$$

where $\mathbb{C}^*$ is identified with the group $\mathrm{Aut}(L^2)$.

The commutator $[(\eta,\varphi),(\eta',\varphi')]$ of any two elements of G(L) belongs to the center, and induces (after composing it with log) the Weyl bilinear form

$$e^L: {}_2A \times {}_2A \to \mathbb{F}_2.$$

Thus, as an abstract group, G(L) isomorphic to the set

$$H(g) = \mathbb{C}^* \times \mathbb{F}_2^g \times \mathbb{F}_2^g$$

with the group law

$$(\lambda,\varepsilon,\varepsilon')\circ(\lambda',\eta,\eta') = ((-1)^{{}^t\varepsilon\cdot\eta' + {}^t\eta\cdot\varepsilon'}\lambda\lambda', \varepsilon+\eta, \varepsilon'+\eta').$$

The homomorphism $(\lambda,\varepsilon,\varepsilon') \to (\varepsilon,\varepsilon')$ defines an extension of groups:

$$1 \to \mathbb{C}^* \to H(g) \to \mathbb{F}_2^{2g} \to 1.$$

An isomorphism

$$\alpha: H(g) \xrightarrow{\sim} G(L)$$

is called a (level 2) theta structure on L.

By projection to $\mathbb{F}_2^{2g}$, α defines a level 2 structure

$$\bar{\alpha}: \mathbb{F}_2^{2g} \to {}_2A$$

on A. Two theta structures α and α' defining the same level 2 structure differ by a character

$$\chi: \mathbb{F}_2^{2g} \to \mu_2,$$

that is, $\alpha' \circ \alpha^{-1}$ is an isomorphism of H(g) given by

$$(\lambda,\varepsilon,\varepsilon') \to (\chi(\varepsilon,\varepsilon')\lambda,\varepsilon,\varepsilon').$$

A choice of an isomorphism $(A,L) \to (A_\tau,L_\tau)$ for some matrix $\tau \in \mathfrak{H}_g$ defines a theta structure. One verifies that it is independent of a choice of τ modulo the action of the subgroup

$$\Gamma_g(2,4) = \{M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g(2)\colon (AB)_0 \equiv (CD)_0 \equiv 0 \bmod 4\}.$$

In this way

$$\mathfrak{A}_g(2,4) = \mathfrak{H}_g/\Gamma_g(2,4)$$

parametrizes the isomorphism classes of ppav with theta structure (A,L,α).

Next we observe that G(L) has a natural linear representation in the space

$$V_A = H^0(A,L^2) \cong \mathbb{C}^{2^g}.$$

It is given by the formula

$$((x,\varphi)s)(a) = \varphi_a(s(a+x)),$$

where $a \in A$, $\varphi_a\colon t_x^*(L^2)_a = (L^2)_{a+x} \to (L^2)_a$.

On the other hand, H(g) acts linearly in the space $V(g) = \mathrm{Maps}(\mathbb{F}_2^g,\mathbb{C})$ by the formula

$$((\lambda,\varepsilon,\varepsilon')f)(v) = \lambda(-1)^{{}^t\varepsilon\cdot\varepsilon' + {}^tv\cdot\varepsilon'} f(v+\varepsilon), \quad v \in \mathbb{F}_2^g.$$

This defines a linear representation of H(g). One checks that it is an irreducible representation. It is known that all irreducible representations of dimension > 1 of the group H(g) are isomorphic. Hence, there exists an isomorphism of linear representations

$$\varphi_\alpha\colon V(g) \to V_A$$

which, by Schur's lemma, is defined uniquely up to a scalar multiplication.

We denote by $H_2(g)$ the subgroup of H(g) of elements $(\lambda,\varepsilon,\varepsilon')$ with $\lambda^2 = 1$. It is a nontrivial central extension

$$1 \to \mu_2 \to H_2(g) \to \mathbb{F}_2^{2g} \to 1.$$

By restriction, V(g) is a linear representation of $H_2(g)$.

For every $v \in \mathbb{F}_2^g$ let

$$Z_v \in V(g)$$

be the characteristic function of the subset $\{v\}$. We have

$$(\lambda,\varepsilon,\varepsilon')Z_v = \lambda(-1)^{{}^t(\varepsilon+v)\cdot\varepsilon'} Z_{v+\varepsilon}.$$

After fixing an order in $\mathbb{F}_2^g$, we obtain a canonical basis

$$\{Z_v\}_{v\in\mathbb{F}_2^g}$$

of $V(g)$. Under the isomorphism φ_α this basis corresponds to the basis

$$\{\vartheta\begin{bmatrix}\varepsilon\\0\end{bmatrix}(2z;2\tau)\}_{\varepsilon\in\mathbb{F}_2^g}$$

of V_{A_τ}.

Let

$$Th_{A_\tau}: A_\tau \to \mathbb{P}(V_{A_\tau}^*)$$

be the map

$$z \to (\ \dots\ \vartheta\begin{bmatrix}\varepsilon\\0\end{bmatrix}(2z;2\tau)\dots)$$

given by the linear system $|L_\tau^2|$ and the basis $\{\vartheta\begin{bmatrix}\varepsilon\\0\end{bmatrix}(2z;2\tau)\}_{\varepsilon\in\mathbb{F}_2^g}$. Composing this map with the unique $G(L_\tau)$-$H(g)$-equivariant isomorphism

$$({}^t\varphi_\alpha): \mathbb{P}(V_{A_\tau}{}^*) \to \mathbb{P}(V(g)^*),$$

obtained from φ_α by transposing and passing to the projectivization, we get a map

$$Th_{A_\tau,\alpha}: A_\tau \to \mathbb{P}(V(g)^*).$$

Note that the group ${}_2A_\tau = G(L)/\mathbb{C}^*$ acts by translations on A_τ and the group $\bar{H}_2(g) = H(g)/\mathbb{C}^* = \mathbb{F}_2^{2g}$ acts on $\mathbb{P}(V(g)^*)$ by the projectivization of the dual representation of $H_2(g)$ on $V(g)^*$. These actions are compatible in the sense that

$$Th_{A_\tau,\alpha}(x+\eta) = \bar{\alpha}(\eta)Th_{A_\tau,\alpha}(x) \text{ for any } \eta\in{}_2A_\tau,\ x\in A_\tau.$$

Taking the value of $Th_{A_\tau,\alpha}$ at the origin allows us to define a map

$$Th^{(2)}: \mathcal{A}_g(2,4) \to \mathbb{P}(V(g)^*) \cong \mathbb{P}_{2^g-1},$$

$$(A_\tau,\alpha) \to Th_{A_\tau,\alpha}(0).$$

Note that $\Gamma_g(2,4)$ is a normal subgroup of $\Gamma_g(2)$ with quotient isomorphic to $\mathbb{F}_2^{2g}$. Thus there is a natural action of this group on $\mathfrak{A}_g(2,4)$ with quotient space isomorphic to $\mathfrak{A}_g(2)$. The map $Th^{(2)}$ is an equivariant map with respect to this action. Now there is more symmetry involved. Let

$$\Gamma_g/\Gamma_g(2,4) = \tilde{S}p(2g,\mathbb{F}_2).$$

Using an isomorphism $\Gamma_g/\Gamma_g(2) \cong Sp(2g,\mathbb{F}_2)$, we have an extension of groups

$$1 \to \mathbb{F}_2^{2g} \to \tilde{S}p(2g,\mathbb{F}_2) \to Sp(2g,\mathbb{F}_2) \to 1.$$

The group $\tilde{S}p(2g,\mathbb{F}_2)$ acts naturally on $\mathfrak{A}_g(2,4)$ with quotient isomorphic to $\mathfrak{A}_g$. Using an isomorphism

$$\tilde{S}p(2g,\mathbb{F}_2) \cong \{\sigma\in Aut(H(g))\mid \sigma|Cent(H(g)) = 1\}$$

under which $\mathbb{F}_2^{2g}$ is mapped onto the subgroup of inner automorphisms, and applying Schur's lemma, we see that the same group acts on $\mathbb{P}(V(g)^*)$. In this way we obtain a $\tilde{S}p(2g,\mathbb{F}_2)$-equivariant map

$$Th^{(2)}: \mathfrak{A}_g(2,4) \to \mathbb{P}(V(g)^*).$$

Note that the projective linear representation of $\tilde{S}p(2g,\mathbb{F}_2)$ on $\mathbb{P}(V(g)^*)$ comes from a linear representation of a certain extension of $\tilde{S}p(2g,\mathbb{F}_2)$ described for example in **[Gri 2]**.

Lemma 6. Let $Sym^k(V(g))$ be the k-th symmetric power of the linear representation $V(g)$ of $H_2(g)$, and $(Sym^k(V(g)))^{H_2(g)}$ denote the subspace of $H_2(g)$-invariant elements. Then

(i) $(Sym^k(V(g)))^{H_2(g)} = \{0\}$ for $k \le 4$;

(ii) $Sym^4(V(g)) = \oplus(Sym^4(V(g)))_\chi$, where $\chi\in X(H_2(g)) = Hom(H_2(g),\mathbb{C}^*)$

$(S^4(V(g)))^{H_2(g)}$ has a basis consisting of $\frac{1}{3}(2^g+1)(2^{g-1}+1)$ polynomials

$$P_I = \sum_{v\in\mathbb{F}_2^g} Z_v Z_{v+v'} Z_{v+v''} Z_{v+v'+v''}, \quad I = \{0,v',v'',v'+v''\} \subset \mathbb{F}_2^g.$$

Proof. This can be easily verified (cf. **[vG 2]**).

We denote by $\mathfrak{I}(g)$ the space $(Sym^4(V(g)))^{H_2(g)}$. A choice of an order in the spanning set $\{P_I\}$ of $\mathfrak{I}(g)$ defines a $H_2(g)$-equivariant map:

$$\mathbb{P}(V(g)^*) \to \mathbb{P}(\mathfrak{I}(g)^*).$$

Its composition with the map

factors to a $Sp(2g,\mathbb{F}_2)$-equivariant map

$$Th^{(4)}: \mathfrak{A}_g(2) \to \mathbb{P}(\mathfrak{I}(g)^*).$$

The following lemma explains our notation for the maps $Th^{(2)}$ and $Th^{(4)}$.

Lemma 7. Let

$$X_\varepsilon(z;\tau) = \vartheta\begin{bmatrix}\varepsilon\\0\end{bmatrix}(2z;2\tau), \quad \varepsilon\in\mathbb{F}_2^g.$$

Then

$$\theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(z;\tau)^2 = \sum_{\sigma\in\mathbb{F}_2^g} (-1)^{{}^t\sigma\cdot\varepsilon'} X_{\varepsilon+\sigma}(0;\tau)X_\sigma(z;\tau).$$

Proof. See **[Ig 1]**, IV.1,Theorem.2.

It follows from this formula that under the identification of the spaces $V(g)$ and V_{A_τ} the composition of $Th^{(2)}$ with the Veronese map is given by the squares of even theta constsants $\theta\begin{bmatrix}\varepsilon\\\varepsilon'\end{bmatrix}(0;\tau)^2$ (considered as modular forms of weight 1 with respect to $\Gamma(2,4)$). Also the map $Th^{(4)}$ is given by the fourth powers of even theta constants (considered as modular forms of weight 2 with respect to $\Gamma(2)$). In fact, one can show that these forth powers span the space $\mathfrak{I}(g)$ (see **[vG 2]** for details).

Example. Assume $g = 2$. Then the map

$$Th^{(2)}: \mathfrak{A}_2(2,4) \to \mathbb{P}(V(2)^*) \cong \mathbb{P}_3$$

is a birational isomorphism, and the map

$$Th^{(4)}: \mathfrak{A}_2(2) \to \mathbb{P}(\mathfrak{I}(2)^*) \cong \mathbb{P}_4$$

is a birational isomorphism onto a quartic hypersurface isomorphic to a level 2 modular quartic 3-fold V_4. In particular, we see that

$$V_4 \cong \mathbb{P}(V(2)^*)/H(2)$$

is rational. By brute computation, it can be verified that $\mathbb{P}(V(3)^*)/H(3)$ is rational (D. Ortland). It is not known whether the quotient $\mathbb{P}(V(g)^*)/H(g)$ is rational for $g \geq 4$. The proof of the corresponding statement in **[Bog]** is false.

Proposition 5. Assume $g = 3$. Then

$$Th^{(2)}: \mathfrak{A}_3(2,4) \to \mathbb{P}(V(3)^*) \cong \mathbb{P}_7$$

is a birational map onto a hypersurface of degree 16, and

$$Th^{(4)}: \mathfrak{A}_3(2) \to \mathbb{P}(\mathcal{T}(3)^*) \cong \mathbb{P}_{14}$$

is a birational map onto its image.

Proof. The first assertion is proven in **[vG-vdG]** (cf.**[Co 8]**, p.487). The second one easily follows from it.

5. Kummer-Wirtinger varieties.

The Kummer-Wirtinger variety of a ppav A of dimension g is defined as the quotient Kum(A) of A by the involution $x \to -x$. Fix a theta structure on A, and choose a basis $\{X(z,\tau)\}$ in the space $H^0(A,L^2)$ corresponding to a basis $\{Z_v\}$ in the space V(g) as in the previous section. Assume that A is indecomposable i.e. is not equal to a product of abelian varieties of smaller dimension. Then the corresponding map

$$A \to \mathbb{P}(H^0(A,L^2)^*) \cong \mathbb{P}(V(g)^*)$$

factors through Kum(A) and defines an embedding:

$$i: \mathrm{Kum}(A) \hookrightarrow \mathbb{P}(V(g)^*)$$

which is ${}_2A$-$\bar{H}_2(g)$-equivariant (the first group acts on A, and hence on Kum(A), by translations by points of order 2, and the second acts on $\mathbb{P}(V(g)^*)$ via its linear representation in V(g)). It is easy to compute the degree of i(Kum(A)) and get

$$\deg(i(\mathrm{Kum}(A)) = \tfrac{1}{2}(2\Theta)^g = 2^{g-1}g!.$$

Note also that Kum(A) has 2^{2g} singular points locally isomorphic to the vertex of the affine cone over the Veronese variety $v_g(\mathbb{P}^{g-1})$.

Proposition 6. Assume $g = 2$. Then i(Kum(A)) is given by an equation:

$$\alpha_0(Z_{00}^4+Z_{01}^4+Z_{10}^4+Z_{11}^4)+2\alpha_1(Z_{00}^2Z_{10}^2+Z_{01}^2Z_{11}^2)+2\alpha_2(Z_{00}^2Z_{01}^2+Z_{10}^2Z_{11}^2)+$$
$$+2\alpha_3(Z_{00}^2Z_{11}^2+Z_{10}^2Z_{01}^2)+4\alpha_4 Z_{00}Z_{01}Z_{10}Z_{11} = 0,$$

where the coefficients $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4$ satisfy the equation:

$$(*) \qquad \alpha_0^3-\alpha_0(\alpha_1^2+\alpha_2^2+\alpha_3^2-\alpha_4^2)+2\alpha_1\alpha_2\alpha_3 = 0.$$

Conversely, every quartic surface given by such an equation is isomorphic to a Kummer surface Kum(A), provided the coefficients α_i satisfy (*) and also the following 15 inequalities:

$$\alpha_0 \neq \alpha_i, \quad \alpha_0 \neq \pm\alpha_i \pm \alpha_4 - \alpha_j - \alpha_k, \quad i = 1,2,3, \quad j,k \in \{1,2,3\} \setminus \{i\}, \quad j \neq k.$$

Proof. We know that i(Kum(A)) is an $\bar{H}_2(2)$-invariant quartic surface. This implies that its equation can be given by a quartic polynomial belonging to some eigensubspace $Sym^4(V(2)^*)_\chi$ with respect to the action of $H_2(2)$. By Lemma 1 this equation is as above if the eigenvalue χ is equal to 1. If $\chi \neq 1$, it is easy to verify that the fixed line of some involution $\sigma \in \bar{H}_2(2)$ must lie on the surface. However, each such a line meets the surface in a set of 4 points $\bar{y}$, the images of the points $y \in A$ such that $2y = x$, $t(y) = y$, where $x \in {}_2Jac(A)$ corresponds to σ, and t is the translation of A corresponding to σ (cf. **[Co 1]**, p.103).

Finally, the condition on the coefficients is necessary and sufficient for the quartic to have a node (and hence 16 of them since the quartic is $H_2(2)$-invariant) (see **[Jes]**, p.99).

Now we observe that the cubic hypersurface in $\mathbb{P}_4$ given by the equation

$$\alpha_0^3 - \alpha_0(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2) + 2\alpha_1\alpha_2\alpha_3 = 0$$

is isomorphic to the Segre cubic primal V_3. In the notation of Lecture I, the projective transformation defining the isomorphism is given by the formulae:

$$\alpha_0 = t_0, \ \alpha_1 = t_0 - 2t_2, \ \alpha_2 = t_0 - 2t_1, \ \alpha_3 = -t_0 + 2t_3 - 2t_4,$$
$$\alpha_4 = -2t_0 + 2t_1 + 2t_2 - 2t_3 - 2t_4.$$

This shows that when (A,τ) varies in $\mathfrak{a}_2(2,4)$, the coefficiens $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ define a map:

$$\mathfrak{a}_2(2,4) \to \mathbb{P}(S^4(V(2)^*)^{H(2)}) = \mathbb{P}(\mathfrak{T}(2)^*) \subset |\mathcal{O}_{\mathbb{P}(V(2))}(4)|$$

which factors through a map

$$Th^{(4)*}: \mathfrak{a}_2(2) \to \mathbb{P}(\mathfrak{T}(2)^*).$$

Recall that earlier we defined a map:

$$Th^{(4)}: \mathfrak{a}_2(2) \to \mathbb{P}(\mathfrak{T}(2))$$

given by the invariant quartic polynomials corresponding to the coefficients $\alpha_0,\dots,\alpha_4$. The image of the first map is projectively isomorphic to the Segre cubic primal V_3, and the image of the second map is isomorphic to its dual hypersurface which is a level 2 quartic modular 3-fold V_4. Thus the cubic equation satisfied by the coeffients of the equation of Kum(A) expresses the condition that Kum(A) is the inverse image of a tangent hyperplane of V_4 under a map:

$$f:\mathbb{P}(V(2)) \to \mathbb{P}(\mathcal{J}(2)) \cong \mathbb{P}_4$$

given by the linear system of $H_2(2)$-invariant quartic polynomials. Note that the image of a Kummer surface Kum(A) under this map is isomorphic to the quotient space

$$K = \mathrm{Kum}(A)/H(2).$$

The action of $H_2(2)$ on Kum(A) corresponds to the action of the group ${}_2A$ on A by translations. The quotient $A/{}_2A$ is canonically isomorphic to the image of A under the isogeny $x \to 2x$, hence is isomorphic to A. This easily implies that

$$K \cong \mathrm{Kum}(A).$$

Thus we obtain that

$$\mathrm{Kum}(A) = f^{-1}(H) \cong V_4 \cap H$$

for some hyperplane H in $\mathbb{P}_4$. The variety V_4 has 15 double lines, hence $H \cap V_4$ has 16 singular points if H is tangent to V_4 at some nonsingular point $a \in V_4$. This gives another explanation why H must be a tangent plane to V_4, and hence the coefficients of the equation of a Kummer surface must satisfy a cubic equation.

Now recall that the nonsingular points of V_4 parametrize principally polarized abelian surfaces with level 2 structure. It is natural to ask whether the tangent hyperplane H to V_4 at a nonsingular point $a \in V_4$ cuts out the Kummer surface of the abelian surface from the isomorphism class defined by the point a. The answer is yes. We refer to **[Co 1]**, p.141, and **[vdG]** for verification of this fact.

Let us now see some analogs of the previous facts in the case g = 3.

Proposition 7. Let A be a principally polarized abelian variety of dimension 3 with a fixed theta structure. Then i(Kum(A)) is contained in the singular locus of a unique hypersurface C_4 given by an $H_2(3)$-invariant quartic polynomial L_4 on V(3):

$$C_4 : L_4 = \Sigma \alpha_i P_i = 0.$$

Moreover, if A = Jac(X), where X is a nonsingular nonhyperelliptic curve of genus 3, then $i(A) = \mathrm{Sing}(C_4)$.

Proof. The embedding of Kum(A) into $\mathbb{P}_7 = \mathbb{P}(H(3))$ is given by the subsystem of $|2\Theta|$ corresponding to symmetric divisors, i.e. their corresponding theta functions on $\mathbb{C}^3$ are even. This easily implies that the restriction of the complete linear system of cubic hypersurfaces in $\mathbb{P}_7$ on K = i(Kum(A)) cuts out on K a linear system of symmetric divisors from $|6\Theta|$. Its dimension is equal to 111. This immediately implies that there is a cubic hypersurface vanishing on K. In fact, since K is projectively normal in $\mathbb{P}_7$, there are 8 linearly independent cubic polynomials vanishing on K. Suppose that a cubic polynomial

$$F = \Sigma a_{ijk} Z_{v_i} Z_{v_j} Z_{v_k}$$

vanishes on K. Since K is $\bar{H}_2(3)$-invariant, and $\bar{H}_2(3)$ contains projective transformations which change the sign of each coordinate Z_v, it is easy to see that one such F defines another where the indices v_i, v_j, v_k corresponding to a nonzero coefficient a_{ijk} add up to some $v \in \mathbb{F}_2^3$. This easily implies that we have 8 cubic polynomials F_v vanishing on K, each corresponding to a vector $v \in \mathbb{F}_2^3$. The group $H_2(3)$ permutes these polynomials. Further computations (see **[Co 1]**, p.105) show that each F_v must be the Z_v-partial of the $H_2(3)$-invariant quartic polynomial

$$L_4 = \Sigma Z_v F_v = \Sigma \alpha_I P_I.$$

This proves the existence and uniqueness (up to a scalar factor) of a $H_2(3)$-invariant quartic polynomial containing K in its singular locus. On the other hand, it is shown in **[N-R]**, that the moduli space $SU_X(2)$ of semi-stable rank 2 vector bundles with trivial determinant on a nonsingular nonhyperelliptic curve X of genus 3 is naturally isomorphic to a quartic hypersurface in $\mathbb{P}_7$ with singular locus equal to Kum(Jac(X)). Note that this isomorphism is $H_2(3)$-equivariant, where $H_2(3)$ acts on $SU_X(2)$ via the action of ${}_2Jac(X)$ defined by the tensor product $E \to L_\eta \otimes E$, where L_η is the line bundle associated to $\eta \in {}_2Jac(X)$. This shows that $SU_X(2)$ must be equal to our quartic C_4 given by $L_4 = 0$.

We propose to call the quartic hypersurface corresponding to an indecomposable ppav of dimension 3

$$C_4: L_4 = 0$$

the Coble quartic of A.

Its equation looks like the equation of a Kummer surface:

$$\alpha_0(Z_0^4+\ldots+Z_7^4)+2\alpha_1(Z_0^2Z_1^2+Z_2^2Z_6^2+Z_3^2Z_5^2+Z_4^2Z_7^2)+\ldots+2\alpha_7(Z_0^2Z_7^2+Z_1^2Z_4^2+Z_2^2Z_5^2+Z_3^2Z_6^2)+$$
$$+4\alpha_8(Z_{00}Z_4Z_2Z_3+Z_7Z_1Z_5Z_6)+\ldots+4\alpha_{14}(Z_0Z_4Z_5Z_6+Z_7Z_1Z_2Z_3) \quad = \quad 0,$$

where $Z_0 = Z_{000}$, $Z_1 = Z_{100}$, $Z_2 = Z_{010}$, $Z_3 = Z_{001}$, $Z_4 = Z_{011}$, $Z_5 = Z_{101}$, $Z_6 = Z_{110}$, $Z_7 = Z_{111}$.

Proposition 8. Let $\varepsilon \in H(3)/\mathbb{C}^* \cong \mathbb{F}_2^6$, $\varepsilon \neq 1$. Then the set of fixed points of ε in $V(3)$ is the union of two disjoint linear subspaces H_ε^+ and H_ε^- of dimension 3. Let C_4 be the Coble quartic of a ppav A. The intersections

$$C_4 \cap H_\varepsilon^{\pm}$$

are isomorphic to a Kummer surface K. Making the identification of ε with the corresponding 2-torsion point of ${}_2A$, K is isomorphic to the Kummer surface of the Prym variety of (A,ε).

Proof. This together with the definition of the Prym variety can be found in [vG 3].

Corollary. The coefficients $\alpha_0,\ldots,\alpha_{14}$ of the Coble quartic satisfy 63 cubic equations.

Thus, as in the case of genus 2, the coefficients $\alpha_0,\ldots,\alpha_{14}$ of the Coble quartic define a map

$$Th^{(4)*}: \mathfrak{a}_3(2) \to \mathbb{P}(\mathfrak{T}(3))$$

whose image lies in the intersection of 63 cubics. This map is injective on the complement of the hyperelliptic locus.

Remark 5. If A = Jac(X), where X is a hyperelliptic curve, then the Coble quartic is equal to a double quadric (see [vG 3]). Compare this with the fact that in the case where A is a decomposable ppav of dimension 2 the map $Kum(A) \to \mathbb{P}_3$ is a double cover onto a quadric. Note that the cubic hypersurface defined by the equation (*) from Proposition 7 has 10 nodes:

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) \quad = \quad (1,1,1,0),(1,1,1,-1,0),(1,1,-1,1,0),(1,-1,1,1,0),$$
$$= (0,0,0,1,\pm1),(0,0,1,0,\pm1),(0,1,0,0,\pm1).$$

The corresponding quartic surface is a double quadric. Recall that the nodes of the Segre cubic primal correspond to the 10 Humbert surfaces in $\mathfrak{A}_2(2)$ parametrizing decomposable abelian surfaces. It seems plausible, though we have not checked this, that the image of each decomposable abelian surface in $\mathbb{P}_3$ is given by the corresponding quadric equation.

6. Cayley dianode surfaces.

We have already defined these surfaces in Chapter VII as the ramification divisor of the map $V(x) \to \mathbb{P}(1,1,1,2)$, where $x \in (\mathbb{P}_2^7)^{gen}$. Here we interrelate these surfaces with Kummer-Wirtinger 3-folds.

Let $K = i(\mathrm{Kum}(A)) \subset \mathbb{P}_{2^g-1}$, where A is the Jacobian variety of a nonhyperelliptic curve C of genus g. For every $\varepsilon \in {}_2A$ let $\Theta_\varepsilon = \Theta + \varepsilon$ denote the translation of the Poincare divisor Θ of A. The image of Θ_ε under the map $i: A \to \mathbb{P}_{2^g-1}$ given by $|2\Theta|$ is a trope $T(\varepsilon)$ of K, i.e. a subvariety of K which is cut out by a hyperplane everywhere tangent to K. It follows from Chapter VIII that each of the 2^{2g} tropes passes through at least $2^{g-1}(2^g-1)$ singular points of K.

Clearly each trope $T(\varepsilon)$ is isomorphic to the quotient $\Theta/(\tau)$, where τ is the involution $x \to -x$ of A.

Assume now that $g = 3$. Let $\mathfrak{N}$ be a net of quadrics through 8 points $x^1,\dots,x^8$ in $\mathbb{P}_3$ such that C is equal to its Hessian curve. Denote by

$$f: V \to \mathbb{P}_2 = |\mathfrak{N}|^*$$

the elliptic fibration defined on the blowing-up V of $x^1,\dots,x^8$. Fix a section of f defined by the exceptional divisor E_8 blown up from x^8. Let Δ be the discriminant curve of f, i.e. the curve in $\mathbb{P}_2$ of points z such that the fibre $f^{-1}(z)$ is singular. It is easy to see that Δ is isomorphic to the dual curve of C. The open subset $f^{-1}(\mathbb{P}_2 \setminus \Delta)$ is an abelian scheme over $\mathbb{P}_2 \setminus \Delta$. Let D be the closure in X of the set of non-trivial points of order 2 in fibres of this scheme. The projection

$$\pi: D \to \mathbb{P}_2$$

is a 3-sheeted cover branched along Δ.

Now we recall from the geometric proof of Torelli's theorem for curves (see [G-H]) that the Gauss map from Θ to $\mathbb{P}_2$ factors through $T = \Theta/(\tau)$ and defines a 3-sheeted cover of $\mathbb{P}_2$ ramified along the dual curve of C. This suggests that D and T are birationally isomorphic.

Theorem 4. Both of the varieties D and T are birationally isomorphic to a Cayley dianode surface R corresponding to the point set $(x^1,\dots,x^7)$.

Proof. Let us show first that D and R are isomorphic. Let X' be the blow-up of $x^1,\dots,x^7$ and

$$\varphi: X' \to \mathbb{P}_6$$

be the map given by the linear system $|-K_{X'}|$ represented by quartics through $x^1,\dots,x^7$. It is easy to see that each such quartic can be written in the form:

$$F = \lambda F_4 + \Sigma a_{ij} Q_i Q_j = 0,.$$

where $Q_1=0, Q_2=0, Q_3=0$ are the quadrics spanning the net $\mathfrak{N}$ and $F_4=0$ is a quartic not passing through x^8 (called a dianode quartic, see **[Ca]**). The fibres of the elliptic fibration $f: X \to \mathbb{P}_2$ are the base curves of pencils of quadrics from $\mathfrak{N}$. Thus each quartic $F = 0$ cuts out a divisor of degree 2 on the fibres of f, and hence φ induces a map φ' of degree 2 from X to the image Y and blows down to a point the exceptional divisor E_8. Obviously the ramification divisors of φ and φ' are birationally isomorphic. The first divisor is a Cayley dianode surface, the second is the surface D. Note that $\varphi(Y)$ is isomorphic to $\mathbb{P}(1,1,1,2)$ embedded naturally into $\mathbb{P}_6$. The divisor E_8 is blown down to the vertex of $\mathbb{P}(1,1,1,2)$. The situation is quite similar to the case of the bi-anticanonical map of a Del Pezzo surface of degree 1.

Now let us see that D is birationally isomorphic to T. Obviously

$$T \cong W_2/(\tau'),$$

where $W_2 \subset \mathrm{Pic}^2(C)$ is the hypersurface of effective divisors of degree 2, and τ' is the involution of $\mathrm{Pic}^2(C)$ given by $d \to K_C - d$. In the Steinerian embedding of C a canonical divisor can be represented by 4 nodes $q_1,\dots,q_4$ of the singular quadrics $Q_1,\dots,Q_4$ in some pencil ℓ of $\mathfrak{N}$. Let $E(\ell)$ be the base curve of this pencil. We assume that ℓ is such that $E(\ell)$ is nonsingular. Let us show that each of the three nontrivial points $a_i(\ell)$ of order 2 naturally corresponds to a pair of opposite edges of tetrahedron formed by the points $q_1,\dots,q_4$.

Let l_i denote the line of the quadric Q_i passing through the point x^8 and let $D_i=x^8+p_i$ denote the divisor of degree 2 that it cuts out on $E(\ell)$. A hyperplane in $\mathbb{P}_3$ cuts out a divisor H of degree 4 on $E(\ell)$ such that $2D_i \sim H$. Since we are assuming that $E(\ell)$ is nonsingular, the quadrics Q_i do not contain a common line. Thus the divisors D_i are all distinct, and since they all contain the point x^8, they must also lie in distinct divisor classes. Thus the divisors of degree zero $\varepsilon_i = D_i - D_4$ for i =

1,2,3, and $\varepsilon_4 = 0$ are all distinct and of order 2. We also have that $\varepsilon_i+\varepsilon_j \sim \varepsilon_k+\varepsilon_r$ for distinct indices.

Now consider the planes:

$$H_{ij} = \langle q_i, q_j, x^8\rangle = \langle l_i, l_j\rangle,$$

$$H_{kr} = \langle q_k, q_r, x^8\rangle = \langle l_k, l_r\rangle,$$

which contain opposite edges of the tetrahedron spanned by the q_i, and consider the divisors that they cut out on $E(\ell)$:

$$H_{ij}\cap E(\ell) = x^8+p_i+p_j+p,$$

$$H_{kr}\cap E(\ell) = x^8+p_k+p_r+p'.$$

Since $D_i+D_j = 2x^8+p_i+p_j \sim H+\varepsilon_i+\varepsilon_j$ we find that

$$p \sim H-x^8-p_i-p_j \sim x^8+\varepsilon_i+\varepsilon_j.$$

Similarly,

$$p' \sim x^8+\varepsilon_k+\varepsilon_r,$$

which shows that p and p' are the same points on $E(\ell)$. Moreover, we have that

$$2p \sim 2x^8,$$

which shows that p is one of the points $a_i(\ell)$ of order 2 on $E(\ell)$. This ends the proof of the theorem.

Remark 6. It is easy to see that the surface T is isomorphic to the branch divisor of $X' \to \mathbb{P}(1,1,1,2)$. It is a hypersurface of degree 6 in $\mathbb{P}(1,1,1,2)$ isomorphic to a canonical model of a nonsingular regular surface of general type with $p_g = 3$ and $K^2 = 3$. It has 28 nodes, and the projection map $R \to T$ is a resolution of singular points of T whose exceptional curves are the proper inverse transforms in R of the lines joining pairs of points from $\{x^1,\dots,x^8\}$. The projection $\bar{R}$ of R in $\mathbb{P}_3$ is a surface of degree 6 with 7 triple points at $x^1,\dots,x^7$. This is another birational model of the surfaces R, T, and D. Note that $\bar{R}$ may be obtained directly from the theta divisor $\Theta \subset Jac(C)$ by mapping it to $\mathbb{P}_3$ via a subsystem of $|4\Theta|_\Theta|$ that depends on a choice of an Aronhold set on C. The seven singular points are ordered via the Aronhold set. This gives another way to reconstruct a point set $(x^1,\dots,x^7)\in P_3^7 \cong P_2^7$ without appealing to the fact that the Abelian variety is the Jacobian of some curve. We refer to **[Sch 1]** for the corresponding construction or to an account of his work in **[Co 1]**, S47.

7. Göpel functions.

In this section we construct a $Sp(6,\mathbb{F}_2)$-equivariant map

$$P_2^7 \to \mathbb{P}(\mathfrak{T}(3))$$

whose image is contained in the intersection of 63 cubic hypersurfaces, and is an analog of the Segre cubic primal in the case g = 3. It is asserted in **[Co 1]** that there is a commutative diagram:

$$\begin{array}{ccc} P_2^7 & \rightrightarrows & \mathfrak{A}_3(2)\backslash\overline{\mathfrak{Syp}_3(2)} \\ & \searrow \quad \swarrow & \\ & \mathbb{P}(\mathfrak{T}(3)). & \end{array}$$

However, we postpone the proof of this until we understand it better.

Let Q be the root lattice of type E_7 isomorphic to the root lattice Q_B of the canonical root basis of type 2 in H_7. Recall that in section 1 we constructed a natural isomorphism of symplectic spaces:

$$\varphi: \mathbb{F}_2^6 \to \bar{Q}',$$

where $\bar{Q}' = (Q/2Q)/\text{Radical}$. We denote by $\bar{v}$ the image of $v \in Q$ in $\bar{Q}'$.

Lemma 8. For every $\varepsilon \in \mathbb{F}_2^6 \backslash \{0\}$, there exists a unique positive root $\alpha \in Q$ such that

$$\varphi(\varepsilon) = \bar{\alpha}.$$

The correspondence $\varepsilon \to \alpha$ is a bijection between the set $\mathbb{F}_2^6\backslash\{0\}$ and the set of positive roots in Q which preserves the orthogonality relations.

Proof. Note that both sets consist of 63 elements. Thus it is enough to verify that

$$\bar{\alpha} = \bar{\beta} \Leftrightarrow \alpha = \beta$$

for any two positive roots $\alpha,\beta \in Q$. Assume $\bar{\alpha} = \bar{\beta}$, then under the canonical map

$$O(Q) \to Sp(6,\mathbb{F}_2)$$

the images of the reflections s_α and s_β are the same. By Lemma 3 this implies that $s_\alpha = s_\beta$, hence $\alpha = \beta$. Suppose that α and β correspond to orthogonal vectors ε and ε' respectively. Then $\alpha\cdot\beta = 0 \bmod 2$, and

$$s_\alpha(\beta) = \beta + (\alpha\cdot\beta)\alpha \equiv \beta \bmod 2Q.$$

This implies that the roots β and $s_\alpha(\beta)$ correspond to the same vector from $\mathbb{F}_2^6$. By the previous argument this implies that

$$\beta+(\alpha\cdot\beta)\alpha = \beta \text{ , or } \beta+(\alpha\cdot\beta)\alpha = -\beta.$$

In the first case $(\alpha\cdot\beta) = 0$, and we are done. In the second case, $\alpha = -\frac{1}{2}(\alpha\cdot\beta)\beta$, which is absurd.

Let $L \subset \mathbb{F}_2^6$ be a maximal isotropic subspace (a <u>Göpel subspace</u> in old terminology). It contains 7 nonzero elements. Denote by $R(L)$ (the <u>Göpel subset</u>) the subset of the corresponding 7 orthogonal positive roots in Q. We use the notations $\alpha(i,j), \alpha(i,j,k), \alpha(i)$ to denote the positive roots

$$e_i-e_j,\ 1\le i<j\le 7;\ e_0-e_i-e_j-e_k,\ 1\le i<j<k\le 7,\ 2e_0-e_1-\ldots-e_7+e_i,\ i=1,\ldots,7$$

respectively.

There are 135 different Göpel subsets $R(L)$,

90 of type $\{\alpha(1,2,3),\alpha(1,4,5),\alpha(2,4,6),\alpha(3,5,6),\alpha(1,6,7),\alpha(2,6,7),\alpha(3,5,7)\}$

45 of type $\{\alpha(1),\alpha(1,2,3),\alpha(1,4,5),\alpha(1,6,7),\alpha(2,3),\alpha(4,5),\alpha(6,7)\}$.

For each Göpel set $R(L)$ we define a <u>Göpel function</u> $F_L \in (R_2^7)_3$:

90 of type $F_L = (123)(145)(246)(356)(167)(267)(357)$,

45 of type $F_L = d_1(123)(145)(167)$,

where

$$d_1 = (347)(567)(235)(246)-(357)(467)(234)(256)$$

expresses the condition that the point set $(x^2,\ldots,x^7)$ lies on a conic (see Chapter 1).

Proposition 9. The Göpel functions F_L span a 15-dimensional subspace T in $(R_2^7)_3$, and the group $Sp(6,\mathbb{F}_2)$ acts linearly on T via its action on Göpel subspaces L. The representation T of $Sp(6,\mathbb{F}_2)$ is irreducible. For every $\varepsilon\in\mathbb{F}_2^6\setminus\{0\}$, the set of all Göpel functions F_L with $\varepsilon\in L$ span a 5-dimensional subspace and satisfy a cubic relation.

<u>Proof</u>. Let $l \subset \mathbb{F}_2^6$ be an isotropic plane. It is easy to see that there are exactly 3 Göpel subspaces L which contain l. Let F_1, F_2 and F_3 be the correponding Göpel functions. We prove that they satisfy a linear relation of the type:

$$F_1 \pm F_2 \pm F_3 = 0.$$

Since $Sp(6,\mathbb{F}_2)$ acts transitively on the set of isotropic subspaces of the same dimension, it suffices to check it for one subspace l. Choose l in such a way that its three nonzero vectors correspond to the positive roots:

$$\alpha(1) = 2e_0-e_2-\ldots-e_7,\ \alpha(6,7) = e_6-e_7,\ \alpha(1,6,7) = e_0-e_1-e_6-e_7.$$

Then the three Göpel functions F_i are given by:

$$F_1 = d_1(167)(123)(145),\quad F_2 = d_1(167)(124)(135),\quad F_3 = d_1(167)(125)(134)$$

Factoring out $d_1(167)$, we have to verify that

$$(123)(145)-(124)(135)+(125)(134) = 0.$$

We consider the Göpel functions as functions on P_2^7. Without loss of generality we may assume that the point x^1 has coordinates (1,0,0). Then the functions (123)(145),(124),(135),(125)(134) satisfy the same relations as the monomials (23)(45), (24)(35), (25)(34) from R_1^4. This relation is the straightening relation:

$$(25)(34) = (24)(35)-(23)(45).$$

Let $V_1 \cong \mathbb{C}^{315}$ be the permutation representation of $Sp(6,\mathbb{F}_2)$ arising from its action on the set of isotropic planes l, and let $V_2 \cong \mathbb{C}^{135}$ be the similar representation corresponding to Göpel spaces. It is clear that the space T spanned by the Göpel functions F_L is a representation of $Sp(6,\mathbb{F}_2)$ isomorphic to a quotient of V_2 by the subspace isomorphic to V_1, the cokernel of the map

$$V_1 \to V_2,\ l \to L_1 \pm L_2 \pm L_3$$

of the representations. Decomposing V_1 and V_2 into irreducible representations we find that V_2 contains an irreducible representation of dimension 15 which is not isomorphic to an irreducible component of V_1. On the other hand, T is, as easily seen, isomorphic to an irreducible representation of the Weyl group $W(E_7)$ given by a construction from **[McD 2]**, cf. Remark 7 below. This shows that T is an irreducible representation of dimension 15.

Let $\varepsilon \in \mathbb{F}_2^6 \setminus \{0\}$, and α be a positive root in Q corresponding to ε. Without loss of generality we may assume that

$$\alpha = \alpha(7) = 2e_0-e_1-\dots-e_6.$$

Then each Göpel function F_L with $\varepsilon \in L$ has the form:

$$F_L = d_7(ij7)(kl7)(mn7),$$

where $i<j, k<l, m<n$, $\{1,\dots,6\} = \{i,j\} \amalg \{k,l\} \amalg \{m,n\}$. Factoring out d_7, we have to check that the functions (ij7)(kl7)(mn7) span a 5-dimensional subspace and satisfy a cubic relation. Following the same argument as above, we may assume that $x^7=(1,0,0)$. Then the functions (ij7)(kl7)(mn7) satisfy the same relations as the functions from $(R_1^6)_1$. It follows from Chapter 1 that they span a 5-dimensional space and define a map of P_1^6 onto a cubic 3-fold isomorphic to the Segre cubic primal. This proves the assertion.

Theorem 5. Let

$$\varphi\colon P_2^7 \dashrightarrow \mathbb{P}(T^*) = \mathbb{P}_{14}$$

be the $Sp(6,\mathbb{F}_2)$-equivariant rational map given by the linear system $|T|$. Then φ is defined outside the union of 35 subvarieties of dimension 3 representing point sets lying on two lines. Its image is isomorphic to a subvariety V defined by 63 cubic equations. Each of them corresponds to a cubic relationship between Göpel functions F_L with $\varepsilon \in L$, $\varepsilon \in \mathbb{F}_2^6$. The induced map $\varphi'\colon P_2^7 \dashrightarrow V$ is birational.

Proof. The assertion about the set of definition of φ is verified directly from the definition of Göpel functions. The only nontrivial assertion is the birationality of φ'. We have to show that a generic point set $(x^1,\dots,x^7)$ can be reconstructed uniquely from a point of V. Assume that the first 4 points are normalized in the usual way. Let (t_0,t_1,t_2) be the coordinates of x^5. Then we observe that

$$\frac{t_1}{t_2} = \frac{(124)(135)}{(125)(134)} = \frac{d_1(124)(135)(167)}{d_1(125)(134)(167)}$$

$$\frac{t_2}{t_0} = \frac{(125)(234)}{(124)(135)} = \frac{d_2(125)(234)(267)}{d_2(124)(235)(267)}$$

$$\frac{t_0}{t_1} = \frac{(134)(235)}{(135)(234)} = \frac{d_3(134)(235)(367)}{d_3(135)(234)(367)}.$$

We verify that the relation

$$\left(\frac{t_1}{t_2}\right)\left(\frac{t_2}{t_0}\right)\left(\frac{t_0}{t_1}\right) = 1$$

is equivalent to the cubic relation

$$(d_1(124)(135)(167))(d_2(125)(234)(267))(d_3(134)(235)(367))-$$

$$-(d_1(125)(134)(167))(d_2(124)(235)(267))(d_3(135)(234)(367))$$

between the Göpel functions. It turns out that this relation follows from the cubic relation between Göpel functions corresponding to the fixed root $\alpha(6,7)$. We refer to **[Co 1]**, p. 195 for this verification and also for the completion of this proof.

Note that the 15-dimensional irreducible representations T^* and $\mathcal{T}(3)$ of $Sp(6,\mathbb{F}_2)$ are isomorphic. Thus the image V of P_2^7 under the map φ given by Göpel functions and the image V' of $\bar{\mathfrak{a}}_2(2)$ under the map f given by the coefficients of the Coble quartics are birationally isomorphic $Sp(6,\mathbb{F}_2)$-invariant subvarieties of $\mathbb{P}(T^*) \cong \mathbb{P}(\mathfrak{T}(3))$. Let $\varepsilon \in \mathbb{F}_2^6 \setminus \{0\}$ and let G_ε be the isotropy subgroup of $Sp(6,\mathbb{F}_2)$ in its natural action on $\mathbb{F}_2^6$. Then

$$G_\varepsilon \cong W(D_6) \cong (\mathbb{Z}/2)^5 \rtimes \Sigma_6.$$

The restriction of T^* to G_ε contains a unique irreducible subrepresentation $L(\varepsilon)$ of dimension 5 which factors through Σ_6. The projective subspace $\mathbb{P}(L(\varepsilon))$ intersects V and V' along a variety isomorphic to the Segre cubic primal. This is the image of a boundary component of $\bar{\mathfrak{a}}_2(2)$ and of discriminant component of P_2^7 under f and φ respectively. Coble claims in **[Co 1]**, p.197, that V is projectively isomorphic to V'. Unfortunately, in our opinion his proof is not complete.

Remark 7. The construction of the representation T of $Sp(6,\mathbb{F}_2)$ is a special case of MacDonald's construction of some irreducible representations of Weyl groups (see **[McD 2]**). In fact, let $\mathfrak{h}$ be a Cartan algebra of a simple Lie algebra of type E_7. Consider every root α of $W(E_7)$ as a linear function on $\mathfrak{h}$. For every subset S of the set of positive roots which form a root basis in the root lattice Q denote

$$F_S = (\prod_{\alpha \in S} \alpha) \in Sym^{\#S}(\mathfrak{h}^*).$$

Let V_S be the subspace of $Sym^{\#S}(\mathfrak{h}^*)$ spanned by the functions $F_{w(S)}$, $w \in W(E_7)$. Then V_S is an irreducible representation of $W(E_7)$. If S consists of seven orthogonal positive roots, we obtain a representation isomorphic to T.

Let $\mathbb{P}_6 = \mathbb{P}(\mathfrak{h})$. There is a canonical $W(E_7)$-equivariant birational morphism

$$s: \mathbb{P}_6 \to (P_2^7)^{un}$$

which is obtained from the identification of $\mathbb{P}_6$ with the variety of projective equivalence classes of point sets $(x^1,\dots,x^7)$ lying in the set of nonsingular points of a fixed cuspidal cubic (cf. **[Pi]**). As is easy to see the pull-back of Göpel functions to $\mathbb{P}_6$ under the map s spans a subspace of the space of homogeneous polynomials of degree 7 which is isomorphic to the Macdonald representation V_S corresponding to a set S of seven orthogonal positive roots. Thus the map

$$\mathbb{P}(\mathfrak{h}) \to |V_S|^* = \mathbb{P}_{14}$$

given by the linear system $|V_S| \subset |\mathcal{O}_{\mathbb{P}_6}(7)|$ factors through the map

$$(P_2^7)^{un} \to \mathbb{P}_{14}$$

given by Göpel functions, and hence has the same image in $\mathbb{P}_{14}$. In particular we see that the Macdonald functions from V_S satisfy 315 three-term linear relations and also 63 cubic relations. This of course can be verified directly without using Göpel functions.

Since every net of cubics through a generic set of 7 points in $\mathbb{P}_2$ contains exactly 21 cuspidal cubics, the map s is of degree 21. By the Chevalley theorem (**[Bo]**):

$$\mathbb{P}_6/W(E_7) \cong \mathbb{P}_6.$$

This defines a rational map of degree 21:

$$\mathbb{P}_6 \to (P_2^7)^{un}/W(E_7) \underset{bir}{\cong} \mathfrak{a}_3(2)/Sp(6,F_2) \underset{bir}{\cong} \mathfrak{M}_3.$$

This is a rational map from $\mathbb{P}_6$ to $\mathfrak{M}_3$ of smallest degree known so far.

8. Final remarks.

The relations between point sets in $\mathbb{P}_n$ and moduli varieties of curves goes a little further. It is easy to extend some of the results from this chapter to the case of curves of genus 4 with a vanishing theta constant. We obtain that the moduli variety of such curves is isomorphic to $(P_2^8)^{un}$. One of the possible relationships between these varieties is seen via Del Pezzo surfaces of degree 1. (see **[Co 1]**, Chapter V). There must also be some interesting connections between some special point sets in the sense of Chapter VI and certaion moduli varieties of curves. Some indications to this can be found in Chapter VI of Coble's book. Further connections have still to be explored. The call of Coble (see the last page of **[Co 1]**) to find curves of genus larger than 4 associated to some point sets is still unanswered.

We refer to **[C-D 3]**, **[Cos]**, where some other interesting relations between Cayley decads and Enriques surfaces are discussed.

In **[Ki 1, Ki 2]** F. Kirwan gives a method for computation of Betti numbers of the orbit spaces. This can be applied to our spaces P_n^m or $\hat{P}_n^m$.

Many topics from Coble's book have not been covered in these notes. One of the reasons for this is our failure to fully understand what is going on there. For example, Coble gives some formulas, due to Schottky, which express the Göpel functions in terms of theta constants (**[Co 1]**,S28). The derivation of these formulae looks rather formal and is not very illuminating. Note also that Coble omits many other interesting developments closely related to the topic of his book. For example, he does not even mention Frobenius work **[Fr 2]** in which an equation of a canonical curve of genus 3 is given explicitly in terms of theta constants or in terms of the equations of an Aronhold set of bitangents. A modern exposition of a part of this work is given in **[vG-vdG]**. We hope to return to all of this later.

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Notations

Index

Dans ce volume sont traités les liens entre la géométrie algébrique classique et la théorie des invariants des ensembles finis ordonnés de points dans les espaces projectifs, des transformations de Cremona et des fonctions thêta. La majeure partie du contenu se trouve dans la littérature, notamment dans le livre de A. Coble intitulé "Algebraic geometry and theta functions". Néanmoins nous traitons ici ce sujet d'un point de vue moderne. On y a inclus les discussions des constructions classiques de l'ensemble des 27 droites d'une surface cubique, de l'ensemble des 28 bitangentes à une courbe plane quartique, des surfaces de Kummer et de del Pozzo ainsi que de leurs analogues en dimensions supérieures, des réseaux de quadriques et des surfaces dianodes de Cayley associées, des involutions birationnelles de Bertini et Geyser. Tout ceci est relié à des sujets plus récents tels que les quotients géométriques, les tableaux standards, les systèmes de racines infinis et leur groupe de Weyl, les représentations de groupes, les groupes d'automorphismes des surfaces rationnelles, les espaces de modules des variétés abéliennes, etc.

This volume is concerned with some topics in classical algebraic geometry concentrated around the theory of invariants of finite ordered point sets in projective spaces, Cremona transformations and theta functions. Most of the material can be found in classical literature, and especially, in a book of A.Coble, however we treat this subject from a modern point of view. Among other things it discusses some famous classical constructions like the set of 27 lines on a cubic surface , the set of 28 bitangents to a plane quartic curve, Del Pezzo and Kummer surfaces and their higher-dimensional analogs, nets of quadrics and Cayley dianode surfaces associated to them, Bertini and Geiser birational involutions. This is interrelated with such modern topics as the geometric quotients, standard tableaux, infinite root systems and their Weyl groups, group representations, automorphism groups of rational surfaces, moduli spaces of abelian varieties and others.